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APPROXIMATED EXPONENTIAL INTEGRATORS FOR THE STOCHASTIC MANAKOV EQUATION

André Berg[1], David Cohen[*2] and Guillaume Dujardin[3]

[1]Department of Mathematics and Mathematical Statistics,
Umeå University
90187 Umeå, Sweden

[2]Department of Mathematical Sciences,
Chalmers University of Technology and University of Gothenburg,
41296 Gothenburg, Sweden

[3]Univ. Lille, Inria, CNRS, UMR 8524 - Laboratoire Paul Painlevé,
F-59000 Lille, France

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Abstract. This article presents and analyzes an approximated exponential integrator for the (inhomogeneous) stochastic Manakov system. This system of SPDE occurs in the study of pulse propagation in randomly birefringent optical fibers. For a globally Lipschitz-continuous nonlinearity, we prove that the strong order of the time integrator is $1/2$. This is then used to prove that the approximated exponential integrator has convergence order $1/2$ in probability and almost sure order $1/2^-$, in the case of the cubic nonlinear coupling which is relevant in optical fibers. Finally, we present several numerical experiments in order to support our theoretical findings and to illustrate the efficiency of the approximated exponential integrator as well as a modified version of it.

1. **Introduction.** Optical fibers play an important role in our modern communication society [1]. In order to model the light propagation over long distance in randomly varying birefringent optical fibers, the Manakov PMD equation was derived from Maxwell's equations in [34]. As noted in [20], polarization mode dispersion (PMD) is one of the main limiting effects of high bit rate transmission in optical fiber links. In addition, the work [20] proves that the asymptotic dynamics of the Manakov PMD equation is given by a stochastic nonlinear evolution equation in the Stratonovich sense: the stochastic Manakov equation, see below. In the present article, we perform a numerical analysis of this stochastic partial differential equation (SPDE).

Let us now briefly survey relevant works on time discretizations of the stochastic Manakov equation. The work [22], see also [21], investigates numerically the effect of noise on Manakov (wave-train) solitons by the following numerical schemes: the nonlinearly implicit Crank–Nicolson scheme, the linearly implicit relaxation

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* Corresponding author: David Cohen.

scheme, and a Fourier split-step scheme (Lie–Trotter splitting). For instance, it is conjectured that, in the small-noise regime and over short distances, solitons are not strongly destroyed and are stable. The Crank–Nicolson scheme has been proved to converge in probability with order $1/2$, see [23] and also [21]. Like the exact solution to the stochastic Manakov equation, this numerical integrator was shown to preserve the $\mathbb{L}^2$-norm. Additionally, in these references, the relaxation scheme and the split-step scheme were numerically shown to almost-surely converge with order $1/2$. A theoretical analysis of the splitting scheme was very recently presented in the companion paper [6], see below for further details.

Exponential integrators for the time integration of deterministic or stochastic (partial) differential equations are nowadays widely used and studied as witnessed by the recent works [12, 19, 28, 26, 14, 29, 15, 35, 16, 3, 13, 2, 4, 9] and references therein. In particular, exponential integrators have been analyzed when applied to (stochastic) Schrödinger equations, see [8, 7, 12, 26, 19, 14, 13, 9, 2, 27] for instance.

As the stochastic Schrödinger equation is closely related to the stochastic Manakov equation, our main goal will thus be to present and analyze a linearly implicit approximated exponential integrator for the time discretization of the stochastic Manakov equation. The main results of this paper can be summarized as follow:

- For the stochastic Manakov equation with a globally Lipschitz continuous nonlinearity, Theorem 3.2 below states that the proposed time integrator converges with order $1/2$ in the $L^p(\Omega, \mathbb{H}^1(\mathbb{R};\mathbb{C}))$ sense for all $p \in [1,\infty)$, where $\mathbb{H}^1$ denotes a Sobolev space, see the next section for definitions of these spaces.
- For the stochastic Manakov equation with a cubic nonlinearity, we prove convergence with order $1/2$ in probability and with order $1/2^-$ in the almost sure sense. These results are stated in Proposition 3.3 and Proposition 3.4.
- The main drawback of the proposed time integrator is that it does not preserve the $\mathbb{L}^2$-norm, see the next section and Section 4. This fact was already observed for classical exponential integrators when applied to the deterministic cubic Schrödinger equation [12] or [9]. Inspired by the works [12, 13], we modify the proposed time integrator and obtain an implicit-in-time conservative version of it for stochastic Manakov problems where the $\mathbb{L}^2$-preservation is an issue, see Proposition 4.1.

Let us add two additional benefits of the proposed time integrator:

- Beside having the same orders of convergence as the nonlinearly implicit Crank–Nicolson scheme studied in [23], the proposed time integrator is only linearly implicit. Therefore, in addition to being easy to implement, the time integrator introduced and analyzed below has a low computational cost for the stochastic Manakov equation, in comparison, for example, to the Crank–Nicolson scheme, see Section 4.
- At this point, one should mention another popular strategy to derive efficient time integrators for (stochastic) Schrödinger equations, namely a splitting strategy. In essence, the main principle of splitting integrators is to decompose the vector field of the original differential equation in several parts, such that the arising subsystems are exactly integrated, see for instance [30]. In [22, 21, 23], a Fourier split-step scheme for the stochastic Manakov equation has been numerically investigated. A theoretical analysis of this splitting scheme was very recently presented in the companion paper [6]. Note also that such a splitting strategy is suitable in the case of a cubic (or, more generally, power law) nonlinearity, because the nonlinear deterministic part of the equation

is explicitly integrable. However, it may fail to be so easy to use as the approximated exponential integrator studied in the present work. This is for example the case, when considering inhomogeneous fibers resulting in non-autonomous Manakov systems. We discuss these issues in more details in Section 4.

We conclude the introduction by noting that splitting and exponential-type integrators share similarities in their design, implementation and analysis.

2. **Setting and notation.**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a three-dimensional standard Brownian motion $W(t) := (W_1(t), W_2(t), W_3(t))$ is defined. We endow the probability space with the complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by $W(t)$. In the present paper, we consider the nonlinear stochastic Manakov system [23]

$$i\mathrm{d}X + \partial_x^2 X\,\mathrm{d}t + |X|^2 X\,\mathrm{d}t + i\sqrt{\gamma}\sum_{k=1}^{3}\sigma_k\partial_x X \circ \mathrm{d}W_k = 0, \tag{1}$$

where $X = X(t,x) = (X^1, X^2)$ is the unknown function with values in $\mathbb{C}^2$ with $t \geq 0$ and $x \in \mathbb{R}$, the symbol $\circ$ denotes the Stratonovich product, $\gamma \geq 0$ measures the intensity of the noise, the nonlinear coupling is denoted by $|X|^2 = |X^1|^2 + |X^2|^2$, and σ_1, σ_2 and σ_3 are the classical Pauli matrices defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The mild form of the stochastic Manakov system (1) reads

$$X(t) = U(t,0)X^0 + i\int_0^t U(t,s)F(X(s))\,\mathrm{d}s, \tag{2}$$

where X^0 denotes the initial value of the problem, $U(t,s)$ for $t \geq s$ with $s,t \in \mathbb{R}_+$ is the random unitary propagator defined as the unique solution to the linear part of (1), and $F(X) = |X|^2 X$.

Let $p \geq 1$. We define $\mathbb{L}^p := \mathbb{L}^p(\mathbb{R}) := (L^p(\mathbb{R};\mathbb{C}))^2$ the Lebesgue spaces of functions with values in $\mathbb{C}^2$. We equip $\mathbb{L}^2$ with the real scalar product $(u,v)_2 = \sum_{j=1}^{2} \operatorname{Re}\left(\int_{\mathbb{R}} u_j \overline{v}_j\,\mathrm{d}x\right)$. Further, for $m \in \mathbb{N}$, we denote $\mathbb{H}^m := \mathbb{H}^m(\mathbb{R})$ the space of functions in $\mathbb{L}^2$ with their m first derivatives in $\mathbb{L}^2$. The norm in $\mathbb{H}^m$ is denoted by $\|\cdot\|_m = \|\cdot\|_{\mathbb{H}^m} = \|\cdot\|_{\mathbb{H}^m(\mathbb{R})}$.

We now recall the local existence and uniqueness result for solutions to (1) obtained in [18] (see also [21]).

Theorem 2.1 (Theorem 1.2 in [18])**.** *Consider the initial value $X^0 \in \mathbb{H}^1$, then there exists a maximal stopping time $\tau^*(X^0,\omega)$ and a unique solution X (in the probabilistic sense) to (1) such that $X \in C([0,\tau^*[, \mathbb{H}^1)$ $\mathbb{P}$-a.s. Furthermore, the $\mathbb{L}^2$-norm is almost surely preserved: $\|X(t)\|_{\mathbb{L}^2} = \left\|X^0\right\|_{\mathbb{L}^2}$ for $t \in [0,\tau^*[$. Moreover, the following alternative holds for the maximal existence time of solutions to (1):*

$$\tau^*(X^0,\omega) = +\infty \quad \text{or} \quad \limsup_{t \nearrow \tau^*(X^0,\omega)} \|X(t)\|_{\mathbb{H}^1} = +\infty.$$

Finally, if the initial value X^0 belongs to $\mathbb{H}^m$ for some $m \geq 1$, then the corresponding solution also belongs to $\mathbb{H}^m$ almost surely.

As seen above, the $\mathbb{L}^2$-norm of the solution is preserved just as for the deterministic Manakov equation (i. e. (1) with $\gamma = 0$). Furthermore, as noted by [23], the occurrence of blow-up in the stochastic Manakov equation (1) remains an open question.

One has to face two issues when discretizing the stochastic Manakov system (1) in time. First, the random unitary propagator $U(t,s)$, generated by the linear part of the problem, is not easy to compute exactly. This is because the Pauli matrices do not commute and thus, $U(t,s)$ is not the product of the stochastic semi-groups associated to each Brownian motion with the group generated by $i\partial_x^2$. Second, the nonlinear coupling term $|X|^2X$ present in the SPDE (1) is generally dealt with an implicit discretization, as in the Crank–Nicolson scheme from [23]. Such implicit discretizations result in an increase of computational costs.

Therefore, we propose to discretize the stochastic Manakov equation with an approximated exponential integrator, that we now define. Let $T > 0$ be a fixed time horizon and consider an integer $N \geq 1$. We define the step size by $h = T/N$ and denote discrete times by $t_n = nh$, for $n = 0, \ldots, N$. Discretizing the integral present in the mild form (2) (by an explicit Euler step) as well as the random propagator (by a midpoint rule), one gets the following approximated exponential integrator

$$X^{n+1} = U_{h,n}\left(X^n + ihF(X^n)\right), \tag{3}$$

where $U_{h,n} = \left(Id + \frac{1}{2}H_{h,n}\right)^{-1}\left(Id - \frac{1}{2}H_{h,n}\right)$, with Id is the identity operator and $H_{h,n} = -ihI_2\partial_x^2 + \sqrt{\gamma h}\sum_{k=1}^{3}\sigma_k\chi_k^n\partial_x$. Here, I_2 is the 2×2 identity matrix and $\sqrt{h}\chi_k^n = W_k((n+1)h) - W_k(nh)$, for $k = 1,2,3$, are i.i.d. Wiener increments. The approximated exponential integrator (3) thus approximates solutions to the stochastic Manakov equation (1), $X(t_n) \approx X^n$, at the grid points $t_n = nh$.

Since the operator $U_{h,n}$ is an approximation of the exponential of the linear random differential operator in (1), we choose to name the scheme (3) approximated exponential integrator. Observe that the operator $H_{h,n}$ is skew-symmetric. This implies that $Id + \frac{1}{2}H_{h,n}$ is always one-to-one, no matter the choices of $h > 0$ and the (real) increments $\sqrt{h}\chi_k^n$, for $k = 1,2,3$. Furthermore, the operator $U_{h,n}$ is almost-surely unitary from $\mathbb{H}^m$ to $\mathbb{H}^m$, as is proved in the Appendix in Section 5. In particular, when implementing the approximated exponential integrator, one does not need to truncate the random variables χ_k^n. Observe that, unlike for example the Lie–Trotter splitting scheme analyzed in [6], the approximated exponential integrator above does not preserve the $\mathbb{L}^2$-norm when applied to (1). However, unlike the Lie–Trotter splitting scheme, which relies on the exact integration of the nonlinear part of (1), the approximated exponential integrator (3) may also be applied to a wider class of Manakov systems, for which the convergence results of this paper are still valid. Some examples are Manakov systems including damping and balance terms (see [32] for example) as well as those with inhomogeneities along the evolution variable (see [31]): In these cases, such a direct integration is not always possible and the $\mathbb{L}^2$-norm preservation is not true at the continuous level, hence not the main issue for a numerical scheme. An illustration of this is presented in the numerical experiments below.

Iterating the recursion (3), one gets the discrete mild form for the approximated exponential scheme

$$X^n = \mathcal{U}_h^{n,0} X^0 + ih \sum_{\ell=0}^{n-1} \mathcal{U}_h^{n,\ell} F(X^\ell), \tag{4}$$

where $\mathcal{U}_h^{n,\ell} := U_{h,n-1} \cdot \ldots \cdot U_{h,\ell}$.

Thanks to Proposition 5.1, this linearly implicit method is well-defined for all $n \geq 0$, and one has that for all $n \in \mathbb{N}$, $X^n \in \mathbb{H}^1$ (respectively $\mathbb{H}^2$, resp $\mathbb{H}^6$) provided that $X^0 \in \mathbb{H}^1$ (resp. $\mathbb{H}^2$, resp. $\mathbb{H}^6$). Moreover, if one assumes that F is bounded by M on $\mathbb{H}^1$, then one has almost surely for all $n \in \mathbb{N}$ and $h > 0$ such that $nh \leq T$, $\|X^n\|_1 \leq \|X^0\|_1 + TM$.

In the following, in the proofs of our results, we denote by C a positive constant that may change from one line to the other, but that does not depend on the parameters indicated in the results' statements.

3. **Convergence analysis of the approximated exponential integrator.**

This section presents the main results of the article and gives the corresponding proofs. We start by considering the stochastic Manakov equation (2), where the nonlinearity F is assumed to be globally Lipschitz-continuous. We show strong order of convergence $1/2$ for the approximated exponential scheme (3) in that case. Then, we analyze the case of a cubic nonlinearity, i. e. $F(X) = |X|^2X$, which is of course *not* globally Lipschitz-continuous, and we show order of convergence in probability $1/2$, as well as order of convergence $1/2^-$ almost surely, for the approximated exponential scheme (3). The main steps of the proofs use similar arguments as in [23, 5, 13, 6] as well as other works on the numerical analysis of SPDEs. However, some technical details are handled differently in this paper (see for example the estimation of J_3^n and J_4^n below).

3.1. **The Lipschitz-continuous case.**

We present a strong convergence analysis of the approximated exponential integrator (3) when applied to the stochastic Manakov equation (2) when F is globally Lipschitz-continuous on $\mathbb{H}^1$. This can be done, for example, by introducing a cut-off function for the cubic nonlinearity in (1): Let $R > 0$ and $\theta \in \mathcal{C}^\infty(\mathbb{R}_+)$, with $\theta \geq 0$, $\text{supp}(\theta) \subset [0, 2]$ and $\theta \equiv 1$ on $[0, 1]$. For $x \geq 0$, we set $\theta_R(x) = \theta(\frac{x}{R})$ and define $F_R(X) = \theta_R(\|X\|^2_{\mathbb{H}^1})|X|^2X$. We thus obtain a bounded globally Lipschitz-continuous function F_R from $\mathbb{H}^1$ to $\mathbb{H}^1$, which sends bounded subsets of $\mathbb{H}^2$ to bounded subsets from $\mathbb{H}^2$, resp. of $\mathbb{H}^6$ to $\mathbb{H}^6$. For ease of presentation, we denote the stochastic processes $X(t)$ and X^n solutions to the continuous problem and to the discrete problem, instead of using the notation $X_R(t)$ and X^n_R that we will use later in the paper, to point out the difference between truncated and untruncated problems and solutions.

For the convenience of the readers, we recall some useful results from [21, 23].

Lemma 3.1.

Regularity property of the semi-group S associated to the linear Schrödinger equation $\partial_t Y(t) = C_\gamma \partial_x^2 Y(t)$, with $C_\gamma = i + \frac{3}{2}\gamma$, similar to [21, Proof of Lemma 4.2.1]*: For all $\gamma \geq 0$, there exists a $C(\gamma) > 0$ such that for all $m \in \mathbb{N}$ and $f \in \mathbb{H}^{m+1}$, one has for all $t \geq 0$*

$$\|(S(t) - Id)\, f\|_{\mathbb{H}^m} \leq C(\gamma) t^{1/2} \|f\|_{\mathbb{H}^{m+1}}\,.$$

Uniform boundedness of the truncated solution, see [23, Lemma 5.3]*: Let $X^0 \in \mathbb{H}^6$, then for all $R, T > 0$ there exists a positive constant $C\left(R, T, \|X^0\|_{\mathbb{H}^6}\right)$ such* *that a.s. for every* $t \in [0,T]$*:*

$$\|X_R(t)\|_{\mathbb{H}^6} \leq C\left(R,T,\left\|X^0\right\|_{\mathbb{H}^6}\right).$$

Moreover, the function $T \mapsto C\left(R,T,\left\|X^0\right\|_{\mathbb{H}^6}\right)$ *is a continuous function from* $\mathbb{R}_+$ *to* $\mathbb{R}_+$ *and then is bounded on every compact set of* $\mathbb{R}_+$*. We denote by* $\tilde{C}\left(R,T,\left\|X^0\right\|_{\mathbb{H}^6}\right)$ *a constant such that, a.s. for every* t *in* $[0,T]$*,* $\|F(X_R(t))\|_{\mathbb{H}^6} \leq \tilde{C}\left(R,T,\left\|X^0\right\|_{\mathbb{H}^6}\right)$*.*

Temporal regularity of the truncated solution, see [23, Lemma 5.4]*: For any* $p \geq 1$*,* $R > 0$*, if* $X^0 \in \mathbb{H}^2$*, then there exists a positive constant* C *such that for all* $\ell = 1,\dots,N$*, one has*

$$\mathbb{E}\left[\sup_{t_{\ell-1}\leq t\leq t_\ell} \|X_R(t)-X_R(t_{\ell-1})\|_{\mathbb{H}^1}^{2p}\right] \leq C\left(R,T,p,\gamma,\left\|X^0\right\|_{\mathbb{H}^2}\right)h^p.$$

Theorem 3.2. *Let* $T \geq 0$*,* $p \geq 1$*, and* $X^0 \in \mathbb{H}^6$*. Consider a bounded Lipschitz nonlinearity* F*, defined as above, in the stochastic Manakov equation* (2)*. Then, there exist a constant* $C = C(F,\gamma,T,p,\left\|X^0\right\|_{\mathbb{H}^6})$ *such that the approximated exponential integrator* (3) *has strong order of convergence* $1/2$*: There exists* $h_0 > 0$ *such that*

$$\forall h \in (0,h_0), \qquad \mathbb{E}\big[\max_{n=0,1,\dots,N}\|X^n - X(t_n)\|_{\mathbb{H}^1}^{2p}\big] \leq Ch^p.$$

Proof. The proof relies on a suitable decomposition of the error in five terms. Then, a bound on each term is obtained using estimates and inequalities such as that recalled in Lemma 3.1 to obtain a "short time" error analysis. In the end, the estimate is extended to large times. Let us denote the difference $X^n - X(t_n)$ by e^n. Using the definitions of the numerical and exact solutions, we thus obtain

$$\begin{aligned}
\|e^n\|_1 &= \left\|\mathcal{U}_h^{n,0}X^0 + ih\sum_{\ell=0}^{n-1}\mathcal{U}_h^{n,\ell}F(X^\ell) - U(t_n,0)X^0 - i\int_0^{t_n} U(t_n,s)F(X(s))\,\mathrm{d}s\right\|_1 \\
&\leq \left\|\left(\mathcal{U}_h^{n,0} - U(t_n,0)\right)X^0\right\|_1 \\
&\quad + \left\|\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}\left(\mathcal{U}_h^{n,\ell}F(X^\ell) - U(t_n,s)F(X(s))\right)\mathrm{d}s\right\|_1 \\
&=: I_1^n + I_2^n.
\end{aligned}$$

We estimate the term I_2^n first using

$$\begin{aligned}
I_2^n = &\left\|\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}\Big(\mathcal{U}_h^{n,\ell}F(X^\ell) - \mathcal{U}_h^{n,\ell}F(X(t_\ell)) + \mathcal{U}_h^{n,\ell}F(X(t_\ell)) - \mathcal{U}_h^{n,\ell}F(X(s))\right. \\
&\left. + \mathcal{U}_h^{n,\ell}F(X(s)) - U(t_n,t_\ell)F(X(s)) + U(t_n,t_\ell)F(X(s)) - U(t_n,s)F(X(s))\Big)\,\mathrm{d}s\right\|_1 \\
\leq &\left\|\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}\mathcal{U}_h^{n,\ell}\left(F(X^\ell) - F(X(t_\ell))\right)\mathrm{d}s\right\|_1 \\
&+ \left\|\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}\mathcal{U}_h^{n,\ell}\left(F(X(t_\ell)) - F(X(s))\right))\,\mathrm{d}s\right\|_1 \\
&+ \left\|\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}\left(\mathcal{U}_h^{n,\ell} - U(t_n,t_\ell)\right)F(X(s))\,\mathrm{d}s\right\|_1
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}\left(U(t_n,t_\ell)-U(t_n,s)\right)F(X(s))\,\mathrm{d}s\right\|_1\\
&=: J_1^n+J_2^n+J_3^n+J_4^n.
\end{aligned}
$$

We next bound the expectation of each of the four terms above to the power $2p$. Using the fact that the nonlinearity F is globally Lipschitz-continuous from $\mathbb{H}^1$ to $\mathbb{H}^1$, and that $\mathcal{U}_h^{n,\ell}$ is unitary on all $\mathbb{H}^s$ (see Appendix 5), the first term can be estimated as follows

$$
\begin{aligned}
&\mathbb{E}\left[\max_{n=0,1,\dots,N}(J_1^n)^{2p}\right]\\
&\leq \mathbb{E}\left[\max_{n=0,1,\dots,N}\left(\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}\mathrm{d}s\left\|\mathcal{U}_h^{n,\ell}\left(F(X^\ell)-F(X(t_\ell))\right)\right\|_1\right)^{2p}\right]\\
&\leq CT^{2p}\mathbb{E}\left[\max_{\ell=0,1,\dots,N}\left\|X^\ell-X(t_\ell)\right\|_1^{2p}\right]=CT^{2p}\mathbb{E}\left[\max_{\ell=0,1,\dots,N}\left\|e^\ell\right\|_1^{2p}\right].
\end{aligned}
$$

Similarly, for the second term we obtain

$$
\begin{aligned}
\mathbb{E}\left[\max_{n=0,1,\dots,N}(J_2^n)^{2p}\right]&\leq C\mathbb{E}\left[\max_{n=0,1,\dots,N}\left(\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}\|X(t_\ell)-X(s)\|_1\,\mathrm{d}s\right)^{2p}\right]\\
&\leq C\mathbb{E}\left[\left(\sum_{\ell=0}^{N-1}\sup_{t_\ell\leq s\leq t_{\ell+1}}\|X(t_\ell)-X(s)\|_1\,h\right)^{2p}\right].
\end{aligned}
$$

Using Hölder's inequality, one then gets

$$
\begin{aligned}
&\mathbb{E}\left[\max_{n=0,1,\dots,N}(J_2^n)^{2p}\right]\\
&\leq Ch^{2p}\mathbb{E}\left[\left(\sum_{\ell=0}^{N-1}1^{2p/(2p-1)}\right)^{2p-1}\sum_{\ell=0}^{N-1}\left(\sup_{t_\ell\leq s\leq t_{\ell+1}}\|X(t_\ell)-X(s)\|_1\right)^{2p}\right]\\
&\leq Ch^{2p}N^{2p-1}\sum_{\ell=0}^{N-1}\mathbb{E}\left[\sup_{t_\ell\leq s\leq t_{\ell+1}}\|X(t_\ell)-X(s)\|_1^{2p}\right]\leq Ch^{2p}N^{2p}h^p\leq Ch^p,
\end{aligned}
$$

where we have used the estimate from Lemma 3.1 (temporal regularity of the mild solution).

Using Lemma 3.1 (uniform boundedness of the mild solution in $\mathbb{H}^6$), the fact that F sends bounded sets of $\mathbb{H}^6$ to bounded sets of $\mathbb{H}^6$, and Proposition 2.2 (strong convergence for the linear stochastic Manakov equation) in [23], we conclude that one has

$$
\forall s\in[0,T],\qquad \mathbb{E}\left(\max_{n\in\{0,\dots,N\}}\max_{\ell\in\{0,\dots,n\}}\left\|\left(\mathcal{U}_h^{n,\ell}-U(t_n,t_\ell)\right)F(X(s))\right\|_1^{2p}\right)\leq Ch^p.
$$

Therefore, we estimate the third term, using Hölder's inequality, as follows

$$
\begin{aligned}
&\mathbb{E}\left[\max_{n=0,1,\dots,N}(J_3^n)^{2p}\right]\\
&\leq \mathbb{E}\left[\max_{n=0,1,\dots,N}\left(\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}\left\|\left(\mathcal{U}_h^{n,\ell}-U(t_n,t_\ell)\right)F(X(s))\right\|_1\mathrm{d}s\right)^{2p}\right]
\end{aligned}
$$

$$\begin{aligned}
&\leq \mathbb{E}\left[\left(\int_0^T \max_{n=0,1,\dots,N}\max_{\ell=0,\dots,n}\left\|\left(\mathcal{U}_h^{n,\ell}-U(t_n,t_\ell)\right)F(X(s))\right\|_1 \mathrm{d}s\right)^{2p}\right]\\
&\leq CT^{2p-1}\mathbb{E}\left[\int_0^T \max_{n=0,1,\dots,N}\max_{\ell=0,\dots,n}\left\|\left(\mathcal{U}_h^{n,\ell}-U(t_n,t_\ell)\right)F(X(s))\right\|_1^{2p} \mathrm{d}s\right]\\
&\leq CT^{2p-1}\int_0^T \mathbb{E}\left[\max_{n=0,1,\dots,N}\max_{\ell=0,1,\dots,n}\left\|\left(\mathcal{U}_h^{n,\ell}-U(t_n,t_\ell)\right)F(X(s))\right\|_1^{2p}\right]\mathrm{d}s\\
&\leq Ch^p.
\end{aligned}$$

To bound the last term, we first use the isometry property of the continuous random propagator and Hölder's inequality to get

$$\begin{aligned}
&\mathbb{E}\left[\max_{n=0,1,\dots,N}(J_4^n)^{2p}\right]\\
&\leq \mathbb{E}\left[\max_{n=0,1,\dots,N}\left(\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}\|(U(t_n,t_\ell)-U(t_n,s))\,F(X(s))\|_1\,\mathrm{d}s\right)^{2p}\right]\\
&\leq \mathbb{E}\left[\max_{n=0,1,\dots,N}\left(\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}1\times\|(Id-U(s,t_\ell))F(X(s))\|_1\,\mathrm{d}s\right)^{2p}\right]\\
&\leq Ch^{2p}\mathbb{E}\left[\left(\left(\sum_{\ell=0}^{N-1}1^{\frac{2p}{2p-1}}\right)^{\frac{2p-1}{2p}}\left(\sum_{\ell=0}^{N-1}\sup_{t_\ell\leq s\leq t_{\ell+1}}\|(Id-U(s,t_\ell))\,F(X(s))\|_1^{2p}\right)^{\frac{1}{2p}}\right)^{2p}\right]\\
&\leq Ch^{2p}N^{2p-1}\sum_{\ell=0}^{N-1}\mathbb{E}\left[\sup_{t_\ell\leq s\leq t_{\ell+1}}\|(Id-U(s,t_\ell))\,F(X(s))\|_1^{2p}\right]. \qquad (5)
\end{aligned}$$

In order to estimate the expectation above, we write this term as

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t_\ell\leq s\leq t_{\ell+1}}\|(Id-U(s,t_\ell))\,(F(X(t_\ell))-F(X(t_\ell))+F(X(s)))\|_1^{2p}\right]\\
&\quad\leq C\mathbb{E}\left[\sup_{t_\ell\leq s\leq t_{\ell+1}}\|(Id-U(s,t_\ell))\,F(X(t_\ell))\|_1^{2p}\right]\\
&\qquad+C\mathbb{E}\left[\sup_{t_\ell\leq s\leq t_{\ell+1}}\|(Id-U(s,t_\ell))\,(F(X(s))-F(X(t_\ell)))\|_1^{2p}\right]. \qquad (6)
\end{aligned}$$

The first term in the equation above, $(Id-U(s,t_\ell))\,F(X(t_\ell))$, is the exact solution to the linear SPDE $i\mathrm{d}Z(t)+\dfrac{\partial^2 Z(t)}{\partial x^2}\mathrm{d}t+i\sqrt{\gamma}\sum_{k=1}^{3}\sigma_k\dfrac{\partial Z(t)}{\partial x}\circ\mathrm{d}W_k(t)=0$ with initial value $F(X(t_\ell))$ at initial time t_ℓ which has the mild Ito form

$$Z(t)-F(X(t_\ell))=(S(t-t_\ell)-Id)\,F(X(t_\ell))+i\sqrt{\gamma}\sum_{k=1}^{3}\int_{t_\ell}^{t}S(t-u)\sigma_k\partial_x Z(u)\,\mathrm{d}W_k(u),$$

where $S(t)$ is the semi-group solution to the Schrödinger equation $\partial_t Y=C_\gamma\partial_x^2 Y$, where $C_\gamma=i+\frac{3\gamma}{2}$. Owing at the regularity property of the group S (Lemma 3.1), the fact that the exact solution X is almost surely bounded in $\mathbb{H}^2$, that F sends bounded sets of $\mathbb{H}^2$ to bounded sets of $\mathbb{H}^2$, and Burkholder–Davis–Gundy's inequality (for the second term), one obtains the following bound for the first term in (6)

$$\mathbb{E}\left[\sup_{t_\ell\leq s\leq t_{\ell+1}}\|(Id-U(s,t_\ell))\,F(X(t_\ell))\|_1^{2p}\right]\leq Ch^p.$$

Using the fact that the random propagator U is unitary, that F is globally Lipschitz-continuous, and the regularity property of the exact solution X, one gets the estimate

$$\mathbb{E}\left[\sup_{t_\ell\leq s\leq t_{\ell+1}}\|(Id-U(s,t_\ell))\,(F(X(s))-F(X(t_\ell)))\|_1^{2p}\right]$$
$$\leq C\mathbb{E}\left[\sup_{t_\ell\leq s\leq t_{\ell+1}}\|F(X(s))-F(X(t_\ell))\|_1^{2p}\right]\leq Ch^p,$$

for the second term in (6).

Using all these estimates, we arrive at the bound

$$\mathbb{E}\left[\max_{n=0,1,\ldots,N}(J_4^n)^{2p}\right]\leq Ch^{2p}N^{2p-1}\sum_{\ell=0}^{N-1}h^p\leq Ch^p.$$

Altogether we thus obtain

$$\mathbb{E}\left[\max_{n=0,1,\ldots,N}\|e^n\|_1^{2p}\right]\leq C\mathbb{E}\left[\max_{n=0,1,\ldots,N}(I_1^n)^{2p}\right]+CT^{2p}\mathbb{E}\left[\max_{n=0,1,\ldots,N}\|e^n\|_1^{2p}\right]+Ch^p$$
$$\leq Ch^p+CT^{2p}\mathbb{E}\left[\max_{n=0,1,\ldots,N}\|e^n\|_1^{2p}\right],$$

using once again [23, Proposition 2.2].

For $T=T_1$ small enough, i. e. such that $CT_1^{2p}<1$, the inequality above gives

$$\mathbb{E}\left[\max_{n=0,1,\ldots,N}\|e^n\|_1^{2p}\right]\leq\frac{C}{1-CT_1^{2p}}h^p,$$

on $[0,T_1]$. Then, by a classical argument, we can split the interval $[0,T]$ in a finite number M of intervals of size T_1 sufficiently small in the sense above. Since the error constant above, that we denote by C_E, can be used repeatedly and since the exact flow of (1) is locally Lipschitz from $\mathbb{H}^1$ to itself (with constant C_L on the appropriate ball centered at the origin), we infer that

$$\mathbb{E}\left[\max_{n=0,1,\ldots,N}\|e^n\|_1^{2p}\right]\leq C_Eh^p+C_LC_Eh^p+C_L^2C_Eh^p+\ldots+C_L^{M-1}C_Eh^p\lesssim Ch^p,$$

since M does not depend on h. This concludes the proof of the theorem. □

3.2. **The non-Lipschitz continous case.** Using the above result as well as ideas from [33, 17, 23, 5, 13], we prove convergence in probability of order $1/2$ and almost sure convergence of order $1/2^-$ for the approximated exponential integrator (3) when applied to the stochastic Manakov equation (1).

For technical reasons, we shall assume in the next results that the set of time steps $\{h\}$ is a discrete sequence converging to zero, $\{h\}=\{h_n\}_{n\in\mathbb{N}}$, for instance $h_n=2^{-n}T$. Other examples of sequences of time steps can be found in [17] for instance.

Proposition 3.3. *Let $X^0 \in \mathbb{H}^6$ and $T > 0$. Denote the strong adapted solution of the stochastic Manakov equation (1) by $X(t)$. Denote the maximum stopping time for the existence of this solution by $\tau^* = \tau^*(X_0, \omega)$. Then, for all stopping times such that $\tau < \tau^* \wedge T$ a.s. there exists $h_0 > 0$ such that we have*

$$\forall h \in (0, h_0), \qquad \lim_{C\to\infty} \mathbb{P}\left(\max_{0\leq n\leq N_\tau} \|X^n - X(t_n)\|_{\mathbb{H}^1} \geq Ch^{1/2} \right) = 0,$$

where X^n denotes the numerical solution given by the approximated exponential integrator (3) with time step h and $N_\tau = \lceil \frac{\tau}{h} \rfloor$.

Proof. Consider the stochastic Manakov equation (2) with a truncated nonlinearity F_R, $R > 0$, and let X_R, resp. X_R^n, denote the exact, resp. numerical, solutions. We denote by κ a positive constant such that for all $Y \in \mathbb{H}^1$, $\left\||Y|^2 Y\right\|_1 \leq \kappa \|Y\|_1^3$.

Fix $X^0 \in \mathbb{H}^6$, $T > 0$, $\varepsilon \in (0,1)$. Let τ be a stopping time such that a.s. $\tau < \tau^* \wedge T$. By Theorem 2.1, there exists an $R_0 > 1$ such that $\mathbb{P}\left(\sup_{t\in[0,\tau]} \|X(t)\|_1 \geq R_0 - 1 \right) \leq \varepsilon/2$. We have the inclusion

$$\left\{ \max_{0\leq n\leq N_\tau} \|X^n - X(t_n)\|_1 \geq \varepsilon \right\} \subset \left\{ \max_{0\leq n\leq N_\tau} \|X(t_n)\|_1 \geq R_0 - 1 \right\}$$
$$\cup \left(\left\{ \max_{0\leq n\leq N_\tau} \|X^n - X(t_n)\|_1 \geq \varepsilon \right\} \cap \left\{ \max_{0\leq n\leq N_\tau} \|X(t_n)\|_1 < R_0 - 1 \right\} \right).$$

One then takes probabilities in the above equation and obtains

$$\mathbb{P}\left(\left\{ \max_{0\leq n\leq N_\tau} \|X^n - X(t_n)\|_1 \geq \varepsilon \right\} \right) \leq \varepsilon/2$$
$$+ \mathbb{P}\left(\left\{ \max_{0\leq n\leq N_\tau} \|X^n - X(t_n)\|_1 \geq \varepsilon \right\} \cap \left\{ \max_{0\leq n\leq N_\tau} \|X(t_n)\|_1 < R_0 - 1 \right\} \right).$$

In preparation of estimating the right-hand side terms, we define the random variable $n_\varepsilon := \min\{n \in \{0, \dots, N_\tau\} \colon \|X^n - X(t_n)\|_1 \geq \varepsilon\}$, with the convention that $n_\varepsilon = N_\tau + 1$ if the set is empty. If $\max_{0\leq n\leq N_\tau} \|X(t_n)\|_1 < R_0 - 1$ then we have by triangle inequality

$$\max_{0\leq n\leq n_\varepsilon - 1} \|X^n\|_1 = \max_{0\leq n\leq n_\varepsilon - 1} \|X^n - X(t_n) + X(t_n)\|_1 \leq \varepsilon + R_0 - 1 \leq R_0.$$

By definition of the approximated exponential integrator (3), for $h < \frac{3}{R_0^2 \kappa}$, we have

$$\|X^{n_\varepsilon}\|_1 \leq \left\|X^{n_\varepsilon - 1}\right\|_1 + h \left\||X^{n_\varepsilon-1}|^2 X^{n_\varepsilon-1}\right\|_1 \leq R_0 + \kappa h \left\|X^{n_\varepsilon-1}\right\|_1^3 \leq R_0 + \kappa h R_0^3 \leq 4R_0,$$

in this case and thus $X^n = X_{4R_0}^n$ for $0 \leq n \leq n_\varepsilon$.

If $n_\varepsilon \leq N_\tau$, by virtue of the definition of n_ε, one gets that $\left\|X_{4R_0}^{n_\varepsilon} - X_{4R_0}(t_{n_\varepsilon})\right\|_1 \geq \varepsilon$. Hence, one obtains that $\max_{0\leq n\leq N_\tau} \left\|X_{4R_0}^n - X_{4R_0}(t_n)\right\|_1 \geq \varepsilon$. Furthermore, again thanks to the definition of n_ε, one has $\left\{ \max_{0\leq n\leq N_\tau} \|X^n - X(t_n)\|_1 \geq \varepsilon \right\} \cap \{n_\varepsilon > N_\tau\} = \varnothing$. We then deduce that

$$\left\{ \max_{0\leq n\leq N_\tau} \|X^n - X(t_n)\|_1 \geq \varepsilon \right\} = \left\{ \max_{0\leq n\leq N_\tau} \|X^n - X(t_n)\|_1 \geq \varepsilon \right\} \cap \{n_\varepsilon \leq N_\tau\}.$$

Combining the above results, using Markov's inequality and Theorem 3.2, since $\tau < T$ a.s., there exists $C > 0$ such that

$$\begin{aligned}&\mathbb{P}\left(\max_{0\le n\le N_\tau}\|X^n - X(t_n)\|_1 \ge \varepsilon, n_\varepsilon \le N_\tau, \max_{0\le n\le N_\tau}\|X(t_n)\|_1 < R_0 - 1\right)\\ &\le \mathbb{P}\left(\max_{0\le n\le N_\tau}\left\|X^n_{4R_0} - X_{4R_0}(t_n)\right\|_1 \ge \varepsilon\right)\\ &\le \frac{1}{\varepsilon^{2p}}\mathbb{E}\left[\max_{0\le n\le N_\tau}\left\|X^n_{4R_0} - X_{4R_0}(t_n)\right\|_1^{2p}\right] \le \frac{1}{\varepsilon^{2p}}Ch^p.\end{aligned}$$

This last term in the expression above is smaller than $\varepsilon/2$ for h small enough. All together we obtain

$$\mathbb{P}\left(\max_{0\le n\le N_\tau}\|X^n - X(t_n)\|_1 \ge \varepsilon\right) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows convergence in probability.

To get the order of convergence in probability, we first choose $R_1 \ge R_0 - 1$ such that for all $h > 0$ small enough, $\mathbb{P}\left(\max_{0\le n\le N_\tau}\|X^n\|_1 \ge R_1\right) \le \frac{\varepsilon}{2}$. As above, for all positive real number C, we have

$$\begin{aligned}&\left\{\max_{0\le n\le N_\tau}\|X^n - X(t_n)\|_1 \ge Ch^{1/2}\right\} \subset \left\{\max_{0\le n\le N_\tau}\|X(t_n)\|_1 \ge R_1\right\}\\ &\cup\left(\left\{\max_{0\le n\le N_\tau}\|X^n - X(t_n)\|_1 \ge Ch^{1/2}\right\} \cap \left\{\max_{0\le n\le N_\tau}\|X(t_n)\|_1 < R_1\right\}\right).\end{aligned}$$

We then take probabilities in the above equation, use again Markov's inequality and Theorem 3.2, and obtain

$$\begin{aligned}&\mathbb{P}\left(\left\{\max_{0\le n\le N_\tau}\|X^n - X(t_n)\|_1 \ge Ch^{1/2}\right\}\right)\\ &\le \frac{\varepsilon}{2} + \mathbb{P}\left(\left\{\max_{0\le n\le N_\tau}\left\|X^n_{4R_1} - X_{4R_1}(t_n)\right\|_1 \ge Ch^{1/2}\right\}\right)\\ &\le \frac{\varepsilon}{2} + \frac{K(4R_1, \gamma, T, p, \|X_0\|_6)}{C^{2p}},\end{aligned}$$

since $\tau \le T$ almost surely. For C large enough, we infer

$$\mathbb{P}\left(\left\{\max_{0\le n\le N_\tau}\|X^n - X(t_n)\|_1 \ge Ch^{1/2}\right\}\right) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

uniformly for $h < h_0$. Finally, the order of convergence in probability of the approximated exponential integrator is $1/2$. □

Using the above proposition, one can establish that the approximated exponential integrator has almost sure convergence order $1/2^-$.

Proposition 3.4. *Under the assumptions of Proposition 3.3, for all $\delta \in \left(0, \frac{1}{2}\right)$ and $T > 0$, there exists a random variable $K_\delta(T)$ such that for all stopping time τ with $\tau < \tau^* \wedge T$, we have*

$$\max_{n=0,\dots,N_\tau}\|X^n(\omega) - X(t_n,\omega)\|_{\mathbb{H}^1} \le K_\delta(T,\omega)h^\delta \qquad \mathbb{P}-a.s.,$$

for small enough time steps $h < h_0(\omega)$, see the proof for the precise definition of the random variable h_0.

Proof. Let τ be a stopping time such that $\tau < \tau^* \wedge T$ almost surely. Fix $R > 0$, $p > 1$ and $\delta \in (0, \frac{1}{2})$ and recall that $N_\tau \leq N$. Using the strong error estimate from Theorem 3.2 and Markov's inequality, one gets positive h_0 and C, which does not depend on τ itself, such that

$$\forall h \in (0, h_0), \qquad \mathbb{P}\left(\max_{0\leq n\leq N_\tau} \|X_R^n - X_R(t_n)\|_1 > h^\delta\right) \leq Ch^{p(1-2\delta)}.$$

Using Borel–Cantelli's lemma, see for instance [33, Lemma 2.8], one then obtains that, choosing $p > 1$ sufficiently large to ensure that $p(1-2\delta) > 1$, there exists a positive random variable $K_\delta(R, \gamma, T, p, \cdot)$ such that

$$\mathbb{P} - a.s., \quad \forall h \in (0, h_0), \qquad \max_{0\leq n\leq N_\tau} \|X_R^n - X_R(t_n)\|_1 \leq K_\delta(R, \gamma, T, p, \omega)h^\delta. \quad (7)$$

We now continue as in the proof of Proposition 3.3. We know that, since $\tau < \tau^*$ a.s., there exists a random variable R_0 such that

$$\sup_{0\leq t\leq \tau} \|X(t)\|_1 \leq R_0(\omega) \quad \mathbb{P} - \text{a.s.}.$$

Let now $\varepsilon \in (0, 1)$ and h small enough ($h \leq 3R_0^{-2}(\omega)\kappa^{-1}$, where we recall that κ was defined in the proof of Proposition 3.3). Assume by contradiction that

$$\max_{0\leq n\leq N_\tau} \|X^n - X(t_n)\|_1 \geq \varepsilon.$$

Define $n_\varepsilon := \min\{n\colon \|X^n - X(t_n)\|_1 \geq \varepsilon\}$. By definition of $R_0(\omega)$ and h, we have that $\|X^n\|_1 \leq R_0(\omega)$ a.s. for $0 \leq n < n_\varepsilon - 1$. Hence, $\|X^{n_\varepsilon}\|_1 \leq 4R_0(\omega)$ and so the numerical solution equals to the numerical solution of the truncated equation $X^n = X_{4R_0}^n$ for $n = 0, 1, \ldots, n_\varepsilon$. We thus obtain that $\max\limits_{0\leq n\leq N_\tau} \left\|X_{4R_0}^n - X_{4R_0}(t_n)\right\|_1 \geq \varepsilon$ for h small enough. This contradicts (7) with $R = 4R_0(\omega)$. This proves almost-sure convergence.

Similarly as in the proof of Proposition 3.3, we now show the order of almost sure convergence. From the above, we have that for all $\varepsilon > 0$ and for ω in a set of probability one, there exists $h_0(\omega) > 0$ such that for all $h \leq h_0(\omega)$, $\max\limits_{0\leq n\leq N_\tau} \|X^n - X(t_n)\|_1 \leq \varepsilon$. Therefore, there exists $R_1(\omega) > R_0(\omega)$ such that $\max\limits_{0\leq n\leq N_\tau} \|X^n\|_1 \leq R_1(\omega)$.

If now $h \leq 3R_1^{-2}(\omega)\kappa^{-1} := h_0(\omega)$, we obtain from (7) that

$$\max_{0\leq n\leq N_\tau} \|X^n - X(t_n)\|_1 = \max_{0\leq n\leq N_\tau} \left\|X_{R_1}^n - X_{R_1}(t_n)\right\|_1 \leq K_\delta(R_1, \gamma, T, p, \omega)h^\delta.$$

We can therefore conclude that the approximated exponential integrator convergences a.s. with order $1/2^-$. □

4. Numerical experiments.

This section presents several numerical experiments illustrating the main properties of the approximated exponential integrator (3), denoted by SEXP below. We will compare this numerical scheme with the following ones:

- The nonlinearly implicit Crank–Nicolson scheme from [23]

$$X^{n+1} = X^n - H_{h,n}X^{n+1/2} + ihG(X^n, X^{n+1}), \quad \text{(CN)}$$

  where $G(X^n, X^{n+1}) = \frac{1}{2}\left(|X^n|^2 + |X^{n+1}|^2\right)X^{n+1/2}$ and $X^{n+1/2} = \frac{X^{n+1}+X^n}{2}$.

- The Lie–Trotter splitting scheme presented in [23]

$$X^{n+1} = U_{h,n+1}\left(X^n + i\int_{t_n}^{t_{n+1}} F(Y^n(s))\,\mathrm{d}s\right), \tag{LT}$$

where we recall that $F(X) = |X|^2 X$, Y^n is the exact solution to the nonlinear differential equation $i\mathrm{d}Y^n + F(Y^n)\,\mathrm{d}t = 0$ with the initial condition $Y^n(t_n) = X^n$. For this cubic nonlinearity, the exact solution to the nonlinear differential equation reads $Y^n(t_n + h) = \exp(ih|Y^n(t_n)|^2)Y^n(t_n)$. A convergence analysis for the Lie–Trotter scheme (LT), similar to the one presented here yet more intricate, is carried out in [6].

- The relaxation scheme presented in [23]

$$i\left(X^{n+1} - X^n\right) + H_{h,n}\left(\frac{X^{n+1}+X^n}{2}\right) + \Phi^{n+1/2}\left(\frac{X^{n+1}+X^n}{2}\right)h = 0, \tag{Relax}$$

where $\Phi^{n+1/2} = 2|X^n|^2 - \Phi^{n-1/2}$ with $\Phi^{-1/2} = |X^0|^2$.

We will consider the SPDE (1) on an interval $[-a, a]$ with a sufficiently large $a > 0$ with homogeneous Dirichlet boundary conditions. For a fixed mesh size, denoted by Δx, we consider a spatial discretization of this SPDE using a centered finite differences method.

If not otherwise specified, the initial condition is given by the soliton of the deterministic Manakov equation [25] which reads

$$X^0 = X(x,0) = \begin{pmatrix}\cos(\theta/2)\exp(i\phi_1)\\ \sin(\theta/2)\exp(i\phi_2)\end{pmatrix}\eta\,\mathrm{sech}(\eta x)\exp(-i\kappa(x-\tau)+i\alpha), \tag{8}$$

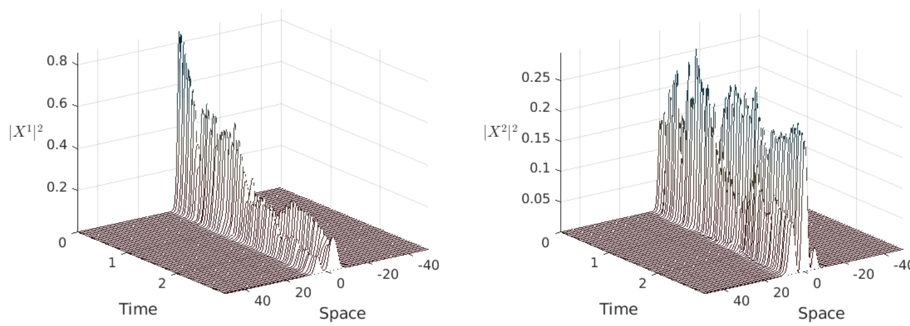
with the parameters $\alpha = \tau = \phi_1 = \phi_2 = \kappa = 0$ and $\theta = \pi/4$, $\eta = 1$.

It has to be noted that the occurrence of blow-up in the stochastic Manakov equation (1) remains an open question, see for instance [23]. We did not observe blow-up in the numerical experiments proposed below.

4.1. **Evolution plots.** In the first numerical experiment, we solve the stochastic Manakov equation (1) with $a = 50$ on the time interval $[0, 3]$ and discretization parameters $h = 3/625$ and $\Delta x = 1/4$. Figure 1 displays the space-time evolution of the numerical intensities $|X^1|^2$ and $|X^2|^2$ along solutions given by the approximated exponential integrator (3). An energy exchange due to the stochastic perturbation and the nonlinearity can be observed. This produces small amplitude perturbations at the basis of the soliton, leading to the formation of further solitons.

4.2. **Strong convergence.** In order to illustrate the strong rate of convergence of the approximated exponential integrator (3) stated in Theorem 3.2, we discretize the stochastic Manakov equation (1) with $a = 50$ and mesh size $\Delta x = 0.4$. We compute the errors $\mathbb{E}\left[\left\|X^N - X_{\mathrm{ref}}(T)\right\|_{\mathbb{H}^1}^2\right]$ at the time $T = 1$ for time steps ranging from $h = 2^{-6}$ to $h_{\mathrm{ref}} = 2^{-16}$ and report these in a loglog plot in Figure 2. The reference solution is computed using the approximated exponential integrator and the expected values are approximated by computing averages over $M_s = 250$ samples. That is, we compute the sample averages

$$\overline{e_N} = \frac{1}{M_s}\sum_{i=1}^{M_s}\left\|X^N(\omega_i) - X_{\mathrm{ref}}(T,\omega_i)\right\|_{\mathbb{H}^1}^2,$$



(A) A 3d perspective of the intensity.

(B) The intensities $|X^1|^2$ (left) and $|X^2|^2$ (right) as functions of time (t) and space (x).

FIGURE 1. $3d$ space-time evolution (Figure 1a) and colormap (Figure 1b) of the intensity of the first component (left) and the second component (right) of the numerical solution to the stochastic Manakov system (1) with initial value (8) computed using the approximated exponential integrator (3). The discretization parameters are $h = 3/625$ and $\Delta x = 1/4$.

and Monte Carlo errors (where $s(e_N)$ denotes the sample standard deviation)

$$\frac{s(e_N)}{\sqrt{M_s}} = \frac{1}{M_s}\sqrt{\sum_{i=1}^{M_s}\left(\overline{e_N} - \|X^N(\omega_i) - X_{\text{ref}}(T,\omega_i)\|_{\mathbb{H}^1}^2\right)^2}.$$

In the numerical simulations reported in Figure 2, the Monte Carlo errors of all time integrators are less than 5% of each respective sample average. This corresponds to adding $\log(1.05) \sim 0.021$ in the logarithmic scale. The Monte Carlo error is thus negligible.

4.3. **Computational costs.** The goal of this numerical experiment is to compare the computational cost of the approximated exponential integrator introduced in this paper to that of numerical methods from the literature. We run all numerical

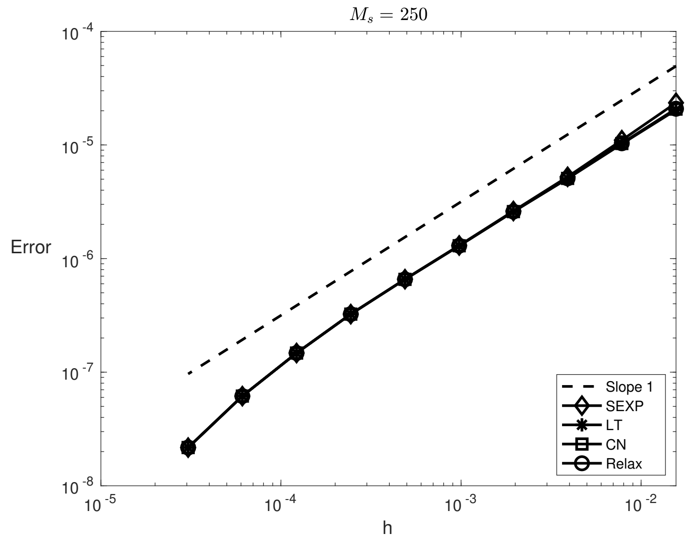


FIGURE 2. Strong rates of convergence for the stochastic Manakov system (1) with initial value (8): Error $\mathbb{E}\left[\left\|X^N - X_{\mathrm{ref}}(T)\right\|_{\mathbb{H}^1}^2\right]$ at the time $T = 1$ as a function of the time step h in loglog scale. The discretization parameters are h from 2^{-6} to 2^{-16} and $\Delta x = 0.4$.

schemes over the time interval $[0, 0.5]$ for the stochastic Manakov equation (1) with $\gamma = 1$. We discretize the spatial domain with $a = 50$, using a mesh of size $\Delta x = 0.2$. We run 500 samples for each numerical scheme. For each scheme and each sample, we run several time steps and compare the $\mathbb{L}^2$-error at the final time with a reference solution provided for the same sample by the same scheme for a very small time step $h = 2^{-17}$. Figure 3 displays the total computational time for all the samples, for each numerical scheme and each time step, as a function of the averaged final error. One observes that the performance of the Crank–Nicolson scheme is a little bit inferior than the performance for the other numerical schemes. The scheme with the label modEXP is presented below.

4.4. **Preservation of the $\mathbb{L}^2$-norm.** The next numerical experiment illustrates the preservation of the $\mathbb{L}^2$-norm along one sample path of the above numerical schemes. For this, we consider $a = 50$, $\gamma = 1$, time interval $[0, 3]$ and discretization parameters $h = 0.006$ and $\Delta x = 0.25$. The results are displayed in Figure 4. Exact preservation of the (squared) $\mathbb{L}^2$-norm for the Crank–Nicolson, the Lie–Trotter and the relaxation schemes is observed. A small drift is observed for the approximated exponential scheme.

4.5. **$\mathbb{L}^2$-preserving exponential integrators.** As seen above, the proposed approximated exponential integrator unfortunately does not preserve the $\mathbb{L}^2$-norm. This can be fixed using ideas from [12, 13]. We thus propose the following modified exponential method for the numerical discretization the stochastic Manakov equation (1)

$$F_* = F\left(U_{h,n}X^n + i\frac{h}{2}F_*\right)$$

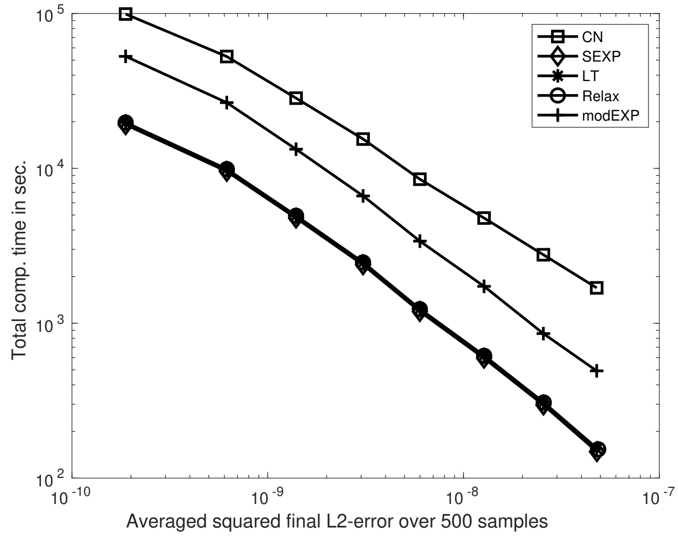


FIGURE 3. Efficiency of the time integrators for the stochastic Manakov system (1) with initial value (8): Computational time as a function of the averaged final error $\mathbb{E}\left[\left\|X^N - X_{\mathrm{ref}}(T)\right\|_{\mathbb{L}^2}^2\right]$ at $T = 0.5$ over 500 samples (loglog scale). The discretization parameters are h from 2^{-9} to 2^{-17} and $\Delta x = 0.2$.

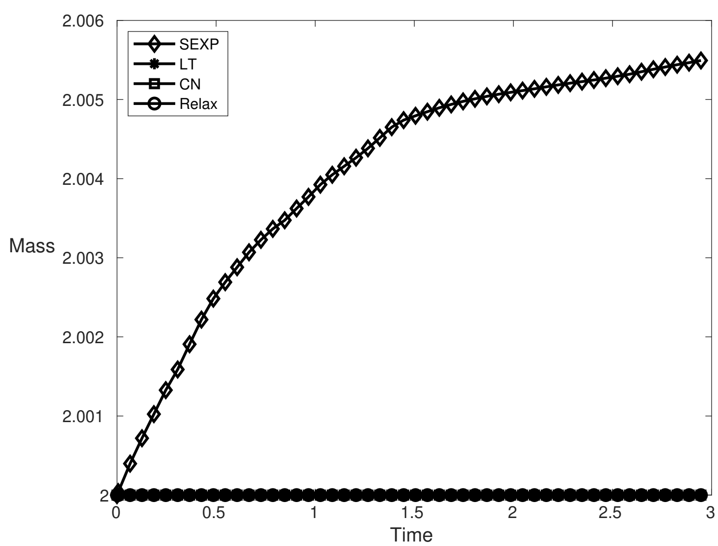


FIGURE 4. Evolution of the (squared) $\mathbb{L}^2$-norm along the numerical solutions of the four numerical integrators applied to the stochastic Manakov system (1) with initial values (8). The discretization parameters are $h = 0.006$ and $\Delta x = 0.25$.

$$X^{n+1} = U_{h,n}X^n + ihF_*, \qquad \text{(modEXP)}$$

where we define $F(X) = |X|^2X$ for the nonlinearity.

As seen in the introduction, the exact solution to the stochastic Manakov equation (1) preserves the $\mathbb{L}^2$-norm. The following proposition states that the modified exponential method enjoys the same property.

Proposition 4.1. *The modified exponential integrator* (modEXP) *preserves the* $\mathbb{L}^2$*-norm.*

Proof. The time integrator (modEXP) turns out to be a composition of midpoint maps, which preserve quadratic invariants, see e.g. [24, Section IV.2]. Thus the time integrator (modEXP) preserves the $\mathbb{L}^2$-norm. □

We now numerically illustrate this property with the same parameters as in the previous numerical experiment. Figure 5 (left) shows the exact preservation of the $\mathbb{L}^2$-norm by the modified exponential scheme (modEXP). Figure 5 (right) numerically illustrates the rate of convergence of the modified exponential scheme (the parameters for the numerical experiments are the same as above). The Monte Carlo error is less than 5% of each respective sample average. The rate of strong convergence is the same as for the approximated exponential scheme.

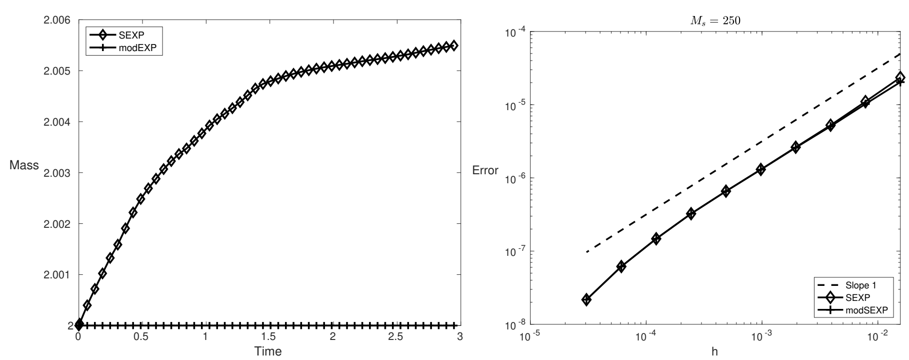


FIGURE 5. Left: Evolution of the (squared) $\mathbb{L}^2$-norm along the numerical solutions of both exponential schemes applied to the stochastic Manakov system (1) with initial values (8). The discretization parameters are $h = 0.006$ and $\Delta x = 0.25$. Right: Strong rates of convergence for the stochastic Manakov system (1) with initial value (8): Error $\mathbb{E}\left[\left\|X^N - X_{\text{ref}}(T)\right\|^2_{\mathbb{H}^1}\right]$ at the time $T = 1$ as a function of the time step h in loglog scale. The discretization parameters are h from 2^{-6} to 2^{-16} and $\Delta x = 0.4$.

In Figure 3, we compare the computational cost of the modified time integrators with the other schemes and observe that it is slightly slower than the original scheme, the Lie–Trotter scheme and the relaxation scheme but faster than the Crank–Nicolson scheme.

It would be of interest to prove the orders of convergence of the $\mathbb{L}^2$-preserving exponential integrator (modEXP). This is however out of the scope of this publication since it seems that one would need to use other techniques than that used in the proofs of the proposed explicit approximated exponential integrator (3). For example, one would need to use estimates on fixed points of random functions and detailed Lipschitz properties of random fixed point functions with parameters.

4.6. **Inhomogeneous stochastic Manakov equation.** To show the versatility of the proposed approximated exponential integrators, we apply the scheme (3) to a stochastic version of the inhomogeneous Manakov equation from [31], namely:

$$i\mathrm{d}X + \partial_x^2 X\,\mathrm{d}t + q(t)|X|^2 X\,\mathrm{d}t + ip(t)X\,\mathrm{d}t + i\sqrt{\gamma}\sum_{k=1}^{3}\sigma_k\partial_x X \circ \mathrm{d}W_k = 0. \quad (9)$$

This corresponds to Equation (1) with an additional term $ip(t)$ modelling gain (when $p(t) \leq 0$) or absorption effects (when $p(t) \geq 0$), and an inhomogeneous Kerr nonlinearity response $q(t)|X(t)|^2X(t)$ corresponding to normal (when $q(t) \leq 0$) and anormal (when $q(t) \geq 0$) regimes.

Using the same parameters as in the corresponding above numerical experiments, with in addition $p(t) = 0.1(\sin(5t) + \mathrm{e}^{-t^2})$ and $q(t) = \cos(0.7t)$, we plot in Figure 6 the strong errors of the approximated exponential integrator as well as the evolution of the (squared) $\mathbb{L}^2$-norm of the numerical solution. The Monte Carlo error is less than 4% of each sample average. The expected rate of convergence provided by Theorem 3.2 is observed (with usual modifications to adapt to a smooth non-autonomous setting). In addition, due to the coefficient $p(t)$ in the above SPDE, one does not have conservation of the $\mathbb{L}^2$-norm in this case at the continuous level. Finally, observe that in the present setting, because of the form of the coefficients $p(t)$ and $q(t)$, the deterministic system of differential equations

$$\dot{X}(t) + q(t)|X(t)|^2X(t) + ip(t)X(t) = 0,$$

cannot be integrated exactly. This would then add an additional error source if one would want to use a Lie–Trotter splitting (LT) strategy.

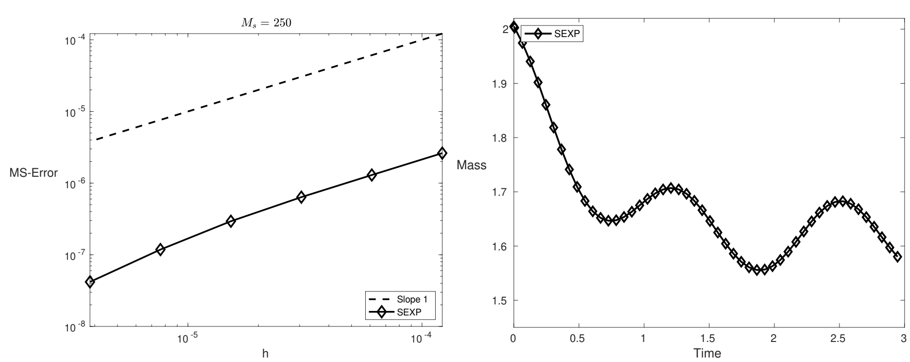


FIGURE 6. Strong rate of convergence (left) and evolution of the (squared) $\mathbb{L}^2$-norm (right) of the approximated exponential scheme when applied to the inhomogeneous stochastic Manakov equation (9). The discretization parameters are h from 2^{-13} to 2^{-19} and $\Delta x = 0.4$ (left) and $h = 0.006$ and $\Delta x = 0.25$ (right).

4.7. **Weak convergence.** Since weak convergence is another important way of measuring the accuracy of a numerical integrator, we conclude this section with an illustration of the weak rates of convergence for the proposed scheme as well as for the Lie–Trotter scheme and the Crank–Nicolson scheme. The other time integrators were not included due to long computational time. The parameters for this numerical experiments are: the time interval $[0, 0.125]$, the time step for the reference solution given by the proposed method is $h = 2^{-13}$, $\gamma = 0.5$ and $M_s = 250000$ samples. The other parameters are as above.

Figure 7 presents the weak rates for the test functions $\phi_1(X) = \|X_1\|_{L^2}$ and $\phi_2(X) = \exp(-\|X_2\|_{L^2}^2)$, also used for instance in [10]. A weak order one is observed for all time integrators. This seems to indicate that the weak rate is twice the strong rate of convergence. Proving these weak rates of convergence is not easy (see for example [11] for the Allen-Cahn model with space-time white noise) and is not the goal of this article. For the test function ϕ_2, see the right panel of Figure 7, the Monte Carlo error for the approximated exponential integrator (3) is less than 40% of each sample average. This corresponds to adding $\log(1.40) \sim 0.146$ in the logarithmic scale. The Monte Carlo error is thus small. For the test function ϕ_1, see the left panel of Figure 7, the Monte Carlo error for the approximated exponential integrator (3) is less than 13% of each sample average. This corresponds to adding $\log(1.13) \sim 0.053$ in the logarithmic scale. The Monte Carlo error is thus negligible. Nothing can be concluded for the smallest time steps since the reference solution is not accurate enough.

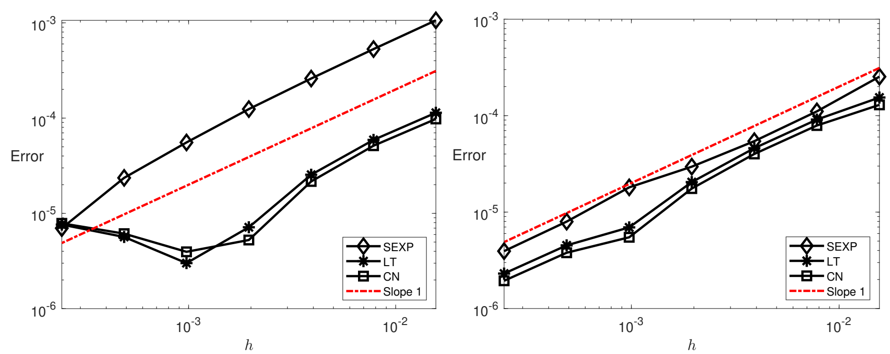


FIGURE 7. Weak rates of convergence of three time integrators when applied to the inhomogeneous stochastic Manakov equation (9). The test functions are $\phi_1(X) = \|X_1\|_{L^2}$ (left) and $\phi_2(X) = \exp(-\|X_2\|_{L^2}^2)$ (right). The discretization parameters are h from 2^{-6} to 2^{-13} and $\Delta x = 0.4$.

5. Appendix.

We prove here that the operator $U_{h,n}$ defined after equation (3) is a unitary operator from $\mathbb{H}^m(\mathbb{R})$ to itself for all $h > 0$, all $m \in \mathbb{N}$ and all realization of the random variable. In the following, $n \in \mathbb{N}$ plays no actual role, but we kept it for consistency of notations.

Proposition 5.1. *Let $m \in \mathbb{N}$, $(u_0, v_0) \in \mathbb{H}^m(\mathbb{R})$, $h > 0$, $\chi_1, \chi_2, \chi_3 \in \mathbb{R}$ and define a distribution $(u_1, v_1) \in (\mathcal{S}'(\mathbb{R}))^2$ as the solution of*

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = U_{h,n} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{10}$$

One has $(u_1, v_1) \in \mathbb{H}^m(\mathbb{R})$ and $\|(u_1, v_1)\|_{\mathbb{H}^m(\mathbb{R})} = \|(u_0, v_0)\|_{\mathbb{H}^m(\mathbb{R})}$. Moreover, the operator $U_{h,n}$ is surjective from $\mathbb{H}^m$ to itself. Hence it is a unitary operator on $\mathbb{H}^m$.

Proof. The operator $H_{h,n}$ defined after (3) acts on the Fourier transform of the couples of functions at frequency $\xi \in \mathbb{R}$ *via* the complex-valued 2×2 matrix $ih\xi^2 I_2 +$

$i\xi\sqrt{\gamma h}(\chi_1\sigma_1+\chi_2\sigma_2+\chi_3\sigma_3)$. This matrix reads $iS(\xi)$ where $S(\xi)$ is an hermitian 2×2 matrix. Therefore, the matrix $S(\xi)$ is diagonalizable in an orthonormal basis of $\mathbb{C}^2$ with real eigenvalues $\lambda_1(\xi)$ and $\lambda_2(\xi)$. We infer that there exists a unitary matrix $P(\xi)$ such that $S(\xi)=P(\xi)^\star D(\xi)P(\xi)$, where $D(\xi)$ is the diagonal matrix with $\lambda_1(\xi)$ and $\lambda_2(\xi)$ on the diagonal. Hence, relation (10) is equivalent to

$$\forall\xi\in\mathbb{R},\qquad P(\xi)\begin{pmatrix}\hat{u}_1(\xi)\\ \hat{v}_1(\xi)\end{pmatrix}=\begin{pmatrix}\frac{1-i\lambda_1(\xi)/2}{1+i\lambda_1(\xi)/2} & 0\\ 0 & \frac{1-i\lambda_2(\xi)/2}{1+i\lambda_2(\xi)/2}\end{pmatrix}P(\xi)\begin{pmatrix}\hat{u}_0(\xi)\\ \hat{v}_0(\xi)\end{pmatrix}. \tag{11}$$

Since the diagonal elements in the diagonal matrix above have modulus 1 and $P(\xi)$ is unitary, we infer that

$$\forall\xi\in\mathbb{R},\qquad |\hat{u}_1(\xi)|^2+|\hat{v}_1(\xi)|^2=|\hat{u}_0(\xi)|^2+|\hat{v}_0(\xi)|^2. \tag{12}$$

This proves that $(u_1,v_1)\in\mathbb{H}^m(\mathbb{R})$ since $(u_0,v_0)\in\mathbb{H}^m(\mathbb{R})$, and the $\mathbb{H}^m(\mathbb{R})$-norm of these two couples of functions is the same. To prove that $U_{h,n}$ is surjective, consider $(u_1,v_1)\in\mathbb{H}^m$, and define $(\hat{u}_0,\hat{v}_0)$ by inverting (11). Relation (12) holds for almost all $\xi\in\mathbb{R}$. This proves that the distributions u_0,v_0, defined by inverse Fourier transform, are in $\mathbb{H}^m$ and solve (10). □

6. **Conclusion and future works.** In this article, we introduce and analyze a novel numerical integrator for the time discretization of the stochastic (homogeneous as well as inhomogeneous) Manakov system (1). For the inhomogeneous problem, this numerical integrator is easier to implement than the Lie–Trotter splitting scheme developed and analyzed in [6]. We prove that the proposed approximated exponential integrator has strong order 1/2 in the case of a globally Lipschitz nonlinearity. For the case of non-Lipschitz nonlinearity, like the cubic case, we show order of convergence 1/2 in probability and 1/2− almost-surely.

The conservation of the mass is an important feature of the stochastic homogeneous Manakov system. Therefore, we propose a modified exponential method which preserves the mass at the discrete level. In addition, we demonstrate numerically that the strong order of convergence of the modified exponential method is 1/2. To prove such a strong convergence result could be the subject of a future work.

Other perspectives of future work include proving strong order of convergence for such a time integrator for the stochastic Manakov system with cubic nonlinearity. This would be a very challenging task. In particular, it would require more information about the exact solution to the stochastic Manakov system as well as moment bounds on the norms of the exact and numerical solutions. For instance, to the best of our knowledge, results on strong order of convergence for numerical schemes for the Schrödinger equation with white noise dispersion and a power-law nonlinearity (a closely related and well-studied model to the stochastic Manakov system) are not available.

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