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Weak solutions to gamma-driven stochastic differential equations

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Abstract

We study a stochastic differential equation driven by a gamma process, for which we give results on the existence of weak solutions under conditions on the volatility function. To that end we provide results on the density process between the laws of solutions with different volatility functions.

Keywords: Gamma process; Stochastic differential equation

1. Introduction

The goal of the present paper is to give conditions such that the Lévy-driven stochastic differential equation

$$\mathrm{d}X_t = \sigma(X_{t-})\,\mathrm{d}L_t,\ X_0 = 0 \tag{1}$$

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has a weak solution that is unique in law. Here L is a gamma process with $L_0 = 0$ and therefore L is a subordinator, i.e. a stochastic process with monotonous sample paths. Furthermore, L has a Lévy measure ν admitting the Lévy density

$$v(x) = \alpha x^{-1} \exp(-\beta x),\ x > 0, \tag{2}$$

where α and β are two positive constants. The process L has independent increments, and $L_t - L_s$ has a Gamma$(\alpha(t-s), \beta)$ distribution for $t > s$. Recall that the Gamma(a, b) distribution has a density given by $x \mapsto \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$ for $x > 0$. In view of financial applications, the function σ will be referred to as volatility function.

Rather surprisingly, there is little to find in the literature on properties of Eq. (1) with a gamma process as the driver, even when it comes down to existence of solutions, which is in sharp contrast to the situation for Brownian motion driven equations. Under the condition that σ is Lipschitz continuous (which is stronger than what we will require in this paper), it is known that (1) has a unique strong solution, see Protter [14, Theorem V.6], or Jacod and Mémin [4]. There are some references on the existence of weak solutions (the latter also called solution-measures) as well, but in very general situations with general semimartingales as driving processes. For example, Jacod and Mémin [6] has existence of weak solutions on a complicated product space (see Theorem 1.8 there), but not of uniqueness in law and under a continuity condition on σ. Similar remarks apply to the results of Jacod and Mémin [5] (also Theorem 1.8 in this reference), where even the assumption that σ is bounded has been made. Under the assumption that the volatility function σ is measurable, positive and satisfies a linear growth condition, we will see in Theorem 5 that Eq. (1) admits a weak solution that is unique in law. This is the main result of the present paper. Note that our assumptions are weaker than those in the just mentioned references.

We will now briefly outline the relevance of gamma processes and gamma-driven stochastic differential equations. They form a special class of Lévy processes (see, e.g., Kyprianou [11]), are a fundamental modelling tool in several fields, e.g. reliability (see van Noortwijk [16]) and risk theory (see Dufresne et al. [3]). Since the driving gamma process L in (1) has non-decreasing sample paths and the volatility function σ is non-negative, also the process X has non-decreasing sample paths. Such processes find applications across various fields. A reliability model as in (1) has been thoroughly investigated from a probabilistic point of view in Wenocur [17], and constitutes a far-reaching generalisation of a basic gamma model. Furthermore, non-decreasing processes are ideally suited to model revenues from an innovation: in Chance et al. [2], the authors study the question of pricing options on movie box office revenues that are modelled through a gamma-like stochastic process. Another potential application is in modelling the evolution of forest fire sizes over time, as in Reed and McKelvey [15].

Any practical application of the model (1) would require knowledge of the volatility function σ, that has to be inferred from observations on the process X. This is a statistical problem to which we present a nonparametric Bayesian approach in Belomestny et al. [1]. The obtained results in that paper assume either a piecewise constant volatility function σ or a Hölder continuous one. In both cases one needs existence of weak solutions to (1) and a likelihood ratio. The present paper covers those two cases and provides the probabilistic foundations of the statistical analysis. For a survey of other contributions to statistical inference for Lévy-driven SDEs we also refer to Belomestny et al. [1].

2. Absolute continuity and likelihood

In the proof our main result, Theorem 5, we need the likelihood ratio between different laws of solutions to (1). In this section we give the relevant results.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and let $(L_t)_{t\geq 0}$ be a gamma process adapted to $\mathbb{F}$, whose Lévy measure admits the density v given by (2). Assume that X is a (weak) solution to (1). For convenience of the reader we recall here the definition of a weak solution.

Definition 1. A weak solution to Eq. (1) is by definition a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, together with càdlàg processes X and L defined on this space, both adapted to the filtration $\mathbb{F}$, such that L is a gamma process and such that (1) is valid for all $t \geq 0$.

We assume that X is observed on an interval $[0, T]$. We denote by $\mathbb{P}^\sigma_T$ its law. Here we follow the set up commonly used in the literature, which assumes that the paths of X and L, and all other processes involved, are elements of the Skorokhod space $D[0,\infty)$, i.e. the space of càdlàg functions, endowed with the Skorokhod metric, see Jacod and Shiryaev [7, Chapter 6]. The measure $\mathbb{P}^\sigma_T$ is then a probability distribution on $D[0, T]$. The precise form of the underlying space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is at this stage less relevant. Later on, when we treat existence of weak solutions, we will specify the corresponding Skorokhod space as well, and then X will be the canonical process, so $X_t(\omega) = \omega(t)$ and the filtration needed will become the canonical filtration, $\mathcal{F}_t = \sigma(X_u, u \leq t)$.

In agreement with the notation $\mathbb{P}^\sigma_T$, we let $\mathbb{P}^1_T$ be the law of X when $\sigma \equiv 1$, in which case $X_t = L_t$, $t \in [0, T]$. The measure $\mathbb{P}^1_T$ will serve as a reference measure. The choice $\sigma = 1$ for obtaining a reference measure is natural, but also arbitrary. Many other choices for the function σ are conceivable, in particular other constant functions. The question we are going to investigate first is under which conditions the laws $\mathbb{P}^\sigma_T$ and $\mathbb{P}^1_T$ are equivalent. Suppose that the process $\sigma(X_{t-})$, $t \in [0, T]$ is strictly positive and define

$$v^\sigma(t,x) = \frac{1}{\sigma(X_{t-})} v\left(\frac{x}{\sigma(X_{t-})}\right). \tag{3}$$

First, we will show that for X, as weak solution satisfying (1), the compensated jump measure under $\mathbb{P}^\sigma_T$ is determined by (3). Recall that the jump measure μ^X associated with an adapted process X is determined by the integral $\int_{[0,t]\times\mathbb{R}} f(s,x)\mu^X(\mathrm{d}s, \mathrm{d}x) = \sum_{s\leq t} f(s, \Delta X_s)\mathbf{1}_{\{\Delta X_s \neq 0\}}$ for all f for which the integral is finite, see also Jacod and Shiryaev [7, Section II.1 and Proposition II.1.16]. By the càdlàg property of X, the sum here contains at most countably many terms. Note that μ^X is an integer valued *random* measure. It has a compensator, a predictable random measure ν, essentially determined by the property that the integral processes $\int_{[0,\cdot]\times\mathbb{R}} W\,\mathrm{d}\mu^X$ and $\int_{[0,\cdot]\times\mathbb{R}} W\,\mathrm{d}\nu$ differ by a local martingale for all 'good' nonnegative predictable processes W. Note that the underlying probability measure $\mathbb{P}$ comes in because of the local martingale property. We refer to Jacod and Shiryaev [7, Section II.1] for all details and properties of these random measures. The weak solution process X to (1) turns out to be a semimartingale, hence it is endowed with the so-called triplet of characteristics, of which, by absence of a continuous local martingale part and a 'drift', only the third component is present, that component being the compensated jump measure. This jump measure parallels to a considerable extent the third characteristic of the triplet of a Lévy process, which is a special case of a semimartingale, although in the latter case the third characteristic is nonrandom and homogeneous in time. In fact, if X is a Lévy process with Lévy measure ν, then $\nu(\mathrm{d}t, \mathrm{d}x) = v(x)\,\mathrm{d}x\mathrm{d}t$ in case there exists a Lévy density v. And this ν is also the third characteristic of X considered as a semimartingale. See Jacod and Shiryaev [7, Section II.2] for details on characteristics. Below we assume to work on the canonical space $D[0,T]$, on which the measures $\mathbb{P}^\sigma_T$ and $\mathbb{P}^1_T$ are defined.

Lemma 2. *Assume that (1) admits a weak solution for a given measurable function $\sigma : [0,\infty) \to [0,\infty)$ with $\sigma(X_{s-}) > 0$ a.s. for all $s \geq 0$. Under the measure $\mathbb{P}^\sigma_T$, the third characteristic of the semimartingale X, its compensated jump measure $\nu^\sigma(\mathrm{d}x, \mathrm{d}t)$, is given by $\nu^\sigma(\mathrm{d}x, \mathrm{d}t) = v^\sigma(t,x)\,\mathrm{d}x\mathrm{d}t$.*

Proof. Let f be a measurable function and let X be given by (1). For technical reasons, see Jacod and Shiryaev [7], Proposition II.1.28 and related results, we assume that f is such that $\sum_{s\leq t} f^2(\Delta X_s)$ defines a locally integrable process. Then (the summations are only for those s with $\Delta X_s > 0$, and $t \in [0,T]$ is arbitrary) with μ^X being the jump measure of X and μ^L being the jump measure of L,

$$\begin{aligned}\int_0^t f(x)\,\mu^X(\mathrm{d}x,\mathrm{d}s) &= \sum_{s\leq t} f(\Delta X_s)\\ &= \sum_{s\leq t} f(\sigma(X_{s-})\Delta L_s)\\ &= \int_0^t \int_{(0,\infty)} f(\sigma(X_{s-})z)\mu^L(\mathrm{d}z,\mathrm{d}s),\end{aligned}$$

which is the sum of a local martingale M under $\mathbb{P}$, adapted to $\mathbb{F}$, and the predictable process $\int_0^t \int_{(0,\infty)} f(\sigma(X_{s-})z)v(z)\,\mathrm{d}z\mathrm{d}s$. As the latter expression only depends on the process X, the local martingale M is also adapted to the filtration generated by X, $\mathbb{F}^X = \{\mathcal{F}^X_t, t \geq 0\}$ with $\mathcal{F}^X_t = \sigma(X_s, s \leq t)$. By a simple change of variable, the double integral equals to

$$\int_0^t \int_{(0,\infty)} f(x)\frac{1}{\sigma(X_{s-})}v\Big(\frac{x}{\sigma(X_{s-})}\Big)\,\mathrm{d}x\mathrm{d}s.$$

Following the *discussion* on page 192 of Karatzas and Shreve [9], we can consider (with slight abuse of notation) $\mathbb{P}^\sigma_T$, the law of X for X given by (1), also as a probability measure on $\mathcal{F}^X_T$. This convention will be followed throughout the paper where needed.

As the expression in the display is depending on X only, this expression is also the $\mathbb{F}^X$-compensator under $\mathbb{P}^\sigma_T$ of $\sum_{s\leq t} f(\Delta X_s)$.

Indeed, the $\mathbb{F}^X$-compensator ν^σ of the jump measure of X, μ^X, satisfies for all relevant functions f the property that

$$M_t := \sum_{s\leq t} f(\Delta X_s) - \int_0^t \int_{(0,\infty)} f(x)\nu^\sigma(\mathrm{d}x,\mathrm{d}s)$$

defines a local martingale, adapted to $\mathbb{F}^X$ and under the law $\mathbb{P}^\sigma_T$. As, apart from the integrability assumptions, f is arbitrary, it follows that ν^σ is given by its density as in (3). □

Let

$$Y(t,x) := \frac{v^\sigma(t,x)}{v(x)} = \frac{1}{\sigma(X_{t-})}v\Big(\frac{x}{\sigma(X_{t-})}\Big)/v(x).$$

Absolute continuity of $\mathbb{P}^\sigma_T$ w.r.t. $\mathbb{P}^1_T$ is guaranteed, see Jacod and Shiryaev [7, Theorem III.5.34], under the condition

$$H_T = \int_0^T \int_0^\infty (\sqrt{Y(t,x)} - 1)^2 v(x)\,\mathrm{d}x\,\mathrm{d}t < \infty, \qquad \mathbb{P}^\sigma_T\text{-a.s.} \tag{4}$$

Here H_T has been derived from Equation (5.7) in Jacod and Shiryaev [7, Chapter III]. As the driving process L is a gamma process with Lévy density v given by (2), one has in fact $Y(t,x) = \exp\Big(-\beta x\Big(\frac{1}{\sigma(X_{t-})} - 1\Big)\Big)$. Hence, one obtains

$$\begin{aligned} H_T &= \alpha \int_0^T \int_0^\infty \Big(\exp\Big(-\frac{1}{2}\beta x\Big(\frac{1}{\sigma(X_{t-})} - 1\Big)\Big) - 1\Big)^2 x^{-1} \exp(-\beta x)\,\mathrm{d}x\,\mathrm{d}t \\ &= \alpha \int_0^T \int_0^\infty \Big(\exp\Big(-\frac{1}{2}\frac{\beta x}{\sigma(X_{t-})}\Big) - \exp\Big(-\frac{1}{2}\beta x\Big)\Big)^2 x^{-1}\,\mathrm{d}x\,\mathrm{d}t \\ &=: \alpha \int_0^T h_t\,\mathrm{d}t. \end{aligned} \tag{5}$$

Clearly, conditions on σ have to be imposed to have absolute continuity, or even equivalence. These are given now. Of course, we still have to assume that a weak solution to (1) exists. As already announced, sufficient conditions for this will be presented in Theorem 5. Below, the jump measure of X is denoted μ^X.

Proposition 3. *Assume that σ is a positive locally bounded measurable function on $[0,\infty)$ such that (1) admits a weak solution unique in law. It is furthermore assumed that σ is lower bounded by a constant $\sigma_0 > 0$. Then the laws $\mathbb{P}^\sigma_T$ and $\mathbb{P}^1_T$ are equivalent, and one has*

$$\frac{\mathrm{d}\mathbb{P}^\sigma_T}{\mathrm{d}\mathbb{P}^1_T} = \mathcal{E}_T\left(\int_0^\cdot \int_{(0,\infty)} (Y(t,x) - 1)(\mu^X(\mathrm{d}x,\mathrm{d}t) - v(x)\,\mathrm{d}x\mathrm{d}t)\right), \tag{6}$$

where $\mathcal{E}_T$ is the Doléans exponent at time T of the process within the parentheses. In other words, $Z_T := \frac{\mathrm{d}\mathbb{P}^\sigma_T}{\mathrm{d}\mathbb{P}^1_T}$ is the solution at time T to the SDE

$$\mathrm{d}Z_t = Z_{t-} \int_{(0,\infty)} (Y(t,x) - 1)(\mu^X(\mathrm{d}x,\mathrm{d}t) - v(x)\,\mathrm{d}x\,\mathrm{d}t), \quad Z_0 = 1. \tag{7}$$

Proof. Split the integrand h_t in (5) into two integrals, for $x \in [0,1]$ and $x \in (1,\infty)$, call them $h^<_t$ and $h^>_t$ respectively. For $h^<_t$ we use the elementary inequality $(e^{-ax} - e^{-bx})^2 \le (b-a)^2 x^2$ for $a, b, x \ge 0$ to obtain the bound $h^<_t \le \frac{\beta^2}{4}(\frac{1}{\sigma(X_{t-})} - 1)^2$ (here we also used $x \le 1$), which is bounded by the finite constant $\frac{\beta^2}{2}(\frac{1}{\sigma_0^2} + 1)$. To treat the integral $h^>_t$ we use the elementary inequality $(a-b)^2 \le 2(a^2 + b^2)$, which leads us to study, also using $x \ge 1$,

$$\int_1^\infty \Big(\exp\Big(-\beta\frac{x}{\sigma(X_{t-})}\Big) + \exp(-\beta x)\Big)\,\mathrm{d}x = \exp\Big(-\frac{\beta}{\sigma(X_{t-})}\Big)\frac{\sigma(X_{t-})}{\beta} + \frac{1}{\beta}\exp(-\beta).$$

Here the first term on the right-hand side is bounded by $\sigma(X_{t-})/\beta$. As X is increasing, X_{t-} is between zero and X_T, which is finite $\mathbb{P}^\sigma_T$-a.s. By the local boundedness of σ, also $\sup_{t \le T} \sigma(X_{t-}) \le \sup_{x \le X_T} \sigma(x)$ is finite $\mathbb{P}^\sigma_T$-a.s. From the obtained bounds on $h^<_t$ and $h^>_t$ it follows that H_T is a.s. bounded under $\mathbb{P}^\sigma_T$, so the condition (4) is satisfied. The expression for the likelihood ratio as a Doléans exponential in (6) follows from Theorem III.5.19 in Jacod and Shiryaev [7]. $\square$

The next proposition gives an explicit expression for the Radon–Nikodym derivative in Proposition 3. This is useful when computations with this Radon–Nikodym derivative have to be done, for instance for likelihood based inference in a statistical analysis.

Proposition 4. *Let the conditions of Proposition 3 hold. Then the solution Z to (7) has at any time $T > 0$ the explicit representation*

$$Z_T = \exp\Big(\int_0^T \int_0^\infty \log Y(t,x)\, \mu^X(\mathrm{d}x, \mathrm{d}t) - \int_0^T \int_0^\infty (Y(t,x) - 1)\nu(x)\, \mathrm{d}x\, \mathrm{d}t\Big), \tag{8}$$

Proof. It follows from Lemma 18.8 in Liptser and Shiryaev [13], under the condition that the process F defined by $F = \int_0^\cdot \int_{(0,\infty)} (Y(t,x) - 1)(\mu^X(\mathrm{d}x, \mathrm{d}t) - \nu(x)\, \mathrm{d}x\, \mathrm{d}t)$ is a process of finite variation, that the explicit expression in (8) holds. We proceed by showing that F has a.s. finite variation over any interval $[0, T]$. Note first that the variation of F over $[0, T]$ is $\|F\|_T := \int_{(0,T]} \int_{(0,\infty)} |Y(t,x) - 1|(\mu^X(\mathrm{d}x, \mathrm{d}t) + \nu(x)\, \mathrm{d}x\, \mathrm{d}t)$. In view of Proposition II.1.28 in Jacod and Shiryaev [7], it is sufficient to check that $\int_{(0,T]} \int_{(0,\infty)} |Y(t,x) - 1|\nu(x)\, \mathrm{d}x\, \mathrm{d}t$ is finite. We consider the inner integral, split into two integrals, one for $x \geq 1$, one for $0 < x < 1$. Consider first

$$\begin{aligned}
\int_{x\geq 1} |Y(t,x) - 1|\nu(x)\, \mathrm{d}x &= \int_1^\infty \Big|\exp(-\beta x(\frac{1}{\sigma(X_{t-})} - 1)) - 1\Big|\frac{\alpha}{x} e^{-\beta x}\, \mathrm{d}x \\
&\leq \alpha \int_1^\infty \Big|\exp\Big(-\frac{\beta x}{\sigma(X_{t-})}\Big) - \exp(-\beta x)\Big|\, \mathrm{d}x \\
&= \frac{\alpha}{\beta}\Big|\sigma(X_{t-}) \exp\Big(-\frac{\beta}{\sigma(X_{t-})}\Big) - \exp(-\beta)\Big|.
\end{aligned}$$

We find that $\int_{(0,T]} \int_{x\geq 1} |Y(t,x) - 1|\nu(x)\, \mathrm{d}x\, \mathrm{d}t$ is finite a.s., as σ is assumed to be a locally bounded function. For the other inner integral we need the elementary inequality for $p, q > 0$,

$$\int_0^1 \frac{|e^{-px} - e^{-qx}|}{x}\, \mathrm{d}x \leq |p - q|. \tag{9}$$

To see that (9) holds true, we assume w.l.o.g. $p > q$. Then we have, using $e^{-ux} \leq 1$ below,

$$\begin{aligned}
\int_0^1 \frac{|e^{-px} - e^{-qx}|}{x}\, \mathrm{d}x &= \int_0^1 \frac{e^{-qx} - e^{-px}}{x}\, \mathrm{d}x \\
&= \int_0^1 \int_q^p e^{-ux}\, \mathrm{d}u\, \mathrm{d}x \\
&\leq p - q,
\end{aligned}$$

which shows (9). Using now (9) and that σ is lower bounded by σ_0, we find

$$\begin{aligned}
\int_{0<x<1} |Y(t,x) - 1|\nu(x)\, \mathrm{d}x &= \int_0^1 \Big|\exp(-\beta x(\frac{1}{\sigma(X_{t-})} - 1)) - 1\Big|\frac{\alpha}{x} e^{-\beta x}\, \mathrm{d}x \\
&= \int_0^1 \Big|\exp\Big(-\frac{\beta x}{\sigma(X_{t-})}\Big) - \exp(-\beta x)\Big|\frac{\alpha}{x}\, \mathrm{d}x \\
&\leq \alpha\beta\Big|\frac{1}{\sigma(X_{t-})} - 1\Big| \leq \alpha\beta\left(\frac{1}{\sigma_0} + 1\right).
\end{aligned}$$

Hence also $\int_{(0,T]} \int_{0<x<1} |Y(t,x) - 1|\nu(x)\, \mathrm{d}x\, \mathrm{d}t$ is finite. $\square$

3. Weak solutions

We will use a variation of Proposition 3 to establish existence of a weak solution to (1) under a growth condition on σ. The precise result follows.

Theorem 5. *Assume that $\sigma : [0,\infty) \to [0,\infty)$ is measurable, lower bounded by a constant $\sigma_0 > 0$, and satisfies a linear growth condition, i.e. there exists $K > 0$ such that for all $x \geq 0$ it holds that $\sigma(x) \leq K(1+x)$. Then, on the interval $[0,\infty)$, Eq. (1) admits a weak solution that is unique in law.*

Proof. This proof is inspired by Section 5.3B of Karatzas and Shreve [9] for a similar problem in a Brownian setting. Fix $T > 0$ and consider a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ on which X is defined as the gamma process. We choose Ω to be the Skorohod space, $\mathcal{F} = \mathcal{F}^X = \sigma(X_t, t \geq 0)$, and X the coordinate process. Furthermore we use the filtration $\mathbb{F}^X = \{\mathcal{F}_t^X, t \geq 0\}$. The restriction of $\mathbb{Q}$ to $\mathcal{F}_T^X$ is denoted $\mathbb{Q}_T$. As a semimartingale, under $\mathbb{Q}$, X has third characteristic $\nu^{X,\mathbb{Q}}(\mathrm{d}x, \mathrm{d}t) = v(x)\mathrm{d}x\mathrm{d}t$ with v as in (2). Define L by $\mathrm{d}L_t = \frac{1}{\sigma(X_{t-})}\,\mathrm{d}X_t$ and $L_0 = 0$. Since σ is bounded from below and measurable, the process L is well-defined. We again take

$$Y(t,z) = \frac{1}{\sigma(X_{t-})}\frac{v(\frac{z}{\sigma(X_{t-})})}{v(z)},$$

and make a measure change on $\mathcal{F}_T^X$, parallel to Proposition 3,

$$\frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\mathbb{Q}_T} = Z_T := \mathcal{E}_T\left(\int_0^{\cdot}\int_{(0,\infty)}(Y(t,z)-1)(\mu^X(\mathrm{d}z,\mathrm{d}t) - \nu^{X,\mathbb{Q}}(\mathrm{d}z,\mathrm{d}t))\right).$$

Provided that $\mathbb{P}_T$ is a probability measure on $\mathcal{F}_T^X$, which happens if $\mathbb{E}_{\mathbb{Q}} Z_T = 1$, the third characteristic of X under $\mathbb{P}_T$ is, similar to Lemma 2,

$$\nu^{X,\mathbb{P}}(\mathrm{d}z,\mathrm{d}t) = Y(t,z)\nu^{X,\mathbb{Q}}(\mathrm{d}z,\mathrm{d}t) = v^{X,\mathbb{P}}(t,z)\mathrm{d}z\mathrm{d}t,$$

where $v^{X,\mathbb{P}}(t,z) = \frac{1}{\sigma(X_{t-})}v(\frac{z}{\sigma(X_{t-})})$, so $v^{X,\mathbb{P}}(t,z) = v^\sigma$ with v^σ as in (3). By the arguments in the proof of Lemma 2, one obtains that under $\mathbb{P}_T$ the process L has third characteristic $\nu^{L,\mathbb{P}}(\mathrm{d}z,\mathrm{d}t) = v^{L,\mathbb{P}}(t,z)\mathrm{d}z\mathrm{d}t$, with $v^{L,\mathbb{P}}(t,z) = v^{X,\mathbb{P}}(t, z\sigma(X_{t-}))\sigma(X_{t-})$, which is nothing else but $v(z)$, implying that under $\mathbb{P}_T$, L is a gamma process on $[0,T]$, and it also holds that $\mathrm{d}X_t = \sigma(X_{t-})\,\mathrm{d}L_t$. We conclude that under $\mathbb{P}_T$, X is a solution of the SDE, where L is a gamma process with Lévy density v. What remains to be shown for existence of a weak solution is that $\mathbb{P}_T$ is a probability measure on $\mathcal{F}_T^X$. We use Theorem IV.3 of Lépingle and Mémin [12], this will be done via a detour as a direct application does not give the desired results. First we compute

$$\int_0^\infty (y(\sigma(x),z)\log y(\sigma(x),z) - y(\sigma(x),z) + 1)v(z)\,\mathrm{d}z,$$

where

$$y(\sigma,z) = \frac{1}{\sigma}v(\frac{z}{\sigma})/v(z) = \exp(-\beta z/\sigma + \beta z).$$

Consider $f(\sigma) = \int_0^\infty (y(\sigma,z)\log y(\sigma,z) - y(\sigma,z) + 1)v(z)\,\mathrm{d}z$. As an intermezzo we now show that

$$f(\sigma) = \alpha(\sigma - 1 - \log\sigma). \tag{10}$$

To see this we need the following computation for $b, a > 0$ with an application of Fubini's theorem,

$$\begin{aligned}\int_0^\infty \frac{e^{-az} - e^{-bz}}{z}\,\mathrm{d}z &= \int_0^\infty \int_a^b e^{-uz}\,\mathrm{d}u\,\mathrm{d}z\\ &= \int_a^b \int_0^\infty e^{-uz}\,\mathrm{d}z\,\mathrm{d}u = \log\frac{b}{a}.\end{aligned}$$

Then we get

$$\begin{aligned}f(\sigma) &= \int_0^\infty (y(\sigma,z)\log y(\sigma,z) - y(\sigma,z) + 1)\nu(z)\,\mathrm{d}z\\ &= \int_0^\infty \big(\exp(-\frac{\beta z}{\sigma} + \beta z)(-\frac{\beta z}{\sigma} + \beta z) - \exp(-\frac{\beta z}{\sigma} + \beta z) + 1\big)\frac{\alpha}{z}e^{-\beta z}\,\mathrm{d}z\\ &= \alpha\int_0^\infty \big(\exp(-\frac{\beta z}{\sigma})(-\frac{\beta}{\sigma} + \beta) - \frac{\exp(-\frac{\beta z}{\sigma}) - \exp(-\beta z)}{z}\big)\,\mathrm{d}z\\ &= \alpha\big(\frac{\sigma}{\beta}(-\frac{\beta}{\sigma} + \beta) - \log\sigma\big)\\ &= \alpha\big(-1 + \sigma - \log\sigma\big).\end{aligned}$$

Having established (10), we simply find

$$\int_{(0,\infty)} (Y(t,z)\log Y(t,z) - Y(t,z) + 1)\nu(z)\mathrm{d}z = \alpha(\sigma(X_{t-}) - 1 - \log\sigma(X_{t-})).$$

Fix an integer N, to be judiciously chosen later, and $\delta = 1/N$. Put $T_n = n\delta T$, for $n = 0, \ldots, N$. Then $T_N = T$, and $T_n - T_{n-1} = \delta T$ for $n = 1, \ldots, N$. Let Z^n be the solution to

$$\mathrm{d}Z_t^n = Z_{t-}^n\left(\int_{(0,\infty)} \mathbf{1}_{(T_{n-1},T_n]}(t)(Y(t,x) - 1)(\mu^X(\mathrm{d}x,\mathrm{d}t) - \nu^{X,\mathbb{Q}}(\mathrm{d}x,\mathrm{d}t))\right), \tag{11}$$

with $Z_0^n = 1$, for $n = 1, \ldots, N$. If the Z^n are martingales, not just local martingales, under $\mathbb{Q}$ w.r.t. the filtration $\mathbb{F}^X$, then $\mathbb{E}_{\mathbb{Q}}[Z_{T_n}^n|\mathcal{F}_{T_{n-1}}^X] = Z_{T_{n-1}}^n$. Consider again the stochastic differential equation for Z^n, which can, in obvious notation, be abridged to $\mathrm{d}Z_t^n = Z_{t-}^n\mathrm{d}M_t^n = Z_{t-}^n\mathbf{1}_{(T_{n-1},T_n]}(t)\mathrm{d}M_t$. Note that $M_t^n = 0$ for $t \le T_{n-1}$. It follows that $Z_t^n = 1$ for $t \le T_{n-1}$, and hence $\mathbb{E}_{\mathbb{Q}}[Z_{T_n}^n|\mathcal{F}_{T_{n-1}}^X] = 1$. Since $Z_T = \prod_{n=1}^N Z_{T_n}^n$, one obtains

$$\mathbb{E}_{\mathbb{Q}}Z_T = \mathbb{E}_{\mathbb{Q}}\prod_{n=1}^{N-1} Z_{T_n}^n\mathbb{E}_{\mathbb{Q}}[Z_{T_N}^N|\mathcal{F}_{T_{N-1}}] = \mathbb{E}_{\mathbb{Q}}\prod_{n=1}^{N-1} Z_{T_n}^n = \mathbb{E}_{\mathbb{Q}}Z_{T_{N-1}},$$

which can be seen equal to one by an induction argument. To see that the Z^n are martingales, we use Theorem IV.3 of Lépingle and Mémin [12], i.e. the aim is to show

$$\mathbb{E}_{\mathbb{Q}}\exp\left(\int_0^T\int_{[0,\infty)} \mathbf{1}_{(T_{n-1},T_n]}(t)(Y(t,z)\log Y(t,z) - Y(t,z) + 1)\nu(z)\mathrm{d}z\mathrm{d}t\right) < \infty.$$

In view of the computations above this amounts to showing that

$$\mathbb{E}_{\mathbb{Q}}\exp\left(\int_{T_{n-1}}^{T_n} \alpha(\sigma(X_{t-}) - 1 - \log\sigma(X_{t-}))\mathrm{d}t\right) < \infty.$$

Using that σ is lower bounded and satisfies the growth condition, and that X is an increasing process under $\mathbb{Q}$, we have that the integrand above is upper bounded by $\alpha(K(1 + X_t) - 1 -$

$\log\sigma_0) \leq C + \alpha K X_T$, for some $C > 0$. Hence, $\int_{T_{n-1}}^{T_n} \alpha(\sigma(X_{t-}) - 1 - \log\sigma(X_{t-}))\mathrm{d}t \leq \delta\alpha K T X_T + C'$, where C' is another positive constant, so it is sufficient to prove that $\mathbb{E}_{\mathbb{Q}}\exp(\delta\alpha K T X_T) < \infty$.

At this point we note, as can be verified by a simple computation of an integral, that for a random variable X having a Gamma(a, b) distribution, it holds that $\mathbb{E}\exp(cX) = (\frac{b}{b-c})^a$ for $c < b$. Using this property and recalling that, under $\mathbb{Q}$, X_T has a Gamma$(\alpha T, \beta)$ distribution, the expectation $\mathbb{E}_{\mathbb{Q}}\exp(\delta\alpha K T X_T)$ is seen to be finite if $\delta < \beta/(\alpha K T)$, equivalently $N > \alpha K T/\beta$. For such a choice of N, we obtain that the Z^n are martingales and hence $\mathbb{E}_{\mathbb{Q}} Z_T = 1$. As a consequence $\mathbb{P}_T$ is a probability measure on $(\Omega, \mathcal{F}_T^X)$ for every $T > 0$. In fact the $\{\mathbb{P}_T\}$ form a consistent family of probability measures, and hence there exists a probability measure $\mathbb{P}$ on $\mathcal{F}$ such that $\mathbb{P}|_{\mathcal{F}_T^X} = \mathbb{P}_T$, see Lemma 18.18 in Kallenberg [8]. This shows that there exists a weak solution on the entire time interval $[0, \infty)$. Finally, we turn to uniqueness in law of a weak solution. Consider two possible weak solutions X^i, or rather (X^i, L^i) for $i = 1, 2$, on an interval $[0, T]$, defined on their own filtered probability spaces $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i)$. Consider changes of measures $\mathrm{d}\tilde{\mathbb{P}}^i = Z_T^i \mathrm{d}\mathbb{P}^i$. Here we take $T = t$ in

$$Z_t^i = \mathcal{E}_t\left(\int_0^{\cdot}\int_{(0,\infty)}(\tilde{Y}^i(s,z) - 1)(\mu^{X^i}(\mathrm{d}z, \mathrm{d}s) - \nu^{X^i,\mathbb{P}^i}(\mathrm{d}z, \mathrm{d}s))\right),$$

where

$$\tilde{Y}^i(s,z) = \frac{\sigma(X_{s-}^i)v(z)}{v(z/\sigma(X_{s-}^i))} = \tilde{y}(\sigma(X_{s-}^i), z),$$

with $\tilde{y}(\sigma, z) = \frac{1}{y(\sigma,z)} = \exp(\beta z/\sigma - \beta z)$. We assume for a while that the Z_T^i have expectations one under $\mathbb{P}^i$ so that the $\tilde{\mathbb{P}}^i$ are probability measures on $\mathcal{F}_T^i$, equivalent to the $\mathbb{P}^i$. By the arguments used earlier in this proof, under $\tilde{\mathbb{P}}^i$ the processes X^i are gamma processes with parameters α, β. Hence the distributions, for $i = 1, 2$, of the X^i are identical under the probability measures $\tilde{\mathbb{P}}^i$. Consider then samples $X^i(n) = (X_{t_1}^i, \ldots, X_{t_n}^i)$, $0 \leq t_1 \leq, \ldots, \leq t_n \leq T$, and Borel sets B of $\mathbb{R}^n$. Then

$$\mathbb{P}^i(X^i(n) \in B) = \mathbb{E}_{\tilde{\mathbb{P}}^i}\frac{1}{Z_T^i}\mathbf{1}_{\{X^i(n)\in B\}}. \tag{12}$$

On the right hand side of Eq. (12), all random quantities are defined in terms of X^i, hence the expectations in (12) are the same for $i = 1, 2$. This shows that the finite dimensional distributions of X^1 and X^2 are identical and hence the laws of X^1 and X^2 are the same as well. It is left to show that the Z_T^i have expectations one under $\mathbb{P}^i$. We follow the same path as above, we first compute $\tilde{F}(x) := \int(\tilde{y}(\sigma(x), z)\log\tilde{y}(\sigma(x), z) - \tilde{y}(\sigma(x), z) + 1)\frac{v(z/\sigma)}{\sigma}\mathrm{d}z$. To that end we consider $\tilde{f}(\sigma) = \int(\tilde{y}(\sigma, z)\log\tilde{y}(\sigma, z) - \tilde{y}(\sigma, z) + 1)\frac{v(z/\sigma)}{\sigma}\mathrm{d}z$. It turns out that $\tilde{f}(\sigma) = \alpha(\frac{1}{\sigma} - 1 + \log\sigma)$, so $\tilde{f}(\sigma) = f(\frac{1}{\sigma})$, and hence $\tilde{F}(x) = \alpha(\frac{1}{\sigma(x)} - 1 + \log\sigma(x))$. From here we continue to solve the SDE for Z^i on intervals $(T_{n-1}, T_n]$, similar to (11), with $T_n = n\delta T$, resulting in processes $Z^{i,n}$ that are martingales with $Z_{T_{n-1}}^{i,n} = 1$. To show the martingale property for $Z^{i,n}$, we use again Theorem IV.3 of Lépingle and Mémin [12], i.e. we show

$$\mathbb{E}_{\mathbb{P}^i}\exp\left(\int_{T_{n-1}}^{T_n}\alpha\Big(\frac{1}{\sigma(X_{t-}^i)} - 1 + \log\sigma(X_{t-}^i)\Big)\mathrm{d}t\right) < \infty.$$

Here the integrand is bounded by $\alpha(\frac{1}{\sigma_0} - 1 + \log K + \log(1 + X_T^i))$. Hence for a constant C, depending on T, the exponent is less than or equal to $C(1 + X_T^i)^{\delta\alpha T}$, and we have to show for a well chosen $\delta > 0$ that $\mathbb{E}_{\mathbb{P}^i}(1 + X_T^i)^{\delta\alpha T} < \infty$, equivalently $\mathbb{E}_{\mathbb{P}^i}(X_T^i)^{\delta\alpha T} < \infty$. The conditions in Proposition 4.1 of Klebaner and Liptser [10] (on their operator $L_s(x_{s-})$) are satisfied by the linear growth condition on σ, and as a result one has $\mathbb{E}_{\mathbb{P}^i}(X_T^i)^2 < \infty$. Therefore we take $\delta < 2/\alpha T$. $\square$

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