Vanishing theorems for Shimura varieties at unipotent level

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Abstract. We show that the compactly supported cohomology of Shimura varieties of Hodge type of infinite $\Gamma_1(p^\infty)$-level (defined with respect to a Borel subgroup) vanishes above the middle degree, under the assumption that the group of the Shimura datum splits at p. This generalizes and strengthens the vanishing result proved in [A. Caraiani et al., Compos. Math. 156 (2020)]. As an application of this vanishing theorem, we prove a result on the codimensions of ordinary completed homology for the same groups, analogous to conjectures of Calegari–Emerton for completed (Borel–Moore) homology.

Keywords. Locally symmetric spaces, perfectoid spaces, p-adic automorphic forms

1. Introduction

This paper proves a generalization of the main geometric result of [7], and gives an application to the bounds on the codimensions of ordinary completed cohomology groups for certain Shimura varieties. Along the way we prove results on finite group quotients of adic spaces and diamonds, and a Poincaré duality spectral sequence for ordinary completed cohomology, which we consider to be of independent interest. Before giving a brief introduction to our results, we refer the interested reader to the introduction of [7] for further context.

Fix a prime p. We give an overview of the setup, referring to the main text for details. Let G be a connected reductive group over $\mathbb{Q}$ admitting a Shimura datum of Hodge type. Assume that G is split at p and choose a split model over $\mathbb{Z}_p$. Choose a Borel subgroup B

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of G over $\mathbb{Z}_p$ and let $U \subseteq B$ be its unipotent radical. If $K \subseteq G(\mathbb{Z}_p)$ is a compact open subgroup, we write X_K for the complex Shimura variety for G of level K at p and some fixed tame level[1] $K^p \subseteq G(\mathbb{A}_f^p)$, viewed as an algebraic variety. We write $X_K(\mathbb{C})$ for the corresponding complex manifold. We may state our main vanishing theorem as follows:

Theorem 1.1 (Corollary 4.1.2, Remark 4.1.3). *Let d be the complex dimension of the Shimura varieties for G. Let $H \subseteq U(\mathbb{Z}_p)$ be a closed subgroup. Then*

$$\varinjlim_{K \supseteq H} H_c^i(X_K(\mathbb{C}), \mathbb{Z}/p^r) = 0 \quad \text{for all } r \geq 1 \text{ and all } i > d.$$

The cohomology here is singular cohomology with compact support. This theorem generalizes [7, Theorem 1.1.2] in two ways. First, the latter only treats Shimura varieties for quasi-split (general) unitary and symplectic groups over totally real fields – here we generalize this to all Hodge type Shimura varieties (in both cases assuming the same splitness conditon at p). Second, in [7, Theorem 1.1.2] we require the subgroup H to be contained in the $\mathbb{Z}_p$-points of the unipotent radical of the Siegel parabolic; this is a stronger assumption than the containment $H \subseteq U(\mathbb{Z}_p)$ in Theorem 1.1.

The method of proof is a variation of that of [7, Theorem 1.1.2], and we refer to the introduction of [7] for a more elaborate sketch. Choosing an embedding $\mathbb{C} \hookrightarrow C$ into an algebraically closed nonarchimedean field C we may base change the X_K to C and then analytify to get rigid analytic varieties $\mathcal{X}_K$. As in [28], we make use of compactifications $\overline{\mathcal{X}}_K$ of the $\mathcal{X}_K$ which are closely related to the minimal compactifications. Through a string of comparison theorems, one reduces Theorem 1.1 to proving

$$H_{\text{ét}}^i(\overline{\mathcal{X}}, j_!(\mathcal{O}_{\mathcal{X}}^+/p))^a = 0 \quad \text{for } i > d,$$

where

$$j \colon \mathcal{X}_H := \varprojlim_{K \supseteq H} \mathcal{X}_K \hookrightarrow \overline{\mathcal{X}}_H := \varprojlim_{K \supseteq H} \overline{\mathcal{X}}_K$$

is the inclusion, the inverse limits are taken as diamonds in the sense of [29], and $-^a$ denotes the corresponding almost module with respect to $\mathcal{O}_C$ and its maximal ideal. As in [7], $H_{\text{ét}}^i(\overline{\mathcal{X}}_H, j_!(\mathcal{O}_{\mathcal{X}}^+/p))^a$ is analyzed using the Leray spectral sequence for a descent π of the Hodge–Tate period map which goes from $\overline{\mathcal{X}}_H$ to a quotient of a partial flag variety $\mathscr{F}\ell_{G,\mu}$ for G, and the "Bruhat" stratification of $\mathscr{F}\ell_{G,\mu}$ into Schubert cells for the action of B.

Apart from the fact that we treat more general Shimura varieties, there are two principal differences between the argument presented here and that of [7] that we wish to point out. The first is that we need to adapt the results of [7, §4] on the Schubert cells for the Siegel parabolic on $\mathscr{F}\ell_{G,\mu}$ to the Schubert cells for B. In fact, it turns out that the arguments flow more naturally in this setting. The second is that our analysis of the

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[1] K^p is assumed to be sufficiently small in a way that we make precise in §3.2.

fibers of π uses some new and different techniques. The argument in [7] relies heavily on a general result about the existence of invariant rational neighborhoods for profinite group actions on affinoid adic spaces [7, Proposition 5.2.1]. Here, we instead make use of some new results on quotients of diamonds by finite groups, which we consider to be of independent interest. A corollary is a strengthening of [14, Theorem 1.4] showing that quotients of affinoid perfectoid spaces are affinoid perfectoid (Theorem 2.1.2). We discuss the differences between our argument and the argument of [7] in more detail in the introduction to §4 and in Remark 4.2.8.

We give one application of Theorem 1.1 in this paper. Hida's theory of the *ordinary projector* and *ordinary automorphic forms* has played a key role in the p-adic study of automorphic forms since its introduction in the 1980's. Hida's constructions come in different flavors, with the most general being in terms of the singular cohomology of locally symmetric spaces. It was later realized by Emerton [11] that the ordinary projector is closely related to the right adjoint of the parabolic induction functor in the mod p and p-adic representation theory of p-adic reductive groups. Moreover, Hida's construction can be recovered[2] by applying this right adjoint to Emerton's *completed cohomology*, which plays a prominent role in the p-adic Langlands program at present (see [6, 13] for surveys).

In this paper we follow Hida's approach and look at the ordinary "completed"[3] (Borel–Moore) homology groups

$$\widetilde{H}_i^{(\mathrm{BM}),\mathrm{ord}} := \varprojlim_{K \supseteq U(\mathbb{Z}_p)} H_i^{(\mathrm{BM})}(X_K(\mathbb{C}), \mathbb{Z}_p)^{\mathrm{ord}},$$

where on the right hand side $H_i^{(\mathrm{BM})}$ denotes i-th (Borel–Moore) homology and the superscript $-^{\mathrm{ord}}$ denotes the ordinary part (which is the image of the ordinary projector, i.e. the direct summand where certain U_p-like operators for G act invertibly[4]). These are direct summands of the corresponding completed cohomology groups

$$\widetilde{H}_i^{(\mathrm{BM})}(U(\mathbb{Z}_p)) := \varprojlim_{K \supseteq U(\mathbb{Z}_p)} H_i^{(\mathrm{BM})}(X_K(\mathbb{C}), \mathbb{Z}_p)$$

at unipotent level. The module $\widetilde{H}_i(U(\mathbb{Z}_p))$ is dual, in an appropriate sense, to the direct limits appearing in Theorem 1.1 for $H = U(\mathbb{Z}_p)$; see (5.5.1) for the precise statement. If $T \subseteq B$ is a maximal split torus, then $\widetilde{H}_i^{(\mathrm{BM}),\mathrm{ord}}$ is a finitely generated module over the Iwasawa algebra $\mathscr{D}(T_0)$, where $T_0 := T(\mathbb{Z}_p)$. Our second main result is then the following.

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[2]This statement needs some care to be made precise. Since this statement is only for context and will not be used in the paper, we will not elaborate on it.

[3]As is now common, we will occasionally use the term "completed (co)homology" to refer to more general (co)limits of (co)homology groups of locally symmetric spaces, in the spirit of the constructions in [6].

[4]We refer to §5.4 for a precise definition.

Theorem 1.2 (Theorem 5.5.1). *We have the following:*

(1) $\widetilde{H}_i^{\mathrm{BM,ord}} = 0$ *for* $i > d$. *In fact more is true: Let* $H \subseteq U(\mathbb{Z}_p)$ *be a closed subgroup. Then*

$$\varprojlim_{K \supseteq H} H_i^{\mathrm{BM}}(X_K(\mathbb{C}), \mathbb{Z}_p) = 0 \quad \text{for all } i > d.$$

(2) *We have*

$$\mathrm{codim}_{\mathscr{D}(T_0)} \widetilde{H}_i^{\mathrm{ord}} \geq d - i \quad \text{for all } 0 \leq i \leq d.$$

The codimension function may be defined as

$$\mathrm{codim}_{\mathscr{D}(T_0)} \widetilde{H}_i^{\mathrm{ord}} = \inf_j \{ j \mid \mathrm{Ext}_{\mathscr{D}(T_0)}^j(\widetilde{H}_i^{\mathrm{ord}}, \mathscr{D}(T_0)) \neq 0 \};$$

we discuss it further in §5.5. Theorem 1.2 is an analogue of an important conjecture of Calegari and Emerton for completed homology and completed Borel–Moore homology [6, Conjecture 1.5]. The analogue of Theorem 1.2 in that setting was proved by Scholze for Shimura varieties of Hodge type [28, Corollary 4.2.3]. The main ingredients in the proof of Theorem 1.2 are Theorem 1.1 and a "Poincaré duality" spectral sequence relating ordinary completed Borel–Moore homology to ordinary completed homology, which we consider to be of independent interest. Such a result has previously been announced by Emerton, though relying on a different method than ours.

The conjecture [6, Conjecture 1.5] is symmetric when swapping the roles of homology and Borel–Moore homology. However, we remark that one will need more care when formulating (conjectural) versions of Theorem 1.2 with homology and Borel–Moore homology swapped, or in analogus situations like eigenvarieties. When swapping homology and Borel–Moore homology, one has to factor in contributions from ordinary boundary homology. In particular, we remark that Theorem 1.1 fails if one replaces compactly supported cohomology with cohomology, and Theorem 1.2 (1) should fail if one replaces Borel–Moore homology with homology due to the presence of boundary homology. For eigenvarieties, conjectures on codimensions have been given by Urban [33, Conjecture 5.7.5], with partial results by Hansen and Newton [15] that are somewhat orthogonal to our Theorem 1.2.

Let us now give a short overview of the paper. Section 2 discusses results on quotients of diamonds by (pro)finite groups which are used in the analysis of the fibers of the Hodge–Tate period map. Section 3 introduces the Shimura varieties we consider in this paper and proves the perfectoidness results needed for Theorem 1.1, and section 4 is devoted to the proof of Theorem 1.1. Finally, Section 5 introduces ordinary completed (Borel–Moore) homology and proves Theorem 1.2.

2. Preliminaries on diamonds

In this section, we will prove some results on diamonds and v-sheaves that will be used later in the paper. We use the notation, terminology and conventions regarding diamonds and v-sheaves of [29]. Unless otherwise specified, v-sheaves are assumed to be on the category Perf of perfectoid spaces of characteristic p.

2.1. Quotients of diamond spectra and perfectoid spaces by finite groups

Proposition 2.1.1. *Let A be a complete Tate $\mathbb{Z}_p$-algebra with a continuous left action of a finite group G, and let A^+ be an open and integrally closed subring of A that is preserved by G. Let*

$$X = \mathrm{Spd}(A, A^+) \quad and \quad X_G = \mathrm{Spd}(A^G, A^{+G}).$$

Then $X \times \underline{G} \rightrightarrows X$ is a presentation of X_G as a v-sheaf.

Proof. We need to show that $X \to X_G$ and $X \times \underline{G} \to X \times_{X_G} X$ are surjections of v-sheaves.

The diamonds X, X_G, $X \times \underline{G}$, and $X \times_{X_G} X$ are spatial (in the last case we use [29, Corollary 11.29]), hence qcqs. So $X \to X_G$ and $X \times \underline{G} \to X \times_{X_G} X$ are qc. So by [29, Lemma 12.11], it suffices to show that $|X| \to |X_G|$ and $|X \times \underline{G}| \to |X \times_{X_G} X|$ are surjections. The former map is surjective by [14, Theorem 3.1].

To prove that the latter map is surjective, we use the characterization of the topological space of a diamond given in [29, Proposition 11.13]. It suffices to show that if (K, K^+) is a perfectoid field and $\phi_1, \phi_2 : (A, A^+) \to (K, K^+)$ have the same restriction to (A^G, A^{+G}), then ϕ_1, ϕ_2 are related by an element of G. Let $Y = \mathrm{Spec}(A)$ and $Y_G = \mathrm{Spec}(A^G)$. Let y_1, y_2 be points of $Y(K)$ corresponding to ϕ_1, ϕ_2. By [26, Thm. V.4.1 (iii)], $Y \times \underline{G} \to Y \times_{Y_G} Y$ is surjective. It follows that after extending K, y_1 and y_2 become related by an element of G. But then they must already be related by an element of G over K. So ϕ_1 and ϕ_2 are related by an element of G. ∎

Our main motivation for introducing the above result is to prove Proposition 3.2.2, but we also mention the following generalization of [14, Theorem 1.4], which will not be used in the rest of the paper. Recall that the category of adic spaces is, by Huber's definition, a full subcategory of a certain category of locally topologically ringed spaces with valuations on the stalks of the structure sheaf. We follow Kedlaya–Liu in calling such spaces *locally v-ringed spaces*; see [18, Definitions 8.2.1 and 8.2.2].

Theorem 2.1.2. *Let X be a perfectoid space with a right action of a finite group G. Suppose that X has a covering by G-stable open subspaces of the form $\mathrm{Spa}(A, A^+)$ with A perfectoid Tate. Let X/G be the coequalizer of $X \times \underline{G} \rightrightarrows X$ in the category of locally v-ringed spaces. Then X/G is perfectoid.*

The theorem is an immediate consequence of item (2) of the following proposition.

Proposition 2.1.3. *Let (A, A^+) be a perfectoid Tate–Huber pair with a continuous left action of a finite group G. Let $X = \mathrm{Spa}(A, A^+)$, and let $X_G = \mathrm{Spa}(A^G, A^{+G})$. Then:*

(1) *$X \times \underline{G} \rightrightarrows X \to X_G$ represents X_G as a coequalizer in the category of v-sheaves on Perfd.*

(2) *$X \times \underline{G} \rightrightarrows X \to X_G$ represents X_G as a coequalizer in the category of locally v-ringed spaces.*

Proof. In the statement of the proposition, we are implicitly using the fact that A^G is perfectoid [19, Thm. 3.3.25] and that any perfectoid space may be regarded as v-sheaf on Perfd since the v-site is subcanonical [29, Thm. 8.7].

By the same argument as in Proposition 2.1.1, $X \to X_G$ and $X \times \underline{G} \to X \times_{X_G} X$ are v-covers. This proves item (1).

By the argument of [14, Thm. 3.3], to prove item (2), it is enough to show that if $U = X_G(T/s)$ is a rational subset of X_G, then the natural map $A^G\langle T/s\rangle \to A\langle T/s\rangle^G$ is an isomorphism.

By item (1), in the category of v-sheaves on Perfd, $\mathrm{Spa}(A\langle T/s\rangle^G, A\langle T/s\rangle^{+G})$ is the coequalizer of $(X \times_{X_G} U) \times \underline{G} \rightrightarrows X \times_{X_G} U$. In a topos, coequalizers commute with base change, so this v-sheaf is canonically isomorphic to $X_G \times_{X_G} U = U = \mathrm{Spa}(A^G\langle T/s\rangle, A^G\langle T/s\rangle^+)$. Therefore $A^G\langle T/s\rangle \to A\langle T/s\rangle^G$ must be an isomorphism. ∎

Remark 2.1.4. The above argument seems to indicate that for general A, the map $A^G\langle T/s\rangle \to A\langle T/s\rangle^G$ must be "totally inseparable" (since v-sheafifying generally loses information about nilpotents and totally inseparable field extensions), but any "totally inseparable" extension of perfectoid rings must be an isomorphism, giving the result.

This observation led us to find a more direct proof of Proposition 2.1.3 (2), which we now sketch. Again we show that $A^G\langle T/s\rangle \to A\langle T/s\rangle^G$ is an isomorphism. By the tilting correspondence, it is enough to consider the case where A has characteristic p. Since $A^G\langle T/s\rangle \to A\langle T/s\rangle^G$ induces a bijection of adic spectra and $A^G\langle T/s\rangle$ is uniform, the map is injective and $A^G\langle T/s\rangle$ has the subspace topology. Let p^m be the largest power of p dividing the order of G. Any $a \in A\langle T/s\rangle^G$ is a limit of elements of $A[1/s]$; then $\binom{|G|}{p^m}a^{p^m}$ is a limit of the p^mth elementary symmetric polynomials in the translates of each element. So $\binom{|G|}{p^m}a^{p^m}$ is a limit of elements of $A^G[1/s]$; hence $\binom{|G|}{p^m}a^{p^m} \in A^G\langle T/s\rangle$. Since $\binom{|G|}{p^m}$ is not divisible by p, $a^{p^m} \in A^G\langle T/s\rangle$, implying $a \in A^G\langle T/s\rangle$ since $A^G\langle T/s\rangle$ is perfect and $A\langle T/s\rangle^G$ is reduced.

2.2. *Inverse limits of surjections of v-sheaves*

This short subsection consists of a single lemma we will need later. While we state the lemma in its natural generality, in practice we will only need the case when the indexing system I has a cofinal subsystem isomorphic to $(\mathbb{Z}_{\geq 1}, \leq)$.

Lemma 2.2.1. *Suppose we have cofiltered inverse systems of v-sheaves $X_i, Y_i, i \in I$, with compatible morphisms $X_i \to Y_i$ that are qcqs and surjective. Let $X := \varprojlim_i X_i$ and $Y := \varprojlim_i Y_i$. Then $X \to Y$ is surjective.*

Proof. It suffices to show that for any qcqs Z with a map $Z \to Y$, $Z \times_Y X \to Z$ is surjective. For each i, let $Z_i := Z \times_{Y_i} X_i$; then $Z \times_Y X = \varprojlim_i Z_i$. For each $i < j$, the map $Z_j \to Z_i$ is qcqs since it is a base change of $X_j \to X_i \times_{Y_i} Y_j$, and $X_j \to Y_j$ is qcqs and $X_i \to Y_i$ is qs. By the argument of [29, Lemma 12.17], we can find a diagram W_i of spatial diamonds with compatible qcqs surjections $W_i \to Z_i$. Then each $W_i \to Z$ is qcqs

and surjective. By the argument of loc. cit., $\varprojlim_i W_i \to Z$ is surjective. So $\varprojlim_i Z_i \to Z$ must be surjective. ∎

2.3. Subsheaves and quotients

Recall that if S is any topological space, then $\underline{S}$ denotes the v-sheaf $Z \mapsto \mathrm{Map}_{\mathrm{cts}}(|Z|, S)$ on Perf. In the following, subsets of topological spaces will be equipped with the subspace topology.

Lemma 2.3.1. *Let X be a spatial v-sheaf, and let S be a qc and generalizing subset of $|X|$. Then:*

(1) $X \times_{|X|} \underline{S} = \varprojlim_{U \supseteq S} U$, *where U runs over qc open sheaves of X containing S.*

(2) $X \times_{|X|} \underline{S}$ *is spatial.*

(3) *If X is a diamond, then $X \times_{|X|} \underline{S}$ is a diamond.*

Proof. The first claim follows from [29, Proposition 12.9] (since $S = \bigcap |U|$), and then the remaining claims follow immediately from the first claim, [29, Lemma 12.17] and [29, Lemma 11.22], respectively. ∎

The following lemma will be key to the arguments of this paper.

Lemma 2.3.2. *Let X be a spatial v-sheaf with a right action of a profinite group G. Let X/G denote the quotient of X by G in the category of v-sheaves, and let $\pi\colon X \to X/G$ be the quotient map. Let $S \subset |X|$ be a qc and generalizing subset. Suppose that the multiplication map $S \times G \to S \cdot G$ is a bijection. Then the natural map $X \times_{|X|} \underline{S} \to X/G \times_{|X/G|} \underline{\pi(S)}$ is an isomorphism of v-sheaves.*

Proof. The product $X/G \times_{|X/G|} \underline{\pi(S)}$ can be identified with $(X \times_{|X|} \underline{S \cdot G})/G$. Since $((X \times_{|X|} \underline{S}) \times \underline{G})/G \cong X \times_{|X|} \underline{S}$, it is enough to show that $(X \times_{|X|} \underline{S}) \times \underline{G} \to X \times_{|X|} (\underline{S \cdot G})$ is an isomorphism, for which it is enough to show that $\underline{S} \times \underline{G} \to \underline{S \cdot G}$ is an isomorphism. This, in turn, reduces to showing that $S \times G \to S \cdot G$ is a homeomorphism. But this map is easily seen to be spectral and generalizing, and it is bijective by assumption. It is then a homeomorphism by [31, Tag 09XU]. ∎

3. Shimura varieties

In this section we define the Shimura varieties that we will work with and prove the perfectoidness results and results on flag varieties that we will need.

3.1. Setup

We start by setting up some notation and assumptions. We let $(G, \mathcal{H})$ be a Shimura datum of Hodge type, and we assume (crucially!) that G is *split* over $\mathbb{Q}_p$. Being split over $\mathbb{Q}_p$,

G has a natural split reductive model over $\mathbb{Z}_p$ [9, Theorem 6.1.16], and we will use the letter G for this model as well, or sometimes $G_{\mathbb{Z}_p}$ for emphasis. Let $E = E(G, \mathcal{H})$ be the reflex field of $(G, \mathcal{H})$. Since G is split over $\mathbb{Q}_p$, $E_{\mathfrak{p}} = \mathbb{Q}_p$ for every prime $\mathfrak{p}$ above p (the local reflex field is the localization of the global reflex field). From now on we fix such a prime $\mathfrak{p}$ of E above p, or equivalently an embedding $E \hookrightarrow \mathbb{Q}_p$. The remainder of this subsection will be devoted to constructing a particular embedding $(G, \mathcal{H}) \hookrightarrow (\widetilde{G}, \widetilde{\mathcal{H}})$ into a Siegel Shimura datum $(\widetilde{G}, \widetilde{\mathcal{H}})$ with certain convenient properties. We start by recalling a lemma from [21].

Lemma 3.1.1. *Let G be a reductive group over $\mathbb{Z}_p$. Let V be a finite-dimensional vector space over $\mathbb{Q}_p$, and let $\rho\colon G_{\mathbb{Q}_p} \hookrightarrow \mathrm{GL}(V)$ be a closed embedding of algebraic groups. Then there exists a lattice $\Lambda \subset V$ such that ρ extends to a map $G \to \mathrm{GL}(\Lambda)$.*

Proof. This is proved in [21, Lemma 2.3.1]. The statement of that lemma includes some additional hypotheses when $p = 2$, but these are only used to guarantee that the map of integral models is a closed embedding, which we do not need. ∎

Next we recall a version of Zarkhin's trick.

Lemma 3.1.2 (Zarkhin's trick, [34, §2]). *Let V be a finite-dimensional symplectic $\mathbb{Q}$-vector space and let $(\mathrm{GSp}(V), \mathcal{H}_V)$ be the associated Siegel Shimura datum. Let $\Lambda \subset V$ be a lattice. Then there is a symplectic form on $W = V^4 \oplus (V^\vee)^4$ preserved up to scaling by $\mathrm{GSp}(V)$ that makes the lattice $\Lambda^4 \oplus (\Lambda^\vee)^4 \subset V^4 \oplus (V^\vee)^4$ self-dual. Moreover, the closed embedding $\mathrm{GSp}(V) \hookrightarrow \mathrm{GSp}(W)$ induces a closed embedding*

$$(\mathrm{GSp}(V), \mathcal{H}_V) \hookrightarrow (\mathrm{GSp}(W), \mathcal{H}_W)$$

of Siegel Shimura data.

For convenience, we make the following definition.

Definition 3.1.3. Let R be a principal ideal domain, and let Λ be a finite free module over R equipped with a symplectic form. Let $\mu\colon (\mathbb{G}_m)_R \to \mathrm{GSp}(\Lambda)$ be a cocharacter. Recall [26, Proposition I.4.7.3] that μ induces a decomposition $\Lambda = \bigoplus_{n \in \mathbb{Z}} \Lambda_n$, where μ acts on Λ_n by $\mu(z)(v) = z^n v$.

We say that μ is *standard* if Λ_0 and Λ_1 are nonzero and all other Λ_i are zero.

For any cocharacter $\mu\colon (\mathbb{G}_m)_R \to \mathrm{GSp}(\Lambda)$, the composition of μ with the similitude factor $\mathrm{GSp}(\Lambda) \to (\mathbb{G}_m)_R$ must be of the form $z \mapsto z^n$ for some integer n. Then for each $i \in \mathbb{Z}$, the symplectic form pairs Λ_i with Λ_{n-i}. If μ is standard, then the nondegeneracy of the symplectic form forces $n = 1$ and Λ_0 and Λ_1 to be maximal isotropic.

We now return to our Hodge type Shimura datum $(G, \mathcal{H})$ and the split integral model $G_{\mathbb{Z}_p}$. Let us fix a choice of Hodge cocharacter μ for G, viewed as a cocharacter over $\mathbb{Z}_p$. If $\rho'\colon (G, \mathcal{H}) \hookrightarrow (\mathrm{GSp}(V), \mathcal{H}_V)$ is any closed embedding into a Siegel Shimura datum, then $\rho' \circ \mu_{\mathbb{Q}_p}$ is standard (i.e. a Hodge cocharacter for $(\mathrm{GSp}(V), \mathcal{H}_V)$, over $\mathbb{Q}_p$). The following proposition summarizes the extra conditions we would like to put on our embedding and shows that they are possible to achieve.

Proposition 3.1.4. *Let* $(G, \mathcal{H})$, $G_{\mathbb{Z}_p}$ *and* μ *be as above. Then there exists a symplectic* $\mathbb{Q}$-*vector space* W *and a closed embedding* $\rho_{\mathbb{Q}}: (G, \mathcal{H}) \to (\mathrm{GSp}(W), \mathcal{H}_W)$ *satisfying the following conditions: There exists a self-dual* $\mathbb{Z}_p$-*lattice* $\Lambda \subset W \otimes_{\mathbb{Q}} \mathbb{Q}_p$ *such that* $\rho_{\mathbb{Q}_p}$ *extends to a homomorphism* $\rho_{\mathbb{Z}_p}: G_{\mathbb{Z}_p} \to \mathrm{GSp}(\Lambda)$, *and* $\rho_{\mathbb{Z}_p} \circ \mu$ *is standard.*

Under these conditions, let P_μ *and* $P_{\widetilde{\mu}}$ *be the parabolic subgroups of* G *and* $\widetilde{G} := \mathrm{GSp}(\Lambda)$ *corresponding to* μ *and* $\widetilde{\mu} := \rho_{\mathbb{Z}_p} \circ \mu$, *respectively, as defined in* [8, §2.1] (*we remark that these are the parabolics opposite to the ones defined by the Hodge filtration*). *Then* $G(\mathbb{Z}_p) = G(\mathbb{Q}_p) \cap \widetilde{G}(\mathbb{Z}_p)$ *and* $P_\mu(\mathbb{Z}_p) = P_\mu(\mathbb{Q}_p) \cap \widetilde{P}_\mu(\mathbb{Z}_p)$.

Proof. Choose an arbitrary closed embedding $\rho': (G, \mathcal{H}) \to (\mathrm{GSp}(V), \mathcal{H}_V)$ into a Siegel Shimura datum. By Lemma 3.1.1, we can find a $\mathbb{Z}_p$-lattice $\Lambda_{V_{\mathbb{Q}_p}} \subset V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such that $\rho'_{\mathbb{Q}_p}$ extends to a map $G_{\mathbb{Z}_p} \to \mathrm{GL}(\Lambda_{V_{\mathbb{Q}_p}})$. There exists a $\mathbb{Z}$-lattice $\Lambda_V \subset V$ such that $\Lambda_{V_{\mathbb{Q}_p}} = \Lambda_V \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Set $W = V^4 \oplus (V^\vee)^4$, $\Lambda = \Lambda_{V_{\mathbb{Q}_p}}^4 \oplus (\Lambda_{V_{\mathbb{Q}_p}}^\vee)^4$ and $\Lambda_W = \Lambda_V^4 \oplus (\Lambda_V^\vee)^4$. Applying Lemma 3.1.2, we can choose a symplectic form on W giving us an embedding of Shimura data $\rho_{\mathbb{Q}}: (G, \mathcal{H}) \to (\mathrm{GSp}(W), \mathcal{H}_W)$ such that Λ_W is self-dual. Then Λ is also self-dual. The composition

$$G_{\mathbb{Z}_p} \to \mathrm{GL}(\Lambda_{V_{\mathbb{Q}_p}}) \to \mathrm{GL}(\Lambda)$$

maps $G_{\mathbb{Q}_p}$ to $\mathrm{GSp}(W)$, so since $G_{\mathbb{Q}_p}$ is dense in the reduced scheme $G_{\mathbb{Z}_p}$, the image of $G_{\mathbb{Z}_p}$ is contained in $\mathrm{GSp}(\Lambda)$. This gives the $\rho_{\mathbb{Z}_p}$ in the statement of the theorem. The cocharacter $\widetilde{\mu} := \rho_{\mathbb{Z}_p} \circ \mu$ is then standard since its generic fiber $\rho_{\mathbb{Q}_p} \circ \mu_{\mathbb{Q}_p}$ is.

Now set $\widetilde{G} := \mathrm{GSp}(\Lambda)$. To prove the final part of the proposition, first note that it is clear that $G(\mathbb{Z}_p) \subseteq G(\mathbb{Q}_p) \cap \widetilde{G}(\mathbb{Z}_p)$. Equality must then hold since $G(\mathbb{Z}_p)$ is a maximal compact subgroup of $G(\mathbb{Q}_p)$. Then we compute

$$P_\mu(\mathbb{Z}_p) = G(\mathbb{Z}_p) \cap P_\mu(\mathbb{Q}_p) = \widetilde{G}(\mathbb{Z}_p) \cap P_\mu(\mathbb{Q}_p) = \widetilde{G}(\mathbb{Z}_p) \cap G(\mathbb{Q}_p) \cap P_{\widetilde{\mu}}(\mathbb{Q}_p)$$
$$= G(\mathbb{Q}_p) \cap P_{\widetilde{\mu}}(\mathbb{Z}_p),$$

finishing the proof. ∎

From now on, we fix an embedding $(G, \mathcal{H}) \hookrightarrow (\widetilde{G}, \widetilde{\mathcal{H}})$ into a Siegel Shimura datum satisfying the conditions of Proposition 3.1.4. As with G, we will use $\widetilde{G}$, or sometimes $\widetilde{G}_{\mathbb{Z}_p}$ for emphasis, to denote the split model of $\widetilde{G}$ over $\mathbb{Z}_p$ given by Proposition 3.1.4. Composing our fixed μ with $G_{\mathbb{Z}_p} \to \widetilde{G}_{\mathbb{Z}_p}$ gives a Hodge cocharacter $\widetilde{\mu}$ for $\widetilde{G}$, which by Proposition 3.1.4 is conjugate over $\mathbb{Z}_p$ to the standard cocharacter $z \mapsto \left(\begin{smallmatrix} zI & 0 \\ 0 & I \end{smallmatrix} \right)$ in $\widetilde{G}_{\mathbb{Z}_p}$. We let P_μ and $P_{\widetilde{\mu}}$ be the parabolic subgroups of G and $\widetilde{G}$, respectively, that are defined in Proposition 3.1.4.

3.2. The anticanonical tower

We now start discussing Shimura varieties. Our notation and definitions will be similar to those of [7], so we will occasionally be rather brief. From now on, we fix a complete algebraically closed extension C of $\mathbb{Q}_p$. To start with, it will be more convenient to indicate the full level subgroup in our notation but later we will fix the tame level and only

specify the level at p. For any compact open subgroup $K \subseteq G(\mathbb{A}_f)$, always assumed to be neat throughout this paper, we let X_K denote the canonical model (defined over the reflex field E) of the Shimura variety of $(G, \mathcal{H})$ of level K. We set

$$\mathcal{X}_K := (X_K \otimes_E C)^{\mathrm{an}},$$

the analytification (as an adic space over C) of the base change of X_K to C, via our fixed embedding $E \to \mathbb{Q}_p$. If $H \subseteq G(\mathbb{A}_f)$ is an arbitrary compact subgroup, we set

$$\mathcal{X}_H := \varprojlim_{K \supseteq H} \mathcal{X}_K^{\diamond},$$

where $-^{\diamond}$ denotes the diamondification functor on analytic adic spaces over $\mathbb{Z}_p$ [29, Definition 15.5], the inverse limit is taken over all open $K \subseteq G(\mathbb{A}_f)$ containing H, and the inverse limit exists in the category of diamonds and is locally spatial by [29, Lemma 11.22]. We note that this is a mild abuse of notation when H itself is open; see [7, Remark 3.2.8 (1)] for more details. We define Shimura varieties $\widetilde{X}_{\widetilde{K}}$ (over E, not $\mathbb{Q}$), $\widetilde{\mathcal{X}}_{\widetilde{K}}$ and $\widetilde{\mathcal{X}}_{\widetilde{H}}$ for $\widetilde{G}$ completely analogously whenever $\widetilde{K} \subseteq \widetilde{G}(\mathbb{A}_f)$ is a compact open subgroup and $\widetilde{H} \subseteq \widetilde{G}(\mathbb{A}_f)$ is a compact subgroup.

Next, we introduce compactifications. Whenever $K \subseteq \widetilde{K}$, there is a natural finite étale map $X_K \to \widetilde{X}_{\widetilde{K}}$, which extends to a finite map $X_K^* \to \widetilde{X}_{\widetilde{K}}^*$ of minimal compactifications. When $K = \widetilde{K} \cap G(\mathbb{A}_f)$ by [10, Proposition 1.15], the map $X_K \to \widetilde{X}_{\widetilde{K}}$ is a closed immersion but the extension $X_K^* \to \widetilde{X}_{\widetilde{K}}^*$ need not be. Following [28, §4], we define the ad hoc compactification $X_K^* \to \overline{X}_K$ to be the universal finite map over which all the $X_K^* \to \widetilde{X}_{\widetilde{K}}^*$ factor (for varying $\widetilde{K}$ satisfying $K = \widetilde{K} \cap G(\mathbb{A}_f)$); as noted by Scholze, $\overline{X}_K$ is the scheme-theoretic image of $X_K^* \to \widetilde{X}_{\widetilde{K}}^*$ for sufficiently small $\widetilde{K}$. The right action of $G(\mathbb{A}_f)$ on the tower $(X_K)_K$ extends to an action on $(\overline{X}_K)_K$. We may then analytify: Set $\overline{\mathcal{X}}_K = (\overline{X} \otimes_E C)^{\mathrm{an}}$ and, for $H \subseteq G(\mathbb{A}_f)$ compact, set $\overline{\mathcal{X}}_H = \varprojlim_{K \supseteq H} \overline{\mathcal{X}}_K^{\diamond}$. The latter is a spatial diamond by [29, Lemma 11.22]. We also define $\widetilde{\mathcal{X}}_{\widetilde{K}}^*$ and $\widetilde{\mathcal{X}}_{\widetilde{H}}^*$ analogously for $\widetilde{G}$ (we use minimal compactifications here). We collect some facts about these diamonds.

Note that any compact subgroup $H \subseteq G(\mathbb{A}_f)$ is also a compact subgroup of $\widetilde{G}(\mathbb{A}_f)$.

Lemma 3.2.1. (1) *Let $H_2 \subseteq H_1$ be neat compact subgroups of $G(\mathbb{A}_f)$, with H_2 normal in H_1. Then*

$$|\overline{\mathcal{X}}_{H_1}| = |\overline{\mathcal{X}}_{H_2}|/(H_1/H_2).$$

Moreover, $|\overline{\mathcal{X}}_H|$ is the closure of $|\mathcal{X}_H|$ inside $|\widetilde{\mathcal{X}}_H^*|$ and $\overline{\mathcal{X}}_H = \underline{|\overline{\mathcal{X}}_H|} \times_{\underline{|\widetilde{\mathcal{X}}_H^*|}} \widetilde{\mathcal{X}}_H^*$, for any neat compact subgroup $H \subseteq G(\mathbb{A}_f)$.

(2) *Let $\widetilde{H}_2 \subseteq \widetilde{H}_1$ be neat compact subgroups of $G(\mathbb{A}_f)$, with $\widetilde{H}_2$ normal in $\widetilde{H}_1$. Then*
$$|\widetilde{\mathcal{X}}_{\widetilde{H}_1}^*| = |\widetilde{\mathcal{X}}_{\widetilde{H}_2}^*|/(\widetilde{H}_1/\widetilde{H}_2).$$

Proof. Part (2) is [7, Lemma 3.2.11]. For part (1), the first statement is essentially [7, Proposition 3.2.15] and the second statement is essentially [7, Corollary 3.2.14]; in both

cases the proofs of the cited results only use an embedding into a Siegel Shimura datum and go through verbatim in our situation. ∎

Proposition 3.2.2. (1) *For any neat compact subgroups $H_2 \subseteq H_1 \subseteq G(\mathbb{A}_f)$ with H_2 normal in H_1, $\overline{\mathcal{X}}_{H_2} \times \underline{H_1/H_2} \rightrightarrows \overline{\mathcal{X}}_{H_2}$ is a presentation of $\overline{\mathcal{X}}_{H_1}$ as a v-sheaf.*

(2) *For any neat compact subgroups $\widetilde{H}_2 \subseteq \widetilde{H}_1 \subseteq \widetilde{G}(\mathbb{A}_f)$ with $\widetilde{H}_2$ normal in $\widetilde{H}_1$, $\widetilde{\mathcal{X}}^*_{\widetilde{H}_2} \times \underline{\widetilde{H}_1/\widetilde{H}_2} \rightrightarrows \widetilde{\mathcal{X}}^*_{\widetilde{H}_2}$ is a presentation of $\widetilde{\mathcal{X}}^*_{\widetilde{H}_1}$ as a v-sheaf.*

Proof. We first prove (2). Consider the case where $\widetilde{H}_1$ and $\widetilde{H}_2$ are open subgroups of $\widetilde{G}(\mathbb{Z}_p)$. Then we can consider $\widetilde{\mathcal{X}}^*_{\widetilde{H}_1}$ and $\widetilde{\mathcal{X}}^*_{\widetilde{H}_2}$ as adic spaces. By [7, Lemma 3.2.2], $\widetilde{\mathcal{X}}^*_{\widetilde{H}_2} \to \widetilde{\mathcal{X}}^*_{\widetilde{H}_1}$ identifies $\widetilde{\mathcal{X}}^*_{\widetilde{H}_1}$ with $\widetilde{\mathcal{X}}^*_{\widetilde{H}_2}/(\widetilde{H}_1/\widetilde{H}_2)$ (here we take the quotient in the category of locally v-ringed spaces). This morphism is finite, so we can cover $\widetilde{\mathcal{X}}^*_{\widetilde{H}_1}$ with affinoids whose pullbacks to $\widetilde{\mathcal{X}}^*_{\widetilde{H}_2}$ are affinoids. Then the result follows from Proposition 2.1.1 and [14, Theorem 1.2].

In the general case, choose an open compact subgroup $\widetilde{K}_0$ containing $\widetilde{H}_1$ and let $\{\widetilde{K}_i\}_{i \geq 0}$ be a shrinking family of open normal subgroups of $\widetilde{K}_0$ with $\bigcap_i \widetilde{K}_i = \{1\}$. For each i, we have shown that $\widetilde{\mathcal{X}}^*_{\widetilde{K}_i \widetilde{H}_2} \times \underline{\widetilde{K}_i \widetilde{H}_1/\widetilde{K}_i \widetilde{H}_2} \rightrightarrows \widetilde{\mathcal{X}}^*_{\widetilde{K}_i \widetilde{H}_2}$ is a presentation of $\widetilde{\mathcal{X}}^*_{\widetilde{K}_i \widetilde{H}_1}$ as a v-sheaf. The diamonds $\widetilde{\mathcal{X}}^*_{\widetilde{K}_i \widetilde{H}_j}$ are spatial, so by many applications of [29, Corollary 11.29], the relevant maps are qcqs. Applying Lemma 2.2.1 then finishes the proof of part (2).

We now prove part (1). By Lemma 3.2.1, $\overline{\mathcal{X}}_{H_1} = |\overline{\mathcal{X}}_{H_1}| \times_{|\widetilde{\mathcal{X}}^*_{H_1}|} \widetilde{\mathcal{X}}^*_{H_1}$. Since coequalizers commute with fiber products, we conclude from part (2) that

$$|\overline{\mathcal{X}}_{H_1}| \times_{|\widetilde{\mathcal{X}}^*_{H_1}|} \widetilde{\mathcal{X}}^*_{H_2} \times \underline{H_1/H_2} \rightrightarrows |\overline{\mathcal{X}}_{H_1}| \times_{|\widetilde{\mathcal{X}}^*_{H_1}|} \widetilde{\mathcal{X}}^*_{H_2}$$

is a presentation of $\overline{\mathcal{X}}_{H_1}$ as a v-sheaf. It remains to show that $|\overline{\mathcal{X}}_{H_1}| \times_{|\widetilde{\mathcal{X}}^*_{H_1}|} \widetilde{\mathcal{X}}^*_{H_2} = \overline{\mathcal{X}}_{H_2}$, which in turns reduces to showing that $|\overline{\mathcal{X}}_{H_1}| \times_{|\widetilde{\mathcal{X}}^*_{H_1}|} |\widetilde{\mathcal{X}}^*_{H_2}| = |\overline{\mathcal{X}}_{H_2}|$. But this follows directly from Lemma 3.2.1. ∎

We will now start to only indicate the level at p in the notation for our Shimura varieties. Let $\widetilde{K}^p \subseteq \widetilde{G}(\widehat{\mathbb{Z}}^p)$ be a neat open subgroup, which we assume[5] to be contained inside the principal congruence subgroup of level N for some $N \geq 3$, $p \nmid N$. The choice of $\widetilde{K}^p$ is arbitrary but fixed, unless otherwise indicated. If $\widetilde{H} \subseteq \widetilde{G}(\mathbb{Z}_p)$ is a closed subgroup, we now write $\widetilde{\mathcal{X}}^*_{\widetilde{H}}$ for what was previously denoted by $\widetilde{\mathcal{X}}^*_{\widetilde{K}^p \widetilde{H}}$, and so on. We make similar conventions for the Shimura varieties, with $K^p \subseteq G(\mathbb{A}_f)$ a compact open subgroup contained in $\widetilde{K}^p$, and $H \subseteq G(\mathbb{Z}_p)$ a closed subgroup.

---

[5]This (rather mild) assumption is imposed to be able to apply the perfectoidness results from [28] later. It is possible to remove this assumption using the results of [14], but we do not go into this. Also recall that the proof of Proposition 3.1.4 furnishes us with a split model of $\widetilde{G}$ over $\mathbb{Z}$, so it makes sense to talk about $\widetilde{G}(\widehat{\mathbb{Z}}^p)$ and its principal congruence subgroups.

Recall the parabolic subgroups $P_\mu \subseteq G$ and $P_{\widetilde{\mu}} \subseteq \widetilde{G}$ defined at the end of §3.1. Let $\widetilde{K}_0(p)$ denote the parahoric subgroup of $\widetilde{G}(\mathbb{Z}_p)$ with respect to the *opposite* parabolic $\overline{P}_{\widetilde{\mu}}$ of $P_{\widetilde{\mu}}$. The Shimura variety $\widetilde{X}_{\widetilde{K}_0(p)}$ is the moduli space of principally polarized abelian varieties (A, λ) together with a $\widetilde{K}^p$-level structure and a subspace $W \subseteq A[p]$ which is Lagrangian with respect to the λ-Weil pairing. For any $\epsilon \in [0, 1/2)$, we let $\widetilde{\mathcal{X}}_{\widetilde{K}_0(p)}(\epsilon)_a \subseteq \widetilde{\mathcal{X}}_{\widetilde{K}_0(p)}$ denote the anticanonical locus of level $\widetilde{K}_0(p)$ and radius of overconvergence ϵ, which is defined in [28, Theorem 3.2.15 (iii)].[6] We then set

$$\widetilde{\mathcal{X}}_{\widetilde{H}}(\epsilon)_a := \widetilde{\mathcal{X}}_{\widetilde{K}_0(p)}(\epsilon)_a \times_{\widetilde{\mathcal{X}}_{\widetilde{K}_0(p)}} \widetilde{\mathcal{X}}_{\widetilde{H}}$$

for all closed subgroups $\widetilde{H} \subseteq \widetilde{K}_0(p)$ and similarly for minimal compactifications. Furthermore, whenever $H \subseteq G(\mathbb{Z}_p) \cap \widetilde{K}_0(p)$ is a closed subgroup, we set

$$\mathcal{X}_H(\epsilon)_a := \widetilde{\mathcal{X}}_{\widetilde{K}_0(p)}(\epsilon)_a \times_{\widetilde{\mathcal{X}}_{\widetilde{K}_0(p)}} \mathcal{X}_H$$

and similarly for the ad hoc compactifications. We then have the following basic perfectoidness results.

Theorem 3.2.3. *If $\widetilde{H} \subseteq \overline{P}_{\widetilde{\mu}}(\mathbb{Z}_p)$ is a closed subgroup, then the diamond $\widetilde{\mathcal{X}}^*_{\widetilde{H}}(\epsilon)_a$ is affinoid perfectoid, and the boundary $\widetilde{\mathcal{Z}}_{\widetilde{H}}(\epsilon)_a := \widetilde{\mathcal{X}}^*_{\widetilde{H}}(\epsilon)_a \setminus \widetilde{\mathcal{X}}_{\widetilde{H}}(\epsilon)_a$ is Zariski closed.*

Proof. This is [7, Corollary 3.2.17], up to a minor difference in the level structure. In [7] the anticanonical locus is defined on the Shimura variety whose level is contained in the parahoric subgroup corresponding to the parabolic $P_{\widetilde{\mu}}$ (following [28]), but $P_{\widetilde{\mu}}$ and $\overline{P}_{\widetilde{\mu}}$ are conjugate (by the longest element of the Weyl group) and the anticanonical loci correspond, so we may conjugate to get the result. ∎

Corollary 3.2.4. *If $H \subseteq \overline{P}_\mu(\mathbb{Z}_p)$, then the diamond $\overline{\mathcal{X}}_H(\epsilon)_a$ is affinoid perfectoid, and the boundary $\mathcal{Z}_H(\epsilon)_a := \overline{\mathcal{X}}_H(\epsilon)_a \setminus \mathcal{X}_H(\epsilon)_a$ is Zariski closed.*

Proof. This follows from Theorem 3.2.3 in exactly the same way as [7, Theorem 3.2.18] follows from [7, Corollary 3.2.17]. ∎

We remark that at this stage we have not proved that $\overline{\mathcal{X}}_H(\epsilon)_a$ is nonempty, but the result and its proof still make sense. In fact we will not need to separately prove the nonemptiness; it follows from Theorem 3.4.1.

3.3. Flag varieties and the Hodge–Tate period map

We begin by briefly recalling some material from [7, §4]. Let G be a split connected reductive group over $\mathbb{Q}_p$ with a split maximal torus T and a Borel subgroup B $\supseteq$ T. Let $\Phi = \Phi(G, T)$ be the roots of G with respect to T, let $\Phi^+ \subseteq \Phi$ (resp. $\Phi^- \subseteq \Phi$) denote the

---

[6]Up to a minor difference in level structures; see the proof of Theorem 3.2.3 for more details.

positive (resp. negative) roots with respect to B, and let $\Delta \subseteq \Phi^+$ be the simple roots. Let $P = P_I \supseteq B$ be the standard parabolic corresponding to a subset $I \subseteq \Delta$, with unipotent radical $N = N_I$, and let $\Phi_P \subseteq \Phi$ be the root system of the Levi factor of P with respect to (the image of) T. Let $W := W(G, T)$ and $W_P := W(P/N, T)$ be the respective Weyl groups.

Let $U \subseteq B$ be the unipotent radical. As before, we use overlines to denote opposites: We have $\overline{B}$, the opposite Borel of B, with unipotent radical $\overline{U}$. Recall that $\overline{B} = w_0 B w_0$, where $w_0 \in W$ is the longest element. We will look at the stratification of the flag variety G/P into orbits for $\overline{B}$. To this end, we recall the generalized Bruhat decomposition

$$G = \bigsqcup_{w \in W/W_P} BwP$$

from [5, Corollaire 5.20]. From this one easily deduces the decomposition

$$G = \bigsqcup_{w \in W/W_P} \overline{B}wP$$

using the fact that $\overline{B} = w_0 B w_0$. We note that $\overline{B}P$ is the "big cell", which is open in G. The following is [7, Lemma 4.3.1].

Lemma 3.3.1. *For any* $w \in W$, $\overline{B}wP \subseteq w\overline{B}P$. *In particular, we have the open cover*

$$G = \bigcup_{w \in W/W_P} w\overline{B}P.$$

We now introduce some more notation. We let $\mathrm{Fl}_G := G/P$ be the partial flag variety of parabolics conjugate to P. For $w \in W$, we have affine open subsets $w\overline{B}P/P \subseteq \mathrm{Fl}_G$, whose stabilizer is $\overline{P}_w := w\overline{P}w^{-1}$ (since $\overline{B}P = \overline{P}P$). The following is the analogue of [7, Lemma 4.3.2]; it is key to proving the stronger vanishing theorem in this article.

Lemma 3.3.2. *We have* $\dim \overline{U} - \dim(\overline{U} \cap \overline{P}_w) + \dim(\overline{B}wP/P) = \dim \overline{N}$.

Proof. We start by observing that $\dim \overline{U} = \#\Phi^-$. Then, note that the second term $\dim(\overline{U} \cap \overline{P}_w)$ is equal to $\#(\Phi^- \cap w(\Phi^- \cup \Phi_P))$. For the third term, we first observe that

$$\overline{B}wP/P \cong \overline{B}/(\overline{B} \cap wPw^{-1})$$

and then the latter has dimension

$$(\dim T + \#\Phi^-) - (\dim T + \#(\Phi^- \cap w(\Phi^+ \cup \Phi_P))) = \#\Phi^- - \#(\Phi^- \cap w(\Phi^+ \cup \Phi_P)).$$

The left hand side of the equality we want to prove is then

$$\#\Phi^- - \#(\Phi^- \cap w(\Phi^- \cup \Phi_P)) + \#\Phi^- - \#(\Phi^- \cap w(\Phi^+ \cup \Phi_P)),$$

which is equal to

$$\#(\Phi^- \setminus w(\Phi^- \cup \Phi_P)) + \#(\Phi^- \setminus w(\Phi^+ \cup \Phi_P)).$$

But this is just $\#(\Phi^- \setminus w(\Phi_P))$, and this is independent of w, since $\alpha \in \Phi_P$ if and only if $-\alpha \in \Phi_P$, and hence precisely one of $w(\alpha)$ and $w(-\alpha) = -w(\alpha)$ will be negative. In particular, if we set $w = 1$, we get $\#(\Phi^- \setminus \Phi_P)$, which is equal to the dimension of $\overline{N}$ as desired.   ■

Next, we leave the setting above and discuss the Hodge–Tate period map for our Shimura varieties. Let $1 \subseteq G(\mathbb{Z}_p) \subseteq \widetilde{G}(\mathbb{Z}_p)$ denote the trivial subgroup, and set $\mathscr{Fl}_{G,\mu} := (G/P_\mu \otimes_{\mathbb{Q}_p} C)^{\mathrm{an}}$ and $\mathscr{Fl}_{\widetilde{G},\widetilde{\mu}} := (\widetilde{G}/P_{\widetilde{\mu}} \otimes_{\mathbb{Q}_p} C)^{\mathrm{an}}$. These flag varieties have natural models over $\mathbb{Q}_p$, and there is a natural Zariski closed embedding $\mathscr{Fl}_{G,\mu} \subseteq \mathscr{Fl}_{\widetilde{G},\widetilde{\mu}}$. We have Hodge–Tate period maps

$$\pi_{\mathrm{HT}} \colon \mathcal{X}_1 \to \mathscr{Fl}_{G,\mu}, \quad \widetilde{\pi}_{\mathrm{HT}} \colon \widetilde{\mathcal{X}}_1^* \to \mathscr{Fl}_{\widetilde{G},\widetilde{\mu}}$$

constructed in [8, 28]. When $K^p \subseteq \widetilde{K}^p$, we have a commutative diagram

$$
\begin{array}{ccc}
\overline{\mathcal{X}}_1 & \xrightarrow{\pi_{\mathrm{HT}}} & \mathscr{Fl}_{G,\mu} \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{X}}_1^* & \xrightarrow{\widetilde{\pi}_{\mathrm{HT}}} & \mathscr{Fl}_{\widetilde{G},\widetilde{\mu}}
\end{array}
$$

By [28, Theorem 4.1.1], $\overline{\mathcal{X}}_1$ and $\widetilde{\mathcal{X}}_1^*$ are perfectoid spaces. The existence of the commutative diagram follows by combining the proof of [8, Theorem 2.1.3 (i)] with the proof of [7, Theorem 3.3.1]. We remark that π_{HT} is equivariant for the action of $G(\mathbb{Q}_p)$, where $\overline{\mathcal{X}}_1$ is given the standard right $G(\mathbb{Q}_p)$-action and $\mathscr{Fl}_{G,\mu}$ is given the right $G(\mathbb{Q}_p)$-action that is inverse to the standard left action (the analogous remark applies to $\widetilde{\pi}_{\mathrm{HT}}$). Next, we define some "topological" Hodge–Tate period maps, as in [7, §4.5]. By Lemma 3.2.1 we may, for any closed subgroup $H \subseteq G(\mathbb{Z}_p)$, define a map

$$|\pi_H| \colon |\overline{\mathcal{X}}_H| \to |\mathscr{Fl}_{G,\mu}|/H$$

by quotienting out the map $|\pi_{\mathrm{HT}}|$ by H. Moreover, we define a morphism

$$\pi_H \colon (\overline{\mathcal{X}}_H)_{\mathrm{\acute{e}t}} \to |\mathscr{Fl}_{G,\mu}|/H$$

of sites by precomposing $|\pi_H|$ with the natural map $(\overline{\mathcal{X}}_H)_{\mathrm{\acute{e}t}} \to |\overline{\mathcal{X}}_H|$.

3.4. Perfectoid loci

Choose a maximal split torus and a Borel subgroup $T \subseteq B \subseteq P_\mu$ of G, all over $\mathbb{Q}_p$. Let W be the Weyl group of G with respect to T and let $W_\mu := W_{P_\mu}$ in the notation of the previous subsection. We will define two open covers of $\mathscr{Fl}_{G,\mu}$. The first is by Zariski open affine subsets. We set

$$V_w := (w\overline{B}P_\mu/P_\mu \otimes_{\mathbb{Q}_p} C)^{\mathrm{an}}$$

for any $w \in W$; this is an open cover by Lemma 3.3.1 and V_w only depends on the coset wW_μ. The second cover is the analogous cover by open affinoid subsets $\mathcal{V}_w$, $w \in W/W_\mu$. One way to define $\mathcal{V}_w$ is as the rigid generic fiber of the formal completion along $p = 0$ of the $\mathcal{O}_C$-scheme

$$(w\overline{B}_{\mathbb{Z}_p} P_{\mu,\mathbb{Z}_p}/P_{\mu,\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} \mathcal{O}_C,$$

where we have added the subscript $\mathbb{Z}_p$ to emphasize that we are considering the models of these algebraic groups over $\mathbb{Z}_p$ (all parabolic subgroups of G over $\mathbb{Q}_p$ extend uniquely to parabolic subgroups of $G_{\mathbb{Z}_p}$). Set

$$\gamma := \mu(p) \in G(\mathbb{Q}_p).$$

Then, by the definition of P_μ, it follows that

$$V_1 = \bigcup_{k \geq 0} \mathcal{V}_1 \gamma^{-k}.$$

Moreover, the open subsets $\mathcal{V}_1 \gamma^k$, for $k \leq 0$, form a basis of open neighborhoods of the base point in $\mathscr{F}\!\ell_{G,\mu}$. For any closed subgroup $H \subseteq \overline{P}_\mu(\mathbb{Z}_p)$, we set $\overline{\mathcal{X}}_{H,1} := \pi_H^{-1}(|V_1|/H)$; this is a locally spatial diamond. Note that $\overline{\mathcal{X}}_1$ is nonempty since its translates by elements in W cover $\overline{\mathcal{X}}_1$. It follows that $\overline{\mathcal{X}}_{H,1}$ is nonempty as well, for all $H \subseteq \overline{P}_\mu(\mathbb{Z}_p)$. Our next result generalizes [7, Theorem 4.5.2].

Theorem 3.4.1. *For any closed subgroup $H \subseteq \overline{P}_\mu(\mathbb{Z}_p)$, the locally spatial diamond $\overline{\mathcal{X}}_{H,1}$ is a perfectoid space. More precisely, $|\overline{\mathcal{X}}_{H,1}|$ is covered by the increasing union of quasi-compact open subsets $|\overline{\mathcal{X}}_1(\epsilon)_a|\gamma^{-k}/H$ for $k \in \mathbb{Z}_{\geq 0}$ (and sufficiently small $\epsilon > 0$), and the corresponding spatial diamonds are affinoid perfectoid with Zariski closed boundary.*

Proof. We may identify $|\overline{\mathcal{X}}_{H,1}|$ with $|\overline{\mathcal{X}}_{1,1}|/H$. The first step is to show that the $|\overline{\mathcal{X}}_1(\epsilon)_a|\gamma^{-k}$ cover $|\overline{\mathcal{X}}_{1,1}|$. By [7, Proposition 3.3.4], $|\widetilde{\mathcal{X}}_1^*(\epsilon)_a|\gamma^{-k}$ cover $|\widetilde{\mathcal{X}}_{1,1}^*|$. Then we just need to observe that $|\overline{\mathcal{X}}_1(\epsilon)_a|\gamma^{-k} = |\widetilde{\mathcal{X}}_1^*(\epsilon)_a|\gamma^{-k} \cap |\overline{\mathcal{X}}_1|$ and $|\overline{\mathcal{X}}_{1,1}| \subseteq |\widetilde{\mathcal{X}}_{1,1}^*| \cap |\overline{\mathcal{X}}_1|$. After this, it remains to show that the open subdiamonds of $\overline{\mathcal{X}}_H$ given by the open subsets $|\overline{\mathcal{X}}_1(\epsilon)_a|\gamma^{-k}/H$ are affinoid perfectoid with Zariski closed boundary. But γ^k induces an isomorphism

$$\gamma^k \colon \overline{\mathcal{X}}_H \xrightarrow{\sim} \overline{\mathcal{X}}_{\gamma^{-k} H \gamma^k}$$

which identifies $|\overline{\mathcal{X}}_1(\epsilon)_a|\gamma^{-k}/H$ with $|\overline{\mathcal{X}}_{\gamma^{-k} H \gamma^k}(\epsilon)_a|$, and $\overline{\mathcal{X}}_{\gamma^{-k} H \gamma^k}(\epsilon)_a$ is affinoid perfectoid with Zariski closed boundary by Corollary 3.2.4, since

$$\gamma^{-k} H \gamma^k \subseteq \gamma^{-k} \overline{P}_\mu(\mathbb{Z}_p) \gamma^k \subseteq \overline{P}_\mu(\mathbb{Z}_p). \qquad \blacksquare$$

We now consider the situation for general w. In this case, the parabolic $\overline{P}_{\mu,w} := w\overline{P}_\mu w^{-1}$ stabilizes V_w. Note that $\overline{P}_{\mu,w}(\mathbb{Z}_p) = w\overline{P}_\mu(\mathbb{Z}_p)w^{-1}$ since G is split over $\mathbb{Q}_p$

and we have chosen the natural split model over $\mathbb{Z}_p$, so w has a representative in $G(\mathbb{Z}_p)$. For any closed subgroup $H \subseteq \overline{P}_{\mu,w}(\mathbb{Z}_p)$, we may define

$$\overline{\mathcal{X}}_{H,w} := \pi_H^{-1}(|V_w|/H).$$

Then these spaces are perfectoid.

Corollary 3.4.2. *For any closed subgroup $H \subseteq \overline{P}_{\mu,w}(\mathbb{Z}_p)$, the locally spatial diamond $\overline{\mathcal{X}}_{H,w}$ is a perfectoid space. More precisely, $|\overline{\mathcal{X}}_{H,w}|$ is covered by the increasing union of quasi-compact open subsets $|\overline{\mathcal{X}}_1(\epsilon)_a|\gamma^{-k}w^{-1}/H$ for $k \in \mathbb{Z}_{\geq 0}$ (and sufficiently small $\epsilon > 0$), and the corresponding spatial diamonds are affinoid perfectoid with Zariski closed boundary.*

Proof. This follows from Theorem 3.4.1 by looking at the commutative diagram

$$
\begin{array}{ccc}
(\overline{\mathcal{X}}_{w^{-1}Hw})_{\text{ét}} & \xrightarrow{\;\;w^{-1}\;\;} & (\overline{\mathcal{X}}_H)_{\text{ét}} \\
\pi_{w^{-1}Hw} \downarrow & & \downarrow \pi_H \\
|\mathscr{F}\ell_{G,\mu}|/w^{-1}Hw & \xrightarrow{\;\;w^{-1}\;\;} & |\mathscr{F}\ell_{G,\mu}|/H
\end{array}
$$

where the horizontal maps are isomorphisms, since $V_w = V_1 w^{-1}$ so $\mathcal{X}^*_{w^{-1}Hw,1} w^{-1} = \mathcal{X}^*_{H,w}$. ∎

In this paper, we will only use Corollary 3.4.2 in the situation when $H \subseteq \overline{U}_w(\mathbb{Z}_p)$, where $\overline{U}_w = \overline{U} \cap \overline{P}_{\mu,w}$ and we recall that $\overline{U}$ is the unipotent radical of $\overline{B}$.

4. The vanishing theorem

In this section we prove Theorem 4.1.1, our main result. Our arguments follow those of [7, §5] closely, with a few differences. As these are somewhat technical, we discuss them in Remark 4.2.8 when all necessary objects have been introduced. For the sake of readability, we have elected to reproduce some arguments that appear in identical form in [7].

4.1. First reductions

We start by stating our main theorem. To state it, we define, for $m \in \mathbb{Z}_{\geq 1}$,

$$K_1(p^m) := \{g \in G(\mathbb{Z}_p) \mid (g \bmod p) \in \overline{U}(\mathbb{Z}/p^m)\}.$$

Note that $\bigcap_{m \geq 1} K_1(p^m) = \overline{U}(\mathbb{Z}_p)$. We let d be the dimension of the Shimura varieties for our Shimura datum $(G, \mathcal{H})$; we have $d = \dim \overline{N}_\mu$.

Theorem 4.1.1. *Let $K \subseteq G(\mathbb{Z}_p)$ be an open compact subgroup. Then*

$$\varinjlim_m H_c^i(X_{K \cap K_1(p^m)}(\mathbb{C}), \mathbb{Z}/p^r) = 0 \quad \text{for all } r \geq 1 \text{ and all } i > d.$$

Here the cohomology is singular cohomology (with compact supports) of the complex manifold $X_{K \cap K_1(p^m)}(\mathbb{C})$. The following more general version follows directly.

Corollary 4.1.2. *Let $H \subseteq \overline{U}(\mathbb{Z}_p)$ be a closed subgroup. Then*

$$\varinjlim_{K \supseteq H} H_c^i(X_K(\mathbb{C}), \mathbb{Z}/p^r) = 0 \quad \text{for all } r \geq 1 \text{ and all } i > d.$$

Remark 4.1.3. This is Theorem 1.1. In the formulation of Theorem 1.1, the U used there was the unipotent radical of an arbitrarily chosen Borel subgroup of G over $\mathbb{Z}_p$, whereas Corollary 4.1.2 is formulated in terms of a specific choice (depending on the choice of μ). The two formulations are easily seen to be equivalent, since all Borel subgroups over $\mathbb{Z}_p$ are conjugate by elements of $G(\mathbb{Z}_p)$ (see e.g. [9, Corollary 5.2.13]). Equivalently, it is possible to conjugate the choice of μ to make $\overline{U}$ arbitrary.

In this subsection we make a series of arguments, as in [7, §5.1], to reduce Theorem 4.1.1 to a particular statement in p-adic geometry. First, note that by dévissage it suffices to treat the case $r = 1$, and that by applying comparison theorems (between singular and étale cohomology of varieties over $\mathbb{C}$, between étale cohomology of varieties over C and their analytification, and also invariance of algebraically closed field for étale cohomology of varieties), Theorem 4.1.1 is equivalent to

$$\varinjlim_m H_{\text{ét}}^i(\overline{\mathcal{X}}_{K \cap K_1(p^m)}, j_! \mathbb{F}_p) = 0$$

for $i > d$. Here and in the rest of this section we write j for any open immersion $Y \to \overline{Y}$ where $\overline{Y}$ is a locally spatial diamond with a (fixed) map to $\overline{\mathcal{X}}_{G(\mathbb{Z}_p)}$, $Y = \overline{Y} \times_{\overline{\mathcal{X}}_{G(\mathbb{Z}_p)}} \mathcal{X}_{G(\mathbb{Z}_p)}$ and $Y \to \overline{Y}$ is the projection onto the first factor. Applying the primitive comparison theorem [27, Theorem 3.13] and [7, Lemma 5.1.3] gives us

$$\varinjlim_m H_{\text{ét}}^i(\overline{\mathcal{X}}_{K \cap K_1(p^m)}, j_! \mathbb{F}_p)^a \otimes_{\mathbb{F}_p} \mathcal{O}_C/p = \varinjlim_m H_{\text{ét}}^i(\overline{\mathcal{X}}_{K \cap K_1(p^m)}, j_!(\mathcal{O}_{\mathcal{X}_{K \cap K_1(p^m)}}^+/p))^a,$$

where $-^a$ denotes the almost setting with respect to the maximal ideal of $\mathcal{O}_C$. Applying the almost version of [7, Proposition 2.2.1] and some results from [7, §2.3] (see [7, paragraph after Proposition 5.1.4] for more details), we get

$$\varinjlim_m H_{\text{ét}}^i(\overline{\mathcal{X}}_{K \cap K_1(p^m)}, j_!(\mathcal{O}_{\mathcal{X}_{K \cap K_1(p^m)}}^+/p))^a = H_{\text{ét}}^i(\overline{\mathcal{X}}_{K \cap \overline{U}(\mathbb{Z}_p)}, j_!(\mathcal{O}_{\mathcal{X}_{K \cap \overline{U}(\mathbb{Z}_p)}}^+/p))^a.$$

From here on we make the following convention: For any quasi-pro-étale $Y \to \mathcal{X}_{G(\mathbb{Z}_p)}$, we write $\mathcal{O}_{\mathcal{X}}^+/p$ for the étale sheaf $\mathcal{O}_Y^+/p$; this is somewhat justified by [7, Lemmas 2.3.2 and 2.3.3].

From now on, we set $H_w := K \cap \overline{U}_w(\mathbb{Z}_p)$ for any $w \in W/W_\mu$, and we let $H := H_1 = K \cap \overline{U}(\mathbb{Z}_p)$ for simplicity. We consider the morphism

$$\pi_H \colon (\overline{\mathcal{X}}_H)_{\text{ét}} \to |\mathscr{F}\ell_{G,\mu}|/H$$

and its Leray spectral sequence

$$E_2^{rs} = H^r(|\mathscr{F}\ell_{G,\mu}|/H, R^s \pi_{H,*} j_!(\mathcal{O}_{\mathcal{X}}^+/p)^a) \implies H_{\text{ét}}^{r+s}(\overline{\mathcal{X}}_H, j_!(\mathcal{O}_{\mathcal{X}}^+/p)^a).$$

Define, for $w \in W/W_\mu$,

$$\mathscr{F}\ell_{G,\mu}^w := (\overline{B}wP_\mu/P_\mu \otimes_{\mathbb{Q}_p} C)^{\text{an}}.$$

These are generalized Schubert cells, and they form a Zariski stratification of $\mathscr{F}\ell_{G,\mu}$. Note also that they are stable under H. The following is the key result.

Theorem 4.1.4. *Let* $w \in W/W_\mu$ *and let* $x \in |\mathscr{F}\ell_{G,\mu}^w|/H$. *Then*

$$(R^i \pi_{H,*} j_!(\mathcal{O}_{\mathcal{X}}^+/p)^a)_x = 0 \quad \text{for } i > d - \dim \mathscr{F}\ell_{G,\mu}^w.$$

Proof of Theorem 4.1.1. By the Leray spectral sequence for π_H it suffices to prove that $H^r(|\mathscr{F}\ell_{G,\mu}|/H, R^s \pi_{H,*} j_!(\mathcal{O}_{\mathcal{X}}^+/p)^a) = 0$ for $r + s > d$. Fix r and assume that $s > d - r$. Let S_r be the set of $w \in W/W_\mu$ for which $\dim \mathscr{F}\ell_{G,\mu}^w < r$, set $Y_r := \bigcup_{w \in S_r} \mathscr{F}\ell_{G,\mu}^w$ and let $\overline{Y}_r$ be the Zariski closure of Y_r. Since Y_r is $\overline{U}(\mathbb{Z}_p)$-invariant and has dimension $< r$, the same holds for $\overline{Y}_r$. By [3, Lemma 3.2.3], $|\overline{Y}_r|/H$ is a spectral space of dimension $< r$. Set $\mathcal{F}^s := R^s \pi_{H,*} j_!(\mathcal{O}_{\mathcal{X}}^+/p)^a$; we claim that it is supported on $|\overline{Y}_r|/H$. Take $x \notin |\overline{Y}_r|/H$; by construction $x \notin |\mathscr{F}\ell_{G,\mu}^w|/H$ for all $w \in S_r$. It follows that $x \in |\mathscr{F}\ell_{G,\mu}^{w'}|/H$ for some w' such that $\dim \mathscr{F}\ell_{G,\mu}^{w'} \geq r$. Hence $\mathcal{F}_x^s = 0$ by Theorem 4.1.4 since $s > d - r \geq d - \dim \mathscr{F}\ell_{G,\mu}^{w'}$, so $\mathcal{F}^s$ is indeed supported on $|\overline{Y}_r|/H$. By [25, Corollary 4.6],

$$H^r(|\mathscr{F}\ell_{G,\mu}|/H, \mathcal{F}^s) = H^r(|\overline{Y}_r|/H, \mathcal{F}^s) = 0,$$

since $|\overline{Y}_r|/H$ is a spectral space of dimension $< r$.   ∎

4.2. *Proof of Theorem* 4.1.4

The rest of this section is devoted to the proof of Theorem 4.1.4. Our argument differs from that in [7, §5.3] in that we make no use of [7, Proposition 5.2.1]. Instead we use the results from §2 of this paper.

Fix w and x as in the statement of Theorem 4.1.4. As in the proof of Theorem 4.1.1, we write $\mathcal{F}^i$ for $R^i \pi_{H,*} j_!(\mathcal{O}_{\mathcal{X}}^+/p)^a$. Recall the notion of a rank 1 point on the topological space of a locally spatial diamond from [3, Definition 3.2.1]; this means that the point has no proper generalizations. We further recall further a slight generalization of some comments from the end of [7, §2.1]. Let Y be a spatial diamond, S a spectral space and $q\colon |Y| \to S$ a spectral map. If $T \subseteq S$ is quasicompact and closed under generalizations, then the preimage $q^{-1}(T)$ carries a natural structure of a spatial diamond, defined as follows: The conditions on T are equivalent to $T = \bigcap_{T \subseteq U} U$, where U runs through the quasicompact open subsets of S containing T, by [31, Tag 0A31]. We may then define

$$q^{-1}(T) := \varprojlim_{T \subseteq U} q^{-1}(U).$$

As mentioned, this naturally yields a spatial diamond by [29, Proposition 11.18, Lemma 11.22], with an injective quasi-pro-étale morphism $q^{-1}(T) \to Y$. When $T = \mathrm{Gen}(s)$ is the set of generalizations of a point $s \in S$, we simply write $q^{-1}(s)$ for $q^{-1}(\mathrm{Gen}(s))$. By [7, Proposition 2.2.5, Lemma 2.2.6], we have

$$\mathcal{F}_x^i \cong H_{\text{ét}}^i(\pi_H^{-1}(x), j_!(\mathcal{O}_{\mathcal{X}}^+/p)^a).$$

Lemma 4.2.1. *It suffices to prove Theorem* 4.1.4 *for points x with no proper generalizations.*

Proof. This is the precise analogue of [7, Lemma 5.3.1] in our situation, and the same proof works. ∎

From now on, we assume that $x \in |\mathscr{F}\ell_{G,\mu}^w|/H$ has no proper generalizations. Let $\tilde{x} \in |\mathscr{F}\ell_{G,\mu}^w|$ be any lift of x; this is a rank 1 point. By Lemma 3.3.1, we have $\mathscr{F}\ell_{G,\mu}^w \subseteq V_w$ and we let $x_w := \tilde{x} H_w \in |\mathscr{F}\ell_{G,\mu}^w|/H_w$ (recall that $H_w \subseteq H$). Now consider the set $(\tilde{x}H)/H_w \subseteq |\mathscr{F}\ell_{G,\mu}^w|/H_w$ (which contains x_w), equipped with the subspace topology. On the one hand, $(\tilde{x}H)/H_w$ consists only of points with no proper generalizations in the spectral space $|\mathscr{F}\ell_{G,\mu}^w|/H_w$, so it is Hausdorff. It is also compact, since it is the image of $\tilde{x}H \subseteq |\mathscr{F}\ell_{G,\mu}|$. Finally, it is spectral, since it is the preimage of x under the map $|\mathscr{F}\ell_{G,\mu}^w|/H_w \to |\mathscr{F}\ell_{G,\mu}^w|/H$. Therefore it is profinite, e.g. by [31, Tag 0905, Lemma 5.23.7]. By our earlier remarks, $\pi_{H_w}^{-1}((\tilde{x}H)/H_w)$ is naturally a spatial diamond. Our first goal is to prove that $\pi_{H_w}^{-1}((\tilde{x}H)/H_w)$ is an affinoid perfectoid space. As emphasized by a referee, the fact that $\mathscr{F}\ell_{G,\mu}^w \subseteq V_w$ implies that

$$\pi_{H_w}^{-1}((\tilde{x}H)/H_w) \subseteq \overline{\mathcal{X}}_{H_w,w} = \pi_{H_w}^{-1}(|V_w|/H_w).$$

The latter is a perfectoid space by Corollary 3.4.2, and that corollary moreover shows that $\pi_{H_w}^{-1}((\tilde{x}H)/H_w)$ is contained in some affinoid perfectoid open subset of $\overline{\mathcal{X}}_{H_w,w}$. However, proving that $\pi_{H_w}^{-1}((\tilde{x}H)/H_w)$ itself is affinoid perfectoid requires a more intricate argument. We prove this in Corollary 4.2.7 below, but first we require some preliminary results.

Lemma 4.2.2. *Let Y be a spatial diamond, let S be a profinite set and assume that we have a spectral map $q: |Y| \to S$. If $q^{-1}(s)$ is affinoid perfectoid for all $s \in S$, then Y is affinoid perfectoid.*

Proof. This is a direct consequence of [29, Lemma 11.27]. ∎

As $x_w \in (\tilde{x}H)/H_w$ is arbitrary by construction, it therefore suffices to prove that $\pi_{H_w}^{-1}(x_w)$ is affinoid perfectoid. To show this, we will use the following simple group-theoretical lemma.

Lemma 4.2.3. *In this lemma and its proof only, let G be a group, let $H \subseteq G$ be a subgroup and let $Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_r = G$ be a sequence of normal subgroups of G such that Z_{i+1}/Z_i is central in G/Z_i. Then $HZ_0 \subseteq HZ_1 \subseteq \cdots \subseteq HZ_r = G$ is a sequence of subgroups of G, and HZ_i is normal in HZ_{i+1}.*

Proof. That the HZ_i are subgroups is clear. For the final assertion, take $z_1 \in Z_i$, $z_2 \in Z_{i+1}$ and $h_1, h_2 \in H$. Since Z_{i+1}/Z_i is central in G/Z_i, we can write $z_2 h_1 = h_1 z_2 w$ for some $w \in Z_i$. Then we compute

$$(h_2 z_2)(h_1 z_1)(h_2 z_2)^{-1} = h_2 h_1 (z_2(w z_1) z_2^{-1}) h_2^{-1},$$

which is in HZ_i as desired. ∎

Lemma 4.2.4. *There is a sequence of normal subgroups $1 = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_r = \overline{U}$ such that Z_{i+1}/Z_i is central in $\overline{U}/Z_i$ and each Z_i is a product (as a subscheme, not necessarily as a subgroup) of root subgroups of G with respect to T. Moreover, the product $\overline{U}_w Z_i$ is a product of root subgroups and $\overline{U}_w Z_i$ is normal in $\overline{U}_w Z_{i+1}$, and $\overline{U}_w Z_{i+1}/\overline{U}_w Z_i$ is isomorphic to $\mathbf{G}_a^{d_i}$ for $d_i = \dim \overline{U}_w Z_{i+1}/\overline{U}_w Z_i$.*

Proof. The first part follows from [9, Proposition 5.1.16 (2)]; one can take the Z_i to be the groups called $U_{\geq n}$ in there, with $\Psi = \Phi^+$ in the notation of [9]. The fact that Z_{i+1}/Z_i is central in $\overline{U}/Z_i$ follows from the same calculation that shows that $U_{\geq n}/U_{\geq n+1}$ is abelian in [9, proof of Proposition 5.1.16].

For the second part, first note that $\overline{U}_w Z_i$ is a product of root subgroups as a subscheme since $\overline{U}_w$ and Z_i are, and it is a subgroup since Z_i is normal. The normality of $\overline{U}_w Z_i$ in $\overline{U}_w Z_{i+1}$ then follows by applying Lemma 4.2.3 to the functors of points. Finally, $\overline{U}_w Z_{i+1}/\overline{U}_w Z_i$ is isomorphic to $Z_{i+1}/(Z_{i+1} \cap \overline{U}_w Z_i) \subseteq Z_{i+1}/Z_i$ and the latter is isomorphic, as groups, to the product of the root subgroups in Z_{i+1} but not in Z_i by construction (i.e. [9, Proposition 5.1.16 (2)]), so $\overline{U}_w Z_{i+1}/\overline{U}_w Z_i$ is isomorphic as a group to the product of root subgroups in Z_{i+1} but not in $\overline{U}_w Z_i$. ∎

Proposition 4.2.5. *Let $\Gamma_{x,w} := \mathrm{Stab}_{H_w}(\widetilde{x})$ and consider the point $\widetilde{x}_\Gamma = \widetilde{x}\Gamma_{x,w} \in |\mathscr{Fl}_{G,\mu}|/\Gamma_{x,w}$. Then the natural map*

$$\pi_{\Gamma_{x,w}}^{-1}(\widetilde{x}_\Gamma) \to \pi_{H_w}^{-1}(x_w)$$

is an isomorphism.

Proof. We will apply Lemma 2.3.2 repeatedly (once is not enough, since $\Gamma_{x,w}$ might not be normal in H_w). The group H_w is nilpotent, so let $Z_0 = 1 \subseteq Z_1 = Z(H_w) \subseteq Z_2 \subseteq \cdots \subseteq Z_r = H_w$ be the upper central series of H_w and set $H_i = \Gamma_{x,w} Z_i$. Consider the points $\widetilde{x} H_i \in |\mathscr{Fl}_{G,\mu}|/H_i$. We claim that the natural maps

$$\pi_{H_i}^{-1}(\widetilde{x} H_i) \to \pi_{H_{i+1}}^{-1}(\widetilde{x} H_{i+1})$$

are isomorphisms for all $i = 0, \ldots, r-1$. Since the composition of all these maps is the map in the statement of the proposition, this suffices. But this is a direct application of Lemma 2.3.2, setting (in the notation of that lemma) $G = H_{i+1}/H_i$, $X = \overline{\mathcal{X}}_{H_i}$, $S = |\pi_{H_i}^{-1}(\widetilde{x} H_i)|$, $X \times_{|X|} S = \pi_{H_i}^{-1}(\widetilde{x} H_i)$ (note that H_i is normal in H_{i+1} by Lemma 4.2.3, and that X/G can be identified with $\overline{\mathcal{X}}_{H_{i+1}}$ by Lemma 3.2.2). ∎

We now study $\pi_{\Gamma_{x,w}}^{-1}(\widetilde{x}_\Gamma)$. First, consider $\pi_1^{-1}(\widetilde{x})$. Recall the standard affinoid open $\mathcal{V}_1 \subseteq \mathscr{Fl}_{G,\mu}$ from §3.4. By Corollary 3.4.2 and the definitions, we have

$$\pi_1^{-1}(\widetilde{x}) \subseteq \pi_1^{-1}(V_w) = \bigcup_{k' \geq 0} \pi_1^{-1}(\mathcal{V}_1 \gamma^{-k'} w^{-1}) = \bigcup_{k \geq 0} \overline{\mathcal{X}_1}(\epsilon)_a \gamma^{-k} w^{-1}.$$

It follows that we may choose k and k' such that $\widetilde{x} \subseteq \mathcal{V}_1 \gamma^{-k'} w^{-1}$ and

$$\pi_1^{-1}(\widetilde{x}) \subseteq \overline{\mathcal{X}_1}(\epsilon)_a \gamma^{-k} w^{-1} \subseteq \pi_1^{-1}(\mathcal{V}_1 \gamma^{-k'} w^{-1}).$$

Using that $\mathcal{V}_1$ is stable under $\overline{P}_\mu(\mathbb{Z}_p)$ and that $\gamma^{k'} \overline{P}_\mu(\mathbb{Z}_p) \gamma^{-k'} \supseteq \overline{P}_\mu(\mathbb{Z}_p)$, we see that $\mathcal{V}_1 \gamma^{-k'} w^{-1}$ is stable under $\overline{P}_{\mu,w}(\mathbb{Z}_p)$, and hence a fortiori stable under H_w and $\Gamma_{x,w}$. Since $\Gamma_{x,w}$ stabilizes the rank 1 point $\widetilde{x}$, we may choose a basis of open neighborhoods U_t, $t \in T$ (some index set), of $\widetilde{x}$ such that, for all t, U_t is a $\Gamma_{x,w}$-stable rational subset of $\mathcal{V}_1 \gamma^{-k'} w^{-1}$ and $\pi_1^{-1}(U_t) \subseteq \overline{\mathcal{X}_1}(\epsilon)_a \gamma^{-k} w^{-1}$. (Any rational subset of $\mathcal{V}_1 \gamma^{-k'} w^{-1}$ has finitely many $\Gamma_{x,w}$-translates by [30, Lemma 2.2] or [7, Lemma 5.2.2], so their intersection is again a rational subset.) Then we see that

$$\widetilde{x}_\Gamma = \bigcap_{t \in T} |U_t| / \Gamma_{x,w}$$

and hence

$$\pi_{\Gamma_{x,w}}^{-1}(\widetilde{x}_\Gamma) = \varprojlim_{t \in T} \pi_{\Gamma_{x,w}}^{-1}(|U_t| / \Gamma_{x,w}),$$

so to prove that $\pi_{\Gamma_{x,w}}^{-1}(\widetilde{x}_\Gamma)$ is affinoid perfectoid it suffices to prove that each $\pi_{\Gamma_{x,w}}^{-1}(|U_t| / \Gamma_{x,w})$ is affinoid perfectoid. For convenience, we introduce the principal congruence subgroups

$$K(p^m) := \{g \in G(\mathbb{Z}_p) \mid (g \bmod p) = 1\}.$$

Note that we may write

$$\pi_1^{-1}(U_t) = \varprojlim_{m \geq 0} \pi_{\Gamma_{x,w} \cap K(p^m)}^{-1}(|U_t| / (\Gamma_{x,w} \cap K(p^m))),$$

and that $\pi_1^{-1}(U_t)$ is a rational subset of $\overline{\mathcal{X}_1}(\epsilon)_a \gamma^{-k} w^{-1}$ by construction. Since rational subsets come from finite level in an inverse limit, it follows that there is an m such that $\pi_{\Gamma_{x,w} \cap K(p^m)}^{-1}(|U_t| / (\Gamma_{x,w} \cap K(p^m)))$ is a rational subset of $|\overline{\mathcal{X}_1}(\epsilon)_a| \gamma^{-k} w^{-1} / (\Gamma_{x,w} \cap K(p^m))$ (which is affinoid perfectoid by Corollary 3.4.2), hence affinoid perfectoid. Let $G_m := \Gamma_{x,w} / (\Gamma_{x,w} \cap K(p^m))$; this is a finite group. Note that $\pi_{\Gamma_{x,w}}^{-1}(|U_t| / \Gamma_{x,w})$, being a quasicompact open subset of $|\mathcal{X}_1^*(\epsilon)_a| \gamma^{-k} w^{-1} / \Gamma_{x,w}$, is a quasicompact perfectoid space.

Proposition 4.2.6. $\pi_{\Gamma_{x,w}}^{-1}(|U_t| / \Gamma_{x,w})$ *is the quotient of* $\pi_{\Gamma_{x,w} \cap K(p^m)}^{-1}(|U_t| / (\Gamma_{x,w} \cap K(p^m)))$ *by the finite group* G_m. *Hence it is affinoid perfectoid.*

Proof. The first claim follows from Proposition 3.2.2, and then the second follows from Proposition 2.1.1 and [19, Theorem 3.3.25]. ∎

We now summarize our discussion so far.

Corollary 4.2.7. $\pi_{H_w}^{-1}((\tilde{x}H)/H_w)$ *is affinoid perfectoid with Zariski closed boundary.*

Proof. We begin by showing that $\pi_{H_w}^{-1}((\tilde{x}H)/H_w)$ is affinoid perfectoid, summarizing the arguments above. First, by Lemma 4.2.2 it is enough to show that $\pi_{H_w}^{-1}(x_w)$ is affinoid perfectoid, which by Proposition 4.2.5 is equivalent to showing that $\pi_{\Gamma_{x,w}}^{-1}(|U_t|/\Gamma_{x,w})$ is affinoid perfectoid, and this is Proposition 4.2.6. That the boundary is Zariski closed is then immediate. ∎

Remark 4.2.8. Let us now make a few remarks on the differences between the arguments of this paper and those of [7]. Thanks to the results in §2, our results on the fibers of $\pi_{H_w}^{-1}$ are slightly stronger in the sense that we can prove Corollary 4.2.7; in [7] we could essentially only prove that it is affinoid perfectoid after a possible modification of the boundary, which was enough to deduce the cohomological consequences that we needed. We also note that Proposition 4.2.5, where Lemma 2.3.2 is the key ingredient, allows us to bypass the technical but powerful [7, Proposition 5.2.1]. Indeed, for our argument here it suffices to produce neighborhoods of points which are stable under the stabilizer of the point rather than stable under the whole group. The former is almost trivial in comparison to the latter.

From here, we wish to compute cohomology on $\pi_H^{-1}(x)$ using the cover $\pi_{H_w}^{-1}((\tilde{x}H)/H_w)$. Again, we are faced with the issue that H_w may not be normal in H, but the fact that H is nilpotent comes to the rescue. Recall that, by Lemma 4.2.1 and the paragraph preceding it, we need to show that

$$H_{\text{ét}}^i(\pi_H^{-1}(x), j_!(\mathcal{O}_{\mathcal{X}}^+/p)^a) = 0$$

for all $i > d - \dim \mathscr{H}_{G,\mu}^w$. Choose a series of subgroups $1 = Z_0 \subseteq \cdots \subseteq Z_r = \overline{U}$ as in Lemma 4.2.4, and set $H_{w,i} = H \cap \overline{U}_w(\mathbb{Z}_p)Z_i(\mathbb{Z}_p)$ for $i = 0, \ldots, r$. Note that we have $H_{w,0} = H_w$ and $H_{w,r} = H$. Moreover, $H_{w,i}$ is normal in $H_{w,i+1}$ and $H_{w,i+1}/H_{w,i}$ embeds naturally as a finite index subgroup of $(\overline{U}_w Z_{i+1}/\overline{U}_w Z_i)(\mathbb{Z}_p) = \overline{U}_w Z_{i+1}(\mathbb{Z}_p)/\overline{U}_w Z_i(\mathbb{Z}_p)$ (this equality follows by looking at the decomposition into root spaces), so it is isomorphic to $\mathbb{Z}_p^{d_i}$ (in the notation of Lemma 4.2.4). By [7, Theorem 2.2.7], we have Hochschild–Serre spectral sequences with E_2-page

$$H_{\text{cts}}^s(H_{w,i+1}/H_{w,i}, H_{\text{ét}}^t(\pi_{H_{w,i}}^{-1}(\tilde{x}H/H_{w,i}), j_!(\mathcal{O}_{\mathcal{X}}^+/p)^a))$$
$$\implies H_{\text{ét}}^{s+t}(\pi_{H_{w,i+1}}^{-1}(\tilde{x}H/H_{w,i+1}), j_!(\mathcal{O}_{\mathcal{X}}^+/p)^a)$$

for all $i = 0, \ldots, r - 1$. By Corollary 4.2.7 and [7, Proposition 5.1.4], we have

$$H_{\text{ét}}^t(\pi_{H_w}^{-1}(\tilde{x}H/H_w), j_!(\mathcal{O}_{\mathcal{X}}^+/p)^a) = 0 \quad \text{for all } t > 0.$$

Remark 4.2.9. Here we use the fact that the notions of Zariski closed and strongly Zariski closed subsets of affinoid perfectoid spaces are now known to agree, by [2, Remark 7.5]. It is perhaps worth noting that, just as in [7], one may prove directly that the sets relevant to

our arguments are strongly Zariski closed; the general result from [2] is strictly speaking not needed.

From this, we deduce by induction using the Hochschild–Serre spectral sequences above and the fact that $H_{w,i+1}/H_{w,i} \cong \mathbb{Z}_p^{d_i}$ has cohomological dimension d_i for continuous cohomology that

$$H^i_{\text{ét}}(\pi_H^{-1}(x), (j_! \mathcal{O}_{\mathcal{X}}^+/p)^a) = 0$$

for all $i > d_0 + \cdots + d_{r-1} = \dim \overline{U} - \dim \overline{U}_w$. We then finish the proof of Theorem 4.1.4 by noting that, by Lemma 3.3.2,

$$\dim \overline{U} - \dim \overline{U}_w = \dim \overline{N}_\mu - \dim \mathscr{F}\ell_{G,\mu}^w = d - \dim \mathscr{F}\ell_{G,\mu}^w.$$

5. Bounds on codimensions for ordinary parts

In this section, we deduce an application of Theorem 4.1.1, namely we bound the codimension over the Iwasawa algebra of the ordinary part of completed homology of the Shimura varieties for G. The application is Theorem 5.5.1 and it relies on a Poincaré duality spectral sequence for the ordinary part of the homology of locally symmetric spaces; most of the work in this section is in constructing this spectral sequence.

We work with a group $G/\mathbb{Q}$ which is split at p since we can only prove our main result, Theorem 5.5.1 under this assumption (and the additional assumption that G admits a Shimura variety of Hodge type), though we believe that the weaker assumption that G is only quasi-split at p should suffice for Sections 5.1 to 5.4. We also do not need to know that G admits a Shimura variety of Hodge type until we appeal to Theorem 4.1.1, as we only need to consider the locally symmetric spaces associated to G.

We note that the existence of such a spectral sequence has been previously announced by Emerton, relying on his theory of ordinary parts [11,12]. In this paper, we give a different construction relying on computations of (co)homology using singular and simplicial chains, in the style of Ash–Stevens [1], as well as ideas of Hill [16].

5.1. Completed cohomology and distributions

We begin by recalling some standard material; the reader is referred to [4] for more details and to [24, §3.1] for a useful summary. Let G be a connected reductive group over $\mathbb{Q}$ and assume it admits a flat affine model over $\mathbb{Z}$ which is split at p. Let X denote the symmetric space for $G(\mathbb{R})$ in the sense of [4, §2]. Let $D := \dim_{\mathbb{R}} X$.

For simplicity, we will consider neat compact open subgroups $K \subset G(\mathbb{A}_f)$ which are of the form $K = \prod_\ell K_\ell$, where ℓ runs over finite primes and $K_\ell \subseteq G(\mathbb{Z}_\ell)$. Let G^{ad} denote the adjoint group of G, and let $G^{\text{ad}}(\mathbb{R})^+$ denote the identity component of $G^{\text{ad}}(\mathbb{R})$ in the real topology. We let $G(\mathbb{R})_+$ denote the preimage of $G^{\text{ad}}(\mathbb{R})^+$ under the natural map $G(\mathbb{R}) \to G^{\text{ad}}(\mathbb{R})$, and we set $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$. For K as above, we define

a locally symmetric space

$$X_K := G(\mathbb{Q})_+ \backslash X \times G(\mathbb{A}_f)/K.$$

As is well known, the fact that K is neat implies that X_K is a smooth manifold; X_K is orientable because X is and $G(\mathbb{R})_+$ acts by orientation-preserving maps.

Remark 5.1.1. The reader familiar with the literature on adelic locally symmetric spaces will know that other sources use slightly different definitions. For example, it is common to quotient out on the left either by the full $G(\mathbb{Q})$ or the smaller $G(\mathbb{Q}) \cap G(\mathbb{R})^+$, where $G(\mathbb{R})^+ \subseteq G(\mathbb{R})$ is the identity component. Roughly speaking, the difference between these definitions is only in the structure of the set of components and the theory developed here could be developed for either of these choices (at least for sufficiently small K). The reason for our convention is that, when G admits a Shimura datum of Hodge type, X_K is exactly the complex points of the Shimura variety for G with level K. This follows from [22, Lemma 5.11] and the fact that in this case, the maximal $\mathbb{R}$-split torus in the center of G is $\mathbb{Q}$-split.

Following [24], we will also make use of the space

$$\mathfrak{X} := G(\mathbb{Q})_+ \backslash X \times G(\mathbb{A}_f),$$

where $G(\mathbb{A}_f)$ is given the discrete topology prior to taking the quotient. As a topological space, this is an uncountable disjoint union of copies of X.

The symmetric space X admits a partial (Borel–Serre) compactification X^{BS}, to which the $G(\mathbb{Q})$-action extends, and we set

$$X_K^{\mathrm{BS}} := G(\mathbb{Q})_+ \backslash X^{\mathrm{BS}} \times G(\mathbb{A}_f)/K;$$

this is a compactification of X_K and the inclusion $X_K \hookrightarrow X_K^{\mathrm{BS}}$ is a homotopy equivalence. We also consider the boundaries $\partial X^{\mathrm{BS}} := X^{\mathrm{BS}} \setminus X$ and $\partial X_K^{\mathrm{BS}} := X_K^{\mathrm{BS}} \setminus X_K$. We also consider the space

$$\mathfrak{X}^{\mathrm{BS}} := G(\mathbb{Q})_+ \backslash X^{\mathrm{BS}} \times G(\mathbb{A}_f)$$

where again $G(\mathbb{A}_f)$ is given the discrete topology. We set $\partial\mathfrak{X}^{\mathrm{BS}} = \mathfrak{X}^{\mathrm{BS}} \setminus \mathfrak{X}$, and note that all these spaces carry right actions of $G(\mathbb{A}_f)$, whose restrictions to neat compact opens $K \subseteq G(\mathbb{A}_f)$ are free. In other words, $\mathfrak{X} \to X_K$ etc. are K-covers. We let $C_\bullet$ and $\partial C_\bullet$ denote the complexes of singular chains with $\mathbb{Z}$-coeffcients on $\mathfrak{X}^{\mathrm{BS}}$ and $\partial\mathfrak{X}^{\mathrm{BS}}$ respectively; these are right $\mathbb{Z}[G(\mathbb{A}_f)]$-modules. There is a natural morphism of complexes $\partial C_\bullet \to C_\bullet$ and we define $C_\bullet^{\mathrm{BM}}$ to be the cone of this morphism. Given a left $\mathbb{Z}[K]$-module M, we define

$$C_{\mathbb{A},\bullet}(K, M) := C_\bullet \otimes_{\mathbb{Z}[K]} M$$

and

$$C_{\mathbb{A},\bullet}^{\mathrm{BM}}(K, M) := C_\bullet^{\mathrm{BM}} \otimes_{\mathbb{Z}[K]} M.$$

These complexes compute the homology, respectively the Borel–Moore homology, of X_K with coefficients in the local system $\widetilde{M}$ determined by M via the K-cover $\mathfrak{X}^{\mathrm{BS}} \to X_K^{\mathrm{BS}}$; we record this as a proposition.

Proposition 5.1.2. *Let M be a left K-module. There are canonical isomorphisms*

$$H_*(X_K, \widetilde{M}) \simeq H_*(C_{\mathbb{A},\bullet}(K, M)) \quad \text{and} \quad H_*^{\text{BM}}(X_K, \widetilde{M}) \simeq H_*(C_{\mathbb{A},\bullet}^{\text{BM}}(K, M)).$$

Remark 5.1.3. Choose a finite (combinatorial) triangulation of X_K^{BS} such that ∂X_K^{BS} is a simplicial subcomplex (X_K^{BS} is homeomorphic to a smooth compact manifold with boundary [4, §11.1], so this can be done using e.g. [23, Theorem 10.6]). Pulling back these triangulations to $\mathfrak{X}^{\text{BS}}$ and $\partial \mathfrak{X}^{\text{BS}}$ we obtain bounded complexes $F_\bullet$ and $\partial F_\bullet$ of finite free right $\mathbb{Z}[K]$-modules which are homotopy equivalent to $C_\bullet$ and $\partial C_\bullet$, respectively. We let $F_\bullet^{\text{BM}}$ denote the cone of $\partial F_\bullet \to F_\bullet$, which is then homotopy equivalent to $C_\bullet^{\text{BM}}$.

This allows us to construct complexes $C_\bullet(K, M)$ and $C_\bullet^{\text{BM}}(K, M)$ which have good finiteness properties and are homotopy equivalent to $C_{\mathbb{A},\bullet}(K, M)$ and $C_{\mathbb{A},\bullet}^{\text{BM}}(K, M)$, respectively,[7] by setting

$$C_\bullet^{(\text{BM})}(K, M) := F_\bullet^{(\text{BM})} \otimes_{\mathbb{Z}[K]} M.$$

Here and elsewhere we write (BM) to mean that one can either make the same construction for the complexes with no superscript or with the BM superscript.

Analogous constructions can be made for cohomology and cohomology with compact support. If M is a right K-module, define

$$C_{\mathbb{A}}^{\bullet,(\text{BM})}(K, M) := \text{Hom}_{\mathbb{Z}[K]}(C_\bullet^{(\text{BM})}, M).$$

We have the following analogue of Proposition 5.1.2, which once again is an instance of descent.

Proposition 5.1.4. *Let M be a right K-module. There are canonical isomorphisms*

$$H^*(X_K, \widetilde{M}) \simeq H^*(C_{\mathbb{A}}^\bullet(K, M)) \quad \text{and} \quad H_c^*(X_K, \widetilde{M}) \simeq H^*(C_{\mathbb{A}}^{\bullet,\text{BM}}(K, M)).$$

We also have complexes $C^{\bullet,(\text{BM})}(K, M) = \text{Hom}_{\mathbb{Z}[K]}(F_\bullet^{(\text{BM})}, M)$ with good finiteness properties as in Remark 5.1.3.

We now discuss the action of $G(\mathbb{A}_f)$ on the adelic complexes we have defined above. Let S be a finite set of finite primes, not necessarily containing p. Assume that M is a left $\mathbb{Z}[G(\mathbb{A}^S) \times K_S]$-module. For any $g \in G(\mathbb{A}^S)$, we have an isomorphism

$$g_*: C_{\mathbb{A},\bullet}^{(\text{BM})}(K, M) \xrightarrow{\sim} C_{\mathbb{A},\bullet}^{(\text{BM})}(g^{-1}Kg, M), \quad \sigma \otimes m \mapsto \sigma g \otimes g^{-1}m.$$

This can be translated into a Hecke action of the double coset $[KgK]$ for $g \in G(\mathbb{A}^S)$ by taking the composition

$$C_{\mathbb{A},\bullet}^{(\text{BM})}(K, M) \to C_{\mathbb{A},\bullet}^{(\text{BM})}(K \cap gKg^{-1}, M) \xrightarrow{g_*} C_{\mathbb{A},\bullet}^{(\text{BM})}(g^{-1}Kg \cap K, M) \to C_{\mathbb{A},\bullet}^{(\text{BM})}(K, M),$$
$$(5.1.1)$$

---

[7]The resulting complexes are called Borel–Serre complexes in [15], though they are constructed slightly differently there.

where the first morphism is the trace map

$$\sigma \otimes m \mapsto \sum \sigma k \otimes k^{-1}m,$$

where the sum runs over a set of coset representatives k for $K/(K \cap gKg^{-1})$, and the last morphism is restriction $\sigma \otimes m \mapsto \sigma \otimes m$. One can check that this recovers the usual Hecke action on homology / Borel–Moore homology.

Assume now that M is a right $\mathbb{Z}[G(\mathbb{A}^S) \times K_S]$-module. For any $g \in G(\mathbb{A}^S)$, we have an isomorphism

$$g^*: C_{\mathbb{A}}^{\bullet,(\mathrm{BM})}(K, M) \xrightarrow{\sim} C_{\mathbb{A}}^{\bullet,(\mathrm{BM})}(gKg^{-1}, M), \quad g^*(\phi)(\sigma) = \phi(\sigma g)g^{-1}.$$

This can be translated into a Hecke action of the double coset $[KgK]$ for $g \in G(\mathbb{A}^S)$ by taking the composition

$$C_{\mathbb{A}}^{\bullet,(\mathrm{BM})}(K, M) \to C_{\mathbb{A}}^{\bullet,(\mathrm{BM})}(K \cap g^{-1}Kg, M) \xrightarrow{g^*} C_{\mathbb{A}}^{\bullet,(\mathrm{BM})}(gKg^{-1} \cap K, M)$$
$$\to C_{\mathbb{A}}^{\bullet,(\mathrm{BM})}(K, M), \quad (5.1.2)$$

where the first morphism is restriction and the third morphism is the trace map

$$\phi \mapsto \psi(\sigma) := \sum \phi(\sigma k)k^{-1},$$

where the sum runs over a set of coset representatives k for $K/(K \cap g^{-1}Kg)$. One can check that this recovers the usual Hecke action on cohomology / cohomology with compact support.

If R is a commutative ring and M, N are R-modules, we have a canonical isomorphism

$$\mathrm{RHom}_R(C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K, M), N) \simeq C_{\mathbb{A}}^{\bullet,(\mathrm{BM})}(K, \mathrm{RHom}_R(M, N)), \quad (5.1.3)$$

from the adjunction between tensor products and homomorphisms. For example, if $R = N = \mathbb{Z}$, we obtain the universal coefficient isomorphism between homology and cohomology, respectively between Borel–Moore homology and cohomology with compact support. One can check from the explicit descriptions (5.1.1) and (5.1.2) that the universal coefficient isomorphism is equivariant for the Hecke action of $[KgK]$.

Our goal is now to use these explicit adelic complexes to describe ordinary completed (co)homology. Let $K'_p \subseteq K_p$ be a compact open subgroup; set $K' := K^p K'_p$, this is a compact open subgroup of K.

Lemma 5.1.5. (1) *For any $n \in \mathbb{Z}_{\geq 1}$, there is a canonical isomorphism*

$$C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K', \mathbb{Z}/p^n\mathbb{Z}) \simeq C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K, \mathbb{Z}/p^n\mathbb{Z}[K_p/K'_p]),$$

where K^p acts trivially and K_p acts by left translation on $\mathbb{Z}/p^n\mathbb{Z}[K_p/K'_p]$.

(2) *As K^p varies, this isomorphism is equivariant for the action of $g^p \in G(\mathbb{A}_f^p)$.*

(3) *If $g \in G(\mathbb{Q}_p)$ and $K_{i,p} \subseteq K_p$ for $i = 1, 2$ with $g^{-1} K_{1,p} g \subseteq K_{2,p}$ and $K_i := K^p K_{i,p}$, the morphism*

$$g_*: C_{\mathbb{A},\bullet}^{(BM)}(K_1, \mathbb{Z}/p^n) \to C_{\mathbb{A},\bullet}^{(BM)}(K_2, \mathbb{Z}/p^n), \quad g_*(\sigma \otimes \lambda) = \sigma g \otimes \lambda,$$

corresponds to the morphism

$$g_*: C_{\mathbb{A},\bullet}^{(BM)}(K, \mathbb{Z}/p^n[K_p/K_{1,p}]) \to C_{\mathbb{A},\bullet}^{(BM)}(K, \mathbb{Z}/p^n[K_p/K_{2,p}]),$$

$$g_*(\sigma \otimes \lambda k) = \sigma k g \otimes \lambda.$$

(4) *If $K_{2,p} \subseteq K_{1,p} \subseteq K_p$, and $K_i := K^p K_{i,p}$ for $i = 1, 2$, the trace morphism $C_{\mathbb{A},\bullet}^{(BM)}(K_1, \mathbb{Z}/p^n) \to C_{\mathbb{A},\bullet}^{(BM)}(K_2, \mathbb{Z}/p^n)$ corresponds to the morphism*

$$C_{\mathbb{A},\bullet}^{(BM)}(K, \mathbb{Z}/p^n[K_p/K_{1,p}]) \to C_{\mathbb{A},\bullet}^{(BM)}(K, \mathbb{Z}/p^n[K_p/K_{2,p}])$$

induced by the natural trace map $\mathrm{tr}: \mathbb{Z}/p^n[K_p/K_{1,p}] \to \mathbb{Z}/p^n[K_p/K_{2,p}]$.

Proof. The first assertion is a simple adjunction, but to make the verification of formulas easier for the reader we give explicit isomorphisms. So, define morphisms

$$\iota: C_{\mathbb{A},\bullet}^{(BM)}(K', \mathbb{Z}/p^n) \to C_{\mathbb{A},\bullet}^{(BM)}(K, \mathbb{Z}/p^n[K_p/K_p'])$$

given by $\sigma \otimes \lambda \mapsto \sigma \otimes \lambda$ and

$$\eta: C_{\mathbb{A},\bullet}^{(BM)}(K, \mathbb{Z}/p^n[K_p/K_p']) \to C_{\mathbb{A},\bullet}^{(BM)}(K', \mathbb{Z}/p^n)$$

given by $\sigma \otimes \lambda k \mapsto \sigma k \otimes \lambda$. It is not hard to check that ι and η are well-defined, and that they are mutual inverses. The Hecke equivariance away from p is immediate, and the Hecke equivariance at p can be checked by direct computation:

$$(\iota_2 \circ g_* \circ \eta_1)(\sigma \otimes \lambda k) = \sigma k g \otimes \lambda.$$

Finally, the last assertion can also be checked by direct computation. ∎

We now begin to describe ordinary completed homology in this language. We fix the tame level K^p and denote all locally symmetric spaces of this fixed tame level by X_{K_p}, where $K_p \subseteq G(\mathbb{Z}_p)$ is a compact open subgroup. Similarly, we will often omit the tame level when using the complexes defined above; for example, we will write $C_{\mathbb{A},\bullet}^{(BM)}(K_p, M)$ for $C_{\mathbb{A},\bullet}^{(BM)}(K^p K_p, M)$. Recall the assumption that $G_{\mathbb{Z}_p}$ is split. Choose a split Borel subgroup $B \subset G_{\mathbb{Z}_p}$, with Levi decomposition $B = T \ltimes U$, where T is the split torus and U is the unipotent subgroup, with opposite unipotent subgroup $\overline{U}$. Let $T_0 := T(\mathbb{Z}_p)$, and, for $i \in \mathbb{Z}_{\geq 1}$, set $T_i := \ker(T(\mathbb{Z}_p) \to T(\mathbb{Z}/p^i))$. Similarly, for $j \in \mathbb{Z}_{\geq 1}$, define $N_j := \ker(U(\mathbb{Z}_p) \to U(\mathbb{Z}/p^j))$ and $\overline{N}_j := \ker(\overline{U}(\mathbb{Z}_p) \to \overline{U}(\mathbb{Z}/p^j))$. For $j \geq 0$ and $i \geq \max(j, 1)$, set $K_{ij} := \overline{N}_i T_j N_1 \subset G(\mathbb{Z}_p)$. To simplify notation, we will also set $K_1 := K_{10}$.

Define the completed homology / Borel–Moore homology at level N_1 by

$$\widetilde{H}_*^{(\mathrm{BM})}(N_1) := \varprojlim_{i,j,n} H_*^{(\mathrm{BM})}(X_{K_{ij}}, \mathbb{Z}/p^n).$$

This is equipped with a Hecke action of $[K^p g^p K^p]$ for $g^p \in G(\mathbb{A}_f^p)$ in the usual way. There is also a Hecke action at p. Let

$$T^+ := \{t \in T(\mathbb{Q}_p) \mid tN_1t^{-1} \subseteq N_1\}.$$

Also define

$$T^- := \{t \in T(\mathbb{Q}_p) \mid t\overline{N}_1 t^{-1} \subseteq \overline{N}_1\}.$$

For $t \in T^+$, set $\overline{N}_t := t^{-1}\overline{N}_1 t$, and $N_t := tN_1 t^{-1}$. For any $t \in T^+$, we have a Hecke action of $[N_1 t N_1]$ on $\widetilde{H}_*^{(\mathrm{BM})}(N_1)$ given by the composition

$$\widetilde{H}_*^{(\mathrm{BM})}(N_1) \xrightarrow{\mathrm{tr}} \widetilde{H}_*^{(\mathrm{BM})}(N_t) \xrightarrow[\sim]{t_*} \widetilde{H}_*^{(\mathrm{BM})}(N_1),$$

where the first map is the natural trace map. Here $\widetilde{H}_*^{(\mathrm{BM})}(N_t)$ is defined as

$$\widetilde{H}_*^{(\mathrm{BM})}(N_t) := \varprojlim_{i,j,n} H_*^{(\mathrm{BM})}(X_{tK_{ij}t^{-1}}, \mathbb{Z}/p^n).$$

The Hecke action above is compatible with the Hecke action of $[K_{ij} t K_{ij}]$ on each $H_*^{(\mathrm{BM})}(X_{K_{ij}}, \mathbb{Z}/p^n\mathbb{Z})$ as described in (5.1.1).

We wish to describe $\widetilde{H}_*^{(\mathrm{BM})}(N_1)$ with its Hecke actions in terms of the adelic complexes defined above, with coefficients in a certain algebra of $\mathbb{Z}_p$-valued distributions. If X is a profinite set, let $\mathscr{D}(X)$ denote the space of continuous $\mathbb{Z}_p$-valued distributions on X; if X is a profinite group then this carries a natural $\mathbb{Z}_p$-algebra structure. In particular, we have

$$\mathscr{D}(K_1/N_1) = \mathbb{Z}_p[\![K_1/N_1]\!] = \varprojlim_{i,j,n} \mathbb{Z}/p^n[K_1/K_{ij}]. \qquad (5.1.4)$$

Set $K_t' := K_1 \cap tK_1t^{-1}$ and $K_t := K_1 \cap t^{-1}K_1 t$. For $t \in T^+$, we have an isomorphism

$$\iota_{K_t, K_1} \circ t_* \circ \eta_{K_1, K_t'} \colon C_{\mathbb{A}, \bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_t)) \xrightarrow{\sim} C_{\mathbb{A}, \bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_1)), \qquad (5.1.5)$$

which we describe explicitly as the composition of three isomorphisms. The first is an isomorphism

$$\eta_{K_1, K_t'} \colon C_{\mathbb{A}, \bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_t)) \xrightarrow{\sim} C_{\mathbb{A}, \bullet}^{(\mathrm{BM})}(K_t', \mathscr{D}(K_t'/N_t)).$$

Choose a set S of coset representatives for K_1/K_t'. Any $\mu \in \mathscr{D}(K_1/N_t)$ can be written uniquely as $\sum_{s \in S} s\mu_s$ with $\mu_s \in \mathscr{D}(K_t'/N_t)$. Set

$$\eta_{K_1, K_t'}(\sigma \otimes \mu) := \sum_{s \in S} \sigma s \otimes \mu_s.$$

This is well-defined, independent of the choice of S, and it is an isomorphism since it has the inverse $\iota_{K_t', K_1}(\sigma \otimes \mu) := \sigma \otimes \mu$. The second is the isomorphism

$$t_*: C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_t', \mathscr{D}(K_t'/N_t)) \xrightarrow{\sim} C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_t, \mathscr{D}(K_t/N_1))$$

given by $\sigma \otimes \mu \mapsto \sigma t \otimes t^{-1}\mu t$. Finally, the third is the isomorphism

$$\iota_{K_t, K_1}: C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_t, \mathscr{D}(K_t/N_1)) \xrightarrow{\sim} C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_1)).$$

Note that $K_1/K_t' \simeq N_1/N_t$. We obtain

$$\iota_{K_t, K_1} \circ t_* \circ \eta_{K_1, K_t'}(\sigma \otimes \mu) = \sum_{s \in N_1/N_t} \sigma s t \otimes t^{-1}\mu_s t.$$

Theorem 5.1.6. *There exist canonical isomorphisms*

$$\widetilde{H}_*^{(\mathrm{BM})}(N_1) \simeq H_*\big(C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_1))\big).$$

These are equivariant for the Hecke action of $[K^p g^p K^p]$ for all $g^p \in G(\mathbb{A}_f^p)$. For $t \in T^+$, the Hecke action of $[N_1 t N_1]$ on $\widetilde{H}_*^{(\mathrm{BM})}(N_1)$ is induced from the composition

$$C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_1)) \xrightarrow{\mathrm{tr}} C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_t)) \xrightarrow{\sim} C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_1)),$$

where tr *is induced from the natural trace map* $\mathscr{D}(K_1/N_1) \to \mathscr{D}(K_1/N_t)$ *and the isomorphism is given by* $\iota_{K_t, K_1} \circ t_* \circ \eta_{K_1, K_t'}$.

Proof. We claim first that the natural map of complexes

$$C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_1)) \to \varprojlim_{i,j,n} C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathbb{Z}/p^n[K_1/K_{ij}])$$

is a homotopy equivalence. Up to homotopy equivalences,[8] we can replace the above complexes with the corresponding Borel–Serre complexes as in Remark 5.1.3, and we have a natural map

$$C_\bullet^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_1)) \to \varprojlim_{i,j,n} C_\bullet^{(\mathrm{BM})}(K_1, \mathbb{Z}/p^n[K_1/K_{ij}]),$$

which can be seen by inspection to be an isomorphism using (5.1.4).

We now claim that the homology of the complex $\varprojlim_{i,j,n} C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathbb{Z}/p^n[K_1/K_{ij}])$ computes $\widetilde{H}_*^{(\mathrm{BM})}(N_1)$. By combining Lemma 5.1.5 and Proposition 5.1.2, we have an isomorphism

$$H_*^{(\mathrm{BM})}(X_{K_{ij}}, \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\sim} H_*(C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathbb{Z}/p^n[K_1/K_{ij}]))$$

---

[8]These can be chosen to be functorial in the coefficients, and compatible with the transition morphisms between different levels.

for every $n \in \mathbb{Z}_{\geq 1}$ and at each finite level K_{ij}. To conclude, we replace each finite level complex by the corresponding Borel–Serre complex. We obtain complexes of abelian compact Hausdorff groups with continuous differentials. We conclude by noting that the category of abelian compact Hausdorff groups is abelian, and inverse limits exist and are exact in this category.

The Hecke equivariance away from p is clear. To prove that the isomorphism is equivariant for the action of $[N_1 t N_1]$, it is enough to show that it is equivariant for tr and t_*. The equivariance for tr follows from Lemma 5.1.5 (4). The equivariance for t_* follows from Lemma 5.1.5 (1, 3). ∎

5.2. The universal coefficient isomorphism at infinite level

For $t \in T^+$, recall that $K_t = K_1 \cap t^{-1} K_1 t = \overline{N}_t T_0 N_1$. We consider $\mathscr{D}(K_t/N_1)$ as a $\mathscr{D}(K_t) \otimes_{\mathbb{Z}_p} \mathscr{D}(T_0)$-module, where K_t acts by multiplication on the left and T_0 acts by multiplication on the right. $\mathscr{D}(T_0)$ is a semi-local ring and is complete with respect to its J-adic topology, where we let J denote its Jacobson radical. Define

$$\mathscr{C}(K_t/N_1) := \mathrm{Hom}^{\mathrm{cont}}_{\mathscr{D}(T_0)}(\mathscr{D}(K_t/N_1), \mathscr{D}(T_0));$$

this is a $\mathscr{D}(T_0) \otimes_{\mathbb{Z}_p} \mathscr{D}(K_t)$-module. Here $\mathscr{D}(K_t/N_1)$ carries the inverse limit topology from equation (5.1.4), and we give $\mathscr{C}(K_t/N_1)$ the J-adic topology.

Lemma 5.2.1. *We have a natural isomorphism*

$$\mathrm{Hom}_{\mathscr{D}(T_0)}(\mathscr{C}(K_t/N_1), \mathscr{D}(T_0)) \simeq \mathscr{D}(K_t/N_1).$$

and, for all $i > 0$,

$$\mathrm{Ext}^i_{\mathscr{D}(T_0)}(\mathscr{C}(K_t/N_1), \mathscr{D}(T_0)) = 0.$$

(We note that all $\mathscr{D}(T_0)$-linear homomorphisms are automatically continuous for J-adic topology.)

Proof. From the definition, we have

$$\mathscr{C}(K_t/N_1) \cong \varprojlim_{j,k} \varinjlim_{i} \mathrm{Hom}_{\mathscr{D}(T_0/T_j)}\big(\mathscr{D}(K_t/\overline{N}_i T_j N_1), \mathscr{D}(T_0/T_j)\big) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^k. \quad (5.2.1)$$

Since T_0 normalizes $\overline{N}_i$, there is an isomorphism of profinite sets with T_0-action

$$\overline{N}_t/\overline{N}_i \times T_0/T_j \xrightarrow{\sim} K_t/\overline{N}_i T_j N_1$$

where T_0 acts trivially on $\overline{N}_t/\overline{N}_i$. Consequently, as a $\mathscr{D}(T_0)$-module,

$$\mathscr{D}(K_t/\overline{N}_i T_j N_1) \cong \mathscr{D}(\overline{N}_t/\overline{N}_i) \otimes_{\mathbb{Z}_p} \mathscr{D}(T_0/T_j),$$

where $\mathscr{D}(T_0)$ acts trivially on $\mathscr{D}(\overline{N}_t/\overline{N}_i)$. In particular, $\mathscr{D}(K_t/\overline{N}_i T_j N_1)$ is a finite free $\mathscr{D}(T_0/T_j)$-module, hence it is also reflexive. This, together with the explicit description (5.2.1), proves the first claim of the lemma.

For the second claim, observe that $\mathscr{C}(K_t/N_1)$ is the J-adic completion of $\mathscr{D}(T_0)$ of the free $\mathscr{D}(T_0)$-module

$$\varinjlim_i \varprojlim_{j,k} \mathrm{Hom}_{\mathscr{D}(T_0)}\big(\mathscr{D}(K_t/\overline{N}_i T_j N_1), \mathscr{D}(T_0/T_j)\big) \otimes \mathbb{Z}/p^k.$$

Then by [31, Tag 06LE], $\mathscr{C}(K_t/N_1)$ is a flat $\mathscr{D}(T_0)$-module. Hence, by [17, Theorem 1], $\mathrm{Ext}^i_{\mathscr{D}(T_0)}(\mathscr{C}(K_t/N_1), \mathscr{D}(T_0)) = 0$ for $i > 0$. ∎

Corollary 5.2.2. *We have a natural isomorphism*

$$\mathrm{RHom}_{\mathscr{D}(T_0)}\big(C_{\mathbb{A},\bullet}(K_t, \mathscr{C}(K_t/N_1)), \mathscr{D}(T_0)\big) \simeq C_{\mathbb{A}}^\bullet(K_t, \mathscr{D}(K_t/N_1)).$$

Proof. This follows from combining the isomorphism (5.1.3) with Lemma 5.2.1. ∎

5.3. Poincaré duality

We start by recalling Poincaré duality between Borel–Moore homology and cohomology; recall that X_K^{BS} is homeomorphic to a smooth compact orientable manifold with boundary.

Lemma 5.3.1. *Let $K \subseteq G(\mathbb{A}_f)$ be a compact open subgroup. There are homotopy equivalences of complexes of $\mathbb{Z}[K]$-modules*

$$F_\bullet^{\mathrm{BM}}[D] \simeq \mathrm{Hom}_{\mathbb{Z}[K]}(F_\bullet, \mathbb{Z}[K]), \quad F_\bullet[D] \simeq \mathrm{Hom}_{\mathbb{Z}[K]}(F_\bullet^{\mathrm{BM}}, \mathbb{Z}[K]).$$

On the right hand sides, $F_\bullet^{(\mathrm{BM})}$ is viewed as a left K-module by inverting the natural right K-module structure.

Proof. We sketch a proof of the second quasi-isomorphism; the proof of the first is completely analogous. As mentioned in Remark 5.1.3, the complex $F_\bullet$ comes from a combinatorial triangulation T of X_K^{BS}, which can even be chosen such that $\partial X_K^{\mathrm{BS}}$ is a simplicial subcomplex. For such a T, we can construct the dual $T^\vee$ of this triangulation (this construction and the remaining assertions in this paragraph seem to be well known in topology, see for example [32, Ch. 14]), which is a CW decomposition of X_K^{BS}. Let $S_\bullet(T)$ denote the simplicial homology complex attached to T and let $S_\bullet(T^\vee, T_\partial^\vee)$ denote the relative cellular homology complex of $T^\vee$ with respect to the boundary $T_\partial^\vee$ (which is a CW subcomplex). There are natural perfect pairings

$$S_{D-k}(T) \times S_k(T^\vee, T_\partial^\vee) \to \mathbb{Z}$$

where a simplex pairs to 1 with its dual cell and to 0 with the other cells. The induced map $S_\bullet(T)[D] \to \mathrm{Hom}_{\mathbb{Z}}(S_\bullet(T^\vee, T_\partial^\vee), \mathbb{Z})$ is an isomorphism of complexes.

Now pull back $T^\vee$ to a K-equivariant CW decomposition of $\mathfrak{X}^{\mathrm{BS}}$, and let $F_\bullet^\vee$ denote the corresponding relative cellular homology complex with respect to the boundary. Note that the pullback of $T^\vee$ is the dual cell decomposition of the pullback of T. Define a pairing

$$F_{D-k} \times F_k^\vee \to \mathbb{Z}[K]$$

by declaring that a simplex σ in F_{D-k} pair to 0 with a cell τ in $F_k^\vee$ unless $\tau = \sigma^\vee k$ for some $k \in K$, in which case we pair them to k^{-1}. From the definitions this pairing induces a map $F_{D-k} \to \mathrm{Hom}_{\mathbb{Z}[K]}(F_k^\vee, \mathbb{Z}[K])$, where we view $F_\bullet^\vee$ as a left K-module by inverting the natural right K-module structure. Using the fact that $S_\bullet(T)[D] \to \mathrm{Hom}_{\mathbb{Z}}(S_\bullet(T^\vee, T_\partial^\vee), \mathbb{Z})$ is an isomorphism, one sees that the maps above form a chain isomorphism $F_\bullet[D] \to \mathrm{Hom}_{\mathbb{Z}[K]}(F_\bullet^\vee, \mathbb{Z}[K])$ of complexes of right K-modules. The proof is then finished by noting that $F_\bullet^\vee$ is chain homotopic to $F_\bullet^{\mathrm{BM}}$, since they come from K-equivariant CW decompositions of the same manifold with boundary. ∎

Corollary 5.3.2. *For any compact open $K \subset G(\mathbb{A}_f)$ and any left K-module M (which we also view as a right K-module by inverting the left K-module structure), there is a natural quasi-isomorphism*

$$C_{\mathbb{A},\bullet}^{\mathrm{BM}}(K, M)[D] \simeq C_{\mathbb{A}}^\bullet(K, M).$$

Proof. Recall that we have homotopy equivalences

$$C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K, M) \simeq F_\bullet^{(\mathrm{BM})} \otimes_{\mathbb{Z}[K]} M.$$

Then the result follows from Lemma 5.3.1 along with the fact that $F_\bullet$, $F_\bullet^{\mathrm{BM}}$ are complexes of finite free $\mathbb{Z}[K]$-modules and, for any finite free $\mathbb{Z}[K]$-module F and any $\mathbb{Z}[K]$-module N, $\mathrm{Hom}_{\mathbb{Z}[K]}(F, N) \cong \mathrm{Hom}_{\mathbb{Z}[K]}(F, \mathbb{Z}[K]) \otimes_{\mathbb{Z}[K]} N$. ∎

Let $w \in G(\mathbb{Z}_p)$ be a representative of the longest element of the Weyl group. We have an involution $t \mapsto w^{-1}t^{-1}w$ of T^+. Let $\tau \colon K_1 \to T_0$ be the map that sends $\bar{n}tn \mapsto t$ for $\bar{n} \in \overline{N}_1$, $t \in T_0$ and $n \in N_1$. Note that τ is not a homomorphism, but it satisfies $\tau(\bar{b}kb) = \tau(\bar{b})\tau(k)\tau(b)$ for any $\bar{b} \in \overline{N}_1 T_0$, $k \in K_1$ and $b \in T_0 N_1$. We consider the pairing

$$\mathscr{D}(K_1/N_1) \times \mathscr{D}(K_1/N_1) \to \mathscr{D}(T_0), \quad \langle k_1 N_1, k_2 N_2 \rangle := \tau(wk_2^{-1}w^{-1}k_1), \forall k_1, k_2 \in K_1,$$

where the K_1 acts on the LHS by left multiplication and on the RHS by left multiplication by the inverse. This induces a morphism of left K_1-modules

$$\kappa \colon \mathscr{D}(K_1/N_1) \to (w^{-1})_* \mathscr{C}(K_1/N_1),$$

where $(w^{-1})_* \mathscr{C}(K_1/N_1)$ denotes $\mathscr{C}(K_1/N_1)$ with the K_1 action twisted such that $k \in K_1$ acts by wkw^{-1}. In turn, κ induces a morphism of complexes

$$\kappa_* \colon C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1)) \to C_{\mathbb{A},\bullet}(K_1, (w^{-1})_* \mathscr{C}(K_1/N_1)) \simeq C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/N_1)).$$

We now consider the composition of morphisms

$$\pi_w \colon C_{\mathbb{A},\bullet}^{\mathrm{BM}}(K_1, \mathscr{D}(K_1/N_1))[D] \xrightarrow{\sim} C_{\mathbb{A}}^\bullet(K_1, \mathscr{D}(K_1/N_1))$$
$$\xrightarrow{\sim} \mathrm{RHom}_{\mathscr{D}(T_0)}(C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/N_1)), \mathscr{D}(T_0))$$
$$\to \mathrm{RHom}_{\mathscr{D}(T_0)}(C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1)), \mathscr{D}(T_0)).$$

The first morphism is the Poincaré duality isomorphism of Corollary 5.3.2, the second is the isomorphism in Corollary 5.2.2 in the case $t = 1$, and the third is the morphism induced by precomposition with κ_*.

Proposition 5.3.3. *For any $t \in T^+$, the morphism*

$$\pi_w \colon C^{\mathrm{BM}}_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1))[D] \to \mathrm{RHom}_{\mathscr{D}(T_0)}(C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1)), \mathscr{D}(T_0))$$

is equivariant for the Hecke action of $[N_1 t N_1]$ on the RHS and the Hecke action of $[N_1 w^{-1} t^{-1} w N_1]$ on the LHS.

The morphism π_w is also equivariant for the Hecke action of $[K^p (g^p)^{-1} K^p]$ on the LHS and the Hecke action of $[K^p g^p K^p]$ on the RHS.

Proof. By translating the Hecke action of $[N_1 w^{-1} t^{-1} w N_1]$ on $C^{\mathrm{BM}}_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1))[D]$, as described in Theorem 5.1.6, under the inverse of the isomorphism

$$\mathrm{RHom}_{\mathscr{D}(T_0)}(C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/N_1)), \mathscr{D}(T_0)) \xrightarrow{\sim} C^{\mathrm{BM}}_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1))[D],$$

we obtain the following composition:

$$C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/N_1)) \xrightarrow[\sim]{\eta_{K_1, K_t}} C_{\mathbb{A},\bullet}(K_t, \mathscr{C}(K_t/N_1))$$

$$\xrightarrow{\iota_{K_t', K_1} \circ (w^{-1} t w)_*} C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/N_t)) \xrightarrow{\mathrm{tr}} C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/N_1)).$$

We have to check that the Hecke action of $[N_1 t N_1]$ on $C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1))$ corresponds to this composition under the morphism

$$C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1)) \to C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/N_1))$$

induced by κ_w. Again, using the explicit description of the Hecke action in Theorem 5.1.6, we obtain the composition

$$C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1)) \xrightarrow{\mathrm{tr}} C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_t)) \xrightarrow{t_* \circ \eta_{K_1, K_t'}} C_{\mathbb{A},\bullet}(K_t, \mathscr{D}(K_t/N_1))$$

$$\xrightarrow[\sim]{\iota_{K_t, K_1}} C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1)).$$

We now define a morphism $\kappa_t \colon \mathscr{D}(K_t/N_1) \to ((t^{-1} w)^{-1})_* \mathscr{C}(K_t/N_1)$ from the pairing

$$\mathscr{D}(K_t/N_1) \times \mathscr{D}(K_t/N_1) \to \mathscr{D}(T_0), \quad \langle k_1 N_1, k_2 N_1 \rangle \mapsto \tau(w k_2^{-1} w^{-1} t k_1 t^{-1}),$$

where note that $w k_2^{-1} w^{-1} t k_1 t^{-1} \in t K_t t^{-1} \subseteq K_1$. We obtain an induced map

$$\kappa_{t*} \colon C_{\mathbb{A},\bullet}(K_t, \mathscr{D}(K_t/N_1)) \to C_{\mathbb{A},\bullet}(K_t, \mathscr{C}(K_t/N_1)).$$

The Hecke equivariance at p now follows from Lemmas 5.3.4 and 5.3.5 below.

The Hecke equivariance away from p is clear, taking into account that the Poincaré duality isomorphism matches $[K^p (g^p)^{-1} K^p]$ on $C^{\mathrm{BM}}_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1))[D]$ with $[K^p g^p K^p]$ on $C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1))$, and that the other morphisms are Hecke equivariant away from p. ∎

Lemma 5.3.4. *The following diagram is commutative:*

$$
\begin{array}{ccc}
C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1)) \xrightarrow{\ \mathrm{tr}\ } C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_t)) \xrightarrow{\ t_* \circ \eta_{K_1,K_t'}\ } C_{\mathbb{A},\bullet}(K_t, \mathscr{D}(K_t/N_1)) \\
\kappa_* \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \kappa_{t*} \\
C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/N_1)) \xrightarrow{\qquad\qquad\qquad \eta_{K_1,K_t} \qquad\qquad\qquad} C_{\mathbb{A},\bullet}(K_t, \mathscr{C}(K_t/N_1))
\end{array}
$$

Proof. We have the Cartesian diagram

$$
\begin{array}{ccc}
X_{K_t} & \xrightarrow{\ t^{-1}w\ } & X_{K_t} \\
t^{-1} \downarrow & & \downarrow 1 \\
X_{K_1} & \xrightarrow{\ w\ } & X_{K_1}
\end{array}
\qquad\qquad (5.3.1)
$$

where the horizontal arrows are isomorphisms, the left vertical arrow is right multiplication by t^{-1} followed by the natural projection, and the right vertical arrow is the natural projection. Using the maps in this diagram, we obtain the following diagram of local systems on X_{K_1}:

$$
\begin{array}{ccc}
\mathscr{D}(K_1/N_1) \xrightarrow{\ \mathrm{tr}\ } \mathscr{D}(K_1/N_t) \xrightarrow{\ t_*\ } (t^{-1})_* \mathscr{D}(K_t/N_1) \\
\kappa \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow (t^{-1})_*(\kappa_t) \\
(w^{-1})_* \mathscr{C}(K_1/N_1) \xrightarrow{\sim} (w^{-1})_* 1_* \mathscr{C}(K_t/N_1) \xrightarrow{\sim} (t^{-1})_*((t^{-1}w)^{-1})_* \mathscr{C}(K_t/N_1)
\end{array}
\qquad (5.3.2)
$$

Note that $(t^{-1})_*((t^{-1}w)^{-1})_* = (w^{-1})_* 1_*$ by (5.3.1). We will prove that this diagram commutes.

To see this, we claim that the map $\mathscr{D}(K_1/N_t) \to (w^{-1})_* \mathscr{C}(K_1/N_1)$ obtained by going clockwise around (5.3.2) is induced from the pairing

$$
\mathscr{D}(K_1/N_t) \times (w^{-1})_* \mathscr{D}(K_1/N_1)
$$

given by

$$
\langle k_1 N_t, k_2 N_1 \rangle = \begin{cases} \tau(w k_2^{-1} w^{-1} k_1) & \text{if } w k_2^{-1} w^{-1} k_1 \in \overline{N}_1 T_0 N_t, \\ 0 & \text{otherwise.} \end{cases}
\qquad (5.3.3)
$$

Indeed, the map $t_* : \mathscr{D}(K_1/N_t) \to (t^{-1})_* \mathscr{D}(K_t/N_1)$ factors as

$$
\mathscr{D}(K_1/N_t) \xrightarrow{\sim} \mathrm{Ind}_{K_t'}^{K_1} \mathscr{D}(K_t'/N_t) \xrightarrow{\sim} (t^{-1})_* \mathscr{D}(K_t/N_1),
$$

where the first map is given by $\mu = \sum_{s \in K_1/K_t'} s\mu_s \mapsto (\mu_s)_{s \in K_1/K_t'}$ and the second map is given by $(\mu_s)_s \mapsto (t^{-1}\mu_s t)_s$. Similarly, the map $\mathscr{C}(K_1/N_1) \xrightarrow{\sim} 1_* \mathscr{C}(K_t/N_1)$ identifies $\mathscr{C}(K_1/N_1)$ with $\mathrm{Ind}_{K_t}^{K_1} \mathscr{C}(K_t/N_1)$. Recall that $K_t' = \overline{N}_1 T_0 N_t = w K_t w^{-1}$. The map

$$
\mathrm{Ind}_{K_t'}^{K_1} \mathscr{D}(K_t'/N_t) \to (w^{-1})_* \mathrm{Ind}_{K_t}^{K_1} \mathscr{C}(K_t/N_1)
$$

obtained by going clockwise around (5.3.2) is simply induced from the map $\mathscr{D}(K_t'/N_t) \to (w^{-1})_* \mathscr{C}(K_t/N_1)$ determined by the pairing

$$\langle k_1 N_t, k_2 N_1 \rangle = \tau(w k_2^{-1} w^{-1} k_1).$$

The claim now follows from the definition of the two inductions.

Now that this claim is established, we notice that any left coset $k_1 N_1$ with $k_1 \in K_1$ gets sent to $[N_1 : N_t]$ left N_t-cosets under tr, and precisely one of these pairs nontrivially with $k_2 N_1$ with $k_2 \in K_1$ under the pairing in (5.3.3). This proves that the diagram (5.3.2) commutes and therefore proves the lemma. ∎

Lemma 5.3.5. *The following diagram is commutative:*

$$
\begin{array}{ccc}
C_{\mathbb{A},\bullet}(K_t, \mathscr{D}(K_t/N_1)) & \xrightarrow{\iota_{K_t,K_1}} & C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_t/N_1)) \\
\downarrow{\scriptstyle \kappa_{t*}} & & \downarrow{\scriptstyle \kappa_*} \\
C_{\mathbb{A},\bullet}(K_t, \mathscr{C}(K_t/N_1)) \xrightarrow{\iota_{K_t',K_1}\circ(w^{-1}tw)_*} C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/N_t)) \xrightarrow{\mathrm{tr}} & C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/N_1))
\end{array}
$$

Proof. The proof is analogous to that of Lemma 5.3.4. ∎

5.4. Ordinary parts

The goal of this section is to show that the morphism π_w in Proposition 5.3.3 induces an isomorphism on ordinary parts.

We start by defining the ordinary part of homology at finite level, through the means of a projector. We focus on what we need; the interested reader may consult [20, §2] for an abstract viewpoint. Let $j \in \mathbb{Z}_{\geq 1}$ and $t \in T^+$. Assume that the compact open subgroup $\overline{N}_t T_j N_1 \subseteq K_1$ admits an Iwahori factorization; this can always be ensured by choosing t large enough with respect to j. For any $s \in T^+$, we let U_s denote the double coset operator $[\overline{N}_t T_j N_1 s \overline{N}_t T_j N_1]$ acting on $C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(\overline{N}_t T_j N_1, \mathbb{Z}_p) \simeq C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1))$.

Lemma 5.4.1. (1) *For $s_1, s_2 \in T^+$ we have $U_{s_1 s_2} = U_{s_1} \circ U_{s_2}$. In particular, all these operators commute.*

(2) *For $t' \geq t$, $j' \geq j$, the restriction morphism*

$$\mathrm{res} : C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_{t'} T_{j'} N_1)) \to C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1))$$

is U_s-equivariant for all $s \in T^+$.

(3) *For any $s \in T^+$, we have a commutative diagram*

$$
\begin{array}{ccc}
C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_{ts} T_j N_1)) & \xrightarrow{\mathrm{res}} & C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1)) \\
\downarrow{\scriptstyle U_s} & {\scriptstyle s_* \circ \mathrm{tr}} & \downarrow{\scriptstyle U_s} \\
C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_{ts} T_j N_1)) & \xrightarrow{\mathrm{res}} & C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1))
\end{array}
$$

Proof. The first part follows from the explicit description of the Hecke operators U_{s_1}, U_{s_2}, and $U_{s_1 s_2}$ (cf. (5.1.1)) and from the same computation as in [11, Lemma 3.1.4]. The second part again follows from the explicit description in (5.1.1) and from the fact that the Iwahori factorization gives a bijection $\overline{N}_t T_j N_1 / \overline{N}_t T_j N_s \xrightarrow{\sim} N_1 / N_s$, which shows that the coset representatives can be chosen independently of j and t.

We now prove the third part. The commutativity of the lower triangle follows from the definition of U_s. For the upper triangle, we write the definition of U_s acting on $C_{\mathbb{A}, \bullet}(K_1, \mathscr{D}(K_1 / \overline{N}_{ts} T_j N_1)) \simeq C_{\mathbb{A}, \bullet}(\overline{N}_{ts} T_j N_1, \mathbb{Z}_p)$. By definition, this is equal to

$$C_{\mathbb{A}, \bullet}(\overline{N}_{ts} T_j N_1, \mathbb{Z}_p) \xrightarrow{\mathrm{tr}} C_{\mathbb{A}, \bullet}(\overline{N}_{ts} T_j N_s, \mathbb{Z}_p) \xrightarrow{s_*} C_{\mathbb{A}, \bullet}(\overline{N}_{ts^2} T_j N_1, \mathbb{Z}_p)$$
$$\xrightarrow{\mathrm{res}} C_{\mathbb{A}, \bullet}(\overline{N}_{ts} T_j N_1, \mathbb{Z}_p),$$

which can be rewritten as

$$C_{\mathbb{A}, \bullet}(\overline{N}_{ts} T_j N_1, \mathbb{Z}_p) \xrightarrow{\mathrm{tr}} C_{\mathbb{A}, \bullet}(\overline{N}_{ts} T_j N_s, \mathbb{Z}_p) \xrightarrow{\mathrm{res}} C_{\mathbb{A}, \bullet}(\overline{N}_t T_j N_s, \mathbb{Z}_p)$$
$$\xrightarrow{s_*} C_{\mathbb{A}, \bullet}(\overline{N}_t T_j N_1, \mathbb{Z}_p). \qquad (5.4.1)$$

We claim that the diagram of locally symmetric spaces with natural projection morphisms

$$
\begin{array}{ccc}
X_{\overline{N}_{ts} T_j N_s} & \longrightarrow & X_{\overline{N}_t T_j N_s} \\
\downarrow & & \downarrow \\
X_{\overline{N}_{ts} T_j N_1} & \longrightarrow & X_{\overline{N}_t T_j N_1}
\end{array}
$$

is Cartesian. To see this, it is enough to see that the morphism

$$X_{\overline{N}_{ts} T_j N_s} \to X_{\overline{N}_{ts} T_j N_1} \times_{X_{\overline{N}_t T_j N_1}} X_{\overline{N}_t T_j N_s}$$

induces an isomorphism of the fibers over $X_{\overline{N}_{ts} T_j N_1}$. Indeed, the fibers on the RHS can be identified with $\overline{N}_{ts} T_j N_1 / \overline{N}_{ts} T_j N_s$ and fibers on the LHS can be identified with $\overline{N}_t T_j N_1 / \overline{N}_t T_j N_s$ [7, Lemma 6.2.1]. The claim now follows from the Iwahori factorization, since all the fibers are identified with N_1 / N_s. Using proper base change, we rewrite (5.4.1) as

$$C_{\mathbb{A}, \bullet}(\overline{N}_{ts} T_j N_1, \mathbb{Z}_p) \xrightarrow{\mathrm{res}} C_{\mathbb{A}, \bullet}(\overline{N}_t T_j N_1, \mathbb{Z}_p) \xrightarrow{\mathrm{tr}} C_{\mathbb{A}, \bullet}(\overline{N}_t T_j N_s, \mathbb{Z}_p)$$
$$\xrightarrow{s_*} C_{\mathbb{A}, \bullet}(\overline{N}_{ts} T_j N_1, \mathbb{Z}_p).$$

This completes the proof. ∎

We call $s \in T^+$ a *controlling element* if $\bigcap_{i \geq 1} s^i N_1 s^{-i} = \{1\}$. Such a controlling element always exists, for example by taking $s = \prod_\alpha p^\alpha$, where the product runs over the positive coroots of G.

Lemma 5.4.2. *Let $s \in T^+$ be a controlling element. For any $s_1 \in T^+$, there exists $i \in \mathbb{Z}_{\geq 1}$ such that $s_1 s_2 = s^i$ for some $s_2 \in T^+$.*

Proof. Since $\bigcap_{i \geq 1} s^i N_1 s^{-i} = \{1\}$, there exists $i \in \mathbb{Z}_{\geq 1}$ such that $s^i N_1 s^{-i} \subseteq s_1 N_1 s_1^{-1}$. This shows that $s_2 := s_1^{-1} s^i \in T^+$. ∎

Let s_0 be a controlling element. Using homotopy equivalences, we transport U_{s_0} on $C_{\mathbb{A},\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1) \otimes \mathbb{Z}/p^k)$ to an operator $\widetilde{U}_{s_0}$ on the corresponding Borel–Serre complex $C_{\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1) \otimes \mathbb{Z}/p^k)$, which acts as U_{s_0} up to homotopy and, in particular, induces the same action on homology. Since the latter is a complex of finite projective $\mathbb{Z}/p^k$-modules, $\widetilde{U}_{s_0}^{N!}$ stabilizes to an idempotent. We denote the corresponding direct summand by $C_{\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1) \otimes \mathbb{Z}/p^k)^{T^+\text{-ord}}$. We define

$$C_{\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1))^{T^+\text{-ord}} := \varprojlim_k C_{\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1) \otimes \mathbb{Z}/p^k)^{T^+\text{-ord}}.$$

By [20, Lemma 2.13], this is a direct summand of $C_{\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1))$ whose homology recovers the ordinary part of homology / Borel–Moore homology with respect to s_0.

Note that the homotopy equivalences between the adelic and the Borel–Serre complexes are functorial in the coefficients. Therefore, the formation of the direct summands is compatible with the transition morphisms between different levels, and we can also define the ordinary part

$$C_{\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_1))^{T^+\text{-ord}} := \varprojlim_{j,t} C_{\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1))^{T^+\text{-ord}}$$

of $C_{\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_1))$ with respect to s_0. This is a direct summand of $C_{\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/N_1))$ and its homology recovers $H_*^{(\mathrm{BM})}(N_1)^{T^+\text{-ord}}$ by [20, Lemma 2.13] and Theorem 5.1.6.

Using Lemma 5.4.2, one can check that all this is independent of the choice of controlling element s_0. By Lemmas 5.4.2 and 5.4.1 (1), we see that U_s acts as a quasi-isomorphism on $C_{\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1))^{T^+\text{-ord}}$ for any $s \in T^+$. As a result, we obtain the following horizontal control theorem.

Proposition 5.4.3. *For any $s \in T^+$, the transition morphisms*

$$C_{\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_{ts} T_j N_1))^{T^+\text{-ord}} \to C_{\bullet}^{(\mathrm{BM})}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1))^{T^+\text{-ord}}$$

are quasi-isomorphisms.

Proof. It is enough to show that the transition morphisms induce an isomorphism on the ordinary part of homology / Borel–Moore homology. Surjectivity follows from the commutativity of the lower triangle in Lemma 5.4.1 (3), since U_s acts as an isomorphism on homology. Injectivity follows from the commutativity of the upper triangle in Lemma 5.4.1 (3). ∎

Assume again that $\overline{N}_t T_j N_1$ admits an Iwahori factorization. Set

$$\mathscr{C}(K_1/\overline{N}_t T_j N_1) := \operatorname{Hom}_{\mathscr{D}(T_0)}\big(\mathscr{D}(K_1/\overline{N}_t T_j N_1), \mathscr{D}(T_0/T_j)\big).$$

We also have an action of $s \in T^-$ on each $C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/\overline{N}_t T_j N_1))$ via a double coset operator U_s. We define the analogous notion of controlling element and use it to define the ordinary part $C_\bullet(K_1, \mathscr{C}(K_1/\overline{N}_t T_j N_1))^{T^--\mathrm{ord}}$ with respect to T^-. In this setting, we have the following horizontal control theorem.

Proposition 5.4.4. *For any $s \in T^-$, the transition morphisms*

$$C_\bullet(K_1, \mathscr{C}(K_1/\overline{N}_t T_j N_1))^{T^--\mathrm{ord}} \to C_\bullet(K_1, \mathscr{C}(K_1/\overline{N}_{ts} T_j N_1))^{T^--\mathrm{ord}}$$

are quasi-isomorphisms.

Proof. This is proved analogously to Proposition 5.4.3. ∎

Set $\mathscr{C}(K_1/\overline{N}_t T_j N_1, \mathbb{Z}/p^k) := \mathscr{C}(K_1/\overline{N}_t T_j N_1) \otimes_{\mathbb{Z}} \mathbb{Z}/p^k$. We have

$$\mathscr{C}(K_1/N_1) \simeq \varprojlim_{j,k} \varinjlim_t \mathscr{C}(K_1/\overline{N}_t T_j N_1, \mathbb{Z}/p^k),$$

where the inverse limit runs over $j, k \in \mathbb{Z}_{\geq 1}$ and the direct limit runs over those $t \in T^+$ that are sufficiently large with respect to j. We define

$$C_\bullet(K_1, \mathscr{C}(K_1/N_1))^{T^--\mathrm{ord}} := \varprojlim_{j,k} \varinjlim_t C_\bullet(K_1, \mathscr{C}(K_1/\overline{N}_t T_j N_1, \mathbb{Z}/p^k))^{T^--\mathrm{ord}},$$

the ordinary part of $C_\bullet(K_1, \mathscr{C}(K_1/N_1))$ with respect to T^-.

Also set

$$(w^{-1})_* \mathscr{C}(K_1/\overline{N}_t T_j N_1) := \operatorname{Hom}_{\mathscr{D}(T_0)}\big((w^{-1})_* \mathscr{D}(K_1/\overline{N}_t T_j N_1), \mathscr{D}(T_0/T_j)\big).$$

For each $j \in \mathbb{Z}_{\geq 1}$, the map $\kappa: \mathscr{D}(K_1/N_1) \to (w^{-1})_* \mathscr{C}(K_1/N_1)$ induces a finite level map

$$\kappa_j: \mathscr{D}(K_1/\overline{N}_t T_j N_1) \to (w^{-1})_* \mathscr{C}(K_1/\overline{N}_t T_j N_1),$$

whenever $t \in T^+$ is such that $\overline{N}_t T_j N_1$ admits an Iwahori factorization.

Lemma 5.4.5. *The morphism*

$$(\kappa_j)_*: C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1)) \to C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/\overline{N}_t T_j N_1))$$

induces a quasi-isomorphism

$$(\kappa_j)_*^{\mathrm{ord}}: C_\bullet(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1))^{T^+-\mathrm{ord}} \xrightarrow{\sim} C_\bullet(K_1, \mathscr{C}(K_1/\overline{N}_t T_j N_1))^{T^--\mathrm{ord}}.$$

Proof. We have the following finite-level version of (5.3.2):

$$\mathscr{D}(K_1/\overline{N}_t T_j N_1) \longrightarrow \mathscr{D}(K_1/\overline{N}_1 T_j N_t) \xrightarrow{\ t_*\ \sim\ } (t^{-1})_* \mathscr{D}(K_t/\overline{N}_t T_j N_1)$$

$$\downarrow{\scriptstyle \kappa_j} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle (t^{-1})_*(\kappa_{j,t})}$$

$$(w^{-1})_* \mathscr{C}(K_1/\overline{N}_t T_j N_1) \xrightarrow{\sim} (w^{-1})_* 1_* \mathscr{C}(K_t/\overline{N}_t T_j N_1) \xrightarrow{\sim} (t^{-1})_* ((t^{-1}w)^{-1})_* \mathscr{C}(K_t/\overline{N}_t T_j N_1)$$

$$(5.4.2)$$

where the top row is part of the definition of the double coset operator corresponding to t. More precisely, in order to get the double coset operator U_t acting on $C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1))$, one needs to apply $C_{\mathbb{A},\bullet}(K_1, \ )$ to the top row and compose with the isomorphism

$$\iota_{K_t, K_1}: C_{\mathbb{A},\bullet}(K_t, \mathscr{D}(K_t/\overline{N}_t T_j N_1)) \xrightarrow{\sim} C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1)).$$

We claim that the right vertical arrow in (5.4.2) is an isomorphism. For this, observe that the natural map $K_t/\overline{N}_t T_j N_1 \to T_0/T_j$ is a bijection, so the pairing

$$\mathscr{D}(K_t/\overline{N}_t T_j N_1) \times ((t^{-1}w)^{-1})_* \mathscr{D}(K_t/\overline{N}_t T_j N_1) \to \mathscr{D}(T_0/T_j),$$

$$\langle t_1 \overline{N}_t T_j N_1, t_2 \overline{N}_t T_j N_1 \rangle = w t_2^{-1} w^{-1} t t_1 t^{-1} T_j = w t_2^{-1} w^{-1} t_1 T_j,$$

is perfect. All of this implies that $(\kappa_j)_*$ is the composite of the double coset operator U_t acting on $C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1))$ with an isomorphism.

In order to prove the lemma, it suffices to check that $(\kappa_j)_*$ induces an isomorphism on the ordinary part of homology. The map induced by $(\kappa_j)_*$ on homology induces a morphism of ordinary parts of homology (where the ordinary part is taken with respect to T^+ on the RHS and T^- on the LHS), which factors as U_t composed with an isomorphism. Since U_t acts as an isomorphism on the ordinary part of homology, the lemma follows. ∎

Proposition 5.4.6. *The map*

$$\kappa_*: C_{\mathbb{A},\bullet}(K_1, \mathscr{D}(K_1/N_1)) \to C_{\mathbb{A},\bullet}(K_1, \mathscr{C}(K_1/N_1))$$

induces a quasi-isomorphism

$$\kappa_*^{\mathrm{ord}}: C_\bullet(K_1, \mathscr{D}(K_1/N_1))^{T^+\text{-ord}} \xrightarrow{\sim} C_\bullet(K_1, \mathscr{C}(K_1/N_1))^{T^-\text{-ord}}.$$

Proof. This follows by combining Lemma 5.4.5 and Propositions 5.4.3 and 5.4.4. ∎

Theorem 5.4.7. *The morphism π_w induces a quasi-isomorphism of ordinary parts*

$$\pi_w^{\mathrm{ord}}: C_\bullet^{\mathrm{BM}}(K_1, \mathscr{D}(K_1/N_1))^{T^+\text{-ord}}[D]$$

$$\to \mathrm{RHom}_{\mathscr{D}(T_0)}\big(C_\bullet(K_1, \mathscr{D}(K_1/N_1))^{T^+\text{-ord}}, \mathscr{D}(T_0)\big).$$

Proof. By Proposition 5.4.6, we have a quasi-isomorphism

$$\mathrm{RHom}_{\mathscr{D}(T_0)}\big(C_\bullet(K_1, \mathscr{C}(K_1/N_1))^{T^--\mathrm{ord}}, \mathscr{D}(T_0)\big)$$
$$\xrightarrow{\sim} \mathrm{RHom}_{\mathscr{D}(T_0)}\big(C_\bullet(K_1, \mathscr{D}(K_1/N_1))^{T^+-\mathrm{ord}}, \mathscr{D}(T_0)\big).$$

For each $k, j \in \mathbb{Z}_{\geq 1}$, Poincaré duality and the universal coefficient isomorphism (at each level $\overline{N}_t T_j N_1$) induce quasi-isomorphisms

$$\varprojlim_t C_\bullet^{\mathrm{BM}}(K_1, \mathscr{D}(K_1/\overline{N}_t T_j N_1) \otimes \mathbb{Z}/p^k)^{T^+-\mathrm{ord}}[D] \xrightarrow{\sim}$$

$$\mathrm{RHom}_{\mathscr{D}(T_0/T_j) \otimes \mathbb{Z}/p^k}\Big(\varinjlim_t C_\bullet(K_1, \mathscr{C}(K_1/\overline{N}_t T_j N_1) \otimes \mathbb{Z}/p^k)^{T^--\mathrm{ord}}, \mathscr{D}(T_0/T_j) \otimes \mathbb{Z}/p^k\Big),$$

where the transition morphisms in both the inverse and the direct limit are isomorphisms by the horizontal control theorems. The morphism is a quasi-isomorphism, since on perfect complexes of $\mathbb{Z}/p^k$-modules, the ordinary part is simply the largest direct summand on which U_{s_0} acts invertibly, and this commutes with

$$\mathrm{RHom}_{\mathscr{D}(T_0/T_j) \otimes \mathbb{Z}/p^k}(-, \mathscr{D}(T_0/T_j) \otimes \mathbb{Z}/p^k).$$

Taking inverse limits with respect to j, k, we obtain a quasi-isomorphism

$$C_\bullet^{\mathrm{BM}}(K_1, \mathscr{D}(K_1/N_1))^{T^+-\mathrm{ord}}[D] \xrightarrow{\sim} \mathrm{RHom}_{\mathscr{D}(T_0)}\big(C_\bullet(K_1, \mathscr{C}(K_1/N_1))^{T^--\mathrm{ord}}, \mathscr{D}(T_0)\big).$$

The theorem follows. ∎

5.5. *An application of Theorem* 4.1.1

We will give some implications of Theorem 4.1.1 for completed homology and completed Borel–Moore homology groups. In the cases we will be interested in below, T is split over $\mathbb{Z}_p$ and hence $T_0 \cong (\mathbb{Z}_p^\times)^{\dim T}$. The ring $\mathscr{D}(T_0)$ is then easily seen to be a semi-local complete intersection ring. For any such ring A, and a finitely generated A-module M, one may define

$$\mathrm{codim}_A M = \inf_j \{j \mid \mathrm{Ext}_A^j(M, A) \neq 0\}.$$

Geometrically, $\mathrm{codim}_A M$ is the minimum of the codimensions of the support of M at the maximal ideals of A. For a closed subgroup $H \subset G(\mathbb{Z}_p)$, we let

$$\widetilde{H}_i(H) := \varprojlim_{n, K \supseteq H} H_i(X_K, \mathbb{Z}/p^n).$$

Theorem 5.5.1. *Assume that G admits a Shimura datum of Hodge type and that $G_{\mathbb{Q}_p}$ is split. Recall that $d = D/2$ is the complex dimension of the Shimura varieties for G.*

(1) *Let $H \subseteq U(\mathbb{Z}_p)$ be a closed subgroup. Then*

$$\varprojlim_{K \supseteq H} H_i^{\mathrm{BM}}(X_K, \mathbb{Z}/p^r) = 0 \quad \text{for all } r \geq 1 \text{ and } i > d.$$

(2) *We have*

$$\operatorname{codim}_{\mathscr{D}(T_0)} \tilde{H}_i(N_1)^{T^+\text{-ord}} \geq d - i \quad \text{for all } 0 \leq i \leq d.$$

Remark 5.5.2. The slightly unusual level N_1 was chosen for convenience. The groups $\tilde{H}_i(N)^{T^+\text{-ord}}$ for N a compact open subgroup of $U(\mathbb{Z}_p)$ are all isomorphic. For $N' \subseteq N \subseteq U(\mathbb{Z}_p)$, the trace map $\tilde{H}_i(N)^{T^+\text{-ord}} \to \tilde{H}_i(N')^{T^+\text{-ord}}$ is an isomorphism for the following reason. We can find $s \in T^+$ so that $sNs^{-1} \subseteq N'$. Consider the diagram

$$\tilde{H}_i(N)^{T^+\text{-ord}} \xrightarrow{\text{tr}_1} \tilde{H}_i(N')^{T^+\text{-ord}} \xrightarrow{\text{tr}_2} \tilde{H}_i(sNs^{-1})^{T^+\text{-ord}} \xrightarrow{\text{tr}_3} \tilde{H}_i(sN's^{-1})^{T^+\text{-ord}}.$$

The maps $\text{tr}_2 \circ \text{tr}_1 = [NsN]s^{-1}$ and $\text{tr}_3 \circ \text{tr}_2 = [N'sN']s^{-1}$ are isomorphisms. Then $(\text{tr}_2 \circ \text{tr}_1)^{-1} \circ \text{tr}_2$ is both a left and right inverse of tr_1 since

$$\text{tr}_1 \circ (\text{tr}_2 \circ \text{tr}_1)^{-1} \circ \text{tr}_2 = (\text{tr}_3 \circ \text{tr}_2)^{-1} \circ \text{tr}_3 \circ \text{tr}_2 \circ \text{tr}_1 \circ (\text{tr}_2 \circ \text{tr}_1)^{-1} \circ \text{tr}_2 = \text{id}.$$

Lemma 5.5.3. *For each compact open subgroup $K \subseteq G(\mathbb{Z}_p)$, there is an isomorphism of $\mathbb{Z}/p^r$-modules*

$$\operatorname{Hom}_{\mathbb{Z}/p^r}(H_c^\bullet(X_K, \mathbb{Z}/p^r), \mathbb{Z}/p^r) \simeq H_\bullet^{\text{BM}}(X_K, \mathbb{Z}/p^r),$$

and these isomorphisms are compatible with changing the level.

Proof. The result can be obtained by applying the functor $\operatorname{Hom}_{\mathbb{Z}/p^r}(-, \mathbb{Z}/p^r)$ to the universal coefficient isomorphism

$$H_c^\bullet(X_K, \mathbb{Z}/p^r) \simeq \operatorname{Hom}_{\mathbb{Z}/p^r}(H_\bullet^{\text{BM}}(X_K, \mathbb{Z}/p^r), \mathbb{Z}/p^r),$$

and using the fact that any finite $\mathbb{Z}/p^r$-module is naturally isomorphic to its double dual.

The universal coefficient isomorphism can be proved as follows. We have isomorphisms

$$H_*^{\text{BM}}(X_K, \mathbb{Z}/p^r) \simeq H_*(F_\bullet^{\text{BM}} \otimes_{\mathbb{Z}[K^p K]} \mathbb{Z}/p^r),$$
$$H_c^*(X_K, \mathbb{Z}/p^r) \simeq H^*(\operatorname{Hom}_{\mathbb{Z}[K^p K]}(F_\bullet^{\text{BM}}, \mathbb{Z}/p^r)).$$

By the adjunction between tensor products and homomorphisms,

$$\operatorname{Hom}_{\mathbb{Z}[K^p K]}(F_\bullet^{\text{BM}}, \mathbb{Z}/p^r) \simeq \operatorname{Hom}_{\mathbb{Z}/p^r}(F_\bullet^{\text{BM}} \otimes_{\mathbb{Z}[K^p K]} \mathbb{Z}/p^r, \mathbb{Z}/p^r).$$

Then the universal coefficient isomorphism follows by taking cohomology and observing that $\mathbb{Z}/p^r$ is an injective $\mathbb{Z}/p^r$-module. All of these isomorphisms are compatible with changing the level. ∎

Proof of Theorem 5.5.1. To prove the first claim, we use Lemma 5.5.3 to write

$$\varprojlim_{K \supseteq H} H_i^{\text{BM}}(X_K, \mathbb{Z}/p^r) \simeq \operatorname{Hom}_{\mathbb{Z}/p^r}\left(\varinjlim_{K \supseteq H} H_c^i(X_K, \mathbb{Z}/p^r), \mathbb{Z}/p^r\right) \qquad (5.5.1)$$

and then apply Corollary 4.1.2. From Theorems 5.4.7 and 5.1.6, we obtain the Poincaré duality spectral sequence

$$\mathrm{Ext}^i_{\mathscr{D}(T_0)}(\tilde{H}_j(N_1)^{T^+\text{-ord}}, \mathscr{D}(T_0)) \implies \tilde{H}^{\mathrm{BM}}_{2d-i-j}(N_1)^{T^+\text{-ord}}.$$

Then the second claim follows from the first claim and the above spectral sequence by the same argument as in [28, Corollary 4.2.3] (note that D from Theorem 5.4.7 can be identified with $2d$). ∎

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