Fast Simulation of 3D Elastic Wheel-Rail Contact Using Proper Generalized Decomposition

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Abstract: To increase computational efficiency, we adopt the Proper Generalized Decomposition (PGD) to solve a reduced-order problem of the displacement field for a three-dimensional rail head exposed to different contact scenarios. The three-dimensional solid rail head is modeled as a two-dimensional cross-section, with the coordinate along the rail being treated as a parameter in the PGD approximation. A novel feature is this allows us to solve the full three-dimensional model with a near two-dimensional computational effort. Additionally, we incorporate the distributed contact load predicted from dynamic vehicle-track simulations as extra-coordinates in the PGD formulation, using a semi-Hertzian contact model. The problem formulation in particular considers the treatment of parameters with linear influence and situations where certain parameters become invariant. We assess the accuracy and efficiency of the proposed strategy through a series of verification examples. It is shown that the PGD solution converges towards the FE solution with reduced computational cost. Furthermore, solving for the PGD approximation based on the load parameterization at an offline stage allows for expedient handling of the wheel-rail contact problem online.

Keywords: Elastic wheel-rail contact, Finite Element Method, Model Reduction, Proper Generalized Decomposition, Separated Representation, Distributed Load, Parametric Modeling.

1. Introduction
Rails in curved tracks are exposed to many contact scenarios resulting from large and varying vehicle loads as well as wheel and rail rim geometries. These scenarios can lead to rail damage, causing traffic interruptions, train delays, and expensive maintenance costs. Therefore, numerical computations are necessary to capture different contact scenarios for cost-effectively assessing rail head degradation. To evaluate the degradation of the rail head over time, it is essential to model various overrolling events that induce damage mechanisms such as wear, plastic deformation, and surface (or subsurface) initiated cracks due to rolling contact fatigue. However, performing these simulations can be computationally demanding, which emphasizes the need for methodologies that can reduce computational costs.

To calculate the long-term damage to the rail surface, there exists a framework [1–5] that considers multiple steps, applied iteratively between dynamic vehicle-track interaction for a given traffic situation, elastic-plastic wheel-rail contact, and accumulated rail damage due to plasticity and surface wear to update the rail profile. Also, surface rolling contact fatigue crack initiation is predicted. In this work, the simulation of the elastic-plastic contact is restricted to a meta-modeling strategy [1], and the subsequent analysis of the evolution of plastic deformation is reduced to a 2D analysis to allow for efficient calculations. However, the assumptions pertinent to a two-dimensional analysis of the rail under contact are quite restrictive, and the contact load amplitude has to be scaled to obtain an "equivalent" load case.
The present study aims towards alleviation of the restrictions pertinent to two-dimensional analysis in the preceding work [5] while maintaining a computationally efficient procedure. As a first step, the elastic response of a 3D rail head is investigated for different contact load scenarios. Thereby, we account for the actual contact stress distribution and Poisson effects in the rail. Furthermore, contact stresses can act longitudinal along the rail. Even though the response is linear, traditional 3D Finite Element (FE) methods are computationally intensive, and often impractical for considering very large amounts of loading scenarios, i.e., when the 3D solution needs to be solved many times.

To reduce computational complexity, some work has been done using the model reduction called Proper Generalized Decomposition (PGD). Unlike some other methods for reduced order models, such as Proper Orthogonal Decomposition (POD) [6, 7] and Reduced Basis (RB) [8], PGD is an a priori technique and does not require that the (approximate) solution to the complete problem is known [9]. The use of PGD allows for efficient computations, since the offline solution is computed only once, and the online solution can be determined efficiently. The offline solution is obtained using a successive enrichment strategy to give a numerical approximation of the unknown fields in a separate form for all solutions of the parameters within their respective intervals. The online phase is achieved efficiently because the inverse analysis only involves a postprocessing step of the pre-calculated parametric solution for a desired setup of the parameter values. This method stems from a space-time separation in the 1980s by Ladavèze et al. [10–13], but was further developed and generalized in the early 2000s by Chinesta et al. [9, 14, 15]. It can be applied to a variety of problems, including high-dimensional problems discussed in [14–17], parametric modeling [18–23] when there are many solutions to a problem, or when a quick solution is needed.

Bognet et al. and Giner et al. [21, 24, 25] proposed a domain decomposition to separate the displacement field \( u(y, x) \) for the in-plane \( y = (y, z) \) and out-of-plane \( x \) coordinates as

\[
    u(y, x) \approx \sum_{n=1}^{N} Y_n(y) X_n(x).
\]  

This separation allows for the representation of \( u(y, x) \) as a finite sum of unknown functions, also known as modes, \( Y_n \) and \( X_n \), respectively. This method reduces the 3D problem to a 2D computational complexity, as the computation of the 1D functions \( X_n \) is negligible compared to the computation of the 2D functions \( Y_n \). In this work, the same approach is used to solve the displacement field of a 3D solid rail head.

The PGD formulation allows for extra-coordinates because the extra dimension does not impact the solvability of the problem. Examples of extra-coordinates in parametric models may include material parameters as in [18–21] or boundary conditions considered in [9, 22]. However, in this work, we would like to incorporate a distributed surface load to account for different contact conditions in the PGD framework. This type of load can be challenging to incorporate into the PGD framework since the optimality of the method depends on the separability of the solution. Cueto et al. and Zou et al. [20, 23] previously addressed this challenge in the context of a moving unit load. However, in this case, we need to develop a similar approach but with a distributed surface load since a contact area arises when two bodies in contact are pressed together.

There are many ways to model the wheel-rail contact, with Hertzian theory of contact [26] being one of the most commonly used approaches. The Hertzian contact has an elliptical contact area described by a constant curvature and a parabolic pressure distribution. It is simple and fast, but may not always apply to wheel-rail contacts that are locally conformal. Using the finite element method or CONTACT
[27, 28] provides higher accuracy but comes with a higher computational cost. A semi-Hertzian approach called STRIPES [29, 30] offers a compromise between similar accuracy as CONTACT, but with lower computational cost. In STRIPES, the contact area is estimated from the interpenetration area and is discretized in the lateral direction. Then, a Hertzian-based formula is applied in the longitudinal direction to determine the stress distribution in each strip. Thus, this method allows for non-Hertzian conditions in the lateral direction since the lateral curvature is not constant in the contact area. This is advantageous when the curvature of the contacting bodies varies or when the track is curved since flange contacts can occur at the wheel which violates the conformal contact condition. The semi-Hertzian approach also facilitates a higher degree of separability of the solution in the context of PGD.

In this paper, we will employ the PGD formulation to compute the displacement field of a 3D elastic rail head subjected to different contact scenarios. The study consists of two parts, the first focusing on 3D modeling and the second on the parameterization of the distributed surface load. In the first part, a 2D model is used to represent the rail cross-section, while the rail coordinate serves as a parameter in the PGD approximation. To validate the results, a 3D FE problem is solved for a known Hertzian load. The second part deals with the parameterization of the distributed surface load, which is treated as extra coordinates to account for different contact scenarios. The STRIPES approach [29, 30] is used to describe the surface load. The proposed method allows for an efficient solution for different contact scenarios since the PGD is solved only once for all specific parameter values within their respective intervals.

2. Problem description - 3D solid rail head analysis
The 3D solid rail head Ω with linear elastic material properties shown in Figure 1 is studied. An elliptical Hertz load [26] acts on the surface ΓN but with a restricted motion in the z-direction. The displacements at the bottom of the rail ΓD are fixed. We are searching for the displacement field \( u(x, y, z) \) at any spatial coordinate \( [x, y, z] \in \Omega \) in the rail head.

![Figure 1: Illustration of 3D rail section Ω with depth d and width w. The rail is subjected to a Hertz load [26] with semi-axes a and b. The load can move in the position s along the upper surface ΓN but with a restricted motion in the z-direction. The maximum traction of the contact surface is defined as \( p_n \), \( p_t \), and \( p_x \) in the corresponding directions \( e_n \), \( e_t \) and \( e_x \). The basis vectors are defined locally from the rail profile at the center position of the contact area s (marked with a blue circle). The displacements are fixed at the bottom of the rail ΓD.](image-url)
We define the trail function space as $U := \{v \in [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_d\}$, where $H^1(\Omega)$ is the space of functions on $\Omega$ with square integrable derivatives of order zero and one. The weak form of the problem reads: find $u \in U$ such that

$$a(u, \delta u) = l(\delta u) \quad \forall \delta u \in U,$$

where we define the bilinear and linear forms $a(\cdot, \cdot)$ and $l(\cdot)$, respectively, as

$$a(u, \delta u) = \int_{\Omega} \epsilon[\delta u] : E : \epsilon[u] \, d\Omega, \quad l(\delta u) = \int_{\Gamma_N} t \cdot \delta u \, d\Gamma.$$

The strain tensor $\epsilon$ is related to the displacement $u$ as $\epsilon = [u \otimes \nabla]^\text{sym}$ under the infinitesimal deformation assumptions. The stress tensor $\sigma$ and the strain tensor $\epsilon$ are related to the 4th order elasticity tensor $E$ by Hooke’s law $\sigma = E : \epsilon[u]$. A standard FE discretization can be applied to the weak form by introducing the FE-subspace $U_h \subset U$.

The non-zero part of the traction $t$ for an elliptic Hertz contact load [26] is expressed as

$$t(y, x) = \left( p_n e_n + p_t e_t + p_x e_x \right) \sqrt{1 - \left( \frac{y - s}{a} \right)^2 - \left( \frac{x}{b} \right)^2}, \quad \left( \frac{y - s}{a} \right)^2 + \left( \frac{x}{b} \right)^2 < 1,$$

where $s$ is the center position of the contact area on the rail, and $p_n$, $p_t$, and $p_x$ are the maximum surface contact traction components corresponding to the normal ($e_n$), lateral ($e_t$) and longitudinal ($e_x$) directions at the rail surface, see Figure 1. Also, $a$ and $b$ are the semi-axes of the elliptical contact patch in- and out-of-plane, respectively.

We approximate the shear stress to be proportional to the contact pressure, pertinent to the case of full slip conditions in Coulomb friction\footnote{Full slip in the Coulomb model would be characterized by the shear traction $\sqrt{(t_t^2 + (t_x^2)} = \mu (t_n)$, with $\mu$ being the coefficient of friction.}. However, here we will let the amplitude of the shear stress components be governed by the amplitude of the applied load. Hence, the amplitudes $p_n$, $p_t$ and $p_x$ are independent variables.

Finally, we introduce the energy norm,

$$||u||_a = \sqrt{a(u, u)},$$

which will be used for measurement of the solution, and its error, in subsequent sections.

3. 3D elastic PGD analysis of the rail for a known load

In this section, we outline the adopted PGD approach with the solution ansatz in terms of multiplicative separated in- and out-of-plane modes. In this development a linear elasticity model is assumed for a known load, where, the Hertzian load described in Section 2 is considered for fixed surface load parameters $p_n$, $p_t$, $p_x$, $a$ and $b$.\footnote{Full slip in the Coulomb model would be characterized by the shear traction $\sqrt{(t_t^2 + (t_x^2)} = \mu (t_n)$, with $\mu$ being the coefficient of friction.}
3.1. In- and out-of-plane separated PGD approximation

To speed up the computation of the weak form (2) of the 3D problem, we introduce a PGD approximation of the parametric solution with a plate decomposition, similar to what was done in [21, 24, 25]. The PGD approach is defined as a finite sum of separable functions to approach a solution. It is assumed that the solution for the parameterized problem converges when approaching the approximation of the superposition of $N$ modes

$$u(y, x) \approx u^\text{PGD}(y, x) = \sum_{n=1}^{N} Y_n(y) \, X_n(x),$$

where $y = (y, z) \in \hat{\Omega}$ are the in-plane coordinates and $x \in I_x = [-d/2, d/2]$ is the out-of-plane coordinate of the rail where the depth is defined as $d$. The modes $Y_n(y)$ and $X_n(x)$ represent the unknown separated functions for the $n$th mode and depend on the in- and out-of-plane parameters, respectively. Thus, we have a $y - x$ separated representation of the displacement field. The main influence of the track curvature stems from the tangential load component in the $y$-direction. Hence, we assume that the rail geometry is straight out-of-plane.

We can now decompose the geometry into $\Omega = (\hat{\Omega} \times I_x)$, $\Gamma_N = (\hat{\Gamma}_N \times I_x) \cup \hat{\Omega} \times \{-d/2\} \cup \hat{\Omega} \times \{d/2\}$. Since $t = 0$ at the end surfaces ($x = \pm d/2$), we can restate the forms in (2) and (3) as

$$a(v, w) = \int_{I_x} \int_{\hat{\Omega}} \epsilon[w] : E : \epsilon[v] \, d\hat{\Omega} \, dx, \quad l(v) = \int_{I_x} \int_{\hat{\Omega}} t \, v \, d\hat{\Gamma} \, dx.$$ (7)

Furthermore, we consider $\Gamma_D = \hat{\Gamma}_D \times I_x$. Hence, the homogeneous Dirichlet boundary condition can be expressed as $u = 0$ on $\hat{\Gamma}_D$, which is independent of $x$.

Assuming the first $N - 1$ first terms have already been computed as

$$u^\text{PGD}_{N-1}(y, x) = \sum_{n=1}^{N-1} Y_n(y) \, X_n(x),$$

we are seeking the modes $Y_N(y) \in \mathbb{Y}$ and $X_N(x) \in \mathbb{X}$ to obtain the further enriched PGD solution

$$u^\text{PGD}_N(y, x) = u^\text{PGD}_{N-1}(y, x) + Y_N(y) \, X_N(x).$$ (9)

Hence, we seek updates in the spaces

$$\mathbb{Y} := \{v \in [H^1(\hat{\Omega})]^3 : v = 0 \text{ on } \hat{\Gamma}_d\}, \quad \mathbb{X} := H^1(I_x),$$

whereby we note that the product $Y_N X_N \in \mathbb{U}$.

In the spirit of Galerkin’s method, the equations for determining modes $Y_N$ and $X_N$ are now established by testing (2) with $\delta u(y, x) = \delta u^\text{PGD}(y, x) = \delta Y(y) \, X_N(x) + Y_N(y) \, \delta X(x)$ for $\delta Y, \delta X \in \mathbb{Y} \times \mathbb{X}$.

Inserting the PGD approximation (9) into the weak form we obtain the problem of seeking $Y_N, X_N \in \mathbb{Y} \times \mathbb{X}$ such that

$$a(Y_N X_N, \delta Y X_N) = l(\delta Y X_N) - a(u^\text{PGD}_{N-1}, \delta Y X_N) \quad \forall \delta Y \in \mathbb{Y},$$ (11a)

$$a(Y_N X_N, Y_N \delta X) = l(Y_N \delta X) - a(u^\text{PGD}_{N-1}, Y_N \delta X) \quad \forall \delta X \in \mathbb{X}.$$ (11b)
In order to solve (11) efficiently, we want to decompose the integration of the integrals as much as possible. We note that the strain appearing in (7) depends on both the in- and out-of-plane coordinates of the displacement. It can be separated as

$$
\varepsilon[u^{PGD}(y, x)] = \sum_{n=1}^{N} \hat{\varepsilon}[Y_n(y)] X_n(x) + \varepsilon_{X}[Y_n(y)] \frac{dX_n(x)}{dx},
$$

(12)

where $\hat{\varepsilon}[Y]$ and $\varepsilon_{X}[Y]$ are defined as

$$
\hat{\varepsilon}[Y(y)] := [Y(y) \otimes \hat{\nabla}]^{sym}, \quad \varepsilon_{X}[Y(y)] := [Y(y) \otimes e_{x}]^{sym},
$$

(13)

where $\hat{\nabla} = [I - e_{x} \otimes e_{x}] \cdot \nabla$ is the in-plane gradient and $e_{x}$ is the out-of-plane basis vector, see Figure 1. When the arguments of the bilinear form $a(\cdot, \cdot)$ are products of modes, it can be written as

$$
a(YX, Y^*X^*) = \sum_{l=1}^{4} m_{l}(X, X^*) \ a_{l}(Y, Y^*),
$$

(14)

with the bilinear forms on the separated domains, $m_{l}$, and $a_{l}$ are defined according to

$$
m_{1}(X, X^*) = \int_{I_{k}} X \ X^* \ dx, \quad a_{1}(Y, Y^*) = \int_{\Omega} \hat{\varepsilon}[Y] : E : \hat{\varepsilon}[Y^*] \ d\hat{\Omega},
$$

(15a)

$$
m_{2}(X, X^*) = \int_{I_{k}} \frac{dX}{dx} \ X^* \ dx, \quad a_{2}(Y, Y^*) = \int_{\Omega} \hat{\varepsilon}[Y] : E : \varepsilon_{X}[Y^*] \ d\hat{\Omega},
$$

(15b)

$$
m_{3}(X, X^*) = \int_{I_{k}} X \ \frac{dX^*}{dx} \ dx, \quad a_{3}(Y, Y^*) = \int_{\Omega} \varepsilon_{X}[Y] : E : \hat{\varepsilon}[Y^*] \ d\hat{\Omega},
$$

(15c)

$$
m_{4}(X, X^*) = \int_{I_{k}} \frac{dX}{dx} \ \frac{dX^*}{dx} \ dx, \quad a_{4}(Y, Y^*) = \int_{\Omega} \varepsilon_{X}[Y] : E : \varepsilon_{X}[Y^*] \ d\hat{\Omega}.
$$

(15d)

This allows for computing the integrals separately. This separation is however not possible for $l(Y^*X^*)$

$$
l(Y^*X^*) = \int_{I_{k}} \int_{F_{N}} t(y, x) \cdot Y^* \ X^* \ d\hat{\Gamma} \ dx,
$$

(16)

because $t(y, x)$ depends on $y$ and $x$ in an in-separable way, cf. (4). Finally, (11) can be explicitly written as finding $X_{N} \in \mathbb{X}$ and $Y_{N} \in \mathbb{Y}$ such that

$$
\sum_{l=1}^{4} m_{l}(X_{N}, X_{N}) \ a_{l}(Y_{N}, \delta Y) = l(\delta Y \ X_{N}) - \sum_{n=1}^{N-1} \sum_{l=1}^{4} m_{l}(X_{N}, X_{N}) a_{l}(Y_{N}, \delta Y) \quad \forall \delta Y \in \mathbb{Y},
$$

(17a)

$$
\sum_{l=1}^{4} m_{l}(X_{N}, \delta X) \ a_{l}(Y_{N}, Y_{N}) = l(Y_{N} \delta X) - \sum_{n=1}^{N-1} \sum_{l=1}^{4} m_{l}(X_{N}, \delta X) a_{l}(Y_{N}, Y_{N}) \quad \forall \delta X \in \mathbb{X}.
$$

(17b)

Here we can see that (17a) and (17b) describe the 2D and 1D problems, respectively.
3.2. Fixed-point algorithm for the YX-coupled problem

Since \( Y_N(y) \) and \( X_N(x) \) appear in a coupled product, the problem is non-linear and must be solved in a suitable iterative scheme. To solve the mode at enrichment step \( N \) we will adopt a fixed-point alternating algorithm so \( Y_N^{(k)} \) can be computed at iteration \( k \) assuming \( X_N^{(k-1)} \) is known, then \( X_N^{(k)} \) can be updated from \( Y_N^{(k)} \). This process is repeated until convergence is reached. Each iteration in the fixed-point algorithm consists of:

1. Compute \( Y_N^{(k)} \) from the previous out-of-plane mode \( X_N^{(k-1)} \), where the weak form (11a) is approximated to find \( Y_N^{(k)} \in \mathbb{Y} \) such that
   \[
a(Y_N^{(k)} X_N^{(k-1)}, \delta Y X_N^{(k-1)}) = l(\delta Y X_N^{(k-1)}) - a(u_{N-1}^{PGD}, \delta Y X_N^{(k-1)}) \quad \forall \delta Y \in \mathbb{Y}. \tag{18}
   \]

2. Compute the mode \( X_N^{(k)} \) from the newly evaluated in-plane displacement mode \( Y_N^{(k)} \), whereby the weak form (11b) is approximated to find \( X_N^{(k)} \in \mathbb{X} \) such that
   \[
a(Y_N^{(k)} X_N^{(k-1)}, Y_N^{(k)} \delta X) = l(Y_N^{(k)} \delta X) - a(u_{N-1}^{PGD}, Y_N^{(k)} \delta X) \quad \forall \delta X \in \mathbb{X}. \tag{19}
   \]

At each enrichment step, the initial guesses for \( X_N^{(0)}(x) \) in the fixed-point iteration are specified. In this paper, we initialized this in two ways: (1) with an arbitrary start guess and (2) with a start guess that has an orthogonal function concerning the previous normalized mode shapes. The 2\(^{nd} \) start guess is a way to expand the solution domain since the direction of the mode shape is forced.

Each enrichment step requires multiple fixed-point iterations. The fixed-point iterations continue until the weighted difference \( \Delta \) between two iteration steps is smaller than a tolerance \( \epsilon_{FP} \), i.e., until
   \[
   \Delta := \sqrt{|\Delta \alpha_N|^2 + ||\Delta Y_N||_Y^2 + ||\Delta X_N||_X^2} < \epsilon_{FP}, \tag{20}
   \]
where \( \Delta f := f^{(k)} - f^{(k-1)} \) denotes the iterative update of any quantity \( f = \alpha_N, \tilde{Y}_N \) and \( \tilde{X}_N \). Here, \( \alpha_N \) and \( \tilde{Y}_N, \tilde{X}_N \) are the amplitude and the normalized mode shapes, respectively,

\[
\alpha_N = ||Y_N||_Y ||X_N||_X, \tag{21a}
\]

\[
\tilde{Y}_N = \frac{Y_N}{||Y_N||_Y}, \quad ||Y||_Y := \sqrt{\sum_{I=1}^{4} [a_I(Y, Y)]^2}, \tag{21b}
\]

\[
\tilde{X}_N = \frac{X_N}{||X_N||_X}, \quad ||X||_X := \sqrt{\sum_{I=1}^{4} [m_I(X, X)]^2}. \tag{21c}
\]

Here, the norms \( || \cdot ||_Y \) and \( || \cdot ||_X \) are chosen such that they provide an upper bound of the energy norm defined in (5) as follows:

\[
||YX||_a = \sqrt{\sum_{I=1}^{4} m_I(X, X) a_I(Y, Y)} \leq \sqrt{\sum_{I=1}^{4} [m_I(X, X)]^2} \sqrt{\sum_{I=1}^{4} [a_I(Y, Y)]^2} = ||Y||_Y \cdot ||X||_X, \tag{22}
\]

for any product \( YX \).
The stopping criterion for the enrichment process is defined as
\[
\frac{\alpha_N}{\alpha_1} < \epsilon, \tag{23}
\]
when the ratio between the amplitude of mode \( N \) and the first mode becomes smaller than the tolerance \( \epsilon \) or the desired amount of modes is obtained. Therefore, the computational cost is dominated by the total number of iterations of the 2D problem. The matrix structure of the problem is defined in Appendix A. A schematic illustration of PGD algorithm for this separated representation can be seen in Figure 2.

**Figure 2:** The PGD algorithm for the in- and out-of-plane separated representation. The enrichment process continues until the stopping criterion is met, or the desired amount of modes is reached. The fixed-point iteration is conducted for \( k \) iterations in each enrichment step.

### 3.3. Verification against 3D FE simulation

To validate the accuracy of the PGD formulation as well as highlight the benefits when it comes to computational time and memory resources, we compare the PGD solution with a standard 3D FE solution for an equivalent in- and out-of-plane discretization as shown in Figure 3. The matrix structure of the 3D FE problem and its relation to the matrix definition of the PGD problem is explained in Appendix B.

The 3D rail head has linear elastic material properties and a contact scenario that is prescribed with Hertz-distributed load in the elastic region. The material parameters and the load settings are described in Table 1. The depth of the rail is set to \( d = 100 \) mm to approximate an infinite rail part.

The 3D FE solution is performed using wedge elements for the discretization of \( 762 \times 50 = 22860 \) DOF. The discretization is refined close to the contact, both in- and out-of-plane, which is evident from Figure 3. The PGD solution contains an equivalent discretization with 762 in-plane DOF composed of 3-noded triangular elements with linear shape functions and 50 out-of-plane DOF for 1D linear elements. Here, the spatial coordinate \( x \) may be regarded as a parameter for the \( X_N \) modes in the PGD approximation. The in- and out-of-plane discretization was chosen to get accurate solutions while not having a computationally
Figure 3: Illustration of how the 3D rail head is modeled with 3D FE wedge elements. For the PGD solution, 2D linear elements are used in $\hat{\Omega}$, and 1D linear elements are used in $I_x$.

Table 1: Input data of prescribed Hertz load shown in Figure 1 for a contact scenario within the elastic region.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus</td>
<td>$E$ 210 GPa</td>
</tr>
<tr>
<td>Poission’s ratio</td>
<td>$\nu$ 0.3 [-]</td>
</tr>
<tr>
<td>Width, Depth</td>
<td>$w$, $d$ [70,100] mm</td>
</tr>
<tr>
<td>Load position</td>
<td>$s$ 10 mm</td>
</tr>
<tr>
<td>Semi-axes</td>
<td>$a$, $b$ [15,10] mm</td>
</tr>
<tr>
<td>Normal surface contact stress</td>
<td>$p_n$ 212 MPa</td>
</tr>
<tr>
<td>Lateral surface contact stress</td>
<td>$p_l$ 21 MPa</td>
</tr>
<tr>
<td>Longitudinal surface contact stress</td>
<td>$p_x$ 0 MPa</td>
</tr>
</tbody>
</table>

demanding 3D FE solution. Also, when more modes are added to the separated representation, the discretization should be able to capture higher-order mode forms.

3.3.1. Convergence rate

Although the PGD approach is an a priori method, the appropriate tolerance for the fixed-point iterations and enrichment process requires investigation for each specific problem.

To examine the impact of the tolerance in the fixed-point algorithm, different values were assigned. A lower value on the tolerance results in higher number of iterations, but does not necessarily improve the quality of the solution. Moreover, for certain enrichment steps, the stopping criterion (20) may not become smaller than the tolerance. Hence, we set the maximum number of iterations to 20 and the tolerance to $10^{-2}$.

The start guesses $X_0^N$ provided to the fixed-point algorithm can impact the number of fixed-point iterations required. To investigate this, two different types of start guesses were used: (1) arbitrary, and (2) orthogonal start guesses. The arbitrary start guesses resulted in 115 total iterations on average to solve 10 modes. The start guesses that had an orthogonal function concerning the previous normalized mode shapes converged within 87 iterations. Additionally, the orthogonal start guesses were able to converge within the 20 iterations at each enrichment step, which is not the case for the arbitrarily chosen ones. However, it is important to note that the solution is not affected by the choice of the start guesses, but the orthogonal start guesses will be used due to the faster convergence of the fixed-point algorithm.
To investigate an acceptable tolerance for the enrichment process and assess the rate of convergence of the PGD solution, the relative error in energy norm $e$ between the PGD and the reference 3D FE solution is computed as

$$e = \frac{||u_{FE} - u_{PGD}||_a}{||u_{FE}||_a},$$

(24)

where the energy norm was defined in (5) and $u_{PGD}$ and $u_{FE}$ are the nodal displacements of the PGD and 3D FE solution, respectively. Figure 4a shows the error as a function of the number of modes, while Figure 4b displays the vertical displacement across the rail section for different numbers of modes included in the solution. As the number of terms included in the separated representation increases, the enriched solution improves gradually. However, the contribution of the first mode to the solution is more significant than that of the subsequent modes. At 15 modes and 160 total iterations, the solution achieves an acceptable accuracy of 0.2 %, which should roughly correspond to the average von Mises stress error in the body. The maximum displacement error is 0.3 %. At 15 modes, the ratio between the amplitudes (23) is smaller than $4 \cdot 10^{-3}$, which is used as the tolerance for the enrichment loop when the discretization is changed in Section 3.3.4.

![Figure 4](image.png)

**Figure 4:** (a) The number of modes is displayed against the relative energy norm error. (b) How the solution changes for an increasing amount of modes included in the solution. The solution has converged with only a few modes included in the separated representation.

### 3.3.2. Mode shapes

The PGD solution is constructed from a sum of $N$ modes. Figure 5 displays the five first normalized mode shapes of $X_N$ and $Y_N$. It is worth noting that the first mode shape of $Y_N$ resembles the 2D solution obtained from standard 2D plane theories, while the other mode shapes $X_N$ and $Y_N$ represent the 3D effects. It is emphasized that the in-plane mode $Y_N$ is a vector field that contains both in-plane and out-of-plane displacement components along with the multiplicative $X_N$ correction.

Additionally, mode shapes 1,2,3, and 5 of $X_n$ are symmetric around the $x$-axis, resulting in a very small out-of-plane component of $Y$, which is expected from the fact that the solution is symmetric about the $x = 0$ plane. However, the more significant 4th mode shape is non-symmetric, corresponding to larger...
Figure 5: First five normalized mode shapes of (a) $\hat{X}_N$ and (b) $\hat{Y}_N$. The grey color represents the local magnitudes of the out-of-plane component normalized modes.
out-of-plane values for $\hat{Y}$. As the mode number increases, the order of the mode shape for $\hat{X}_N$ increases. Therefore, it is necessary to have a discretization with sufficiently small increments to capture higher-order mode forms.

3.3.3. von Mises stress

Figure 6 shows the von Mises stress for a section of the rail and the error in von Mises stress compared to the 3D FE solution for the converged solution. The stress is concentrated under the contact load, with a maximum value of 285 MPa, while the edges are nearly stress-free. The error in von Mises stress compared to the 3D FE solution is highest around the edges of the contact patch, reaching 6%, which is related to the discontinuity in the load. However, the average error in the body is about 0.4%, which is similar to the relative error in energy norm discussed in Section 3.3.1. Hence, the PGD solution shows very promising results.

![Figure 6: Illustration of the (a) von Mises stress and (b) the von Mises stress error when comparing the converged PGD solution to the 3D FE results.](image)

3.3.4. CPU time and memory allocation

The PGD formulation offers a significant advantage in terms of Central Processing Unit (CPU) time and memory allocation compared to 3D FE analysis as the Number of Degrees of Freedom (NDOF) increases. This is showcased in Figure 7, where NDOF was changed both in- and out-of-plane. For a small amount of DOF, the 3D FE analysis is slightly faster. As the NDOF increases, the CPU time increases exponentially, while memory allocation increases linearly. In contrast, PGD solution’s CPU time and allocated memory remain nearly constant as NDOF increases without compromising the accuracy of the solution. At the extreme corner, the PGD solution converges within 15 modes, with a relative error in energy norm of 0.2%, while consuming only 3% of the CPU time and memory compared to the 3D FE solution.

The CPU time $C$ for the 3D FE problem is proportional to the NDOF in the in-plane and out-of-plane directions, i.e., $C \propto [NDOF_y \times NDOF_x]^{\alpha}$. For the PGD approximation, the problem is instead solved for $i$ total iterations for the in-plane 2D problem and for the out-of-plane 1D problem, resulting in the relation $C \propto i \times [(NDOF_y)^{\alpha} + (NDOF_x)^{\alpha}]$. Therefore, the cost of the 1D problem is negligible compared to the 2D problem. This explains why there is not a significant increase in CPU time for the PGD solution even when the NDOF increases, since the number of modes and iterations required to represent the solution does not significantly increase.
4. 3D elastic PGD analysis of the rail for parameterized discrete load

In this section, we extend the analysis to seek a parameterized solution in terms of a set of load parameters. The goal is to establish a PGD approximation that can be solved and stored from an “offline” computation and evaluated later on near-instantaneously in an “online” stage. For greater generality, we shall consider a more complex loading situation than that described by the Hertzian load.

4.1. Semi-Hertizan contact using STRIPES

To address the distributed surface load, a discrete approach is taken for the contact stresses. The PGD formulation can accommodate extra-coordinates since the extra dimension of the problem does not affect the solvability. Although a Hertzian contact load was initially applied, it may not be appropriate when the contact occurs at the flange corner of the wheel profile or when there are irregularities in the profiles of the contacting bodies, then the conformal contact and constant curvature assumptions of Hertz contact are violated [26]. Therefore, the semi-Hertzian approach called STRIPES, as described in [29, 30], is employed. This method discretizes the contact area parallel to the out-of-plane direction and locally determines the contact stresses in each strip by applying a Hertzian-based formula. The STRIPES approach allows for a more general load distribution, with non-constant lateral curvature and without assuming full slip. It also allows multiple contacts to be active simultaneously, and the separability of the load is greater than with a Hertzian contact.

As shown in Figure 8, the surface load in-plane is discretized with \( m = [1, \ldots, M] \) parallel contact strips that act piece-wise constant for a width of \( \Delta s \). Each strip \( m \) has an assigned value for the maximum surface contact traction \( p_n^m, p_t^m, \) and \( p_x^m \) applied in the corresponding normal (\( e_n^m \)), lateral (\( e_t^m \)), and longitudinal (\( e_x^m \)) direction to the rail profile at each strip.

Describing the contact load with STRIPES, there is a resemblance to Hertz’s contact theory in (4) related to the out-of-plane direction, since the strips have the parabolic distribution \( \rho(x, b^m) \) defined as

\[
\rho(x, b^m) = \begin{cases} 
\sqrt{1 - \left(\frac{x}{b^m}\right)^2}, & |x| < b^m, \\
0, & |x| > b^m,
\end{cases}
\]

(25)
In-plane

Out-of-plane

Figure 8: Equivalent discrete load handling of the contact tractions. The example is shown for a Hertzian contact but can be adapted to more general shapes. The contact traction is subdivided into a number of independent strips with the width $\Delta s$ in-plane. Out-of-plane the load distribution is dictated parabolically.

for the semi-axes $b^m$ in each strip. With this load representation, the traction $t$ is be modeled as

$$t(y, x, \{b^m\}_m, \{p^m_\beta\}_m, \beta) = \sum_{m=1}^{M} \rho(x, b^m) \sum_{\beta} p^m_\beta \phi^m(y) e^m_\beta(y), \quad (26)$$

where $\beta \in A = \{n, t, x\}$ is the index for the normal $n$, lateral $l$, and longitudinal $x$ direction and $\phi(y)$ is defined as

$$\phi^m(y) = \begin{cases} 1 & \text{on } \Gamma_m, \\ 0 & \text{else}, \end{cases} \quad (27)$$

where $\Gamma_m$ is the surface on the rail of the active strip. Please note that the discrete representation of the contact tractions makes the parameters separate for each strip. Also, the surface contact tractions are entirely separated within a strip. However, a relation between $x$ and $b^m$ remains due to the inseparable definition of $\rho(x, b^m)$.

4.2. The load-parametrized PGD approximation

Beyond the separated representation of the spatial coordinates $(y, x)$, the load parameters $p = (\{b^m\}_m, \{p^m_\beta\}_m, \beta)$ are considered as extra-coordinates to the PGD solution outlined in Section 3. Hence, we seek to determine an approximation of the parametric solution $u(y, x, \{b^m\}_m, \{p^m_\beta\}_m, \beta)$. We assume each parameter $b^m \in I_b$ and each $p^m_\beta \in I^\beta_p$, where the intervals are the same for all strips $m$, in order to include the parameter space

$$I_p = [I_b]^M \times [I^\beta_p]^M.$$

The hyper-dimensional weak form of the full problem reads as that of finding $u \in \mathcal{U}$ such that

$$A(u, \delta u) = L(\delta u) \quad \forall \delta u \in \mathcal{U}, \quad (28)$$

with the bilinear and linear forms $A(\cdot, \cdot)$ and $L(\cdot)$ defined as

$$A(u, \delta u) = \int_{I_p} a(u, \delta u) \, dp, \quad L(\delta u) = \int_{I_p} l(p, \delta u) \, dp, \quad (29)$$

\footnote{Here, we introduce the notation $\times$ for the product over all components $\beta \in A = \{n, t, x\}$, i.e., $[I_b]^M \times [I^\beta_p]^M = [I_b]^M \times [I_p]^M \times [I^\beta_p]^M = [I_b]^M$.}
representing the original problem\(^3\) as described in Section 2 integrated over the parameter domains
\[ I_p = [I_b]_\beta^M \times [I_p^\beta]_\beta^M \] with
\[ d\mathbf{p} = (\prod_m d\mathbf{b}^m) \left( \prod_{m, \beta} d\mathbf{p}_\beta^m \right). \]

The trial- and test-space describing functions pertinent to the weak form (28) reads
\[ \mathcal{U} = \left\{ \mathbf{v}(\mathbf{y}, x, \{ b^m \}_m, \{ p^{m, \beta}_\beta \}_{m, \beta}) : \int_{I_p} \| \mathbf{v}(\bullet, \bullet, \{ b^m \}_m, \{ p^{m, \beta}_\beta \}_{m, \beta}) \|_a^2 d\mathbf{p} < \infty \right\}. \] (30)

It becomes evident that, even in discretized form, the high dimensionality of the problem makes the explicit solution to (28) intractable in practice.

For the load-parametrized problem, we shall now seek the PGD approximation of the displacement in the extended separated form to include the load parameters. This is formulated as
\[ \mathbf{u}^{\text{PGD}}_N(\mathbf{y}, x, \{ b^m \}_m, \{ p^{m, \beta}_\beta \}_{m, \beta}) = \sum_{n=1}^N \mathbf{Y}_n(\mathbf{y}) \mathbf{X}_n(x) \prod_{m=1}^M B^m_n(b^m) \prod_{\beta} P^{m, \beta}_n(p^{m, \beta}_\beta), \] (31)

where \( b^m \in I_b \) and \( p^{m, \beta}_\beta \in I^\beta_p \).

In order to set up the PGD problem of finding the \( N^{\text{th}} \) mode, the test function \( \delta \mathbf{u}^{\text{PGD}} \) can be written as
\[
\delta \mathbf{u}^{\text{PGD}}(\mathbf{y}, x, \{ b^m \}_m, \{ p^{m, \beta}_\beta \}_{m, \beta}) = X_N \prod_{m} B^m_n \prod_{\beta} P^{m, \beta}_n \delta \mathbf{Y} + Y_N \prod_{m} B^m_n \prod_{\beta} P^{m, \beta}_n \delta \mathbf{X}

+ \sum_{m=1}^M \mathbf{Y}_N \mathbf{X}_N \left( \prod_{q \neq m} B^q_n \right) \left( \prod_{q, \beta} P^{q, \beta}_n \right) \delta B^m + \sum_{m=1}^M \sum_{\beta \in \mathcal{A}} \mathbf{Y}_N \mathbf{X}_N \left( \prod_{q} B^q_n \right) \left( \prod_{q \neq m \gamma \neq \beta} P^{q, \gamma}_n \right) \delta P^{m, \beta}_n \] (32)

for variations \( \delta \mathbf{Y}, \delta \mathbf{X}, \{ \delta B^m \}_m, \{ \delta P^{m, \beta}_n \}_{m, \beta} \in \mathcal{Y} \times \mathcal{X} \times [\mathcal{B}]_\beta^M \times [\mathcal{P}_\beta]_\beta^M \). Hence, we seek updates in the spaces \( \mathcal{Y} \) and \( \mathcal{X} \), defined in (10), and \( \mathcal{B} = L_2(I_b), \mathcal{P}_\beta = L_2(I^\beta_p) \). Here, \( L_2(\bullet) \) denotes the space of square-integrable functions.

With the test function given above, we are now in the position to state the PGD problem for mode \( N \) in the expansion. The problem is then to find \( \mathbf{Y}_N, \mathbf{X}_N, \{ B^m_n \}_m, \{ \delta P^{m, \beta}_n \}_{m, \beta} \in \mathcal{Y} \times \mathcal{X} \times [\mathcal{B}]_\beta^M \times [\mathcal{P}_\beta]_\beta^M \) such that
\[ A(\mathbf{Y}_N \mathbf{X}_N \prod_{m} B^m_n \prod_{\beta} P^{m, \beta}_n, \delta \mathbf{u}^{\text{PGD}}) = L(\delta \mathbf{u}^{\text{PGD}}) - A(\mathbf{u}^{\text{PGD}}_{N-1}, \delta \mathbf{u}^{\text{PGD}}), \] (33)

for any \( \delta \mathbf{u}^{\text{PGD}} \) on the form given in (32).

\(^3\)In this section, the loading \( l(\mathbf{p}, \bullet) \) follows from the traction described in (26).
In order to solve the PGD approximation from (33) efficiently, we make use of separation of the forms as

\[
\sum_{I=1}^{4} F_{I,N,N}^{(Y)} a_I(Y_N, \delta Y) = \int_{\Gamma_N} \hat{t}_N(y) \cdot \delta Y \, d\Gamma - \sum_{n=1}^{N-1} \sum_{I=1}^{4} F_{I,N,n}^{(Y)} a_I(Y_n, \delta Y) \quad \forall \delta Y \in \mathbb{Y},
\]

(34a)

\[
\sum_{I=1}^{4} F_{I,N,N}^{(X)} m_I(X_N, \delta X) = \int_{I_x} t_{x,N}(x) \delta X \, dx - \sum_{n=1}^{N-1} \sum_{I=1}^{4} F_{I,N,n}^{(X)} m_I(X_n, \delta X) \quad \forall \delta X \in \mathbb{X},
\]

(34b)

\[
F_{N,N}^{(B),m} m_b(B^m, \delta B^m) = \int_{I_b} g^m_{N}(b^m) \delta B^m \, db^m - \sum_{n=1}^{N-1} F_{N,n}^{(B),m} m_b(B^m, B^m_n)
\]

\[
\forall \delta B^m \in \mathbb{B}, \quad m \in \{1, \ldots, M\},
\]

(34c)

\[
F_{N,N}^{(P),m,\beta} m_p(P^m, \delta P^{m,\beta}) = \int_{I_p} f_{0,N}^{m,\beta} + f_{1,N}^{m,\beta} p_{\beta} \delta P^{m,\beta} \, dp_{\beta} - \sum_{n=1}^{N-1} F_{N,n}^{(P),m,\beta} m_p(P^m, P^{m,\beta})
\]

\[
\forall \delta P^{m,\beta} \in \mathbb{P}_{\beta}, \quad m, \beta \in \{1, \ldots, M\} \times \mathbb{A}.
\]

(34d)

Here \(m_1\) and \(a_1\) defined in (15), whereas \(m_b\) and \(m_p\) are stated as

\[
m_b(B^m, B^{m*}) = \int_{I_b} B^m B^{m*} \, db^m, \quad m_p(P^{m,\beta}, P^{m*}) = \int_{I_p} P^{m,\beta} P^{m*} \, dp_{\beta}.
\]

(35)

In Appendix C, the detailed expressions for the components in equations (34a)-(34d) can be found. It is important to note that all coefficients denoted by \("F^m\) are not affected by the mode of "its own" parameter. For instance, \(F_{1,N,N*}^{(Y)}\) is independent of \(Y_N\) and \(F_{N,N*}^{(B),m}\) is independent of \(B^m_N\).

The loading functions \(\hat{t}_N(y), t_{x,N}(x)\) and \(g^m_{N}(b^m)\) are in general nonlinear functions with respect to their respective parameters \((y, x\) and \(b^m\)), depending on the modes not solved for in the pertinent equation\(^4\). Hence, the solutions \(Y_N(y), X_N(x)\) and \(B^m_N(b^m)\) are solved for from (34a)-(34c). In practice, the solutions are obtained numerically by using the finite element method.

However, considering (34d), we note that \(f_{0,N}^{m,\beta}\) and \(f_{1,N}^{m,\beta}\) are constants with respect to \(p^m_{\beta}\), and define a linear loading term. As a result, the exact solution to the equation can be expressed explicitly in strong form as

\[
P^{m,\beta}_{N} = \frac{f_{0,N}^{m,\beta}}{F_{N,N}^{(P),m,\beta}} + \frac{f_{1,N}^{m,\beta}}{F_{N,N}^{(P),m,\beta}} p^m_{\beta} - \sum_{n=1}^{N-1} \frac{F_{N,n}^{(P),m,\beta}}{F_{N,N}^{(P),m,\beta}} P^{m,\beta}_{n}.
\]

(36)

From induction, we see that \(P^{m,\beta}_{N}\) will be linear for \(N = 1, 2, \ldots\). In conclusion, we seek discrete solutions using a fine discretization of the spaces \(\mathbb{Y}, \mathbb{X},\) and \(\mathbb{B}\). However, it is sufficient to represent linear functions in \(\mathbb{P}_{\beta}\) to obtain the exact solution.

The problem is solved with the same fixed-point strategy explained in Section 3.1, but also including orthogonal start guesses for \(F_{N}^{m,(0)}\) and \(F_{N}^{m,\beta,(0)}\) at the beginning of each enrichment step. However, the stopping criteria for the fixed-point iterations (20) now also includes the difference between two iteration steps for the modes involving the load parameters

\[
\Delta = \sqrt{||\Delta \alpha||^2 + ||\Delta \hat{Y}||^2_Y + ||\Delta \hat{X}||^2_X + \sum_{m=1}^{M} (||\Delta \hat{B}^m||^2_B + \sum_{\beta} ||\Delta \hat{P}^m_{\beta}||^2_{P}) < \epsilon_{FP},
\]

(37)

\(^4\)\(\hat{t}_N(y)\) does not depend on \(Y_N\), \(t_{x,N}(x)\) does not depend on \(X_N\) and \(g^m_{N}(b^m)\) does not depend on \(B^m_N\),
where $\epsilon_{FP}$ is the tolerance, and $\hat{B}^m$ and $\hat{P}^{m,\beta}$ are the normalized modes shapes. The amplitude $\alpha_N$ and norms $\| \cdot \|_B$ and $\| \cdot \|_P$ of the load parameters read

$$\alpha_N = \| Y_N \| U \| X_N \| X \prod_{m=1}^{M} \| B^m_N \|_B \prod_{\beta} \| P^{m,\beta}_N \|_P,$$  

(38a)

$$\| B \|_B = \sqrt{\frac{1}{I_B} m_B(B, B)}, \quad \| P_{\beta} \|_P = \sqrt{\frac{1}{I_P} m_P(P_{\beta}, P_{\beta})}.$$  

(38b)

Each norm is divided with the length of the interval $I_B$ and $I_P$ to supply a numerically stable solution when the number of strips increases. When the solution is available, the displacement field can be generated for any setup of the contact load within their respective intervals.

**Remark:** The detailed equations for the fixed-point iterations are omitted for brevity. They follow explicitly from the (non-linear) PGD equations (34a)-(34d) analogously as the fixed-point iterations (18) and (19) follow from (11) in Section 3. This means that the coefficients $F(\cdot)$, $f_{0,\beta}^{m,\beta}$ and $f_{1,\beta}^{m,\beta}$, and the functions $t_N(y)$, $t_{x,N}(x)$ and $g_N^m(b^m)$ will depend on various iteration of the modes $N$ rather than the converged values. In particular, the fact that $P_{N,\beta}^{m,\beta}$ is linear will also hold for any iterative solution $P_{N}^{m,\beta(k)}$.

### 4.3. Verification for different Hertz loads

In this Section, we will verify the PGD parameterization of the discrete load by comparing the resulting PGD approximation, presented in (31), to different reference 3D FE solutions for selected realizations of the load ($p$). The discretization for the in- and the out-of-plane meshes is the same as the one used in Section 3.3.

To generate relevant loading cases on the form given in (26), we choose parameters $b^m$ and $p_{\beta}^{m}$ such that the piece-wise constant representation approximates a Hertzian load. Importantly, though, the reference FE solution will always correspond to the discrete traction form.

The reference Hertz load cases are obtained by placing the contact patch center at $s = 0$ mm and using a semi-axis of $a_H = 10$ mm. Therefore, the strips will be discretized between $[-10, 10]$ mm, while varying the semi-axis $b_H$. The discrete Hertzian semi-axis $b^m$ and surface contact traction magnitudes $p_{n}^{m}$, $p_{l}^{m}$, and $p_{x}^{m}$ are determined through interpolation, see Figure 9. We consider nine different samples of $b_H = [6, 10, 14]$ mm and $p_{n,H} = [75, 150, 225]$ MPa, as shown in Figure 10b. The lateral and longitudinal traction is set to 15% of the normal surface traction magnitude, pertinent to full-slip conditions.

![Figure 9](image-url)  

**Figure 9:** The discrete load parameter $b^m$, $p_{n}^{m}$, $p_{l}^{m}$, $p_{x}^{m}$ are identified through interpolation at the points marked with black circles for a theoretical Hertzian distribution, defined by parameters $a_H$, $b_H$ and $p_{n,H}$.  

17
For the PGD approximation, the load parameters can vary within the ranges specified in Table 2. These intervals were determined based on different contact scenarios from the vehicle-track interaction generated from the load sequence described in [5]. The load magnitudes are limited so that they stay around the elastic region of the rail material. Each parameter is discretized with a uniform 1D FE mesh of linear continuous elements, allowing the solution to be interpolated to any value within the intervals.

**Table 2:** Parameterized load for discrete contact strips. The intervals were found by generating the load sequence explained in [5]. The load magnitudes are limited to the elastic regions.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semi-axis</td>
<td>([0,16]) mm</td>
</tr>
<tr>
<td>Normal surface contact stress</td>
<td>([0,300]) MPa</td>
</tr>
<tr>
<td>Lateral surface contact stress</td>
<td>([-50,50]) MPa</td>
</tr>
<tr>
<td>Longitudinal surface contact stress</td>
<td>([-50,50]) MPa</td>
</tr>
</tbody>
</table>

4.3.1. Influence of discretization of load parameters and settings for fixed-point algorithm

As previously demonstrated in Section 4.2, it is sufficient to employ only one linear element, i.e., two points, for the discretization along each load amplitude \(p_m^\beta\). For the discretization along the semi-axis \(b_m\), a finer discretization will enable the representation of higher-order mode forms. Therefore, for \(b_m\), we adopt a discretization with 40 points, which provides a good balance between accuracy and computational cost.

The accuracy of the PGD approximation varies in the parameter space, as seen in Figure 10, where different solutions are displayed against the relative error in the energy norm for five strips. Nevertheless, the general trend in all cases is that the error decreases as more modes are included in the solution.

![Figure 10](a) The relative energy norm error for the PGD approximation shown against the number of modes for different load parameters, and (b) illustration of the sampled load parameters. The number of strips is \(m=5\), and the PGD discretization of \(b_m\) pertains to 40 points.
The number of iterations required by the fixed-point algorithm to converge also affects the CPU time. As explained in Section 3.3.1, the convergence rate is dependent on the specified tolerance for the fixed-point iteration. In this case, when the load parameters are treated as extra-coordinates, the tolerance is set to $\epsilon_{FP} = 0.1$ to accommodate the higher value of (37). The order in which the parameters are solved within the fixed-point algorithm can also impact the number of iterations required for convergence, but that is not the case in this problem.

### 4.3.2. Influence of number of strips

To see how the number of strips $M$ influences the results, the strips are divided over the contact patch area. The number of strips is limited by the in-plane mesh size. The width of the strips $\Delta y$ cannot be smaller than the largest distance between the nodes on the upper edge within the contact patch region. For the in-plane discretization, the largest width is about 2.3 mm on the upper edge. Thus, with the selected semi-axis value $a_H = 10$ mm, a maximum of eight strips can be applied.

The impact of the number of strips $M$ on the convergence rate, CPU time, and memory allocation are shown in Figure 11. The Root Mean Square Error (RMSE) is computed instead of displaying the error for each load setup. As the number of strips increases, more values are treated as coordinates. As a result, the solution becomes more complex, and the convergence decrease. Thus, more modes must be included to maintain the same accuracy when more strips are considered. This is also evident in Figure 12, where the accuracy of the RMSE value is 2 %.

![Figure 11: How the number of strips $M$ affects the (a) convergence rate in terms of RMSE and (b) CPU time, and memory allocation.](image)

The memory usage remains relatively stable as the number of strips increases, while the CPU time $C$ is affected, shown in Figure 11b. This is because the CPU time of the PGD problem can be estimated as $C \propto i \times [M \times (n_b)^\alpha + (NDOF_y)^\alpha + NDOF_x)^\alpha]$, if the separated representation includes additional coordinates in each strip, where $i$ is the total number of iterations and $n_b$, is the out-of-plane discretization of the semi-axis. Therefore, the computational effort for the 1D problems is not negligible, unlike in Section 3.3, where only the out-of-plane coordinate was involved. In contrast, when solving the same problem using the 3D FE method, the CPU effort is $C \propto [(n_b)^M \times (NDOF_y \times NDOF_x)^\alpha]$, which is significantly more computationally demanding and less memory efficient. Even though the solution
Figure 12: To remain a 2% RMSE accuracy more modes are necessary to include as the number of strips increases.

becomes less sufficient as more strips are added, we can allow for an increase in CPU time for the PGD problem since a single offline solution solves the displacement field for any parameter setup within its respective interval. The online solution can be generated for many load scenarios as a computationally inexpensive postprocessing step.

In addition to the number of strips, the CPU time is also affected by the number of modes solved in the PGD problem. As shown in Figure 13, the CPU time increases exponentially with the number of modes. This is because the sum $\alpha(u_{N-1}^{\text{PGD}}, \delta u^{\text{PGD}})$ in (17) requires more computations when more terms are included in the sum, increasing the time required to solve the problem.

Figure 13: CPU time increases exponentially for the number of modes that are included in the separated representation.
4.4. Result of applying a semi-Hertzian contact

So far, the solution has been validated assuming a Hertzian load distribution. However, one of the advantages of the discrete contact approach is its flexibility when it comes to variations in the shape of the contact area in the lateral direction and variations in the magnitude of the load. In addition, the approach allows for the simultaneous application of multiple contact points. To demonstrate these advantages, 20 strips were applied across the entire load region $[-15, 35]$ mm, found from the generated load sequence [5]. Each strip has a width of 2.5 mm, which is wider than the in-plane mesh size. The load parameters can vary within the ranges specified in Table 2.

Nine representative load scenarios were generated using the commercial software Simpack v.2022, based on the load sequence described in [5]. In Simpack, the discrete elastic contact model was used, which employs the STRIPES method outlined in [29, 30]. The contact model in Simpack functions similarly to how the load is parameterized in the PGD model. However, in Simpack, the width of the strips is 0.5 mm, whereas the strips generated from the PGD model are 2.5 mm wide. As a result, the input to the PGD model consists of the average values for the load parameters obtained from five strips.

One of the nine load cases is illustrated in Figure 15, where each load parameter is normalized by $\hat{b}_m = b_m/I_b$ and $\hat{p}_\beta = p_{\beta m}/I_p^\beta$ for their respective interval length given in Table 2.

4.4.1. Convergence rate - Influence of invariant parameters

Compared to the Hertz contact load discussed in Section 4.3, the discrete load setup only loads certain strips, as shown in Figure 15. For such a strip $m$, where $p_{q m} = p_{x m} = p_{z m} = 0$, it is evident from the traction format in (26) that $b_m$ cannot influence the exact solution. In fact, for the studied load scenario, and the pertinent reference FE solution, no value for $b_m$ is available for those strips. However, this invariance is not explicitly apparent in the PGD approximation. Hence, we have to assign a value for $b_m$ for all strips even though the traction component is zero.

In the following, we shall investigate two approaches for selecting $b_m$ for those strips $m$ on which no load $p_{q m}^\beta$ acts:

$$B_n^m(b_m) = \begin{cases} B_n^m(\tilde{b}) & \text{or}, \\ \frac{1}{b_n^m} \int_{b_n^m} B_n^m(b_m) \, \text{d}b_m, & \forall m \in \{q : p_{q m}^\beta = 0 \land \forall \beta \in A\}, \end{cases}$$

for all modes $n = 1, \ldots, N$. Thus, the two alternatives are either to set the parameter to a preset default $b$ or to integrate the average mode over the interval.

Figure 14 displays the RMSE for the PGD approximation as a function of the number of modes, adopting the different strategies, and different choices of default values for $\tilde{b}$. The evaluation is performed for nine load scenarios, and the results are compared against the 3D FE solution. The Figure clearly shows that, unlike the exact solution, the PGD solution is not invariant to the values of $b_m$. Thus, the accuracy of the PGD solution is sensitive to the adopted strategy chosen for $B_n^m(b_m)$.

From Figure 14, it appears that the solution cannot converge when a small value of $\tilde{b}$ is assigned to the unloaded strips ($\tilde{b} = 4$ mm). This behavior may be related to the relationship between the semi-axis and the out-of-plane coordinate $x$. When $\tilde{b}$ is small, more $X$-modes are needed to obtain an accurate solution. Conversely, setting the value too high ($\tilde{b} = 14$ mm) will also result in lower accuracy. By averaging the mode contributions, or when $\tilde{b} = 6$ mm, the solution initially converges, but the accuracy deteriorates after 600 modes. The most accurate solution is approximately in the middle of the interval, for $\tilde{b} = 10$ mm, which is subsequently used to evaluate the results.
It is also clear from Figure 14 that the convergence rate has slowed down for this number of strips, which was also shown in Figure 11. At 1000 modes, the RMSE is 6.3 %.

![Figure 14](image.png)

**Figure 14:** RMSE of PGD solution of discrete contact compared to 3D FE solution for nine different load scenarios given by a generated load sequence. The RMSE value is shown for different values on \( \tilde{b} \) for the unloaded strips. In one case, the mode contribution is averaged.

### 4.4.2. von Mises stress for one load scenario

Figure 15 displays the load parameters in each strip for one of the nine load scenarios evaluated in Section 4.4.1. The Figure shows that three different contact points are active, that the contact area is no longer elliptical, and that the traction in-plane of the surface contact is no longer parabolic. In addition, each load parameter is normalized by dividing it by its corresponding interval length given in Table 2, i.e., \( \tilde{b}^m = b^m / I_b \) and \( \tilde{p}_\beta^m = p^m_\beta / I_\beta \).

![Figure 15](image.png)

**Figure 15:** Given load parameters for each strip \( m \). The loading scenario consists of three different contact points and in total eight active strips. Each load parameter is normalized for its respective interval length given in Table 2.
For the given load scenario, the PGD solution exhibits a relative error of 6.12 % for the energy norm and an error of 5.6 % for the maximum displacement at 1000 modes. Figure 16 displays the von Mises stress and the error of the von Mises stress compared to the 3D FE solution for this loading scenario. The maximum von Mises stress for the PGD solution is 295 MPa, while the reference has the highest von Mises stress at 313 MPa. Therefore, the PGD solution slightly underestimates the stresses in the rail head, resulting in a maximum error of 14.45 %. Unlike the von Mises stress generated for the in- and out-of-plane separation shown in Figure 6, the error now occurs in the middle of the contact. However, the accuracy of the von Mises stress is 1.26 % over the entire body. Increasing the number of modes in the solution would further reduce the error. When working with more parameters in the separated representation, it is crucial to find a suitable balance between accuracy and computational time.

5. Conclusions

In this paper, a Proper Generalized Decomposition (PGD) formulation is proposed to efficiently solve various contact scenarios on a three-dimensional elastic rail head. First, the spatial domain is separated into a two-dimensional in-plane discretization of the rail cross-section and a one-dimensional out-of-plane discretization, which constitutes a parameter in the PGD approximation. Comparing the PGD solution with a three-dimensional finite element solution, using the same discretization, demonstrates the accuracy of the PGD approximation, measured as the relative error in the energy norm. In addition, this method provides a large reduction in computational cost and memory allocation.

The second part of the paper concerns the extension to a parametric problem, whereby we seek solutions to the elastic problem for a large variety of contact scenarios. To model the various contact scenarios, the STRIPES approach is utilized to implement a distributed load that is piece-wise constant in-plane and parabolic out-of-plane for a specific width of the strips. This approach enables a separate representation of the traction where different variations of contact shape and magnitude distribution can be solved, and multiple contact points can be active simultaneously. The load parameters are treated as extra-coordinates within their intervals, including the semi-axes of a strip out-of-plane \( b^m \) and the traction magnitudes of the contact area in the normal \( p^m_n \), lateral \( p^m_t \), and longitudinal \( p^m_x \) directions for each strip \( m \). In this way, the PGD formulation can handle many variables, some of which exhibit linear dependence. It is important to treat the special case where we know that the exact solution is invariant with respect to a
given parameter.

The more parameters treated as coordinates, the more complex the solution becomes, as more modes are required to accurately capture a loading setup, resulting in longer computational times. However, at an “off-line” stage, a single solution for the displacement field is generated that contains all solutions of \( b^m, p^n, p_i, p^m \) within their respective intervals. At the “on-line” stage, the solution can be quickly obtained as a post-processing step of the pre-computed parametric solution for a desired load scenario. Hence, the approach can be highly effective, even in the case the ”off-line” solution pertinent to build the PGD approximation is computationally demanding.

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**A. Matrix structure of the separated representation**

The weak forms of the PGD approximation (18) and (19) can be rewritten to matrix form by employing (linear) finite elements. The FE-subspaces are introduced as \( \mathcal{Y}_h \subset \mathcal{Y} \) and \( \mathcal{X}_h \subset \mathcal{X} \), where the spaces were defined in (10). The nodal approximations for \( Y_N(y) \) and \( X_N(x) \) with the \( N^i(y) \) and \( N^i(x) \) are FE shape functions read

\[
Y_N(y) \approx \sum_{i=1}^{\text{NDOF}_y} N_i^i(y)\left[Y_N\right]_i \in \mathcal{Y}_h, \quad \delta Y(y) \approx \sum_{i=1}^{\text{NDOF}_y} N_i^i(y)(\delta Y)_i, \quad \text{(A.1a)} \\
X_N(x) \approx \sum_{i=1}^{\text{NDOF}_x} N_i^i(x)\left[X_N\right]_i \in \mathcal{X}_h, \quad \delta X(x) \approx \sum_{i=1}^{\text{NDOF}_x} N_i^i(x)(\delta X)_i \quad \text{(A.1b)} \\
dX_N(x)/dx \approx \sum_{i=1}^{\text{NDOF}_x} B_i(x)\left[X_N\right]_i \in \mathcal{X}_h, \quad d\delta X(x)/dx \approx \sum_{i=1}^{\text{NDOF}_x} B_i(x)(\delta X)_i, \quad \text{(A.1c)}
\]

where vectors \( Y_N \) and \( X_N \) contain nodal values in the FE mesh of the in- and out-of-plane parameters, respectively, whereas \( \delta Y \) and \( \delta X \) contain the corresponding arbitrary parameters. The sum goes to the NDOF of each parameter. The corresponding FE-approximations of the strains (12) are

\[
\varepsilon[Y_N(y)] \approx \sum_{i=1}^{\text{NDOF}_y} \varepsilon_i[N_i^i(y)]\left[Y_N\right]_i = \sum_{i=1}^{\text{NDOF}_y} B_i^{(\Omega)}(y)\left[Y_N\right]_i, \quad \varepsilon[\delta Y(y)] \approx \sum_{i=1}^{\text{NDOF}_y} B_i^{(\Omega)}(y)(\delta Y)_i, \quad \text{(A.2a)} \\
\varepsilon_X[Y_N(y)] \approx \sum_{i=1}^{\text{NDOF}_y} \varepsilon_X_i[N_i^i(y)]\left[Y_N\right]_i = \sum_{i=1}^{\text{NDOF}_y} B_i^{(x)}(y)\left[Y_N\right]_i, \quad \varepsilon_X[\delta Y(y)] \approx \sum_{i=1}^{\text{NDOF}_y} B_i^{(x)}(y)(\delta Y)_i. \quad \text{(A.2b)}
\]

Using the FE-approximations (A.1) and (A.2) in the fixed-point algorithm equation (18) and (19), results
in the discrete form of the problem
\[
\tilde{a}(Y_N X_N, \delta Y_X N) = [X_N]^T F \delta Y - \sum_{n=1}^{N-1} A(Y_n, X_n, \delta Y X_n), \quad (A.3a)
\]
\[
\tilde{a}(Y_N X_N, \delta X) = [\delta X]^T F Y_N - \sum_{n=1}^{N-1} \tilde{a}(Y_n, X_n, \delta X), \quad (A.3b)
\]
where the FE-discretized bilinear form \( \tilde{a}(\cdot, \cdot) \) from (14) is defined as
\[
\]
The global stiffness matrices \( K \), mass matrices \( M \) and external force matrix \( F \) reads
\[
(K_{\Omega})_{kl} = \int_{\Omega} B^{\Omega}_k(y) : E : B^{\Omega}_l(y) \, d\hat{\Omega}, \quad (M_{\Omega})_{kl} = \int_{\Gamma} N^k(x) \cdot N^l(x) \, dx, \quad (A.5a)
\]
\[
(K_{\Omega X})_{kl} = \int_{\Omega} B^{\Omega}_k(y) : E : B^{(x)}_l(y) \, d\hat{\Omega}, \quad (M_{\Omega X})_{kl} = \int_{\Gamma} B_k(x) \cdot N^l(x) \, dx, \quad (A.5b)
\]
\[
(K_{X})_{kl} = \int_{\Omega} B^{(x)}_k(y) : E : B^{(x)}_l(y) \, d\hat{\Omega}, \quad (M_{X})_{kl} = \int_{\Gamma} B_k(x) \cdot B_l(x) \, dx, \quad (A.5c)
\]
\[
(F)_{kl} = \int_{\Gamma} \int_{\Gamma N} N^k(x) N_l(y) \cdot t \, d\hat{\Gamma} \, dx. \quad (A.5d)
\]
The advantage of the matrix structure of the problem is that \( K \), \( M \) and \( F \) only need to be computed once. The FE-discrete form of the nodal displacements \( u^{\text{PGD}}_N \) read
\[
u^{\text{PGD}}_N = \sum_{n=1}^{N} \alpha_n \tilde{Y}_n \tilde{X}_n, \quad (A.6)
\]
where \( \alpha \) and \( \tilde{Y}, \tilde{X} \) are the amplitude and normalized mode shapes, respectively, defined from (21).

**B. Matrix structure of 3D FE solution**

Taking advantage of the separated representation of the displacements and strain in the PGD formulation, the 3D FE approximations are defined as
\[
u(y, x) \approx \sum_{n=1}^{N_{\text{DOF}}^x} \sum_{k=1}^{N_{\text{DOF}}^y} N^y_k(y)(\nu_n)_k N^x_n(x), \quad (B.7)
\]
\[
\delta \nu(y, x) \approx \sum_{n=1}^{N_{\text{DOF}}^x} \sum_{k=1}^{N_{\text{DOF}}^y} N^y_k(y)(\delta \nu_n)_k N^x_n(x), \quad (B.8)
\]
where \( \nu \) are the nodal displacements whereas \( \delta \nu \) is the corresponding arbitrary parameter. \( N^y_i(y) \) and \( N^x_i(x) \) are the FE shape functions that was first defined in (A.1). The same separation of the strain (12)
is used for the 3D FE, where the discrete form of the strains reads

\[
\varepsilon[u(y, x)] \approx \sum_{n=1}^{NDOF_y} \sum_{k=1}^{NDOF} \varepsilon[N_i^y(y)](u_n)_k N_n(x) = \sum_{n=1}^{NDOF_y} \sum_{k=1}^{NDOF} B_i^{(y)}(y)(u_n)_k N_n(x),
\]

(B.9a)

\[
\varepsilon[\delta u(y, x)] \approx \sum_{n=1}^{NDOF_y} \sum_{k=1}^{NDOF} B_i^{(y)}(y)(\delta u_n)_k N_n(x),
\]

(B.9b)

\[
\varepsilon_X[u(y, x)] \approx \sum_{n=1}^{NDOF_y} \sum_{k=1}^{NDOF} \varepsilon_X[N_i^y(y)](u_n)_k N_n(x) = \sum_{n=1}^{NDOF_y} \sum_{k=1}^{NDOF} B_i^{(x)}(y)(u_n)_k N_n(x),
\]

(B.9c)

\[
\varepsilon_X[\delta u(y, x)] \approx \sum_{n=1}^{NDOF_y} \sum_{k=1}^{NDOF} B_i^{(x)}(y)(\delta u_n)_k N_n(x). \tag{B.9d}
\]

For these FE approximations, the FE-discrete form of the problem reads

\[
\sum_{n=1}^{NDOF_y} \delta u_n K_{nm} u_m = \sum_{n=1}^{NDOF_y} \delta u_n f_n, \tag{B.10}
\]

where the external force vector \( f_n \), which is the \( n \)th column of \( F \), and the global stiffness matrix \( K_{nm} \) are defined as

\[
(f_n)_k = \int_{x_0}^{x} \int_{\Gamma} N_n(x)N_k^y(y) \cdot t \ d\Gamma \ dx, \tag{B.11a}
\]

\[
K_{nm} = (M_\Omega)_{nm} K_{\Omega} + (M_{\Omega X})_{nm} K_{\Omega X} + (M_{X\Omega})_{nm} K_{X\Omega} + (M_X)_{nm} K_X, \tag{B.11b}
\]

here the separated representations of \( K \) and \( M \) are used and where defined in (A.5). It should be noted that \( K_{nm} = 0 \) for \( |m-n| > 1 \). Thus, the problem in matrix form is written as

\[
\begin{bmatrix}
K_{11} & K_{12} & 0 & \cdots & 0 \\
K_{21} & K_{22} & K_{23} & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_{NDOF_y-1} \\
u_{NDOF_y} \\
\end{bmatrix}
= \begin{bmatrix}
f_1 \\
f_2 \\
\vdots \\
f_{NDOF_y-1} \\
f_{NDOF_y} \\
\end{bmatrix},
\]

(B.12)

where \( NDOF_x \) is the NDOF out-of-plane.

### C. Separated representation of load-parametrization

In order to solve the PGD approximation from (33) efficiently, we make use of the separation of variables. The separated representation of the bilinear and linear forms \( A(\cdot, \cdot) \) and \( L(\cdot) \) reads

\[
A(YX \prod_{m} B^{m} \prod_{m,\beta} P^{m,\beta}, Y^* X^* \prod_{m} B^{m*} \prod_{m,\beta} P^{m,\beta})
= \sum_{l=1}^{4} m^l(X, X^*) a_l(Y, Y^*) \prod_{m} m^b_{m}(B^{m}, B^{m*}) \prod_{m,\beta} m^\beta_{m}(P^{m,\beta}, P^{m,\beta*}), \tag{C.13a}
\]

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\[
L(Y^* X^* \prod_{m} B_{m,\beta}^{\text{m}*} \prod_{m,\beta} \prod_{m^\beta}) = \sum_{m=1}^{M} \sum_{\beta \in \mathbb{A}} \int_{I_0} \int_{I_\mathbb{B}} \rho(x, b^m) X^* B_{m,\beta}^{\text{m}*} \, dx \, db^m \times \ldots
\]
\[
\ldots \times \int_{\mathbb{B}} p_{\beta} \prod_{m,\beta} \prod_{m} P_{m,\beta}^{m,*} \prod_{m} \prod_{\gamma \neq \beta} P_{\gamma,\gamma}^{m,*} \, dp_{\gamma}, \quad (C.13b)
\]
with \( m_1 \) and \( a_1 \) defined in (15) and \( m_b \) and \( m_p \) stated in (35).

Using the separation of the forms, we repeat (34) for completeness and seek \( Y_N, X, \{B_{m}^{N}\}_m, \{\delta P_{m,\beta}^{m^\beta}\}_{m,\beta} \in \mathbb{Y} \times \mathbb{X} \times [\mathbb{B}]^{M} \times [\mathbb{P}_{\beta}]^{M} \) such that

\[
\sum_{I=1}^{4} F_{I,N,N}^{(Y)} a_I(Y_N, \delta Y) = \int_{I_N} \mathbf{t}_N(y) \cdot \delta Y \, d\Gamma - \sum_{n=1}^{N-1} \sum_{I=1}^{4} F_{I,N,n}^{(Y)} a_I(Y_N, \delta Y) \quad \forall \delta Y \in \mathbb{Y}, \quad (C.14a)
\]
\[
\sum_{I=1}^{4} F_{I,N,N}^{(X)} m_I(X_N, \delta X) = \int_{I_\mathbb{B}} t_{x,N}(x) \delta X \, dx - \sum_{n=1}^{N-1} \sum_{I=1}^{4} F_{I,N,n}^{(X)} m_I(X_N, \delta X) \quad \forall \delta X \in \mathbb{X}, \quad (C.14b)
\]
\[
F_{N,N}^{(B^m)} m_b(B_{N}^{m}, \delta B^m) = \int_{I_0} g_N(b^m) \delta B^m \, db^m - \sum_{n=1}^{N-1} F_{N,n}^{(B^m)} m_b(B_{n}^{m}, B_{N}^{m}) \quad (C.14c)
\]
\[
\forall \delta B^m \in \mathbb{B}, \, m \in \{1, \ldots, M\},
\]
\[
F_{N,N}^{(P^\beta)} m_p(P_{N}^{m,\beta}, \delta P^{m,\beta}) = \int_{I_0} f_0^{m,\beta} + f_{1,N}^{m,\beta} P_{\beta}^m \delta P^{m,\beta} \, dp_{\beta}^m - \sum_{n=1}^{N-1} F_{N,n}^{(P^m,\beta)} m_p(P_{n}^{m,\beta}, P_{N}^{m,\beta}) \quad (C.14d)
\]
\[
\forall \delta P^{m,\beta} \in [\mathbb{P}_{\beta}]^{M}, \, m, \beta \in \{1, \ldots, M\} \times \mathbb{A},
\]
where "\( F_{n}^{m^\beta} \)" are independent of the corresponding mode and are defined as

\[
F_{I,N,N}^{(Y)} = m_I(X_N^*, X_N) \prod_{m} m_b(B_{N}^{m}, B_{N}^{m}) \prod_{\beta} m_{P}^{m,\beta}(P_{N}^{m,\beta}, P_{N}^{m,\beta}), \quad (C.15a)
\]
\[
F_{I,N,N}^{(X)} = a_I(Y_N^*, Y_N) \prod_{m} m_b(B_{N}^{m}, B_{N}^{m}) \prod_{\beta} m_{P}^{m,\beta}(P_{N}^{m,\beta}, P_{N}^{m,\beta}), \quad (C.15b)
\]
\[
F_{N,N}^{(B^m)} = \sum_{I=1}^{4} a_I(Y_N^*, Y_N) m_I(X_N^*, X_N) \prod_{q \neq m} m_{q}(B_{N}^{q}, B_{N}^{q}) \prod_{q,\beta} m_{P}^{q,\beta}(P_{N}^{q,\beta}, P_{N}^{q,\beta}), \quad (C.15c)
\]
\[
F_{N,N}^{(P^\beta)} = \sum_{I=1}^{4} a_I(Y_N^*, Y_N) m_I(X_N^*, X_N) \prod_{q} m_{q}(B_{N}^{q}, B_{N}^{q}) \prod_{q,\gamma \neq m,\beta} m_{\gamma}(P_{N}^{q,\gamma}, P_{N}^{q,\gamma}). \quad (C.15d)
\]
Furthermore, the loading terms become

\[ i_N(y) = \sum_{m=1}^{M} \sum_{\beta \in A} i_{m,\beta} N \phi_m(y) e_\beta^m(y), \quad \text{(C.16a)} \]

\[ i_{m,\beta} N = \int_{I_b} \int_{I_x} \rho(x, b^m) X_N B_{N,\beta}^m \, dx \, db^m \int_{I_p} p_{\beta}^m P_{N,\beta}^m \, dp_{\beta}^m \times \ldots \quad \text{(C.16b)} \]

\[ \ldots \times \prod_{q \neq m} \int_{I_b} B_{N,\beta}^q \, db^q \prod_{q \neq m} \int_{I_p} \gamma \beta \prod_{q \neq m} \int_{I_p} P_{N,\gamma}^q \, dp_{\gamma}, \]

\[ t_{x,N}(x) = \sum_{m=1}^{M} \sum_{\beta \in A} i_{m,\beta} x, \int_{I_b} \rho(x, b^m) B_N \, db^m, \quad \text{(C.16c)} \]

\[ i_{m,\beta} x, \int_{I_b} \phi_m(y) e_\beta^m(y) \cdot Y_N \, d\Gamma \int_{I_p} p_{\beta}^m P_{N,\beta}^m \, dp_{\beta}^m \times \ldots \quad \text{(C.16d)} \]

\[ \ldots \times \prod_{q \neq m} \int_{I_b} B_{N,\beta}^q \, db^q \prod_{q \neq m} \int_{I_p} \gamma \beta \prod_{q \neq m} \int_{I_p} P_{N,\gamma}^q \, dp_{\gamma}, \]

\[ g_m^m(b^m) = g_{N,0} + C_m \int_{I_x} \rho(x, b^m) X_N \, dx, \quad \text{(C.16e)} \]

\[ g_{N,0} = \sum_{q \neq m} \sum_{\beta \in A} \int_{I_N} \phi_q(y) e_\beta^q(y) \cdot Y_N \, d\Gamma \int_{I_p} p_{\beta}^q P_{N,\beta}^m \, dp_{\beta}^m \times \ldots \quad \text{(C.16f)} \]

\[ \ldots \times \prod_{r \neq \{m, q\}} \int_{I_b} B_{N,\beta}^r \, db^r \prod_{r \neq \{m, q\}} \int_{I_p} \gamma \beta \prod_{r \neq \{m, q\}} \int_{I_p} P_{N,\gamma}^r \, dp_{\gamma}, \]

\[ G_m^m = \sum_{\beta \in A} \int_{I_N} \phi_m(y) e_\beta^m(y) \cdot Y_N \, d\Gamma \int_{I_p} p_{\beta}^m P_{N,\beta}^m \, dp_{\beta}^m \times \ldots \quad \text{(C.16g)} \]

\[ \ldots \times \prod_{q \neq m} \int_{I_b} B_{N,\beta}^q \, db^q \prod_{q \neq m} \int_{I_p} \gamma \beta \prod_{q \neq m} \int_{I_p} P_{N,\gamma}^q \, dp_{\gamma}, \]

\[ f_{0,N}^{m,\beta} = \sum_{q \neq m} \sum_{\gamma \neq \beta} \int_{I_N} \phi_q(y) e_\gamma^q(y) \cdot Y_N \, d\Gamma \int_{I_b} \rho(x, b^q) X_N B_{N,\gamma}^q \, dx \, db^q \times \ldots \quad \text{(C.16h)} \]

\[ \ldots \times \int_{I_p} p_{\gamma}^q P_{N,\gamma}^q \, dp_{\beta}^m \times \prod_{r \neq \{m, q\}} \int_{I_b} B_{N,\beta}^r \, db^r \prod_{r \neq \{m, q\}} \int_{I_p} \gamma \beta \prod_{r \neq \{m, q\}} \int_{I_p} P_{N,\gamma}^r \, dp_{\gamma}, \]

\[ f_{1,N}^{m,\beta} = \int_{I_N} \phi_m(y) e_\beta^m(y) \cdot Y_N \, d\Gamma \int_{I_b} \rho(x, b^m) X_N B_{N,\gamma}^m \, dx \, db^m \times \ldots \quad \text{(C.16i)} \]

\[ \ldots \times \prod_{q \neq m} \int_{I_b} B_{N,\beta}^q \, db^q \prod_{q \neq m} \int_{I_p} \gamma \beta \prod_{q \neq m} \int_{I_p} P_{N,\gamma}^q \, dp_{\gamma}, \]
References


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