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Citation for the original published paper (version of record):
Berg, M., Persson, D. (2023). Massive theta lifts. Journal of High Energy Physics, 2023(5).
http://dx.doi.org/10.1007/JHEP05(2023)062

Published for SISSA by Springer

Received: *February 5, 2023*
Accepted: *April 4, 2023*
Published: *May 8, 2023*

Massive theta lifts

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Abstract: We use Poincaré series for massive Maass-Jacobi forms to define a "massive theta lift", and apply it to the examples of the constant function and the modular invariant j-function, with the Siegel-Narain theta function as integration kernel. These theta integrals are deformations of known one-loop string threshold corrections. Our massive theta lifts fall off exponentially, so some Rankin-Selberg integrals are finite without Zagier renormalization.

Keywords: String Duality, Superstrings and Heterotic Strings

ArXiv ePrint: 2212.11957

https://doi.org/10.1007/JHEP05(2023)062

JHEP05(2023)062

Contents

JHEP05(2023)062

1 Introduction

The theory of automorphic forms is a wide-ranging area of mathematics, as demonstrated, for example, through Wiles's proof of Fermat's last theorem. This is part of a "grand unified theory of mathematics" called the *Langlands program*, which relates automorphic forms to group representation theory, geometry and number theory in deep and unexpected ways. Many interesting properties of automorphic forms are captured by their Fourier coefficients. Classically, these coefficients often carry a wealth of interesting information such as eigenvalues of Hecke operators and counts of rational points of elliptic curves. For higher-rank groups, such Fourier coefficients play a crucial role in analyzing automorphic L-functions by the Langlands-Shahidi method (see, e.g., [1]).

The moduli space of toroidal string compactifications is a symmetric space $G(\mathbb{R})/K$. String scattering amplitudes that preserve maximal supersymmetry are functions on this space that are covariant (equivariant) under a discrete subgroup $G(\mathbb{Z}) \subset G(\mathbb{R})$ (U-duality group). Physical considerations reveal that these functions are highly constrained: (i) they must be eigenfunctions of the Laplacian on $G(\mathbb{R})/K$ with specific eigenvalues; (ii) they must have a small number of constant terms with specific growth conditions, and (iii) their Fourier coefficients are supported only on a subspace of all possible 'instanton charges'. Mathematically this means that these functions are automorphic forms attached to certain small automorphic representations. The vanishing of the associated Fourier coefficients precisely matches with the expected physical properties of quartic gravitational scattering amplitudes in string compactifications.

The focus of the present work is to study *massive theta lifts*. Theta lifts are given by certain integrals of automorphic forms, producing new automorphic objects. They have a representation-theoretic origin, generally known as *theta correspondences*, in which an automorphic form φ attached to a group G is transferred to an automorphic form φ' on another group G'. If $G \subset G'$ we call this a *lifting* of the automorphic form. In practice, we theta-lift by integrating the automorphic form φ against a theta function. A theta function is an example of a modular form corresponding to a *small* representation, as mentioned above. Theta lifts play an important role in mathematics, in particular through their explicit realization of the transfer of automorphic representations, known as the *principle of functoriality* in the Langlands program. On the other hand, theta lifts also appear naturally in string theory where they correspond to certain scattering amplitudes. More precisely string S-matrix elements are naturally computed as integrals over worldsheet moduli space, which is the double coset space $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. Integrating a Siegel-Narain theta function against this moduli space produces an automorphic form on the orthogonal group $\mathrm{SO}(d,d)$. The prototypical example of a theta lift of this form is given by

$$\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} \Theta_{d,d}(\tau;g)\frac{d^2\tau}{\tau_2^2}, \qquad g \in \mathrm{SO}(d,d), \tag{1.1}$$

where $\Theta_{d,d}(\tau;g)$ is a Siegel-Narain theta function on $\mathbb{H} \times \mathrm{SO}(d,d)/(\mathrm{SO}(d) \times \mathrm{SO}(d))$, with $\mathbb{H}$ being the standard upper half plane. This is a theta lift of the constant function 1, i.e. the trivial representation, and produces an automorphic form on $\mathrm{SO}(d,d)$ attached to the minimal representation. If we insert a non-trivial modular form $f(\tau)$ we can obtain a wider class of representations.

JHEP05(2023)062

The purpose of this paper is to analyze a "massive deformation" of the theta lift. Massive modular forms were introduced in [3, 4] (and references therein), with motivation from string theory. They are building blocks of string loop amplitudes in gravitational plane wave backgrounds. Plane wave geometries arise taking a limit of any curved background, where the curvature is determined by a free parameter μ. On the worldsheet, the parameter μ appears as a mass. For more details, see also the original paper on the BMN limit [5].

From a mathematical point of view, this is a one-parameter family of deformations of modular forms. We will consider theta lifts of these massive deformations. Rather than developing a general theory, our aim is to work out a few examples in detail. It is interesting to inquire about the role of representation theory here. We offer some comments on this question in the conclusions.

This paper is organized as follows. In section 2 we introduce some necessary background on modular forms (Jacobi forms and Maass forms), after which we introduce their massive deformations. We then consider the explicit Fourier expansion of the massive Eisenstein series. This result is needed later for the massive theta lifts. Section 3 introduces the Rankin-Selberg-Zagier transform, which is used to evaluate standard theta lifts. We consider the explicit examples of lifting both the trivial function 1 as well as the modular invariant j-function. The massive theta lift is introduced in section 4. Here we consider various approaches to computing the massive theta lift of the constant function. The following section 5 then treats the case of the massive theta lift of the j-function. We end in section 6 with some conclusions and suggestions for future work. The paper also contains several appendices with calculational details and various digressions.

2 Massive Maass-Jacobi forms

In this section we first introduce the necessary background on modular forms and their massive deformations (sections 2.1 and 2.2). In section 2.3 we then present the complete Fourier expansion of the massive non-holomorphic Eisenstein series. This result will be used in subsequent sections when we compute the massive theta lift.

2.1 Some preliminaries

A *modular form* of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ on the complex upper half-plane $\mathbb{H}$, that transforms according to $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ with respect to modular transformations. Modular forms admit a Fourier expansion $f(\tau) = \sum_{n\geq 0} c(n)q^n$, where $q = e^{2\pi i\tau}$. In this context the Fourier expansion is also referred to as a *q-expansion* since it can be written as a power series in q. As we will see below, this is not the case when we relax the condition of holomorphy, which will be crucial for the project. The $n = 0$ part of the Fourier sum is called the *constant term*.

As already alluded to, one can also consider modular forms that are not holomorphic; these are known as *Maass forms*. A classical example is the non-holomorphic Eisenstein series

$$E_s(\tau) = \frac{1}{2} \sum_{\gcd(m,n)=1} \frac{y^s}{|m+n\tau|^{2s}}, \qquad \tau = x+iy \in \mathbb{H}. \tag{2.1}$$

JHEP05(2023)062

This function is invariant under $\mathrm{SL}(2,\mathbb{Z})$ (i.e. of weight 0), and converges absolutely for $\Re(s) > 1$, but can be analytically continued to a meromorphic function of all $s \in \mathbb{C}$ away from a simple pole at $s = 1$. It is an eigenfunction of the Laplacian on $\mathbb{H}$ with eigenvalue $s(s-1)$. The non-holomorphic Eisenstein is a prototype of the kind of functions that are relevant for this paper. Let us therefore analyze its Fourier coefficients in some detail. Since it is a non-holomorphic function, the Fourier expansion is not a simple q-expansion.

The Fourier coefficients $c(m;\tau)$ of $E_s(\tau)$ now depend on $\tau = \tau_1 + i\tau_2$ and are indexed by $m \in \mathbb{Z}$. The non-constant coefficients ($m \neq 0$) are given by

$$c(m;\tau) = \int_0^1 E_s(\tau+u) e^{-2\pi i m u} du = \frac{2}{\xi(2s)} \tau_2^{1/2} |m|^{s-1/2} \sigma_{1-2s}(m) K_{s-1/2}(2\pi|m|\tau_2) e^{2\pi i \tau_1}, \tag{2.2}$$

where K_t is the modified Bessel function, $\xi(t)$ is the completed Riemann zeta function and $\sigma_t(m) = \sum_{d|m} d^t$ is the sum over divisors of m. The K-Bessel function appears as a solution to the Laplacian equation that respects the moderate-growth condition.

The non-holomorphic Eisenstein series discussed above is an example of a *Maass form*. A Maass form is a smooth, $\mathbb{C}$-valued function on the upper half plane $\mathbb{H}$, with prescribed transformation properties with respect to $\mathrm{SL}(2,\mathbb{Z})$ and at most polynomial growth at the cusps. It is furthermore required to be an eigenfunction with respect to certain differential operators on $\mathbb{H}$.

A *Jacobi form of weight k and index m* is a function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ that transforms as follows with respect to $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$:

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\nu}{c\tau+d}\right) = \exp\left[2\pi i \left(-c\frac{(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z\right)\right] (c\tau+d)^k \phi(\tau, z). \tag{2.3}$$

Examples include the Jacobi theta functions.

2.2 Massive deformations

Following [4], a *massive Maass form of weight k* is, for each $\mu \in \mathbb{R}_{\geq 0}$, a function $f_\mu : \mathbb{H} \to \mathbb{C}$ that transforms as a Maass form of weight k with respect to $\mathrm{SL}(2,\mathbb{Z})$, and satisfies the differential equation

$$\Delta_{\tau,k} f_\mu(\tau) = \Big(g_2(\mu)\partial_\mu^2 + g_1(\mu)\partial_\mu + g_0(\mu)\Big) f_\mu(\tau), \tag{2.4}$$

for certain functions $g_j : \mathbb{R}_{\geq 0} \to \mathbb{C}$, $j = 0, 1, 2$. In general, $f_\mu(\tau)$ should also satisfy a certain higher order differential equation, but we focus on the property (2.4) of $f_\mu(\tau)$. See [4] for more details. We say that $f_\mu(\tau)$ is a massive deformation of the weight k Maass form $f(\tau) := \lim_{\mu \to 0+} f_\mu(\tau)$.

In order to introduce massive Maass-Jacobi forms we consider the following differential operator

$$\Delta_{z,k,m} := 2\tau_2 \partial_z \partial_{\bar{z}} + 8\pi i \tau_2 m \partial_{\bar{z}} - 2\pi i m. \tag{2.5}$$

A *massive Maass-Jacobi form of weight k and index m* is, for each $\mu \in \mathbb{R}_{\geq 0}$, a function $\phi_\mu : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$, that transforms like a Jacobi form of weight k and index m and furthermore satisfies the condition

$$\Delta_{z,k,m}\phi_\mu(\tau,z) = -\Big(G_2(\mu)\partial_\mu^2 + G_1(\mu)\partial_\mu + G_0(\mu)\Big)\phi_\mu(\tau,z), \tag{2.6}$$

for certain functions $G_j : \mathbb{R}_{\geq 0} \to \mathbb{C}$, $j = 0, 1, 2$.

As an example, consider the following function

$$\mathcal{E}_{s,\mu}(\tau) := \frac{2\pi^s}{\Gamma(s)}(\mu\tau_2)^{s/2} \sum_{(m,n)=1}' \left(\frac{\sqrt{\mu\tau_2}}{|m\tau+n|}\right)^s K_s\left(2\pi\sqrt{\frac{\mu}{\tau_2}}|m\tau+n|\right). \tag{2.7}$$

The prime on the sum indicates that the term $(m,n) = (0,0)$ is absent, and we only sum over coprime m and n. This is trivially a massive Maass-Jacobi form of weight 0 and index 0 (it has no elliptic parameter z), and in particular (2.7) is a massive Maass form. It will be our main test case, but we believe that the calculations below can be of interest also for more general classes of massive Maass-Jacobi forms than (2.7).

In appendix B we also discuss massive deformations of some holomorphic modular forms.

2.3 Fourier expansion of massive Eisenstein series

It is convenient to rewrite the massive Eisenstein series as a Poincaré series as follows:

$$\mathcal{E}_{s,\mu}(\tau,z) = \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}(2,\mathbb{Z})} \sigma_\mu(\gamma\cdot\tau_2), \tag{2.8}$$

where we have defined the *massive seed function*

$$\sigma_\mu(\tau_2) := \frac{2\pi^s}{\Gamma(s)}(\mu\tau_2)^{s/2} K_s(2\pi\sqrt{\mu/\tau_2}). \tag{2.9}$$

Just like for ordinary non-holomorphic Eisenstein series the seed function has no τ_1 dependence, which simplifies the computation of the Fourier modes.

The constant term (in τ_1) is the sum of the seed $\sigma_\mu(\tau_2)$ and the $n = 0$ Fourier mode:

$$\begin{aligned} f_{\mu,0}(\tau_2) &= \sigma_\mu(\tau_2) + \frac{\pi^s}{\Gamma(s)}\sum_{c>0} S(0,0;c)\int_{\mathbb{R}} 2\left(\frac{\mu\tau_2}{c^2(\tilde\omega^2+\tau_2^2)}\right)^{s/2} K_s\left(2\pi\sqrt{\frac{\mu}{\tau_2/(c^2(\tilde\omega^2+\tau_2^2))}}\right)d\tilde\omega \\ &= \sigma_\mu(\tau_2) + \frac{\pi^s}{\Gamma(s)}\sum_{c>0}\varphi(c)c^{-s-1/2}\cdot 2\tau_2^{3/4-s/2}\mu^{s/2-1/4}K_{s-1/2}(2\pi c\sqrt{\mu\tau_2}). \end{aligned}$$

Viewed as a deformation of the usual Eisenstein series at $\mu = 0$, we can attempt to series expand in μ. Note, however, that this ruins the exponential suppression of the Bessel function at large arguments, and that even for small μ, there will be some large c or large τ_2 where the series is not a good approximation to the original expression. If we proceed formally anyway, the sum has the following series expansion in μ:

$$f_{\mu,0}(\tau_2) - \sigma_\mu(\tau_2) = \frac{\xi(2s-1)}{\xi(2s)}\tau_2^{1-s} - \frac{\pi^2\xi(4-2s)}{(s-1)\xi(3-2s)}\mu\tau_2^{2-s} + \frac{\pi^4\xi(6-2s)}{2(s-2)(s-1)\xi(5-2s)}\mu^2\tau_2^{3-s} + \ldots$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, and we used ($\varphi$ is the Euler totient function)

$$\sum_{c>0}\frac{S(0,0;c)}{c^{2s}} = \sum_{c>0}\frac{\varphi(c)}{c^{2s}} = \frac{\zeta(2s-1)}{\zeta(2s)}. \tag{2.10}$$

In particular, there are a priori odd powers of c, that come with odd powers of $\sqrt{\mu}$, but by eq. (2.10), they sum to zero. Note that the pole at $s = 1$ now receives corrections order by order in μ. For the $s = 1$ case, note that $(s-1)\zeta(3-2s)$ is finite, but all higher terms contribute to the residue.

For the nonzero Fourier modes we find:

$$\begin{aligned} f_{\mu,n}(\tau_2) &= \frac{\pi^s}{\Gamma(s)} \sum_{c>0} S(0,n;c) \int_{\mathbb{R}} e^{-2\pi i n\tilde{\omega}}\, 2\left(\frac{\mu\tau_2}{c^2(\tilde{\omega}^2+\tau_2^2)}\right)^{s/2} K_s\left(2\pi\sqrt{\frac{\mu}{\tau_2/(c^2(\tilde{\omega}^2+\tau_2^2))}}\right) d\tilde{\omega} \\ &= \frac{\pi^s}{\Gamma(s)} \sum_{c>0} S(0,n;c) c^{-2s} \cdot 2\tau_2^{3/4-s/2}(n^2\tau_2+\mu c^2)^{s/2-1/4} K_{s-1/2}(2\pi\sqrt{\tau_2}\sqrt{n^2\tau_2+\mu c^2}) \end{aligned}$$

where we have used the Bessel identity in appendix A. Now, if we write each Fourier coefficient as

$$f_{\mu,n}(\tau_2) = f_n^{(0)}(\tau_2) + f_n^{(1)}(\tau_2)\mu + f_n^{(2)}(\tau_2)\mu^2 + f_n^{(3)}(\tau_2)\mu^3 + \dots \tag{2.11}$$

we have

$$\begin{aligned} f_n^{(0)}(\tau_2) &= \frac{2}{\xi(2s)} \tau_2^{1/2} |n|^{s-1/2} \sigma_{1-2s}(n) K_{s-1/2}(2\pi|n|\tau_2) \\ f_n^{(1)}(\tau_2) &= \frac{\pi}{(s-1)\xi(3-2s)} \tau_2^{-1/2} |n|^{s-5/2} \sigma_{3-2s}(n) \\ &\quad \times \left(c_-^{(1)} K_{s-1/2}(2|n|\pi\tau_2) + c_+^{(1)} K_{s+1/2}(2\pi|n|\tau_2)\right) \\ f_n^{(2)}(\tau_2) &= \frac{\pi^2}{4(s-1)(s-2)\xi(5-2s)} \tau_2^{-3/2} |n|^{s-9/2} \sigma_{5-2s}(n) \\ &\quad \times \left(c_-^{(2)} K_{s-1/2}(2|n|\pi\tau_2) + c_+^{(2)} K_{s+1/2}(2\pi|n|\tau_2)\right) \\ f_n^{(3)}(\tau_2) &= \frac{\pi^3}{24(s-1)(s-2)(s-3)\xi(7-2s)} \tau_2^{-5/2} |n|^{s-13/2} \sigma_{7-2s}(n) \\ &\quad \times \left(c_-^{(3)} K_{s-1/2}(2|n|\pi\tau_2) + c_+^{(3)} K_{s+1/2}(2\pi|n|\tau_2)\right) \end{aligned} \tag{2.12}$$

where the coefficients of $K_{s-1/2}$ and $K_{s+1/2}$ at each order in μ are, with the shorthand $x = 2\pi|n|\tau_2$,

$$\begin{aligned} c_-^{(1)} &= 1-2s & c_+^{(1)} &= -x \\ c_-^{(2)} &= 4(s-2)s + x^2 + 3\,, & c_+^{(2)} &= (3-2s)x\,. \\ c_-^{(3)} &= (2s-3)(4(s-3)s + 2x^2 + 5)\,, & c_+^{(3)} &= -x(4(s-4)s + x^2 + 15)\,. \end{aligned} \tag{2.13}$$

Here we used that the sum $S(0,n;c)$ produces (σ is the divisor function)

$$\sum_{c>0} \frac{S(0,n;c)}{c^{2s}} = \frac{\sigma_{1-2s}(n)}{\zeta(2s)}\,. \tag{2.14}$$

Both here and for the constant term, we the kept the coefficient $f_n^{(0)}(\tau_2)$ of the zeroth order term in its original form, but for the higher-order terms we used the functional relation for ζ, so that the argument of the zeta function increases with order in μ.

JHEP05(2023)062

A minor comment is that the order μ term can be simplified to a single Bessel function $K_{s-3/2}$, but we have chosen to keep this form to exhibit a pattern that persists to higher order: the expansion in μ can be arranged as an expansion of the coefficients of $K_{s-1/2}$ and $K_{s+1/2}$.

The zeroth order (in μ) coefficient $f_n^{(0)}(\tau_2)$ vanishes as $s \to 0$. We remind the reader that the usual Eisenstein series has an apparently ill-defined Poincaré series (2.8) as $s \to 0$, whereas the Fourier expansion in τ_1 gives the value 1 for the ordinary massless Eisenstein series as $s \to 0$.

The polynomials that are generated for $s \to 0$ are reverse Bessel polynomials that are generated by

$$\frac{1}{\sqrt{1-2t}} e^{x(1-\sqrt{1-2t})} = \sum_{n=0}^{\infty} \theta_n(x) \frac{t^n}{n!} \,, \tag{2.15}$$

with $\theta_n(x) = \sqrt{2/\pi} x^{n+1/2} e^x K_{n+1/2}(x)$.

For the $s \to 1$ case, note as above that $(s-1)\zeta(3-2s)$ is finite at $s=1$, but all higher terms contribute to the residue. The residue can be subtracted order by order in μ, as in the massless case, that produces a finite remainder with a Fourier series that converges rapidly due to the Bessel functions with n in the argument. It is somewhat complicated and we do not reproduce it here since it is not needed: as we already remarked, it is better to keep the Bessel function unexpanded.

One final comment: for $\mu = 0$ the seed $\sigma_{\mu=0}$ is a power of τ_2, and the integral and summation over c for the constant term is a power of τ_2, whereas the nonzero modes produce a Bessel function $K_{s-1/2}$. Here for $\mu \neq 0$, the seed is a Bessel function K_s and result of the integration and summation over c for the nonzero is a Bessel function $K_{s-1/2}$ both for zero mode and the nonzero modes. So for the massive Eisenstein series, the distinction between zero modes and nonzero modes is less pronounced than for $\mu = 0$.

3 Theta lifts and the Rankin-Selberg-Zagier transform

The Rankin-Selberg method is a general approach for constructing new automorphic L-functions by integrating the product of two modular forms over a group. The Rankin-Selberg-Zagier (RSZ) transform makes use of this idea as a way of regularizing a theta-type integral. The basic idea is to deform an integral by inserting a non-holomorphic Eisenstein series in the integrand, thus creating a family of integrands parametrized by $s \in \mathbb{C}$. The original integral is obtained by analytically continuing the result to $s = 0$ or $s = 1$. We start this section by giving a brief summary of the RSZ-transform, after which we show how it can be applied to calculate theta lifts. In subsequent sections we will then generalize these constructions to the massive theta lifts.

3.1 The RSZ-transform

In this section we consider the basic integral of a modular invariant function $F(\tau)$ over the fundamental domain $\mathcal{F} = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$:

$$\int_{\mathcal{F}} F(\tau) \, \frac{d\tau d\bar{\tau}}{\tau_2^2} \,. \tag{3.1}$$

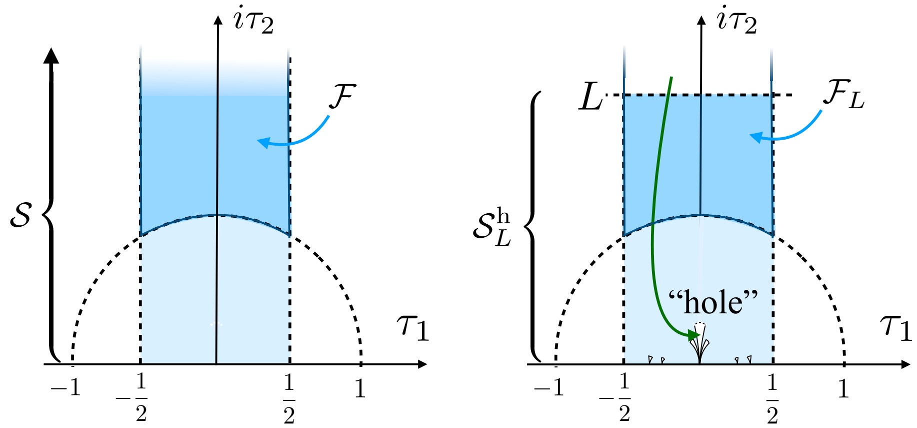

Figure 1. Left: full strip $\mathcal{S} : -1/2 < \tau_1 < 1/2$. Right: cutoff fundamental domain $\mathcal{F}_L$. The region $\tau_2 > L$ above the cutoff maps under Γ to "holes" in the full strip. The strip $\mathcal{S}$ minus the holes is the "holey" strip $\mathcal{S}^{\rm h}_L$. A few representative holes are drawn.

For functions that are not of rapid decay, Zagier [2] introduced a cutoff fundamental region $\mathcal{F}_L$ with $\tau_2 \le L$, as in figure 1. (See also [6] for a nice physics-oriented review.) The RSZ transform of an $\mathrm{SL}(2,\mathbb{Z})$-invariant function $F(\tau)$ is given by introducing an Eisenstein series $E_s(\tau)$ into the integral (for Re $s > 1$) and writing it as a Poincaré series that can be realized as a single term by unfolding the integral to the strip:

$$R_L(F,s) = \int_{\mathcal{F}_L} F(\tau)E_s(\tau)\,\frac{d\tau d\bar\tau}{\tau_2^2} = \int_{\mathcal{F}_L} F(\tau) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\mathrm{Im}(\gamma\cdot\tau))^s\,\frac{d\tau d\bar\tau}{\tau_2^2} = \int_{\mathcal{S}^{\rm h}_L} F(\tau)(\mathrm{Im}\,\tau)^s\,\frac{d\tau d\bar\tau}{\tau_2^2}\,. \tag{3.2}$$

For $L \to \infty$, the domain of integration is unfolded to the full strip $\mathcal{S}$ as in the left panel of figure 1. However, for finite L, the unfolded domain $\mathcal{S}^{\rm h}_L$ is *not* simply the strip cut off at the top, since it has "holes" due to the cutoff. We can expand in $1/L$ to isolate divergences, and subtract them to construct a renormalized $R^{\rm ren}_L(F,s)$. If we now take $L \to \infty$, the holes are filled in.[1]

We assume that the modular invariant function $F(\tau)$ has a Fourier expansion in τ_1:

$$F(\tau) = f(\tau_2) + \sum_n g_n(\tau_2)e^{2\pi i n\tau_1}\,. \tag{3.3}$$

Since the seed function $(\mathrm{Im}\,\tau)^s = \tau_2^s$ is independent of τ_1, and $\int_{-1/2}^{1/2} e^{2\pi i n\tau_1} d\tau_1 = 0$ for $n \neq 0$, the τ_1 integral projects out the non-zero mode sum, and we only have the zero mode left:

$$R_\infty(F,s) = \int_0^\infty \tau_2^{s-2} f(\tau_2)d\tau_2 \tag{3.4}$$

which is a Mellin transform of the zero-mode f of F. Recall from above that Re $s > 1$. Note also that the Fourier projection $\int_{-1/2}^{1/2} e^{2\pi i n\tau_1} d\tau_1 = 0$ for $n \neq 0$ applies for the *full* strip $\mathcal{S}$, not for the "holey" strip $\mathcal{S}^{\rm h}_L$.

[1] We thank Federico Zerbini for discussions on this point.

If we are only interested in the original integral eq. (3.1) over the fundamental domain, the clever Rankin-Selberg trick is that the Eisenstein series has a pole at $s = 1$ with τ_2-independent residue, so we can extract the value of the integral (3.1) from the residue of eq. (3.4). However, for some purposes (e.g. theta lifts) the full RSZ transform $R(F, s)$ (i.e. including the Eisenstein series E_s in the integrand) is of interest, so we do not view the residue of eq. (3.4) as the only interesting property of the RSZ transform $R(F, s)$.

In this paper, we mainly integrate functions F that decay exponentially as $\tau_2 \to \infty$, except possibly for some degenerate piece that can be handled separately. We can then take the $L \to \infty$ limit with less effort than for more general falloff conditions. Nevertheless, in a few places we will find it useful to compare to Zagier's approach.

3.2 Theta lift

By the motivations discussed in the introduction, we want to Rankin-Selberg transform the $d = 2$ Siegel-Narain theta function:

$$\Theta_{2,2}(\tau) = \tau_2 \sum_{k\in\mathbb{Z}^{2*},n\in\mathbb{Z}^2} \exp\Big(-\pi\tau_2\underbrace{((k+Bn)G^{-1}(k+Bn)+nGn)}_{\mathcal{M}}+2\pi i\tau_1\langle k,n\rangle\Big)\,, \tag{3.5}$$

with real 2×2 matrices G (symmetric) and B (antisymmetric). The zero mode $f(\tau_2)$ of $F = \Theta_{2,2}$ consists of the terms (k^1, k^2, n_1, n_2) for which $\langle k, n\rangle = k^1n_1 + k^2n_2 = 0$. The Poincaré series seed function for E_s with our normalization is simply τ_2^s, so we have:

$$R_L(\Theta, s) = \int_{\mathcal{F}_L} \Theta_{2,2}(\tau)E_s(\tau)\,\frac{d\tau d\bar\tau}{\tau_2^2} = \int_{\mathcal{S}_L^{\rm h}} \Theta_{2,2}(\tau)\tau_2^s\,\frac{d\tau_1 d\tau_2}{\tau_2^2}. \tag{3.6}$$

If Re $s > 1$, we can send $L \to \infty$ and perform the τ_1 integral:

$$R_\infty(\Theta, s) = \sum_{\substack{k\in\mathbb{Z}^{2*},n\in\mathbb{Z}^2\\ \langle k,n\rangle=0}} \int_0^\infty \tau_2^{s-1}e^{-\pi\tau_2\mathcal{M}}d\tau_2 = \frac{\Gamma(s)}{\pi^s}\sum_{\substack{k\in\mathbb{Z}^{2*},n\in\mathbb{Z}^2\\ \langle k,n\rangle=0}}\frac{1}{\mathcal{M}^s} =: E_{V,s}\,, \tag{3.7}$$

with $E_{V,s}$ the *lattice Eisenstein series*. Here $\mathcal{M}$ implicitly depends on complex moduli T and U that come from the real matrices G and B: we will make the moduli dependence of $E_{V,s}$ (and some of its generalizations) explicit in eq. (3.14) below. Here, we only need the property (3.25) below: for Re $s > 1$, the lattice Eisenstein series $E_{V,s}$ factors into one factor depending on the modulus T and one factor depending on the modulus U:

$$E_{V,s}(T, U) = \pi^{-s}\Gamma(s)\zeta(2s)E_s(T)E_s(U)\,, \tag{3.8}$$

which representation-theoretically comes from $O(2,2) \sim \mathrm{SL}(2)\times\mathrm{SL}(2)\ltimes\mathbf{Z}_2$. We can take the $s \to 1$ limit by Kronecker's first limit formula: each E_s factor has a single pole at $s = 1$, which in our normalization is eq. (10.10) in the book [15]:

$$E_s(\tau) = \frac{3}{\pi(s-1)} - \frac{6}{\pi}(\log(\sqrt{\tau_2}|\eta(\tau)|^2 + \text{constant}) + \mathcal{O}(s-1) \tag{3.9}$$

so by combining eqs. (3.8) and (3.9), the single pole of the Rankin-Selberg transform at $s = 1$ arises as a cross term:

$$E_{V,s} = \frac{3}{2\pi(s-1)^2} - \frac{3}{\pi(s-1)}(\log(\sqrt{T_2U_2}|\eta(T)\eta(U)|^2) + \text{constant}) + \mathcal{O}((s-1)^0) \tag{3.10}$$

and by the Rankin-Selberg trick mentioned above, the simple pole gives the moduli dependence of the classic Dixon-Kaplunovsky-Louis threshold correction $\Delta_{\rm DKL}$ [23]:

$$\Delta_{\rm DKL}(T,U) = \frac{\pi}{3}\operatorname*{Res}_{s=1} R_\infty(\Theta,s) = -\frac{1}{2}\log(T_2U_2|\eta(T)\eta(U)|^4) + \text{constant}\,, \tag{3.11}$$

that only differs from the original DKL result by overall normalization and an additive renormalization-scheme dependent constant.

To summarize, since we allowed for generic s, i.e. allow for poles in the transform, there was no need for cutoff L in the RSZ transform of the Siegel-Narain theta function. This is analogous to dimensional regularization.

Alternatively, we could have fixed $s=1$ and continued with the cutoff L, with the result that $\Gamma(s)$ would have been replaced by $\Gamma(s)-\Gamma(s,\pi L\mathcal{M})$, where $\Gamma(s,x)$ is the (lower) incomplete gamma function and $\Gamma(1,x)=e^{-x}$. But then the application of the factorization (3.8) is less direct.

3.3 Selberg-Poincaré series

The Siegel-Narain Θ is a sum of exponentially suppressed terms and converges quickly. In the previous section we encountered its theta lift: the lattice Eisenstein series $E_{\rm V,s}$. For later use, we will introduce an integer superscript ℓ and refer to the previous case as $\ell=0$:

$$\int \Theta(T,U,\tau)E_s^{(\ell)}(\tau)\frac{d^2\tau}{\tau_2^2} = E_{{\rm V},s}^{(\ell)}(T,U) \tag{3.12}$$

where the Selberg-Poincaré series [30] is a Poincaré series with seed $\tau_2^s q^{-\ell}$

$$E_s^{(\ell)}(\tau) = \frac{1}{2\zeta(s)}\sum_{c,d}\frac{\tau_2^s}{|c\tau+d|^{2s}}(\widetilde{q})^{-\ell} \tag{3.13}$$

with $\widetilde{q}=e^{2\pi i\widetilde{\tau}}=e^{2\pi i(a\tau+b)/(c\tau+d)}$, and a,b are determined from $ad-bc=1$. Clearly for $\ell=0$, $E_s^{(0)}(\tau)=E_s(\tau)$ reduces to the ordinary nonholomorphic Eisenstein series. For $\ell=1$, the Selberg-Poincaré series $E_s^{(1)}(\tau)$ provides a non-holomorphic regularization of Klein's modular invariant j function as a Poincaré series, that becomes holomorphic as $s\to 0$. For $\ell>1$, $E_s^{(\ell)}(\tau)$ can be obtained from Hecke operators acting on j [6].[2] The result of unfolding the Siegel-Narain Θ against the Selberg-Poincaré series $E_s^{(\ell)}$ in (3.12) is the quadruple sum constrained lattice Eisenstein series

$$E_{{\rm V},s}^{(\ell)}(T,U) = \sum_{k\in\mathbb{Z}^{2*},n\in\mathbb{Z}^2}\frac{\delta(|p_{\rm L}|^2-|p_{\rm R}|^2-4\ell)}{(|p_{\rm L}|^2+|p_{\rm R}|^2-4\ell)^s}\,,\qquad \operatorname{Re}s>2 \tag{3.14}$$

where the momenta $p_{\rm L}$, $p_{\rm R}$ give the dependence on the moduli T,U as

$$p_{\rm R} = \frac{n_2-Un_1+Tk^1+TUk^2}{\sqrt{T_2U_2}}\,,\qquad p_{\rm L} = \frac{n_2-Un_1+\overline{T}k^1+\overline{T}Uk^2}{\sqrt{T_2U_2}}\,. \tag{3.15}$$

As noted, the quadruple sum in (3.14) generically only converges for Re $s>2$. In particular, unlike for the ordinary nonholomorphic Eisenstein series, if we are interested in $s=1$, we

[2] For example for $s=0$, we have $E_0^{(\ell)}=T_\ell j+12\sigma_1(\ell)$ (see [6] for more details).

cannot set $s = 1+\epsilon$ with ϵ small for convergence. In the following section we view (3.12) as an integral representation that analytically continues the Selberg-Poincaré series (3.14) to all s.

In the remainder of this section, we review how the constraint $|p_\mathrm{L}|^2 - |p_\mathrm{R}|^2 - 4\ell = 0$ in (3.14) arises, how to perform the sum, and give the moduli dependence in more detail. In complex coordinates T, U, the Siegel-Narain theta function (3.5) becomes

$$\Theta(T,U,\tau) = \tau_2 \sum_{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2} q^{\frac{1}{4}|p_\mathrm{L}|^2} \bar{q}^{\frac{1}{4}|p_\mathrm{R}|^2} = \tau_2 \sum_{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2} e^{\frac{\pi i \tau_1}{2}(|p_\mathrm{L}|^2-|p_\mathrm{R}|^2)} e^{-\frac{\pi\tau_2}{2}(|p_\mathrm{L}|^2+|p_\mathrm{R}|^2)} \tag{3.16}$$

and we note that the coefficient of $\pi i \tau_1/2$ in the exponent is

$$|p_\mathrm{L}|^2 - |p_\mathrm{R}|^2 = -4\langle k, n\rangle = -4(k^1 n_1 + k^2 n_2)\,. \tag{3.17}$$

The τ_1 integral in (3.12) becomes (essentially giving the ℓth Fourier coefficient of τ_1 of the Siegel-Narain theta function Θ)

$$\begin{aligned}\int_{-1/2}^{1/2} d\tau_1\, e^{2\pi i\ell\tau_1}\Theta(\tau) &= \int_{-1/2}^{1/2} d\tau_1\, \tau_2 e^{2\pi i \ell\tau_1} e^{2\pi i \tau_1\langle k,n\rangle} \sum_{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2} \tau_2 e^{-\frac{\pi}{2}\tau_2(|p_\mathrm{L}|^2+|p_\mathrm{R}|^2)} \\ &= \sum_{\langle k,n\rangle=-\ell} \tau_2 e^{-\frac{\pi}{2}\tau_2(|p_\mathrm{L}|^2+|p_\mathrm{R}|^2)}\end{aligned} \tag{3.18}$$

so we have a quadruple sum over (k^1, k^2, n_1, n_2) with a single constraint $\langle k, n\rangle = k^1 n_1 + k^2 n_2 = -\ell$. We can perform the quadruple sum while enforcing this condition, or we can solve the constraint as discussed below.

Finally including also the Selberg-Poincaré series in (3.12) we have

$$\int \Theta(T,U,\tau) E_s^{(\ell)}(\tau)\frac{d^2\tau}{\tau_2^2} = \int_0^\infty \tau_2^{s-1} e^{2\pi\ell\tau_2} \sum_{\langle k,n\rangle=-\ell} e^{-\frac{\pi}{2}\tau_2(|p_\mathrm{L}|^2+|p_\mathrm{R}|^2)} d\tau_2. \tag{3.19}$$

As already emphasized, we cannot move the sum out of the integral for $s \le 2$, which for many purposes is the region of interest. We discuss analytic continuation in appendix C.

3.3.1 Degenerate constraint $\langle k, n\rangle = 0$

We need to solve the summation constraint $\langle k, n\rangle = k^1 n_1 + k^2 n_2 = 0$. There are two cases:

Case 1. If $k^1 = k^2 = 0$, n_1, n_2 are unconstrained. We find

$$|p_\mathrm{L}|^2 + |p_\mathrm{R}|^2 = \frac{2}{T_2 U_2}|n_1 U - n_2|^2\,. \tag{3.20}$$

This gives a contribution to (3.19) that is

$$\begin{aligned}&\int_0^\infty \tau_2^{s-1}\sum_{n_1,n_2} e^{-\frac{\pi}{2}\tau_2(|p_\mathrm{L}|^2+|p_\mathrm{R}|^2)} d\tau_2 = \int_0^\infty \tau_2^{s-1} \sum_{n_1,n_2} e^{-\frac{\pi\tau_2}{T_2U_2}|n_1U-n_2|^2} \\ &\overset{*}{=} \pi^{-s}\Gamma(s) T_2^s \sum_{n_1,n_2} \frac{U_2^s}{|n_1U-n_2|^{2s}} = \pi^{-s}\Gamma(s)\zeta(2s) T_2^s E_s(U)\end{aligned} \tag{3.21}$$

where $\stackrel{*}{=}$ assumes $\mathrm{Re}\, s > 1$. The $\zeta(2s)$ comes in because our $E_s(U)$ is defined for coprime summation integers. Now, (3.21) is only a single term in the (k^1, k^2) double sum, so naturally it has the convergence properties of a double sum in (n_1, n_2), as opposed to a quadruple sum over (k^1, k^2, n_1, n_2). In appendix C we review the textbook calculation of giving an analytic continuation for double sums, instead of the step labelled "$\stackrel{*}{=}$".

Case 2. View n_1 and n_2 as fixed. If they are not coprime, we factor out the biggest common factor $(n_1, n_2) = c(n_1^*, n_2^*)$, where c can be positive or negative. Now $k^1 n_1 + k^2 n_2 = 0$ becomes $k^1 n_1^* + k^2 n_2^* = 0$ (notice this is not true for the nondegenerate constraint $\langle k, n\rangle = -\ell \neq 0$), and we have

$$k^1 = -\frac{k^2 n_2^*}{n_1^*}\,. \tag{3.22}$$

Since (n_1^*, n_2^*) have no common factors, for this to be integer we must have $k^2/n_1^* = d$ integer for $d \geq 1$. (Including also $k^2 = -dn_1^*$ would overcount, and $d = 0$ is covered by Case 1.)

To summarize, $(k^1, k^2) = d(-n_2^*, n_1^*)$ and $(n_1, n_2) = c(n_1^*, n_2^*)$. We find that the exponent factorizes:

$$|p_{\mathrm{L}}|^2 + |p_{\mathrm{R}}|^2 = \frac{2}{T_2 U_2}|n_2^* U - n_1^*|^2 |dT - c|^2. \tag{3.23}$$

This gives a contribution to (3.19) that is

$$\begin{aligned}
&\int_0^\infty \tau_2^{s-1} \sum_{d\geq 1, c} \sum_{\gcd(n_1^*, n_2^*)=1} e^{-\frac{\pi\tau_2}{T_2 U_2}|n_2^* U - n_1^*|^2 |dT - c|^2} d\tau_2 \\
&\stackrel{*}{=} \pi^{-s}\Gamma(s) \sum_{d\geq 1, c} \frac{T_2^s}{|dT - c|^{2s}} \sum_{\gcd(n_1^*, n_2^*)=1} \frac{U_2^s}{|n_2^* U - n_1^*|^{2s}}
\end{aligned} \tag{3.24}$$

where again $\stackrel{*}{=}$ only holds for $\mathrm{Re}\ s > 1$. Adding (3.21) and (3.24) fills in the missing $d = 0$ term to make $E_s(T)$ and the total is

$$\int \frac{d^2\tau}{\tau_2^2} \Theta(T, U, \tau) E_s^{(0)}(\tau) = E_{\mathrm{V},s}^{(0)}(T, U) = \pi^{-s}\Gamma(s)\zeta(2s) E_s(T) E_s(U) = E_s^\star(T) E_s(U)\,, \tag{3.25}$$

where the factor $\zeta(2s)$ came from (3.21), and similarly for (3.24).

We can also consider the completed transform, that includes a factor of $\xi(2s) = \pi^{-s}\Gamma(s)\zeta(2s)$ in front, that makes both Eisenstein series completed:

$$\xi(2s) \int \Theta(T, U, \tau) E_s^{(0)}(\tau) \frac{d^2\tau}{\tau_2^2} = \xi(2s) E_{\mathrm{V},s}^{(0)}(T, U) = E_s^\star(T) E_s^\star(U)\,. \tag{3.26}$$

The additional factor $\xi(2s)$ supplies a double pole at $s = 0$. The expression is now symmetric under application of the functional relation: $E_{1-s}^\star = E_s^\star$, which implies for the lattice Eisenstein series that

$$\xi(2(1-s)) E_{\mathrm{V},1-s}^{(0)}(T, U) = \xi(2s) E_{\mathrm{V},s}^{(0)}(T, U). \tag{3.27}$$

JHEP05(2023)062

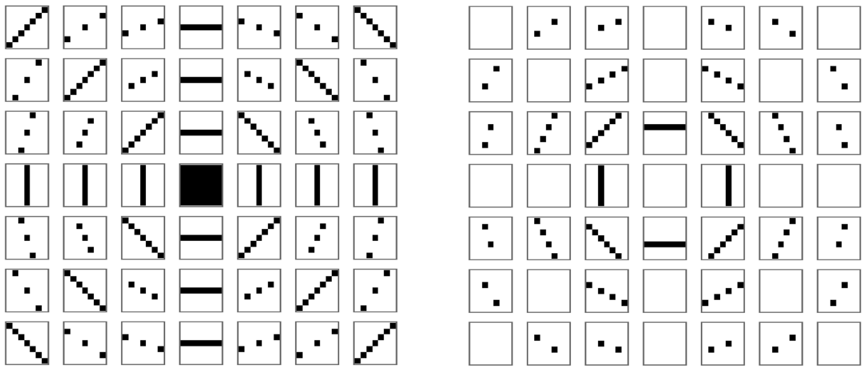

Figure 2. Each element in the 7×7 (n_1, n_2) matrix is a 7×7 (k^1, k^2) matrix. The black squares are where the constraint is satisfied: $\langle k, n\rangle = 0$ in the left panel and $\langle k, n\rangle = -1$ in the right panel.

Using the functional relation of the completed Riemann zeta function $\xi(s) = \xi(1-s)$ we can also write this in the more familiar form from the $\mathrm{SL}(2,\mathbb{R})$ non-holomorphic Eisenstein series:

$$E^{(0)}_{\mathrm{V},s}(T,U) = \frac{\xi(2s-1)}{\xi(2s)} E^{(0)}_{\mathrm{V},1-s}(T,U). \tag{3.28}$$

3.3.2 Non-degenerate case $\langle k, n\rangle = -\ell$ for $\ell > 0$

In the non-degenerate case $\langle k, n\rangle = -\ell$ we solve the constraint as follows.

Given coprime (n_1^*, n_2^*), the solution of $\langle k, n\rangle = -\ell$ is $n_1 = n_1^* + Mk^2$, $n_2 = n_2^* - Mk^1$, since

$$k^1 n_1 + k^2 n_2 = k^1(n_1^* + Mk^2) + k^2(n_2^* - Mk^1) = k^1 n_1^* + k^2 n_2^* = -\ell. \tag{3.29}$$

From the point of view of solving, this is quite different from above. We will see in the next section that the fiducial (starred) solution appears in the inconsequential upper elements of the $\mathrm{SL}(2,\mathbb{Z})$ image, whereas above, they were summed over and of consequence as the $\mathrm{SL}(2,\mathbb{Z})$ sum producing the Eisenstein series of U.

(The opposite constraint $\langle k, n\rangle = \ell$ formally has the same solution $k_1 = k_1^* + Mn^2$, $k_2 = k_2^* - Mn^1$ as above, but the numbers that come out are different.)

In figure 2 we show the entries that are included. The constraint $\langle k, n\rangle = -\ell$ is a plane in 4 dimensions. In each truncated little 7×7 block in the figure, the constraint is satisfied along a straight line. The angle of this plane is different for different blocks, hence the appearance of "circles". We will have occasional use of truncations of our infinite sums, and these graphical representation can provide some visual checks that truncated summations include a sufficient number of terms in the right range. For more details, see appendix D.

3.4 Theta lift of the j-function

Now we want to RSZ-transform $j(\tau)\Theta_{2,2}(\tau)$:

$$j(\tau)\Theta_{2,2}(\tau) = \\ \tau_2 \sum_{\ell=-1}^{\infty} c(\ell) e^{-2\pi\ell\tau_2} e^{2\pi i\ell\tau_1} \sum_{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2} \exp\Big(-\pi\tau_2 \underbrace{\big((k+Bn)G^{-1}(k+Bn) + nGn\big)}_{\mathcal{M}} + 2\pi i\tau_1\langle k, n\rangle\Big), \tag{3.30}$$

so the zero mode is not given by $\langle k,n\rangle = 0$ as it was without the j, but now by $\langle k,n\rangle = -\ell$. Note that the $\ell = 0$ term is the same as before up to an overall constant, so we can set $c(0) = 0$ for now, and j with $c(0) = 0$ is denoted J. (We return to $c(0)$ below.) We find

$$R_L(J\Theta, s) = \int_{\mathcal{F}_L} J(\tau)\Theta_{2,2}(\tau)E_s(\tau)\,\frac{d\tau d\bar{\tau}}{\tau_2^2} = \int_{\mathcal{S}_L^{\mathrm{h}}} J(\tau)\Theta_{2,2}(\tau)\tau_2^s\frac{d\tau_1 d\tau_2}{\tau_2^2}\,. \tag{3.31}$$

Taking $L \to \infty$, and provided we can use the q expansion of J:

$$\int_0^\infty \sum_{\ell=-1}^{\infty} c(\ell)e^{-2\pi\ell\tau_2}\tau_2 \sum_{\substack{k\in\mathbb{Z}^{2*},n\in\mathbb{Z}^2\\ \langle k,n\rangle=-\ell}} e^{-\pi\tau_2\mathcal{M}}\tau_2^{s-2}d\tau_2 = \sum_{\ell=-1}^{\infty} c(\ell) \sum_{\substack{k\in\mathbb{Z}^{2*},n\in\mathbb{Z}^2\\ \langle k,n\rangle=-\ell}} \int_0^\infty \tau_2^{s-1}e^{-\pi\tau_2(\mathcal{M}+2\ell)}d\tau_2 \tag{3.32}$$

The $\ell = -1$ term (the $1/q$ pole in j) appears to cause the integral to diverge, but the minimum value of $\mathcal{M}$ for specific moduli is 2, so for generic moduli $\mathcal{M} > 2$, eq. (3.32) still falls off rapidly at $\tau_2 \to \infty$ as it did without the J function. However, the $\tau_2 \to 0$ side is unsuppressed.

There is the *singular theta lift* to take care of the new divergences compared to case of the theta lift of the constant function [22, 24, 25]. From the physics point of view, this leads to the Harvey-Moore threshold correction (see e.g. the book [15], section 13.4)

$$\Delta_{\mathrm{HM}}(T,U) = \frac{\pi}{3}\operatorname*{Res}_{s=0} R_\infty(J\Theta, s) = -2\log|J(T)-J(U)|^2\,. \tag{3.33}$$

This is not regularized for $T = U$.

An explicit way to regularize the theta lift and give meaning to (3.32) is to use Niebur-Poincaré series [17]. In section 5, we offer a few comments on how Niebur-Poincaré series compare to what we do here.

4 Massive theta lift

After this brief review of standard theta lifts, we are ready to introduce the massive theta lift. We consider the theta integral with the insertion of a massive Eisenstein series, in the spirit of the RSZ transform. We start by discussing the general structure of such integrals, after which we explicitly calculate the lift of the constant function, and in a later section a j-lift.

4.1 Massive RSZ in general

We replace the Eisenstein series E_s in the standard RSZ transform (3.6) with the massive Eisenstein series $\mathcal{E}_{s,\mu}$ from [4]. Like for massless, massive RSZ unfolds the integral to the strip with holes $\mathcal{S}_L^{\mathrm{h}}$:

$$R_{L,\mu}(F,s) = \int_{\mathcal{F}_L} F(\tau)\mathcal{E}_{s,\mu}(\tau,\bar{\tau})\,\frac{d\tau d\bar{\tau}}{\tau_2^2} = \frac{\pi^s}{\Gamma(s)}\int_{\mathcal{S}_L^{\mathrm{h}}} F(\tau)2(\mu\tau_2)^{s/2}K_s\left(2\pi\sqrt{\mu/\tau_2}\right)\frac{d\tau_1 d\tau_2}{\tau_2^2} \tag{4.1}$$

and if we take the $L \to \infty$ limit, we can trivially perform the τ_1 integral, like above:

$$R_{\infty,\mu}(F,s) = \frac{2\pi^s\mu^{s/2}}{\Gamma(s)}\int_0^\infty \tau_2^{s/2-2}K_s\left(2\pi\sqrt{\mu/\tau_2}\right)f(\tau_2)d\tau_2$$

This is somewhat like a Bessel transform, rather than a Mellin transform, but note that τ_2 is in the denominator in the argument of the Bessel function.

A few words on asymptotics. For $\tau_2 \to 0$, because τ_2 appears in the denominator in the argument of the Bessel function, the μ deformation improves the behavior of the seed: it goes as $e^{-2\pi\sqrt{\mu/\tau_2}}\tau_2^{s/2-7/2}$ rather than diverging. For $\tau_2 \to \infty$, on the other hand, and for fixed μ, the deformation does not help, and the seed function diverges as τ_2^s, like it does for the usual massless case $\mu = 0$. Subleading terms now depend on μ, and depending on s they are either divergent or not.

Note also that although the seed is finite at $\tau_2 \to 0$, it is nonanalytic in τ_2 due to the $e^{-2\pi\sqrt{\mu/\tau_2}}$ factor. A theme in this paper is that nonanalyticity can make some calculations appear unfamiliar, but if it helps regulate some divergences, it could be worth it.

4.2 The massive theta lift

With nonzero mass parameter μ, we will not need to dimensionally regularize to complex d, or zeta-regularize to complex s. This can be an advantage in some contexts, for example since dimensional regularization generically does not respect supersymmetry.

The Siegel-Narain theta function is modular invariant. (This is more obvious in the Poisson-resummed form, [15] eq. (13.135)). Again, the zero mode $f(\tau_2)$ of $F = \Theta_{2,2}$ consists of the terms when $\langle k, n\rangle = 0$:

$$f(\tau_2) = \tau_2 \sum_{\substack{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2 \\ \langle k,n\rangle=0}} e^{-\pi\tau_2\mathcal{M}}. \tag{4.2}$$

One could consider whether Θ should itself be deformed by μ. The DKL correction would then have massive Dedekind η functions of T and U. However, at least in the examples of plane gravitation wave backgrounds [3], the massive worldsheet fields X^m appear in the external directions polarized along the wave (usually viewed to be noncompact), whereas the T, U moduli parametrize an internal torus X^i (which is compact). In this product geometry, the Kaluza-Klein and winding modes along the internal torus X^i are unaffected by the mass deformation in the noncompact directions. One could consider a non-product geometry, but that is beyond the scope of this work.

On the other hand, as we will see, even though we do not begin with a massive Θ, we end up with a massive function of the spacetime moduli.

The massive Rankin-Selberg transform of $F = \Theta_{2,2}$ is

$$R_{L,\mu}(\Theta, s) = \int_{\mathcal{F}_L} \Theta_{2,2}(\tau)\mathcal{E}_{s,\mu}(\tau)\,\frac{d\tau d\bar\tau}{\tau_2^2} = \frac{\pi^s}{\Gamma(s)}\int_{\mathcal{S}_L^{\mathrm{h}}} \Theta_{2,2}(\tau)2(\mu\tau_2)^{s/2}K_s\left(2\pi\sqrt{\mu/\tau_2}\right)\frac{d\tau_1 d\tau_2}{\tau_2^2} \tag{4.3}$$

If we send $L \to \infty$, we can integrate over τ_1 and find

$$\begin{aligned} R_{\infty,\mu}(\Theta, s) &= \frac{2\pi^s\mu^{s/2}}{\Gamma(s)}\int_0^\infty \tau_2 \sum_{\substack{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2 \\ \langle k,n\rangle=0}} e^{-\pi\tau_2\mathcal{M}}\tau_2^{s/2-2}K_s\left(2\pi\sqrt{\mu/\tau_2}\right)d\tau_2 \\ &= \sum_{\substack{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2 \\ \langle k,n\rangle=0}} \frac{2\pi^s\mu^{s/2}}{\Gamma(s)}\int_0^\infty \tau_2^{s/2-1}e^{-\pi\tau_2\mathcal{M}}K_s\left(2\pi\sqrt{\mu/\tau_2}\right)d\tau_2 = \end{aligned}$$

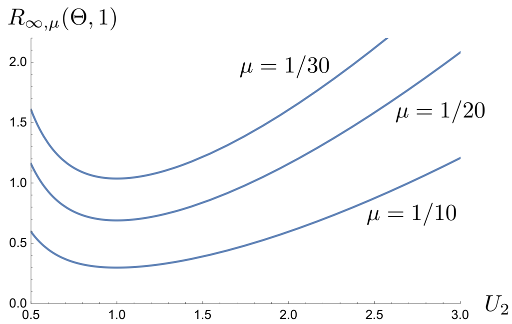

Figure 3. Plot of (4.4) for $T = i$, $U = iU_2$. The sum was truncated to $|k^1|, |k^2|, |n_1|, |n_2| \leq 5$, which with $\langle k, n\rangle = 0$ leaves 832 terms, of which 99 distinct. More terms do not change this plot.

$$= \sum_{\substack{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2 \\ \langle k,n\rangle=0}} \frac{2\pi^s \mu^{s/2}}{\Gamma(s)} \cdot \frac{\pi^s}{2}\mu^{s/2} G^{3,0}_{0,3}(0,0,-s,\pi^3\mu\mathcal{M})$$

$$= \frac{\pi^{2s}}{\Gamma(s)} \sum_{\substack{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2 \\ \langle k,n\rangle=0}} \mu^s G^{3,0}_{0,3}(0,0,-s,\pi^3\mu\mathcal{M})\,. \tag{4.4}$$

Here G is the Meijer G function, a type of Mellin-Barnes representation:

$$G^{3,0}_{0,3}(b_1,b_2,b_3,z) = \frac{1}{2\pi i}\int_C \Gamma(b_1+t)\Gamma(b_2+t)\Gamma(b_3+t)z^{-t}dt \tag{4.5}$$

that we review in appendix E. As described there in greater detail, the contour C avoids the poles. In particular, for the form of the arguments in (4.4) we have

$$G^{3,0}_{0,3}(0,0,-s,z) = \frac{1}{2\pi i}\int_C \Gamma(t)^2\Gamma(t-s)z^{-t}dt\,. \tag{4.6}$$

Here the contour C passes to the right of poles, for example we can pick $t = 3/2 + ix$ and $x \in (-\infty, \infty)$. We plot (4.4) for $s = 1$ in figure 3.

The Meijer G function can also be written as a sum of hypergeometric functions. We review in appendix E why a sum of hypergeometric functions is a less useful representation for our purposes. Briefly, each term in the sum may require separate regularization at special parameter values, whereas the Meijer G function represents a "coherent" regularization among those terms, in a sense. However, probably everything in this paper could be reexpressed in terms of somewhat lengthier expressions with hypergeometric functions.

Although we will not only be interested in small z, for illustration let us consider small z so we can truncate the sum over residues. Again for illustration, set $s = 1$ directly:

$$G^{3,0}_{0,3}(0,0,-1,z) = \operatorname*{Res}_{t=1,0,-1,\ldots} \Gamma(t)^2\Gamma(t-s)z^{-t} = \frac{1}{z} - \frac{1}{2}\log^2 z + (1-3\gamma)\log z + \text{constant} + \mathcal{O}(z) \tag{4.7}$$

Compare (4.7) and (4.4). The argument z in (4.4) contains the mass parameter μ. As already remarked, expanding around small μ ruins some properties of the massive Eisenstein series, so we will only use this expansion to check the connection to the undeformed case. Still, we wanted to illustrate that the expansion (4.7) is explicit and unambiguous for each term in the sum (4.4).

JHEP05(2023)062

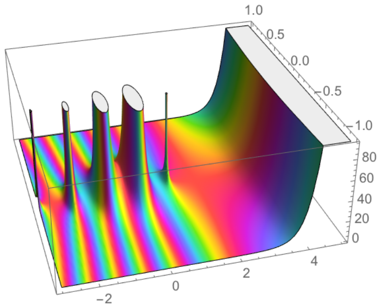

Figure 4. The coefficient $\Gamma(t)^2\Gamma(t-s)$ of z^t in the integrand eq. (4.7), for $s=1$: a single pole at $t=1$ and triple poles at $t=-n$ for $n=0,1,2,\ldots$

Now the more important question what happens at fixed μ and large summation variables, so large $\mathcal{M}$, and therefore large argument z of the Meijer G function. At $z\to\infty$, the Meijer $G^{3,0}_{0,3}$ is exponentially suppressed but nonanalytic, as

$$G^{3,0}_{0,3}(0,0,-s,z) = \frac{4\pi}{\sqrt{3}} z^{-(s+1)/3} e^{-3z^{1/3}+\mathcal{O}(x^{-1/3})} \,. \tag{4.8}$$

In the massless case ($\mu=0$), the result for the theta lift integral was only power-law suppressed $\mathcal{M}^{-s}$, despite the factor $e^{-\pi\tau_2\mathcal{M}}$ in the integrand. As is discussed in greater detail in the appendix, the exponential suppression (4.8) can be understood from a change of variables $\tau_2\mathcal{M}=\hat{\tau}_2$ in the integral (4.4)

$$\int_0^\infty \tau_2^{s/2-1} e^{-\pi\tau_2\mathcal{M}} K_s\left(2\pi\sqrt{\mu/\tau_2}\right) d\tau_2 = \mathcal{M}^{-s/2}\int_0^\infty \hat{\tau}_2^{s/2-1} e^{-\pi\hat{\tau}_2} K_s\left(2\pi\sqrt{\frac{\mu}{\tau_2}}\mathcal{M}^{1/2}\right) d\hat{\tau}_2 \tag{4.9}$$

together with the fact that the Bessel function is exponentially suppressed for large arguments. Without the Bessel function to "catch half" of the $\mathcal{M}$, the change of variables $\tau_2\mathcal{M}=\hat{\tau}_2$ simply brings $\mathcal{M}^{-s}$ out of the integral, leaving the massless integral with only power-law suppression of the quadruple sum over k,n. (In the massless case, as an alternative, we can prevent $\mathcal{M}$ slipping out of the integral by splitting the integration region $[0,\infty]$ to $[0,1]$ and $[1,\infty]$, as in appendix C. While explicit, this has its own challenges, as we will return to in section 4.4.)

For each term in the quadruple sum, we could also consider expanding around a smooth point like $\mu=1$. The function $G^{3,0}_{0,3}(0,0,-s,wz)$ has a smooth series expansion around $\mu=1$, by the multiplication relation

$$G^{3,0}_{0,3}(0,0,-s,wz) = w^{-s}\sum_{n=0}^{\infty}\frac{(w-1)^m}{m!}G^{3,0}_{0,3}(0,0,-s+m,z). \tag{4.10}$$

One way to use this is to write the sum (4.4) as an infinite sum over $E_{V,s}$ for different integer s. Although we will not explore such representations in detail in this paper, we will return to sums of this form in section 4.3.

Finally, we see how the massive theta lift (4.4) reduces to the standard theta lift (3.7) for $\mu=0$ as follows. The factor $\Gamma(t-s)$ has its first pole at $t=s$ (counting from the right), that gives z^{-s}. Noting that there is an extra factor μ in front of the Meijer G function in (4.4), we see that

$$R_{\infty,\mu\to 0}(\Theta,s)=\frac{\Gamma(s)}{\pi^s}\sum_{\substack{k\in\mathbb{Z}^{2*},n\in\mathbb{Z}^2\\ \langle k,n\rangle=0}}\frac{1}{\mathcal{M}^s}=E^{(0)}_{\mathrm{V},s}(T,U) \tag{4.11}$$

as in (3.7).

4.3 Mellin-dual massive theta lift

Here we do something perhaps less obvious. Let us Mellin-transform with respect to the mass parameter μ, following section 4.2 of [4]. (Of course, in the massless case ($\mu=0$) there is no direct analog of this operation.) The variable that is Mellin-dual to the mass parameter μ is called t. We obtain

$$\begin{aligned}\mathbf{M}(R_{L,\mu}(\Theta,s))(t)&=\int_0^\infty \mu^{t-1}\left(\int_{\mathcal{F}_L}\Theta_{2,2}(\tau)\mathcal{E}_{s,\mu}(\tau)\,\frac{d\tau d\bar\tau}{\tau_2^2}\right)d\mu\\ &=\int_{\mathcal{S}_L^{\rm h}}\Theta_{2,2}(\tau)\pi^{-2t}\frac{\Gamma(t)\Gamma(s+t)}{\Gamma(s)}\tau_2^{s+t-2}\frac{d\tau_1 d\tau_2}{\tau_2^2}\\ &=\pi^{-2t}\frac{\Gamma(t)\Gamma(s+t)}{\Gamma(s)}\sum_{\substack{k\in\mathbb{Z}^{2*},n\in\mathbb{Z}^2\\ \langle k,n\rangle=0}}\int_{\mathcal{S}_L^{\rm h}}\tau_2 e^{-\pi\tau_2\mathcal{M}}\tau_2^{s+t-2}\frac{d\tau_1 d\tau_2}{\tau_2^2}\,.\end{aligned} \tag{4.12}$$

It will be useful for later to note that since here s is just an external parameter as far as this integral transform is concerned, there is freedom to pick some s-dependent combination like $s+t$ as the Mellin-dual variable to μ, instead of just t.

Sticking to the choice t and taking $L\to\infty$, we have

$$\mathbf{M}(R_{\infty,\mu}(\Theta,s))(t)=\pi^{-s-3t}\frac{\Gamma(t)\Gamma(s+t)^2}{\Gamma(s)}\sum_{\substack{k\in\mathbb{Z}^{2*},n\in\mathbb{Z}^2\\ \langle k,n\rangle=0}}\mathcal{M}^{-(s+t)} \tag{4.13}$$

$$=\pi^{-2t}\frac{\Gamma(t)\Gamma(s+t)}{\Gamma(s)}E_{V,s+t}(\mathcal{M})\,. \tag{4.14}$$

As expected from the general arguments of [4], in the Mellin-dual variable t, the result looks formally similar to the ordinary massless $\mu=0$ result, except that the latter is divergent for $L\to\infty$ (as L^t) without explicit subtractions.

To Mellin-transform back $t\to\mu$, we should pick a contour. This should in principle reproduce the Meijer G function (4.7) from the previous subsection, but it is not completely obvious at this point how it does, since the Gamma functions appear in a slightly different combination: $\Gamma(t)\Gamma(s+t)^2$ in (5.8) versus $\Gamma(t)\Gamma(t-s)^2$ in (4.7). (Remember the contour integral is with respect to t, not s.)

Viewed as an integral representation of the μ dependence, we of course do not expect it to be unique. As already mentioned, we can change variables of integration to combinations like $t+s$ or $t-s$ instead of t as Mellin-dual variable. A discrete freedom of parameterization is captured by the functional relation for $E_{V,s+t}$, as we will discuss below.

Like in the previous section, but unlike in the massless case, it is possible to set $s=1$ at this stage. We note that this prevents viewing the massive theta lift as "infinitesimal", as we expect it to diverge if we send $\mu \to 0$, from being on top of the double pole of $E_{V,s}$. However, when viewed as a finite deformation this is no problem.

Now, instead of being on top of the $s = 1$ pole, we are faced with $E_{V,1+t}$, that has a double pole at the origin $t = 0$ of the complex t plane. The Mellin contour is *defined* to avoid poles. We can pick for example $t = 1/2 + iy$ and then close at infinity to the left. This gives a residue at $t = 0$, but also at negative integers t, so at $t = -n$ for $n = 0, 1, 2, \ldots$. We have

$$R_{\infty,\mu}(\Theta, 1) = \sum_{n=0}^{\infty} \operatorname*{Res}_{t=-n} \pi^{-2t}\mu^{-t}\Gamma(t)\Gamma(1+t)E_{V,1+t}(\mathcal{M})\,. \tag{4.15}$$

Let us first consider the $t = 0$ pole. Eq. (4.15) has a single pole at $t = 0$ from $\Gamma(t)$, so taking the residue picks out the *constant* term from $E_{V,1+t}$ as $t \to 0$. Contrast the massless case when picking out the *single pole* in s. The constant term contains the "$\log^2$" piece

$$R_{\infty,\mu}(\Theta, 1)\Big|_{\log^2} = \operatorname*{Res}_{t=0} \pi^0\mu^0 t^{-1}\Gamma(1)E_{V,1+t}(\mathcal{M})\Big|_{\log^2} = \frac{6}{\pi}\log(\sqrt{T_2}|\eta(T)|^2)\log(\sqrt{U_2}|\eta(U)|^2)\,. \tag{4.16}$$

This is not a DKL correction. Instead, it is more like a "Sudakov" double logarithm.[3] Similarly to the Fourier expansion in τ_1, for the remaining poles at negative integers $t = -n$ we use the functional relation (reflection formula) eq. (3.27) for $E_{V,1+t}$ to map it to $1 - (1+t) = -t = n$. This produces an additional Gamma function and a zeta function in numerator and denominator. We have for the $t < 0$ $(n > 0)$ terms

$$\begin{aligned} R_{\infty,\mu}(\Theta, 1)\Big|_{t<0} &= \sum_{n=1}^{\infty} \operatorname*{Res}_{t=-n} \pi^{-2t-1/2}\mu^{-t}\frac{\Gamma(-t)\Gamma(t)\Gamma(t+1)\zeta(-2t)}{\Gamma(-t-1/2)\zeta(-2t-1)}E_{\mathrm{V},-t} \\ &= -\sum_{n=1}^{\infty} \operatorname*{Res}_{t=-n} \frac{2^{-2(t+1)}\pi^{1-2t}\mu^{-t}}{t\sin^2\pi t}\frac{\zeta(-2t)}{\zeta(-2t-1)\Gamma(-2t-1)}E_{\mathrm{V},-t}\,. \end{aligned} \tag{4.17}$$

In the last line, we used the functional relation for ζ. This is like a "reflected" version of the smooth $\mu = 1$ expansion (4.10). Explicitly,

$$R_{\infty,\mu}(\Theta,1)\Big|_{t<0} = \frac{\pi^3}{3}\mu E_{\mathrm{V},1} + \frac{\pi^3}{180\zeta(3)^2}\mu^2(c_2E_{\mathrm{V},2} + c_2'E_{\mathrm{V},2}') + \frac{\pi^5}{8505\zeta(5)}\mu^3(c_3E_{\mathrm{V},3} + c_3'E_{\mathrm{V},3}') + \ldots$$

where

$$\begin{aligned} c_1 &= -4\pi^4\zeta'(3) + 360\zeta(3)\zeta'(4) + \pi^4\zeta(3)(2\log(\mu) + 4\gamma - 7 + 4\log(\pi) + \log(16)) \\ c_1' &= 2\pi^4\zeta(3) \\ c_2 &= -4\pi^6\zeta'(5) + 3780\zeta(5)\zeta'(6) + \pi^6\zeta(5)(2\log(\mu) + 4\gamma - 9 + 4\log(\pi) + \log(16)) \\ c_2' &= 2\pi^6\zeta(5) \end{aligned} \tag{4.18}$$

The first term with $E_{\mathrm{V},1}$ is shorthand to use Kronecker's 1st limit formula, which gives a finite result for the order μ term that we computed, but we do not display the expression here.

The total massive theta lift is the sum of the two kinds of contribution:

$$R_{\infty,\mu}(\Theta, 1) = R_{\infty,\mu}(\Theta, 1)\Big|_{t=0} + R_{\infty,\mu}(\Theta, 1)\Big|_{t<0}\,, \tag{4.19}$$

[3]In QED and QCD, a Sudakov double logarithm in momentum arises when there are two mass scales available [16].

where the $t = 0$ term includes the $\log^2$ term, and the $t < 0$ terms the "tail" of higher $E_{V,n}$. This tail is typical for the massive theta lift, and can be thought of as in the smooth $\mu = 1$ expansion (4.10). The residue (4.17) gives an explicit way to compute it without using known properties of Meijer G functions (Mellin-Barnes representations).

Another way to put it is that contour integration calculations like those in this section show some properties of those Mellin-Barnes representations.

4.4 Alternative: cutoff

This section provides an alternative to the main computations above, and can be skipped.

In the previous sections, we defined analytic continuations by contour deformation in the complex plane, which is built into the Meijer G (Mellin-Barnes) representation. An alternative to contour deformation is to define the analytic continuation by a cutoff procedure. In this section, we will illustrate this at the simplest level of a double sum Θ_2 as opposed to the quadruple sum $\Theta_{2,2}$, but the principle is the same.

In appendix C, we review the standard argument that gives analytic continuation of the Eisenstein series $E_s(U)$ as integral representation, following the logic in chapter 1 of Bump's book [11]. The final result (C.6) in the appendix is an explicit analytic continuation, that we repeat here for convenience:

$$\int_0^\infty \tau_2^{s-1}\Theta_2(U,\tau_2)d\tau_2 = {\sum_{n_1,n_2}}' \mathtt{E}_s\left(\frac{\pi}{U_2}|n_2U-n_1|^2\right) + \frac{1}{s-1} + {\sum_{n_1,n_2}}' \mathtt{E}_{1-s}\left(\frac{\pi}{U_2}|n_2U-n_1|^2\right) - \frac{1}{s}, \tag{4.20}$$

where $\mathtt{E}_s$ is the exponential integral

$$\mathtt{E}_s(z) = \int_1^\infty t^{-s}e^{-zt}dt \tag{4.21}$$

that is closely related to the incomplete gamma function, but this form suits our purposes better.

In this section, we mass deform eq. (4.20):

$$\begin{aligned}&\underbrace{2\mu^{s/2}\frac{\pi^s}{\Gamma(s)}\int_0^\infty \tau_2^{s/2-1}K_s\left(2\pi\sqrt{\frac{\mu}{\tau_2}}\right){\sum_{n_1,n_2}}' e^{-\frac{\pi\tau_2}{T_2U_2}|n_2U-n_1|^2}d\tau_2}_{\mathtt{E}_{s,\mu}[T,U]}\\ &+\underbrace{2\mu^{s/2}\frac{\pi^s}{\Gamma(s)}\int_0^\infty \tau_2^{s/2-1}K_s\left(2\pi\sqrt{\frac{\mu}{\tau_2}}\right)d\tau_2}_{\text{“massive pole”}}\end{aligned} \tag{4.22}$$

where the summand in the second line is the massive analog of the exponential integral $\mathtt{E}_s$:

$$\begin{aligned}&{\sum_{n_1,n_2}}' \mathtt{E}_{s,\mu}(\pi|n_2U-n_1|^2/(U_2T_2)) =\\ &2\mu^{s/2}{\sum_{n_1,n_2}}'\frac{T_2^{s/2}U_2^{s/2}}{|n_2U-n_1|^s}\int_0^\infty \hat{\tau}_2^{s/2-1}K_s\left(2\pi\sqrt{\frac{\mu}{\hat{\tau}_2T_2U_2}}|n_2U-n_1|\right)e^{-\pi\hat{\tau}_2}d\hat{\tau}_2\end{aligned} \tag{4.23}$$

where we rescaled the exponent to $-\pi\hat{\tau}_2$. The double sum is a certain massive Eisenstein series of U, but it is not the Poincaré series we consider here, where the double sum is restricted to coprime integers. As above, we will instead perform the integral in each summand, so we obtain a sum over Meijer G functions.

The key point here is the following. For the ordinary massless integral representation in the appendix, since we are free to change variable of integration τ_2 to whatever is in the exponent, the exponential suppression of the double sum "slips out" of the integrand. Here, the Bessel function prevents this "slipping out", so unlike the exponential in the ordinary massless integral, convergence is improved. Note that we are not using any property of the Bessel function other than its exponential suppression for large arguments.

The integral labelled "massive pole" is the massive analog of the pole term $1/s$. This can easily be calculated explicitly. (We only display this for illustration — we will argue in a moment that this expression is not needed.)

$$
\begin{aligned}
2\mu^{s/2}\frac{\pi^s}{\Gamma(s)}\int_0^L \tau_2^{s/2-1}K_s\left(2\pi\sqrt{\frac{\mu}{\tau_2}}\right)d\tau_2 = L &+ (\pi^2(2\gamma-1+2\log\pi)\mu+\pi^2\mu\log\mu)\log L \\
&-\frac{\pi^2}{2}\mu\log^2 L - \frac{\pi^2}{6}\mu(3\log\mu(\log\mu+4\gamma-2+4\log\pi) \\
&+\pi^2+6+12((\gamma-1)\gamma+\log^2\pi+2\gamma\log\pi)-12\log\pi).
\end{aligned}
$$

Like for the pole term in the massless case, we can subtract this counterterm, so its explicit form is not needed when comparing to the finite value in the sum over the Meijer G function.

Computing the sum over the Meijer G function to high truncation is somewhat time-consuming. But we can replace the Meijer G function with the asymptotic expression $\sim z^{-2/3}e^{-z^{1/3}}$ to good accuracy for $\mathcal{M}^2 \gtrsim 10$. A compromise is to use the exact expression up to $|n_1|, |n_2|, |k_1|, |k_2| \leq 5$ (a few hundred terms), and then the asymptotic expression, with which $|n_1|, |n_2|, |k_1|, |k_2| \leq 30$ (about a million terms) is easy to achieve, if desired. Since we are not interested in high accuracy, we will content ourselves with $|n_1|, |n_2|, |k_1|, |k_2| \leq 5$ and only use the asymptotic expression as a check.

Here is a more direct comparison to the $[1,\infty]$ cutoff integrals in the appendix. We introduce a cutoff Meijer G:

$$
G_{s,\mathrm{cutoff}}(X) = 2\mu^{s/2}\frac{\pi^s}{\Gamma(s)}\int_1^\infty \tau_2^{s/2-1}K_s\left(2\pi\sqrt{\frac{\mu}{\tau_2}}\right)e^{-\pi\tau_2 X}d\tau_2 \tag{4.24}
$$

For $\mu \to 0$ and $s=1$, this becomes the special case $\mathtt{E}_0$ of the exponential integral, which is elementary:

$$
G_{\mathrm{cutoff}}(X)\Big|_{s=1} \overset{\mu\to 0}{\to} \frac{e^{-\pi X}}{\pi X} \tag{4.25}
$$

Although $G_{s,\mathrm{cutoff}}$ in eq. (4.24) is explicit enough to compute with, as we now show, it is not a well-known special function. For our purposes we prefer the usual Meijer G function in the preceding (and following) sections.

With our normalization, the factor $\pi^s/\Gamma(s)$ sets the term that was $\mathtt{E}_1$ in the appendix to zero at $s=0$. We only need to compute the second term. For illustration, at $U=i$

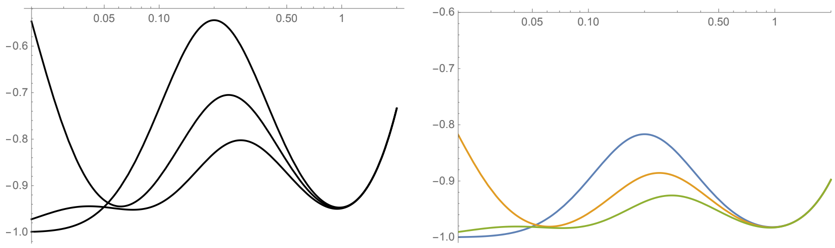

Figure 5. Left panel: same three U_1 values as in the massless cutoff in figure 7, but with our normalization convention. Right panel: deformation to $\mu = 0.1$, same three U_1 values as in previous plot.

we have

$$\sum_{n_1,n_2}{}' E_0\left(|n_2 i - n_1|^2\right)\Big|_{|n_1|\le 2, |n_2|\le 2} = -1+4\frac{e^{-\pi}}{\pi}+2\frac{e^{-2\pi}}{\pi}+\frac{e^{-4\pi}}{\pi}+8\frac{e^{-5\pi}}{5\pi}+\frac{e^{-8\pi}}{2\pi} = -0.943788 \tag{4.26}$$

whereas at the same truncation, we have for the massive version the values in the table.

$$\sum_{n_1,n_2}{}' G_{\mathrm{cutoff},\mu}\left(|n_2 i - n_1|^2\right)\Big|_{|n_1|\le 2, |n_2|\le 2} = \tag{4.27}$$

μ	value
10^{-1}	-0.980804
10^{-2}	-0.954947
10^{-3}	-0.945860
10^{-4}	-0.944096
10^{-5}	-0.943829
10^{-6}	-0.943793

For small μ we approach the massless cutoff value (4.26). We give an illustrative plot in the figure.

5 Massive theta lift of the j-function

We will now calculate the massive theta lift of the modular invariant j-function. The standard case of the theta lift of the j-function was reviewed in section 3.4, and our result may be viewed as a massive deformation of that result.

5.1 The theta lift

The integral we wish to consider is (for later use, we leave $c(0)$ arbitrary)

$$\int_{\mathcal{F}} j(\tau)\Theta_{2,2}(\tau)\mathcal{E}_{s,\mu}(\tau)\,\frac{d\tau d\bar{\tau}}{\tau_2^2}. \tag{5.1}$$

Without the insertion of the massive Eisenstein serie $\mathcal{E}_{s,\mu}(\tau)$ this is the integral of Harvey-Moore. Here with the massive theta lift, we expect not to have to regularize, since as in the massive theta lift without j, the Bessel-K and hence Meijer-G function provide exponential

JHEP05(2023)062

suppression for the quadruple sum.

$$
\begin{aligned}
R_{L,\mu}(\Theta, s) &= \int_{\mathcal{F}_L} j(\tau)\Theta_{2,2}(\tau)\mathcal{E}_{s,\mu}(\tau)\,\frac{d\tau d\bar{\tau}}{\tau_2^2} \\
&= \frac{\pi^s}{\Gamma(s)}\int_{\mathcal{S}_L^{\mathrm{h}}} j(\tau)\Theta_{2,2}(\tau)2(\mu\tau_2)^{s/2}K_s\left(2\pi\sqrt{\mu/\tau_2}\right)\frac{d\tau_1 d\tau_2}{\tau_2^2}.
\end{aligned}
\tag{5.2}
$$

We take $L \to \infty$ to find:

$$
\begin{aligned}
R_{\infty,\mu}(\Theta, s) &= \frac{2\pi^s\mu^{s/2}}{\Gamma(s)}\int_0^\infty c(m)e^{-2\pi m\tau_2}\tau_2 \sum_{\substack{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2\\ \langle k,n\rangle=-m}} e^{-\pi\tau_2\mathcal{M}}\tau_2^{s/2-2}K_s\left(2\pi\sqrt{\mu/\tau_2}\right)d\tau_2 \\
&= \sum_{m=-1}^{\infty} c(m)\sum_{\substack{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2\\ \langle k,n\rangle=-m}} \frac{2\pi^s\mu^{s/2}}{\Gamma(s)}\int_0^\infty \tau_2^{s/2-1}e^{-\pi\tau_2(\mathcal{M}+2m)}K_s\left(2\pi\sqrt{\mu/\tau_2}\right)d\tau_2.
\end{aligned}
\tag{5.3}
$$

At this point, one can be worried about the $1/q$ pole ($m = -1$ term) due to the j function which gives rise to the $\mathcal{M} + 2m$ shift, which for $m = -1$ is $\mathcal{M} - 2$. However, the minimum value of $\mathcal{M}$ is 2, so the Siegel-Narain Θ takes care of the $\tau_2 \to \infty$ behavior. The massless theta lift from the previous section appears in each term in the m sum:

$$
\begin{aligned}
R_{\infty,\mu}(\Theta, s) &= \sum_{m=-1}^{\infty} c(m)\sum_{\substack{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2\\ \langle k,n\rangle=-m}} \frac{2\pi^s\mu^{s/2}}{\Gamma(s)}\cdot\frac{\pi^s}{2}\mu^{s/2}G^{3,0}_{0,3}(\pi^3\mu(\mathcal{M}+2m)|0,0,-s) \\
&= \sum_{m=-1}^{\infty} c(m)\frac{\pi^{2s}}{\Gamma(s)}\sum_{\substack{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2\\ \langle k,n\rangle=-m}} \mu^s G^{3,0}_{0,3}(\pi^3\mu(\mathcal{M}+2m)|0,0,-s).
\end{aligned}
\tag{5.4}
$$

The advantage of a regularization like the above is that we can use the q-series for $j(\tau)$ directly. Without regularization, the "naive" massless j-lift is singular, so the above is not directly a deformation of the naive j-lift. Instead, we can compare it to the Selberg-Poincaré regularization (3.13) for $s = \epsilon > 0$ (where the original holomorphic j is $s = 0$), and include massive regularization. Writing ℓ in the Selberg-Poincaré series, so that the above is the $\ell = 1$ special case, we have

$$
\begin{aligned}
R_{L,\mu}(\Theta, s)(T,U) &= \int_{\mathcal{F}_L} \underbrace{E^{(\ell)}_\epsilon(\tau)}_{\text{unfold}}\Theta_{2,2}(T,U,\tau)\mathcal{E}_{s,\mu}(\tau)\,\frac{d\tau d\bar{\tau}}{\tau_2^2} \\
&= \int_{\mathcal{S}_L^{\mathrm{h}}} \tau_2^\epsilon e^{-2\pi i\ell\tau_1}e^{2\pi\ell\tau_2}\Theta_{2,2}(T,U,\tau)\sum_m f^{(m)}_{s,\mu}(\tau_2)e^{2\pi i m\tau_1}\frac{d\tau_1 d\tau_2}{\tau_2^2}\,.
\end{aligned}
\tag{5.5}
$$

For $L \to \infty$, we have

$$
R_{\infty,\mu}(\Theta, s)(T,U) = \sum_m \sum_{\substack{k\in\mathbb{Z}^{2*}, n\in\mathbb{Z}^2\\ \langle k,n\rangle=\ell-m}} \int_0^\infty \tau_2^{-1+\epsilon}e^{-\pi\tau_2(\mathcal{M}(T,U)-2\ell)}f^{(m)}_{s,\mu}(\tau_2)d\tau_2 \tag{5.6}
$$

where the τ_1 Fourier coefficients $f^{(m)}_{s,\mu}(\tau_2)$ of $\mathcal{E}_{s,\mu}(\tau)$ are given in (2.12). Note that the q-expansion coefficients of j do not occur explicitly in this expression. So if the two versions of the calculation are to agree, those coefficients must be encoded in the right-hand side.

Indeed, the Petersson-Rademacher expansions express $c(m)$ in sums over Kloosterman sums and I Bessel functions. One difference is that the Kloosterman sums that occur for the Eisenstein series E always have one argument zero, whereas the Selberg-Poincaré representation of the j function has one argument equal to one.

We now have two alternative regularizations of the same integral, so the interesting question arises whether they agree for finite μ. To check this, we can consider the simpler case of just $R_{\infty,\mu}(1,s)$ without the Θ:

$$R_{L,\mu}(1,s)=\int_{\mathcal{F}_L} j(\tau)\underbrace{\mathcal{E}_{s,\mu}(\tau)}_{\text{unfold}}\frac{d\tau d\bar{\tau}}{\tau_2^2} \overset{?}{=} \lim_{\epsilon\to 0}\int_{\mathcal{F}_L}\underbrace{E_\epsilon^{(\ell)}(\tau)}_{\text{unfold}}\mathcal{E}_{s,\mu}(\tau)\frac{d\tau d\bar{\tau}}{\tau_2^2} \quad (\ell=1)\,. \tag{5.7}$$

At this point, it might be of use to compare our calculations to the beautiful work on Niebur-Poincaré representations in [7, 17]. There, Selberg-Poincaré sums like $E_\epsilon^{(\ell)}(\tau)$ come with I Bessel functions, since they must have $1/q^\ell$ poles at $q\to 0$, and are deformed to Whittaker functions. Here, massive Eisenstein series do not have $1/q$ poles, and are represented with K Bessel functions, and deform the massless Eisenstein series E_s in the RSZ transform to massive $\mathcal{E}_{s,\mu}$, as in (5.7).

We take $L\to\infty$:

$$2\mu^{s/2}c(0)\int_0^\infty \tau_2^{s/2-2}K_s\left(2\pi\sqrt{\mu/\tau_2}\right)d\tau_2 \overset{?}{=}\lim_{\epsilon\to 0}\int_0^\infty \tau_2^{-1+\epsilon}e^{2\pi\ell\tau_2}f_{s,\mu}^{(\ell)}(\tau_2)d\tau_2 \quad (\ell=1)$$

where again $f_{s,\mu}^{(m)}(\tau_2)$ is given in (2.12). On the left, the q^0 term of $j(\tau)$ was singled out by the τ_1 integral. On the right, the ℓth Fourier mode of $\mathcal{E}_{s,\mu}(\tau)$ was singled out, since $E_\epsilon^{(\ell)}(\tau)$ contributes $e^{-2\pi i\ell\tau_1}$, so $e^{2\pi i m\tau_1}$ only gives nonzero τ_1 integral for $m=\ell$. For $j(\tau)$, we have $\ell=1$.

The relation (5.1) effectively fixes the value of $c(0)$. Of course, $j(\tau)$ is modular invariant irrespective of the value of $c(0)$. But Selberg-Poincaré series regularization of $j(\tau)$ fixes $c(0)=12$ in one convention. So it is not surprising that there is also a value fixed here.

5.2 Mellin-dual massive j-theta lift

As in section 4.3, since we find that the massive theta lift is naturally a Mellin integral representation, we could also have compute the Mellin forward transformation $\mu\to t$ directly:

$$\begin{aligned}
\mathbf{M}(R_{\infty,\mu}(\Theta,s))(t) &= \int_0^\infty \mu^{t-1}\left(\int_{\mathcal{F}_L} j(\tau)\Theta_{2,2}(\tau)\mathcal{E}_{s,\mu}(\tau)\frac{d\tau d\bar{\tau}}{\tau_2^2}\right)d\mu\\
&= \int_{\mathcal{S}} j(\tau)\Theta_{2,2}(\tau)\pi^{-2t}\frac{\Gamma(t)\Gamma(s+t)}{\Gamma(s)}\tau_2^{s+t-2}\frac{d\tau_1 d\tau_2}{\tau_2^2}\\
&= \sum_\ell c(\ell)\pi^{-2t}\frac{\Gamma(t)\Gamma(s+t)}{\Gamma(s)}\sum_{\substack{k\in\mathbb{Z}^{2*},n\in\mathbb{Z}^2\\ \langle k,n\rangle=-\ell}}\int_{\mathcal{S}}\tau_2 e^{-\pi\tau_2(\mathcal{M}+2\ell)}\tau_2^{s+t-2}\frac{d\tau_1 d\tau_2}{\tau_2^2}\\
&= \sum_\ell c(\ell)\pi^{-s-3t}\frac{\Gamma(t)\Gamma(s+t)^2}{\Gamma(s)}\sum_{\substack{k\in\mathbb{Z}^{2*},n\in\mathbb{Z}^2\\ \langle k,n\rangle=-\ell}}(\mathcal{M}+2\ell)^{-(s+t)}\\
&= \sum_\ell c(\ell)\pi^{-2t}\frac{\Gamma(t)\Gamma(s+t)}{\Gamma(s)}E_{V,s+t}^{(\ell)}
\end{aligned} \tag{5.8}$$

where $E_V^{(\ell)}$ is a shifted ($\mathcal{M} \to \mathcal{M} + 2\ell$) non-degenerately constrained ($\langle k, n\rangle = 0 \to \langle k, n\rangle = -\ell$) lattice Eisenstein series. Apart from that, each term is exactly like without the j function. As before, we can fix $s = 1$ and pick out poles for $t = -n$ with $n = 0, 1, 2, \ldots$.

$$R_{\infty,\mu}(j\Theta, 1) = \sum_{\ell} c(\ell) \sum_{n=0}^{\infty} \operatorname*{Res}_{t=-n} \pi^{-2t}\mu^{-t}\Gamma(t)\Gamma(1+t)E^{(\ell)}_{V,1+t}\,. \tag{5.9}$$

The drawback with this representation in this context is that the constrained sums give no indication of how differential operators act on this composite object. On the other hand, we see in [6] that the sum with constraint $\langle k, n\rangle = -\ell$ is found from the first constrained sum $\ell = 1$ from acting with Hecke operators.

6 Conclusions and outlook

In this paper we have taken the first steps toward analyzing theta lifts of massive modular forms. Our work points to several interesting questions that we leave for future research. Below we provide a brief summary.

We have obtained explicit expressions for theta lifts, but a satisfactory interpretation of this is still lacking. As mentioned in the introduction, a key aspect of the theta lift is to transfer automorphic forms (or automorphic representations) from one Lie group to another. This fact plays an important role also in string theory, where the amount of supersymmetry constrains the relevant automorphic representation. It is therefore natural to wonder if there is a corresponding representation-theoretic statement underlying the massive theta lift. In physics, non-perturbative solutions (solitons) are known to correspond to certain massive (BPS) representations of the (super-)Poincaré algebra. Plane-wave limits of any solution from a perturbative viewpoint corresponds to fields that satisfy a harmonic oscillator differential equation in spacetime [41]. For maximal supersymmetry, these BPS-representations are intimately related to small automorphic representations (see e.g. [15]). It would be very interesting to understand whether there exists corresponding massive automorphic representations underlying the image of the massive theta lift.

On a related note, string loop corrections in plane wave backgrounds as in [3] are of interest in themselves. It is our hope that some of the objects discussed here could contribute building blocks for such computations in the future.

In this paper we have treated the mass μ as an external parameter, i.e. the massive modular forms really correspond to families of modular forms parameterized by $\mu \in \mathbb{R}_{\geq 0}$. It would be interesting to investigate whether it is possible to embed this construction into a larger Lie group, where μ would be promoted to an honest modulus. One natural conjecture is that the massive Eisenstein series occur as Fourier coefficients of some automorphic form on $\mathrm{SL}(3, \mathbb{R})$. For example, it would be very interesting to understand whether there is a relation to very similar formulas in [44–46].

We have considered theta lifts from $\mathrm{SL}(2, \mathbb{R})$ to $\mathrm{SO}(d, d)$. The theta correspondence works however in far greater generality, and allows for transferring automorphic representations between a wide class of groups, known as *dual reductive pairs*. For example, in the context of the pair $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(d, d)$ one can also consider theta "lifts" from $\mathrm{SO}(d, d)$ to

JHEP05(2023)062

SL(2, $\mathbb{R}$). In practise this involves integrating over $\Gamma\backslash\mathrm{SO}(d,d)/(\mathrm{SO}(d)\times\mathrm{SO}(d))$ to produce a modular form on SL(2, $\mathbb{R}$). For certain special choices this is well known in mathematics as the *Siegel-Weil formula*. This type of integral was also considered recently in string theory by Maloney and Witten [26], as a way of averaging over Narain moduli space. A similar type of integral would be straightforward to study in the massive case, using the results in this paper. We hope to return to this question in the future.

Finally, let us note that calculations such as those in this paper can also apply to condensed matter physics. One application for the kind of integration performed here is that the area of a closed subregion gives the Berry phase in topological systems [8]. Another long-standing topic in condensed matter physics is the statistical physics of the Ising model away from criticality, as in recent work by one of us [43]. In fact, methods of this kind were used in statistical physics before they made it to high-energy physics [42].

Acknowledgments

We are grateful to Dan Bump, YoungJu Choie, Jens Funke, Joe Minahan, Federico Zerbini and Sol Friedberg for helpful discussions and correspondence. We would also like to thank the Isaac Newton Institute of Mathematical Sciences, Cambridge, for support and hospitality during the programme "New connections in number theory and physics" when work on this paper was undertaken. D.P. was supported by the Swedish Research Council (grant no. 2018-04760), and the Wallenberg AI, Autonomous Systems and Software Program (WASP) funded by the Wallenberg Foundation (grant no. 2020.0173).

A Fourier-transforming Bessel functions with square roots

The Fourier integrals (zero mode and nonzero mode) are of the form [40] (1.13.45):

$$\begin{aligned}&\int_0^\infty dx\,\cos(xy)(x^2+\beta^2)^{-\nu/2}K_\nu(\alpha\sqrt{x^2+\beta^2}) =\\ &\sqrt{\frac{\pi}{2}}\alpha^{-\nu}\beta^{1/2-\nu}(y^2+\alpha^2)^{\nu/2-1/4}K_{\nu-1/2}\left(\beta\sqrt{y^2+\alpha^2}\right)\end{aligned} \tag{A.1}$$

for Re $\alpha > 0$, Re $\beta > 0$.

At the time of writing, Mathematica does not perform the integral directly, but it can do the following two steps. Using the following Mellin representation

$$(x^2+\beta^2)^{-\nu/2}K_\nu(\alpha\sqrt{x^2+\beta^2}) = \int_0^\infty dt\; 2^{-\nu-1}\alpha^{-\nu-1}e^{-\frac{\alpha^2(x^2+\beta^2)}{4t}-t}\,, \tag{A.2}$$

we can easily Fourier transform the Mellin integrand:

$$\int_0^\infty dx\;\cos(xy)\,2^{-\nu-1}\alpha^{-\nu-1}e^{-\frac{\alpha^2(x^2+\beta^2)}{4t}-t} = \sqrt{\pi}\,2^{-\nu-1}\alpha^{\nu-1}t^{-\nu-1/2}e^{\frac{\alpha^2\beta^2}{4t}-\frac{ty^2}{\alpha^2}-t}\,. \tag{A.3}$$

Now undoing the Mellin representation by performing the t integral gives eq. (A.1) above, where for our purposes it is sufficient to make the stronger assumption $\alpha > 0$, $\beta > 0$, $\nu > 0$.

JHEP05(2023)062

B Massive deformations of holomorphic objects

The massive Maass-Jacobi form $\mathcal{E}_{s,\mu}(\tau,z)$ is nonholomorphic, but there is a sense in which it also provides deformations of holomorphic objects. We do not expect the deformation to preserve holomorphy, since part of the definition of the mass-deformed objects is that they satisfy other differential equations than the Cauchy-Riemann equations. Let us briefly explain this.

The mass-deformed Maass form $\mathcal{E}_{\mu,s}(\tau)$ descends from the Jacobi-Maass form [4]

$$\mathcal{E}_{\mu,s}(z;\tau) = 2\sum_{m,n}{}' \left(\frac{\sqrt{\mu\tau_2}}{|m\tau+n|}\right)^s K_s\left(2\pi\sqrt{\frac{\mu}{\tau_2}}|m\tau+n|\right) e^{\frac{2\pi i}{\tau_2}\mathrm{Im}((m\tau+n)\bar{z})} \tag{B.1}$$

which is a deformation of the massless ($\mu=0$) nonholomorphic Kronecker-Eisenstein series

$$E_s(z;\tau) = \sum_{m,n}{}' \frac{\tau_2^s}{|m\tau+n|^{2s}} e^{\frac{2\pi i}{\tau_2}\mathrm{Im}((m\tau+n)\bar{z})} = \sum_{m,n}{}' \frac{\tau_2^s}{(m\tau+n)^s(m\bar{\tau}+n)^s} e^{\frac{\pi}{\tau_2}((m\tau+n)\bar{z}-(m\bar{\tau}+n)z)}. \tag{B.2}$$

Setting $s=2k\in 2\mathbb{Z}$, differentiating $2k$ times with respect to z, we obtain

$$\partial_z^{2k} E_s(z;\tau) = \pi^{2k}\sum_{m,n}{}' \frac{1}{(m\tau+n)^{2k}} e^{\frac{2\pi i}{\tau_2}\mathrm{Im}((m\tau+n)\bar{z})} \tag{B.3}$$

so setting $z=0$, we have the ordinary holomorphic Eisenstein series

$$E_{2k}^{\mathrm{hol}}(\tau) = \frac{1}{\pi^{2k}}\partial_z^{2k} E_{2k}(z;\tau)\Big|_{z=0} = \sum_{m,n}{}' \frac{1}{(m\tau+n)^{2k}}\,. \tag{B.4}$$

The "generating function" (B.3) suggests to define a massive deformation of $E_{2k}^{\mathrm{hol}}(\tau)$ by μ as

$$\mathcal{E}_{\mu,2k}^{\mathrm{hol}}(\tau) = \frac{1}{\pi^{2k}}\partial_z^{2k}\mathcal{E}_{\mu,2k}(z;\tau)\Big|_{z=0} = 2\sum_{m,n}{}'(\mu\tau_2)^k\left(\frac{m\bar{\tau}+n}{m\tau+n}\right)^k K_{2k}\left(2\pi\sqrt{\frac{\mu}{\tau_2}}|m\tau+n|\right) \tag{B.5}$$

with the massive Jacobi-Maass form from (B.1). Perhaps the notation $\mathcal{E}_{\mu,2k}^{\mathrm{hol}}$ is not optimal since although it is a deformation of a holomorphic object $E_{2k}^{\mathrm{hol}}(\tau)$, the deformation $\mathcal{E}_{\mu,2k}^{\mathrm{hol}}$ itself has no reason to be holomorphic. On the other hand, unlike (B.1), one specific term in (B.5) coming from the series expansion of the Bessel function in μ will be holomorphic, and there will formally be a "tail" of nonholomorphic contributions, much like the "shadow" of Niebur-Poincaré series. This expansion is only formal, since the double sum of the series representation does not converge. We can either simply avoid series-expanding the Bessel function, or use the integral representation of (B.1) instead of the double sum.

Note that while $\mathcal{E}_{\mu,s}(0;\tau)$ has an obvious Poincaré series representation, the deformation (B.5) of E_{2k} does not have an obvious Poincaré series generated purely from τ_2. This means that the calculation of the Fourier expansion does not apply and would need to be generalized. For the purposes of this discussion we can stick to (B.5) as a definition of $\mathcal{E}_{\mu,2k}^{\mathrm{hol}}(\tau)$.

From (B.5) we can define a massive deformation of the j-function

$$j_\mu(\tau) = 1728\frac{\mathcal{E}_{\mu,4}^{\mathrm{hol}}(\tau)^3}{\mathcal{E}_{\mu,4}^{\mathrm{hol}}(\tau)^3-\mathcal{E}_{\mu,6}^{\mathrm{hol}}(\tau)^2} \tag{B.6}$$

JHEP05(2023)062

that evidently by (B.4) reduces for $\mu = 0$ to the usual j-function

$$j_{\mu=0}(\tau) = 1728 \frac{E_4^3(\tau)}{E_4^3(\tau) - E_6^2(\tau)} . \tag{B.7}$$

Of course, when dealing with deformations, we should explore whether this deformation $j_\mu(\tau)$ is equivalent, in the sense of [4], to other possible deformations of the j function. One possibility for expressing the massive theta lifts in terms of these types of objects, including also the discriminant function

$$\Delta_\mu(\tau) = (\eta_\mu(\tau))^{24} = \mathcal{E}^{\rm hol}_{\mu,4}(\tau)^3 - \mathcal{E}^{\rm hol}_{\mu,6}(\tau)^2 . \tag{B.8}$$

For basic numerical checks, the Fourier series converges much faster than the double sum. (Alternatively, the integral representation is also exponentially convergent.)

Incidentally, if we don't set $s = 2k$ above, we can define the more general mixed deformation

$$\mathcal{E}^{\rm mixed}_{\mu,s,2k}(\tau) = \frac{1}{\pi^{2k}} \partial_z^{2k} \mathcal{E}_{\mu,s}(z;\tau)\Big|_{z=0} \tag{B.9}$$

so for $\mu = 0$ we generate the mixed object

$$\mathcal{E}^{\rm mixed}_{\mu=0,s,2k}(\tau) = \sum_{m,n}{}' \frac{(m\bar{\tau}+n)^{2k}}{|m\tau+n|^{2s}} = \sum_{m,n}{}' \frac{1}{(m\tau+n)^{2k}|m\tau+n|^{2s-2k}} . \tag{B.10}$$

C Integral representation as analytic continuation

In the main text we encounter

$$\int \frac{d^2\tau}{\tau_2^2} \Theta(T,U,\tau) E_s^{(\ell)}(\tau) = E^{(\ell)}_{{\rm V},s}(T,U) \tag{C.1}$$

In the degenerate case $\ell = 0$, then $E^{(0)}_{{\rm V},s}(T,U)$ is an $SO(2,2)$ vector representation Langlands-Eisenstein series (see for example eq. (4.135) in [15]), hence the "V" for vector.

Degenerate lattice Eisenstein series: Θ_2

We first review the standard argument that gives analytic continuation of $E_s(U)$ as integral representation. We follow Siegel's Tata notes [29]. Some other references are appendix E in [10], appendix A in [4] and Bump's book [11] where this is Exercise 1.6.2. (Presumably the book's author would require more rigor for full credit than the meager offering on display here.)

By scaling $\tau_2/T_2 \to \tau_2$, which moves the constant T_2^s out front,

$$\Theta_2(U,\tau_2) = \sum_{n_1,n_2} e^{-\frac{\pi\tau_2}{U_2}|n_2 U - n_1|^2} \tag{C.2}$$

(as opposed to the quadruple sum $\Theta_{2,2}(T,U,\tau)$) so $\Theta_2(U,\tau_2) = \tau_2^{-1}\Theta_2(U,1/\tau_2)$, analogously to the familiar $\vartheta_3(i\tau_2) = \tau_2^{-1/2}\vartheta_3(i/\tau_2)$. As in Riemann's paper with the integral representation of $\zeta(s)$, we split up the vertical strip integral into $[0,1]$ and $[1,\infty]$:

$$\int_0^\infty \tau_2^{s-1}\Theta_2(U,\tau_2)d\tau_2 = \underbrace{\int_0^1 \tau_2^{s-1}\Theta_2(U,\tau_2)d\tau_2}_{I_1(s)} + \underbrace{\int_1^\infty \tau_2^{s-1}\Theta_2(U,\tau_2)d\tau_2}_{I_2(s)} \tag{C.3}$$

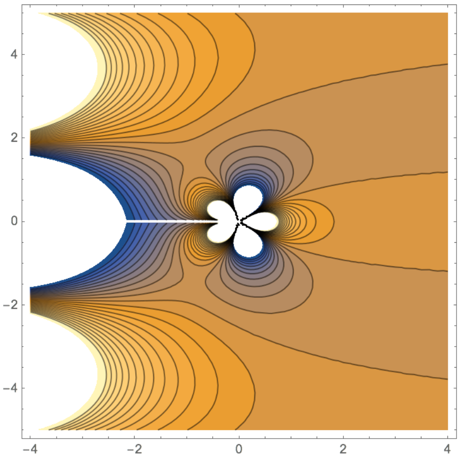

Figure 6. Analytic continuation of the exponential integral: Re $\mathtt{E}_{-3/2}(z)$.

and we use $\Theta_2(U,\tau_2) = \tau_2^{-1}\Theta_2(U,1/\tau_2)$ in the $[0,1]$ piece I_1 and change variable of integration $\tilde{\tau}_2 = 1/\tau_2$:

$$I_1(s) = \int_0^1 \tau_2^{s-1}\tau_2^{-1}\Theta_2(U,1/\tau_2)d\tau_2 = \int_1^\infty \tilde{\tau}_2^{2-s}\Theta_2(U,\tilde{\tau}_2)\tilde{\tau}_2^{-2}d\tilde{\tau}_2$$
$$= \int_1^\infty \tilde{\tau}_2^{-s}\Theta_2(U,\tilde{\tau}_2)d\tilde{\tau}_2 = I_2(1-s)$$

Now both terms are over the interval $[1,\infty]$, and the sums converge exponentially. Separating out the term $(n_1,n_2) = (0,0)$, for $s \geq 0$ we should put an upper cutoff L to be able to integrate it. Denoting the sum without $(n_1,n_2) = (0,0)$ with a prime, we find:

$$I_2(s)\Big|_L = \int_1^\infty \tau_2^{s-1}\sum_{n_1,n_2}{}' e^{-\frac{\pi\tau_2}{U_2}|n_2U-n_1|^2}d\tau_2 + \int_1^L \tau_2^{s-1}d\tau_2 = \sum_{n_1,n_2}{}' \mathtt{E}_{1-s}\left(\frac{\pi}{U_2}|n_2U-n_1|^2\right) + \frac{L^s-1}{s} \tag{C.4}$$

where $\mathtt{E}_s$ is the exponential integral

$$\mathtt{E}_s(z) = \int_1^\infty t^{-s}e^{-zt}dt \tag{C.5}$$

which for large real arguments $z = x \to \infty$ is bounded by $e^{-x}\ln(1+1/x)$, so the sum converges quickly. The exponential integral is related to the incomplete gamma function as $\mathtt{E}_s(z) = x^{s-1}\Gamma(1-s,z)$, and is a special case of the Kummer U function as $\mathtt{E}_s(z) = e^{-z}\mathrm{U}(1,1,z)$.

We now subtract the term L^s/s, take the limit $L \to \infty$, and the remaining total is

$$\int_0^\infty \tau_2^{s-1}\Theta_2(U,\tau_2)d\tau_2 = \sum_{n_1,n_2}{}' \mathtt{E}_s\left(\frac{\pi}{U_2}|n_2U-n_1|^2\right) + \frac{1}{s-1} + \sum_{n_1,n_2}{}' \mathtt{E}_{1-s}\left(\frac{\pi}{U_2}|n_2U-n_1|^2\right) - \frac{1}{s}, \tag{C.6}$$

which provides the desired analytic continuation to all s away from the poles at $s=0$ and $s=1$. Reflection symmetry s to $1-s$ is manifest from switching the two pairs of terms.

Contrast this with the direct integration for Re $s>1$:

$$\int_0^\infty \tau_2^{s-1}\Theta_2(U,\tau_2) = \pi^{-s}\Gamma(s)\sum_{n_1,n_2}\frac{U_2^s}{|n_2U-n_1|^{2s}} = \pi^{-s}\Gamma(s)2\zeta(2s)E_s(U) =: E_s^\star(U) \quad \text{(C.7)}$$

where $E_s^\star(U)=\xi(2s)E_s(U)$ and $\xi(2s)=\pi^{-s}\Gamma(s)\zeta(2s)$. Formally the above observation of the symmetry under $s\leftrightarrow 1-s$ verifies that

$$E_{1-s}^\star(U) = E_s^\star(U) \tag{C.8}$$

but of course, the double sum representation is never valid on both sides of this functional relation simultaneously. For later, it is useful to give the first equality (C.7) in more explicitly: we move the sum out of the integral, absorb the exponent's U dependence into τ_2 as $\hat\tau_2=\frac{\tau_2}{U_2}|n_2U-n_1|^2$ and re-emit it in front:

$$\int_0^\infty \tau_2^{s-1}\sum_{n_1,n_2}e^{-\frac{\pi\tau_2}{U_2}|n_2U-n_1|^2}\,d\tau_2 = \sum_{n_1,n_2}\frac{U_2^s}{|n_2U-n_1|^{2s}}\int_0^\infty \hat\tau_2^{s-1}e^{-\pi\hat\tau_2}\,d\hat\tau_2 \tag{C.9}$$

and the remaining $\hat\tau_2$ integral produces $\pi^{-s}\Gamma(s)$. Again, this requires Re $s>1$ for the double sum to converge.

As a concrete example that the double sums in (C.6) above provide a sensible truncation prescription, we want to compare to Kronecker's first limit formula (for more on normalizations, see appendix F)

$$E_s(\tau) = \frac{3}{\pi(s-1)} + \frac{6}{\pi}\left(12\log A - \log(4\pi) - \log(\sqrt{\tau_2}|\eta(\tau)|^2)\right) + \dots \tag{C.10}$$

Let us brutally truncate to $|n_1|\le 1, |n_2|\le 1$. We find at $U=i$ that

$$\left(\text{r.h.s. of eq. } (C.6) - \frac{1}{s-1}\right)_{|n_1|,|n_2|\le 1} = 4(\mathtt{E}_1(\pi)+\mathtt{E}_1(2\pi)) + \frac{4}{\pi}(e^{-\pi}+\frac{1}{2}e^{-2\pi}) - 1 = -0.89912\dots \tag{C.11}$$

whereas (C.10) gives, using $\eta(i)=\Gamma(1/4)^4/(16\pi^3)$, the exact result

$$\left(E_s(i) - \frac{3}{\pi(s-1)}\right)_{s\to 1} = \gamma + \log(4\pi) - 4\log\Gamma(1/4) = -0.89912\dots \tag{C.12}$$

so agreement is good for such a simple truncation. We plot them for a few values of U in figure 7.

Degenerate lattice Eisenstein series: $\Theta_{2,2}$

For the quadruple sum with $\ell=0$, there one factor in the exponent for U and one factor for T, and only a single variable of integration τ_2, so the summand appears coupled as in (3.24):

$$\int_0^\infty \tau_2^{s-1}\Theta_{2,2}^{(\ell=0)}(T,U,\tau_2) = \int_0^\infty \tau_2^{s-1}\sum_{c,d}\sum_{n_1^*,n_2^*}e^{-\frac{\pi\tau_2}{T_2U_2}|n_2^*U-n_1^*|^2|dT-c|^2}\,d\tau_2\,. \tag{C.13}$$

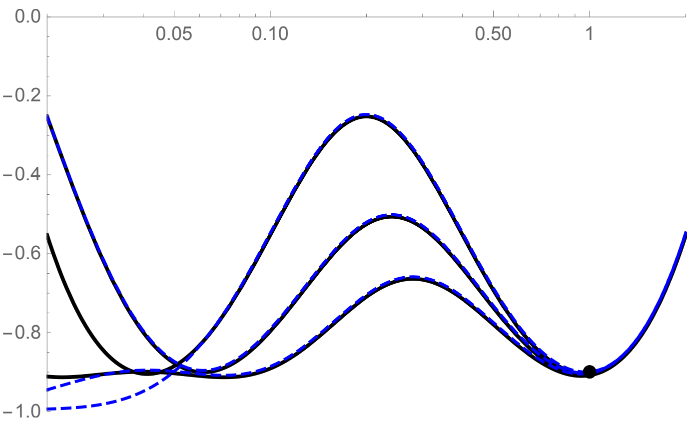

Figure 7. The truncation eq. (C.11) to $|n_1| \leq 4, |n_2| \leq 4$ in solid black, and eq. (C.12) in dashed blue. Here $U = 0.2, 0, 24, 0.28$. The black dot is -0.899 at $U_2 = 1$. For $|n_1| \leq 5, |n_2| \leq 5$ the curves become indistinguishable.

For Re $s > 1$, the point of (C.9) was to make it evident that they both just come out front, which was (3.25). We see that it is because there is just a power τ_2^{s-1} that allows it to factor. (In the massive case, there is not just a power and it will not factor.)

$$E_s^\star(T)E_s^\star(U) = \left(\mathtt{E}_s[T] + \frac{1}{s-1} + \mathtt{E}_{1-s}[T] - \frac{1}{s}\right)\left(\mathtt{E}_s[U] + \frac{1}{s-1} + \mathtt{E}_{1-s}[U] - \frac{1}{s}\right) \quad \text{(C.14)}$$

with the shorthand $\mathtt{E}_s[U] = \sum' \mathtt{E}_s(\pi|n_2U - n_1|^2/U_2)$. So the completed integral over the quadruple sum must split into these 16 terms.

D Truncated sums

To compute numerical values and make the plots in this paper, we were led to consider some basic issues in numerical analysis. Detailed bounds and estimates are beyond the scope of this paper, but here are three examples of truncation issues that might be of use to readers who are interested in numbers and plots.

One general aspect is that modular invariance requires infinite sums. Truncated (partial) sums are not sufficient in general. This means that some equations in this paper do not automatically hold for partial sums. The point of this appendix is that certain equations can still be *arranged* to hold.

Example 1. If we tried to trace the analytic continuation in (C.6) back to the original integral (C.3), we find an exponential integral $\mathtt{E}(1/\tau_2)$ that is not integrable on the required interval. This is the usual physics approach to regularization: the original expression is in fact divergent, and is only to be viewed as a "sketch". The analytic continuation (C.6) can then be thought of as a "definition", that gives meaning to (C.3). As we show in the example (C.11), this cutoff analytic continuation can in fact be truncated to just a few terms that give a decent approximation.

Note that for truncated sums, this is different from the usual analytic continuation, where we subtract a pole as $E_s - 1/(s-1)$. The double sum representation for E_s, although it converges for $s > 1$, converges slower and slower for $s \to 1$. Only for infinite summation does the pole appear that we should subtract. For our purposes, eq. (C.6) is more useful for truncation. In (C.12) we were only able to get a number from pole subtraction followed by truncation because we had the exact expression from Kronecker's first limit formula.

(In the main text we use contour deformation, not cutoff as in this example. From the point of view of truncations, both contour deformation and cutoff are sufficiently explicit. A comparison is given in section 4.4. See also appendix E.)

Example 2. Adding (3.21) and (3.24) should give the factorization property (3.25). But in (3.21), $E_s(U)$ arises as an unconstrained double sum, whereas in (3.24), $E_s(U)$ arises as a sum over coprime integers. These two sum representations for $E_s(U)$ are only proportional for infinite summation. If truncated as they first arise, the sums run over different subsets, and generically factorization will not hold for partial sums. This would seem problematic, because convergence of the (quadruple) lattice Eisenstein series for $s > 1$ (as opposed to $s > 2$) relies on the factorization property.

But this is easy to resolve: in the final equality of (3.21), we have our E_s, which we have defined by coprime summation. As long as we truncate this expression, and not the unconstrained sum before the last equality sign, the factorization property will be consistently represented on the partial sums.

In particular, the argument in the previous example applies: we could use the integral representation (C.6) to efficiently compute truncated values for E_s, and hence by (3.25) for the degenerate ($\ell = 0$) lattice Eisenstein series.

Example 3. For the nondegenerate ($\ell \neq 0$) lattice Eisenstein series, we pick a fiducial solution (n_1^*, n_2^*) of the constraint, that for fixed (k^1, k^2) generates (n_1, n_2) by varying a single integer M. If truncated carelessly, the sum may become asymmetrically truncated in different directions in the (k^1, k^2, n_1, n_2) lattice, in which case plots of truncated sums may not be representative, and change significantly if more terms are included. This can be avoided for example by first picking outer truncation boundaries $|n_1|, |n_2| \leq n_{\text{trunc}}$, and then truncating $|M| \leq M_{\text{trunc}}$ to a range that we increase until it "fills in" the range $|n_1|, |n_2| \leq n_{\text{trunc}}$, for example by inspecting a graphical representation as in figure 2.

We have included an ancillary file that shows in a few steps how the solution set is filled in by increasing the value of M_{trunc}.

E Contour integrals and the Meijer G function

Contour integration is the first way that Riemann in his 1859 paper[4] provided an analytic continuation of $\zeta(s)$, the second way being the type of analytic continuation we discussed in a previous appendix, with ϑ in place of Θ. So it is natural to consider the first point of view here as well.

The Meijer G function is defined from a Mellin-Barnes representation [32]

$$G^{m,n}_{p,q}\left(\begin{matrix}\mathbf{a}_p \\ \mathbf{b}_q\end{matrix}, z, r\right) = \frac{r}{2\pi i}\int_C \frac{\prod_{j=1}^m \Gamma(b_j - rs)\prod_{j=1}^n \Gamma(1 - a_j + rs)}{\prod_{j=m+1}^q \Gamma(1 - b_j + rs)\prod_{j=n+1}^p \Gamma(a_j - rs)} z^s ds \tag{E.1}$$

[4]For a discussion of Riemann's paper, see [31].

where $\mathbf{a}_p = (a_1, \ldots, a_p)$, $\mathbf{b}_q = (b_1, \ldots, b_q)$ and the contour C separates the poles of $\Gamma(b_1 - rs)$, $\ldots \Gamma(b_1 - rs)$ from poles of $\Gamma(1 - a_1 + rs)$, $\ldots \Gamma(1 - a_n + rs)$. We give an example below. Note that q and p are in reverse order, and that the original references consider the case $r = 1$. When we omit r, we intend $r = 1$.[5]

In view of the definition (E.1), it is not surprising that the Mellin transform of the Meijer G function is simply gamma functions of the parameters and a power:

$$\int_0^\infty z^{s-1} G^{m,n}_{p,q}\left(\begin{matrix}\mathbf{a}_p \\ \mathbf{b}_q\end{matrix}, wz\right) dz = \frac{\prod_{j=1}^m \Gamma(b_j + s) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - s) \prod_{j=n+1}^p \Gamma(a_j + s)} w^{-s} \tag{E.2}$$

A physicist might view the integration along the real axis (E.2) as a more natural starting point for a definition than (E.1), but the real axis is no less arbitrary as a contour than the Mellin contour above.

The set of Meijer G functions is closed under convolution:

$$\begin{aligned}&\int_0^\infty G^{m',n'}_{p',q'}\left(\begin{matrix}\mathbf{a}_{p'} \\ \mathbf{b}_{q'}\end{matrix}, yz\right) G^{m,n}_{p,q}\left(\begin{matrix}\mathbf{a}_p \\ \mathbf{b}_q\end{matrix}, xz\right) dz \\ &= G^{m+n',n+m'}_{p+q',q+p'}\left(\begin{matrix}-b_{1'}, \ldots, -b_{m'}, \mathbf{a}_p, -b_{m'+1}, \ldots -b_{q'} \\ -a_{1'}, \ldots, -a_{n'}, \mathbf{b}_q, -a_{n'+1}, \ldots, -a_{p'}\end{matrix}, \frac{x}{y}\right)\end{aligned} \tag{E.3}$$

The x/y introduces an asymmetry, but there is an equivalent representation with y/x using the inversion property

$$G^{m,n}_{p,q}\left(\begin{matrix}\mathbf{a}_p \\ \mathbf{b}_q\end{matrix}\middle| z\right) = G^{n,m}_{q,p}\left(\begin{matrix}1 - \mathbf{b}_q \\ 1 - \mathbf{a}_p\end{matrix}\middle| z^{-1}\right) \tag{E.4}$$

One advantage compared to a hypergeometric function is that inverting the argument is still a single Meijer-G.

Representations of known functions include

$$\begin{aligned}&G^{1,0}_{0,1}\left(\begin{matrix}- \\ b_1\end{matrix}, z\right) = z^{b_1} e^{-z}, \qquad G^{0,1}_{1,0}\left(\begin{matrix}a_1 \\ -\end{matrix}, z\right) = z^{a_1-1} e^{-1/z}, \\ &G^{2,0}_{0,2}\left(\begin{matrix}- \\ b_1, b_2\end{matrix}, z\right) = 2z^{(b_1+b_2)/2} K_{b_1-b_2}(2\sqrt{z})\end{aligned} \tag{E.5}$$

and

$$G^{2,1}_{1,2}\left(\begin{matrix}a_1 \\ b_1, b_2\end{matrix}, z\right) = \Gamma(a_1 + b_1 + 1)\Gamma(a_1 + b_2 + 1)\mathrm{U}(a_1 + b_1 + 1, a_1 + b_1 + 1, z) \tag{E.6}$$

where U is the Kummer confluent hypergeometric function, related to the Whittaker function as

$$W_{a,b}(z) = e^{-z/2} z^{b+1/2} \mathrm{U}(b - a + 1/2, 1 + 2b, z). \tag{E.7}$$

[5]Mathematica uses $r \neq 1$ for its most basic functions, for example try `MeijerGReduce[BesselK[s,z],z]` which gives $r = 1/2$ as default. A representation of $K_s(z)$ with $r = 1$ is given in (E.5). Also, Mathematica's s is minus that in equation (E.1), which does not change the result since s is a variable of integration, but it changes the explicit representation of the contour in the sense of orientation, including whether a contour is "to the left" or "to the right" of poles.

These examples provide an illustration of the integration (E.3)

$$\int_0^\infty G^{1,0}_{0,1}\left(\begin{matrix}-\\ b_{1'}\end{matrix}, z\right) G^{2,0}_{0,2}\left(\begin{matrix}-\\ b_1, b_2\end{matrix}, xz\right) dz = G^{2+0,0+1}_{0+1,2+0}\left(\begin{matrix}-b_{1'}\\ b_1, b_2\end{matrix}, x\right) = G^{2,1}_{1,2}\left(\begin{matrix}-b_{1'}\\ b_1, b_2\end{matrix}, x\right) \tag{E.8}$$

which if we write it out using (E.5) and (E.6) corresponds to

$$\begin{gathered}\int_0^\infty (z^{b_{1'}} e^{-z})(2(xz)^{(b_1+b_2)/2} K_{b_1-b_2}(2\sqrt{xz}))\, dz = \\ \Gamma(b_1 + b_{1'} + 1)\Gamma(b_{1'} + b_2 + 1)\mathrm{U}(b_1 + b_{1'} + 1, b_1 + b_{1'} + 1, x)\end{gathered} \tag{E.9}$$

provided $b_1 + b_{1'} > -1$, $b_2 + b_{1'} > -1$, $(b_1 + b_2)/2 + b_{1'} > -1$. The result (E.9) can be checked directly using power series representations of the Bessel $I_s(z)$ and the Kummer $M(a, b, z)$, that $K_s(z)$ and $\mathrm{U}(a, b, z)$ are linear combinations of, respectively.

We can easily map example (E.8) to similar formulas that might be less familiar. For example, using (E.4) we can map between the two elementary examples $G^{0,1}_{1,0}$ and $G^{1,0}_{0,1}$ in (E.8):

$$(z^{-1})^{b_1} e^{-z^{-1}} = G^{1,0}_{0,1}\left(\begin{matrix}-\\ b_1\end{matrix}, z^{-1}\right) = G^{0,1}_{1,0}\left(\begin{matrix}1 - b_1\\ -\end{matrix}, z\right) = z^{(1-b_1)-1} e^{-1/z} = z^{-b_1} e^{-1/z} \,. \tag{E.10}$$

Note that we did not quite obtain $G^{0,1}_{1,0}(b_1, z^{-1})$, since the arguments $\mathbf{a}_p$, $\mathbf{b}_q$ are acted upon by the inversion (E.4). Mapping $G^{1,0}_{0,1}$ in example (E.8) to $G^{0,1}_{1,0}$ we find our second integration example

$$\int_0^\infty G^{0,1}_{1,0}\left(\begin{matrix}a_{1'}\\ -\end{matrix}, z\right) G^{2,0}_{0,2}\left(\begin{matrix}-\\ b_1, b_2\end{matrix}, xz\right) dz = G^{2+1,0+0}_{0+0,2+1}\left(\begin{matrix}-\\ -a_{1'}, b_1, b_2\end{matrix}, x\right) = G^{3,0}_{0,3}\left(\begin{matrix}-\\ -a_{1'}, b_1, b_2\end{matrix}, x\right) \tag{E.11}$$

which if we write it out using (E.5) and (E.6) corresponds to

$$\int_0^\infty (z^{a_{1'}-1} e^{-1/z})(2(xz)^{(b_1+b_2)/2} K_{b_1-b_2}(2\sqrt{xz}))\, dz = G^{3,0}_{0,3}\left(\begin{matrix}-\\ -a_{1'}, b_1, b_2\end{matrix}, x\right) \tag{E.12}$$

which unlike (E.8) is not a single hypergeometric function, since neither m nor n is equal to one (cf. (E.15) below). To check (E.11), we can compare to 3.16.3.9 in [34], Vol.4.

Note that the difference between (E.8) and (E.11) is not merely a change of variable of integration z: the inversion (E.10) only acted on the first factor in the integrand, whereas a change of variables of course acts on both factors, and indeed the right hand sides of (E.8) and (E.11) are not the same. The replacement of $z^{b_{1'}}$ by $z^{a_{1'}-1}$ can be absorbed by relating the parameters, but replacing e^{-z} with $e^{-1/z}$ changes the analytic properties of the integrand, making it more like a combination of ${}_0F_2$ hypergeometric functions, whereas U is a combination of ${}_1F_1$ hypergeometric functions. This inversion in the exponent can come from S-transforming a lattice theta function, so in this sense Meijer $G^{3,0}_{0,3}$ is “dual” to Kummer U.

JHEP05(2023)062

One comment about representation: we see from definition (E.1) that a power can always be absorbed in a shift of all arguments

$$z^{\mu} G^{m,n}_{p,q}\left(\begin{matrix}\mathbf{a}_p\\ \mathbf{b}_q\end{matrix}, z\right) = G^{m,n}_{p,q}\left(\begin{matrix}\mathbf{a}_p+\mu\\ \mathbf{b}_q+\mu\end{matrix}, z\right) \tag{E.13}$$

where $\mathbf{a}_p + \mu$ means adding μ to all components of the vector $\mathbf{a}_p$. For example, in the special case $\mathbf{b}_q = (0, 0, -s)$ of the integral (E.11) that appears in the main text, we have two alternative representations:

$$G^{3,0}_{0,3}\left(\begin{matrix}-\\ -s,0,0\end{matrix}, z\right) = z^{-s/2} G^{3,0}_{0,3}\left(\begin{matrix}-\\ -\frac{s}{2},\frac{s}{2},\frac{s}{2}\end{matrix}, z\right)\,. \tag{E.14}$$

To end this appendix, we can compare the definition (E.1) with the hypergeometric function ${}_pF_q$, that always has a power series representation. The Meijer $G^{m,n}_{p,q}$ does not necessarily have a single power series representation, but neither does K_s or U — each is a sum of two terms. Meijer $G^{m,n}_{p,q}$ has some other useful properties as discussed above. Meijer G and hypergeometric functions are put on the same footing by the Barnes representation of hypergeometric functions, where Meijer $G^{m,n}_{p,q}$ is specialized to one of the upper arguments m and n being one, say $m = 1$, and $n = p$:

$${}_pF_q\left(\begin{matrix}\mathbf{a}_p\\ \mathbf{b}_q\end{matrix}, z\right) = \frac{\Gamma(\mathbf{b}_q)}{\Gamma(\mathbf{a}_p)} G^{1,p}_{p,q+1}\left(\begin{matrix}1-\mathbf{a}_p\\ 0,1-\mathbf{b}_q\end{matrix}, -z\right)\,, \tag{E.15}$$

and $G^{2,1}_{1,2}$ in (E.8) is related to $G^{1,2}_{2,1}$ (which is in the form (E.15) for $p = 2$, $q = 0$, i.e. ${}_0F_2$) by the inversion (E.4), so the argument is inverted to $-z^{-1}$.

A perhaps more familiar example of (E.15) is $p = 2$, $q = 1$ which gives ${}_2F_1$ in the Barnes representation:

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds\,. \tag{E.16}$$

Now, other Meijer G functions like $G^{3,0}_{0,3}$, where neither m nor n are equal to one as in (E.15), are not hypergeometric, but they can be sums of hypergeometric functions. To make the connection to the main text, we note that if the integral (E.1) converges, and the m factors $\Gamma(b_j - s)$ have no confluent poles, evaluation of $G^{m,n}_{p,q}$ by residues gives m terms with hypergeometric functions. For $G^{3,0}_{0,3}$ this means that if the differences between the arguments (b_1, b_2, b_3) are not integers, $G^{3,0}_{0,3}$ has a representation as a sum of $m = 3$ terms with ${}_0\widetilde{F}_2$ hypergeometric functions:[6]

$$\begin{aligned} G^{3,0}_{0,3}\left(\begin{matrix}-\\ b_1,b_2,b_3\end{matrix}, x\right) = \pi^2\Bigg(&\frac{z^{b_1}{}_0\widetilde{F}_2(b_1-b_2+1,b_1-b_3+1,-z)}{\sin(\pi(b_2-b_1))\sin(\pi(b_3-b_1))}\\ &+\frac{z^{b_2}{}_0\widetilde{F}_2(1-b_1+b_2,b_2-b_3+1,-z)}{\sin(\pi(b_1-b_2))\sin(\pi(b_3-b_2))}\\ &+\frac{z^{b_3}{}_0\widetilde{F}_2(1-b_1+b_3,1-b_2+b_3,-z)}{\sin(\pi(b_1-b_3))\sin(\pi(b_2-b_3))}\Bigg) \end{aligned} \tag{E.17}$$

[6]See Wolfram Functions, Meijer G.

JHEP05(2023)062

where the standard regularized hypergeometric function is

$$ {}_0\widetilde{F}_2(b_1,b_2,z)=\frac{{}_0F_2(b_1,b_2,z)}{\Gamma(b_1)\Gamma(b_2)} \tag{E.18}$$

that is finite for finite values of its arguments. Conversely, ${}_0F_2$ itself has poles for example if an argument becomes zero, that is divided out in (E.18).

In our example $(b_1,b_2,b_3)=(0,0,-s)$, to use the specialized expression (E.17) we would have to introduce regularization by hand, but we emphasize that this is not in the spirit of the definition (E.1) where the contour avoids poles by definition. Still, it might be of some use to state what regularization of (E.17) by hand would mean. For example, we can set $b_2=\delta>0$ to prevent the first two denominators in (E.17) from vanishing as b_1-b_2. Additionally we are interested in $s=0$ and $s=1$, and set $s=\epsilon$ or $s=1+\epsilon$ with $\epsilon>0$, respectively, and then set $\delta=\epsilon$, we have simple and double poles in ϵ. The choices $b_2=\delta$ and $\delta=\epsilon$ represent regularization ambiguities. However for our purposes we will not need (E.17), we only provided it for context.

A final comment: one might ordinarily think of a Bessel function with a square root in its argument $K_s(2\sqrt{x})$ as more complicated than one without a square root, but to get a Bessel function *without* a square root we need to go one step *higher* in the Meijer-G hierarchy, from $p=0$ in (E.6) to $p=1$:

$$G^{2,0}_{1,2}\left(\begin{matrix}1/2\\ s,-s\end{matrix},z\right)=\pi^{-1/2}e^{-x/2}K_s(x/2)\,. \tag{E.19}$$

So the Bessel function with a square root in the argument $K_s(2\sqrt{x})$, as we get from the massive Eisenstein series, is more basic than $K_s(x/2)$ in the sense that the former requires one less gamma function in the Mellin-Barnes expression (E.1). The I Bessel function without a square root in the argument is similar to (E.19) with $m=2,n=0$ replaced by $m=1,n=1$:

$$G^{1,1}_{1,2}\left(\begin{matrix}1/2\\ s,-s\end{matrix},z\right)=\pi^{1/2}e^{-x/2}I_s(x/2)\,. \tag{E.20}$$

This representation and the algorithmic convolution in (E.3) could perhaps be of use to combine the massive Eisenstein series with Niebur-Poincaré series.

F Three normalizations of $\mathrm{SL}(2,\mathbb{Z})$ Eisenstein series

We use the normalization from the book [15], Ch. 10 (Ch. 11 in the arXiv version), where the seed is just τ^s with no additional factor. In particular,

$$E_s=\tau_2^s+\frac{\xi(2s-1)}{\xi(2s)}\tau_2^{1-s}+\frac{2}{\xi(2s)}\tau_2^{1/2}\sum_{m\neq 0}|m|^{s-1/2}\sigma_{1-2s}(m)K_{s-1/2}(2\pi|m|\tau_2)e^{2\pi i\tau_1} \tag{F.1}$$

with the completed zeta function $\xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$. The pole at $s=1$ comes from the second term (constant $\tau_2^{1-s}=\tau_2^0$ in the limit $s=1$) and the residue is $3/\pi$.

JHEP05(2023)062

We are typically interested in the first subleading term near a pole. In the same reference [15] section 10.1,

$$E_s(\tau) = \frac{3}{\pi(s-1)} + \frac{6}{\pi}\left(12\log A - \log(4\pi) - \log(\sqrt{\tau_2}|\eta(\tau)|^2)\right) + \dots \tag{F.2}$$

where A is the Glaisher constant $A = e^{1/12-\zeta'(1)}$.

If we change overall normalization, there is both a multiplicative factor and an additive constant in the subleading term. Explicitly, if the new normalization $N(s)$ is finite at $s=1$ (as will be the case in the examples below):

$$\begin{aligned} N(s)E_s(\tau) &= (N(1) + N'(1)(s-1) + \dots)\left(\frac{3}{\pi(s-1)} + E_s'(\tau)\Big|_{s=1} + \dots\right) \\ &= N(1)\left(\frac{3}{\pi(s-1)} + E_s'(\tau)\Big|_{s=1}\right) + \frac{3}{\pi}\cdot N'(1)\,. \end{aligned} \tag{F.3}$$

The first common example of normalization other than (F.1) is if the lattice sum is not restricted to mutually prime m, n, that gives a change of normalization by $N = 2\zeta(2s)$. We have from (F.3) that

$$2\zeta(2s)E_s(\tau) = \frac{\pi}{(s-1)} + 2\pi\left(\gamma - \log(2) - \log(\sqrt{\tau_2}|\eta(\tau)|^2)\right) + \dots \tag{F.4}$$

where γ is the Euler-Mascheroni constant. The factor in front of the subleading term is $N(1)\cdot(6/\pi) = 2\zeta(2)\cdot(6/\pi) = 2\pi$, the shift is $\frac{N'(1)}{N(1)} = \frac{2\zeta'(2)}{\zeta(2)}$ and (F.2) and (F.4) are consistent because

$$12\log A - \log(4\pi) = \gamma - \log(2) - \frac{\zeta'(2)}{\zeta(2)}\,. \tag{F.5}$$

A third common normalization is the completed version:

$$E_s^\star = \xi(2s)E_s = \pi^{-s}\Gamma(s)\zeta(2s)E_s \tag{F.6}$$

with the completed zeta function ξ given at the beginning of this section. In particular $\xi(2) = \pi^{-1}\Gamma(1)\zeta(2) = \pi^{-1}\pi^2/6 = \pi/6$. (Note $\xi(2)/\zeta(2) = \pi^{-1}$.)

For $E_s^\star$, we can check from (F.1) that the coefficient of τ_2^{1-s} is

$$\pi^{-s}\Gamma(s)\cdot\zeta(2s)\cdot\frac{\xi(2s-1)}{\xi(2s)} = \pi^{1/2-s}\Gamma(s-1/2)\zeta(2s-1) = \frac{1}{2(s-1)} + \dots \tag{F.7}$$

so E_s^* has residue $1/2$ at $s=1$.

G Ramanujan duality

If we know how an automorphic form transforms under Weyl reflection, this will imply some summation identity for the Fourier sum. For example, if we Fourier expand the identity $E_s(\tau_2) = E_s(1/\tau_2)$ we find an identity that according to [38] was given by Ramanujan:

$$\begin{aligned} 4\sqrt{x}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s/2}}K_{s/2}(2\pi n x) &= \frac{4}{\sqrt{x}}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s/2}}K_{s/2}\left(\frac{2\pi n}{x}\right) \\ &\quad + \xi(-s)(x^{(1+s)/2} - x^{(1+s)/2}) - \xi(s)(x^{(1-s)/2} - x^{(s-1)/2}) \end{aligned} \tag{G.1}$$

which for $s=1$, when the Bessel functions reduce to exponential functions, implies the modular transformation of $|\eta(\tau)|$. We will refer to the summation (G.1) as "Ramanujan duality" $x \leftrightarrow 1/x$.

In [39], Ramanujan duality for Dirichlet series and L-functions is studied, and we find a similar duality $r \leftrightarrow c$ as follows:

$$\sum_{n=0}^{\infty} \frac{\sigma_k(n)}{(c^2+n)^{\nu/2}} K_\nu(4\pi r\sqrt{c^2+n}) = -\frac{\delta_{1,k}}{4\pi r c^{\nu-1}} K_{\nu-1}(4\pi rc) + \frac{1}{r^\nu c^{\nu-k-1}} \sum_{n=0}^{\infty} \frac{(-1)^{(k+1)/2}\sigma_k(n)}{(r^2+n)^{(k+1-\nu)/2}} K_{k+1-\nu}(4\pi c\sqrt{r^2+n})\,. \tag{G.2}$$

Another is for χ a primitive character modulo q:

$$\sum_{n=0}^{\infty} \frac{n\chi(n)}{(c^2+n^2/(2q))^{\nu/2}} K_\nu(4\pi r\sqrt{c^2+n^2/(2q)}) = -\frac{i\tau(\chi)}{r^\nu c^{\nu-3/2}\sqrt{q}} \sum_{n=0}^{\infty} \frac{n\bar{\chi}(n)}{(r^2+n^2/(2q))^{(3/2-\nu)/2}} K_{3/2-\nu}(4\pi c\sqrt{r^2+n^2/(2q)})\,. \tag{G.3}$$

Both (G.2) and (G.3) are reminiscient of the summation identity that follows from equating Fourier coefficients for $\mathcal{E}_s(\tau_2)$ and $\mathcal{E}_s(1/\tau_2)$. It is natural to expect that this equivalence would yield similar summation identities as in (G.2) and (G.3), but we leave the details of this for future work.

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JHEP05(2023)062

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JHEP05(2023)062

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JHEP05(2023)062