Writing the History of Mathematics: Interpretations of the Mathematics of the Past and Its Relation to theMathematics of Today

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Writing the History of Mathematics: Interpretations of the Mathematics of the Past and Its Relation to the Mathematics of Today

Johanna Pejlare and Kajsa Bråting

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Abstract

In the present chapter, interpretations of the mathematics of the past are problematized, based on examples such as archeological artifacts, as well as written sources from the ancient Egyptian, Babylonian, and Greek civilizations. The distinction between *history* and *heritage* is considered in relation to Euler's function concept, Cauchy's sum theorem, and the Unguru debate. Also, the distinction between the *historical past* and the *practical past*, as well as the distinction between the *historical* and the *nonhistorical* relations to the past, are made concrete based on Torricelli's result on an infinitely long solid from the seventeenth century. Two complementary but different ways of analyzing

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the mathematics of the past are the *synchronic* and *diachronic* perspectives, which may be useful, for instance, regarding the history of school mathematics. Furthermore, recapitulation, or the belief that students' conceptual development in mathematics is paralleled to the historical epistemology of mathematics, is problematized emphasizing the important role of culture.

Keywords
History of mathematics · Epistemology of mathematics · Interpretations · History and heritage · Synchronic and diachronic perspectives · Recapitulation

Introduction

Why should we study the history of mathematics? That is a question often encountered among students of mathematics, as well as among people within other areas. One reason for studying the history of mathematics is to throw some light on the nature of the subject; to truly appreciate mathematics, it may be necessary to be acquainted with its history. The historical development of mathematics is also very closely linked to the cultural development of society: when studying the history of mathematics, at the same time, we learn something about ourselves and the society of today.

Based on a variety of examples, the intention of this chapter is to problematize interpretations of the mathematics of the past. The chapter begins by considering interpretations of ancient artifacts, as well as written sources. Probably already the first humans had a need for counting; by investigating surviving artifacts, such as notched bones, we can learn about the mathematical ideas they developed. However, it is difficult to make fair interpretations of isolated archeological findings. Also, it can for various reasons be problematic to interpret written sources from early civilizations such as ancient Egypt, Babylonia, and ancient Greece. From Egypt there are very few preserved papyri containing mathematics, which makes it difficult to derive an adequate interpretation of the mathematical knowledge of the ancient Egyptians. From Babylonia there is a large amount of preserved clay tablets containing mathematics, but the cuneiform symbols have been difficult to decode. There are also rather few original sources containing mathematics preserved from ancient Greece; instead, commentaries and translations of older texts in order to learn about their mathematical knowledge must be studied.

After considering the problems of interpreting ancient mathematics, we turn to different postures one can take toward the mathematics of the past. One important question within the research of history of mathematics concerns how historical texts should be interpreted in a proper way. The two approaches *history* and *heritage* are considered, where history deals with what happened in the past regardless of the modern situation, and heritage refers to the impact of a certain mathematical notion upon later work. Also, the distinction between *historical past* and *practical past*, both being experiences in the present dealing with artifacts from the past preserved to the present, are considered. Historical past is understood as the distinctness from the present, while practical past considers the past in terms of present values, needs, and ideas. Depending on the intention of the person who is exploring the past, both a more *historical* and *nonhistorical* relation to the past may be beneficial. Furthermore, the *synchronic* and *diachronic* perspectives, which are two complementary ways of analyzing, for example, the history of mathematics, are considered. The diachronic approach refers to the development along the time axis, while the synchronic approach refers to mathematics at a specific moment in time, without taking its history or further development into account.

The chapter will be concluded by considering the so-called recapitulation theory: the belief of a parallel between the historical development of mathematics and students' conceptual development in mathematics. The recapitulation theory was for a long time accepted among mathematicians as well as among mathematics educators. By considering the concrete example of the historical development of negative numbers, it is argued that local and cultural ideas about mathematics influence this development, and therefore it becomes problematic to assume that students of today, with very different cultural conditions compared to different periods and cultures in the history, would recapitulate the historical development.

Traces of Mathematics of the First Humans

Humanity has a long history. Several million years ago, humanity emerged when it separated from the common ancestor we have with the chimpanzees. Homo sapiens originated in Africa about 200,000 years ago and eventually replaced Homo erectus and Homo neanderthalensis. Typical recognizable human characteristics, involving the development of language and abstract thought, are believed to have arisen more than 40,000 years ago, marking the beginning of the Upper Paleolithic. Presumably already the first humans had a need for counting and developed mathematical thinking. There are obviously no written sources left from the Upper Paleolithic; instead, archeological artifacts have to be interpreted in order to say something about the mathematical knowledge of that time. However, it is not always easy to make credible interpretations.

The earliest known mathematical artifact is the Lebombo bone, estimated to have originated from around 37,000 years ago. It is a small piece of a fibula of a baboon, marked with 29 clearly defined notches, found in the Border cave between Swaziland and South Africa. The bone has been interpreted as a tally stick used for counting and may thus be the first hint of the emergence of calculation in human history. Generally, the notches are thought to represent counting by the principle of one-to-one correspondence, i.e., by pairing; each notch could represent one object, one person, or one day.

Since the Lebombo bone resembles the calendar stick still in use during the twentieth century by Bushman clans in Namibia, it is also believed that it may have been used as a lunar phase counter. With the 29 notches, in this interpretation, humans would be able to predict when the moon will be full. The close link between mathematics and astronomy has a long history. It is not unbelievable to suppose that

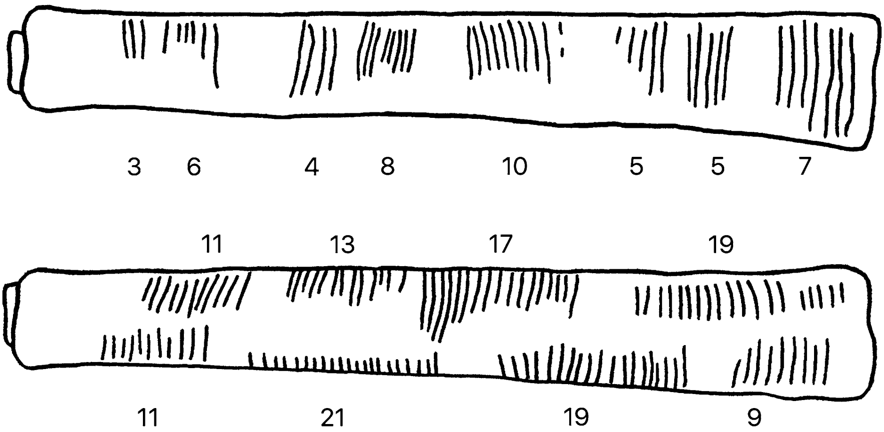

Fig. 1 A drawing of two sides of the Ishango bone showing the grouped notches

early humans felt the need to record the passage of time out of practical necessity or just out of curiosity. A lunar count, as, for example, a count from full moon to full moon or from new moon to new moon, would be the simplest possible early system of time reckoning. Reflecting about that women would benefit from keeping track of the menstrual cycle, which requires a lunar calendar, it is not difficult to draw the conclusion that the first mathematicians probably were women.

Another interesting prehistoric artifact is the Ishango bone, found at Ishango, on the shore of Lake Edward in the Democratic Republic of Congo. This bone was found during the excavation led by the geologist and archeologist Jean de Heinzelin (1920–1998) in the 1950s. It was first estimated to be about 8500 years old, but a re-dating of the archeological site where it was found suggests that it may be more than 20,000 years old. At one end of the bone is a piece of quartz, most probably for engraving purposes. There are asymmetrically grouped notches cut in the bone in three rows (see Fig. 1). The number of notches in each group in the three rows is, from left to right, as follows:

First row: 3, 6, 4, 8, 10, 5, 5, and 7
Second row: 11, 13, 17, and 19
Third row: 11, 21, 19, and 9

The carvings of this bone are more complex than the ones of the Lebombo bone, and there have been several interpretations of this bone's markings. The initial interpretation suggests that the notches are tally marks used for counting objects. However, the grouping of the notches indicate that they are more than a tally. One interpretation is that the bone indicates knowledge of simple arithmetic (De Heinzelin 1962). The first row indicates duplications, i.e., multiplication by two, among the first four groups of notches: the double of three is six and the double of four is eight. Furthermore, the ten in the first row has been interpreted to be divided into two equal groups of five, i.e., the two groups of five notches that follow the group of ten notches. The third row indicates the numbers 10 and 20 plus and minus 1, which may indicate a link to the number 10 which today is a well-known number base. The second row, however, has caused a lot of speculations about early humans' advanced knowledge of numbers, since it has been interpreted as a table of prime numbers, i.e., positive integers having exactly two positive integer factors (one and itself). However, to develop the concept of prime numbers, you first have to develop the concept of division, and there are no indications that this was understood at this time; it has been suggested that the concept of division was developed after 10,000 BC and that the concept of prime numbers was understood around 500 BC. Therefore, it is disputable that humans would have this knowledge already 20,000 years ago.

Pletser and Huylebrouck (1999) have proposed that the Ishango bone provides evidence for early humans' use of the base 12 and the subbases 3 and 4. Their main argument is that the numbers in the left and right column add up to 60 and the numbers in the middle column add up to 48, and since $60 = 5 \cdot 12$ and $48 = 4 \cdot 12$, they suggest that the bone indicates a numerical system in base 12. Also, the duplications of the 3 and the 4 in the middle column indicates that these numbers may be used as subbases, and since $3 \cdot 4 = 12$, this furthermore indicates a base 12. The base 12 interpretation could also explain the numbers in the left column: they can be seen as 12 ± 1 and 18 ± 1. Furthermore, the base 12 is, just as the base 10, obvious considering the anatomy of human hands: on one hand we can count the phalanges of the four fingers with the thumb. This way of counting is still practiced today by some populations. If we also keep track of the number of dozens on the five fingers of the other hand, the total will be 60.

Another interpretation is that the Ishango bone, just as the Lebombo bone, may be a lunar calendar. In a microscopic investigation of the bone, Marshack (1972) found additional markings and suggested that there is a strong evidence of a correspondence between the markings on the bone and the phases of the moon.

There is also evidence of early use of mathematics in Europe. In Dolni Vestonice in the Czech Republic, a 33,000-year-old wolf bone, believed to have been a tally stick, was found. In it 55 notches are carved, in groups of five. This suggests a use of a base five counting system. This special attention to five is probably due to the five fingers of one hand and has in fact been found in many primitive cultures.

The development of other research fields, for example, linguistics, may be helpful in learning more on early mathematical ideas. By studying how the numerals are structured in different languages, traces of how humans in past times may have comprehended numbers can be found. For example, the problem of a number base for handling bigger numbers has not been solved in the same way in all cultures. Many languages have numerals clearly in base ten, probably as a result of our ten fingers on our two hands. Based on their numerals, some people seem to have instead collected objects to be counted in groups of five, just like the humans of Dolni Vestonice possibly did 33,000 years ago. Thus, the human hand provided an important model for the structuring of numbers. One example of this is found when the Api language, spoken on the New Hebrides, is studied. The numerals from one to ten in the Api language are the following:

Tai	1	Otai ("the new one")	6
Lua	2	Olua ("the new two")	7
Tolu	3	Otula ("the new three")	8
Vari	4	Ovari ("the new four")	9
Luna ("hand")	5	Lualuna ("two hands")	10

It is noted, however, that the word for hand is not mentioned in the words for the numbers from six to nine – this notion seems to be already implied in the presence of the morpheme "new."

Also, in many African languages, the word for 10 means "two hands" or "two fives," and 15 may be expressed as "two hands and one hand." In some languages the word for 20 means "take one man," probably referring to the ten fingers and ten toes of a man. For example, in the Central African Banda language, the word for 15 means "three fists," and the word for 20 means "take one person." In this sense 20 could be seen as a number base. Traces of the base 20 can also be found in other cultures. For example, in the Inuit language, the numerals are structured with a base 20 and with subbases 5, 10, and 15.

History of Ancient Mathematics: The First Written Sources

Archeological discoveries such as notched bones provide evidence that the idea of numbers is much older than the art of writing and technological advances such as the use of metals. The humans using tally sticks were likely nomadic hunter-gatherers. When they transitioned to an agricultural lifestyle about 10,000 years ago, it allowed for a great social change: the opportunity to begin the development of permanent settlements and stable civilizations. The rise of civilizations first took place in river valleys, such as those in Egypt, Mesopotamia, China, and India. In ancient Egypt and Mesopotamia, forms of writing had developed before the end of the fourth millennium BC. Written sources can reveal knowledge of mathematics in ancient civilizations, but it can also be problematic to interpret historical texts, for different reasons. For example, there are very few mathematical texts from ancient Egypt preserved to our days. On the other hand, the Babylonians have left us many written sources containing mathematics, but the cuneiform symbols have been difficult to decode. From ancient Greece there are very few original sources preserved; instead commentaries and translations of older texts must be interpreted.

The ancient Egyptian society is the oldest culture of which it is known a bit more about their mathematical knowledge due to preserved written sources containing mathematics. Egypt had become a unified state when the Pharaoh Menes unified the Upper and Lower Nile valleys around 3150 BC, and it lasted until the Roman conquest in 30 BC with Cleopatra as the last pharaoh. The knowledge we have today of Egyptian mathematics mainly originates from two important sources in hieratic writing: the Rhind papyrus (also called the Ahmes papyrus, approximately 1550 BC) and the Moscow papyrus (approximately 1850 BC). The Rhind papyrus is a mathematical text resembling a practical handbook and contains 85 mathematical problems with solutions. The Moscow papyrus contains another 25 problems with solutions. All problems in these two papyri are numerical, and most of them have a practical origin. From these two papyri, it is known that the ancient Egyptians had knowledge of, for example, arithmetic, fractions, and basic geometry. However, it is not known in what context these papyri were used. Also, since papyri are a very fragile material, it is not known how many papyri containing mathematics that have not been preserved to our time. Therefore, it is difficult to make an adequate interpretation of the mathematical knowledge of the ancient Egyptians.

Around 2000 BC the Babylonians invaded Mesopotamia and defeated the Sumerians. The Sumerians had developed writing based on cuneiform symbols, and the Babylonians adopted the same style of writing. They wrote on soft clay tablets using the triangular end of a reed stalk, and the tablets were then baked in the sun or in ovens. Clay tablets are far less vulnerable to the influence of time; hundreds of thousands of these clay tablets have survived and are well preserved to our days. Many hundreds of these have been identified as strictly mathematical, containing mathematical tables and mathematical problems. For example, the tablets contain multiplication tables, tables of squares and cubes, and tables of square and cube roots, possibly with the aim to aid in calculations. During a long time period, several failed attempts were made to decode the cuneiform symbols. Finally, in the 1840s, the English archeologist Sir Henry Creswicke Rawlinson (1810–1895) succeeded.

The Egyptian and the Babylonian number systems are strikingly different. The Egyptians used an additive hieroglyphic number system with different symbols for powers of ten (see Fig. 2). In the hieratic script (which is found in the Rhind and Moscow papyri), a different numeral system was utilized. This was also an additive number system but included individual symbols for the numbers 1 to 10, multiples of 10 from 10 to 90, multiples of 100 from 100 to 900, and so on. In this way large numbers could be written with fewer symbols than when written with hieroglyphic number symbols. Also, the hieratic script seems to be better suited to the use of writing with pen and ink on papyrus. The Babylonians used a cuneiform positional number system with the base 60 and with the subbase 10 within each position. That way they only needed two symbols: a vertical wedge representing one and a horizontal wedge representing ten. Possibly it was the inflexibility of the cuneiform

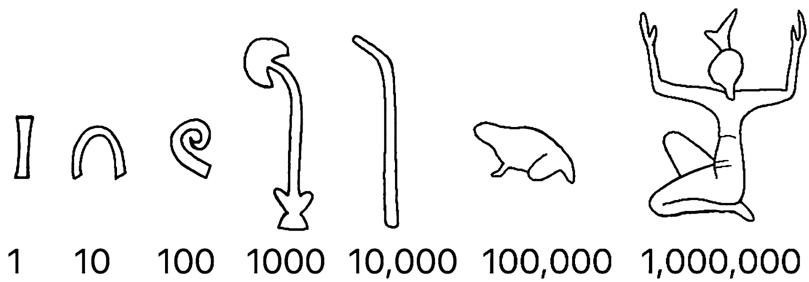

Fig. 2 The Egyptian hieroglyphic numerals

writing in clay that prompted an insight that the two symbols for one and ten together with the positional notation were sufficient for representing any integer without too much repetitiveness.

The ancient Greek civilization is impressive and most influential in the development of the modern Western culture, in particular regarding mathematics. Ancient Greek can be divided into three periods: the Archaic period (c. 800–c. 500 BC), the Classical period (lasting to the death of Alexander the Great 323 BC), and the Hellenistic period (ending with the emergence of the Roman Empire as signified by the Battle of Acticum in 31 BC). During the Archaic period, the Greek alphabet developed and replaced various hieroglyphic systems of writing. Traditionally, the beginning of Greek mathematics is placed at the time of the pre-Socratic philosopher and mathematician Thales of Miletus (c. 624–548 BC). The Greeks were highly influenced by Egyptian and Babylonian mathematics. However, the Greeks moved beyond the use of mathematics developed only for practical purposes, toward a theoretical approach. For example, Thales is presumably the first mathematician to have used deductive reasoning when he derived Thales theorem, stating that if a triangle is inscribed in a circle where one side of the triangle is the diameter of the circle, then the angle opposite to that side is a right angle.

Until about 450 BC, the Greeks had an oral tradition on passing on knowledge through their students. When they began to write their texts, they used papyrus rolls as well as wooden writing boards and wax tablets. These materials are very fragile and easily become damaged. Therefore, it is not surprising that no original manuscripts of the most important Greek mathematicians, for example, Euclid and Apollonius, are preserved to our days. The main sources for Greek mathematics that are preserved to our days are Byzantine Greek manuscripts written as commentaries to older texts, as well as Arabic translations and Latin versions derived from Arabic works (Kline 1972). When these texts are studied, it is not known what changes the editors and translators may have done. Therefore, it is problematic when, for example, Euclid's *Elements* is studied. The *Elements* originates from about 300 BC and is considered to be the most successful and influential mathematics book ever written. Euclid's version of the *Elements* is believed to have been written on a papyrus roll, and has not been preserved to our days, but it is believed that, to a large extent, it was a compilation of texts by earlier Greek mathematicians. Most certainly, the *Elements* was continually copied, but in this process, errors could have been made, and material could also have been added. It is known that the Alexandrian Greek Heron, in his edition of Euclid's *Elements*, made numerous changes, such as giving alternative proofs. Also, Theon of Alexandria in his edition of Euclid's *Elements* made additions and attempted to remove difficulties that students studying the text may have experienced. Modern editions of the *Elements* are often based on the Vatican Euclid, dating from the ninth century, and discovered at the Vatican in the nineteenth century. It is widely believed that this manuscript is the closest to Euclid's original text.

Two of the main sources to our knowledge of the classical and Alexandrian Greek mathematics are the important commentaries by Pappus (c. 290–c. 350) and Proclus (412–485). Pappus wrote, among others, the mathematical *Collection* which is an exposition of the classical and Alexandrian mathematics from Euclid to Ptolemy. Unfortunately, this manuscript is not preserved in complete form. Proclus was a Greek Neoplatonist philosopher, and most knowledge we have today on the classic Greek mathematics originates from him. The principal source about the early history of Greek geometry is his commentary which deals historically and critically with Book I of Euclid's *Elements*.

There are still gaps in our knowledge on the history of Greek mathematics as well as in our knowledge of the history of mathematics of other ancient civilizations. With the aim to reconstruct the history of Greek mathematics, scholars proceed to study preserved manuscripts and their relation to each other in order to interpret how the original texts may have been structured. When doing this, historians of mathematics can, depending on purpose, adopt different perspectives from which the manuscripts can be interpreted in different ways. Of course, this is not only valid when manuscripts from ancient Greek are considered; also, when historical mathematical texts from other time periods are studied, a variety of perspectives when interpreting the texts can be adopted. In the following sections, this will problematized.

History of Mathematics or Heritage of Mathematics?

During the last few decades, the legitimacy of historical research in mathematics has been debated. One crucial question concerns how historical texts in mathematics should be interpreted in a proper and historically correct way. An influential model is composed by Ivor Grattan-Guinness (1941–2014) who has considered this issue on the basis of the following two approaches: *history* and *heritage* (Grattan-Guinness 2004). The distinction between history and heritage, both theoretically and by means of concrete examples, will be described in this section.

History focuses on what happened in the past and pays no intention to the modern situation. In order to study a specific mathematical theory, definition, theorem, concept, etc., history concentrates on the details of its development, its prehistory, the chronology of progress, and its impact in the years immediately following. As Grattan-Guinness points out, history addresses the question "what happened in the past?" but also the question "what did not happen in the past?" In order to answer the corresponding questions "why?", history gives descriptions and also attempts explanations. History may also consider differences between the historical notion and more modern notions that are seemingly similar.

An example of interpreting historical texts by means of history is the following: if Leonhard Euler's (1707–1783) function concept is studied, the focus would be to analyze his function concept on the basis of the mathematical context at that particular time period, without being influenced by the modern function concept. The historical interpretation must be based on the fact that Euler neither had access to the modern function concept nor did he struggle to formulate it. Instead, he formulated a definition of a function within his conceptual framework which was suitable for dealing with problems in mathematics of interest at that particular time.

Thus, for the historical interpreter, it is essential to be aware of the mathematical context that Euler had at hand. The fact that the function concept during the eighteenth century was quite different compared to today would not be a main point.

Let us discuss this example a bit further. In 1748 Euler defined a function in the following way:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities. (Euler 1748, p. 18)

Apparently, to understand Euler's function concept, it is also necessary to study variables from the mid-eighteenth century which differs significantly from how variables are defined today. Most mathematicians during the eighteenth century regarded variables as being quantities that vary – that change in magnitude. For instance, the mathematician Guillaume de l'Hospital (1661–1704) defined variables in the following way:

> We call variable quantities those that increase or decrease continuously; and to the contrary we call constant quantities those that remain the same while the others change. (l'Hospital 1696, p. 1)

Certainly, l'Hospital's definition differs significantly from the modern variable concept in the sense that today a variable is viewed as a symbol representing an arbitrary element of a specific set. Hence, analyzing Euler's view of the function concept includes an investigation of the whole context at that particular time.

Now, let us turn to the heritage approach. Heritage refers to the impact of a certain mathematical notion, for example, a mathematical theory, definition, theorem, and concept, upon later work. Often the main focus is the modern form of the notion studied, with attention paid to the course of its development. Sometimes, when appropriate, the modern notions are inserted into the notion studied. As Grattan-Guinness formulates it: "Heritage addresses the question 'how did we get here?' and often the answer reads like 'the royal road to me'" (Grattan-Guinness 2004, p. 165). A typical example of the utilization of heritage is review articles where motivation, cultural background, and historical complications are usually left out, but names, dates, and references are given frequently.

In order to better understand the difference between history and heritage, let us again consider Euler's function concept. Euler, as well as several other mathematicians during this time period, expanded functions as power series (Kline 1983). In this way, Euler could easily determine a function's derivative and integral by simply differentiating and integrating the series term by term. However, Euler and his contemporaries were manipulating infinite power series in the same way as usual polynomials without any consideration to problems that may arise when infinity is involved. Sometimes this led to ambiguous results which the following example from Euler illustrates. Consider the power series expansion:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots \qquad (1)$$

Differentiation of this series term by term, and after changing the sign, gives

$$\frac{1}{(1+\mathrm{x})^2} = 1 - 2\mathrm{x} + 3\mathrm{x}^2 - 4\mathrm{x}^3 + 5\mathrm{x}^4 \cdots$$

For Euler there never was a problem to divide a number with 0, since the result became infinity, where infinity was considered as a number. Therefore, when Euler substituted $x = -1$ in the series above, he received the following result:

$$\infty = 1 + 2 + 3 + 4 + 5 + \cdots \tag{2}$$

In the same way, Euler substituted $x = -2$ in the original series (1) above and obtained

$$-1 = 1 + 2 + 4 + 8 + 16 + \cdots \tag{3}$$

The terms in the series (3) are clearly larger than the corresponding terms in the series (2), and therefore Euler concluded that $\infty < -1$, and since $1 < \infty$, Euler concluded that $1 < \infty < -1$. Within a heritage perspective, this result is incorrect since we obviously arrive at a contradiction. The reason why Euler arrived at this kind of result was his usage of power series expansions of functions outside their convergence domains. However, utilizing a history perspective, conclusions through the application of modern knowledge cannot be drawn. Instead, Euler's result has to be interpreted using the tools that Euler had at hand. Euler argued that infinity separates positive and negative numbers, just as 0 does. Perhaps he considered the number line as an infinitely large circle where the both halves, i.e., the positive and negative number lines, were tied up at zero and at the infinity?

During the nineteenth century, there was an increasing concern to make the theory of series rigorous, not the least through the French mathematician Augustin Louis Cauchy's (1789–1857) pioneering contributions. Cauchy rejected the methods used by Euler and his contemporaries who, as we have seen above, applied rules for finite expansions when manipulating infinite expansions. However, during Cauchy's attempts to rigorize the theory of series, his mathematical sophistication sometimes blinded him to counterexamples that were lurking. One example is the famous *Cauchy's sum theorem* first formulated in 1821, where Cauchy claimed that the sum function of a convergent series of real-valued continuous functions is continuous:

> When the different terms of the series $(u_0 + u_1 + u_2 + \cdots + u_n + \cdots)$ are functions of the same variable x, continuous with respect to that variable in the vicinity of a particular value for which the series is convergent, the sum s of the series is also a continuous function of x in the vicinity of this particular value. (Cauchy 1821, pp. 131–132)

The proof was relatively imprecise which led to that contemporary mathematicians criticized the theorem and came up with counterexamples. For instance, the Norwegian mathematician Niels Henrik Abel (1802–1829) showed that the trigonometric series

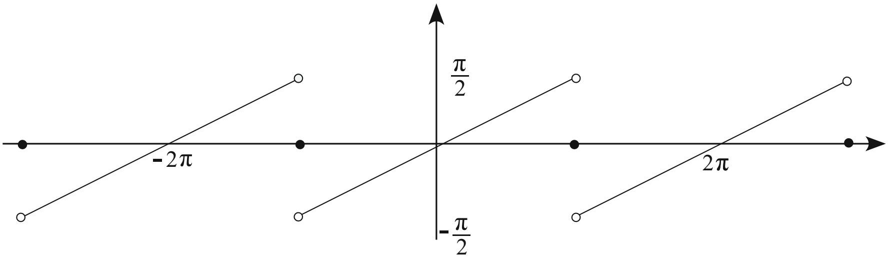

Fig. 3 A graphical representation of the sum of the series $\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots$

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$

is an exception to Cauchy's theorem (Abel 1826). Although the series is convergent and consists of continuous functions, the sum of the series is discontinuous at $x = (2k + 1)\pi$, for each integer k (see Fig. 3). Abel also added that there exist several other counterexamples similar to this (see Sørensen 2005).

Apparently, the mathematical theory had reached a point where the convergence condition was not precise enough to exclude counterexamples such as Abel's. In 1853 Cauchy modified his theorem by adding the stronger convergence condition "always convergent" to his 1821 version.

Let us discuss this example in terms of the distinction between history and heritage. It is tempting to think that Cauchy's convergence conditions from 1821 and 1853 are the same as the modern notions of pointwise and uniform convergence. Certainly, such a conclusion would be hasty, since pointwise and uniform convergence depend on the modern function concept which Cauchy and his contemporaries did not have at hand. As Grattan-Guinness (2000) points out, during the nineteenth century, there was a problem to distinguish between the expressions "for all x there is a y such that…" and "there is a y such that for all x…" which is needed to express the modern convergence conditions. However, considering Cauchy's distinction as an attempt to reach the modern convergence concepts or investigating which impact Cauchy's conditions had upon later work on convergence would be typical examples of utilizing a heritage perspective (e.g., Spalt 2002).

There are several examples of research where Cauchy's sum theorem has been analyzed from a history perspective. For instance, Sørensen has studied the development of Cauchy's sum theorem by analyzing which role Abel's counterexamples played in the transition between a formula-centered analysis and a concept-centered analysis (Sørensen 2005). Another example is the investigation of the Swedish nineteenth-century mathematician Emanuel Gabriel Björling's (1808–1872) contribution to the development of Cauchy's stronger convergence condition "always convergent" from 1853 (Bråting 2007; Grattan-Guinness 1986). By analyzing Björling's (1853) own distinction between "convergence for every value of x" and "convergence for every x," where the latter is a stronger condition than the former, we can get a better understanding of what problems mathematicians during the nineteenth century were facing.

It is important to point out that the utilization of heritage often results in a modernization of old results in order to show their current place in mathematics, but the historical context is not always taken into consideration. Grattan-Guinness states that this is perfectly legitimate, as long as heritage is not confused with history; one should be aware of that mathematicians of the past based on their definitions within the conceptual framework available at that particular time and not assume that they strived for the modern definitions of today. An example of confusing history with heritage is to claim that Euclid was a "geometric algebraist," in the sense that he was handling geometrical notions but that he actually was practicing common algebra. However, to insert modern notions of algebra into Euclid's work is certainly acceptable and would be a typical example of using heritage. It becomes problematic if one argues that algebra is discovered in Euclid's work. Let us now consider this example in more detail.

During the 1970s there was a debate within the historiography of mathematics concerning whether it is correct to claim that Euclid was a geometric algebraist, in the sense that he was handling geometrical notions but was actually practicing common algebra (Corry 2013). Within the research field of history of mathematics, this debate is sometimes referred to as the Unguru debate: did the Greeks have an algebra? Sabetai Unguru, historian of science and mathematics, argued that the received opinion, which was based on research by among others Paul Tannery (1843–1904) and Hieronymus George Zeuthen (1839–1920), was incorrect and based on an anachronistic reading of ancient Greek texts as they were translated into a modern algebraic notation. According to Unguru, algebra was imposed on the Greek texts rather than discovered in them. Regarding the use of modern algebraic notions in Euclid's *Elements*, Unguru stated the following:

> History? Perhaps, but certainly not sound, acceptable history. It is rather 'logical history', i.e., in more cases than not, non-history. It is history as it should be rather than an honest attempt to establish it as it was; it is, in other words, a logical rather than a historical reconstruction. (Unguru 1975, p. 92)

The leading mathematician André Weil (1906–1998) dismissed Unguru's critique by accusing Unguru of not knowing enough mathematics and claimed, without much justification, that Euclid just used a somewhat cumbersome notation in his algebra (Weil 1978). Today Weil's claim is sometimes regarded as a scandal in the field of history of mathematics.

Grattan-Guinness discusses the interpretation that Euclid's *Elements* contains algebra. He considers, among other things, Book II, Proposition 4 of Euclid's *Elements*: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments" (Heath 1956, p. 380). This proposition is today often referred to as completing the square and in algebraic notation is described as $(a + b)^2 = a^2 + 2ab + b^2$. A modern illustration is given in Fig. 4.

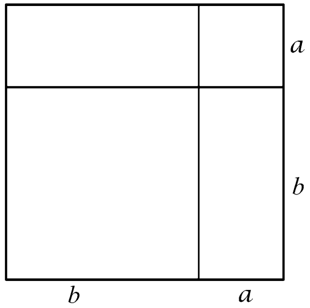

Fig. 4 A modern illustration of Book II, Proposition 4 of Euclid's *Elements*

Grattan-Guinness suggests some problems resulting from interpreting Euclid as a "geometric algebraist" based on the proposition above. First, the expression above is a piece of algebra which Euclid did not use; in fact, his picture did not contain the letters *a* and *b*. One should be aware of that Euclid's theorem concerned geometry, about the large square being composed of four parts, with two smaller squares and two rectangles. Furthermore, *a* and *b* are associated with numbers and lengths, but Euclid worked with lines and regions, i.e., Euclid did not have at hand any arithmetized analogues such as lengths or areas. Grattan-Guinness even claims that this is already a historical distortion since Euclid never multiplied geometrical magnitudes of any kind.

As a matter of fact, algebra was not inserted into Euclid's *Elements* until the seventeenth century. The English mathematician William Oughtred (1574–1660) was one of the first to exemplify theorems of classic geometry using algebra (Stedall 2002). In *Clavis mathematicæ* from 1631, he demonstrated the 14 propositions of Book II of Euclid's *Elements* with his analytical method, which means that he used François Viète's (1540–1603) algebraic notation, as presented in Viète's symbolic algebra. During the end of the sixteenth century, Viète was inspired by Diophanto's work and used capital letters instead of abbreviations as symbols for the unknown and known entities. Also, the Swedish mathematician Anders Gabriel Duhre (c. 1680–1739) formulated and proved some of the propositions in Book II of Euclid's *Elements* using algebra (Pejlare 2017). He did this using both René Descarte's (1596–1650) notation and Oughtred's notation. In this way Duhre's proofs can be seen as proofs of algebraic identities where operations are performed on algebraic expressions. Duhre considered the algebraic notation to be both "clear and convenient for the sense"; with algebra he could obtain convenience in calculations, since complicated expressions can be transformed into simpler ones. Geometrical results can, with algebra, also be generalized to different kinds of quantities, since unknowns do not necessarily have to be, for example, lines.

The conclusion here – which should be seen as an interpretation of the Unguru debate on the basis of history and heritage – is that Euclid did not have an algebra, in the sense that he did not use any symbolic treatment such as the algebraic expression above. Therefore, interpreting Euclid as a geometric algebraist is a typical example of confusing history with heritage. In the next section, further perspectives of interpreting historical texts will be considered.

Further Views of the Past and Its Relation to the Present

The distinction between history and heritage is a useful tool to analyze how mathematics of the past is treated. However, it does not explain in an explicit way how these different approaches to the mathematics of the past are truly different views of the past itself. In order to understand the nature of the history of mathematics, we first have to understand what it means for history of mathematics to be history and what relation it has to the present. The historian's materials, such as books, manuscripts, and other artifacts, are things that have made their way into the present, and therefore, in order to study the history, it is necessary to refer to the present to a certain extent. However, it is not only the past and present that are essential but also how these are treated.

The philosopher and historian Michael Oakeshott (1901–1990) theorized about how the past can be experienced in various ways (Oakeshott 1933). In particular, he distinguishes between *historical past* and *practical past*. He argues that both the historical past and the practical past are experiences in the present, since both approaches deals with artifacts from the past preserved to the present. Furthermore, practical past considers the past in terms of present values, need, and ideas, i.e., the practical past depends on the present. Historical past, on the other hand, is a past that is understood as its distinctness from the present, i.e., the subject of historical past is the past in its own particularity. Thus, to experience the past in the present as historical past, the past must be considered unconditionally.

Oakeshott was clearly influenced by the British historian Herbert Butterfield (1900–1979), who coined the concept of the *Whig interpretation of history* (Butterfield 1931). Whig history refers to a history that studies the past with reference to the present; it presents the past as an inevitable progression toward the present, in particular toward enlightenment, liberal democracy, and the British constitutional settlement. Whig historians seek in the past what is useful for the present, emphasizing the seemingly inevitable success of the victors. Thus, adopting Oakeshott's term, Whig historians treat the past similar to a practical past. Also, heritage, as described by Grattan-Guinness, resembles a Whig interpretation of the mathematics of the past. However, when Oakeshott and Grattan-Guinness accepts practical past and heritage as legitimate, Butterfield rejects the Whig interpretation as illegitimate: a Whig interpretation distorts the past by reading modern conceptions and intentions into the writings of, for example, mathematicians of the past. Thus, a Whig historian would interpret Euclid's *Elements* with the help of algebra. Furthermore, Whig history also forces the past "through a sieve" to only let through ideas that can be related to modern mathematics and keep out those that are foreign to us today.

Another example of a Whig interpretation of history involves Nicholas Oresme (c. 1320–1382) in our search for the function concept (Fried 2001). Oresme was interested in the nature of motion and change and questioned the Aristotelian definitions of time and space. Today it is easy to interpret Oresme's graphical representation of motion as a representation of a function, since we know what we are looking for, and how important the concept is in modern mathematics. But when this interpretation is made, the Aristotelian context and Oresme's interest in motion and change are lost. When we insist on our own understanding in a historical text that originally had a different purpose, the result is that we let Oresme think our own thoughts.

Fried (2018) discusses the nature of the history of mathematics and the different relationships or postures toward the mathematics of the past. He points out that both *historical* and *nonhistorical* relations to the past may be beneficial, depending on the intention of the person exploring it. For example, a mathematician may enter the world of past mathematics through a "mathematical door" without any intention of doing history, but rather to use the past as a resource for gaining insight into her present mathematical research. A mathematician may consider mathematicians of the past as her contemporaries or colleagues; she may consider herself working on the same problems within the same frameworks as the mathematicians from the past. A historian of mathematics may instead enter the world of past mathematics through a "historical door." For the historian of mathematics, there is a discontinuity between mathematical thought of the past and mathematical thought of the present. For example, the historian of mathematics does not try to coordinate a historical text with the mathematics of the present, since she does not intend to transform the past into present experience. Instead, she intends to make the pastness of past mathematical thought stand out, showing to which extent past ideas were unlike modern ideas, with the intention to understand the past in its own right.

Let us consider the different postures toward the mathematics of the past in relation to the Italian mathematician Evangelista Torricelli's (1608–1647) surprising result that an infinitely stretched out hyperboloid has a finite volume and an infinite surface area (Mancosu 1996). If a branch of the Apollonian hyperbola (in modern terminology: $xy = a^2$) is revolved around one of the asymptotes, a solid of infinite length in the direction of the axis of revolution is obtained. If the solid is cut by a plane perpendicular to the axis of revolution, a solid of infinite length and surface area and finite volume is obtained (see Fig. 5). Torricelli improved Bonaventura Cavalieri's (1598–1647) theory of indivisibles to prove that this infinite long solid has the same volume as a finite cylinder by establishing a one-to-one correspondence between surface areas of cylinders inscribed in the infinite hyperboloid and the circles "making up" the finite cylinder. In modern terminology the volume would be calculated as a generalized integral:

$$V = \pi \int_1^\infty \left(\frac{a^2}{y}\right)^2 dy = \pi \lim_{R\to\infty} \int_1^R \left(\frac{a^2}{y}\right)^2 dy = -\pi \lim_{R\to\infty} \frac{a^4}{y}\bigg|_1^R$$

$$= -\pi \lim_{R\to\infty} \left(\frac{a^4}{R} - \frac{a^4}{1}\right) = \pi a^4.$$

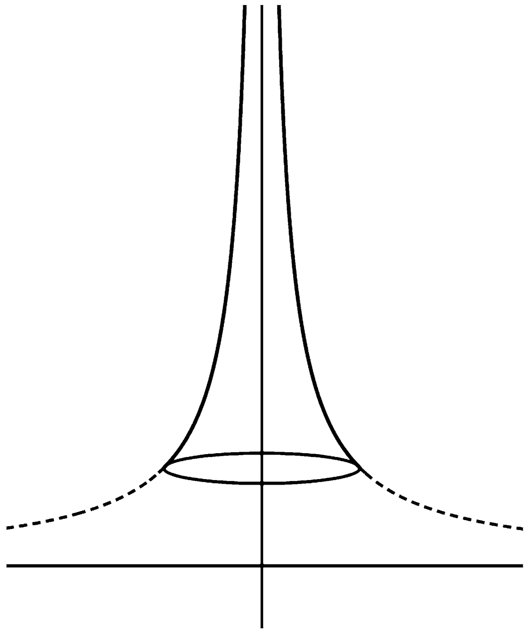

Fig. 5 Torricelli's infinitely long solid

In particular, if $a = 1$, the volume will be π cubic units.

Torricelli's infinitely long solid is one of the first examples which challenged the ancient dictum that there could be no ratio between the finite and the infinite. As described by Mancosu (1996), the result was widely debated in the 1640s and beyond. For example, Isaac Barrow (1630–1677) was worried about whether the universe could contain such an object of infinite length, since he considered the reality of geometrical entities to be grounded in their material existence. Another typical reaction at that time was by Thomas Hobbes (1588–1679) who insisted that all knowledge should involve a set of self-evident truths known by the "natural light." According to Hobbes, we can only have ideas of what we sense or of what we can construct out of ideas so sensed. Thus, since we only can experience finite things and repeated composition of such ideas of finite things cannot produce the idea of an infinite thing, he rejected infinite objects such as Torricelli's infinitely long solid.

Today examples like Torricelli's infinitely long solid are looked upon in a less naturalistic and more conventional way. For example, in modern mathematics there are no difficulties with the infinitely long solid. Unlike Barrow, we do not try to fit the infinitely long solid into our universe, and, unlike Hobbes, we do not reject the infinite because we cannot sense it. The modern mathematician, entering the world of Torricelli's mathematics through the mathematical door, may consider the debate on Torricelli's result as obsolete but may instead interpret the new techniques developed by Torricelli in terms of modern integral calculus. The historian of mathematics, entering the same world through the historical door, may not transfer

Torricelli's result into present experience by interpreting it in terms of modern integral calculus. Instead she may be concerned with the historical debate by Barrow and Hobbes in order to make the pastness of Torricelli's result palpable and bring out its own identity.

This section will be closed by mentioning the *synchronic* and *diachronic* perspectives borrowed from the Swiss linguist Ferdinand de Saussure (1857–1913), who studied semiotic systems. Saussure suggests that the synchronic and diachronic viewpoints are two different, but also complementary, ways of analyzing languages. The diachronic approach refers to the development of a language along a time axis, meanwhile, the synchronic approach considers a language at a specific moment in time, often the present, without taking its history into account. Saussure viewed human languages as systems of meaning which can be understood both as products of history and as a-historical systems with invariant structures. Here Saussure used the analogy of chess; the synchronic perspective focuses on how the pieces interact at a given time, while the diachronic perspective focuses on the evolution of the game, how the value of the pieces changes during the game. Saussure points out that a person who has watched the entire game has no advantage over a newly arrived person in understanding the existing position at that particular moment.

The synchronic and diachronic perspectives cannot only be applied on languages but also on mathematics. As Fried (2007) points out, in the same way as languages, mathematics can also be understood as a system of meaning. In order to concretize how the synchronic and diachronic perspectives can be used in a mathematical context, an example will now be discussed. So far, this text has considered interpretations of the history of mathematics by means of the development of specific mathematical concepts, findings from old civilizations, and the work of famous mathematicians throughout history. However, in this example focus will not be on mathematics at the front edge during a specific time period but instead on the formation and development of school mathematics. Such studies, leaving the traditional historiography of mathematics, provide important knowledge of what kind of mathematics that has been relevant for the community at large.

The aim of the study in this example is to identify how the content in Swedish school algebra has been formed and developed during the last 50 years (see Bråting and Pejlare 2019). Within the study, curriculum documents and textbooks from elementary school up until upper secondary school are used as empirical material. During the time period, there have been five curriculum reforms in Sweden: in 1962, 1969, 1980, 1994, and 2011. In the study, content patterns in the curriculum documents as well as the textbooks are identified by means of mathematical content, degree of difficulty, and contextualization. The study applies both a diachronic as well as a synchronic perspective. The former perspective refers to the development of school algebra along the time axis, i.e., from 1962 until today. Meanwhile, the latter perspective refers to what actually exists at each school level at particular moments in time. For instance, the curriculum documents from 1980 are investigated separately for each school level without taking its history or future into account (the synchronic perspective). In addition, the curriculum documents from the five reforms are also compared with each other in order to find content patterns, as well as teaching traditions, within Swedish school algebra (the diachronic perspective).

In a diachronic perspective, the results revealed that over the years, the algebraic content had become more integrated with other school subjects, the level of complexity of algebraic expressions in textbooks had decreased, and algebra had more often been considered as a tool for solving practical and everyday problems. Moreover, the results also showed that algebra was introduced earlier during the years (diachronic perspective). For instance, in the 1980 curriculum document, algebraic expressions and proportionality first appear at the secondary school level, while in the 2011 document, it already appears in primary school (synchronic perspectives). It can also be noted that the emphasis on everyday mathematics increases with the implementation of the 1980 curriculum which probably was a reaction to the great focus on abstract mathematics in connection with "New math" from the 1969 curriculum.

In this way, the usage of a diachronic perspective enhances the investigation of the algebraic content in a synchronic perspective. The contrasting effect occurring between different time periods clarifies the algebraic content today as well as how the algebraic content has changed during the years.

Can History Be Recapitulated or Does Culture Matter?

In the research field of mathematics education, there have, for different reasons, been efforts in combining the history of mathematics and the mathematics education. One of the many ways of utilizing the history of mathematics in the research field of mathematics education is the investigation of historical conceptual development to deepen our understanding of mathematical thinking and students' learning of mathematics. Since the end of the twentieth century, the belief of a parallel between the historical development of mathematical concepts and the development of students' understanding of the concept has become a topic largely shared within mathematics education. However, this belief of a parallel development has also been criticized (Bråting and Pejlare 2015). For example, in the history of mathematics, examples where cultural aspects were crucial for the development of certain concepts or areas can be found. One such example is the Babylonian number system, referred to earlier in this chapter (see section "History of Ancient Mathematics: The First Written Sources"): possibly the art of writing on clay tablets with reed stalks prompted the development of a positional number system.

The belief of a parallel between the historical development of certain concepts and individual's conceptual development has its origin, in particular, in the German biologist Ernst Haeckel's (1834–1919) law of biological development, also known as recapitulation. Haeckel is well-known for his claim that "Ontogeny recapitulates phylogeny," believing that the study of embryonic development (or, ontogeny) retold the history of evolution (or, phylogeny). He wrote:

> The series of forms through which the individual organism passes during its development from the ovum to the complete bodily structure is a brief, condensed repetition of the long series of forms which the animal ancestors of the said organism, or the ancestral forms of the species, have passed through from the earliest period of organic life down to the present day. (Haeckel 1912, pp. 2–3)

Although Haeckel's theory of recapitulation is now discredited among biologists, it had a strong influence on social theories, and educational practices developed during the end of the nineteenth century. According to the psychological version of the theory of recapitulation, it is assumed that the present intellectual developments are to some extent a condensed version of those of the past (Radford 1997). This would imply that while developing the understanding of, for example, a mathematical concept, the student recapitulates the historical development of the concept. In general, the theory promises that the origins of the mental, social, and ethical development of humanity can be revealed by studying the learning of mathematics among young children. However, natural selection is presented as a function of the environment against which individuals act, and therefore, for recapitulation to be possible, the environment must remain essentially the same. But in fact, environments do change, and thus it may become difficult to maintain that the intellectual development of a child today will undergo the same process as the one a child would have experienced in the past.

In the late nineteenth century, the law of recapitulation was however adopted among mathematicians and mathematics educators. For example, the French mathematician Henri Poincaré (1854–1912) suggested that the individual's conceptual development should recapitulate the historical development of the concept. He wrote:

> Zoologists claim that the embryonal development of animals summarizes in a very short time all the history of its ancestors of geologic epochs. It seems that the same happens to the mind's development. The educators' task is to make children follow the path that was followed by their fathers, passing quickly through certain stages without eliminating any of them. In this way, the history of science has to be our guide. (Poincaré 1899, p. 159)

Also, the German mathematician Felix Klein (1849–1925) had an interest in educational questions, which he believed were closely connected to the history of mathematics. He was convinced that the road to discovery, and not the formal arguments, was of most importance, since:

> Anyone who wants to enter into mathematics must, step by step, through his own labor mentally recapitulate the entire development; it is by all means impossible to understand even a single mathematical concept without having mastered all the antecedent concepts and their connections that led to its creation. (Klein 1926, p. 1)

Both Poincaré and Klein were strong advocates of intuitive arguments, in contrast to rigor and formal logic, in mathematics. At the end of the nineteenth century, the term *arithmetization* had been introduced in order to describe various programs for providing non-geometric foundations of analysis and other areas of mathematics. Poincaré and Klein reacted against this and emphasized the importance of the interaction between intuition and logical arguments in mathematics. According to them intuition is indispensable in the learning of mathematics, since without intuition, students cannot begin to understand mathematics. However, they also pointed out that pure intuition is not enough in mathematics, arguing that only logic can give us certainty. In particular, both were convinced that logical thinking was preceded by an intuitive stage in the historical development of mathematics as well as in the learning of mathematics.

A more reflected conception on the relations between history of mathematics and mathematics education was established by the German mathematician Otto Toeplitz (1881–1940). In a paper on the genetic approach in the teaching of calculus (Toeplitz 1927), he suggested that proper attention to the history of mathematics in teaching would benefit the students' learning. To use history as a didactical means in the teaching of mathematics is what Toeplitz called the *genetic method*. He argued that the development of mathematical ideas should be taken as a guide for teaching; this would reveal not only the drama of the historical development but also the logic and interconnection of mathematical ideas to the students. He argued that there are two ways in which the genetic method can be used: the "direct" and the "indirect" approach.

- The direct approach, on the one hand, is a direct way of using the history in teaching, where, instead of bridging between rigorous and intuitive approaches when introducing a new topic, students should arrive at mathematical ideas by following the same path by which these ideas followed historically. In this way, rigorous ideas would unfold for the students in the same way as they unfolded historically.
- The indirect approach, on the other hand, is rather a way of analyzing teaching and understanding the teacher as actively reflecting on the history of mathematics and the real meaning or the "true essence" of each concept. In this way, the historical analysis serves to turn the teacher's attention into the right direction in the teaching, but the history itself is not necessarily brought into the classroom.

In the twentieth century, there was an interest among psychologists in the relationship between ontogenesis and phylogenesis. In particular, the views of Piaget and Vygotsky have been influential on the use of history in mathematics education. The concept of genetic development was elaborated by Jean Piaget (1896–1980) and Rolando Garcia (1919–2012) as a reaction against the simplistic psychological version of the recapitulation theory (Piaget and Garcia 1989). They disputed Haeckel's recapitulation theory but believed in a parallel between historical and psychological developments, suggesting that this parallel must be seen not in terms of content but in terms of mechanisms allowing the acquisition of knowledge. According to Piaget and Garcia, these mechanisms are invariable in time and space and do not change, regardless of the period in history and the geographical place of the individuals. In particular, this implies that they cannot be modified by, for example, culture.

Lev Vygotsky (1896–1934) also dealt with the problem of recapitulation, but unlike Piaget he emphasized the epistemological role of culture. He pointed out that the activity of mental functions is modified by the use of tools and artifacts, for example, clay tablets, abacuses, computers, words, and language. Thus, since the tools available differ in different cultures, and sociohistorical conditions vary among different historical periods, recapitulation must be impossible. Furthermore, Vygotsky believed that thinking developed as the result of two different processes: a biological (or, natural) process and a historical (or, cultural) process. Thus, according to Vygotsky, ontogenesis could be considered as proponed both by the sociohistorical conditions where it takes place and by biological phylogenesis. In the history of mathematics, several examples can be found where the role of culture had a significant influence on the development of mathematical concepts. Let us consider one of these examples in greater detail: the development of the concept of negative numbers in ancient China and in Western Europe; due to different cultural and sociohistorical conditions, the development of negative numbers was very different in ancient China compared to Western Europe.

References to negative numbers were first made in ancient China in the practical handbook on mathematics called Jiuzhang Suanshu (*The Nine Chapters on the Mathematical Art*); for more details on Jiuzhang Suanshu, see Lam (1994). This handbook contains mathematical concepts and methods, and it played a fundamental role in the development of Chinese mathematics. The original is lost, but Liu Hui wrote a commentary on Nine Chapters in 263 AD. Among others, a method is given for solving systems of linear equations. The coefficients of the equations are placed in a rectangular array of rod numerals using a decimal place notation, and calculations are performed on the counting rods following an algorithm. Counting rods with different colors were used in ancient China to symbolize subtraction: a black number to be subtracted from a red one. As the red counting rods represented positive numbers, it was not hard to give the black counting rods a meaning as negative numbers. Thus, negatives arose naturally in solving concrete problems, and in Liu's commentary rules for adding and subtracting, positive and negative numbers were carefully explained. The development of an understanding of the negatives was in this way induced by the different colors of the rods that the Chinese had at hand.

In Western Europe, however, the concept of negative numbers was resisted for a long time. The abacus was used for performing arithmetical calculations, but different colors were not used to represent addition and subtraction. A first knowledge of negative numbers and algebraic techniques had reached Western Europe in the thirteenth century. For example, Leonardo of Pisa (1170–1250) handled negatives when they arose in calculations, but he was only able to give negative quantities a meaning in problems concerning money. However, he only gives a few examples with "giving and taking," and he never formulates explicit rules for extending the number system to include also negative numbers.

In the sixteenth century, Girolamo Cardano (1501–1576) gave methods of solution of the cubic and quartic equation. He rejected negative numbers, and therefore he had to describe how to solve 13 distinct cases of cubic equations with only positive coefficients. He did recognize that some of his equations had negative solutions, but he systematically ignored them. Since there was a problem with imaginary numbers when the negatives appeared under square roots, it may have seemed more suitable not to consider the negatives at all.

Negative numbers started being used, not only through the theory of equations but also through the problem of dealing with the correspondence between the terms of arithmetical and geometrical progressions (Thomaidis and Tzanakis 2007). In his book *Arithmetica Integra* (1544), Michael Stifel (1487–1567) examined the correspondence between the two progressions 0, 1, 2, 3, 4, 5, ... and 1, 2, 4, 8, 16, 32, ... He explained how to reduce multiplication and division between the terms of the geometrical progression to addition and subtraction of the arithmetical progression. In order to extend the correlation between the two progressions "to the left," he had to introduce "fictitious" numbers into the arithmetical progression:

$$\begin{array}{cccccccccccc} \ldots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \ldots \\ \\ \ldots & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 & 8 & 16 & \ldots \end{array}$$

Stifel clearly stated that the negative numbers are less than zero, but he did not acknowledge equal status for positive and negative numbers. Instead, he declared positive numbers to be *real* numbers, and negative numbers were ascribed only an imagined existence.

A first clear view of negative numbers in Western Europe should be attributed to John Wallis (1616–1703) and Isaac Newton (1642–1727) in the seventeenth century. For example, Wallis gave a simple and clear definition of positive and negative numbers as contrary signification:

> If + signify Upward, Forward, Gain, Increase, Above, Before, Addition, & c. then – is to be interpreted of Downward, Backward, Loss, Decrease, Below, Behind, Subduction, & c. And if + be understood of these, then – is to be interpreted of the contrary. (Wallis 1685, p. 16)

The historical development of the notion of negative numbers in different cultures illustrates clearly that there are local and cultural ideas about mathematics, and its objects and methods, which influence this development. In China the number rod system had been developed, and its availability made it natural to simply use a different color to represent, for example, debts or loss. However, in Europe addition and subtraction on the abacus were not represented with different colors. Also, the European resistance to negative numbers can be explained with Euclid's *Elements* that for generations had come to define what is and what is not mathematics; in the *Elements* magnitudes and ratios are dealt with, and these concepts do not provide numbers that can be negative or even zero. Thus, for a long time, the full understanding and acceptance of the negatives was kept back. Because of the different roles played by the cultural conditions in ancient China and in Western Europe in the development of negative numbers, it becomes difficult, or even impossible, to assume that students in the society of today would recapitulate the mathematical development of the past.

Concluding Remarks

In this chapter, interpretations of the mathematics of the past, as well as our posture toward mathematics of the past and its relation to mathematics of today, have been problematized. While the aim of this chapter has not been to discuss the integration of history of mathematics in mathematics education, per se, nevertheless, the history of mathematics can play an important role in the learning of mathematics. A permanent issue of debate among historians of mathematics and mathematics educators with an interest to integrate the historical and pedagogical perspectives is which history is suitable and relevant for educational purposes. It is undeniable that history follows a complicated zigzag path, or rather several ditto, sometimes leading to dead ends; throughout the history of mathematics, notations and methods no longer used in mathematics of today can be found. In this context Grattan-Guinness' distinction between history and heritage may be of great relevance to mathematics education: education may benefit from both approaches as long as they are not confused. History may be utilized when the past is considered in its proper context, but also heritage may be utilized, as, for instance, when modern algebra is inserted into Euclid's *Elements*.

The history of mathematics may be profitably utilized in the teaching and learning of mathematics. Besides helping students to sustain interest and excitement in mathematics, history of mathematics can be used to support the actual learning of mathematics, for example, by providing different points of view on certain concepts. A historical approach in the teaching of mathematics may also give mathematics a more human face and help students understand that mathematics is a human activity. Mathematics has evolved in time and space in different ways, and many different cultures have influenced the mathematical epistemology. Knowledge about history of mathematics may not only lead to a better understanding of specific parts of mathematics but may also give us a deeper awareness of what mathematics as a discipline can be.

Cross-References

► Corinna Rossi, Egyptian Architecture and Mathematics
► Sylvie Duvernoy, Classical Greek and Roman Architecture: Mathematical Theories and Concepts

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