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Citation for the original published paper (version of record):
Berman, R. (2024). The probabilistic vs the quantization approach to Kähler–Einstein geometry. Mathematische Annalen, 388(4): 4383-4404. http://dx.doi.org/10.1007/s00208-023-02627-5

Mathematische Annalen
https://doi.org/10.1007/s00208-023-02627-5

Mathematische Annalen

The probabilistic vs the quantization approach to Kähler–Einstein geometry

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Received: 17 February 2022 / Revised: 27 March 2023 / Accepted: 12 April 2023

Abstract
In the probabilistic construction of Kähler–Einstein metrics on a complex projective algebraic manifold X—involving random point processes on X—a key role is played by the partition function. In this work a new quantitative bound on the partition function is obtained. It yields, in particular, a new direct analytic proof that X admits a Kähler–Einstein metrics if it is uniformly Gibbs stable. The proof makes contact with the quantization approach to Kähler–Einstein geometry.

1 Introduction

A complex projective algebraic manifold X admits a Kähler–Einstein metric with positive Ricci curvature if and only if X is a Fano manifold satisfying an algebro-geometric condition called K-stability; this is the content of the solution of the Yau–Tian–Donaldson (YTD) conjecture for Fano manifolds [21]. The proof in [21] is based on a variant of Aubin's method of continuity [1], extended to Aubin's original method in [23]. It involves the following equations for a Kähler metric ω_t, parameterized by "time" t :

$$\mathrm{Ric}\,\omega_t = t\omega_t + (1-t)\mathrm{Ric}\,dV, \tag{1.1}$$

where dV is a fixed a volume form on X, which may be taken to have positive Ricci curvature $\mathrm{Ric}\,dV$ (since X is Fano). The supremum over all $t \in [0, 1]$ for which a solution ω_t exists defines an invariant of X, which is strictly positive, that we shall

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denote by $R(X)$, following [38].[1] As t is increased towards $R(X)$ either ω_t blows-up or it converges towards a Kähler–Einstein metric (in which case $R(X)=1$). The first alternative is precisely what it is shown to be excluded by the condition of K-stability [23]. While it is usually assumed that $t \in [0,1]$ it will in the present work be important to allow t to be any real number.

A probabilistic construction of Kähler–Einstein metrics with negative Ricci curvature was introduced in [4], where the Kähler–Einstein metric emerges from a random point process on X with N points as N tends to infinity (see also [29] for a different probabilistic framework involving random $N \times N$ Hermitian matrices, also inspired by the YTD conjecture). A conjectural extension to Kähler–Einstein metrics with positive Ricci curvature was proposed in [5] and conditional convergence results were given in [6, 8]. In this probabilistic approach the role of K-stability is played by a new type of stability, dubbed Gibbs stability, which amounts to the finiteness of the corresponding partition functions. In the survey [7] connections to the variational proof of the uniform YTD conjecture [13] (involving uniform K-stability) are explained, including non-Archimedean aspects. In the present paper a new quantitative lower bound on the partition functions is obtained, which yields a new direct analytic proof that uniform Gibbs stability implies the existence of a unique Kähler–Einstein metric on X. The proof makes contact with the quantization approach to Kähler geometry and, in particular, with Zhang's new remarkably direct proof of the (uniform) YTD conjecture [45].

1.1 Background on the probabilistic approach

Let X be a Fano manifold. Given a positive integer k we denote by N the dimension of the space of all holomorphic sections of the kth tensor power of the anti-canonical line bundle $-K_X$ (i.e. the top exterior power of the tangent bundle of X):

$$N := \dim H^0(X, -kK_X)$$

(using additive notation for tensor products of line bundles). The Fano assumption on X ensures, in particular, that $N \to \infty$, as $k \to \infty$ (more precisely, $N \sim k^{\dim X}$). Given a basis $s_1^{(k)}, \ldots, s_N^{(k)}$ in $H^0(X, -kK_X)$ denote by $\det S^{(k)}$ the corresponding holomorphic section of the line bundle $-(kK_{X^N}) \to X^N$ defined as the Slater determinant

$$(\det S^{(k)})(x_1, x_2, \ldots, x_N) := \det\left(s_i^{(k)}(x_j)\right). \tag{1.2}$$

Given a volume form dV on X and a parameter $\beta > 0$, the N-fold product X^N is endowed with the following probability measure, introduced in [4]:

$$\mu_\beta^{(N)} := \frac{\left\|\det S^{(k)}\right\|^{2\beta/k} dV^{\otimes N}}{\mathcal{Z}_N(\beta)}, \quad \mathcal{Z}_N(\beta) := \int_{X^{N_k}} \left\|\det S^{(k)}\right\|^{2\beta/k} dV^{\otimes N} \tag{1.3}$$

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[1] the invariant was called β in [41], but here the letter β will be reserved for the parameter appearing in Definition 1.3.

where $\|\cdot\|$ denotes the metric on $-K_X$ (and its tensor powers) induced by dV. In statistical mechanical terms this probability measure represents the equilibrium distribution of N interacting particles on X at inverse temperature β and $\mathcal{Z}_{N_k}(\beta)$ is the corresponding partition function. The probability measure $\mu_\beta^{(N)}$ is, in fact, independent of the choice of bases. It will be convenient to fix a reference volume form dV_0 on X with positive Ricci curvature and a basis $(s_i^{(k)})$ in $H^0(X, -kK_X)$ which is orthonormal with respect to Hermitian product on $H^0(X, -kK_X)$ induced by dV_0.

The probability measure $\mu_\beta^{(N)}$ is symmetric (since the determinant is anti-symmetric) and thus defines a random point process on X with N points $x_1, ..., x_N$. By [4, Thm 5.7] the corresponding *empirical measure* δ_N, i.e. the discrete measure on X defined by

$$\delta_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}, \tag{1.4}$$

converges in probability, as $N \to \infty$, towards a normalized volume form dV_β on X with the property that

$$\omega_\beta := \frac{1}{\beta}\frac{i}{2\pi}\partial\bar{\partial}\log dV_\beta \tag{1.5}$$

is the unique Kähler form solving Aubin's continuity Eq. 1.1 with $t := -\beta$. The convergence of δ_N towards dV_β also implies that the following convergence holds in the weak topology of currents on X :

$$\omega_{k,\beta} := \frac{1}{\beta}\frac{i}{2\pi}\partial\bar{\partial}\left(\log\int_{X^{N-1}}\left\|\det S^{(k)}\right\|^{2\beta/k} dV^{\otimes N-1}\right) \to \omega_\beta, \quad k \to \infty,$$

where $\omega_{k,\beta}$ is a Kähler form, for k sufficiently large (to ensure that $-kK_X$ is very ample).

In fact, the convergence of δ_N towards dV_β was shown to hold at an exponential speed in the sense of Large deviation theory [24]. More precisely, a Large Deviation Principle (LDP) was established, which may be symbolically expressed as

$$(\delta_N)_*\left(\left\|\det S^{(k)}\right\|^{2\beta/k} dV^{\otimes N}\right) \sim e^{-NF_\beta(\mu)}, \quad N \to \infty, \tag{1.6}$$

where the left hand side defines a measure on the space of all probability measures $\mathcal{P}(X)$ on X and $F_\beta(\mu)$ is a free energy type functional on $\mathcal{P}(X)$ (see formula 2.3). Expressing μ as the normalized volume form of a Kähler metric ω in the space $\mathcal{H}$ of all Kähler metrics representing the first Chern class of X, the free energy functional $F_\beta(\mu)$ gets identified with the twisted Mabuchi functional on $\mathcal{H}$ (which is minimized precisely by the unique Kähler metrics ω_β solving Aubin's equation 1.1 with $t = -\beta$) :

$$F_\beta\left(\frac{\omega^n}{V}\right) = \mathcal{M}_\beta(\omega)$$

(cf. formula 2.2). The LDP 1.6 thus implies that

$$\lim_{N\to\infty} -\frac{1}{N}\log \mathcal{Z}_N(\beta) = \inf_{\mathcal{H}} \mathcal{M}_\beta \tag{1.7}$$

1.1.1 The case $\beta < 0$

In the case when β is negative the probability measure $\mu_\beta^{(N)}$ is well-defined for β sufficiently close to 0. The main case of interest is when $\beta = -1$. In this case the measure $\mu_\beta^{(N)}$ is canonically attached to X, i.e. it is independent of the choice of volume form dV (since the contributions from the metric $\|\cdot\|$ on $-K_X$ and the volume form dV on X cancel). Hence, if $\mu_{-1}^{(N)}$ is well-defined when N is sufficiently large, i.e. if $\mathcal{Z}_N(-1) < \infty$—in which case X is called *Gibbs stable*—one obtains canonical random point processes on X with N points. It was conjectured in [5] that the corresponding empirical measures δ_N converge towards a unique Kähler–Einstein metric on X [5], as $N \to \infty$. A conjectural extension of the LDP for positive β in formula 1.6 to any negative β was also put forth in [5]. In a weaker form this conjecture may be formulated as follows:

Conjecture 1.1 *Let X be a Fano manifold endowed with a volume form dV. Given a negative number β_0 the following is equivalent:*

1. *For any given $\beta > \beta_0$ the partition function $\mathcal{Z}_N(\beta)$ is finite, when N is sufficiently large.*
2. *For any given $\beta > \beta_0$ the twisted Mabuchi functional $\mathcal{M}_\beta$ admits a minimizer in $\mathcal{H}$.*

Moreover, if β_0 satisfies the first condition, then for any given $\beta > \beta_0$ the empirical measure δ_N of the ensemble $\left(X^N, \mu_\beta^{(N)}\right)$ converges in probability as $N \to \infty$—after perhaps passing to a subsequence—towards a volume form dV_β such that ω_β, defined by formula 1.5, is a Kähler metric minimizing $\mathcal{M}_\beta$ on $\mathcal{H}$.

In fact, it is enough to show that the limit dV_β is a probability minimizing the functional F_β; it then follows from the regularity results in [11] that ω_β is a Kähler metric and thus minimizes $\mathcal{M}_\beta$ on $\mathcal{H}$. The reason that one has to pass to a subsequence is that a minimizer of $\mathcal{M}_\beta$ need not be uniquely determined, unless dV is assumed to have positive Ricci curvature and $\beta > -1$. The integrability condition $\mathcal{Z}_{N_k}(\beta) < \infty$ is, however, independent of choice of volume form dV. Accordingly, one obtains invariants of the Fano manifold X by setting

$$\gamma_k(X) = \sup_{\gamma>0}\left\{\gamma : \quad \mathcal{Z}_{N_k}(-\gamma) < \infty\right\} \quad \gamma(X) := \liminf_{k\to\infty} \gamma_k(X). \tag{1.8}$$

and X is called *uniformly Gibbs stable if* $\gamma > 1$. This is, a priori, a stronger condition than Gibbs stability (which amounts to the condition that $\gamma_k(X) > 1$ for any sufficiently large k).

The validity of the equivalence "1 $\iff$ 2" in the previous conjecture would, in particular, imply that a Fano manifold X is uniformly Gibbs stable iff X admits a unique Kähler–Einstein metric (in analogy to the uniform version of YTD [13], which, in fact, is equivalent to the ordinary formulation of the YTD [33]). The general equivalence "1 $\iff$ 2" may be reformulated as the following identity:

$$\gamma(X) = \sup_{\gamma>0} \left\{ \gamma : \inf_{\mathcal{H}} \mathcal{M}_{-\gamma} > -\infty \right\}, \tag{1.9}$$

as follows from [2, Thm 1.2] (when restricted to $\beta \geq -1$ the supremum in the right hand side above coincides with the maximal existence time $R(X)$ for Aubin's equations 1.1). Moreover, as shown in [6, Section 7] and [8, Thm 2.3], in order to prove the conjectured convergence towards a minimizer of $\mathcal{M}_\beta$ it is enough to extend the asymptotics 1.7 to $\beta < 0$.

1.2 Main results

For $\beta < 0$ the limsup upper bound in formula 1.7 was established in [5, Thm 6.7] (by combining Gibbs variational principle in statistical mechanics with the asymptotics for transfinite diameters in [9, Thm 6.7]). The main new result in the present work is the following quantitative upper bound that holds for any fixed k, shown using a completely different argument. Henceforth, we set $\gamma := -\beta$.

Theorem 1.2 *There exists a constant $C > 0$ (depending only on the reference volume for dV_0) such that for any $\gamma > 0$ and positive integer k*

$$-\frac{1}{N} \log \mathcal{Z}_N(-\gamma) \leq \frac{k+\gamma}{k+1} \inf_{\mathcal{H}} \mathcal{M}_{-\gamma c_k} + k^{-1}\gamma \left(C + (|1-\gamma| + C) \log \left\| \frac{dV}{dV_0} \right\|_{L^\infty(X)} \right),$$

where $c_k := (1 - Ck^{-1})(k+1)/(k+\gamma)$.

For $\gamma \leq 1$ the first term in the right hand side of the previous inequality may be replaced by the infimum of $\mathcal{M}_{-\gamma(1-Ck^{-1})}$ (see Sect. 2.4).

The previous theorem immediately implies one direction of the conjectured equality 1.9:

Corollary 1.3 *The following inequality holds*

$$\gamma(X) \leq \sup_{\gamma>0} \left\{ \gamma : \inf_{\mathcal{H}} \mathcal{M}_{-\gamma} > -\infty \right\}$$

In other words "1 $\Longrightarrow$ 2" in Conjecture 1.1. In particular, if X is uniformly Gibbs stable, then X admits a unique Kähler–Einstein metric.

As next explained this corollary also follows from combining the algebro-geometric results in [31, Thm 6.7] with the solution of the (uniform) YTD-conjecture in [21, Thm 6.7] (or [13, Thm 6.7]) and its very recent generalization in [45, Thm 6.7] (which applies to general β). More precisely, exploiting that $\gamma_k(X)$ may be realized as the log canonical threshold (lct) of an anti-canonical divisor on X^{N_k} it is shown in [31, Thm 2.5] that $\gamma_k(X)$ is bounded from above by the invariant $\delta_k(X)$ introduced in [31]:

$$\gamma_k(X) \leq \delta_k(X) := \inf_{\Delta_k} \text{lct}\,(\Delta_k), \tag{1.10}$$

where the infimum is taken over all anti-canonical $\mathbb{Q}$-divisors Δ_k on X of k-basis type, i.e. Δ_k is the normalized sum of the N zero-divisors on X defined by the members of a given basis in $H^0(X, -kK_X)$. In particular,

$$\gamma(X) \leq \delta(X) := \limsup_{k\to\infty} \delta_k(X),$$

where the invariant $\delta(X)$ characterizes uniform K-stability; $\delta(X) > 1$ iff X is uniformly K-stable [32] (by [17] the limsup above is, in fact, a limit). Recently, it was shown in [35] that $\delta_k(X)$ coincides with the coercivity threshold of the quantized Ding functional on the symmetric space $GL(N, \mathbb{C})/U(N)$. Combining this result with the quantized maximum principle in [16], it was then shown in [45] that $\delta(X)$ coincides with the coercivity threshold of the Ding functional (as further discussed in Sect. 1.3). Finally, Corollary 1.3 follows from [2, Thm 3.4], which implies that the coercivity thresholds of the Ding and the Mabuchi functionals coincide.

1.2.1 Outline of the proof of Theorem 1.2

The proof of Theorem 1.2 is surprisingly simple. The key new observation is an inequality which, in its simplest form, $\beta = -1$ (i.e. $\gamma = 1$), may be formulated as follows:

$$-\log \mathcal{Z}_N \leq \left(1 + k^{-1}\right) \inf_{\mathcal{H}} \mathcal{D}_k + \frac{1}{kN} \log N \tag{1.11}$$

where the infimum runs over the space $\mathcal{H}$ of all metrics on $-K_X$ with positive curvature and $\mathcal{D}_k$ is a certain functional on $\mathcal{H}$, approximating the twisted Ding functional $\mathcal{D}$ (in the sense that $\mathcal{D}_k$ converges towards $\mathcal{D}$ as $k \to \infty$); see formula 2.5. Next, by an inequality established in [12] (leveraging the positivity of direct image bundles in [15]) there exists a constant C such that

$$\mathcal{D}_k \leq \mathcal{D} - Ck^{-1}\mathcal{E} \;\text{ on }\; \mathcal{H}_0,$$

where $\mathcal{H}_0$ denotes the subspace of all sup-normalized metrics on $\mathcal{H}$ and $\mathcal{E}$ denotes the standard functional on $\mathcal{H}$ defined as the primitive of the Monge–Ampère operator (which is non-positive on $\mathcal{H}_0$). Finally, using the well-known fact that $\mathcal{D}$ is bounded from above by the Mabuchi functional $\mathcal{M}$ this proves Theorem 1.2 when $\gamma = 1$ (by absorbing the error term $-Ck^{-1}\mathcal{E}$ in the subscript γ of the twisted Mabuchi functional $\mathcal{M}_\gamma$). A slight twist of this argument yields the inequality in Theorem 1.2 for a general γ, using the thermodynamical formalism in [2].

1.3 Comparison with the quantization approach

In the quantization approach to Kähler geometry, which goes back to [26, 27, 40, 44], the space $\mathcal{H}(L)$ of all Hermitian metrics on a holomorphic line bundle L over a complex manifold X is approximated by the finite dimensional space $\mathcal{H}_k(L)$ of all Hermitian metrics on the N-dimensional complex vector space $H^0(X, kL)$[2] The space $\mathcal{H}_k(L)$ may be identified with the symmetric space $GL(N, \mathbb{C})/U(N)$. When X is Fano and $L = -K_X$ a quantization of the Ding functional $\mathcal{D}$ on $\mathcal{H}$ was introduced in [10], building on [28], which defines a functional on $\mathcal{H}_k$ that we shall denote by D_k (formula 3.2). Here it is observed (Proposition 3.1) that

$$\inf_{\mathcal{H}_k} D_k = \left(1 + k^{-1}\right) \inf_{\mathcal{H}} \mathcal{D}_k, \tag{1.12}$$

where $\mathcal{D}_k$ is the approximation on $\mathcal{H}$ of the Ding functional $\mathcal{D}$ which appeared in the inequality 1.11. As a consequence,

$$-\log \mathcal{Z}_N \leq \inf_{\mathcal{H}_k} D_k + \frac{1}{kN} \log N. \tag{1.13}$$

A similar inequality holds for a general γ (see Theorem 3.3) which yields a new proof of the inequality 1.10.

This line of reasoning is inspired by K.Zhang's very recent new proof of the uniform YTD conjecture for Fano manifolds [45]. In fact, the author discovered the equality 1.12 while trying to find a conceptual replacement for an inequality used in the proof of [45, Thm 5.1] (involving Tian's α-invariant [39]). One virtue of the present approach is that it directly yields a quantitative estimate on the infimum of D_k over $\mathcal{H}_k$. More generally, denoting by $D_{k,\beta}$ the twisted generalization of D_k (coinciding with D_k for $\beta = -1$),

$$\inf_{\mathcal{H}_k} D_{k,-\gamma} \leq \frac{k+\gamma}{k+1} \inf_{\mathcal{H}} \mathcal{M}_{-\gamma c_k} + k^{-1}\gamma \left(C + (|1-\gamma| + C) \log \left\| \frac{dV}{dV_0} \right\|_{L^\infty(X)} \right), \tag{1.14}$$

as follows from combining formula 1.12 (extended to general γ) with the inequality 1.11 (extended to general γ). As in [45, Thm 5.1] this shows that uniform K-stability of X implies that X admits a unique Kähler–Einstein metric. Indeed, as shown in [35], building on [31, 32], uniform K-stability is equivalent to the existence of some $\epsilon > 0$ such that the infimum of $D_{k,-1-\epsilon}$ on $\mathcal{H}_k$ is finite for k sufficiently large. By the inequality 1.14 this implies that $\mathcal{M}_{-1-\epsilon}$ is bounded from below (or equivalently, that $\mathcal{M}_{-1}$ is coercive) which, in turn, implies that X admits a unique Kähler–Einstein metric (as first shown in [43] using Aubin's method of continuity and then using a direct variational approach in [2, 11], which applies to any γ).

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[2] In physical terms $\mathcal{H}_k(L)$ can be viewed as the quantization of $\mathcal{H}$ with k^{-1} playing the role of Planck's constant in quantum mechanics [26].

1.4 Outlook on converse bounds and exceptional Fano orbifolds

The converse of the inequality 1.14 also holds, as follows from [10, Lemma 7.7]. As a consequence,

$$\delta(X)=\sup_{\gamma>0}\left\{\gamma:\ \inf_{\mathcal{H}}\mathcal{M}_{-\gamma}>-\infty\right\}. \tag{1.15}$$

This identity is equivalent to the result [45, Thm 5.1] (which is formulated in terms of the infimum of $\mathcal{D}_\beta$, but, by [2, Thm 1.1], this infimum coincides with the infimum of $\mathcal{M}_\beta$). It remains, however, to establish a similar lower bound on $-\log\mathcal{Z}_{N,-\gamma}$ or, at least, the missing lower bound on $\gamma(X)$ in the conjectured formula 1.9. By formula 1.15, this amounts to upgrading the inequality between $\gamma(X)$ and $\delta(X)$ in Cor 1.3 to an equality. In contrast, it should be stressed that the inequality 1.10 between $\gamma_k(X)$ and $\delta_k(X)$ is *not* an equality, in general. For example, when X is the Riemann sphere, i.e. the complex projective line $\mathbb{P}^1$,

$$\gamma_k(X)=1-\frac{1}{2k+1},\quad \delta_k(X)=1.$$

[30, 35]. This discrepancy becomes even more pronounced in the more general of setting of Fano orbifolds X, where the role of K_X is played by the orbifold canonical line bundle $K_{X_{orb}}$. All the results in the present paper readily extend to the orbifold setting (using, in particular, the uniform asymptotics for Bergman measures on orbifolds in [22, Thm 1.4] as a replacement for the inequality 2.10). The partition function of the Fano orbifold X then coincide with the partition function of the quasi-regular Calabi-Yau cone over X introduced in [14], in the context of the AdS/CFT correspondence (see the proof of [14, Thm B]).

For example, any Fano orbifold curve is of the form

$$X=\mathbb{P}^1/G$$

where G is the finite group acting on $\mathbb{P}^1$ induced by the action on $\mathbb{C}^2$ of a finite subgroup of $SU(2)$. By the "ADE-trichotomy" such groups fall into the three classes, corresponding to the classification of simply laced Dynkin diagrams; two infinite series $A_n$and D_n and three exceptional cases E_6, E_7 and E_8. As it turns out, the ADE-trichotomy is detected by the corresponding partition functions at the canonical value $\gamma=1$ (as follows from [8, Thm 3.5]):

- (A) $\mathcal{Z}_N(-1)=\infty$ for all N (i.e. $\gamma_k(X)<1$ for all k)
- (D) $\mathcal{Z}_N(-1)<\infty$ for $N\gg 1$, but not all N (i.e. $\gamma_k(X)>1$ for k sufficiently large)
- (E) $\mathcal{Z}_N(-1)<\infty$ for all N (i.e. $\gamma_k(X)>1$ for all).

Moreover, $\gamma_k(X)$ is strictly increasing wrt k. On the hand it can be shown that

$$\delta_k(X)=\delta(X)$$

and thus $\delta(X)=\gamma(X)$, while, $\gamma_k(X)<\delta_k(X)$.

The notion of exceptionality has been extended to general Fano orbifolds [18], motivated by the Minimal Model Program in birational algebraic geometry [36]. A Fano orbifold X is said to be exceptional if $\alpha(X) > 1$, where $\alpha(X)$ denotes Tian's alpha-invariant [39] (which, in algebro-geometric terms, coincides with the global log canonical threshold of X [25]). For example, in [18, Cor 1.1] a finite list of exceptional Fano orbifold surfaces X is given, realized as hypersurfaces in weighted three-dimensional complex projective space. In general, it follows readily from the definitions that

$$\alpha(X) \leq \gamma_k(X)$$

(cf. [6, Lemma 7.1]). As a consequence, if X is exceptional, then $\mathcal{Z}_N$ is finite for *any* N. Does the converse also hold? For Fano orbifold curves this is, indeed, the case, according to the ADE-list above.

2 Proof of Theorem 1.2

2.1 Setup

We will use additive notation for line bundles and metrics. Accordingly, the k the tensor power of a holomorphic line bundle L over an n-dimensional complex manifold X will be denoted by kL and if ϕ is a metric on L then $k\phi$ denotes the induced metric on kL. Accordingly, if s is a holomorphic section of L, i.e. $s \in H^0(X, L)$, the point-wise norm of s with respect to a metric ϕ on L is denoted by $|s|_\phi$ Given a local trivializing section of L we may identify s with a local holomorphic function on X and ϕ with a local smooth function so that

$$|s|_\phi^2 := |s|^2 e^{-\phi}$$

and the normalized curvature of the metric ϕ may be expressed as

$$dd^c\phi := \frac{i}{2\pi}\partial\bar{\partial}\phi.$$

A smooth metric ϕ on L is said to have positive curvature if $dd^c\phi > 0$ and semi-positive curvature of $dd^c\phi \geq 0$ (when identified with an $n \times n$ Hermitian matrix). Equivalently, this means that, locally, ϕ is plurisubharmonic (psh) and strictly psh, respectively. Given a metric ϕ with semi-positive curvature we denote by $MA(\phi)$ the corresponding Monge–Ampère measure, normalized to have unit total mass:

$$MA(\phi) := \frac{1}{V}(dd^c\phi)^n.$$

2.1.1 The anti-canonical setup

Henceforth, the line bundle L will be taken to be the anti-canonical line bundle $-K_X$ of X, i.e. top exterior power of the tangent bundle of X. Then any smooth metric ϕ on $-K_X$ induces a volume form on X that we shall, abusing notation slightly, denote by $e^{-\phi}$. This notation is intended to reflect the fact that if $z_1, ..., z_n$ are local holomorphic coordinates on X and ϕ is locally represented by a function with respect to the local trivialization $\partial/\partial z_1 \wedge \cdots \partial/\partial z_n$ of $-K_X$, then the volume form in question has density $e^{-\phi}$ with respect to the local Euclidean volume form corresponding to $z_1, ..., z_n$.

Given a metric ϕ on $-K_X$ and a volume form μ on X we shall denote by $H^{(k)}(\phi, \mu)$ the corresponding Hermitian metric on the N-dimensional complex vector space $H^0(X, -kK_X)$, defined by

$$H^{(k)}(\phi, \mu)(s, s) := \int_X |s|^2_{k\phi}\mu.$$

The space of all metrics on ϕ on $-K_X$ with positive curvature will be denoted by $\mathcal{H}$. We will fix once and for all a reference metric ψ_0 in $\mathcal{H}$ and denote by dV_0 the corresponding volume form on X :

$$dV_0 := e^{-\psi_0}.$$

Moreover, we fix a basis $s_1^{(k)}, ..., s_N^{(k)}$ in $H^0(X, -kK_X)$ which is orthonormal with respect to the corresponding Hermitian norm $H^{(k)}(\psi_0, dV_0)$ .Accordingly, we can identify a Hermitian metric H on $H^0(X, -kK_X)$ with the corresponding $N \times N$ positive definite Hermitian matrix $H(s_i^{(k)}, s_j^{(k)})$.

2.1.2 Energies

Following (essentially) the notation in [9] we denote by $\mathcal{E}$ the functional on $\mathcal{H}$ uniquely determined by the following conditions:

$$d\mathcal{E}_{|\phi} = MA(\phi), \quad \mathcal{E}(\psi_0) = 0$$

Alternatively, $\mathcal{E}(\phi)$ may be explicitly defined by

$$\mathcal{E}(\phi) := \frac{1}{V(n+1)} \int_X \sum_{j=0}^{n} (\phi - \psi_0)(dd^c\phi)^{n-j} \wedge (dd^c\psi_0)^j$$

Dually, following [10], the pluricomplex energy of a probability measure μ on X (wrt the reference metric ψ_0) is defined by

$$E(\mu) = \sup_{\phi \in \mathcal{H}} \left(\mathcal{E}(\phi) - \int_X (\phi - \psi_0)\mu \right)$$

(however in [10] the functional $\mathcal{E}(\phi)$ is denoted by $E(\phi)$ and the pluricomplex energy is denoted by E^*). We will use the following basic

Lemma 2.1 *There exists a positive constant c_X only depending on X such that*

$$-\mathcal{E}(\phi)+\sup_X(\phi-\psi_0)\leq nE\left(MA(\phi)\right)+c_X. \tag{2.1}$$

Proof This is essentially well-known but for completeness a short proof is provided. First observe that there exists a constant c_X such that $\sup_X(\phi-\psi_0)-c_X$ is bounded from above by the integral of $(\phi-\psi_0)$ against $MA(\psi_0)$. Indeed, this follows directly follows from the submean property of plurisubharmonic functions and the compactness of X. Hence, the proof is concluded by invoking the following basic inequality (see [2, Lemma 2.13]):

$$J(\phi):=-\mathcal{E}(\phi)+\int_X(\phi-\psi_0)MA(\psi_0)\leq nE\left(MA(\phi)\right)$$

2.1.3 The twisted Ding and Mabuchi functional associated to (ϕ_0,γ)

Fix $\gamma>0$. Given a volume form dV on X we will denote by ϕ_0 the corresponding metric on $-K_X$ (i.e. $dV=e^{-\phi_0}$). To the pair (ϕ_0,γ) we attach the *twisted Ding functional* on $\mathcal{H}$ defined by

$$\mathcal{D}_{-\gamma}(\phi):=-\mathcal{E}(\phi)-\frac{1}{\gamma}\log\int_X e^{-(\gamma\phi+(1-\gamma)\phi_0)}.$$

(coinciding with the ordinary Ding functional when $\gamma=-1$). The definition is made so that $\mathcal{D}_{-\gamma}$ is scale invariant, i.e. invariant under $\phi\mapsto\phi+c$ for any $c\in\mathbb{R}$. The corresponding (twisted) Mabuchi functional is usually defined, modulo an additive constant, by demanding that its first variation is proportional to the (twisted) scalar curvature [34, 37], but here it will be convenient to use the thermodynamical formalism introduced in [2, Prop 4.1]:

$$\mathcal{M}_{-\gamma}(\phi):=F_{-\gamma}(\mu),\quad \mu=MA(\phi), \tag{2.2}$$

where $F_\gamma(\mu)$ is the *free energy* of a probability measure μ on X defined by

$$F_{-\gamma}(\mu):=-\gamma\left(E(\mu)+\int_X(\phi_0-\psi_0)\mu\right)+\mathrm{Ent}\ \left(\mu|e^{-(\gamma\psi_0+(1-\gamma)\phi_0)}\right), \tag{2.3}$$

where $\mathrm{Ent}\,(\mu|\nu)$ denotes the entropy of a measure μ on X relative to the measure ν on X (using the sign convention that renders $\mathrm{Ent}\,(\mu|\nu)$ non-negative when μ and ν are both probability measures). By [2, Prop 3.5]

$$\mathcal{D}_{-\gamma}(\phi)\leq\gamma^{-1}\mathcal{M}_{-\gamma}(\phi) \tag{2.4}$$

(moreover, the two functionals $\mathcal{D}_{-\gamma}$ and $\mathcal{M}_{-\gamma}$ have the same infimum over $\mathcal{H}$, but his fact will not be needed here).

Remark 2.2 In the notation of [2], $\mathcal{D}_{-\gamma} = -\mathcal{G}_{-\gamma}$ and the definition of the free energy $F_{-\gamma}$ used here is $-\gamma$ times the definition employed in [2]. When $\gamma = 1$ formula 2.2 is equivalent to the Tian–Chen formula for the Mabuchi functional [20, 42] and the case $\gamma \neq 1$ is closely related to the generalized Mabuchi functional introduced in [37, Def 6.1].

2.2 Two inequalities

The key new observation in the proof of Theorem 1.2 is the following proposition which yields a bound, from below, on the partition function

$$\mathcal{Z}_{N,-\gamma} = \int_{X^N} \left\| \det S^{(k)} \right\|_{k\phi_0}^{-2\gamma/k} (e^{-\phi_0})^{\otimes N},$$

in terms of the infimum over the space of all metrics ϕ on $-K_X$ of the functional $\mathcal{D}_{k,-\gamma}$ defined by

$$\mathcal{D}_{k,-\gamma}(\phi) := -\mathcal{E}_k(\phi) - \frac{1}{\gamma} \log \int_X e^{-(\gamma\phi + (1-\gamma)\phi_0)}, \tag{2.5}$$

with

$$\mathcal{E}_k(\phi) := -\frac{1}{N(k+\gamma)} \log \det H^{(k)} \left(\phi, e^{-(\gamma\phi+(1-\gamma)\phi_0)} \right),$$

where the Hermitian metric $H^{(k)} \left(\phi, e^{-(\gamma\phi+(1-\gamma)\phi_0)} \right)$ has been identified with a Hermitian matrix, as in Sect. 2.1.1. The normalization have been chosen to ensure that

$$\mathcal{E}_k(\phi + c) = \mathcal{E}_k(\phi) + c \ , \forall c \in \mathbb{R}.$$

As a consequence, since $\mathcal{E}_k(\phi)$ is increasing wrt ϕ, its differential $d\mathcal{E}_{k|\phi}$ may be represented by a probability measure on X. For future reference we note that the probability measure in question coincides with the *Bergman measure* associated to the Hermitian metric $H^{(k)} \left(\phi, e^{-(\gamma\phi+(1-\gamma)\phi_0)} \right)$:

$$d\mathcal{L}_{k|\phi} = B_{k\phi} := \rho_{k\phi} e^{-(\gamma\phi+(1-\gamma)\phi_0)}, \quad \rho_{k\phi} := \frac{1}{N} \sum_{i=1}^{N_k} |S_i|^2_{k\phi} \tag{2.6}$$

where S_i denotes any bases in $H^0(X, -kK_X)$ which is orthonormal wrt $H^{(k)} \left(\phi, e^{-(\gamma\phi+(1-\gamma)\phi_0)} \right)$ (as follows from [9, Lemma 2.1]).

Proposition 2.3 *Given* (ϕ_0, γ) *the following inequality holds for any* k*:*

$$-\frac{1}{\gamma N_k}\log \mathcal{Z}_{N_k}(-\gamma) \leq \left(1+\gamma k^{-1}\right)\inf_{\phi}\mathcal{D}_{k,-\gamma}(\phi)+\frac{1}{kN_k}\log N_k$$

where the infimum runs over all smooth metrics ϕ *on* $-K_X$ *(with no restrictions on the curvature).*

Proof Let ϕ be a metric on $-K_X$. Then we can rewrite

$$\begin{aligned}\mathcal{Z}_{N,-\gamma} &:= \int_{X^N}\left\|\det S^{(k)}\right\|_{k\phi_0}^{-2\gamma/k}(e^{-\phi_0})^{\otimes N}\\ &= \int_{X^N}\left\|\det S^{(k)}\right\|_{k\phi}^{-2\gamma/k}\left(e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)^{\otimes N}.\end{aligned}$$

Indeed, locally on each factor of X^N this simply amounts to rewriting

$$\left(e^{-k\phi_0}\right)^{-\gamma/k}e^{-\phi_0}=\left(e^{-k\phi}\right)^{-\gamma/k}e^{-\phi_0}e^{-(\gamma\phi+(1-\gamma)\phi_0)}$$

Now assume that ϕ has the property that $e^{-(\gamma\phi+(1-\gamma)\phi_0)}$ is a probability measure. Then, applying Hölder's inequality with negative exponent $-\gamma/k$ (or Jensen's inequality applied to the convex function $t\mapsto t^{-\gamma/k}$ on $]-\infty,\infty[$) yields

$$\mathcal{Z}_{N,-\gamma}\geq\left(\int_{X^N}\left\|\det S^{(k)}\right\|_{k\phi}^{2}\left(e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)^{\otimes N}\right)^{-\gamma/k}.$$

Taking logarithms this means that

$$-\frac{1}{\gamma N}\log \mathcal{Z}_{N,-\gamma}\leq\frac{1}{kN}\log\int_{X^N}\left\|\det S^{(k)}\right\|_{k\phi}^{2}\left(e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)^{\otimes N}.$$

Now, for any metric ϕ on $-K_X$ we may apply the previous inequality to $\phi+\log\int_X e^{-(\gamma\phi+(1-\gamma)\phi_0)}$ and deduce that $-\frac{1}{\gamma N}\log\mathcal{Z}_{N,-\gamma}$ is bounded from above by

$$\begin{aligned}&(1+\gamma k^{-1})\\ &\left(\frac{1}{(k+\gamma)N}\log\int_{X^N}\left\|\det S^{(k)}\right\|_{k\phi}^{2}\left(e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)^{\otimes N}-\frac{1}{\gamma}\log\int_X e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right).\end{aligned}$$

The proof is thus concluded by invoking the following formula [9, Lemma 5.3], which holds for any volume form μ on X :

$$\int_{X^N}\left\|\det S^{(k)}\right\|_{k\phi}^{2}\mu^{\otimes N}=N!\det\left(H^{(k)}(\phi,\mu)\right)$$

We will also make use of the following slight generalization of the inequality in [12, formula 3.4] (to the case $\gamma \neq 1$),employing the notation

$$\phi^{(\epsilon)} := \phi(1-\epsilon) + \epsilon\psi_0, \tag{2.7}$$

for a given positive number ϵ.

Lemma 2.4 *There exists a constant C_0 depending only on ψ_0 such that the following inequality holds for with $\epsilon := \frac{\gamma-1}{k+\gamma}$:*

$$\begin{aligned}&-\frac{1}{(1-\epsilon)}\mathcal{E}_k\left(\phi^{(\epsilon)}\right)\\ &\leq -\mathcal{E}(\phi) + C_0 k^{-1}\left(-\mathcal{E}(\phi) + \sup_X(\phi - \psi_0)\right) + \frac{|\gamma-1|}{k+1}\left\|\phi_0 - \psi_0\right\|_{L^\infty(X)}.\end{aligned}$$

Proof This is shown in essentially the same way as in the proof of [12, formula 3.4], but to pinpoint the exact dependence on the constant we recall the argument. Let ψ_t be a weak geodesic connecting ϕ (at $t = 1$) with ψ_0 (at $t = 0$) [19]. In particular, this means that $t \mapsto \psi_t$ is a *psh path* (aka a subgeodesic) in the following sense: extending ψ_t to $X \times ([0, 1] \times i\mathbb{R})$, so that ψ_t is independent of the imaginary part of t, the corresponding local function $(z, t) \mapsto \psi_t(z)$ is psh locally on $X \times (]0, 1[\times i\mathbb{R})$. Moreover, it will be convenient to use the following regularity properties [19]: $dd^c\psi_t \in L^\infty_{loc}$ for any fixed t and $t \mapsto \psi_t$ is C^1-differentiable up to the boundary of [0, 1] (but, as explained in [12], for the proof it is enough to use that ψ_t is in L^∞_{loc} for any fixed t). Now,

$$\text{(i)}\, t \mapsto \mathcal{E}(\psi_t) \text{ is affine}, \quad \text{(ii)}\,\text{t} \mapsto \mathcal{E}_k\left(\psi_t^{(\epsilon)}\right) \text{ is concave} \tag{2.8}$$

if ϵ is sufficiently small. In fact, the first statement characterizes the geodesic ϕ_t among all psh paths ϕ_t as above [13, Thm 1.7] and the second one follows from [15], only using that $\psi_t^{(\epsilon)}$ is a psh path. To see this rewrite $-kK_X = (k+1)L + K_X$ for $L = -K_X$. Then, locally, rewriting $e^{-k\phi}e^{-(\gamma\phi+(1-\gamma)\phi_0)} = e^{-(k\phi+\gamma\phi+(1-\gamma)\phi_0)}$ the Hermitian metric $H^{(k)}\left(\phi, e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)$ coincides with the L^2-metric on $H^0(X, (k+1)L + K_X)$ induced by the metric $k\phi + \gamma\phi + (1-\gamma)\phi_0$ on $(k+1) + L_X$. Accordingly, $\mathcal{E}_k(\phi)$ may be identified with the L^2-metric on the determinant line of $H^0(X, (k+1)L + K_X)$. Now replace ϕ with $\psi_t^{(\epsilon)}$ (defined as in formula 2.7 with ϕ replaced by ψ_t) and decompose the corresponding metric on $(k+1)L + K_X$ as

$$k\psi_t^{(\epsilon)} + \gamma\psi_t^{(\epsilon)} + (1-\gamma)\phi_0 = (k+\gamma)(1-\epsilon)\psi_t + ((k+\gamma)\epsilon\psi_0 + (1-\gamma)\phi_0). \tag{2.9}$$

We will first consider the special case that $\phi_0 = \psi_0$. Then the second term above has non-negative curvature on X as soon as $\epsilon \geq \frac{\gamma-1}{k+\gamma}$. Henceforth it will assumed that $\epsilon = \frac{\gamma-1}{k+\gamma}$ (then $(1-\epsilon) = (k+1)(/k+\gamma) > 0$). Since ψ_t is locally psh on $X \times ([0, 1] \times i\mathbb{R})$ the whole expression in formula 2.9 is thus locally psh. Hence, the convexity of $t \mapsto \mathcal{E}_k\left(\psi_t^{(\epsilon)}\right)$ follows from the positivity of direct image bundles in [15], applied the to trivial fibration $X \times (]0, 1[\times i\mathbb{R}) \to]0, 1[\times i\mathbb{R}$. This concludes the proof of the properties in formula 2.8. As a consequence, the function $t \mapsto -\frac{1}{(1-\epsilon)}\mathcal{E}_k\left(\psi_t^{(\epsilon)}\right) + \mathcal{E}(\psi_t)$ is concave, giving,

$$
\begin{aligned}
-\frac{1}{(1-\epsilon)}\mathcal{E}_k\left(\phi^{(\epsilon)}\right) + \mathcal{E}(\phi) &\leq -\mathcal{E}_k\left(\psi_0^{(\epsilon)}\right) \\
&+\mathcal{E}(\psi_0) + \int_X \left(-\frac{d\psi_t}{dt}|_{t=0}\right)\left(\frac{1}{(1-\epsilon)}d\mathcal{E}_k\left(\phi^{(\epsilon)}\right) - d\mathcal{E}\right)_{|\psi_0}
\end{aligned}
$$

Now assume first that ϕ is sup-normalized, i.e. that $\sup_X(\phi - \psi_0) = 0$. Then it follows from the convexity of $t \mapsto \phi_t$ that $\frac{d\psi_t}{dt}|_{t=0} \leq 0$. Next, since we are considering the special case $\phi_0 = \psi_0$ and $\psi_0^{(\epsilon)} = \psi_0$ the term $\mathcal{E}_k\left(\psi_0^{(\epsilon)}\right)$ vanishes and so does $\mathcal{E}(\psi_0)$ (by definition). Moreover, since the differential of the functional

$$
\phi \mapsto \frac{1}{(1-\epsilon)}\mathcal{E}_k\left(\phi^{(\epsilon)}\right)
$$

is given by the Bergman measure B_k associated to the Hermitian metric $H^{(k)}(\psi_0, e^{-\psi_0})$ (by formula 2.6) it follows from Bergman kernel asymptotics [42] that there exists a constant C_0 (depending only on ψ_0) such that

$$
\left(\frac{1}{(1-\epsilon)}d\mathcal{E}_k\left(\phi^{(\epsilon)}\right) - d\mathcal{E}\right)_{|\psi_0} \leq -C_0k^{-1}d\mathcal{E}(\psi_0). \tag{2.10}
$$

(in fact only an *upper* bound on B_k is needed for which there is an elementary proof [12, Prop 2.4]). Hence,

$$
-\frac{1}{(1-\epsilon)}\mathcal{E}_k\left(\phi^{(\epsilon)}\right) + \mathcal{E}(\phi) \leq C_0k^{-1}\int \left(-\frac{d\psi_t}{dt}|_{t=0}\right)(d\mathcal{E})_{|\psi_0} = -C_0k^{-1}\mathcal{E}(\phi),
$$

using, in the last equality (i) in formula 2.8. Replacing a general $\phi \in \mathcal{H}$ with its sup-normalized version $\phi - \sup_X(\phi - \psi_0)$ we deduce that

$$
-\frac{1}{(1-\epsilon)}\mathcal{E}_k\left(\phi^{(\epsilon)}\right) + \mathcal{E}(\phi) \leq C_0k^{-1}\left(-\mathcal{E}(\phi) + \sup_X(\phi - \psi_0)\right).
$$

This concludes the proof when $\phi_0 = \psi_0$. Finally, to handle the case of a general case note that replacing ϕ_0 with ψ_0 in the definition of $\mathcal{E}_k\left(\phi^{(\epsilon)}\right)$ just gives rise to an extra term which, after multiplication by $\frac{1}{(1-\epsilon)}$, may estimated from above by

$$
\begin{aligned}
&\frac{1}{(1-\epsilon)}\frac{1}{k+\gamma}\log e^{(\gamma-1)\sup_X(\phi_0-\psi_0)} \\
&\leq \sup_X|\phi_0 - \psi_0|\frac{1}{(1-\epsilon)(k+\gamma)}|\gamma - 1| = \sup_X|\phi_0 - \psi_0|\frac{1}{k+1}|\gamma - 1|.
\end{aligned}
$$

2.3 Conclusion of the proof of Theorem 1.2

By Proposition 2.3 the following inequality holds for any metric ϕ on $-K_X$ and number satisfying $(1-\epsilon)\geq 0$:

$$-\frac{1}{\gamma N}\log \mathcal{Z}_N(-\gamma)\leq \left(1+\gamma k^{-1}\right)(1-\epsilon)\left(\frac{1}{N(k+\gamma)(1-\epsilon)}\mathcal{E}_k(\phi)\right.\\ \left.-\frac{1}{\gamma(1-\epsilon)}\log\int_X e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)$$

Taking $\epsilon=\frac{\gamma-1}{k+\gamma}$ and replacing ϕ with $\phi^{(\epsilon)}$ (defined as in the previous lemma) and setting $\gamma^{(\epsilon)}:=(1-\epsilon)\gamma$ thus yields (using that $\left(1+\gamma k^{-1}\right)(1-\epsilon)=1+k^{-1})$)

$$-\frac{1}{\gamma N}\log \mathcal{Z}_N(-\gamma)\leq (1+k^{-1})\left(\frac{1}{N(k+\gamma)(1-\epsilon)}\log\det H^{(k)}(\phi^{(\epsilon)},\gamma)\right.\\ \left.-\frac{1}{\gamma^{(\epsilon)}}\log\int_X e^{-\left(\gamma^{(\epsilon)}\phi+(1-\gamma^{(\epsilon)})\phi_0\right)}\right)$$

Next, in order to fix ideas, we first consider the special case when $\phi_0=\psi_0$. Then, by the previous lemma, the right hand side above is bounded from above by

$$(1+k^{-1})\left(\mathcal{D}_{-\gamma^{(\epsilon)}}(\phi)+C_0k^{-1}\left(-\mathcal{E}(\phi)+\sup_X(\phi-\psi_0)\right)\right). \tag{2.11}$$

Since (trivially) $\mathcal{D}_{-\gamma^{(\epsilon)}}(\phi)\leq -\mathcal{E}(\phi)+\sup_X(\phi-\psi_0)$ it follows that

$$-\frac{1}{\gamma N}\log \mathcal{Z}_N(-\gamma)\leq \mathcal{D}_{-\gamma^{(\epsilon)}}(\phi)+(C_0+1)k^{-1}\left(-\mathcal{E}(\phi)+\sup_X(\phi-\psi_0)\right)$$

Invoking the inequality 2.1 thus reveals that there exists a constant C only depending on ψ_0 such that

$$-\frac{1}{N}\log \mathcal{Z}_N(-\gamma)\leq \gamma\mathcal{D}_{-\gamma^{(\epsilon)}}(\phi)+C\gamma k^{-1}E\left(MA(\phi)\right)+C\gamma k^{-1} \tag{2.12}$$

Next, we rewrite the first two terms in the right hand side above as

$$\gamma\mathcal{D}_{-\gamma^{(\epsilon)}}(\phi)+C\gamma k^{-1}E\left(MA(\phi)\right)\\ =(1-\epsilon)^{-1}\left(\gamma^{(\epsilon)}\mathcal{D}_{-\gamma^{(\epsilon)}}(\phi)+C\gamma^{(\epsilon)}k^{-1}E\left(MA(\phi)\right)\right),$$

where the second factor above is is bounded from above by $\mathcal{M}_{-\gamma^{(\epsilon)}(1-Ck^{-1})}$, as follows from the inequality 2.4 and the free energy formula 2.2. Hence,

$$-\frac{1}{N}\log \mathcal{Z}_N(-\gamma)\leq (1-\epsilon)^{-1}\mathcal{M}_{-\gamma^{(\epsilon)}(1-Ck^{-1})}+C\gamma k^{-1},$$

proving the theorem in the case when $\phi_0 = \psi_0$. The general case is shown in essentially the same way, by first including the error term involving ϕ_0 from Lemma 2.4 in formula 2.11 and then, in formula 2.12, estimating

$$E\left(MA(\phi)\right) \leq \left(E\left(MA(\phi)\right) + \int (\phi_0 - \psi_0) MA(\phi)\right) + \|\phi_0 - \psi_0\|_{L^\infty(X)},$$

so that the first term in the right hand side above can be absorbed into the twisted Mabuchi functional, as before.

2.4 The case $\gamma \leq 1$

For $\gamma \leq 1$ the estimate in Theorem 1.2 implies that (after perhaps increasing the constant C) :

$$\begin{aligned} & -\frac{1}{N}\log \mathcal{Z}_N(-\gamma) \\ & \leq \inf_{\mathcal{H}} \mathcal{M}_{-\gamma(1-Ck^{-1})} + Ck^{-1} + k^{-1}\gamma\left(|1-\gamma| + C\right)\log \left\| \frac{dV}{dV_0} \right\|_{L^\infty(X)}. \end{aligned}$$

Indeed, by a simple scaling argument (applied to Lemma 2.4) it it enough to consider the case when $e^{-(\gamma\psi_0 + (1-\gamma)\phi_0)}$ is a probability measure. This implies that the entropy term in the free energy F_γ is non-negative. It then follows readily from the definition that the function

$$T \mapsto \inf T F_{T^{-1}}$$

is increasing in T (where the infimum is taken over a given subset of $\mathcal{P}(X)$). In particular, applied to the present setup at $T_0 = (k+\gamma)/(k+1)$ and $T_1 = 1$ this monotonicity yields (since $T_0 \leq T_1$ when $\gamma \leq 1$)

$$\frac{k+\gamma}{k+1} \inf_{\mathcal{H}} \mathcal{M}_{-\gamma(1-Ck^{-1})(k+1)/(k+\gamma)} \leq \inf_{\mathcal{H}} \mathcal{M}_{-\gamma(1-Ck^{-1})},$$

as desired.

3 Comparison with the quantization approach

Given a holomorphic line bundle L over a compact complex manifold X and a positive integer k denote by $\mathcal{H}_k(L)$ the space of all Hermitian metrics on the N-dimensional complex vector space $H^0(X, kL)$, assuming that $N > 0$. The "Fubini-Study map" FS maps $\mathcal{H}_k(L)$ into the space $\mathcal{H}(L)$ of all metrics on L with positive curvature:

$$FS : \mathcal{H}_k(L) \to \mathcal{H}(L), \quad FS(H) := k^{-1}\log\left(\frac{1}{N}\sum_{i=1}^{N}\left|s_i^H\right|^2\right) \tag{3.1}$$

where (s_i^H) is any basis in $H^0(X, kL)$ which is orthonormal wrt H. The normalization by N used here is non-standard, but it will simplify some of the formulas below. Note also that since we are not assuming that kL is globally generated $FS(H)$ can, in general, be singular.

Henceforth, we shall specialize to the anti-canonical setting in Sect. 2.1.1. Thus X is a Fano manifold and $L = -K_X$. We will abbreviate $\mathcal{H}_k(-kK_X) = \mathcal{H}_k$ and $\mathcal{H}(-K_X) = \mathcal{H}$. Consider now the functional $D_{k,-\gamma}$ on $\mathcal{H}_k$ defined by

$$D_{k,-\gamma}(H_k) := \frac{1}{kN_k} \log \det H_k - \frac{1}{\gamma} \log \int_X e^{-(\gamma FS(H_k)+(1-\gamma)\phi_0)}, \tag{3.2}$$

which is invariant under scaling by positive numbers:

$$D_{k,-\gamma}(e^c H_k) = D_{k,-\gamma}(H_k) \ \forall c \in \mathbb{R}.$$

As is well-known the functional $D_{k,-\gamma}$ on $\mathcal{H}_k$ can be viewed as a quantization of the functional $\mathcal{D}_{-\gamma}$ on $\mathcal{H}$ [10, 35]. The following proposition relates the functional $D_{k,-\gamma}$ on $\mathcal{H}_k$ to the functional $\mathcal{D}_{k,-\gamma}$ on $\mathcal{H}$ defined in formula 2.5.

Proposition 3.1 *For any metric ϕ on $-K_X$*

$$D_{k,-\gamma}\left(H^{(k)}\left(\phi, e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)\right) \leq \left(1+\gamma k^{-1}\right)\mathcal{D}_{k,-\gamma}(\phi)$$

and for any $H \in \mathcal{H}_k$

$$\left(1+\gamma k^{-1}\right)\mathcal{D}_{k,-\gamma}\left(FS(H)\right) \leq D_{k,-\gamma}(H)$$

In particular,

$$\inf_{\mathcal{H}_k} D_{k,-\gamma} = \left(1+\gamma k^{-1}\right) \inf_{\mathcal{H}} \mathcal{D}_{k,-\gamma}$$

Proof To prove the first inequality let ϕ be a given metric on $-K_X$ and set $\psi_k := FS\left(H^{(k)}\left(\phi, e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)\right)$. Then

$$\int_X e^{-(\gamma\psi_k+(1-\gamma)\phi_0)} \geq \left(\int_X e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)^{(1+\gamma/k)}. \tag{3.3}$$

Indeed, rewriting $e^{-(\gamma\psi_k+(1-\gamma)\phi_0)} = e^{-\gamma(\psi_k-\phi)}e^{-(\gamma\phi+(1-\gamma)\phi_0)}$ and using that

$$e^{(\psi_k-\phi)} = \rho_{k\phi}$$

(as follows directly from the definition of $\rho_{k\phi}$ in formula 2.6) gives

$$\int_X e^{-(\gamma\psi_k+(1-\gamma)\phi_0)} = \int_X (\rho_{k\phi})^{-\gamma/k} e^{-(\gamma\phi+(1-\gamma)\phi_0)}$$

$$\geq \left(\int_X (\rho_{k\phi}) e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)^{-\gamma/k} \left(\int_X e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)^{(1+\gamma/k)}$$

using Hölder's inequality with negative exponent $-\gamma/k$ (or Jensen's inequality applied to the convex function $t \mapsto t^{-\gamma/k}$ on $]-\infty,\infty[$). The integral appearing in the first factor in the right hand side above is precisely the integral of the Bergman measure $B_{k\phi}$ (defined in formula 2.6) and thus equal to one, which proves the inequality 3.3. Hence, using $(1+\gamma k^{-1})/(k+\gamma) = 1/k$,

$$\begin{aligned}&(1+\gamma/k)\mathcal{D}_{k,-\gamma}\left(H^{(k)}\left(\phi, e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)\right)\\&\leq \frac{1}{kN}\log\det H^{(k)}\left(\phi, e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right) - \gamma^{-1}\log\left(\int_X e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right),\end{aligned}$$

which proves the first inequality stated in the proposition. To prove the second one first observe that for any H and and volume form μ on X

$$\det\left(H^{(k)}\left(FS(H),\mu\right)\right) \leq \det H \cdot \left(\int_X \mu\right)^N \tag{3.4}$$

Indeed, for any given ϕ in $\mathcal{H}$ and $H \in \mathcal{H}_k$, taking a basis (s_i^H) in $H^0(X,-kK_X)$ which is orthonormal wrt H, we can factorize

$$\det\left(H^{(k)}(\phi,\mu)\right) = \det H \cdot \det\left(H^{(k)}(\phi,\mu)(s_i^H, s_j^H)\right), \tag{3.5}$$

where the second factor arises as the determinant of the change of bases matrix between the reference basis $(s_i^{(k)})$ in $H^0(X,-kK_X)$ and (s_i^H). Next, by the arithmetic/geometric means inequality

$$\left(\det\left(H^{(k)}(\phi,\mu)(s_i^H, s_j^H)\right)\right)^{1/N} \leq N^{-1}\sum_{i=1}^N H^{(k)}(\phi,\mu)(s_i^H, s_i^H).$$

Now assume that $\phi = FS(H)$. Then

$$H^{(k)}(\phi,\mu)(s_i^H, s_i^H) := \int_X \frac{|s_i^H|^2}{N^{-1}\sum_{j=1}^N |s_i^H|^2}\mu.$$

Hence, the second factor in the right hand side in formula 3.5 is bounded from above by the Nth power of the integral of μ, proving the inequality 3.4. Thus, if H is a given element in $\mathcal{H}_k$ which is normalized so that $\int e^{-(\gamma FS(H)+(1-\gamma)\phi_0)} = 1$, then applying the inequality 3.4 to $\mu = e^{-(\gamma FS(H)+(1-\gamma)\phi_0)}$ proves the second inequality for any normalized H in $\mathcal{H}_k$. Finally, since both sides of the inequality in question are invariant under scaling, $H \to e^c H$, this concludes the proof for a general H in $\mathcal{H}_k$.

Remark 3.2 The previous proposition refines a monotonicity result [3, Lemma 2.6], concerning Donaldson's iteration in the anti-canonical setting of [28]. Indeed, applying the first inequality to $\phi = FS(H)$ and then the second inequality reveals that $D_{k,-\gamma}(H)$ is decreasing under Donaldson's map on $\mathcal{H}_k$, defined as the composition of the maps $F \mapsto FS(H)$ and $\phi \mapsto H^{(k)}\left(\phi, e^{-(\gamma\phi+(1-\gamma)\phi_0)}\right)$. As in [3] one gets equality in the first equality in the proposition when $\phi = FS(H)$ iff H is a balanced metric in $\mathcal{H}_k$ in the anti-canonical sense of [10, 28, 35].

Combining Proposition 2.3 and the equality for the infima in Proposition 3.1 (only the upper bound is needed) we thus arrive at the following result:

Theorem 3.3 *Let X be a compact complex manifold and assume that k is a positive integer such that $N_k := \dim H^0(X, -kK_X) > 0$. Given (ϕ_0, γ) the following inequality holds:*

$$-\frac{1}{\gamma N}\log \mathcal{Z}_{N_k}(-\gamma) \le \inf_{\mathcal{H}_k} D_{k,-\gamma} + \frac{1}{kN}\log N_k.$$

As a consequence, if $\mathcal{Z}_{N_k}(-\gamma)$ is finite, then the infimum of $D_{k,-\gamma}$ over $\mathcal{H}_k$ is finite. In other words, the invariant $\gamma_k(X)$ defined by formula 1.8 is smaller than or equal to the coercivity threshold of the functional D_k on $\mathcal{H}_k$ which, by [35], coincides with the invariant $\delta_k(X)$ introduced in [31] (appearing in formula 1.10). We thus arrive at a new proof of the following inequality first shown in [32] (see 1.10 for a reformulation of the proof in terms of non-Archimedean pluripotential theory).

Corollary 3.4 [31, Thm 2.5]. *For a Fano manifold X the following inequality holds:*

$$\gamma_k(X) \le \delta_k(X)$$

Combining the equality for the infima in Prop 3.1 with the argument employed in Sect. 2.3 also yields the following analog of Theorem 1.2:

Theorem 3.5 *There exists a constant $C > 0$ (depending only on the reference volume for dV_0) such that for any $\gamma > 0$ and positive integer k*

$$\begin{aligned}\inf_{\mathcal{H}_k} D_{k,-\gamma} &\le -\frac{1}{N}\log \mathcal{Z}_N(-\gamma)\\ &\le \frac{k+\gamma}{k+1}\inf_{\mathcal{H}} \mathcal{M}_{-\gamma c_k} + k^{-1}\gamma\left(C + (|1-\gamma| + C)\log\left\|\frac{dV}{dV_0}\right\|_{L^\infty(X)}\right),\end{aligned}$$

where $c_k := (1 - Ck^{-1})(k+1)/(k+\gamma)$.

As explained in Sect. 1.3 this inequality is closely related to results in [45].

Acknowledgements Thanks to Bo Berndtsson for comments on a first draft of this manuscript and the referee for helpful remarks. This work was supported by grants from the Knut and Alice Wallenberg foundation, the Göran Gustafsson foundation and the Swedish Research Council.

Funding Open access funding provided by Chalmers University of Technology.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest There is no conflict of interest.

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