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Topological Manin pairs and (n, s)-type series

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Abstract

Lie subalgebras of $L = \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n \mathfrak{g}[x]$, complementary to the diagonal embedding Δ of $\mathfrak{g}[[x]]$ and Lagrangian with respect to some particular form, are in bijection with formal classical *r*-matrices and topological Lie bialgebra structures on the Lie algebra of formal power series $\mathfrak{g}[[x]]$. In this work we consider arbitrary subspaces of *L* complementary to Δ and associate them with so-called series of type (n, s). We prove that Lagrangian subspaces are in bijection with skew-symmetric (n, s)-type series and topological quasi-Lie bialgebra structures on $\mathfrak{g}[[x]]$. Using the classification of Manin pairs we classify up to twisting and coordinate transformations all quasi-Lie bialgebra structures. Series of type (n, s), solving the generalized classical Yang-Baxter equation, correspond to subalgebras of *L*. We discuss their possible utility in the theory of integrable systems.

Keywords Lie bialgebras \cdot quasi-Lie bialgebras \cdot Manin pairs \cdot Yang-Baxter equations \cdot r-matrices \cdot Lie algebra splittings

Mathematics Subject Classification 17B62 · 17B38 (Primary) · 17B80 (Secondary)

Dedicated to the memory of Yuri Manin.

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1 Introduction

Let *F* be an algebraically closed field of characteristic 0 equipped with the discrete topology and \mathfrak{g} be a simple Lie algebra over *F*. We define the Lie algebra $\mathfrak{g}[[x]]$ to be the space $\mathfrak{g} \otimes F[[x]]$ with the bracket

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg$$

and we equip it with the (x)-adic topology. The continuous dual of $\mathfrak{g}[[x]]$ is denoted by $\mathfrak{g}[[x]]'$ and it is endowed with the discrete topology.

A topological Manin pair is a pair $(L, \mathfrak{g}[[x]])$ where

- 1. *L* is a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form *B*;
- 2. $\mathfrak{g}[[x]] \subset L$ is a Lagrangian subalgebra with respect to *B*;
- 3. for any continuous functional $T: \mathfrak{g}[[x]] \to F$ there is $f \in L$ such that T = B(f, -).

Topological Manin pairs were classified in [1] using the tools from [8]. More precisely, if $(L, \mathfrak{g}[x])$ is a topological Manin pair, then *L* is isomorphic, as a Lie algebra with form, to either $L(\infty)$ or $L(n, \alpha)$. In this paper we consider only the "non-degenerate" case, namely $L \cong L(n, \alpha)$.

As a Lie algebra

$$L(n,\alpha) = \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n \mathfrak{g}[x].$$

The bilinear form *B* on $L(n, \alpha)$ is completely determined by the sequence $\alpha = (\alpha_i \in F \mid -\infty < i \le n-2)$. For example, when n = 0 the form is given by

$$B(a \otimes f, b \otimes g) = \kappa(a, b) \operatorname{res}_0 \{ \alpha(x) fg \},\$$

where κ is the Killing form on \mathfrak{g} and $\alpha(x) := 1 + \alpha_{-2}x + \alpha_{-3}x^2 + \cdots \in F((x))$. In case n > 0 the form is given by a similar formula; see Sect. 2.

It was established in [1], that the following objects are in one-to-one correspondence

• Lagrangian subalgebras $W \subseteq L(n, 0), 0 \le n \le 2$, complementary to the diagonal

$$\Delta := \{ (f, [f]) \mid f \in \mathfrak{g}[[x]] \},\$$

i.e. $\Delta + W = L(n, 0);$

- non-degenerate topological Lie bialgebra structures on $\mathfrak{g}[[x]]$ and
- formal solutions to the classical Yang-Baxter equation (CYBE) in the form

$$\frac{y^n\Omega}{x-y} + g(x,y) = \Omega \sum_{k\geq 0} x^{-k-1} y^{k+n} + g(x,y) \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x))[\![y]\!], \quad (1)$$

where $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the quadratic Casimir element and $g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$.

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Furthermore, the proof of the above-mentioned correspondence reveals that series Eq. (1) can be viewed as a generating series for the corresponding subalgebra W. The present paper can be thus considered as a continuation of [1], where we extend the preceding correspondence using series of type (n, s).

To define a series of type (n, s) fix a basis $\{b_i\}_{i=1}^d$ of \mathfrak{g} , orthonormal with respect to its Killing form κ , and interpret $y^n \Omega/(x-y)$ as a series

$$\frac{y^n \Omega}{x - y} = \sum_{k=0}^{\infty} \sum_{i=1}^d w_{k,i} \otimes b_i y^k \in \left((\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n \mathfrak{g}[x]) \otimes \mathfrak{g} \right) \llbracket y \rrbracket.$$
(2)

This expression might be understood as a Taylor series expansion. Elements $w_{k,i} \in \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n \mathfrak{g}[x]$ are presented explicitly in Eq. (20). A series of type (n, s) is a series of the form

$$\frac{s(x)y^{n}\Omega}{x-y} + g(x,y) \in \left((\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^{n}\mathfrak{g}[x]) \otimes \mathfrak{g} \right) [\![y]\!], \tag{3}$$

where $s \in F[[x]]^{\times}$ and $g \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$; See Definition 3.2. For each series *r* of type (n, s) we define another series \overline{r} of the same type as follows

$$\overline{r} := \frac{s(y)x^n\Omega}{x-y} - \tau(g(y,x)),\tag{4}$$

where τ is the F[[x, y]]-linear extension of the map $a \otimes b \mapsto b \otimes a$.

Each series of type (n, s) produces a subspace of $\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n \mathfrak{g}[x]$ complementary to the diagonal embedding Δ of $\mathfrak{g}[[x]]$. The following results generalize the above-mentioned correspondence from [1].

Theorem A Let $n \in \mathbb{Z}_{\geq 0}$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ be an arbitrary sequence with the corresponding series $\alpha(x) := x^{-n} + \alpha_{n-2}x^{-n+1} + \cdots + \alpha_0x^{-1} + \cdots \in F((x))$. For any (n, s)-type series

$$r = \sum_{k=0}^{\infty} \sum_{i=1}^{d} f_{k,i} \otimes b_i y^k \in \left((\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n \mathfrak{g}[x]) \otimes \mathfrak{g} \right) \llbracket y \rrbracket$$
(5)

define the space

$$W(r) := \operatorname{span}_F\{f_{k,i} \mid k \ge 0, \ 1 \le i \le d\} \subseteq \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^n \mathfrak{g}[x].$$
(6)

The following results are true:

- 1. W defines a bijection between series of type $\left(n, \frac{1}{x^n \alpha(x)}\right)$ and subspaces $V \subset L(n, \alpha)$ complementary to the diagonal Δ , i.e. $L(n, \alpha) = \Delta \dotplus V$;
- 2. For any series r of type $\left(n, \frac{1}{x^n \alpha(x)}\right)$ we have $W(r)^{\perp} = W(\overline{r})$ inside $L(n, \alpha)$;

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Fig. 1 Series-subspaces correspondence

3. Any series r of type $\left(n, \frac{1}{x^n \alpha(x)}\right)$ satisfies $\text{GCYB}(r) = \psi$ (see Definition 3.5 for the meaning of GCYB(r)), where $\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x_1, x_2, x_3]]$ is defined by

$$B(v_1 \otimes v_2 \otimes v_3, \psi) = B(v_1, [v_2, v_3])$$

for all $v_1 \in W(\overline{r}), v_2, v_3 \in W(r)$.

In particular, considering the case when r is skew-symmetric, meaning $r = \overline{r}$, or when $\psi = 0$ we get the following correspondences.

Corollary B Let $n \in \mathbb{Z}_{\geq 0}$, $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ and W be the map from *Theorem A. Then*

- 1. W defines a bijection between skew-symmetric $\left(n, \frac{1}{x^n \alpha(x)}\right)$ -type series and Lagrangian subspaces $V \subseteq L(n, \alpha)$, complementary to the diagonal Δ ;
- 2. W defines a bijection between $\left(n, \frac{1}{x^n \alpha(x)}\right)$ -type series solving the GCYBE and subalgebras $V \subseteq L(n, \alpha)$ complementary to the diagonal Δ .

Observe that an (n, s)-type series produces a subspace of $L(n, \alpha)$ for any sequence α . However, to obtain the compatibility with the form, given by α , we need the equality $s(x) = 1/(x^n \alpha(x))$. In this case, the components $f_{k,i}$ and $b_i y^k$ of the series become dual bases for W(r) and Δ respectively.

The requirement on a series *r* of type (n, s) to solve the CYBE is equivalent to being skew-symmetric and to solve the GCYBE. Together with Corollary B this implies that Lagrangian subalgebras $W \subset L(n, \alpha)$, satisfying $W \dotplus \Delta = L(n, \alpha)$, are in bijection with $(n, 1/(x^n\alpha(x)))$ -type series solving the classical Yang-Baxter equation. These correspondences are schematically depicted in Fig. 1.

Remark 1.1 Let *r* be a series of type (n, s). Applying the projection $(a, b) \otimes c \mapsto a \otimes c$ onto the left component to *r* we obtain the series

$$r_{\text{proj}} = \frac{s(x)y^n \Omega}{x - y} + g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]].$$
(7)

Conversely, starting with a series r_{proj} of the form Eq. (7), we can obtain an (n, s)-type series r by taking two Taylor series expansions of r_{proj} at x = 0 and y = 0 respectively and then constructing r by combining the coefficients of $b_i y^k$, $k \ge 0$, in these expansions. These two constructions are inverse to each other and hence both r and its projection r_{proj} contain exactly the same information. Consequently, all the statements made for (n, s)-type series can be stated in terms of their projections onto the left component and vice versa. In cotrast with [1], in this paper we give preference to series of type (n, s) rather than to their projections, because the statement that series of type (n, s) generate subspaces of $L(n, \alpha)$ becomes transparent.

Reinterpreting the results of [1] in terms of (n, s)-type series we see that skewsymmetric series of type $(n, 1/(x^n \alpha(x)))$, that also solve the GCYBE, exist only for n = 0, 1 and n = 2 with $\alpha_0 = 0$.

Lagrangian subalgebras of $L(n, \alpha)$, complementary to Δ , correspond to topological Lie bialgebra structures on $\mathfrak{g}[[x]]$. If we instead consider Lagrangian subspaces (not necessarily subalgebras) of $L(n, \alpha)$, we get so called (non-degenerate) topological quasi-Lie bialgebra structures on $\mathfrak{g}[[x]]$. A topological quasi-Lie bialgebra structure on $\mathfrak{g}[[x]]$ consists of

- a skew-symmetric continuous linear map $\delta : \mathfrak{g}[[x]] \to (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ and
- a skew-symmetric element $\varphi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x, y, z]],$

which are subject to the following three conditions

- $\delta([a, b]) = [a \otimes 1 + 1 \otimes a, \delta(b)] [b \otimes 1 + 1 \otimes b, \delta(a)]$, i.e. δ is a 1-cocycle;
- $\frac{1}{2}$ Alt $((\delta \otimes 1)\delta(a)) = [a \otimes 1 \otimes 1 + 1 \otimes a \otimes 1 + 1 \otimes 1 \otimes a, \varphi];$
- Alt $((\delta \otimes 1 \otimes 1)\varphi) = 0$,

where Alt $(x_1 \otimes \ldots \otimes x_n) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$.

Following [5] we prove the following direct relation between δ , φ and skew-symmetric (n, s)-type series r.

Proposition C There is a bijection between topological quasi-Lie bialgebras and skew-symmetric (n, s)-type series. Let r be the (n, s)-type series corresponding to $(\mathfrak{g}[\![x]\!], \delta, \varphi)$, then, under the identification $\mathfrak{g}[\![x]\!] \cong \Delta$, we have the following identities:

• $[a \otimes 1 + 1 \otimes a, r] = -\delta(a)$ for any $a \in \mathfrak{g}[[x]]$ and

•
$$\operatorname{CYB}(r) = -\varphi$$
.

The same is true if r is interpreted as an element in $(\mathfrak{g} \otimes \mathfrak{g})((x))[[y]]$.

In view of this result we call skew-symmetric (n, s)-type series quasi-r-matrices.

Repeating the ideas from [7] and [5] we show that topological quasi-Lie bialgebras can be twisted similar to topological Lie bialgebras. More precisely, if δ is a quasi-Lie bialgebra structure on $\mathfrak{g}[x]$, given by the Lagrangian subspace W, and s :=

 $\sum_{i} a_i \otimes b^i \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ is an arbitrary skew-symmetric tensor, then

$$W_s := \left\{ \sum_i B(b^i, w) a_i - w \mid w \in W \right\}$$
(8)

is another (twisted) Lagrangian subspace complementary to the diagonal. This observation implies, that in order to classify all topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ up to twisting it is enough to find one single Lagrangian subspace within each $L(n, \alpha)$. Moreover, it was shown in [1] that substitutions of the form $x \mapsto x + a_2x^2 + a_3x^3 + \ldots, a_i \in F$, allow us to assume that α has the form

$$\alpha = (\ldots, 0, \alpha_0, 0, \ldots, 0).$$

Lagrangian subspaces for such $L(n, \alpha)$ are constructed in Sect. 4.1.

Using Theorem A and Proposition C we explain how twisting of a Lagrangian subspace $W \subset L(n, \alpha)$ is seen at the level of δ and the corresponding quasi-*r*-matrix *r*.

Corollary D Let $(\mathfrak{g}[[x]], \delta, \varphi)$ be a topological quasi-Lie bialgebra structure corresponding to the quasi-r-matrix r. If we twist W(r) with a skew-symmetric tensor s we obtain another topological quasi-Lie bialgebra $(\mathfrak{g}[[x]], \delta_s, \varphi_s)$, such that

1. $W(r)_s = W(r - s);$ 2. $\delta_s = \delta + ds;$ 3. $\varphi_s = \varphi + \text{CYB}(s) - \frac{1}{2}\text{Alt}((\delta \otimes 1)s).$

Therefore, to describe all quasi-*r*-matrices up to twisting it is enough to find one single quasi-*r*-matrix for each $L(n, \alpha)$. We achieve that goal in Sect. 4.2 by writing out explicitly series of type (n, s) for subspaces from Sect. 4.1.

The results above, in particular, show that if r is a quasi-r-matrix and $\delta(a) := [a \otimes 1 + 1 \otimes a, r]$, then the condition

$$Alt((\delta \otimes 1 \otimes 1)CYB(r)) = 0$$
⁽⁹⁾

is trivially satisfied.

We conclude the paper by using Theorem A for construction of Lie algebra splittings $\Delta + W = L(n, \alpha)$ and the corresponding (n, s)-type series, which we call generalized *r*-matrices. These constructions are important in the theory of integrable systems because of their use in the Adler-Konstant-Symes (AKS) scheme and the so-called *r*-matrix approach; see [4, 6]. The subalgebra splittings of L(0, 0) as well as their physical applications were considered in e.g. [9, 10].

Our first result shows that in order to obtain new generalized *r*-matrices from subalgebra splittings $L(n, \alpha) = \Delta + W$ with n > 2, the subalgebra W must be unbounded. Otherwise the situation can be reduced to the splitting of $L(2, \alpha)$.

Proposition E Let $L(n, \alpha) = \Delta + W$ for some subalgebra $W \subset L(n, \alpha)$ and n > 2. Assume W is bounded, i.e. there is an integer N > 0 such that

$$x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N\mathfrak{g}[x^{-1}],$$

where W_+ is the projection of $W \subset L(n, \alpha) = \mathfrak{g}((x)) \oplus \mathfrak{g}[x]/x^n \mathfrak{g}[x]$ on the first component $\mathfrak{g}((x))$. Then we have the inclusion

$$\{0\} \times [x^2]\mathfrak{g}[x]/x^n\mathfrak{g}[x] \subseteq W$$

and the image \widetilde{W} under the canonical projection $L(n, \alpha) \to L(2, \alpha)$ is a subalgebra satisfying $L(2, \alpha) = \Delta + \widetilde{W}$.

Despite this result we think that bounded subalgebras $W \subset L(n, \alpha)$ complementary to Δ are still interesting, because in the case $\alpha \neq 0$ they lead to unbounded orthogonal complements W^{\perp} which are also important in view of the AKS scheme. We give examples of subalgebras of $L(n, \alpha)$ with unbounded orthogonal complements.

2 Topological Manin pairs

Let *F* be an algebraically closed field of characteristic 0, \mathfrak{g} be a finite-dimensional simple *F*-Lie algebra and $\mathfrak{g}[[x]] := \mathfrak{g} \otimes F[[x]]$ be the Lie algebra with the bracket defined by

$$[a \otimes f, b \otimes g] := [a, b] \otimes fg,$$

for all $a, b \in \mathfrak{g}$ and $f, g \in F[[x]]$. From now on, we always endow F with the discrete topology and view $\mathfrak{g}[[x]]$ as a topological Lie algebra with the (x)-adic topology.

A *topological Manin pair* is a pair $(L, \mathfrak{g}[x])$, where L is a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form B, such that

- 1. $\mathfrak{g}[[x]] \subseteq L$ is a Lagrangian Lie subalgebra with respect to B;
- 2. for any continuous functional $T: \mathfrak{g}[[x]] \to F$ there exists an element $f \in L$ such that T = B(f, -).

The statements of [8, Proposition 2.9] and [1, Proposition 3.12] give a description of all topological Manin pairs. For precise formulation we need to repeat the definitions of some specific Lie algebras with forms from [1, Section 3.2] and [8, Section 2].

Definition 2.1 We define the Lie algebra $L(\infty) := \mathfrak{g} \otimes A(\infty)$, where $A(\infty)$ is the unital commutative algebra with underlying space $\sum_{i\geq 0} Fa_i + F[[x]]$ and multiplication given by

$$a_i a_j := 0, a_i x^j := a_{i-j}$$
 for $i \ge j$ and $a_i x^j := 0$ otherwise

Let t: $A \to F$ be the functional, given by $t(a_0) := 1, t(a_i) := 0, i \ge 1$ and t(F[[x]]) := 0. We equip $L(\infty)$ with the symmetric non-degenerate invariant bilinear form

$$B\left(a\otimes\left(\sum_{i\geq 0}c_ia_i,\,f(x)\right),\,b\otimes\left(\sum_{i\geq 0}t_ia_i,\,g(x)\right)\right)$$

$$:= \kappa(a,b) \operatorname{t}\left(g(x)\sum_{i\geq 0}c_ia_i + f(x)\sum_{i\geq 0}t_ia_i\right).$$
(10)

Definition 2.2 Let $n \ge 1$ and $\alpha = (\alpha_i \in F \mid -\infty < i \le n-2)$ be an arbitrary sequence. Consider the algebra

$$A(n,\alpha) := F((x)) \oplus F[x]/(x^n).$$

Abusing the notation we denote the element $x^{-n} + \alpha_{n-2}x^{-n+1} + \dots + \alpha_0x^{-1} + \dots \in F((x))$ with the same letter α . Define the functional t: $A(n, \alpha) \to F$ by

$$t(f, [p]) := res_0 \{ \alpha(f - p) \}$$

Taking the tensor product of $A(n, \alpha)$ with \mathfrak{g} we get the Lie algebra

$$L(n,\alpha) := \mathfrak{g} \otimes A(n,\alpha) \tag{11}$$

which we equip with the form

$$B(a \otimes (f, [p]), b \otimes (g, [q])) := \kappa(a, b) \operatorname{t}(fg, [pq]).$$

$$(12)$$

It is known that the bilinear form B is symmetric non-degenerate and invariant.

Definition 2.3 Take an arbitrary sequence $\alpha = (\alpha_i \in F \mid -\infty < i \leq -2)$ and let $A(0, \alpha) := F((x))$. We define the functional t: $A(0, \alpha) \rightarrow F$ by

$$\mathsf{t}(f) := \operatorname{res}_0 \left\{ \alpha f \right\},\,$$

where $\alpha = 1 + \alpha_{-2}x + \cdots \in F((x))$. We equip the Lie algebra $L(0, \alpha) := \mathfrak{g} \otimes A(0, \alpha)$ with the bilinear form

$$B(a \otimes f, b \otimes g) := \kappa(a, b) \operatorname{t}(fg), \tag{13}$$

which is again symmetric non-degenerate and invariant. From now on we identify F((x)) with $F((x)) \times \{0\}$ and write (f, 0) for elements in $A(0, \alpha)$.

Definition 2.4 A series of the form $\varphi = x + a_2x^2 + a_3x^3 + \cdots \in F[[x]]$ is called a *coordinate transformation*. Coordinate transformations form a group $\operatorname{Aut}_0 F[[x]]$ under substitution which we view as a subgroup of automorphisms of F[[x]].

An element $\varphi \in \operatorname{Aut}_0 F[[x]]$ induces an automorphism of $A(n, \alpha)$ by $f/g \mapsto \varphi(f)/\varphi(g)$ and $[p] \mapsto [\varphi(p)]$ that changes the functional t to $t \circ \varphi$. We write $A(n, \alpha)^{(\varphi)}$ for the algebra $A(n, \alpha)$ with the functional $t \circ \varphi$. It is not hard to see that for any $\varphi \in \operatorname{Aut}_0 F[[x]]$ there is a sequence β such that $A(n, \alpha)^{(\varphi)} = A(n, \beta)$.

$$\Delta := \{ (f, [f]) \mid f \in \mathfrak{g}[[x]] \} \subset L(n, \alpha).$$

Moreover, we can assume that all the elements α_i in the sequence α , except maybe α_0 , are 0 by virtue of the following result.

Proposition 2.5 [1, Proposition 3.12] Let $n \ge 0$ and $\alpha = (\alpha_i \in F \mid -\infty < i \le n-2)$ be a sequence. There exists $a \varphi \in \operatorname{Aut}_0 F[[x]]$ such that $A(n, \alpha) \cong A(n, \beta)^{(\varphi)}$, where β is the sequence satisfying $\beta_i = 0$ for all $i \ne 0$ and $\beta_0 = \alpha_0$.

Remark 2.6 Observe that the result of Proposition 2.5 can be interpreted in terms of a formal differential equation. Consider an arbitrary $\alpha(x) = x^{-n} + \alpha_{n-2}x^{-n+1} + \cdots + \alpha_0 x^{-1} + \cdots \in F((x))$ and $\beta(x) = x^{-n} + \alpha_0 x^{-1}$. Then the functionals t_{α} and t_{β} defined on $A(n, \alpha)$ and $A(n, \beta)$ respectively are given by

$$t_{\alpha}(f, [p]) = \operatorname{res}_0\{\alpha(f-p)\} \text{ and } t_{\beta}(f, [p]) = \operatorname{res}_0\{\beta(f-p)\})$$

The equality $A(n, \alpha)^{(\varphi)} = A(n, \beta)$ can be expressed as

$$\operatorname{res}_0\{\beta(x)f(x)\} = \operatorname{res}_0\{\alpha(x)f(\varphi(x))\} = \operatorname{res}_0\{\alpha(\psi(x))f(x)\psi'(x)\}, \quad (14)$$

where $\psi \in \text{Aut}_0(F[[x]])$ is the compositional inverse of φ , i.e. $\varphi(\psi(x)) = x$. Since the residue pairing is non-degenerate on F((x)), we obtain

$$\alpha(\psi(x))\psi'(x) = \beta(x). \tag{15}$$

In particular, the transformation φ is the compositional inverse of the solution to Eq. (15).

3 Series of type (n, s) and subspaces of $L(n, \alpha)$

Let $\{b_i\}_{i=1}^d$ be an othonormal basis of \mathfrak{g} with respect to the Killing form κ . We write Ω for the quadratic Casimir element $\sum_{i=1}^d b_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$. It satisfies the identity $[a \otimes 1 + 1 \otimes a, \Omega] = 0$ for all $a \in \mathfrak{g}$.

In this section we describe a bijection between subspaces $W \subset L(n, \alpha)$ complementary to Δ and certain series. The following definition introduces convenient spaces containing these series.

Definition 3.1 We put $A_1(n, \alpha) := A(n, \alpha) = F((x_1)) \oplus F(x_1)/(x_1^n)$ and then define inductively the algebras

$$A_m(n,\alpha) := A_{m-1}(n,\alpha)((x_m)) \oplus A_{m-1}(n,\alpha)[x_m]/x_m^n A_{m-1}(n,\alpha), \ m > 1. \ (16)$$

The functional t defined on $A(n, \alpha)$ extends inductively to a functional on $A_m(n, \alpha)$. More precisely,

$$t\left(\sum_{k\geq -N} f_k x_m^k, \sum_{\ell=0}^{n-1} [g_\ell x_m^\ell]\right) := \sum_{k\geq -N} t(f_k) t(x_m^k, 0) + \sum_{\ell=0}^{n-1} t(g_\ell) t(0, [x_m]^\ell),$$
(17)

where $f_k, g_\ell \in A_{m-1}(n, \alpha)$. Since $t(x^n F[[x]]) = 0$, the sum on the right-hand side of Eq. (17) is finite and well-defined. This allows us to extend the form *B* on $L(n, \alpha)$ to a symmetric non-degenerate bilinear form on the g-module

$$L_m(n,\alpha) := \mathfrak{g}^{\otimes m} \otimes A_m(n,\alpha) \tag{18}$$

by letting

$$B((a_1 \otimes \ldots \otimes a_m) \otimes f, (b_1 \otimes \ldots \otimes b_m) \otimes g) := \mathsf{t}(fg) \prod_{k=1}^m \kappa(a_k, b_k), \quad (19)$$

for all $a_1, \ldots, a_m, b_1, \ldots, b_m \in \mathfrak{g}$ and $f, g \in A_m(n, \alpha)$.

Fix some integer $n \ge 0$. We interpret the quotient $y^n \Omega/(x - y)$ in the following way

$$\frac{y^{n}\Omega}{x-y} = \sum_{k=0}^{n-1} \sum_{i=1}^{d} b_{i}(0, -[x]^{(n-1)-k}) \otimes b_{i}(y^{k}, [y]^{k}) + \sum_{k=n}^{\infty} \sum_{i=1}^{d} b_{i}(x^{(n-1)-k}, 0) \otimes b_{i}(y^{k}, 0) = \sum_{k=0}^{\infty} \sum_{i=1}^{d} w_{k,i} \otimes b_{i}(y^{k}, [y]^{k}) \in (L(n, \alpha) \otimes \mathfrak{g}) [[(y, [y])]] \subset L_{2}(n, \alpha),$$
(20)

where α is an arbitrary sequence and we write $b_i(x^{\ell}, [x]^m)$ meaning $b_i \otimes (x^{\ell}, [x]^m)$. **Definition 3.2** Since $(L(n, \alpha) \otimes \mathfrak{g})$ $\llbracket (y, [y]) \rrbracket$ is an $F\llbracket x \rrbracket \cong F\llbracket (x, [x]) \rrbracket$ -module and

$$(\mathfrak{g} \otimes \mathfrak{g})\llbracket x, y \rrbracket \cong (\Delta \otimes \mathfrak{g})\llbracket (y, [y]) \rrbracket \subset (L(n, \alpha) \otimes \mathfrak{g})\llbracket (y, [y]) \rrbracket$$

the series

$$r(x, y) = \frac{s(x)y^n\Omega}{x - y} + g(x, y), \qquad (21)$$

where $g \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ and $s \in F[[x]]^{\times}$, is also inside $(L(n, \alpha) \otimes \mathfrak{g})[[(y, [y])]]$. Series of the form Eq. (21) are called *series of type* (n, s). Remark 3.3 Every series

$$r(x, y) = \frac{h(x, y)\Omega}{x - y} + g(x, y) \in L_2(n, \alpha),$$

where $h \in F[[x, y]]$, $h(x, x) \neq 0$ and $g \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$, has a unique representation as a series of type (n, s). Indeed, write $h(x, x) = x^n s(x)$ for some $s \in F[[x]]^{\times}$. Then $h(x, y) - y^n s(x) = (x - y) f(x, y)$ for some $f \in F[[x, y]]$. This implies that we can rewrite r in the (n, s) form

$$r(x, y) = \frac{s(x)y^{n}\Omega}{x - y} + f(x, y)\Omega + g(x, y).$$
 (22)

In the construction of f we are using the fact that for any F-vector space V and any element $h \in V[[x, y]]$

$$h(z, z) = 0 \implies h(x, y) = (x - y)f(x, y)$$
(23)

for some $f \in V[[x, y]]$.

Definition 3.4 For each series *r* of type (n, s) we define another series \overline{r} of the same type (n, s) by

$$\overline{r}(x, y) := \frac{s(y)x^n\Omega}{x - y} - \tau(g(y, x)) \in (L(n, \alpha) \otimes \mathfrak{g}) \llbracket (y, [y]) \rrbracket,$$
(24)

where τ is the F[[x, y]]-linear extension of the map $a \otimes b \mapsto b \otimes a$. To see that this is an (n, s)-type series its enough to apply the argument from Remark 3.3. Series of type (n, s), satisfying $r = \overline{r}$, are called *skew-symmetric*.

Definition 3.5 *The generalized classical Yang-Baxter equation (GCYBE)* is the equation for an (n, s)-type series of the form

$$GCYB(r) := [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] + [r^{13}(x_1, x_3), \overline{r}^{23}(x_2, x_3)] = 0.$$
(25)

Here $(-)^{13}$: $L_2(n, \alpha) \to (U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})) \otimes A_3(n, \alpha)$ is the inclusion map given by

$$a \otimes b \otimes \left(\sum_{k \geq -N} F(x_1, [x_1]) x_2^k, \sum_{m=0}^{n-1} G(x_1, [x_1]) [x_2]^m \right)$$

$$\mapsto a \otimes 1 \otimes b \otimes \left(\sum_{k \geq -N} F(x_1, [x_1]) x_3^k, \sum_{m=0}^{n-1} G(x_1, [x_1]) [x_3]^m \right).$$

Other inclusions are defined in a similar manner. The commutators are then taken in the associative $A_3(n, \alpha)$ - algebra $(U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})) \otimes A_3(n, \alpha)$.

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Before formulating the main theorem of the section we note that if $\alpha = (\alpha_i \in F \mid -\infty < i \le n-2)$ is an arbitrary sequence and $\alpha(x) = x^{-n} + \alpha_{n-2}x^{-n+1} + \cdots + \alpha_0 x^{-1} + \cdots \in F((x))$ is the corresponding series, then $x^n \alpha(x) \in F[[x]]^{\times}$.

Theorem 3.6 Let $n \in \mathbb{Z}_{\geq 0}$ and $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ be an arbitrary sequence with the corresponding series $\alpha(x) \in F((x))$. Consider the map

$$W: L_2(n, \alpha) \longrightarrow \{V \subset L(n, \alpha) \mid V \text{ is a subspace}\}$$

given by

$$\sum_{i,j} b_i \otimes b_j \otimes \left(\sum_{k \ge -N_i} (f_k^{ij}, [p_k^{ij}]) x^k, \sum_{m=0}^{n-1} (g_m^{ij}, [q_m^{ij}]) [x]^m \right)$$

$$\mapsto \operatorname{span}_F \left\{ b_i (f_k^{ij}, [p_k^{ij}]) \mid k \ge -N, 1 \le i, j \le d \right\}.$$

The following results are true:

- 1. W defines a bijection between series of type $\left(n, \frac{1}{x^n \alpha(x)}\right)$ and subspaces $V \subseteq L(n, \alpha)$ complementary to the diagonal Δ , i.e. $L(n, \alpha) = \Delta \dotplus V$;
- 2. For any series r of type $\left(n, \frac{1}{x^n \alpha(x)}\right)$ we have $W(r)^{\perp} = W(\overline{r})$ inside $L(n, \alpha)$;
- 3. Any series r of type $\left(n, \frac{1}{x^n \alpha(x)}\right)$ satisfies $\text{GCYB}(r) = \psi$, where

$$\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})\llbracket (x_1, [x_1]), (x_2, [x_2]), (x_3, [x_3]) \rrbracket$$

is defined by

$$B(v_1 \otimes v_2 \otimes v_3, \psi) = B(v_1, [v_2, v_3])$$

for all $v_1 \in W(\overline{r}), v_2, v_3 \in W(r)$.

Proof Fix an $\left(n, \frac{1}{x^n \alpha(x)}\right)$ -type series inside $(L(n, \alpha) \otimes \mathfrak{g}) \llbracket (y, [y]) \rrbracket$

$$r(x, y) = \frac{1}{x^{n} \alpha(x)} \frac{y^{n} \Omega}{x - y} + g(x, y)$$

= $\sum_{k=0}^{\infty} \sum_{i=1}^{d} s_{k,i} \otimes b_{i}(y^{k}, [y]^{k}) + \sum_{k=0}^{\infty} \sum_{i=1}^{d} g_{k,i} \otimes b_{i}(y^{k}, [y]^{k}).$

It is easy to see that

$$U := \operatorname{span}_F\{w_{k,i} \mid k \ge 0, 1 \le k \le d\} \subset L(n, \alpha),$$

where $w_{k,i}$ are defined in Eq. (20), satisfies the condition $\Delta + U = L(n, \alpha)$. Since $s := \frac{1}{x^n \alpha(x)}$ is invertible, we have $sU + s\Delta = sU + \Delta = L(n, \alpha)$. In other words, the space

$$sU = \operatorname{span}_{F}\{s_{k,i} = sw_{k,i} \mid k \ge 0, 1 \le k \le d\} \subset L(n,\alpha)$$
(26)

is also complementary to the diagonal. Finally, since $g_{k,i} \in \Delta$ the space

$$W(r) = \operatorname{span}_F \{ sw_{k,i} + g_{k,i} \mid k \ge 0, 1 \le k \le d \} \subset L(n,\alpha)$$

is complementary to the diagonal. Conversely, if $V \subset L(n, \alpha)$ satisfies $V \dotplus \Delta = L(n, \alpha)$, then for each $k \ge 0$ and $1 \le i \le d$ we can find a unique $g_{k,i} \in \Delta$ such that $sw_{k,i} + g_{k,i} \in V$. Define the (n, s) series r_V by

$$r_V(x, y) = \sum_{k \ge 0} \sum_{i=1}^d (sw_{k,i} + g_{k,i}) \otimes b_i(y^k, [y]^k).$$

It is now clear, that $W(r_V) = V$. These constructions establish the bijection in part 1.

To prove the second statement, observe that

$$B(sw_{k,i}, b_{i}(y^{\ell}, [y]^{\ell})) = \delta_{i,i}\delta_{k,\ell}.$$
(27)

Furthermore, the straightforward calculation shows that

$$B(sw_{k,i}, sw_{\ell,j}) = \begin{cases} -\operatorname{res}_0 \left\{ sx^{(n-1)-k-\ell-1} \right\} & \text{if } i = j \text{ and } 0 \le k, \ell \le n-1, \\ \operatorname{res}_0 \left\{ sx^{(n-1)-k-\ell-1} \right\} & \text{if } i = j \text{ and } k, \ell \ge n, \\ 0 & \text{otherwise}, \end{cases}$$
$$= \begin{cases} -s_{k+\ell-n+1} & \text{if } i = j, 0 \le k, \ell \le n-1 \text{ and } k+\ell \ge n-1, \\ s_{k+\ell-n+1} & \text{if } i = j \text{ and } k, \ell \ge n, \\ 0 & \text{otherwise}, \end{cases}$$

where $s(x) = \sum_{k=0}^{\infty} s_k x^k$. We write

$$\overline{r}(x, y) = \frac{s(y)x^{n}\Omega}{x - y} - \tau(g(y, x)) = \frac{s(x)y^{n}\Omega}{x - y} - \frac{(s(x)y^{n} - s(y)x^{n})\Omega}{x - y} - \tau(g(y, x))$$
$$= \sum_{k \ge 0} \sum_{i=1}^{d} (sw_{k,i} + \overline{g}_{k,i}) \otimes b_{i}(y^{k}, [y]^{k}).$$

Consider the quotient

$$\frac{(s(x)y^n - s(y)x^n)\Omega}{x - y} = \frac{y^n(s(x) - s(y))\Omega}{x - y} - \frac{s(y)(x^n - y^n)\Omega}{x - y}$$
$$= \sum_{k \ge 0} \sum_{i=1}^d s_k \left(\sum_{\ell=1}^k b_i(x^{k-\ell}, [x]^{k-\ell}) \otimes b_i(y^{(n-1)+\ell}, [y]^{(n-1)+\ell}) - \sum_{\ell=1}^n b_i(x^{n-\ell}, [x]^{n-\ell}) \otimes b_i(y^{k+\ell-1}, [y]^{k+\ell-1}) \right).$$

The coefficient of $b_i(x^k, [x]^k) \otimes b_i(y^\ell, [y]^\ell)$ in the expression above is

$$-s_{k+\ell-(n-1)} \text{ if } 0 \le k, \ell \le n-1 \text{ and } k+\ell \ge n-1,$$

$$s_{k+\ell-(n-1)} \text{ if } k, \ell \ge n,$$

which coincides with $B(sw_{k,i}, sw_{\ell,i})$. If we now expand the coefficients $g_{k,i}$ in the following way

$$g_{k,i} = \sum_{\ell \ge 0} \sum_{j=1}^{d} g_{k,i}^{\ell,j} b_j(x^{\ell}, [x]^{\ell}),$$

the coefficients $\overline{g}_{k,i}$ can be rewritten as

$$\overline{g}_{k,i} = -\sum_{\ell \ge 0} \sum_{j=1}^{d} (g_{\ell,j}^{k,i} + B(sw_{k,i}, sw_{\ell,j})) b_i(x^k, [x]^k) \otimes b_j(y^\ell, [y]^\ell).$$

Combining all the results above we obtain the desired equality

$$B(sw_{k,i} + g_{k,i}, sw_{\ell,j} + \overline{g}_{\ell,j}) = B(sw_{k,i}, sw_{\ell,j}) + B(sw_{k,i}, \overline{g}_{\ell,j}) + B(g_{k,i}, sw_{\ell,j}) + B(g_{k,i}, \overline{g}_{\ell,j}) = B(sw_{k,i}, sw_{\ell,j}) + (-g_{k,i}^{\ell,j} - B(sw_{k,i}, sw_{\ell,j})) + g_{k,i}^{\ell,j} + 0 = 0$$

which completes the proof of the second statement.

Using the same technique as in [2, Section 1], one can prove that

 $\psi := \operatorname{GCYB}(r) \in (\Delta \otimes \mathfrak{g} \otimes \mathfrak{g})\llbracket (x_2, [x_2]), (x_3, [x_3]) \rrbracket$

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for any series *r* of type (n, s). Define $r_{k,i} := sw_{k,i} + g_{k,i}$ and $\overline{r}_{k,i} := sw_{k,i} + \overline{g}_{k,i}$ and rewrite GCYB(r) as

$$\psi = \sum_{k,\ell \ge 0} \sum_{i,j=1}^{d} [r_{k,i}, r_{\ell,j}] \otimes b_i(x_2^k, [x_2]^k) \otimes b_j(x_3^\ell, [x_3]^\ell) + \sum_{k\ge 0} \sum_{i=1}^{d} r_{k,i} \otimes \left([b_i(x_2^k, [x_2]^k) \otimes (1, 1), r(x_2, x_3)] + [(1, 1) \otimes b_i(x_3^k, [x_3]^k), \overline{r}(x_2, x_3)] \right).$$
(28)

Applying $B(\overline{r}_{k_1,i_1} \otimes r_{k_2,i_2} \otimes r_{k_3,i_3}, -)$ to the equation above, we get

$$B(\bar{r}_{k_1,i_1} \otimes r_{k_2,i_2} \otimes r_{k_3,i_3}, \psi) = B(\bar{r}_{k_1,i_1}, [r_{k_2,i_2}, r_{k_3,i_3}]).$$
(29)

This gives the last statement because W(r) and $W(\bar{r})$ are generated by $r_{k,i}$ and $\bar{r}_{k,i}$ respectively.

Corollary 3.7 Let $n \in \mathbb{Z}_{\geq 0}$, $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ and W be as in Theorem 3.6. Then

- 1. W defines a bijection between skew-symmtric $\left(n, \frac{1}{x^n \alpha(x)}\right)$ -type series and Lagrangian subspaces $V \subseteq L(n, \alpha)$ complementary to the diagonal Δ ;
- 2. W defines a bijection between $\left(n, \frac{1}{x^n \alpha(x)}\right)$ -type series solving GCYBE and subalgebras $V \subseteq L(n, \alpha)$ complementary to the diagonal Δ .

As we can see from the proof of Theorem 3.6 the element ψ in GCYB $(r) = \psi$ represents the obstruction for W(r) from being a Lie subalgebra. This observation raises an interesting question that we do not consider in this paper: what elements ψ can appear on the right-hand side of the above-mentioned equation.

Observe that if r is a series of type (n, s) and it satisfies

$$CYB(r) := [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] + [r^{13}(x_1, x_3), r^{23}(x_2, x_3)] = \psi$$
(30)

for some $\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x, y, z]]$, then *r* is automatically skew-symmetric and hence solves the first equation as well. To prove that one can e.g. repeat the argument from [1, Lemma 5.2]. In other words, for a fixed ψ solutions to $CYB(r) = \psi$ form a subclass of solutions to $GCYB(r) = \psi$. In particular, solutions to CYB(r) = 0. are exactly the skew-symmetric solutions to GCYB(r) = 0. We call the equation $CYB(r) = \psi$ *Manin-Yang-Baxter equation*.

Remark 3.8 As our notation suggest, we could have interpreted $y^n \Omega/(x - y)$ as

$$\frac{y^n \Omega}{x - y} = \sum_{k \ge 0} \sum_{i=1}^d b_i x^{-k-1} \otimes b_i y^{n+k} \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]]$$

$$\frac{s(x)y^n\Omega}{x-y} + g(x,y) \in (\mathfrak{g} \otimes \mathfrak{g})(x)[[y]]$$
(31)

we can simply view $s(x) \in F[[x]]^{\times}$ and $g(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ as elements in $F[[(x, [x])]]^{\times}$ and $(\mathfrak{g} \otimes \mathfrak{g})[[(x, [x]), (y, [y])]]$ respectively and reinterpret the singular part $y^n \Omega/(x-y)$ as it was done in Eq. (20).

Conversely, to get a series of the form Eq. (31) from a series of type (n, s) we can just project the latter onto the first component.

In other words, we have a bijection between (n, s)-type series in $L_2(n, \alpha)$ and their projections Eq. (31) onto the first component given by different interpretations of the singular part $y^n \Omega/(x - y)$.

Although, all arithmetic operations can be performed in the form Eq. (31), the construction of W(r) and statements like $\Delta \cap W(r) = 0$ require us to pass to the interpretation Eq. (20). This is our main motivation to work directly with (n, s)-type series in $L_2(n, \alpha)$ instead of their projections.

In view of Remark 3.8, we have a new proof of [1, Corollary 5.5].

Corollary 3.9 Classical (formal) r-matrices, i.e. skew-symmetric elements

$$\frac{s(x)y^n\Omega}{x-y} + g(x,y) = \frac{1}{x^n\alpha(x)}\frac{y^n\Omega}{x-y} + g(x,y) \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]], \quad (32)$$

solving GCYBE, are in bijection with skew-symmetric series of type (n, s) solving GCYBE and hence in bijection with Lagrangian Lie subalgebras of $L(n, \alpha)$ complementary to the diagonal Δ .

The result of [1, Theorem 5.6] can be now formulated in the following way.

Corollary 3.10 Skew-symmetric series of type $\left(n, \frac{1}{x^n \alpha(x)}\right)$ that also solve GCYBE exist only for n = 0, 1 and n = 2 with $\alpha_0 = 0$.

4 Quasi-Lie bialgebra structures on g[[x]]

We remind that *F* is a discrete algebraically closed field of characteristic 0 and $\mathfrak{g}[[x]]$ is an *F*-Lie algebra equipped with the (*x*)-adic topology.

As we now know, series of type $(n, 1/(x^n\alpha(x)))$ solving CYBE Eq. (30) are in bijection with Lagrangian subalgebras $W \subset L(n, \alpha)$ complementary to the diagonal. On the other hand, such Lagrangian subalgebras are in bijection with non-degenerate topological Lie bialgebra structures. See [1] for their definition and classification.

It turns out, that if we drop the condition on *W* being a subalgebra, we get so called (non-degenerate) topological quasi-Lie bialgebras. This section is devoted to their classification up to topological twists and coordinate transformations.

Definition 4.1 A *topological quasi-Lie bialgebra* structure on $\mathfrak{g}[[x]]$ consists of

- a skew-symmetric continuous linear map $\delta : \mathfrak{g}[\![x]\!] \to (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$ and
- a skew-symmetric element $\varphi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x, y, z]],$

which are subject to the following conditions

- 1. $\delta([a, b]) = [a \otimes 1 + 1 \otimes a, \delta(b)] [b \otimes 1 + 1 \otimes b, \delta(a)]$, i.e. δ is a 1-cocycle;
- 2. $\frac{1}{2}$ Alt $((\delta \otimes 1)\delta(a)) = [a \otimes 1 \otimes 1 + 1 \otimes a \otimes 1 + 1 \otimes 1 \otimes a, \varphi];$
- 3. $\overline{Alt}((\delta \otimes 1 \otimes 1)\varphi) = 0$,

where Alt $(x_1 \otimes \ldots \otimes x_n) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$.

Lemma 4.2 There is a one-to-one correspondence between triples $(L, \mathfrak{g}[\![x]\!], W)$, where $(L, \mathfrak{g}[\![x]\!])$ is a topological Manin pair and $W \subset L$ is a Lagrangian subspace satisfying $W + \mathfrak{g}[\![x]\!] = L$, and quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$.

Proof We start with a topological Manin pair $(L, \mathfrak{g}[[x]])$. If $W \subset L$ is a Lagrangian subspace complementary to $\mathfrak{g}[[x]]$, then it is easy to see that $W \cong \mathfrak{g}[[x]]'$. Therefore, we have an isomorphism of vector spaces

$$L \cong \mathfrak{g}\llbracket x \rrbracket \dot{+} \mathfrak{g}\llbracket x \rrbracket'.$$

The form on *L* under this isomorphism becomes standard evaluation form $\langle -, - \rangle$ on $\mathfrak{g}[[x]] + \mathfrak{g}[[x]]'$. We fix such an isomorphism.

Let us define two linear functions

$$p_1: \mathfrak{g}[\![x]\!]' \otimes \mathfrak{g}[\![y]\!]' \to \mathfrak{g}[\![x]\!] \text{ and } p_2: \mathfrak{g}[\![x]\!]' \otimes \mathfrak{g}[\![y]\!]' \to \mathfrak{g}[\![x]\!]'$$

by $[f, g] = p_1(f \otimes g) + p_2(f \otimes g)$. We put

$$\delta := p_2^{\vee} \colon (\mathfrak{g}[\![x]\!]')^{\vee} \cong \mathfrak{g}[\![x]\!] \to (\mathfrak{g}[\![x]\!]' \otimes \mathfrak{g}[\![y]\!]')^{\vee} \cong (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!],$$

and let $\psi \in (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})[[x, y, z]]$ be the unique element satisfying the condition

$$\langle h, [f, g] \rangle = \langle h, p_1(f \otimes g) \rangle = \langle f \otimes g \otimes h, \psi \rangle \text{ for all } f, g, h \in \mathfrak{g}[[x]]'.$$
(33)

The skew-symmetry of p_2 implies the skew-symmetry of δ , whereas the skew-symmetry of p_1 and the invariance of the evaluation form yield the skew-symmetry of ψ .

Next, we observe that for all $a, b \in \mathfrak{g}[[x]]$ and $f, g \in \mathfrak{g}[[x]]'$ we have

$$\langle [a, f], g \rangle = \langle a, [f, g] \rangle = \langle a, p_2(f \otimes g) \rangle = \langle \delta(a), f \otimes g \rangle = \langle (f \otimes 1)\delta(a), g \rangle, \langle [a, f], b \rangle = -\langle f, [a, b] \rangle = -\langle f \circ ad_a, b \rangle.$$

In other words, the invariance of the form forces the following equality to hold

$$[a, f] = -f \circ \operatorname{ad}_a + (f \otimes 1)\delta(a).$$
(34)

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Using Eq. (34) and non-degeneracy of the form we show that δ is a 1-cocycle:

$$\langle \delta([a, b]), f \otimes g \rangle = \langle [a, b], p_2(f \otimes g) \rangle = \langle [a, b], [f, g] \rangle = \langle [[a, b], f], g \rangle$$

$$= \langle -[[b, f], a] - [[f, a], b], g \rangle$$

$$= \langle [f \circ ad_b - (f \otimes 1)\delta(b), a] - [f \circ ad_a - (f \otimes 1)\delta(a), b], g \rangle$$

$$= -\langle a, [f \circ ad_b, g] \rangle + \langle b, [f \circ ad_a, g] \rangle + \langle (f \otimes ad_a)\delta(b), g \rangle$$

$$- \langle (f \otimes ad_b)\delta(a), g \rangle$$

$$= \langle [a \otimes 1 + 1 \otimes a, \delta(b)] - [b \otimes 1 + 1 \otimes b, \delta(a)], f \otimes g \rangle.$$

$$(35)$$

The 1-cocycle condition implies that δ is continuous as it was noted in [1, Remark 3.16].

For conditions 2 and 3 from the definition of a topological quasi-Lie bialgebra consider the Jacobi identity for $f, g, h \in \mathfrak{g}[[x]]'$:

$$0 = [p_1(f \otimes g), h] + [p_1(g \otimes h), f] + [p_1(h \otimes f), g]$$

+ $p_1(p_2(f \otimes g) \otimes h) + p_1(p_2(g \otimes h) \otimes f) + p_1(p_2(h \otimes f) \otimes g)$
+ $p_2(p_2(f \otimes g) \otimes h) + p_2(p_2(g \otimes h) \otimes f) + p_2(p_2(h \otimes f) \otimes g).$ (36)

We denote by \bigcirc the summation over circular permutations of symbols f, g and h, e.g. $\bigcirc \langle p_1(f \otimes g), h \rangle = \langle p_1(f \otimes g), h \rangle + \langle p_1(g \otimes h), f \rangle + \langle p_1(h \otimes f), g \rangle$. Applying $\langle -, a \rangle$ to Eq. (36) for an arbitrary $a \in \mathfrak{g}[x]$ gives

$$\langle p_2(p_2 \otimes 1)(\bigcirc f \otimes g \otimes h), a \rangle = -\langle \bigcirc [p_1(f \otimes g), h], a \rangle$$

$$\langle p_2 \otimes 1(\bigcirc f \otimes g \otimes h), \delta(a) \rangle = \bigcirc \langle -h \circ ad_a, p_1(f \otimes g) \rangle$$

$$\langle \bigcirc f \otimes g \otimes h, (\delta \otimes 1)\delta(a) \rangle = \bigcirc \langle f \otimes g \otimes (-h \circ ad_a), \psi \rangle$$

$$\langle f \otimes g \otimes h, Alt((\delta \otimes 1)\delta(a))/2 \rangle = -\langle f \otimes g \otimes h, [1 \otimes 1 \otimes a + 1 \otimes a \otimes 1 + a \otimes 1 \otimes 1, \psi] \rangle,$$

where the very last identity holds because of the skew-symmetry of ψ . Multiplying this equality by 2 we get the relation

$$\langle f\otimes g\otimes h, \operatorname{Alt}((\delta\otimes 1)\delta(a))+2[1\otimes 1\otimes a+1\otimes a\otimes 1+a\otimes 1\otimes 1,\psi]\rangle=0.$$

Letting $\varphi := -\psi$ we obtain the second identity from the definition of a topological quasi-Lie bialgebra structure. Applying instead $\langle s, - \rangle, s \in \mathfrak{g}[[x]]'$ to the Jacobi identity Eq. (36) we get the desired

$$\operatorname{Alt}((\delta \otimes 1 \otimes 1)\psi) = 0.$$

Therefore, $(\mathfrak{g}[[x]], \delta, \varphi)$ is a topological quasi-Lie bialgebra.

For the converse direction, we put $L := \mathfrak{g}[\![x]\!] + \mathfrak{g}[\![x]\!]'$ with the standard evaluation form; we let p_1 be the unique element in $\operatorname{Hom}_{F-\operatorname{Vect}}(\mathfrak{g}[\![x]\!]' \otimes \mathfrak{g}[\![x]\!]', \mathfrak{g}[\![x]\!])$ satisfying Eq. (33) with $\psi := -\varphi$; we define $p_2 := \delta'$, i.e. the dual map of δ . The Lie bracket between two elements in $\mathfrak{g}[\![x]\!]'$ is given by the sum $p_1 + p_2$. Defining [a, f] as in Eq. (34) the evaluation form becomes invariant and we get a topological Manin pair $(L, \mathfrak{g}[\![x]\!])$ with the Lagrangian subspace $\mathfrak{g}[\![x]\!]'$. These constructions are clearly inverse to each other.

Combining the classification of Manin pairs mentioned in Sect. 2 with Corollary 3.7 and Lemma 4.2 we get the following description of all topological quasi-Lie bialgebra structures on $\mathfrak{g}[x]$.

Lemma 4.3 There is a bijection between topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ and Lagrangian subspaces $W \subset L(n, \alpha)$ or $L(\infty)$ complementary to the diagonal Δ , where $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$ is an arbitrary sequence and $n \geq 0$. Moreover, such Lagrangian subspaces $W \subset L(n, \alpha)$ are in bijection with skew-symmetric sequences of type $(n, 1/(x^n\alpha(x)))$.

In view of this result we call skew-symmetric series of type (n, s) as well as their projections onto the first component *quasi-r-matrices*. Quasi-Lie bialgebra structures can also be described using their associated quasi-*r*-matrices in the following way.

Proposition 4.4 Assume $(\mathfrak{g}[x]], \delta, \varphi)$ is a topological quasi-Lie bialgebra and let $r \in L_2(n, \alpha)$ be the corresponding quasi-r-matrix given by the bijection from Lemma 4.3. Under the identification $\mathfrak{g}[[(x, [x])]] \cong \mathfrak{g}[[x]]$ we have the following identities:

• $[a \otimes 1 + 1 \otimes a, r] = -\delta(a)$ for any $a \in \mathfrak{g}[[x]]$ and

•
$$CYB(r) = -\varphi$$
.

The same is true for the projection $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]]$.

Proof We start, as in the proof of Lemma 4.2, by fixing an identification $L(n, \alpha) = \Delta + W(r) \cong \mathfrak{g}[\![x]\!] + \mathfrak{g}[\![x]\!]'$. Let $\{v_{k,i}\}$ be a basis for $\mathfrak{g}[\![x]\!]'$ dual to $\{\varepsilon_{k,i} := b_i y^k\}$. Then $r = \sum_{k\geq 0} \sum_{i=1}^d v_{k,i} \otimes \varepsilon_{k,i}$ and we have

$$[a \otimes 1 + 1 \otimes a, r] = \sum_{k \ge 0} \sum_{i=1}^{d} [a, v_{k,i}] \otimes \varepsilon_{k,i} + v_{k,i} \otimes [a, \varepsilon_{k,i}]$$
$$= \sum_{k \ge 0} \sum_{i=1}^{d} (-v_{k,i} \circ \operatorname{ad}_{a} + (v_{k,i} \otimes 1)\delta(a)) \otimes \varepsilon_{k,i} + v_{k,i} \otimes [a, \varepsilon_{k,i}].$$

Applying $\langle v_{\ell,j} \otimes v_{m,t}, - \rangle$ to the equality above we get

$$\begin{aligned} \langle v_{\ell,j} \otimes v_{m,t}, [a \otimes 1 + 1 \otimes a, r] \rangle &= \sum_{k \ge 0} \sum_{i=1}^{d} \langle v_{\ell,j} \otimes v_{m,t}, (v_{k,i} \otimes 1) \delta(a) \otimes \varepsilon_{k,i} \rangle \\ &= \langle v_{\ell,j}, (v_{m,t} \otimes 1) \delta(a) \rangle \\ &= \langle v_{\ell,j} \otimes v_{m,t}, -\delta(a) \rangle. \end{aligned}$$

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Applying instead $\langle \varepsilon_{\ell,i} \otimes v_{m,t}, - \rangle$ to the same equality we obtain

$$\langle \varepsilon_{\ell,j} \otimes v_{m,t}, [a \otimes 1 + 1 \otimes a, r] \rangle = \sum_{k \ge 0} \sum_{i=1}^{d} \langle \varepsilon_{\ell,j} \otimes v_{m,t}, (-v_{k,i} \circ \mathrm{ad}_{a}) \otimes \varepsilon_{k,i} \\ + v_{k,i} \otimes [a, \varepsilon_{k,i}] \rangle \\ = -\langle \varepsilon_{\ell,j}, v_{m,t} \circ \mathrm{ad}_{a} \rangle + \langle v_{m,t}, [a, \varepsilon_{\ell,j}] \rangle \\ = 0.$$

This implies the desired equality $[a \otimes 1 + 1 \otimes a, r] = -\delta(a)$. The identity CYB $(r) = -\varphi$ follows from the skew-symmetry of r, Theorem 3.6 and the fact that $\varphi = -\psi$ according to the proof of Lemma 4.2.

Remark 4.5 Assume $r \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]]$ is a series such that

$$[f(x) \otimes 1 + 1 \otimes f(y), r(x, y)] \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$$
(37)

for all $f \in \mathfrak{g}[[x]]$. Write $r = s(x^{-1}, y) + g(x, y)$, where $s \in x^{-1}(\mathfrak{g} \otimes \mathfrak{g})[x^{-1}][[y]]$ and $g \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$. Then, because of Eq. (37), we must have

$$[a \otimes 1 + 1 \otimes a, s(x^{-1}, y)] = 0$$

for all $a \in \mathfrak{g}$. Since the \mathfrak{g} -invariant elements of $\mathfrak{g} \otimes \mathfrak{g}$ are precisely the multiples of the quadratic Casimir element Ω , we have the identity $s(x^{-1}, y) = p(x^{-1}, y)\Omega$ for some $p \in x^{-1}F[x^{-1}][[y]]$. Furthermore, the condition

$$[ax \otimes 1 + 1 \otimes ay, p(x^{-1}, y)\Omega] = [a(x - y) \otimes 1, p(x^{-1}, y)\Omega] \in (\mathfrak{g} \otimes \mathfrak{g})\llbracket x, y\rrbracket$$

implies $(x - y)p(x^{-1}, y) \in F[[x, y]]$, meaning that there exists an $s \in F[[y]]$ such that $p(x^{-1}, y) = s(y)/(x - y)$. In other words, *r* has the form Eq. (21). This result can be considered as another motivation to study series of type (n, s).

Observe that if we know one Lagrangian subspace W_0 inside $L \cong \mathfrak{g}[[x]] + \mathfrak{g}[[x]]'$ then any other Lagrangian subspace can be constructed from W_0 through twisting. More precisely, if $s = \sum_i a_i \otimes b^i \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$ is a skew-symmetric tensor, then we can associate with it a (twisted) Lagrangian subspace

$$W_s := \left\{ \sum_i B(b^i, w) a_i - w \mid w \in W \right\} \subseteq L$$
(38)

complementary to $\mathfrak{g}[[x]]$. The converse is also true; for proof see [3]. In other words, the following statement holds.

Lemma 4.6 There is a bijection between Lagrangian subspaces $W \subseteq L(n, \alpha)$ or $L(\infty)$ and skew-symmetric tensors in $(\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$.

Combining Lemma 4.4, Eq. (38) and the algorithm for constructing a quasi-*r*-matrix from a Lagrangian subspace $W \subset L(n, \alpha)$, $W \dotplus \Delta = L(n, \alpha)$, we obtain the following twisting rules for Lagrangian subspaces, quasi-Lie bialgebra structures and quasi-*r*-matrices.

Lemma 4.7 Let $(\mathfrak{g}[\![x]\!], \delta, \varphi)$ be a topological quasi-Lie bialgebra structure corresponding to the quasi-r-matrix r. If we twist W(r) with a skew-symmetric tensor s as described in Eq. (38) we obtain another topological quasi-Lie bialgebra $(\mathfrak{g}[\![x]\!], \delta_s, \varphi_s)$, such that

1. $W(r)_s = W(r - s);$ 2. $\delta_s = \delta + ds;$ 3. $\varphi_s = \varphi + \text{CYB}(s) - \frac{1}{2}\text{Alt}((\delta \otimes 1)s),$

where $ds(a) := [a \otimes 1 + 1 \otimes a, s]$.

Remark 4.8 Since any quasi-*r*-matrix *r* defines a topological quasi-Lie bialgebra structure $\delta(a) = [a \otimes 1 + 1 \otimes a, r]$ on $\mathfrak{g}[[x]]$, the third condition in Definition 4.1 is trivially satisfied. In other words,

$$\operatorname{Alt}((\delta \otimes 1 \otimes 1)\operatorname{CYB}(r)) = 0$$

for any quasi-*r*-matrix *r*.

Lemma 4.6 and Lemma 4.7 state that, in order to obtain a description of topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ up to twisting it is enough to find a single Lagrangian subspace W_0 , complementary to $\mathfrak{g}[\![x]\!]$, inside $L(\infty)$ and each $L(n, \alpha)$. The same is true for the associated quasi-*r*-matrices

The case $L(\infty)$ is trivial, because by definition $\mathfrak{g}[[x]]' = \bigoplus_{j\geq 0} \mathfrak{g} \otimes a_j \subseteq L(\infty)$ is a Lagrangian subalgebra (see Lemma 2.1). Similar to the Lie bialgebra case, topological quasi-Lie bialgebras corresponding to the Manin pair $(L(\infty), \mathfrak{g}[[x]])$ are called *degenerate*.

Let us now focus on *non-degenerate* topological quasi-Lie bialgebra structures, i.e. the ones corresponding to the Manin pairs $(L(n, \alpha), \Delta)$. By Proposition 2.5 for each Manin pair $(L(n, \alpha), \Delta)$ there exists an appropriate coordinate transformation that makes it into $(L(n, \beta), \Delta)$, where $\beta_0 = \alpha_0$ and all other $\beta_i = 0$. This means, that to classify all non-degenerate topological quasi-Lie bialgebras on $\mathfrak{g}[[x]]$, up to coordinate transformations and twisting, it is enough to construct a Lagrangian subspace W_0 within each $L(n, \alpha_0) := L(n, (..., 0, \alpha_0, 0, ..., 0))$ complementary to Δ . Equivalently, it is enough to find a quasi-*r*-matrix of type (n, α_0) for any $n \ge 0$ and $\alpha_0 \in F$.

4.1 Lagrangian subspaces of $L(n, \alpha_0)$

As before we let $\{b_i\}_{i=1}^d$ be an orthonormal basis for \mathfrak{g} with respect to the Killing form κ . The form *B* on $L(n, \alpha_0)$ has the following explicit form

$$B(a \otimes (f, [p]), b \otimes (g, [q])) = \begin{cases} \kappa(a, b) \{\operatorname{coeff}_{n-1}(fg - pq) - \alpha_0 \operatorname{coeff}_0(fg - pq)\} & \text{if } n \ge 2, \\ \kappa(a, b) \operatorname{coeff}_{n-1}(fg - pq) & \text{if } n = 0, 1. \end{cases}$$

$$(39)$$

We now present an explicit construction for a Lagrangian subspace of $L(n, \alpha_0)$ complementary to Δ for arbitrary $n \ge 0$ and $\alpha_0 \in F$. Using the twisting procedure from Lemma 4.7, this subspace can be twisted in order to obtain all other Lagrangian subspaces of $L(n, \alpha_0)$ complementary to Δ .

n = 0: When n = 0, the subalgebra $W_0 := x^{-1}\mathfrak{g}[x^{-1}] \subseteq \mathfrak{g}(x)$ is known to be Lagrangian.

n = 1: For n = 1 it is easy to see that the subspace

$$W_0 := \operatorname{span}_F\{b_i(1, -1), b_i(x^{-k}, 0) \mid k \ge 1, \ 1 \le i \le d\} \subset L(1, \alpha_0)$$
(40)

is Lagrangian and complementary to the diagonal Δ .

n = 2k: For even $n \ge 2$ and arbitrary $\alpha_0 \in F$ the subspace $W_0 \subset L(n, \alpha_0)$ spanned by the elements

$$\begin{split} b_i \left\{ (x^{(n-1)-m}, 0) - \alpha_0 (x^{2(n-1)-m}, 0) + \alpha_0^2 (x^{3(n-1)-m}, 0) - \alpha_0^3 (x^{4(n-1)-m}, 0) + \dots \right\}, \\ 0 &\leq m \leq \frac{n}{2} - 1, \\ b_i \left(0, -[x]^{(n-1)-\ell} \right), \ \frac{n}{2} \leq \ell < n - 1, \\ b_i \left(0, -1 + \frac{\alpha_0}{2} [x]^{n-1} \right), \\ b_i (x^{-k}, 0), k \geq 1, \end{split}$$

is Lagrangian and complementary to the diagonal.

n = 2k + 1: Modifying slightly the basis for even case we obtain the following basis for $W_0 \subset L(n, \alpha_0)$ with odd $n \ge 3$:

$$\begin{split} b_i \left\{ (x^{(n-1)-m}, 0) - \alpha_0 (x^{2(n-1)-m}, 0) + \alpha_0^2 (x^{3(n-1)-m}, 0) - \alpha_0^3 (x^{4(n-1)-m}, 0) + \dots \right\}, \\ 0 &\leq m \leq \frac{n-1}{2} - 1, \\ b_i \left\{ (x^{\frac{n-1}{2}}, -[x]^{\frac{n-1}{2}}) - \alpha_0 (x^{\frac{3(n-1)}{2}}, 0) + \alpha_0^2 (x^{\frac{5(n-1)}{2}}, 0) - \alpha_0^3 (x^{\frac{7(n-1)}{2}}, 0) + \dots \right\}, \\ b_i (0, -[x]^{(n-1)-\ell}), \ \frac{n-1}{2} + 1 \leq \ell < n-1, \\ b_i \left(0, -1 + \frac{\alpha_0}{2} [x]^{n-1} \right), \\ b_i (x^{-k}, 0), k \geq 1. \end{split}$$

The subspaces above were constructed by "guessing". However, there is an abstract procedure that produces Lagrangian subspaces for arbitrary n and α . We present it here for completeness.

The easiest skew-symmetric (n, s)-type series is given by

$$r(x, y) := \frac{1}{2} \left(\frac{s(x)y^{n}\Omega}{x - y} + \frac{s(y)x^{n}\Omega}{x - y} \right) = \frac{s(x)y^{n}\Omega}{x - y} + \frac{\Omega}{2} \left(\frac{s(y)x^{n} - s(x)y^{n}}{x - y} \right)$$
$$= \frac{s(x)y^{n}\Omega}{x - y} - \frac{1}{2} \sum_{k,\ell=0}^{\infty} \sum_{i,j=1}^{d} B(sw_{k,i}, sw_{\ell,j})b_{i}(x, [x])^{k} \otimes b_{j}(y, [y])^{\ell},$$

where we recall that

$$B(sw_{k,i}, sw_{\ell,j}) = \begin{cases} -s_{k+\ell-n+1} & \text{if } i = j, \ 0 \le k, \ \ell \le n-1 \text{ and } k+\ell \ge n-1, \\ s_{k+\ell-n+1} & \text{if } i = j \text{ and } k, \ \ell \ge n, \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 3.7 the subspace

$$W(r) = \operatorname{span}_{F} \left\{ sw_{k,i} - \frac{1}{2} \sum_{\ell=0}^{\infty} B(sw_{\ell,i}, sw_{k,i})b_{i}(x, [x])^{\ell} \, \middle| \, k \ge 0, \, 1 \le d \le n \right\}$$

$$= \operatorname{span}_{F} \left\{ sw_{k,i} + \frac{1}{2} \left(\sum_{\ell=0}^{n-1} s_{k+\ell-n+1}b_{i}(x, [x])^{\ell} - \sum_{\ell=n}^{\infty} s_{k+\ell-n+1}b_{i}(x, [x])^{\ell} \right) \, \middle| \\ k \ge 0, \, 1 \le d \le n \right\}$$

is Lagrangian and complementary to the diagonal. Here we used the convention that $s_k = 0$ for k < 0. Calculating the basis explicitly for some particular *s* requires some effort and it may not look as friendly as the ones given above.

4.2 Quasi-r-matrices

The goal of this section is to describe the quasi-*r*-matrices corresponding to the Lagrangian subspaces described in the previous section. The twisting procedure from Lemma 4.7 then yields all other quasi-*r*-matrices.

The proof of Theorem 3.6 gives us an algorithm for constructing a series of type $(n, s(x) := 1/(x^n \alpha(x)))$ from a subspace $W \subset L(n, \alpha)$ complementary to the diagonal. More precisely, the desired series is given by

$$\sum_{k\geq 0} \sum_{i=1}^{d} v_{k,i} \otimes b_i(y^k, [y]^k),$$
(41)

where

$$W = \operatorname{span}_F \{ v_{k,i} \mid k \ge 0, \ 1 \le i \le d \}$$
 and $B(v_{k,i}, b_j(y^{\ell}, [y]^{\ell})) = \delta_{i,j} \delta_{k,\ell},$

i.e. $\{v_{k,i}\}$ is a basis of V dual to $\{b_i(y^k, [y]^k)\}$. Indeed, non-degeneracy of the form B then implies that $v_{k,i}$ has the desired form $v_{k,i} = sw_{k,i} + g_{k,i}$ for some $g_{k,i} \in (\mathfrak{g} \otimes \mathfrak{g})[[x, y]]$.

Applying this idea to W_0 's constructed in the preceding section we get the following series.

n = 0: The classical *r*-matrix (equivalently (0, 1)-type series) corresponding to $W_0 := x^{-1}\mathfrak{g}[x^{-1}] \subseteq \mathfrak{g}(x)$ is the Yang's matrix $\Omega/(x - y)$.

n = 1: The quasi-*r*-matrix corresponding to $\operatorname{span}_F\{b_i(1, -1), b_i(x^{-k}, 0) \mid k \ge 1, 1 \le i \le d\} \subset L(1, \alpha_0)$ is

$$\frac{y\Omega}{x-y} + \frac{1}{2} \sum_{i=1}^{d} b_i(1,-1) \otimes b_i(1,1) \in L_2(1,1) \text{ with the projection}$$
$$\frac{y\Omega}{x-y} + \frac{1}{2} \Omega \in (\mathfrak{g} \otimes \mathfrak{g})((x))[[y]].$$

n = 2k: For even $n \ge 2$ and arbitrary $\alpha_0 \in F$ we have the following quasi-*r*-matrix

$$\frac{1}{1+\alpha_0 x^{n-1}} \frac{y^n \Omega}{x-y} + \frac{\Omega}{1+\alpha_0 x^{n-1}} \sum_{0 \le m < \frac{n}{2}} x^{(n-1)-m} y^m + \frac{\alpha_0 \Omega}{(1+\alpha_0 x^{n-1})(1+\alpha_0 y^{n-1})} \left(y^{2(n-1)} + \sum_{\frac{n}{2} \le \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} - \frac{1}{2} x^{n-1} y^{n-1} \right)$$

n = 2k + 1: In the odd case $n \ge 3$ the series corresponding to $W_0 \subset L(n, \alpha_0)$ is

$$\begin{aligned} \frac{1}{1+\alpha_0 x^{n-1}} \frac{y^n \Omega}{x-y} + \frac{\Omega}{1+\alpha_0 x^{n-1}} \left(x^{\frac{n-1}{2}} y^{\frac{n-1}{2}} + \sum_{0 \le m < \frac{n-1}{2}} x^{(n-1)-m} y^m \right) \\ &+ \frac{\alpha_0 \Omega}{(1+\alpha_0 x^{n-1})(1+\alpha_0 y^{n-1})} \left(y^{2(n-1)} + \sum_{\frac{n-1}{2} < \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} - \frac{1}{2} x^{n-1} y^{n-1} \right). \end{aligned}$$

5 Lie algebra splittings of $L(n, \alpha)$ and generalized *r*-matrices

By Corollary 3.7 we have a bijection between subalgebras of $L(n, \alpha)$ and series of type $(n, 1/(x^n\alpha(x)))$ solving GCYBE. Therefore, we can construct new solutions to GCYBE by finding subalgebras of $L(n, \alpha)$ complementary to the diagonal. However, as the following result shows, the most interesting new solutions should arise from unbounded subalgebras of $L(n, \alpha)$, n > 2.

Proposition 5.1 Let $L(n, \alpha) = \Delta + W$ for some subalgebra $W \subset L(n, \alpha)$ and n > 2. Assume W is bounded, i.e. there is an integer N > 0 such that

$$x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N\mathfrak{g}[x^{-1}],$$

where W_+ is the projection of $W \subset L(n, \alpha) = \mathfrak{g}(x) \oplus \mathfrak{g}[x]/x^n \mathfrak{g}[x]$ on the first component $\mathfrak{g}(x)$. Then there is an element $\sigma \in \operatorname{Aut}_{F[x]-\operatorname{LieAlg}}(\mathfrak{g}[x])$ such that

$$\{0\} \times [x^2]\mathfrak{g}[x]/x^n \mathfrak{g}[x] \subseteq (\sigma \times \sigma)W \subseteq x\mathfrak{g}[x^{-1}] \times \mathfrak{g}[x]/x^n \mathfrak{g}[x]$$

and the image \widetilde{W} under the canonical projection $L(n, \alpha) \to L(2, \alpha)$ is a subalgebra satisfying $L(2, \alpha) = \Delta + \widetilde{W}$.

In the language of (n, s)-type series: Let

$$r = \frac{s(x)y^n\Omega}{x-y} + g(x, y)$$

be the generalized *r*-matrix corresponding to a bounded $W \subset L(n, \alpha), n \geq 2$. Then there is $p(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})[x, y]$ of degree at most one in *x* and an element $\sigma \in$ $\operatorname{Aut}_{F[x]-\operatorname{LieAlg}}(\mathfrak{g}[x])$ such that

$$(\sigma(x) \otimes \sigma(y))r(x, y) = y^{n-2} \Big(\underbrace{\frac{s(x)y^2\Omega}{x-y} + p(x, y)}_{r'(x, y)} \Big),$$

where r' is a generalized r-matrix in $L_2(2, \alpha)$.

Proof The condition $x^{-N}\mathfrak{g}[x^{-1}] \subseteq W_+ \subseteq x^N\mathfrak{g}[x^{-1}]$ means exactly that W_+ is an order. Moreover, since W is complementary to the diagonal, we have $W_+ + \mathfrak{g}[x] = \mathfrak{g}[x, x^{-1}]$. It was shown in [11] that such orders, up to the action of some $\sigma \in \operatorname{Aut}_{F[x]-\operatorname{LieAlg}}(\mathfrak{g}[x])$, are contained in a maximal order \mathfrak{M} associated to the so called fundamental simplex Δ_{st} . These maximal orders are explicitly described in [11] and satisfy $\mathfrak{M} \subseteq x\mathfrak{g}[x^{-1}]$. Therefore, we have $\sigma W_+ \subseteq \mathfrak{M} \subseteq x\mathfrak{g}[x^{-1}]$. Moreover, we have the identity

$$(\sigma \times \sigma)W \dotplus \Delta = L(n, \alpha),$$

implying the inclusion $\{0\} \times [x^2]\mathfrak{g}[x]/x^n\mathfrak{g}[x] \subseteq (\sigma \times \sigma)W$. The remaining parts follow straightforward from the construction Theorem 3.6.

Unfortunately, we have not found a new example of an unbounded subalgebra of $L(n, \alpha)$. However, we present an infinite family of bounded subalgebras. We believe these examples are still interesting because their orthogonal complements, which are important in the view of Adler-Kostant-Symes scheme, are unbounded if $\alpha \neq 0$.

Consider the subspaces of $L(n, \alpha_0)$, n > 0:

$$W_0 = \operatorname{span}_F \{ b_i(x^{-k}, 0), b_i(1, 0), b_i(0, -[x]^{\ell}) \mid k \ge 1, \ 1 \le \ell \le n - 1 \}, W_1 = \operatorname{span}_F \{ b_i(x^{-k}, 0), b_i(0, -1), b_i(0, -[x]^{\ell}) \mid k \ge 1, \ 1 \le \ell \le n - 1 \}.$$

These are clearly subalgebras. The corresponding generalizerd r-matrices are

$$\begin{split} r_{0} &= \frac{1}{1 + \alpha_{0} x^{n-1}} \frac{y^{n} \Omega}{x - y} + \frac{y^{n-1} \Omega}{(1 + \alpha_{0} x^{n-1})(1 + \alpha_{0} y^{n-1})} \\ &+ \frac{\alpha_{0} \Omega}{(1 + \alpha_{0} x^{n-1})(1 + \alpha_{0} y^{n-1})} \left(y^{2(n-1)} + \sum_{0 \le \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} \right) \\ &= \frac{y^{n-1}}{1 + \alpha_{0} y^{n-1}} \left(\frac{y\Omega}{x - y} + \Omega \right), \\ r_{1} &= \frac{1}{1 + \alpha_{0} x^{n-1}} \frac{y^{n} \Omega}{x - y} + \frac{\alpha_{0} \Omega}{(1 + \alpha_{0} x^{n-1})(1 + \alpha_{0} y^{n-1})} \\ &\times \left(y^{2(n-1)} + \sum_{0 < \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} \right) \\ &= \frac{1}{1 + \alpha_{0} y^{n-1}} \frac{y^{n} \Omega}{x - y}. \end{split}$$

By considering decompositions $\mathfrak{g} = \mathfrak{s}_1 + \mathfrak{s}_2$ of \mathfrak{g} into direct sums of subalgebras we can get an infinite family of generalized *r*-matrices "in between" r_0 and r_1 . More precisely, let $\{s_{1,i}\}_{i=1}^{d_1}$ and $\{s_{2,j}\}_{j=1}^{d_2}$ be bases for \mathfrak{s}_1 and \mathfrak{s}_2 respectively. Such a decomposition leads to another subalgera of $L(n, \alpha_0)$:

$$W_{01} := \operatorname{span}_{F} \left\{ b_{i}(x^{-k}, 0), s_{1,m}(1, 0), s_{2,j}(0, 1), b_{i}(0, -[x]^{\ell}) \mid k \ge 1, \ 1 \le \ell \le n - 1, \ 1 \le i \le d, \\ 1 \le m \le d_{1}, \ 1 \le j \le d_{2} \right\}.$$

Rewrite the elements b_i in terms of $s_{1,m}$ and $s_{2,j}$:

$$b_i = \sum_{m=1}^{d_1} \lambda_{1,m}^i s_{1,m} + \sum_{j=1}^{d_2} \lambda_{2,j}^j s_{2,j},$$

where $\lambda_{1,m}^i, \lambda_{2,j}^i \in F$. Finding a basis in W_{12} dual to $\{b_i(y^m, [y]^m)\} \subset \Delta$ and then projecting the generating series for W_{01} onto the first component we obtain the following generalized *r*-matrix

$$r_{01} = \frac{1}{1 + \alpha_0 x^{n-1}} \frac{y^n \Omega}{x - y} + \frac{\alpha_0 \Omega}{(1 + \alpha_0 x^{n-1})(1 + \alpha_0 y^{n-1})} \\ \times \left(y^{2(n-1)} + \sum_{0 < \ell < n-1} x^{(n-1)-\ell} y^{(n-1)+\ell} \right) \\ + \frac{y^{n-1}}{1 + \alpha_0 y^{n-1}} \sum_{i=1}^d \sum_{m=1}^{d_1} \lambda_{1,m}^i s_{1,m} \otimes b_i \\ = \frac{y^{n-1}}{1 + \alpha_0 y^{n-1}} \left(\frac{y\Omega}{x - y} + \sum_{i=1}^d \sum_{m=1}^{d_1} \lambda_{1,m}^i s_{1,m} \otimes b_i \right).$$
(42)

Clearly r_{01} coincides with r_0 when $\mathfrak{s}_1 = \mathfrak{g}$ and r_1 if $\mathfrak{s}_2 = \mathfrak{g}$. The corresponding orhogonal complements are

$$W_0^{\perp} = W(\overline{r_0}) = \operatorname{span}_F \left\{ b_i\left(0, [x]^{n-1}\right), b_i\left(\frac{x^{-k(n-1)-m}}{1+\alpha_0 x^{n-1}}, 0\right) \mid k \ge -1, 0 < m < n-1 \right\},\$$

$$W_1^{\perp} = W(\overline{r_1}) = \operatorname{span}_F \left\{ b_i \left(\frac{x^{-k(n-1)-m}}{1 + \alpha_0 x^{n-1}}, 0 \right) \mid k \ge -1, 0 \le m < n-1 \right\},$$

$$W_{01}^{\perp} = W(\overline{r_{01}}) = \mathfrak{s}_{1}^{\perp} \left(\frac{x^{n-1}}{1 + \alpha_{0} x^{n-1}}, 0 \right) \dotplus \mathfrak{s}_{2}^{\perp}(0, [x]^{n-1}) \dashv \operatorname{span}_{F} \left\{ b_{i} \left(\frac{x^{-k(n-1)-m}}{1 + \alpha_{0} x^{n-1}}, 0 \right) \mid k \ge -1, 0 < m < n-1 \right\},$$
(43)

which are unbounded because of the factor $1/(1 + \alpha_0 x^{n-1})$.

Note that a series of type (n, s) defines a subspace inside $L(n, \alpha)$ for any α , because the subalgebra property is not affected by the form. With the previous examples in mind we can prove the following statement.

Lemma 5.2 Let B_0 and B_α be the bilinear forms on L(n, 0) and $L(n, \alpha)$ respectively. For a series r of type (n, s) we have

$$W(r)^{\perp_{B_{\alpha}}} = \frac{1}{x^n \alpha(x)} W(r)^{\perp_{B_0}} \subset L(n,\alpha).$$

$$(44)$$

Proof Set $u(x) := 1/(x^n \alpha(x))$. Write

$$r = \sum_{k \ge 0} \sum_{i=1}^{d} (sw_{k,i} + g_{k,i}) \otimes b_i(y^k, [y]^k) \text{ and } \overline{r} = \sum_{k \ge 0} \sum_{i=1}^{d} (sw_{k,i} + \overline{g_{k,i}}) \otimes b_i(y^k, [y]^k).$$

Then by Theorem 3.6 and definition Eq. (12) $B_{\alpha}(sw_{k,i} + g_{k,i}, u(sw_{\ell,j} + \overline{g_{\ell,j}})) = B_0(sw_{k,i} + g_{k,i}, sw_{\ell,j} + \overline{g_{\ell,j}}) = 0.$

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Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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