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CAUSAL SPARSE DOMINATION OF BEURLING MAXIMAL REGULARITY OPERATORS

By

TUOMAS HYTÖNEN* AND ANDREAS ROSÉN†

Abstract. We prove boundedness of Calderón–Zygmund operators acting in Banach function spaces on domains, defined by the L_1 Carleson functional and L_q ($1 < q < \infty$) Whitney averages. For such bounds to hold, we assume that the operator maps towards the boundary of the domain. We obtain the Carleson estimates by proving a pointwise domination of the operator, by sparse operators with a causal structure. The work is motivated by maximal regularity estimates for elliptic PDEs and is related to one-sided weighted estimates for singular integrals.

1 Introduction

We prove estimates of Calderón–Zygmund operators (CZO) acting on functions $f(t, x)$ defined on a half space $\mathbf{R}_+^{1+n} = \{x = (t, x) : t > 0, x \in \mathbf{R}^n\}$, in function spaces defined by $L_p(\mathbf{R}^n)$ norms of the classical Carleson and non-tangential maximal functionals $(Cf)(x)$ and $(Nf)(x)$ respectively. We recall their definitions in (3) and (2) below. For CZOs to be even locally bounded inside \mathbf{R}_+^{1+n} , it is necessary to modify C and N since these build on $L_1(\mathbf{R}_+^{1+n})$ and $L_\infty(\mathbf{R}_+^{1+n})$ norms respectively. Following Kenig and Pipher [19], Auscher and Axelsson [3], Hytönen and Rosén [17] and Huang [14], we use Whitney L_q averaging $W_q f(t, x)$ with $1 < q < \infty$, as defined in (4) below, and consider function spaces defined by norms

$$\|C(W_q f)\|_p \quad \text{and} \quad \|N(W_q f)\|_p,$$

which encode interior local L_q regularity, transversal L_1 or L_∞ regularity, and L_p regularity along the boundary \mathbf{R}^n . It was shown in [17] that the norms $\|N(W_q f)\|_{p'}$ and $\|C(W_q f)\|_p$ are dual, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$. In applications to elliptic partial differential equations (PDEs) as in [19, 3], the case $q=2$ is particularly important, since for gradients f of weak solutions to elliptic equations, in general we do not have pointwise interior bounds but only local L_2 estimates on Whitney regions.

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†Formerly Andreas Axelsson.

A general CZO on \mathbf{R}_+^{1+n}

$$Sf(\mathbf{x}) = \int_{\mathbf{R}_+^{1+n}} k(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y},$$

where $\mathbf{x} = (t, x)$, $\mathbf{y} = (s, y)$, fails to be bounded in any of the norms above, even with Whitney averaging. A key observation that we make is that for causal CZOs, bounds in some of the norms hold. By causal we mean that we consider either an upward mapping CZO, denoted S^+ , where $k = 0$ for $t < s$, or a downward mapping CZO, denoted S^- , where $k = 0$ for $t > s$. In our main Theorem 5.1, we prove that downward mapping CZOs S^- are bounded in Whitney modified Carleson norms $\|C(W_q f)\|_p$ and that, dually, upward mapping CZOs S^+ are bounded in Whitney modified non-tangential maximal norms $\|N(W_{q'} f)\|_{p'}$.

In Section 6, we discuss motivating examples: maximal regularity operators arising from integration of elliptic PDEs in the variable transversal to the boundary as in [3]. At the end of the present paper, we explain how our Theorem 5.1, in the presence of pointwise kernel bounds, sharpens the estimate of half of the maximal regularity operators in [3]. In the case of the Laplace equation in \mathbf{R}_+^2 , the maximal regularity operators S^\pm appearing are the two causal parts of the Beurling transform, in which case one of the key estimates from [3] independently has been proved by Astala and González [2]. We recall that the kernel $k(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})^{-2}$ of this Beurling transform is symmetric and as a consequence its causal parts S^\pm are L_2 bounded. Hence they are examples of rough CZOs: their kernels are discontinuous across the hypersurface $t = s$. With a slight abuse of notation we refer to such rough CZOs simply as CZOs below.

Our Carleson estimate for S^- uses the method of sparse domination of CZO by A. Lerner. In Section 4 we adapt the proof for general CZO from Lerner [22] and prove a domination of causal CZOs by certain causal sparse operators. Here we take a direction somewhat against the mainstream of recent activity, where the trend has been to establish sparse domination for ever more general classes of operators; in contrast, we deal with operators having additional structure (the causality), and the point is to preserve at least part of this structure in the dominating sparse operators as well. The one-dimensional special case, $n = 0$, of our Theorem 4.1 reads

$$(1) \quad |S^- f(t)| \lesssim \sum_{t \in (a, a+\ell) \in \mathcal{D}_f} \ell^{-1} \int_{a+\ell/2}^{a+2\ell} |f(s)| ds,$$

where the sum is over a sparse collection \mathcal{D}_f of dyadic intervals $(a, a + \ell)$, but the average is over the right half and a right neighbourhood of this interval.

There are a number of previous results dealing with causal (also known as one-sided) operators and related weight classes. Due to the intimate connections of sparse domination and weighted norm inequalities, which have been explored in the recent literature, we briefly comment on these works. Sawyer [28] found that the one-sided maximal operator on the line satisfies weighted estimates if and only if the weight satisfies a one-sided A_p condition. Later, Aimar et al. [1] showed that the one-sided A_p condition is also the right condition for the boundedness of one-sided singular integrals on the line. Versions of these results for operators on a half-line (case $n = 0$ of our setting in \mathbf{R}_+^{1+n}), including extensions to operator-valued kernels and applications to maximal regularity of Cauchy problems, are recently due to Chill and Król [11]. We refer to this paper for an extensive bibliography of other related works.

A sharp form of the one-sided maximal function estimates was found by Martín-Reyes and de la Torre [25]. The corresponding problem (a “one-sided A_2 conjecture”) for one-sided singular integrals remains open, but the analogous result for one-sided martingale transforms has been achieved in Chen, Han and Lacey [10]. At the time of writing, the established methods of reducing continuous singular integrals to their dyadic models (including [20, 21, 22, 23, 24] in the case of regular CZOs and [9, 13] for some rough extensions, to name but a selection of the extensive literature) are not available in a form that would respect the one-sided structure. In the present work we obtain a result that may be seen as a version of sparse domination for one-sided operators. While this version serves our present purposes, it is not strong enough to make progress on the mentioned weighted questions. The problem is that our dominating sparse operators are not strictly one-sided (for that, the integral in (1) should be over $(a + \ell, a + 2\ell)$ only) and cannot possibly be estimated in terms of one-sided A_p weights.

As mentioned above, one-sided weighted estimates for vector-valued singular integrals have been applied to maximal regularity operators for Cauchy problems in [11]. However, our Carleson estimates cannot be viewed as such weighted estimates, but rather correspond to the end-point norms $r = 1$ and $r = \infty$ in the scale of L_r tent spaces from Coifman, Meyer and Stein [12]. Estimates for maximal regularity operators in tent spaces based on the L_2 area functional are in Auscher, Kriegler, Monniaux and Portal [5]. Tent space estimates of horizontally mapping, that is, acting in the \mathbf{R}^n variable, CZOs are in Auscher and Prisuelos-Arribas [6]. In Section 2, we include counterexamples that show that for non-causal \mathbf{R}_+^{1+n} CZOs, as well as horizontally mapping \mathbf{R}^n CZOs, the Carleson estimates considered in this paper fail in general.

2 CZOs on non-tangential and Carleson spaces

We denote cubes on the boundary \mathbf{R}^n by Q, R, \dots , and cubes in the half-space \mathbf{R}_+^{1+n} by boldface $\mathbf{Q}, \mathbf{R}, \dots$. We denote n or $n+1$ dimensional measure by $|\cdot|$ depending on the dimension of the cube, and we denote by cQ , $c > 0$, the cube with the same center as Q and sidelength $\ell(cQ) = c\ell(Q)$. For $\mathbf{Q} \subset \mathbf{R}_+^{1+n}$ we form $c\mathbf{Q}$ in \mathbf{R}_+^{1+n} as above, and define $c\mathbf{Q} \subset \mathbf{R}_+^{1+n}$ as the intersection with \mathbf{R}_+^{1+n} , which may not be a cube if \mathbf{Q} is near \mathbf{R}^n . A Carleson cube in \mathbf{R}_+^{1+n} is a cube of the form

$$\mathbf{Q} = \mathbf{Q}^{\text{ca}} = (0, \ell(Q)) \times Q,$$

where Q is a cube in \mathbf{R}^n with sidelength $\ell(Q)$. The corresponding Whitney region is the top half

$$Q^w = (\ell(Q)/2, \ell(Q)) \times Q$$

of \mathbf{Q}^{ca} , and is the union of 2^n cubes in \mathbf{R}_+^{1+n} of sidelength $\ell(Q)/2$. We denote points in \mathbf{R}^n by x, y, \dots , and points in \mathbf{R}_+^{1+n} by boldface $\mathbf{x} = (t, x), \mathbf{y} = (s, y), \dots$. Indicator functions of cubes and more general sets E are denoted by 1_E . Subsets, not necessarily strict, are denoted $A \subset B$.

In particular we use dyadic cubes, and let $\mathcal{D} = \bigcup_{j \in \mathbf{Z}} \mathcal{D}_j$ denote a system of dyadic cubes in \mathbf{R}^n , with \mathcal{D}_j being the cubes of sidelength $\ell(Q) = 2^{-j}$, such that the dyadic cubes in \mathcal{D} form a connected tree under inclusion. Let $\mathcal{D} = \bigcup_{j \in \mathbf{Z}} \mathcal{D}_j$ denote the associated dyadic system for \mathbf{R}_+^{1+n} , where \mathcal{D}_j consists of dyadic cubes of the form $\mathbf{Q} = (k2^{-j}, (k+1)2^{-j}) \times Q$, $Q \in \mathcal{D}_j$, $k = 0, 1, 2, \dots$. We note that also \mathcal{D} form a connected tree under inclusion where the dyadic Carleson cubes, the cubes touching \mathbf{R}^n , will play the special role.

We consider CZOs acting on functions defined on the half space \mathbf{R}_+^{1+n} , $n \geq 1$, and belonging to endpoint spaces in the scale of tent spaces from [12], whose definitions use the non-tangential maximal and Carleson functionals

$$(2) \quad Nf(x) = \text{ess sup}_{|y-x| < at} |f(t, y)|,$$

$$(3) \quad Cf(x) = \sup_{Q \ni x} |Q|^{-1} \iint_{Q^{\text{ca}}} |f(t, x)| dt dx,$$

where the second supremum is over all cubes $Q \subset \mathbf{R}^n$ containing x , and the Whitney averaging operator

$$(4) \quad W_q f(t, x) = |Q^w|^{-1/q} \|f\|_{L_q(Q^w)},$$

where $Q \subset \mathbf{R}^n$ is the cube with center x and sidelength t . More precisely, we consider the norms $\|N(W_q f)\|_{L_{p'}(\mathbf{R}^n)}$, introduced in [19], and predual norms $\|C(W_q f)\|_{L_p(\mathbf{R}^n)}$

considered in [17]. We exclude $p = 1$, where the former norms are ∞ except for $f = 0$, while the latter norms are $\|W_q f\|_\infty$. To avoid technicalities we also exclude $p = \infty$, although this endpoint case sometimes is of interest and some of our results hold for $p = \infty$, mutatis mutandis. We recall from [17, 18] that up to constants these two scales of norms are independent of the choice of aperture $\alpha > 0$ and the precise shape of the Whitney regions and Carleson cubes. Moreover, for the dyadic functionals

$$\begin{aligned} N_{\mathcal{D}} f(x) &= \sup_{Q \in \mathcal{D}, Q \ni x} f_Q, \\ C_{\mathcal{D}} f(x) &= \sup_{Q \in \mathcal{D}, Q \ni x} |Q|^{-1} \sum_{R \in \mathcal{D}, R \subset Q} |R^w| f_R, \end{aligned}$$

acting on sequences $(f_Q)_{Q \in \mathcal{D}}$, and dyadic Whitney L_q averages

$$(W_{\mathcal{D},q} f)_Q = |Q^w|^{-1/q} \|f\|_{L_q(Q^w)}, \quad Q \in \mathcal{D},$$

we have

$$\|N_{\mathcal{D}} W_{\mathcal{D},q} f\|_{p'} \approx \|N(W_q f)\|_{p'} \quad \text{and} \quad \|C_{\mathcal{D}} W_{\mathcal{D},q} f\|_p \approx \|C(W_q f)\|_p$$

for $1 < p, q < \infty$. Finally, we recall that if $1 < p < \infty$, then $\|Cf\|_p \approx \|Af\|_p$, where

$$Af(x) = \iint_{|y-x| < \alpha t} |f(t, y)| t^{-n} dt dx,$$

but the estimate \gtrsim breaks down at $p = \infty$, and \lesssim breaks down at $p = 1$.

We are interested in boundedness of CZOs in the norms $\|C(W_q f)\|_p$ or equivalently, by duality, in the norms $\|N(W_q f)\|_{p'}$. As discussed in the Introduction, it is necessary to require $1 < q < \infty$, and main applications concern $q = 2$. For the Carleson functional, which builds on L_1 integrals, it is natural to make use of the recent technology of sparse domination of CZOs. We recall the following estimate of A. Lerner [22]. Assume given a standard CZO S on \mathbf{R}_+^{1+n} and a function $f \in L_1(\mathbf{R}_+^{1+n})$ with bounded support. Then there exists a collection $\mathcal{D}_f \subset \mathcal{D}$ of cubes, which is η -sparse, for some $0 < \eta < 1$ independent of f , such that

$$|Sf(x)| \lesssim \sum_{Q \in \mathcal{D}_f, Q \ni x} \int_{3Q} |f(y)| dy$$

holds pointwise almost everywhere. Here η -sparse means that there exists subsets $E_Q \subset Q$, $Q \in \mathcal{D}_f$, such that $|E_Q| \geq \eta|Q|$ for all $Q \in \mathcal{D}_f$, and $E_Q \cap E_R = \emptyset$ if $Q \neq R$.

The method of sparse domination has proven very successful in proving optimal bounds for CZOs on weighted L_p spaces. We apply in this paper the technique

to the scales of Banach function spaces described above. Since N and C involve an L_∞ and L_1 norm respectively however, it is not surprising that in general we have the following counterexamples.

Example 2.1. A sparse operator, that is a sublinear operator of the form

$$\widehat{S}f(\mathbf{x}) = \sum_{Q \in \widehat{\mathcal{D}}, Q \ni \mathbf{x}} \int_Q |f(\mathbf{y})| d\mathbf{y},$$

with a fixed η -sparse collection $\widehat{\mathcal{D}}$ independent of f , is not in general bounded on $\|C(W_q f)\|_p$ for any p, q . An example is as follows. Let $n = 1$ and fix a dyadic Carleson cube $Q_0 = Q_0^{\text{ca}}$ with $\ell(Q_0) = 1$. Consider

$$(5) \quad f_N = 2^N 1_{(0, 2^{-N}) \times Q_0}$$

and let $\widehat{\mathcal{D}}$ be the collection of all dyadic Carleson cubes contained in Q_0 . Then $W_{\mathcal{D}, q} f_N = f_N$ since f_N is constant on Whitney regions, $C_{\mathcal{D}} f_N = 1$ on Q_0 since $\sup_{x \in Q_0} \int_0^1 f_N dt = 1$, and $\|C_{\mathcal{D}} f_N\|_p \lesssim 1$ for any fixed $p > 1$. However, for $\mathbf{x} = (t, x) \in Q^w$ with $Q \subset Q_0$, $\ell(Q) = 2^{-k}$, $0 \leq k \leq N$, we have

$$\widehat{S}f_N(\mathbf{x}) = \sum_{j=0}^k (2^N 2^{-N} 2^{-j}) / (2^{-j})^2 = 2^{k+1} - 1 \approx 1/t.$$

Hence $C_{\mathcal{D}}(\widehat{S}f_N) \gtrsim \int_{2^{-N}}^1 \frac{dt}{t} \approx N$ at each point in Q_0 . It follows that \widehat{S} fails to be bounded in the norm $\|C(W_q f)\|_p$ for any $1 < p, q < \infty$, since $q \rightarrow W_q$ is increasing.

Example 2.2. Although domination of CZOs by sparse operators is considered to be quite sharp, Example 2.1 does not disprove that CZOs are bounded in these Carleson norms. And indeed, by direct calculations one can show that the Beurling transform on the upper half plane

$$(6) \quad Sf(z) = \text{p.v.} \frac{-1}{\pi} \int_{\text{Im } w > 0} \frac{f(w)}{(w - z)^2} |dw|$$

maps f_N from (5) boundedly in the Carleson norms. To find a counterexample for the Beurling transform, it is convenient to consider a weak limit

$$(7) \quad f(t, x) = g(x) \delta_0(t),$$

of functions like f_N , to be modulated by a function $g(x)$ supported on $Q_0 = (0, 1)$ that remains to be chosen. In this limit we have

$$Sf(z) = \text{p.v.} \frac{-1}{\pi} \int_0^1 \frac{g(x)}{(x - z)^2} dx,$$

and for real-valued g we note from Cauchy–Riemann’s equations that

$$|Sf(z)| = |\nabla u(z)|,$$

where u is the Poisson extension of g . As discussed in [18, Intro.], by constructing g via a lacunary Fourier series, it is known that the bound

$$(8) \quad \|C(\nabla u)\|_p \lesssim \|g\|_p$$

cannot hold uniformly for g , for any fixed $1 < p < \infty$. Replacing δ_0 in (7) by $2^N 1_{(0, 2^{-N})}$, it can be shown that S fails to be bounded in the norm $\|C(W_q f)\|_p$ for any $1 < p, q < \infty$. The technical details of this counterexample are found in the Appendix of this paper.

Example 2.3. We next demonstrate that \mathbf{R}^n CZOs acting horizontally, that is pointwise in t , in general are not bounded either in the norms $\|N(W_q f)\|_{p'}$ or $\|C(W_q f)\|_p$. A concrete counterexample is the following. Let

$$Hf(t, x) = \frac{1}{\pi} \text{p.v.} \int_{\mathbf{R}} \frac{f(t, y)}{x - y} dy$$

be the Hilbert transform acting horizontally in \mathbf{R}_+^2 , and consider the function

$$f_N(t, x) = \sum_{j=0}^N \sum_{k=2^j}^{2^{j+1}-1} 2^k 1_{(2^{-k}, 2^{1-k})}(t) 1_{(k2^{-j}-1, (k+1)2^{-j}-1)}(x)$$

with support in $[0, 1]^2$. The terms in this double sum are supported on thin rectangles, with integrals equal to their lengths in the x -direction. Consider a Carleson square \mathcal{Q} with sidelength 2^{-m} . The terms for a given $0 \leq j \leq N$ will have in total at most integral 2^{-m} inside \mathcal{Q} , so $C(W_{\mathcal{D}, \infty} f_N) \lesssim N$ on $[0, 1]$, since the thin rectangles are unions of Whitney regions. We conclude that $\|C(W_{\infty} f_N)\|_p \lesssim N$ for each $p > 1$, since f_N is supported on $[0, 1]^2$. However, an explicit computation reveals that

$$\int_0^1 |Hf_N(t, x)| dx \approx \int_0^1 2^k \ln \left(1 + \frac{2^{-j}}{x} \right) dx \approx 2^{k-j}(j+1)$$

for $t \in (2^{-k}, 2^{1-k})$, $2^j \leq k < 2^{j+1}$, $0 \leq j \leq N$. Therefore

$$\int_{[0, 1]^2} |Hf_N| dt dx \approx \sum_{j=0}^N \sum_{k=2^j}^{2^{j+1}-1} 2^{-j}(j+1) \approx \sum_{j=0}^N (j+1) \approx N^2,$$

so $C(Hf_N) \gtrsim N^2$ on $(0, 1)$. It follows that H fails to be bounded in the norm $\|C(W_q f)\|_p$ for any $1 < p, q < \infty$, since $q \rightarrow W_q$ is increasing.

3 Causal Calderón–Zygmund operators

We consider a Calderón–Zygmund (CZ) kernel $k(\mathbf{x}, \mathbf{y})$ on \mathbf{R}_+^{1+n} . More precisely, we assume kernel bounds

$$(9) \quad |k(\mathbf{x}, \mathbf{y})| \lesssim |\mathbf{x} - \mathbf{y}|^{-(n+1)}$$

and regularity

$$(10) \quad \max(|k(\mathbf{x}, \mathbf{y} + \mathbf{t}) - k(\mathbf{x}, \mathbf{y})|, |k(\mathbf{x} + \mathbf{t}, \mathbf{y}) - k(\mathbf{x}, \mathbf{y})|) \lesssim |\mathbf{t}|^\gamma / |\mathbf{x} - \mathbf{y}|^{n+1+\gamma},$$

for all $|\mathbf{t}| \leq |\mathbf{x} - \mathbf{y}|/2$ and a fixed $0 < \gamma \leq 1$.

We consider a linear operator S^+ which is bounded $L_2(\mathbf{R}_+^{1+n}) \rightarrow L_2(\mathbf{R}_+^{1+n})$ and has rough CZ kernel

$$k^+((t, x), (s, y)) = \begin{cases} k((t, x), (s, y)), & t > s, \\ 0, & t < s. \end{cases}$$

We also consider a linear operator S^- which is bounded $L_2(\mathbf{R}_+^{1+n}) \rightarrow L_2(\mathbf{R}_+^{1+n})$ and has rough CZ kernel

$$k^-((t, x), (s, y)) = \begin{cases} 0, & t > s, \\ k((t, x), (s, y)), & t < s. \end{cases}$$

(Here k is assumed to satisfy (9) and (10) in all \mathbf{R}_+^{1+n} .) Thus S^\pm are (rough) CZOs, where S^+ is upward mapping away from \mathbf{R}^n and S^- is downward mapping towards \mathbf{R}^n . We refer to S^\pm as **causal operators**. They are simple examples of singular integral with rough kernels, in that $k^\pm((t, x), (s, y))$ may be discontinuous on the hyperplane $t = s$ in $\mathbf{R}^{2(1+n)}$, with a simple jump discontinuity when $x \neq y$ and $t = s$.

Lemma 3.1. *The kernels k^\pm satisfy the Hörmander regularity condition*

$$\int_{(3Q)^c} (|k^\pm(\mathbf{x}, \mathbf{y}_1) - k^\pm(\mathbf{x}, \mathbf{y}_2)| + |k^\pm(\mathbf{y}_1, \mathbf{x}) - k^\pm(\mathbf{y}_2, \mathbf{x})|) d\mathbf{x} \lesssim 1,$$

uniformly for all cubes $Q \subset \mathbf{R}_+^{1+n}$ and $\mathbf{y}_1, \mathbf{y}_2 \in Q$.

Proof. For $Q = (a, b) \times Q$, this follows from using (10) for $t < a$ and $t > b$, and using (9) for $a < t < b$. \square

Denote by Mf the standard Hardy–Littlewood maximal function of f . As in [22], we also require a weak L_1 estimate of Lerner’s maximal singular integral

$$M_{S^\pm} f(\mathbf{x}) = \sup_{\mathbf{Q} \ni \mathbf{x}} \|S^\pm(1_{(3\mathbf{Q})^c} f)\|_{L_\infty(\mathbf{Q})}, \quad \mathbf{x} \in \mathbf{R}_+^{1+n},$$

with supremum over dyadic cubes $\mathbf{Q} \subset \mathbf{R}_+^{1+n}$, into which S^\pm maps from the complement of the enlarged (non-dyadic) cube $3\mathbf{Q}$.

Proposition 3.2. *The causal CZOs S^\pm and the maximal singular integrals M_{S^\pm} are all bounded from $L_1(\mathbf{R}_+^{1+n})$ to $L_{1,\infty}(\mathbf{R}_+^{1+n})$.*

Proof. (1) The weak L_1 estimate for S^\pm itself follows from the standard proof employing the Calderón–Zygmund decomposition, since this only requires the Hörmander regularity estimate from Lemma 3.1.

(2) Following the standard proof of Cotlar’s lemma, see for example [26, Sec. 7.7], we write

$$f_1 = 1_{3\mathbf{Q}} f \quad \text{and} \quad f_2 = 1_{(3\mathbf{Q})^c} f,$$

for a cube \mathbf{Q} with center \mathbf{x}_0 . We estimate $S^\pm(1_{(3\mathbf{Q})^c} f) = S^\pm f_2$ at each fixed $\mathbf{x}_1 \in \mathbf{Q}$, by comparing it to the value at a variable point $\mathbf{x} \in \mathbf{Q}$, writing

$$S^\pm f_2(\mathbf{x}_1) = (S^\pm f_2(\mathbf{x}_1) - S^\pm f_2(\mathbf{x})) + S^\pm f(\mathbf{x}) - S^\pm f_1(\mathbf{x}) =: I + II + III.$$

Raising the terms to power $1/p$, $p > 1$, and taking the average over \mathbf{Q} , terms II and III are estimated in the usual way: $\int_{\mathbf{Q}} |S^\pm f|^{1/p} d\mathbf{x} \lesssim M(|S^\pm f|^{1/p})(\mathbf{x}_0)$, and

$$\int_{\mathbf{Q}} |S^\pm f_1|^{1/p} d\mathbf{x} \lesssim |\mathbf{Q}|^{-1/p} \|S^\pm f_1\|_{L_{1,\infty}}^{1/p} \lesssim |\mathbf{Q}|^{-1/p} \|f_1\|_{L_1}^{1/p} \lesssim (Mf(\mathbf{x}_0))^{1/p}$$

by Kolmogorov’s inequality and weak L_1 estimate (1) above.

For I , we define

$$\Phi(\mathbf{z}) = \Phi(u, z) = \begin{cases} (1 + |\mathbf{z}|)^{-(1+n+\gamma)}, & u \in (-\infty, -1) \cup (1, \infty), \\ (1 + |\mathbf{z}|)^{-(1+n)}, & u \in (-1, 1). \end{cases}$$

Using (10) and (9)

$$|(S^\pm f_2(\mathbf{x}_1) - S^\pm f_2(\mathbf{x}))| \lesssim M_\Phi f(\mathbf{x}_0),$$

where M_Φ is the maximal operator

$$M_\Phi f = \sup_{t>0} (|f| * t^{-(n+1)} \Phi(\cdot/t)).$$

By [29, Sec. II.4, Prop. 2], M_Φ is weak-type $(1,1)$. This completes the proof, since we have

$$|M_{S^\pm} f(\mathbf{x}_0)| \lesssim M_\Phi f(\mathbf{x}_0) + M(|S^\pm f|^{1/p})(\mathbf{x}_0)^p + Mf(\mathbf{x}_0)$$

and M is bounded on $L_{p,\infty}$ and weak-type $(1,1)$. □

4 Causal sparse domination of S^-

To state our causal sparse domination, we need the following subsets of the neighbourhood $3Q$. For $Q = (a, a + \ell) \times Q \in \mathcal{D}$, $Q \in \mathcal{D}$, we define the upper and lower halves of Q

$$\begin{aligned} Q^u &= (a + \ell/2, a + \ell) \times Q, \\ Q^l &= (a, a + \ell/2) \times Q, \end{aligned}$$

the parts of $(a, a + 2\ell) \times 3Q$ above and below $t = a + \ell/2$

$$\begin{aligned} Q^T &= (a + \ell/2, a + 2\ell) \times 3Q, \\ Q^L &= (a, a + \ell/2) \times 3Q, \end{aligned}$$

and the part of $3Q \setminus \overline{Q^l}$ above $t = a$

$$Q^\square = ((a, a + 2\ell) \times 3Q) \setminus \overline{Q^l}.$$

For $\mathbf{x} = (t, x) \in Q$ we let

$$Q_x^\square = Q^\square \cap \{(s, y) : s > \min(a + \ell/2, t)\},$$

so that $Q^T \subset Q_x^\square \subset Q^\square$. The following sparse estimate is a causal adaptation of the estimate in [22]. We specifically note that the very argument of [22] seemed more amenable to this adaptation than either its predecessors or successors in the sparse domination literature.

Theorem 4.1. *Let S^- be a downward mapping causal Calderón–Zygmund operator as in Section 3. Let $f \in L_1(\mathbf{R}^{1+n})$ have bounded support. Then there exists a $1/4$ -sparse family of cubes $\mathcal{D}_f \subset \mathcal{D}$ such that*

$$|S^-f(\mathbf{x})| \lesssim \sum_{Q \in \mathcal{D}_f, Q \ni \mathbf{x}} \int_{Q_x^\square} |f(\mathbf{y})| d\mathbf{y}, \quad \text{for a.e. } \mathbf{x} \in \mathbf{R}_+^{1+n}.$$

We remark that an essential point in Theorem 4.1 is that the sum uses the same dyadic system as we start out with. If we allow ourselves to replace \mathcal{D} by a finite number of other dyadic systems, then the result is immediate from [22], but cannot be used to prove our Carleson bounds.

We also remark that the one-dimensional result, $n = 0$, is somewhat cleaner. In this case \mathcal{D} is the standard dyadic intervals and for $Q = (a, a + \ell)$, the average in the sparse sum is over $(a + \ell/2, a + 2\ell)$, independent of $\mathbf{x} \in Q$, as in (1). The reason why Q^T , which is the direct analogue of $(a + \ell/2, a + 2\ell)$ for $n \geq 1$, must be replaced by the larger set Q_x^\square in Theorem 4.1, is the term II_1 appearing in its proof.

The even larger set \mathcal{Q}^\square is too large, since it cannot control the term II_1 appearing in the later proof of Theorem 5.1: it is important that these sets do not touch $t = a$. However, we have found that Theorem 5.1 is valid when \mathcal{Q}_x^\square is enlarged to a set, to be denoted by \mathcal{Q}_x^\square , which goes down to the t coordinate of \mathbf{x} also inside \mathcal{Q} .

Proof. (1) Fix $\mathcal{Q} \in \mathcal{D}$ and assume that the closure of $3\mathcal{Q}$ contains the support of f . Write $f^T := 1_{\mathcal{Q}^c} f$ and define the set

$$E := \left\{ \mathbf{x} \in \mathbf{R}_+^{1+n} : |S^- f^T(\mathbf{x})| > c \int_{\mathcal{Q}^c} |f| d\mathbf{y} \right\} \\ \cup \left\{ \mathbf{x} \in \mathbf{R}_+^{1+n} : |M_{S^- f^T}(\mathbf{x})| > c \int_{\mathcal{Q}^c} |f| d\mathbf{y} \right\}.$$

Let $\mathbf{R}_1, \dots, \mathbf{R}_{2^n}$ denote all the children of \mathcal{Q} (maximal subcubes strictly contained in \mathcal{Q}) contained in \mathcal{Q}^l , and let $\mathcal{Q}_1, \mathcal{Q}_2, \dots$ be an enumeration of all the maximal subcubes of \mathcal{Q}^u such that

$$(11) \quad |\mathcal{Q}_j \cap E| > \alpha |\mathcal{Q}_j|.$$

Here $c > 0$ and $\alpha \in (0, 1)$ are parameters to be fixed below. We obtain a family $\{\mathbf{R}_i, \mathcal{Q}_j\}$ of disjoint subcubes of \mathcal{Q} that give us a disjoint union

$$\mathcal{Q} = \mathcal{Q}^l \cup \mathcal{Q}^u = \left(\bigcup_i \mathbf{R}_i \right) \cup \left(\bigcup_j \mathcal{Q}_j \right) \cup (\mathcal{Q}^u \setminus \bigcup_j \mathcal{Q}_j).$$

Hence, splitting the indicator $1_{\mathcal{Q}}$ outside S^- in the first step and the function f inside S^- in the second one,

$$1_{\mathcal{Q}} S^- f = \sum_i 1_{\mathbf{R}_i} S^- f + \sum_j 1_{\mathcal{Q}_j} S^- f + 1_{\mathcal{Q}^u \setminus \bigcup_j \mathcal{Q}_j} S^- f \\ = \sum_i 1_{\mathbf{R}_i} S^- (1_{3\mathbf{R}_i} f) + \sum_j 1_{\mathcal{Q}_j} S^- (1_{3\mathcal{Q}_j} f) + \\ + \sum_i 1_{\mathbf{R}_i} S^- (1_{3\mathcal{Q} \setminus 3\mathbf{R}_i} f) + 1_{\mathcal{Q}^u \setminus \bigcup_j \mathcal{Q}_j} S^- f^T + \sum_j 1_{\mathcal{Q}_j} S^- (1_{3\mathcal{Q} \setminus 3\mathcal{Q}_j} f^T) \\ =: I_1 + I_2 + II_1 + II_2 + II_3.$$

Using causality for S^- , we have replaced f by f^T in terms II_2 and II_3 .

(2) We next show that terms II are bounded by $\int_{\mathcal{Q}_x^\square} |f| d\mathbf{y}$, for almost every $\mathbf{x} \in \mathcal{Q}$. For II_1 , this follows from (9) and the downward mapping property of S^- since $|\mathbf{x} - \mathbf{y}|^{-(1+n)} \lesssim |\mathcal{Q}_x^\square|^{-1}$ for all $\mathbf{y} \in (3\mathcal{Q}) \setminus (3\mathbf{R}_i)$ and $\mathbf{x} \in \mathbf{R}_i$. For II_2 , it follows from Lebesgue's differentiation theorem that

$$E \cap \mathcal{Q}^u \subset \bigcup_j \mathcal{Q}_j$$

modulo a set of measure zero. The estimate of II_2 then follows from the definition of E .

For II_3 , consider one of the stopping cubes \mathbf{Q}_j . By definition, we have $|\mathbf{Q}_j \cap E| > \alpha |\mathbf{Q}_j|$. We require an upper estimate of the measure of E . It follows from Proposition 3.2 that

$$|E| \lesssim \left(c \int_{\mathbf{Q}^T} |f| d\mathbf{y} \right)^{-1} \int_{\mathbf{R}_+^{1+n}} |f^T| d\mathbf{y} \approx |\mathbf{Q}|/c.$$

Moreover, if \mathbf{Q}_j^p denotes the dyadic parent of \mathbf{Q}_j (the minimal dyadic cube such that $\mathbf{Q}_j^p \supsetneq \mathbf{Q}_j$), then by maximality of \mathbf{Q}_j , its parent does not satisfy (11) and we have

$$|\mathbf{Q}_j \cap E| \leq |\mathbf{Q}_j^p \cap E| \leq \alpha |\mathbf{Q}_j^p| = \alpha 2^{1+n} |\mathbf{Q}_j|.$$

Let $\alpha = 2^{-2-n}$. Then in particular there exists $\mathbf{x} \in \mathbf{Q}_j \setminus E$ since

$$|\mathbf{Q}_j \setminus E| \geq |\mathbf{Q}_j|/2 > 0$$

with this choice of α . From the definition of E it now follows that

$$|S^-(1_{(3\mathbf{Q}) \setminus (3\mathbf{Q}_j)} f^T)| \leq M_{S^-} f^T(\mathbf{x}) \lesssim \int_{\mathbf{Q}^T} |f| d\mathbf{y}$$

on almost all of \mathbf{Q}_j . This proves the estimate of II_3 .

(3) We have shown that for f supported on $3\mathbf{Q}$, we have

$$|1_{\mathbf{Q}} S^- f - \sum_i 1_{\mathbf{R}_i} S^-(1_{3\mathbf{R}_i} f) - \sum_j 1_{\mathbf{Q}_j} S^-(1_{3\mathbf{Q}_j} f)| \lesssim \int_{\mathbf{Q}_x^\cap} |f| d\mathbf{y}$$

at almost all $\mathbf{x} \in \mathbf{Q}$. To conclude the proof, we iterate this estimate for all the subcubes \mathbf{R}_i and \mathbf{Q}_j . To this end, we note that

$$\sum_i |\mathbf{R}_i| = |\mathbf{Q}|/2$$

and, since $\bigcup \mathbf{Q}_j \subset \{M(1_E) > \alpha\}$, that

$$\sum_j |\mathbf{Q}_j| \lesssim \alpha^{-1} \int 1_E d\mathbf{y} = |E|/\alpha \lesssim |\mathbf{Q}|/(c\alpha).$$

Fix c large enough so that $\sum_i |\mathbf{R}_i| + \sum_j |\mathbf{Q}_j| \leq 3|\mathbf{Q}|/4$. Define

$$E_{\mathbf{Q}} := \mathbf{Q} \setminus \left(\bigcup_i \mathbf{R}_i \cup \bigcup_j \mathbf{Q}_j \right),$$

so that $|E_{\mathbf{Q}}| \geq |\mathbf{Q}|/4$.

Finally consider a disjoint union

$$\mathbf{R}_+^{1+n} = \bigcup_k \mathcal{Q}^k,$$

modulo zero sets, where $\mathcal{Q}^k \in \mathcal{D}$ are constructed as follows. Since $\text{supp}(f)$ is bounded and \mathcal{D} is connected, we can choose $\mathcal{Q}^1 \in \mathcal{D}$ containing $\text{supp}(f)$. Then let $\mathbf{P}^1 = \mathcal{Q}^1$ and recursively define \mathbf{P}^{j+1} to be the dyadic parent of \mathbf{P}^j , that is, the smallest dyadic cube such that $\mathbf{P}^{j+1} \supsetneq \mathbf{P}^j$. The siblings of \mathbf{P}^j are the other dyadic subcubes of \mathbf{P}^{j+1} of the same size as \mathbf{P}^j . Now choose $\mathcal{Q}^2, \mathcal{Q}^3, \dots$ to be an ordering of all the siblings of all $\mathbf{P}^j, j = 1, 2, \dots$. Then $\mathcal{Q}^1 \subset 3\mathcal{Q}^k$ for all k , so that

$$S^-f = \sum_k 1_{\mathcal{Q}^k} S^-f = \sum_k 1_{\mathcal{Q}^k} S^-(1_{3\mathcal{Q}^k} f),$$

and the estimates above apply to each $\mathcal{Q} = \mathcal{Q}^k$. We define the family of dyadic cubes \mathcal{D}_f to be $\mathcal{Q}^1, \mathcal{Q}^2, \dots$ along with, for each of \mathcal{Q}^k , all generations of stopping cubes \mathcal{Q}_j and subcubes R_i starting from $\mathcal{Q} = \mathcal{Q}^k$, constructed as in (1) above. It follows that \mathcal{D}_f is $1/4$ sparse and that we have the stated sparse bound of S^-f . \square

5 Bounds for causal CZO

The Carleson bounds established in this section apply to slightly larger causal sparse operators

$$(12) \quad |\hat{S}f(\mathbf{x})| \lesssim \sum_{\mathcal{Q} \in \widehat{\mathcal{D}}, \mathcal{Q} \ni \mathbf{x}} \int_{\mathcal{Q}^\square} |f(\mathbf{y})| d\mathbf{y}, \quad \mathbf{x} \in \mathbf{R}_+^{1+n},$$

where we have replaced the region \mathcal{Q}_x^\square appearing in the sparse operator in Theorem 4.1, by the slightly larger region

$$\mathcal{Q}_x^\square = (\min(a + \ell/2, t), a + 2\ell) \times 3\mathcal{Q},$$

for $\mathbf{x} = (t, x) \in \mathcal{Q} = (a, a + \ell) \times \mathcal{Q}$. We also write

$$\mathcal{Q}^\square = (a, a + 2\ell) \times 3\mathcal{Q}$$

for the union of \mathcal{Q}^T and \mathcal{Q}^L , modulo a set of measure zero. The following theorem is the main result of this paper. Its proof uses regions \mathbf{R}^\square for dyadic cubes $\mathbf{R} \subset \mathcal{Q}^w$, where \mathcal{Q}^w is a dyadic Whitney region. We define enlarged auxiliary Whitney regions

$$\tilde{\mathcal{Q}}^w = (\ell(\mathcal{Q})/2, 3\ell(\mathcal{Q})/2) \times (2\mathcal{Q}),$$

so that $\mathbf{R}^\square \subset \tilde{\mathcal{Q}}^w$ whenever $\mathbf{R} \subset \mathcal{Q}^w$.

Theorem 5.1. *Let S^+ and S^- be causal Calderón–Zygmund operators as in Section 3 and $1 < p, q < \infty$. Then we have the estimates*

$$(13) \quad \|CW_q(S^-f)\|_p \lesssim \|CW_q(f)\|_p,$$

$$(14) \quad \|NW_{q'}(S^+f)\|_{p'} \lesssim \|NW_{q'}(f)\|_{p'}.$$

More precisely, the estimate

$$(15) \quad C_{\mathcal{D}}W_{\mathcal{D},q}(\hat{S}f) \lesssim C_{\mathcal{D}}\tilde{W}_{\mathcal{D},q}(f) + C(f)$$

holds pointwise on \mathbf{R}^n for the causal sparse operator \hat{S} from (12), for any η -sparse collection $\hat{\mathcal{D}} \subset \mathcal{D}$ with $0 < \eta < 1$. Here $\tilde{W}_{\mathcal{D},q}(f)$ denotes the Whitney L_q average over \tilde{Q}^w .

We remark that interior regularity $1 < q < \infty$ is necessary for boundedness of CZOs, and is used to ensure boundedness of the Hardy–Littlewood maximal operator in the proof below, while $p > 1$ is necessary for the Carleson and non-tangential maximal norms to be non-degenerate. The restriction $p < \infty$ is less essential, but is made so that the following lemma holds.

Lemma 5.2. *Fix $1 < p, q < \infty$ and define Banach spaces*

$$Z = \{f \in L_q^{\text{loc}}(\mathbf{R}_+^{1+n}) : \|CW_q(f)\|_p < \infty\} \quad \text{and} \\ X = \{f \in L_{q'}^{\text{loc}}(\mathbf{R}_+^{1+n}) : \|NW_{q'}(f)\|_{p'} < \infty\}.$$

Let Z_0 denote the subspace of functions $f \in L_q(\mathbf{R}_+^{1+n}) \cap Z$ with bounded support, and let X_0 denote the subspace of functions $f \in L_{q'}(\mathbf{R}_+^{1+n}) \cap X$ with bounded support.

Then Z_0 is dense in Z , in the norm $\|CW_q(\cdot)\|_p$, and X_0 is dense in X , in the norm $\|NW_{q'}(\cdot)\|_{p'}$. Moreover, Z is non-reflexive and X equals the dual space Z^ , using the standard $L_2(\mathbf{R}_+^{1+n})$ pairing.*

Proof. The density of Z_0 in Z follows from [17, Lem. 2.5], which proves that even compactly supported L_q functions are dense in Z . Also the non-reflexivity of Z and that $X = Z^*$ was shown in [17, Thm. 3.2].

To see the density of X_0 in X , let $\epsilon > 0$ and $f \in X$ so that $\|N_{\mathcal{D}}W_{\mathcal{D},q'}f\|_{p'} < \infty$. Recall that equivalences of norms allow us to use the dyadic versions of the norms. Since $p > 1$ and \mathcal{D} is connected, there exists $Q_0 \in \mathcal{D}$ such that

$$\int_{\mathbf{R}^n \setminus Q_0} |N_{\mathcal{D}}W_{\mathcal{D},q'}f|^{p'} d\mathbf{x} \leq \epsilon^{p'}.$$

Let \mathcal{D}_f be the set of maximal cubes $Q \subset Q_0$ such that

$$(W_{\mathcal{D},q'}f)_Q > 1/\epsilon.$$

Define functions

$$f_0 = \sum_{Q \in \mathcal{D}_f} f 1_{Q^{\text{ca}}} \quad \text{and} \quad f_1 = f 1_{\mathbf{R}_+^{1+n} \setminus Q_0^{\text{ca}}},$$

and let $f_2 = f - f_0 - f_1$. We have $N_{\mathcal{D}} W_{\mathcal{D}, q'} f_0 = 1_E N_{\mathcal{D}} W_{\mathcal{D}, q'} (f 1_{Q_0^{\text{ca}}})$, where $E = \{x \in \mathbf{R}^n : N_{\mathcal{D}} W_{\mathcal{D}, q'} (f 1_{Q_0^{\text{ca}}}) > \epsilon^{-1}\}$. Since $|E| \rightarrow 0$, it follows from dominated convergence that $\|f_0\|_X \rightarrow 0$ as $\epsilon \rightarrow 0$.

To estimate f_1 , let Q_1 denote the dyadic parent of Q_0 . Then

$$\sup_{Q_0} N_{\mathcal{D}} W_{\mathcal{D}, q'} f_1 \leq \inf_{Q_1 \setminus Q_0} N_{\mathcal{D}} W_{\mathcal{D}, q'} f,$$

and we conclude that $\|f_1\|_X \lesssim \epsilon$ since $N_{\mathcal{D}} W_{\mathcal{D}, q'} f_1 = N_{\mathcal{D}} W_{\mathcal{D}, q'} f$ on $\mathbf{R}^n \setminus Q_0$. Since $f = f_0 + f_1 + f_2$ where $\text{supp } f_2 \subset Q_0$ and

$$\int_{\mathbf{R}_+^{1+n}} |f_2|^{q'} dx = \sum_{Q \subset Q_0, Q \not\subset \bigcup_f \mathcal{D}_f} \int_{Q^w} |f|^{q'} dx \leq \sum_{Q \subset Q_0, Q \not\subset \bigcup_f \mathcal{D}_f} \epsilon^{-q'} |Q^w| \leq \epsilon^{-q'} |Q_0^{\text{ca}}| < \infty,$$

it follows that $f_2 \in X_0$. This shows that X_0 is dense in X . \square

Proof of Theorem 5.1. We first show that it suffices to prove the estimate (15). Indeed, combining this and Theorem 4.1 yields the bound for S^- on Z_0 , using the notation from Lemma 5.2 and equivalences of norms between the different versions of Carleson functionals and Whitney averages from [17, Sec. 3]. By Lemma 5.2, we have a unique extension of S^- by continuity to all Z .

Next consider S^+ on $L_{q'}(\mathbf{R}_+^{1+n}) \supset X_0$, with adjoint $(S^+)^*$ on $L_q(\mathbf{R}_+^{1+n}) \supset Z_0$. For $f \in X_0$ and $g \in Z_0$, we obtain the bound

$$(16) \quad \int_{\mathbf{R}_+^{1+n}} |S^+ f(x)| |g(x)| dx \lesssim \|f\|_X \|g\|_Z$$

by applying the duality to f and $g \operatorname{sgn}((S^+ f)g)$. By Fatou's lemma and the density of Z_0 in Z , (16) continues to hold for all $g \in Z$. Let $h = S^+ f$ and let $K \subset \mathbf{R}_+^{1+n}$ be a compact set, so that $\|NW_{q'}(h 1_K)\|_{p'} < \infty$. It follows from [17, Thm. 3.2] and (16) that $\|h 1_K\|_X \lesssim \|f\|_X$. Exhausting \mathbf{R}_+^{1+n} with K and using Fatou's lemma shows that $S^+ f \in X$ with $\|S^+ f\|_X \lesssim \|f\|_X$, for all $f \in X_0$. By Lemma 5.2, we have a unique extension of S^+ by continuity to all X .

It remains to prove (15). Fix $Q_0 \in \mathcal{D}$ and consider $\int_{Q_0^{\text{ca}}} |W_{\mathcal{D}, q}(\widehat{S}f)| dx$. Consider a Whitney region $Q^w \subset Q_0^{\text{ca}}$, where $Q \subset Q_0$, $Q \in \mathcal{D}$. We write

$$\widehat{S}f(x) = \sum_{R \in \mathcal{D}, R \ni x} \int_{R_x^{\square}} |f| dy = I + II + III,$$

for a.e. $x \in Q^w$, where we split the sum according to the cases $R \subsetneq Q^{\text{ca}}$, $R \supset Q^{\text{ca}}$ and $R \cap Q^{\text{ca}} = \emptyset$, noting that $R, Q^{\text{ca}} \in \mathcal{D}$, with Q^w being the top half of Q^{ca} .

Clearly $III = 0$, for I we have $\mathbf{R} \subset Q^w$ since $\mathbf{R} \ni \mathbf{x}$, and for II we have that $\mathbf{R} = R^{\text{ca}}$ are Carleson boxes since Q^{ca} , and hence \mathbf{R} , touches \mathbf{R}^n .

The local terms I do not require causality and we may replace \mathbf{R}_x^\square by the larger set \mathbf{R}^\square . Standard sparse estimates via duality apply as follows. Let $g \geq 0$ be supported on Q^w with $\int g^{q'} d\mathbf{x} = 1$, $1/q + 1/q' = 1$. Then

$$\begin{aligned} \int_{Q^w} g \left(\sum_{\mathbf{R} \in \widehat{\mathcal{D}}, \mathbf{R} \subset Q^w} 1_{\mathbf{R}} \int_{\mathbf{R}^\square} |f| d\mathbf{y} \right) d\mathbf{x} \\ = \sum_{\mathbf{R} \in \widehat{\mathcal{D}}, \mathbf{R} \subset Q^w} \left(\int_{\mathbf{R}} g d\mathbf{x} \right) \left(\int_{\mathbf{R}^\square} |f| d\mathbf{y} \right) |\mathbf{R}| \\ \lesssim \sum_{\mathbf{R} \in \widehat{\mathcal{D}}, \mathbf{R} \subset Q^w} \int_{E_{\mathbf{R}}} \left(\int_{\mathbf{R}} g d\mathbf{x} \right) \left(\int_{\mathbf{R}^\square} |f| d\mathbf{y} \right) dz \\ \lesssim \int_{Q^w} M(g) M(1_{\tilde{Q}^w} f) dz \lesssim |Q^w|^{1/q} \left(\int_{\tilde{Q}^w} |f|^q dz \right)^{1/q}. \end{aligned}$$

We have used that $|\mathbf{R}| \lesssim |E_{\mathbf{R}}|$, that $\int_{\mathbf{R}} g d\mathbf{x} \leq M(g)$ and $\int_{\mathbf{R}^\square} |f| d\mathbf{y} \leq M(1_{\tilde{Q}^w} f)$ on all $E_{\mathbf{R}}$, Hölder's inequality, and $L_{q'}$ and L_q boundedness of the Hardy–Littlewood maximal operator M . Taking supremum over g gives the pointwise bound $W_{\mathcal{D},q}(I) \lesssim \widetilde{W}_{\mathcal{D},q}(f)$. Integration over Q_0^{ca} yields the desired estimate of I by $C_{\mathcal{D}} \widetilde{W}_{\mathcal{D},q}(f)$.

The non-local terms II do not require sparseness, and we may increase the sum over $R^{\text{ca}} \in \widehat{\mathcal{D}}$ to a sum over $R \in \mathcal{D}$. Note that the term II is essentially constant on Q^w , and we replace the Whitney L_q average by the supremum norm on Q^w , and increase $(R^{\text{ca}})_x^\square$ to

$$(R^{\text{ca}})_Q^\square := (R^{\text{ca}})^\square \cap \{(s, y) : s > \ell(Q)/2\}.$$

We have

$$\begin{aligned} \int_{Q_0^{\text{ca}}} |II| d\mathbf{x} &\lesssim \sum_{Q \in \mathcal{D}, Q \subset Q_0} |Q^w| \sum_{R \in \mathcal{D}, R \supset Q} |R^{\text{ca}}|^{-1} \int_{(R^{\text{ca}})_Q^\square} |f| d\mathbf{y} \\ &= \sum_{R \in \mathcal{D}} \left(\sum_{Q \in \mathcal{D}, Q \subset Q_0, Q \subset R} |Q^w| |R^{\text{ca}}|^{-1} \int_{(R^{\text{ca}})_Q^\square} |f| d\mathbf{y} \right) = II_1 + II_2, \end{aligned}$$

where we split the outer sum according to $R \subset Q_0$ and $R \not\subset Q_0$. For the tail terms II_2 we increase $(R^{\text{ca}})_Q^\square \subset (R^{\text{ca}})^\square = 3R^{\text{ca}}$, where we recall that $3R^{\text{ca}} = 3R^{\text{ca}} \cap \mathbf{R}_+^{1+n}$ according to our definition, and get

$$II_2 \leq |Q_0^{\text{ca}}| \sum_{R \in \mathcal{D}, R \not\subset Q_0} |R^{\text{ca}}|^{-1} \int_{3R^{\text{ca}}} |f| d\mathbf{y} \lesssim |Q_0| \left(\sum_{k=1}^{\infty} 2^{-k} \right) \inf_{Q_0} Cf,$$

because $|3R|/|R^{\text{ca}}| \lesssim 2^{-k}/\ell(Q_0)$ for the k 'th ancestor of Q_0 , and since $|Q_0^{\text{ca}}|/\ell(Q_0) = |Q_0|$. For the mid-range terms II_1 , we estimate

$$\begin{aligned} II_1 &= \sum_{R \in \mathcal{D}, R \subset Q_0} |R^{\text{ca}}|^{-1} \int_{3R^{\text{ca}}} \left(\sum_{Q \in \mathcal{D}, Q \subset R} |Q^w| 1_{\ell(Q) < 2s} \right) |f(s, y)| ds dy \\ &\lesssim \sum_{R \in \mathcal{D}, R \subset Q_0} |R^{\text{ca}}|^{-1} \int_{3R^{\text{ca}}} (s|R|) |f(s, y)| ds dy \\ &\lesssim \int_{3Q_0^{\text{ca}}} \left(\sum_{R \in \mathcal{D}, R \subset Q_0} s \ell(R)^{-1} 1_{3R^{\text{ca}}} \right) |f(s, y)| ds dy \\ &\lesssim \int_{3Q_0^{\text{ca}}} \left(s \sum_{2^{-k} \geq s/3} (2^{-k})^{-1} \right) |f(s, y)| ds dy \lesssim \int_{3Q_0^{\text{ca}}} |f| ds dy. \end{aligned}$$

For the second last estimate, note that at a point $(s, y) \in 3Q_0^{\text{ca}}$ we have $1_{3R^{\text{ca}}} = 0$ if $2^{-k} < s/3$, where $\ell(R) = 2^{-k}$, while at each scale $2^{-k} \geq s/3$ at most 3^n cubes R contribute to the sum. In total we obtain a bound of II by $C(f)$, which completes the proof. \square

6 Motivations from elliptic PDEs

We end this paper by discussing motivations for the estimates in Theorem 5.1 coming from maximal regularity estimates for PDEs. A natural point of departure for the discussion is the parabolic problem

$$\partial_t u_t + Lu_t = g_t,$$

for $u_t(x) = u(t, x)$, $t > 0$, $x \in \mathbf{R}^n$, with $u_0 = 0$. Here $L = -\operatorname{div}_x A(x) \nabla_x$, where $A \in L_\infty(\mathbf{R}^n, \mathcal{L}(\mathbf{C}^n))$ is accretive. The maximal regularity problem for a space \mathcal{H} of functions in \mathbf{R}_+^{1+n} is whether a source $g \in \mathcal{H}$ yields a solution u with Lu (or equivalently $\partial_t u$) in \mathcal{H} . The parabolic maximal regularity operator is

$$(17) \quad Lu_t = \int_0^t L e^{-(t-s)L} g_s ds,$$

which is upward mapping like the operators S^+ considered in this paper. However, the operator $g \mapsto Lu$ does not have the CZO kernel bounds (9), even for $L = -\Delta_x$. Indeed the estimate $|\partial_t(t^{-n/2} e^{-|x|^2/(4t)})| \lesssim (t + |x|)^{-(1+n)}$ fails, and non-tangential estimates are not natural for this problem due to the parabolic scaling $t \sim |x|^2$.

Instead our motivation comes from the analogous considerations for elliptic divergence form equations

$$(18) \quad \operatorname{div}_{t,x} A(t, x) \nabla_{t,x} u(t, x) = 0, \quad t > 0, x \in \mathbf{R}^n,$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in L_\infty(\mathbf{R}_+^{1+n}, \mathcal{L}(\mathbf{C}^{1+n}))$ is accretive. Following [4, 3], we consider the associated first order generalized Cauchy–Riemann system

$$(19) \quad \partial_t f_t + DB_t f_t = 0$$

for the conormal gradient $f_t = [a\partial_t u + b\nabla_x u \quad \nabla_x u]^T$ of u , consisting of the conormal derivative and tangential gradient. This uses the self-adjoint first order differential operator $D = \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix}$ and transformed accretive matrix

$$B = \begin{bmatrix} a^{-1} & -a^{-1}b \\ ca^{-1} & d - ca^{-1}b \end{bmatrix},$$

acting in $L_2(\mathbf{R}^n; \mathbf{C}^{1+n})$ for each $t > 0$. For each $t > 0$, the conormal gradient belongs to the closure of the range of D since $\operatorname{curl}_x(\nabla_x u) = 0$, and it was demonstrated in [4, Sec. 3] that (18) is equivalent to (19), under this tangential curl free constraint on the tangential part of f_t .

Now fix bounded and accretive coefficients $B = B(x)$, which are independent of t , and consider the equation

$$(20) \quad \partial_t f_t + DBf_t = g_t,$$

where f and g are functions of $t > 0$ and take values in $\mathcal{H} = \overline{\mathbf{R}(D)} \subset L_2(\mathbf{R}^n; \mathbf{C}^{1+n})$. The t -independent case when $B_t = B$ and $g = 0$ was studied in [4], whereas perturbation results for t -dependent coefficients B_t were obtained in [3] via Duhamel's principle with $g_t = D(B - B_t)f_t$. As in these works, we make use of the holomorphic functional calculus of the bisectorial operator DB in \mathcal{H} : the symbols

$$\chi^\pm(z) = \begin{cases} 1, & \pm \operatorname{Re} z > 0, \\ 0, & \pm \operatorname{Re} z < 0, \end{cases}$$

yield the spectral projections $E^\pm = \chi^\pm(DB)$, the symbol $|z| = z(\chi^+(z) - \chi^-(z))$ yields the sectorial operator $\Lambda = |DB|$ and the symbols $e^{-t|z|}$ yield the Poisson semigroup $e^{-t\Lambda}$ generated by Λ .

Following [3], we integrate (20) with boundary and decay conditions

$$\lim_{t \rightarrow 0} E^+ f_t = 0 = \lim_{t \rightarrow \infty} E^- f_t$$

and a general source term g_t , and obtain

$$(21) \quad -\partial_t f_t = DBf_t = T^+ g_t + T^- g_t,$$

which uses both an upward mapping maximal regularity operator

$$(22) \quad T^+ g_t = \int_0^t \Lambda e^{-(t-s)\Lambda} E^+ g_s ds$$

and a downward mapping maximal regularity operator

$$(23) \quad T^- g_t = \int_t^\infty \Lambda e^{-(s-t)\Lambda} E^- g_s ds.$$

We saw above that the parabolic maximal regularity problem concerned an upward mapping operator (17), using the heat semigroup $\exp(-tL)$, which is not of the form considered in this paper. The elliptic maximal regularity problem involves both upward and downward mapping operators T^+ and T^- . Both T^\pm use the Poisson semigroup $\exp(-t\Lambda)$, and T^\pm generalize the causal parts of the Beurling singular integral as the following example from [27] shows.

Example 6.1. Consider the Cauchy–Riemann equations for a holomorphic function $f(z)$ in the region above the graph γ of a Lipschitz function $y = \phi(x)$. Parametrizing $z = x + i(\phi(x) + t)$, these equations read

$$\partial_t f + BDf = 0 \quad \text{in } \mathbf{R}_+^2$$

when written in the real basis $\{1 + i\phi'(x), i\}$. Here $B = (1 + i\phi'(x))^{-1}$ is an accretive, t -independent complex multiplier, and $D = -i\partial_x$ is the first order self-adjoint derivative.

Consider now the associated Poisson semigroups and maximal regularity operators in (21), with DB replaced by the similar operator $BD = B(DB)B^{-1}$. As observed in [27, Lem. 3.1], by computing the kernels of resolvents and operators in the functional calculus of BD , the causal operators from (22) and (23) are

$$(24) \quad T^+ g_t(z) = \frac{-1}{2\pi} \int_0^t \int_\gamma \frac{g_s(w)dw}{(w + is - (z + it))^2} ds$$

and

$$(25) \quad T^- g_t(z) = \frac{-1}{2\pi} \int_t^\infty \int_\gamma \frac{g_s(w)dw}{(w + is - (z + it))^2} ds.$$

We parametrize $w = y + i\phi(y)$, $y \in \mathbf{R}$, and consider the bounded and invertible multiplier $dw/dy = 1 + i\phi'(y)$. It follows that $T^\pm = S^\pm(1 + i\phi')$, where S^\pm are defined as in (24) and (25), but replacing dw by dy .

The operators T^\pm , and hence also the operators S^\pm , are bounded on $L_2(\mathbf{R})$. See [3, Thm. 6.5], which also applies to the more general operators (22) and (23) in any dimension. Moreover, we see that the kernels k^\pm for S^\pm are the causal truncations of

$$k((t, x), (s, y)) = \frac{1}{(y + i(\phi(y) + s) - x - i(\phi(x) + t))^2}.$$

Since ϕ is a Lipschitz function, it is readily verified that the CZ bounds (9) and (10) hold for this k , with $\gamma = 1$.

For general elliptic equations (18), the estimate (9) of the distribution kernels of the associated operators T^\pm holds only in an average L_2 sense. Global L_2 bounds

$$\|\Lambda e^{-|t-s|\Lambda} E^\pm\|_{L_2 \rightarrow L_2} \lesssim 1/|t-s|$$

follow from the Kato quadratic estimates, more precisely [8, Thm. 3.1]. Also L_2 off-diagonal estimates for $\Lambda e^{-|t-s|\Lambda} E^\pm$ can be derived from such estimates for the resolvents of DB (see [8, Prop. 5.2]). As for pointwise kernel estimates, we saw in Example 6.1 that for general non-smooth coefficients B , only the first estimate in (10) can hold for DB , and similarly only the second estimate can hold for BD . More importantly, even (9) may fail for $n \geq 2$. For $n = 1$, at least for real coefficients, the pointwise kernel bound (9) follows from the interior regularity estimates for solutions to (18) of De Giorgi, Nash and Moser (which hold for real coefficients in any dimension). Indeed, $f_t = e^{-|BD|} f_0$ solves the first order equation $\partial_t f_t + BDf_t = 0$, where the first component of f satisfies a second order equation (18) and hence have pointwise bounds. See [3, Sec. 3.2]. In general this is not true for the remaining components (conjugate functions) in f , except when $n = 1$, in which case also the second component/conjugate function does satisfy an equation (18), with some conjugate coefficients. See [7, Lem. 5.3]. To summarize, the operators T^\pm from (22) and (23) are causal CZOs, as in Section 3, at least if $n = 1$, if A is real and if we factor out an invertible multiplicative factor.

Coming to estimates of T^\pm , we consider conormal gradients f solving the Cauchy–Riemann system (19) in \mathbf{R}_+^{1+n} . It is known from [3] that on the one hand the function space \mathcal{X} in \mathbf{R}_+^{1+n} with norm

$$\|f\|_{\mathcal{X}} = \|N(W_2 f)\|_2,$$

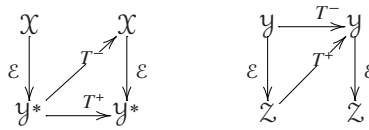
along with the subspace $\mathcal{Y}^* = L_2(\mathbf{R}_+^{1+n}; t^{-1} dt dx)$, are natural for boundary value problems with boundary topology $L_2(\mathbf{R}^n)$ for $\lim_{t \rightarrow 0} f_t$. On the other hand, the function space

$$\mathcal{Y} = L_2(\mathbf{R}_+^{1+n}; t dt dx),$$

along with the subspace \mathcal{Z} with norm $\|f\|_{\mathcal{Z}} = \|C(W_2 f)\|_2$, are natural for boundary value problems with boundary topology $H^{-1}(\mathbf{R}^n)$ for $\lim_{t \rightarrow 0} f_t$ (corresponding to boundary function space $L_2(\mathbf{R}^n)$ for the potential u). It was shown in [17, Thm. 3.2] that $\mathcal{Z}^* = \mathcal{X}$, but that \mathcal{Z} is not reflexive.

Previously known bounds for the causal operators T^\pm in (22) and (23) from [3, Prop. 7.1], for general accretive coefficients B and not assuming pointwise kernel

bounds, can be summarized in the following diagrams:



The estimate

$$(26) \quad \left(\int_{\mathbf{R}_+^2} |f(t, y)|^2 t dt dx \right)^{1/2} \approx \|(A(|f|^2 t))^{1/2}\|_2 \lesssim \|A(W_2 f)\|_2 \approx \|C(W_2 f)\|_2$$

shows that $\mathcal{Z} \subset \mathcal{Y}$, and hence $\mathcal{Y}^* \subset \mathcal{X}$. It was shown in [3, Lem. 5.5] that a multiplication operator \mathcal{E} maps $\mathcal{X} \rightarrow \mathcal{Y}^*$ if it satisfies the Carleson condition

$$(27) \quad \|C(W_\infty \mathcal{E}^2/t)\|_\infty < \infty.$$

Moreover, it follows from [17] that \mathcal{E} maps as in the diagrams if and only if (27) holds.

We solve PDEs $\partial_t f_t + DB_t f_t = 0$ with t -dependent coefficients by freezing the coefficients B_t to B . Writing

$$\mathcal{E}_t = B^{-1}(B - B_t),$$

with $B = \lim_{t \rightarrow 0} B_t$, the PDE becomes $\partial_t f_t + DB_t f_t = DB \mathcal{E}_t f_t$. To prove the existence of boundary values of solutions f_t and representation formulas as in [3, Sec. 8–9], it is needed that

$$(28) \quad (T^+ + T^-)\mathcal{E}$$

is a bounded operator (with small norm to obtain a Cauchy type representation formula of f in terms of the boundary trace f_0). This will yield equivalences of norms $\|f_0\|_{L_2(\mathbf{R}^n)} \approx \|f\|_{\mathcal{X}}$ for solutions $f \in \mathcal{X}$ to (19), and $\|f_0\|_{H^{-1}(\mathbf{R}^n)} \approx \|f\|_{\mathcal{Y}}$ for solutions $f \in \mathcal{Y}$ to (19). Thus two fundamental estimates in the study of boundary problems for elliptic divergence form equations (18) are to bound the operators $T^\pm \mathcal{E}$ in the \mathcal{X} and \mathcal{Y} norms, under an as weak as possible hypothesis on \mathcal{E} , which measures the t -variation of the coefficients.

- **$L_2(\mathbf{R}^n)$ boundary traces of solutions f :** Assuming that pointwise kernel bounds hold as discussed above, so that T^+ is a CZO modulo a trivial multiplicative factor, it follows from Theorem 5.1 that already $T^+ : \mathcal{X} \rightarrow \mathcal{X}$ is bounded. We obtain the new result that only $\mathcal{E} \in L_\infty(\mathbf{R}_+^{1+n})$, which is weaker than (27), is needed for boundedness of $T^+ \mathcal{E} : \mathcal{X} \rightarrow \mathcal{X}$. However, this does not yield any new estimate for $(T^+ + T^-)\mathcal{E}$ as a whole needed for the PDE

application. Indeed, for the downward mapping maximal regularity operator, the best known bound is that $T^- : \mathcal{Y}^* \rightarrow \mathcal{X}$. Similar to below, we conjecture that the Carleson condition (27) is in general necessary for the boundedness of $T^- \mathcal{E} : \mathcal{X} \rightarrow \mathcal{X}$.

- **$H^{-1}(\mathbf{R}^n)$ boundary traces of solutions f :** It is known from [3, 17] that already $T^- : \mathcal{Y} \rightarrow \mathcal{Y}$ is bounded, so only $\mathcal{E} \in L_\infty(\mathbf{R}_+^{1+n})$ is needed for $T^- \mathcal{E} : \mathcal{Y} \rightarrow \mathcal{Y}$. The new result from Theorem 5.1 that $T^- : \mathcal{Z} \rightarrow \mathcal{Z}$ is bounded, is not relevant for the boundedness of $T^- \mathcal{E} : \mathcal{Y} \rightarrow \mathcal{Y}$.

For the upward mapping maximal regularity operator, the best known bound is that $T^+ : \mathcal{Z} \rightarrow \mathcal{Y}$. This follows by duality from [3, Thm. 6.8]. Furthermore, the Carleson condition (27) on multipliers \mathcal{E} is in general necessary for the boundedness of $T^+ \mathcal{E} : \mathcal{Y} \rightarrow \mathcal{Y}$. To see this, by duality and the above estimate $T^- : \mathcal{Y} \rightarrow \mathcal{Y}$, we equivalently consider the boundedness of $\mathcal{E}^*(T^+ + T^-) : \mathcal{Y}^* \rightarrow \mathcal{Y}^*$. In the case of the Beurling transform it was proved in Astala–González [2, Thm. 1] that this latter boundedness holds if and only if \mathcal{E} satisfies (27), but without the Whitney factor W_∞ (due to the pointwise kernel bounds present for the Beurling transform).

6.1 Summary and open problems. Causal maximal regularity operators T^\pm are fundamental in the study of elliptic boundary value problems, as evidenced by [3]. Such operators T^\pm are close to being causal CZOs S^\pm and we have investigated boundedness of such S^\pm in this paper. In regards to this motivation from elliptic PDEs, our results are meager: assuming pointwise kernel bounds, it follows from Theorem 5.1 that the \mathcal{X} norm of $T^+ \mathcal{E}$ is bounded by $\|\mathcal{E}\|_\infty$. However, we believe that the techniques developed in this paper are important and that future work in this direction may result in more substantial applications. We end by formulating some open problems.

- Can the estimates in Theorem 5.1, with $p = q = 2$ at least, be shown for the more general operators T^\pm appearing in (22) and (23), without assuming pointwise kernel bounds? A positive result in this direction are the weighted norm estimates through sparse domination for such generalized CZOs by Bernicot, Frey and Petermichl [9].
- Consider the L_r based tent spaces of Coifman, Meyer and Stein [12], with $1 < r < \infty$ and in particular $r = 2$. The present paper concerns the end point cases $r = 1$ and $r = \infty$, and the duality results [17] have been extended to the scale of tent spaces by Huang [14]. Can sparse estimates, possibly using L_r averages, prove tent space estimates of CZOs, or more general maximal regularity operators as in [5]? A concrete PDE application would be to

reprove the bound $T^+ : \mathcal{Z} \rightarrow \mathcal{Y}$ from [3, Prop. 7.1], using sparse estimates. Note that \mathcal{Z} and \mathcal{Y} are L_1 and L_2 based tent spaces respectively.

- We started this Section by considering the more well known upward mapping maximal regularity operator (17) for parabolic initial value problems, but this does not have the CZ kernel bounds. Can sparse estimates be adapted to show a parabolic analogue of the non-tangential bound (14)?

Appendix A The Beurling counterexample

We supply the details omitted in Example 2.2, which show that no Carleson bound can hold for the Beurling transform S defined in (6). Assuming for contradiction that

$$(29) \quad \|C(W_q(Sf))\|_p \lesssim \|C(W_q f)\|_p,$$

we derive a series of consequences, eventually leading to an obvious impossibility. By [17, Prop. 3.7], we may replace C and W_q by their dyadic versions, but we continue to denote them by the same symbols for simplicity. Since $W_q(Sf) \geq W_1(Sf)$ and $C(W_1(Sf)) = C(Sf)$, our first derived bound will be

$$(30) \quad \|C(Sf)\|_p \lesssim \|C(W_q f)\|_p.$$

As suggested in Example 2.2, we apply (30) to $f_N(t, x) = g(x)\phi_N(t)$, where $\phi_N = 2^N 1_{(0, 2^{-N})}$, and investigate the limit $N \rightarrow \infty$.

We first consider the right-hand side. For a Whitney cube R^w , we have

$$|R^w|^{-1/q} \|f_N\|_{L_q(R^w)} = \begin{cases} 2^N |R|^{-1/q} \|g\|_{L_q(R)}, & \text{if } \ell(R) \leq 2^{-N}, \\ 0, & \text{else.} \end{cases}$$

Let us denote by

$$M_{q, 2^{-N}} g(x) := \sup_{\substack{R: x \in R \in \mathcal{D} \\ \ell(R) \leq 2^{-N}}} |R|^{-1/q} \|g\|_{L_q(R)}$$

the L_q (dyadic) maximal operator restricted to cubes of sidelength $\ell(R) \leq 2^{-N}$. Thus, for all $x \in Q \in \mathcal{D}$, we have

$$\frac{1}{|Q|} \int_{Q^{\text{ca}}} W_q f_N \leq \frac{1}{|Q|} \int_0^{\min(\ell(Q), 2^{-N})} \int_Q 2^N M_{q, 2^{-N}} g(y) dy dt \leq M(M_{q, 2^{-N}} g)(x),$$

and hence $C(W_q f_N) \leq M(M_{q, 2^{-N}} g)$ and

$$\|C(W_q f_N)\|_p \leq \|M(M_{q, 2^{-N}} g)\|_p \lesssim \|M_{q, 2^{-N}} g\|_p, \quad p > 1.$$

If $p > q$, we could further dominate this by $\|g\|_p$, uniformly in N , but we do not wish to impose this restriction. Instead, if g is, say, a continuous compactly supported function (and this will be enough for our eventual counterexample), then $M_{q,2^{-N}}g(x) \rightarrow |g(x)|$ pointwise, as $N \rightarrow \infty$, and $M_{q,2^{-N}}g$ is dominated by a bounded and compactly supported function, uniformly in N . It hence follows from dominated convergence that

$$(31) \quad \limsup_{N \rightarrow \infty} \|C(W_q f_N)\|_p \lesssim \|g\|_p.$$

We then turn to the left-hand side of (30). If $\operatorname{Im} z > 2^{-N}$, there is no singularity in (6) for $f = f_N$, and hence

$$\begin{aligned} Sf_N(z) &= \frac{-1}{\pi} \int_{\operatorname{Im} w > 0} \frac{f_N(w)}{(w-z)^2} |dw|, \quad \operatorname{Im} z > 2^{-N}, \\ &= 2^N \int_0^{2^{-N}} \left(\frac{-1}{\pi} \int_{\mathbf{R}} \frac{g(s)}{(s+it-z)^2} ds \right) dt \xrightarrow{N \rightarrow \infty} \frac{-1}{\pi} \int_{\mathbf{R}} \frac{g(s)}{(s-z)^2} ds. \end{aligned}$$

Since any $z \in \mathbf{C}$ with $\operatorname{Im} z > 0$ satisfies $\operatorname{Im} z > 2^{-N}$ when N is large enough, we have

$$\lim_{N \rightarrow \infty} Sf_N(z) = \frac{-1}{\pi} \int_{\mathbf{R}} \frac{g(s)}{(s-z)^2} ds =: v(z), \quad \operatorname{Im} z > 0.$$

From repeated applications of Fatou's lemma it then follows that

$$\begin{aligned} \frac{1}{|Q|} \int_{Q^{\text{ca}}} |v(z)| |dz| &\leq \liminf_{N \rightarrow \infty} \frac{1}{|Q|} \int_{Q^{\text{ca}}} |Sf_N(z)| |dz| \leq \liminf_{N \rightarrow \infty} C(Sf_N)(x), \quad x \in Q, \\ C(v) &\leq \liminf_{N \rightarrow \infty} C(Sf_N), \\ \|C(v)\|_p &\leq \liminf_{N \rightarrow \infty} \|C(Sf_N)\|_p. \end{aligned}$$

In combination with (31), this shows that (30) implies the estimate

$$\|C(v)\|_p \lesssim \|g\|_p.$$

If g is real-valued, by a direct computation one checks that $|v| = |\nabla u|$, where u is the Poisson extension of g , and the previous estimate can be rewritten as (8). By splitting into real and imaginary parts, it is immediate that if this holds for real g , it also holds for complex g .

It remains to show that estimate (8) cannot hold. Using

$$CF(x) \geq 1_Q(x) \int_{Q^{\text{ca}}} |F| ds dy$$

with $Q = (0, 1)$ and $|\nabla u| \geq |\partial_t u|$, we find that (8) would in particular imply that

$$(32) \quad \int_0^1 \|\partial_t u(t, \cdot)\|_{L_1(0,1)} dt \lesssim \|g\|_p.$$

Let us consider random functions of the form $g = \sum_k \varepsilon_k g_k$, where the ε_k are independent random signs with $P(\varepsilon_k = -1) = P(\varepsilon_k = +1) = \frac{1}{2}$. Denoting by u_k the Poisson extension of g_k , (32) would then imply, using basic properties of random sums (Kahane's contraction principle in the first step, and Khintchine's inequality in the last one; see [15, Prop. 3.2.10] and [16, Thm. 6.1.13, Prop. 6.3.3]), that

$$(33) \quad \int_0^1 \max_k \|\partial_t u_k(t, \cdot)\|_{L_1(0,1)} dt \leq \int_0^1 E \left\| \sum_k \varepsilon_k \partial_t u_k(t, \cdot) \right\|_{L_1(0,1)} dt \\ \lesssim E \left\| \sum_k \varepsilon_k g_k \right\|_p \lesssim \left\| \left(\sum_k |g_k|^2 \right)^{1/2} \right\|_p,$$

where E , appearing only in the intermediate steps, denotes the mathematical expectation. (The advantage of (33) in view of reaching a contradiction is that we have eliminated interactions between different functions g_k ; in particular, on the left, we only need to check that individual terms are large for appropriate parameter values, and we do not need to worry about this largeness being destroyed by the largeness of some other term of opposite sign.)

We will now construct appropriate g_k to see the failure of this derived estimate (33). Let

$$g_k(x) = \phi(x) e^{i2\pi 2^k x},$$

where ϕ is a test function to be specified shortly. Note that $|g_k| = |\phi|$. Using the normalisation $\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-i2\pi x \xi} dx$ for the Fourier transform, the Poisson integral $P_t f(x)$ has Fourier transform $\widehat{P_t f}(\xi) = e^{-2\pi t |\xi|} \hat{f}(\xi)$, and its time derivative the transform $\widehat{\dot{P}_t f}(\xi) = -2\pi |\xi| e^{-2\pi t |\xi|} \hat{f}(\xi)$. Hence

$$\widehat{\dot{g}_k}(\xi) = \hat{\phi}(\xi - 2^k), \quad \widehat{\dot{P}_t g_k}(\xi) = -2\pi |\xi| e^{-2\pi t |\xi|} \hat{\phi}(\xi - 2^k).$$

Let us now choose a Schwartz test function ϕ , not identically zero, such that the support of $\hat{\phi}$ is a compact subset of \mathbf{R}_+ . Thus $\phi(\xi - 2^k)$ is only non-zero for $\xi > 2^k > 0$, and hence we can drop the absolute values on ξ above. We have

$$\begin{aligned} \widehat{\dot{P}_t g_k}(\eta + 2^k) &= -2\pi(\eta + 2^k) e^{-2\pi t(\eta + 2^k)} \hat{\phi}(\eta) \\ &= 2^{-2\pi t 2^k} (-2\pi \eta e^{-2\pi t \eta} \hat{\phi}(\eta) - 2\pi 2^k e^{-2\pi t \eta} \hat{\phi}(\eta)) \\ &= 2^{-2\pi t 2^k} (\widehat{\dot{P}_t \phi}(\eta) - 2\pi 2^k \widehat{P_t \phi}(\eta)). \end{aligned}$$

Thus, inverting the Fourier transform,

$$\dot{P}_t g_k(x) e^{-i2\pi 2^k x} = 2^{-2\pi t 2^k} (\dot{P}_t \phi(x) - 2\pi 2^k P_t \phi(x)),$$

and hence

$$\begin{aligned} |\dot{P}_t g_k(x)| &\geq 2^{-2\pi t 2^k} (2\pi 2^k |P_t \phi(x)| - |\dot{P}_t \phi(x)|), \\ \|\dot{P}_t g_k\|_{L_1(0,1)} &\geq 2^{-2\pi t 2^k} (2\pi 2^k \|P_t \phi\|_{L_1(0,1)} - \|\dot{P}_t \phi\|_{L_1(0,1)}). \end{aligned}$$

We claim that $\|P_t\phi\|_{L_1(0,1)} \geq c_0$ and $\|\dot{P}_t\phi\|_{L_1(0,1)} \leq c_1$ for some positive constants depending only on the choice of ϕ , but not on $t \in (0, 1)$. For the first one, note that $\widehat{P_t\phi}(\xi) = e^{-2\pi i \xi t} \widehat{\phi}(\xi)$ is not identically zero, and its support, the same as that of $\widehat{\phi}$, is a compact subset of \mathbf{R}_+ . Hence $P_t\phi(x)$ is not identically zero on \mathbf{R} , and it extends to an analytic function, hence it is also not identically zero for $x \in (0, 1)$, and thus $\|P_t\phi\|_{L_1(0,1)} > 0$ for every $t \in [0, 1]$. Since $t \mapsto P_t\phi$ is continuous from $[0, 1]$ to $L_1(0, 1)$, the function $t \mapsto \|P_t\phi\|_{L_1(0,1)}$ attains a minimum on $[0, 1]$, and this gives the number $c_0 > 0$. For the second claim, note that

$$\widehat{\dot{P}_t\phi}(\xi) = -2\pi i \xi e^{-2\pi i \xi t} \widehat{\phi}(\xi) = i e^{-2\pi i \xi t} (2\pi i \xi) \widehat{\phi}(\xi) = i \widehat{P_t(\phi')}(\xi),$$

and hence

$$\|\dot{P}_t\phi\|_{L_1(0,1)} \leq \|P_t(\phi')\|_{L_1(\mathbf{R})} \leq \|\phi'\|_{L_1(\mathbf{R})} = c_1.$$

Using these bounds, we conclude that

$$\|\partial_t u_k(t, \cdot)\|_{L_1(0,1)} = \|\dot{P}_t g_k\|_{L_1(0,1)} \gtrsim 2^{-2\pi t 2^k} 2^k, \quad k > k_0, \quad t \in (0, 1).$$

With $k = k_0 + 1, \dots, k_0 + K$, we can now see the failure of (33). For $t \in (2^{-j}, 2^{1-j})$ and $j = k_0 + 1, \dots, k_0 + K$, we find that

$$\max_k \|\partial_t u_k(t, \cdot)\|_{L_1(0,1)} \geq \|\partial_t u_j(t, \cdot)\|_{L_1(0,1)} \gtrsim 2^{-2\pi t 2^j} 2^j \gtrsim 2^j,$$

and hence

$$\int_0^1 \max_k \|\partial_t u_k(t, \cdot)\|_{L_1(0,1)} dt \geq \sum_{j=k_0+1}^{k_0+K} \int_{2^{-j}}^{2^{1-j}} \max_k \|\partial_t u_k(t, \cdot)\|_{L_1(0,1)} dt \gtrsim \sum_{j=k_0+1}^{k_0+K} 1 = K,$$

while, on the other hand, we have

$$\left\| \left(\sum_{k=k_0+1}^{k_0+K} |g_k|^2 \right)^{1/2} \right\|_p = \left\| \left(\sum_{k=k_0+1}^{k_0+K} |\phi|^2 \right)^{1/2} \right\|_p = \|\sqrt{K}|\phi|\|_p \lesssim \sqrt{K}.$$

Thus (33) would imply that $K \lesssim \sqrt{K}$, which is clearly false. This contradiction shows the failure of each of the estimates (8), (29), (30), (32), and (33).

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Tuomas Hytönen

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF HELSINKI
P.O. BOX 68 (PIETARI KALMIN KATU 5)
FI-00014 UNIVERSITY OF HELSINKI, FINLAND
email: tuomas.hytonen@helsinki.fi

Andreas Rosén

DEPARTMENT OF MATHEMATICAL SCIENCES
CHALMERS UNIVERSITY OF TECHNOLOGY
412 96 GÖTEBORG, SWEDEN

and

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF GOTHENBURG
SE-412 96 GÖTEBORG, SWEDEN
email: andreas.rosen@chalmers.se

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