Symmetries and Asymmetries in Branching Processes

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Review

Symmetries and Asymmetries in Branching Processes

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Abstract: As is known in stochastic particle theory, the same random process can be described by two different master equations for the evolution of the probability density, namely, by a forward or a backward master equation. These are the generalised analogues of the direct and adjoint equations of traditional transport theory. At the level of the first moment, these two equations show considerable resemblance to each other, but they become increasingly different with increasing moment order. The purpose of this paper is to demonstrate this increasing asymmetry and to discuss the underlying reasons. It is argued that since the reason of the different forms of the forward and the backward equations lies in the lack of invariance of the process to time reversal, the reason for the increasing asymmetry between the two forms for higher-order moments or processes with several variables (particle types) can be related to the increasing level of the violation of the invariance to time reversal, as is illustrated with some examples.

Keywords: branching processes; stochastic transport; master equations; forward equation; backward equation; Chapman–Kolmogorov

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1. Introduction

Neutron transport in a multiplying (fissile) medium is a branching process. It has been known ever since the theory of the extinction of family trees was formulated [1,2], that branching processes, in which an entity can give rise to a random number of secondaries, might show significant fluctuations. It is also known [3–5], that both the traditional (deterministic) neutron transport equation, as well as the master equations for the probability distribution, or the higher-order moments of the neutron population, can be derived both from a direct (forward) and an adjoint (backward) approach. The corresponding equations are different, even if their solutions, at the level of the Green's function, are identical. The relationship between the forward and backward equations has been discussed substantially in the literature [6–9].

At the level of expectations, i.e., traditional deterministic neutron transport, the direct and adjoint forward equations, although different, still show a considerable resemblance to each other, with expressed symmetry properties. The reason for the differences between the forward and backward, or direct and adjoint forms, is that the underlying physical processes are not invariant for a time reversal, which is a kind of symmetry breaking. As is known, for the deterministic neutron transport equation, the lack of the invariance lies partly in the boundary condition (particles only leave the system but do not enter it from the outside), and partly in the irreversibility of the scattering process. This asymmetry is though "mild", the boundary condition is not even reflected in the equation itself, and therefore, the direct and adjoint equations are reasonably similar to each other.

It is also known, however, that the forward and backward forms become increasingly different in structure, even for purely time-dependent processes, when turning to the higher-order statistical moments of the process. The difference further increases when considering more involved processes, such as a branching process with several different particle types, or considering a branching process on the phase space (i.e., including space,

angle, and/or energy dependence). Unlike for the case of deterministic neutron transport, the reasons for these differences have not been discussed in the literature.

The objective of this paper is twofold. The first is to demonstrate the increasing asymmetry between the forward and backward formalism with increasing moment order or when the process depends on more variables. The second is to relate this increasing asymmetry to the increasing level of the lack of invariance against time reversal, or in other words to the increasing level of symmetry breaking.

2. Preliminaries: The Neutron Transport Equation from a Master Equation

We start out with a comparison between the direct and the adjoint forms of the deterministic neutron transport equation. The adjoint equation is traditionally derived either by variational principles from the direct equation [10], or by heuristic methods [3]. Nevertheless, since in the main body of the paper we will treat neutron fission chains as random processes, we will here reproduce a non-traditional derivation of the forward and adjoint deterministic neutron transport equations by methods of stochastic processes [11].

To this end, we will first briefly recall the basics of the Chapman–Kolmogorov equations of Markovian random processes. For simplicity, for the sake of demonstration we will use a simple discrete random process, indexed with initial states M and final states N. We will then use the principles behind the derivation in order to derive the traditional (deterministic or first moment) linear neutron transport equation. After this, we will switch to the probabilistic description of branching processes, first to the generic branching process with one type of particle, then finally moving to the stochastic description of neutron chains in a nuclear reactor, which consists of a branching process with two particle types (the neutrons and the delayed neutron precursors), the random detection process, and a stationary random extraneous neutron source. We will show how the moment equations in the forward and backward formalism become increasingly more different when moving from simple to more complicated models of branching processes, as well as with the increase in the moment order.

The basic master equation of discrete processes concerns the transition probability

$$P(N, t|\, M, t_0) \tag{1}$$

that, given that the system at the starting (source or initial) time t_0 was in state M (i.e., there were M neutrons in the system), the system will be in state N at the final (detection or terminal) time t. Then, by virtue of the Markovian property, for any intermediate time $t \geq t' \geq t_0$, the following relationship is valid:

$$P(N, t|\, M, t_0) = \sum_L P(N, t|\, L, t') P(L, t'|\, M, t_0) \tag{2}$$

where the summation is performed for all (mutually exclusive) intermediate states L of the system. Although at this point (2) is only a formal identity, it can be converted into a differential equation based on the assumption that we can predict the behaviour of the process for infinitesimal times from physical considerations. There are two alternative ways of obtaining a differential equation from (2): first, by letting $t' \to t$, i.e., $t - t' = \mathcal{O}(dt)$, and second, by $t' \to t_0$, i.e., $t' - t_0 = \mathcal{O}(dt)$. In this way one obtains two differential equations for the same quantity. These are called the forward and backward master equations, or Chapman–Kolmogorov equations.

Similarly to the derivations of differential equations of deterministic physical processes, it can be assumed that for different states $N \neq M$, the transition probability tends linearly to zero with dt. Hence, one can write

$$P(N, t + dt|\, M, t) = w_{N,M} dt; \qquad N \neq M \tag{3}$$

where the $w_{N,M}$ are called the *transition intensities*. At the same time, the probability of remaining in the same state tends to unity as dt tends to zero. This can be expressed by the

fact that the sum of the $w_{N,M}\,dt$ with respect to the final state N is unity, which can be used to express $w_{N,N}$ as

$$w_{N,N}dt = 1 - \sum_{L\neq N} w_{L,N}dt \tag{4}$$

After these preliminaries, the forward and backward master equations are derived as follows.

2.1. Forward Master Equation

The forward master equation is obtained from (2) by choosing $t' = t - dt$. Using the definition (3) and the property (4), after rearranging, dividing by dt, and letting $dt \to 0$, the forward master equation is obtained as

$$\frac{d}{dt}P(N,t|\,M,t_0) = \sum_{L\neq N} w_{N,L}P(L,t|\,M,t_0) - P(N,t|\,M,t_0)\sum_{L\neq N} w_{L,N} \tag{5}$$

One can notice that in this equation all operations are performed on the final (terminal) variables N and t, whereas the initial (source) coordinates M and t_0 are only parameters. The first term on the right-hand side describes the cases when at $t - dt$ the system was in the state L, and during dt there is a transition to the final state N. The second term describes the situation when the system at $t - dt$ is already at the sought state N, and hence any transition to another state will scatter it out from the desired state. In this respect, the equation shows a similarity to the traditional neutron transport equation, in that the right-hand side represents the intensity of the "in-scatter" events minus the intensity of the "out-scatter" events.

2.2. Backward Master Equation

To obtain the backward equation, one uses $t' = t_0 + dt_0$ which, after performing similar steps as above, leads to

$$-\frac{d}{dt_0}P(N,t|\,M,t_0) = \sum_{L\neq M} P(N,t|\,L,t_0)w_{L,M} - P(N,t|\,M,t_0)\sum_{L\neq M} w_{L,M} \tag{6}$$

It is noted that in contrast to the forward Equation (5), in which all operations are performed on the final variables, here the initial or source variables M and t_0 are operated upon, whereas the final coordinates N and t are only parameters.

Although the backward Equation (6) looks formally similar to the forward Equation (5), it cannot be given a similar transparent interpretation in terms of balance between in-scatter and out-scatter processes as in the case of the forward equation. This is because whatever happens in the first infinitesimal time interval dt_0, i.e., transition or no transition to other states, the system still has the full chance to reach the desired final state in the remaining finite time in either case. Accordingly, the first term on the right-hand side corresponds to the cases when during dt_0 the system scatters into state L, after which it has to reach the desired state N from state L until the terminal time t. The second term represents the cases where, during dt_0 no transition takes place, therefore, the system remains in its initial state M, and has to evolve to state N until t. The clarity of this interpretation is blurred by this second term having a negative sign. The reason for this is that the intensity of "no transition" is $1 - \sum_{L\neq N} w_{L,N}dt_0$ (see Equation (4)), out of which the $-\sum_{L\neq N} w_{L,N}dt_0$ remains on the right-hand side, and the unit multiplication factor constitutes a part of the time derivative on the left-hand side. Hence, despite its negative sign, the last term contributes positively to the evolution of the probability. The interpretation of the terms of the backward equation will be more transparent in the subsequent sections, where both the forward and the backward equations will be derived in a more heuristic way.

The fact that in the forward Equation (5) the initial variables are not operated upon, allows for the possibility of summing up both sides with respect to a weight function

that depends on the initial variables. A practical choice of such a weight function is a probability distribution, defining a distributed (i.e., probabilistic) initial condition for the probability $P(N,t|\,M,t_0)$. This means that the forward equation can be used even if the initial conditions are not crisp, i.e., if they do not correspond to a single state, rather they are distributed according a source function. This property of (5) is again a complete analogy with the traditional (forward) transport equation.

In the backward master Equation (6) the opposite is valid, namely, the final variables are not operated upon. This makes it possible to preserve the form of the equation after a weighted summation with respect to the final variables, for instance, with a weight function describing a finite (distributed) detector. Again, the backward master equation is a perfect analogy of the adjoint equation of transport theory (see Equation (23) later), whose solution, the neutron importance, gives the contribution of one neutron (crisp initial variables) to the neutron count in a finite detector (distributed final variables).

We note, finally, that many processes are homogeneous in time (e.g., when the coefficients $w_{L,M}$ are constant). In this case, the derivative with respect to t_0 in (6) can be converted into a derivative with respect to t, since then,

$$\frac{d}{dt_0}P(N,t|\,M,t_0) = -\frac{d}{dt}P(N,t|\,M,t_0) \tag{7}$$

Then, choosing $t_0 = 0$ and dropping notation on it, i.e.,

$$P(N,t|\,M,t_0) \Rightarrow P(N,t|\,M) \tag{8}$$

leads to

$$\frac{d}{dt}P(N,t|\,M) = \sum_{L\neq M} P(N,t|\,L)w_{L,M} - P(N,t|\,M)\sum_{L\neq M} w_{L,M} \tag{9}$$

This form is frequently encountered in the literature. It has to be kept in mind, however, that (9) is a mixed-type equation where both the final and the initial coordinates are operated upon. A corresponding form of the neutron adjoint equation is frequently used, but it is not a pure backward-type equation, which makes its interpretation somewhat confusing. These equations will be used in the determination of the master equations of the neutron population and for the calculation of the variance of the neutron fluctuations.

It is obvious from the derivation of the forward and the backward equations from a common general equation, Equation (2), that their solutions are identical. Some conditions that are necessary to be fulfilled for this equivalence are discussed in [7].

2.3. Derivation of the Neutron Transport Equation

We will now give a dual derivation of the direct and adjoint transport equations from a common master equation, following [11]. We start by deriving an equation for the Green's function in the phase space $\mathbb{R}^6 = \boldsymbol{r} \otimes \boldsymbol{v}$ for a subcritical medium,

$$G(\boldsymbol{r},\boldsymbol{v},t\,|\,\boldsymbol{r}_0,\boldsymbol{v}_0,t_0). \tag{10}$$

This represents the neutron density around $(\boldsymbol{r},\boldsymbol{v})$ at time t, due to one starting neutron at $(\boldsymbol{r}_0,\boldsymbol{v}_0)$ at time t_0.

The key point to note is that if the infinitesimal volume $dV \subset \mathbb{R}^6$ is sufficiently small, then in the first order of dV, there will be at most one particle. Hence,

$$P(N,dV) = \delta_{N,0}\{1 - n(\boldsymbol{r},\boldsymbol{v},t)dV\} + \delta_{N,1}n(\boldsymbol{r},\boldsymbol{v},t)dV + \mathcal{O}(dV^2) \tag{11}$$

Since

$$\langle N(dV)\rangle = \sum_{N=0}^{\infty} N\,P(N,dV) = n(\boldsymbol{r},\boldsymbol{v},t)dV, \tag{12}$$

it is seen that $n(\boldsymbol{r},\boldsymbol{v},t)dV$ is both a probability and an expectation.

The starting master equation is an analogue of (2) for continuous parametric discrete processes in the form

$$G(\boldsymbol{r},\boldsymbol{v},t\,|\,\boldsymbol{r}_0,\boldsymbol{v}_0,t_0)=\int G(\boldsymbol{r},\boldsymbol{v},t\,|\,\boldsymbol{r}',\boldsymbol{v}',t')G(\boldsymbol{r}',\boldsymbol{v}',t'\,|\,\boldsymbol{r}_0,\boldsymbol{v}_0,t_0)\,\mathrm{d}\boldsymbol{r}'\,\mathrm{d}\boldsymbol{v}' \tag{13}$$

The transition intensity can be given by the total macroscopic cross-section Σ and the scattering function $f(\boldsymbol{v}\to\boldsymbol{v}')$, where

$$\int f(\boldsymbol{v}\to\boldsymbol{v}')\,\mathrm{d}\boldsymbol{v}'=\mathrm{c} \tag{14}$$

with c being the expectation of the number of secondaries per reaction, as

$$\begin{aligned}G(\boldsymbol{r}',\boldsymbol{v}',t+dt\,|\,\boldsymbol{r},\boldsymbol{v},t)=(1-v\,\Sigma\,dt)\,\delta(\boldsymbol{v}-\boldsymbol{v}')\delta(\boldsymbol{r}'-\boldsymbol{r}-\boldsymbol{v}dt)+\\ v\,\Sigma\,dt\,f(\boldsymbol{v}\to\boldsymbol{v}')\delta(\boldsymbol{r}'-\boldsymbol{r})\end{aligned} \tag{15}$$

Here, the first term corresponds to the event of no collision during dt, and the second to the case of a collision, these two events being mutually exclusive. Using (15) in (13) and performing the limits $t'\to t$ and $t'\to t_0$, respectively, and using

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\boldsymbol{v}\nabla_{\boldsymbol{r}} \tag{16}$$

leads to the direct and adjoint transport equations. The direct transport equation reads as

$$\begin{aligned}&\frac{\partial}{\partial t}G(\boldsymbol{r},\boldsymbol{v},t\,|\,\boldsymbol{r}_0,\boldsymbol{v}_0,t_0)+\boldsymbol{v}\nabla_{\boldsymbol{r}}G(\boldsymbol{r},\boldsymbol{v},t\,|\,\boldsymbol{r}_0,\boldsymbol{v}_0,t_0)=\\ &-v\Sigma G(\boldsymbol{r},\boldsymbol{v},t\,|\,\boldsymbol{r}_0,\boldsymbol{v}_0,t_0)+\int v'\Sigma' f(\boldsymbol{v}'\to\boldsymbol{v})G(\boldsymbol{r},\boldsymbol{v}',t\,|\,\boldsymbol{r}_0,\boldsymbol{v}_0,t_0)\,\mathrm{d}\boldsymbol{v}'\end{aligned} \tag{17}$$

and the adjoint (backward) transport equation results from the above procedure as

$$\begin{aligned}&-\frac{\partial}{\partial t_0}G(\boldsymbol{r},\boldsymbol{v},t\,|\,\boldsymbol{r}_0,\boldsymbol{v}_0,t_0)+\boldsymbol{v_0}\nabla_{\boldsymbol{r}_0}G(\boldsymbol{r},\boldsymbol{v},t\,|\,\boldsymbol{r}_0,\boldsymbol{v}_0,t_0)=\\ &-v_0\Sigma_0 G(\boldsymbol{r},\boldsymbol{v},t\,|\,\boldsymbol{r}_0,\boldsymbol{v}_0,t_0)+v_0\Sigma_0\int f(\boldsymbol{v_0}\to\boldsymbol{v}')G(\boldsymbol{r},\boldsymbol{v},t\,|\,\boldsymbol{r}_0,\boldsymbol{v}',t_0)\,\mathrm{d}\boldsymbol{v}'.\end{aligned} \tag{18}$$

The boundary conditions for both equations arise naturally from the no incoming flux at the boundary for the forward equation, and no progeny generated for the outgoing directions for the backward equation. The initial condition for both equations is

$$G(\boldsymbol{r},\boldsymbol{v},t_0\,|\,\boldsymbol{r}_0,\boldsymbol{v}_0,t_0)=\delta(\boldsymbol{r}-\boldsymbol{r}_0)\,\delta(\boldsymbol{v}-\boldsymbol{v}_0) \tag{19}$$

The above shows that the Green's function obeys (i.e., can be determined from) either a forward or a backward equation.

The Green's function gives the neutron density in one point of the phase space due to a single starting particle in another point. In practice, one seeks the neutron density or the neutron flux due to a spatially extended source with an energy spectrum, steadily injecting neutrons into the system, or one wants to calculate the detection rate of neutrons in a detector occupying a finite volume of the phase space. These both can be obtained by convoluting the Green's function with the source or with the detector cross-section, but one can also derive equations for these quantities (convolutions) directly. In this process, the differences between the forward and backward forms become more apparent. Including an extraneous source of neutrons instead of a single starting neutron requires an integral of the equations with respect to the initial variables, which is only possible for the forward equation that operates on the final (terminal) variables. Hence, with an extraneous source, only the forward equation can be written down for the flux or the neutron density.

Similarly, the detection rate in a volume of a neutron detector requires an integral over the final variables, hence, only the backward equation is suitable for describing such a situation.

According to the above, adding the initial condition (19) to both equations explicitly, introducing the notations

$$\phi(\boldsymbol{r},\boldsymbol{v},t) = \int v\, G(\boldsymbol{r},\boldsymbol{v},t \,|\, \boldsymbol{r}_0,\boldsymbol{v}_0,t_0)\, S(\boldsymbol{r}_0,\boldsymbol{v}_0)\, \mathrm{d}\boldsymbol{v}_0 \tag{20}$$

for the neutron flux due to an extraneous source with intensity $S(\boldsymbol{r},\boldsymbol{v})$, and

$$\phi^{\dagger}(\boldsymbol{r}_0,\boldsymbol{v}_0,t_0) = \int G(\boldsymbol{r},\boldsymbol{v},t \,|\, \boldsymbol{r}_0,\boldsymbol{v}_0,t_0)\Sigma_d(\boldsymbol{r},\boldsymbol{v})\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{v}, \tag{21}$$

for the adjoint flux yielding the reaction rate of neutrons in a finite detector with a macroscopic cross-section $\Sigma_d(\boldsymbol{r},\boldsymbol{v})$, for the static (time-independent) case, one obtains the equations

$$\begin{aligned} &\boldsymbol{\Omega}\cdot\nabla\phi(\boldsymbol{r},\boldsymbol{\Omega},E) + \Sigma\,\phi(\boldsymbol{r},\boldsymbol{\Omega},E) = \\ &\qquad \int f(\boldsymbol{\Omega}'\to\boldsymbol{\Omega},E'\to E)\phi(\boldsymbol{r},\boldsymbol{\Omega}',E')\Sigma'\,\mathrm{d}\boldsymbol{\Omega}'\,\mathrm{d}E' + S(\boldsymbol{r},\boldsymbol{\Omega},E) \end{aligned} \tag{22}$$

and

$$\begin{aligned} &-\boldsymbol{\Omega}\cdot\nabla\phi^{\dagger}(\boldsymbol{r},\boldsymbol{\Omega},E) + \Sigma\,\phi^{\dagger}(\boldsymbol{r},\boldsymbol{\Omega},E) = \\ &\qquad \int f(\boldsymbol{\Omega}\to\boldsymbol{\Omega}',E\to E')\phi^{\dagger}(\boldsymbol{r},\boldsymbol{\Omega}',E')\Sigma'\,\mathrm{d}\boldsymbol{\Omega}'\,\mathrm{d}E' + \Sigma_d(\boldsymbol{r},E) \end{aligned} \tag{23}$$

where, in order for better resemblance to tradition neutron transport theory, notations on the neutron velocity vector were changed to that on energy E and unit direction vector $\boldsymbol{\Omega}$. Formally, these equations are still similar to each other. However, it has also to be kept in mind that the $\phi(\boldsymbol{r},\boldsymbol{\Omega},E)$ and the $\phi^{\dagger}(\boldsymbol{r},\boldsymbol{\Omega},E)$ are now not the notations for the same quantity, but stand for different physical quantities.

3. The Basic Branching Process

We will consider a basic branching process, started by one single particle which will undergo reactions with exponentially distributed time points, such that at each reaction (collision) a random number of new particles will be generated, whereas the incoming particle, inducing the reaction, will be lost. The intensity of the reactions will be denoted by Q, and the number distribution of the secondaries (the number of new particles generated in a reaction) will be denoted by f_k. It is assumed that the process starts at $t = 0$ with one starting particle in the system. Since we will consider a mixed-type backward equation, in both the forward and mixed backward master equations only the final time t will appear as a variable. Accordingly, we will seek the probability $p_n(t)$ that at time t there will be n particles in the system, given that there was one particle present in the system at $t = 0$.

The two equations will be derived by the "common sense" methodology, by considering the probabilities of the mutually exclusive possible events between $(t, t+dt)$ and $(0, dt)$, respectively. For the forward equation, one can write

$$p_n(t+dt) = p_n(t)(1 - n\,Q\,dt) \; + Q\,dt \sum_{k=0}^{n} f_k\,(n-k+1)p_{n-k+1}(t) \tag{24}$$

where use was made of the fact that in the case where there are n particles in the system, then the probability that any one of them can undergo a reaction during dt is equal to $Q\,n\,dt$. This can be converted into a differential equation by letting $dt \to 0$, leading to

$$\frac{dp_n(t)}{dt} = -Qnp_n(t) + Q\sum_{k=0}^{n} f_k(n-k+1)p_{n-k+1}(t) \tag{25}$$

with the initial condition being $p_n(0) = \delta_{n1}$. Introducing the generating functions

$$g(z,t) = \sum_0^\infty p_n(t)\, z^n \tag{26}$$

and

$$q(z) = \sum_0^\infty f_k\, z^k, \tag{27}$$

one obtains the following linear partial differential equation for the generating function $g(z,t)$ of the branching process

$$\frac{\partial g(z,t)}{\partial t} = Q[q(z) - z]\, \frac{\partial g(z,t)}{\partial z}. \tag{28}$$

with the initial condition

$$g(z,0) = z \tag{29}$$

Following similar considerations, the "common sense" derivation of the mixed-type master equation starts as

$$p_n(t) = (1 - Q\,dt)p_n(t - dt) + Q\,dt \sum_{k=1}^\infty f_k \sum_{n_1+\cdots+n_k=n} \prod_{j=1}^k p_{n_j}(t) \tag{30}$$

The last term on the right-hand side needs some explanation. Here, a crucial assumption was introduced to avoid the so-called closure problem. Since in the first reaction a random number of neutrons are generated simultaneously, each term of the distribution f_k in the sum on the right-hand side should be multiplied with the probability

$$p_n(t|\,k) \tag{31}$$

that the k particles present in the system at time dt will *together* lead to the appearance of n particles in the system at time t. However, this would mean the appearance of new unknowns of the type (31) with different values of k. To avoid the entailing problem, one utilizes the fact that each of the k neutrons will start a branching process independently from the others. This is expressed by using the factorisation property of the process (valid only in a steady medium) in the form

$$p_n(t|k) = \sum_{n_1+\cdots+n_k} \prod_{j=1}^k p_{n_j}(t|1) = \sum_{n_1+\cdots+n_k} \prod_{j=1}^k p_{n_j}(t). \tag{32}$$

Converting (30) into a differential equation and switching to the equation for the generating function, one obtains the mixed backward-type differential equation

$$\frac{\partial g(z,t)}{\partial t} = Q\Big\{q[g(z,t)] - g(z,t)\Big\}, \tag{33}$$

with the initial condition $g(z,0) = z$, which is of course the same as (29).

It is worth noting that (33) is a non-linear equation since, according to the definition (27) of the generating function $q(z)$,

$$q[g(z,t)] = \sum_{k=0}^\infty f_k\, g(z,t)^k, \tag{34}$$

in the first term on the right-hand side $g(z,t)$ appears as the argument of a polynomial whose coefficients are the probabilities f_k. The degree of non-linearity is equal to the maximum number of particles that can be generated in a reaction.

Comparing (28) with (33), the differences between the forward and backward equations of the basic branching process are clear. The forward equation is a linear partial differential equation, with derivatives with respect to both time and the axuiliary variable z of the generating function. The backward equation is an ordinary differential equation, containing a derivative only with respect to time, but on the other hand non-linear in the searched quantity, the generating function of the process itself. The degree of non-linearity is equal to the maximum number of secondaries which can result from a reaction. For neutron transport, this is equal to the maximum number of neutrons generated in fission, which is eight for the most common fissile nuclides. Because of this non-linearity (which for the forward equation concerns the variable z), the equations for the generating functions can only be solved if the non-linearity is quadratic (Yule–Furry process [12]), in which case an analytical solution of both equations is possible [5]. Naturally, the two solutions are identical.

Some further insight can be obtained into the properties of the forward and (mixed) backward equations by deriving the equations for the first two factorial moments of the number of particles from the equations, namely (The conventions of Ref. [5] will be used, i.e. random variables are denoted by bold symbols, whereas their realisations with plain italics),

$$\langle \boldsymbol{n}(t) \rangle \equiv n(t) = \left. \frac{\partial g(z,t)}{\partial z} \right|_{z=1} \tag{35}$$

and

$$\langle \boldsymbol{n}(t) \left(\boldsymbol{n}(t) - 1 \right) \rangle \equiv m(t) = \left. \frac{\partial^2 g(z,t)}{\partial z^2} \right|_{z=1} \tag{36}$$

For the first moment, from both the forward Equation (28) and the mixed backward Equation (33) one obtains the same equation, namely,

$$\frac{d\, n(t)}{d\, t} = Q \left(q_1 - 1 \right) n(t) \tag{37}$$

with the initial condition

$$n(0) = 1, \tag{38}$$

where

$$q_1 \equiv \left. \frac{d\, g(z,t)}{d\, z} \right|_{z=1} = \langle \boldsymbol{\nu} \rangle \tag{39}$$

is the average number of secondaries per reaction (the average number of neutrons in fission, if fission is the only reaction in the system).

The above shows that at the level of the first moment, the forward and mixed backward equations are identical. This also means that although the backward equation is non-linear for the generating function, the equations for the moments are linear. This is to be understood in the sense that for any order factorial moment, the equation is linear in that order of the moment, but contains a non-linear combination of the lower-order moments as an inhomogeneous term. Solving the moment equations sequentially, starting with the first order and progressing upwards, at each level the factorial moments from the lower orders are already available. This will be seen directly in the equations for the second factorial moments.

For the second factorial moment $m(t)$, from the forward form (28) one obtains the equation

$$\frac{d\, m(t)}{d\, t} = Q \left[2 \left(q_1 - 1 \right) m(t) + q_2\, n(t) \right] \tag{40}$$

whereas from the backward equation one obtains

$$\frac{d\, m(t)}{d\, t} = Q \left(q_1 - 1 \right) m(t) + Q\, q_2\, n^2(t) \tag{41}$$

with

$$q_2 \equiv \frac{d^2\, g(z,t)}{d\, z^2}\Big|_{z=1} = \langle \nu(\nu-1) \rangle \tag{42}$$

Here, it is seen that the two equations are not identical, and also that the backward equation is linear in the second factorial moment $m(t)$, but contains the square of the first moment as an inhomogeneous term. Further, one notices that the homogeneous part of the backward Equation (41) is identical to that of the first moment Equation (37). This can be made use of by including the initial condition (38) into (37) by rewriting the latter as

$$\frac{d\, n(t)}{d\, t} = Q\,(q_1 - 1)\, n(t) + \delta(t) \tag{43}$$

with the initial condition

$$n(0_-) = 0. \tag{44}$$

A comparison of (41) with (43) shows that the first moment $n(t)$ serves as the Green's function for the second moment $m(t)$. Due to the initial condition $m(0) = 0$, one can thus write

$$m(t) = q_2\, Q \int_0^t n(t-t')\, n^2(t')\, \mathrm{d}t' \tag{45}$$

This shows a generic property of branching processes, namely, that for the higher-order moments, in the forward approach one has to solve a new differential equation for each moment, whereas in the backward approach one only has to solve one equation, that for the first moment, and thereafter one only needs to perform integrals over already known functions. Regarding the higher-order moments, for the backward approach this incurs performing an increasing number of nested integrals. Still, at least conceptually, this expedites a more straightforward calculation of the higher-order moments as compared to the forward approach, which necessitates the solution of new differential equations for each moment. The difference between the forward and backward moment equations is even more apparent for the case of branching processes with several variables (particle types), as will be demonstrated in the next section.

4. Branching Processes with Several Particle Types and Detection

Finally, we are going to treat a case of practical interest, even if the application itself falls outside the scope of this paper. This is the statistics of the neutron detection in a nuclear reactor, in which the neutrons can be absorbed or induce fission. In a real system, scattering events can also occur in which the energy and the direction of the neutrons is changed. However, for the simplicity of the description, following general practice, we will disregard the energy and space variable, i.e., consider an infinite homogeneous medium in one-speed theory (all neutrons are supposed to have the same energy). The neutrons can be absorbed either in the reactor, or in the detector, and only the latter will lead to a neutron count. The detector is also assumed to be distributed homogeneously in the system.

An important property of the physics of nuclear fission is that although the overwhelming majority of the fission neutrons appear immediately (prompt neutrons account for 99.35% of all neutrons in the thermal fission of ^{235}U), a small fraction of neutrons appear with a delay, through the decay of some fission fragments in an excited state that leads to the emission of a neutron. These fission fragments are called delayed neutron precursors, and in order to keep track of the neutrons appearing after some time of the fission process, the time evolution of the delayed neutron precursors needs to be considered together with the number of neutrons. In practice, there are several types of delayed neutron precursors with different decay times, but these can be lumped into one average group with an average decay time. Hence, in order to calculate the moments of the number of detector counts during a time period, a master equation will be derived for the joint distribution of neutrons, the delayed neutron precursors (considering only one averaged group), and that of the neutron counts in an interval in a subcritical system with an extraneous neutron source.

4.1. Forward Approach

Since in the forward approach one operates on the final (terminal) variables, it will be possible to include the extraneous source directly into the equations, as Equation (22) illustrates for the deterministic case. Moreover, one can assume that the source was switched on at $t = -\infty$, and the detection of neutrons was started at $t = 0$, when the system was already in a stationary state. Hence, at time $t = 0$ and subsequent times, the expectation of the number of neutrons and precursors in the system is constant, whereas the number of detections starts from zero. We will then follow the evolution of the detection process for $t \geq 0$.

Let the random processes $\mathbf{N}(t)$ and $\mathbf{C}(t)$ represent the number of neutrons and the delayed neutron precursors at the time instant $t \geq 0$, respectively, due to an extraneous stationary source emitting neutrons randomly into the system. Likewise, let the random process $\mathbf{Z}(t)$ represent the number of detected neutrons in the time interval $[0, t]$. Obviously,

$$\mathbf{Z}(t) = \begin{cases} \mathbf{Z}(t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Define the probability

$$\mathcal{P}\{\mathbf{N}(t) = N, \mathbf{C}(t) = C, \mathbf{Z}(t) = Z | \ \mathbf{N}(-\infty) = 0, \mathbf{C}(-\infty) = 0, \mathbf{Z}(0) = 0\}$$

$$\equiv P(N, C, Z, t). \tag{46}$$

Here, $P(N, C, Z, t)$ is the joint probability that in the subcritical system driven by an extraneous source, at time t there will be N neutrons and C delayed neutron precursors, and that during the time interval $[0, t]$, a total of Z neutron counts were recorded.

For the derivation of the equation for the evolution of the probability distribution, one also needs the transition probabilities, which in turn are related to the nuclear physics parameters and the intensity of the source. Let λ_c, λ_f, and λ_d denote the intensities ("transition probabilities per unit time", with dimensions 1/s) of capture, fission, and detection, respectively. These are obtained from the macroscopic cross-sections Σ and the neutron velocity v as, e.g., $\lambda_c = v\,\Sigma_c$, etc., where Σ_c (cm^{-1}) is the macroscopic cross-section of capture. Further, λ stands for the decay constant of the delayed neutron precursors, whereas $p_f(n, m)$ is the probability of n neutrons being emitted and m delayed neutron precursors generated in a fission event:

$$\mathcal{P}\{\boldsymbol{\nu_p} = n, \boldsymbol{\nu_d} = m\} = p_f(n, m). \tag{47}$$

Finally, S is the source intensity (neutrons/s), i.e., the probability of injecting a neutron into the system by an extraneous source within dt is equal to $S\,dt$.

Following standard procedures, the forward master equation for $P(N, C, Z, t)$ can be written down by considering the possible changes in the state of the system between t and $t + dt$, leading to [13]

$$\begin{aligned} \frac{dP(N, C, Z, t)}{dt} &= \lambda_c P(N+1, C, Z, t)(N+1) + \\ &\lambda_d P(N+1, C, Z-1, t)(N+1) + \\ \lambda_f \sum_n \sum_m & P(N+1-n, C-m, Z, t)(N+1-n)p_f(n, m) + \\ S\ P(N-1, C, Z, t) &+ \lambda P(N-1, C+1, Z, t)(C+1) - \\ &P(N, C, Z, t)[N(\lambda_f + \lambda_c + \lambda_d) + \lambda C + S]. \end{aligned} \tag{48}$$

The initial condition associated with this equation reads as

$$P(N, C, Z, t = 0) = \delta_{N,N_0}\delta_{C,C_0}\delta_{Z,0}. \tag{49}$$

where N_0 and C_0 are the expectations of the number of the neutrons and delayed neutrons in the stationary system, respectively. These will be obtained from the solution of the first moments equations, with the assumption of stationarity (the time derivatives of the expectations of $\mathbf{N}(t)$ and $\mathbf{C}(t)$ are zero).

By defining the generating functions

$$G(x, y, v, t) = \sum_N \sum_C \sum_Z x^N y^C v^Z P(N, C, Z, t) \tag{50}$$

and

$$g_f(x, y) = \sum_n \sum_m x^n y^m p_f(n, m), \tag{51}$$

the following equation is obtained from (48):

$$\begin{gathered} \frac{\partial G(x, y, v, t)}{\partial t} = \\ \left\{\lambda_f[g_f(x, y) - x] - \lambda_c(x - 1) - \lambda_d(x - v)\right\}\frac{\partial G(x, y, v, t)}{\partial x} + \\ \lambda(x - y)\frac{\partial G(x, y, v, t)}{\partial y} + (x - 1)S\, G(x, y, v, t) \end{gathered} \tag{52}$$

with the initial condition

$$G(x, y, v, t = 0) = x^{N_0}\, y^{C_0}. \tag{53}$$

Equations for the first and second moments of the number of neutrons, precursors, and detector counts can be derived from (52) by the usual way of taking derivatives with respect to the parameters x, y, and v. In the continuation, the following notation is introduced. For the expectation of the random processes $\mathbf{N}(t)$, $\mathbf{C}(t)$, and $\mathbf{Z}(t, 0)$, the notation of the expectation value is omitted, e.g.,

$$\mathbf{E}\{\mathbf{N}(t)\} \equiv \langle \mathbf{N}(t) \rangle \equiv N(t) = \left.\frac{\partial G(x, y, v, t)}{\partial x}\right|_{x=y=v=1}. \tag{54}$$

Further, one has

$$\left.\frac{\partial g_f(x, y)}{\partial x}\right|_{x=y=1} = \sum_n \sum_m n\; p_f(n, m) \equiv \langle \nu_p \rangle \equiv \langle \nu \rangle (1 - \beta), \tag{55}$$

$$\left.\frac{\partial g_f(x, y)}{\partial y}\right|_{x=y=1} = \sum_n \sum_m m\; p_f(n, m) \equiv \langle \nu_d \rangle \equiv \langle \nu \rangle \beta, \tag{56}$$

where $\langle \nu \rangle$ is the average total number of neutrons per fission and β is the effective delayed-neutron fraction. The notation

$$\rho = \frac{\langle \nu \rangle \lambda_f - (\lambda_f + \lambda_c + \lambda_d)}{\langle \nu \rangle \lambda_f} \equiv \frac{\langle \nu \rangle \lambda_f - \lambda_a}{\langle \nu \rangle \lambda_f}, \tag{57}$$

$$\Lambda = \frac{1}{\langle \nu \rangle \lambda_f} \quad \text{and} \quad \epsilon = \frac{\lambda_d}{\lambda_f} \tag{58}$$

is standard, i.e., ρ, Λ, and ϵ stand for reactivity, prompt neutron generation time, and detector efficiency, respectively, and $\lambda_a = \lambda_f + \lambda_c + \lambda_d$ is the total intensity of a reaction of any type.

4.1.1. First Moments

The three first moment equations read as follows:

$$\frac{dN(t)}{dt} = \frac{\rho - \beta}{\Lambda} N(t) + \lambda C(t) + S, \tag{59}$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda} N(t) - \lambda C(t) \tag{60}$$

and

$$\frac{dZ(t)}{dt} = \lambda_d N(t) \equiv \varepsilon \lambda_f \, N(t), \qquad t \geq 0 \tag{61}$$

where the detector efficiency ε was introduced by the last identity.

In a steady subcritical medium ($\rho < 0$) with a steady source switched on at time $t = -\infty$, we find that

$$N(0) = N_0, \qquad C(0) = C_0, \qquad \text{and} \qquad Z(0) = 0 \tag{62}$$

By using these initial conditions, one obtains the following stationary solutions:

$$N_0 = \frac{\Lambda \, S}{-\rho}, \tag{63}$$

$$C_0 = \frac{\beta N}{\lambda \Lambda} = \frac{\beta S}{\lambda(-\rho)} \tag{64}$$

and

$$Z(t) = \varepsilon \, \lambda_f N_0 \, t. \tag{65}$$

It is important to note that the argument of $Z(t)$ in (65), which looks formally like a time instant, actually means the measurement time interval $[0, t]$ and not a specific time point.

4.1.2. Second Moments

For the sake of simplicity, we introduce the modified second moment of the random variables **a** and **b** as follows:

$$\mu_{aa} \equiv \langle \mathbf{a}(\mathbf{a} - 1) \rangle - \langle \mathbf{a} \rangle^2 = \sigma_a^2 - \langle \mathbf{a} \rangle \tag{66}$$

and

$$\mu_{ab} \equiv \langle \mathbf{ab} \rangle - \langle \mathbf{a} \rangle \langle \mathbf{b} \rangle \tag{67}$$

where **a** and **b** stand for any of the variables neutron population **N**, precursor population **C**, or detector count **Z**. Then, taking the auto- and cross-derivatives, one obtains the following six equations:

$$\frac{d\mu_{NN}(t)}{dt} = -2 \, \alpha \, \mu_{NN}(t) + 2\lambda \mu_{NC}(t) + \lambda_f \langle \nu_p(\nu_p - 1) \rangle N(t), \tag{68}$$

$$\frac{d\mu_{NC}(t)}{dt} = -(\alpha + \lambda)\mu_{NC}(t) + \frac{\beta}{\Lambda} \mu_{NN}(t) + \lambda \mu_{CC}(t|t_0) + \lambda_f \langle \nu_p \, \nu_d \rangle N(t), \tag{69}$$

$$\frac{d\mu_{CC}(t)}{dt} = -2\lambda \mu_{CC}(t) + 2\frac{\beta}{\Lambda} \mu_{NC}(t) + \lambda_f \langle \nu_d(\nu_d - 1) \rangle N(t), \tag{70}$$

$$\frac{d\mu_{NZ}(t,0)}{dt} = -\alpha \mu_{NZ}(t,0) + \lambda \mu_{CZ}(t,0) + \varepsilon \lambda_f \mu_{NN}(t), \tag{71}$$

$$\frac{d\mu_{CZ}(t,0)}{dt} = -\lambda \mu_{CZ}(t,0) + \frac{\beta}{\Lambda} \mu_{NZ}(t,0) + \varepsilon \lambda_f \mu_{NC}(t), \tag{72}$$

and

$$\frac{d\mu_{ZZ}(t,0)}{dt} = 2\varepsilon\lambda_f\mu_{NZ}(t,0). \tag{73}$$

In the above, the prompt neutron decay constant α was introduced as

$$\alpha = \frac{\beta-\rho}{\Lambda}. \tag{74}$$

Further, notation for some of the second moments is introduced as follows:

$$\langle \nu_p(\nu_p-1)\rangle = \sum_n\sum_m n(n-1)p_f(n,m), \tag{75}$$

$$\langle \nu_p\nu_d\rangle = \sum_n\sum_m n\,m\,p_f(n,m) = \langle \nu_p\rangle\langle \nu_d\rangle, \tag{76}$$

and

$$\langle \nu_d(\nu_d-1)\rangle = \sum_n\sum_m m(m-1)p_f(n,m). \tag{77}$$

Equation (76) is based on the rather plausible assumption that the number of fission neutrons and that of the delayed neutron precursors are independent random variables. Actually, there are no experimental data available on the statistical dependence or independence of the prompt and delayed neutron numbers, so one is left with assumptions only. The term corresponding to (77) is equal to zero, assuming that only one delayed neutron precursor can be generated in a fission.

It is evident that in a stationary system, the modified variances $\mu_{NN}(t)$, $\mu_{NC}(t)$, and μ_{CC} are constant, i.e.,

$$\mu_{NN}(t) = \mu_{NN}, \qquad \mu_{NC}(t) = \mu_{NC} \quad \text{and} \quad \mu_{CC}(t) = \mu_{CC}$$

while $\mu_{NZ}(t)$, $\mu_{CZ}(t)$, and $\mu_{ZZ}(t)$ remain time-dependent, with zero initial conditions at $t = 0$. Hence, μ_{NN}, μ_{NC}, and μ_{CC} can be derived from three algebraic equations, obtained from (68)–(70) by setting the time derivatives on the left-hand side equal to zero, whereas $\mu_{NZ}(t)$, $\mu_{NC}(t)$, and $\mu_{ZZ}(t)$ are derived by solving the time-dependent equations (71)–(73) with the other three (constant) modified moments being known. However, our goal here is not to solve these equations, but rather to compare these equations for the corresponding equations of the backward approach to the same moments, which will be derived in the next subsection.

4.2. Backward Approach

As was discussed earlier, since the backward equation operates on initial variables, it is not possible to write down directly one single backward-type equation for the distribution of the particles in the system when an extraneous source is present which continuously injects particles into the system in a random manner. In such a case it is necessary to progress in two steps [5]. One starts with a master equation that describes the evolution of the population that was started by one single initial neutron. In order to calculate the distribution of a cascade induced by a steady source of particles injected continuously to the system with an intensity S, one needs to use another master equation, expressing the source-induced distributions with the single-particle-induced ones. This latter equation can then be used to calculate the moments of the source-induced cascade from those of a cascade initiated by a single particle.

Since now two different types of distributions need to be handled, to aid keeping track of the notation, some conventions, introduced in refs. [5,14], will be used. The random variables, their probability distribution, the corresponding generating functions, and the moments belonging to a cascade started by a single particle will be denoted by lower case symbols. The same random variables, distributions, generating functions, and corresponding moments that belong to a cascade induced by a source over a time period,

will be denoted by capital letters. The notation in the previous section, referring to the case of source-induced cascades with the forward approach, conform too with these conventions. Since with the forward approach there is no need to calculate the single-particle-induced quantities, only the capital letter notation was used. Similarly to the previous section, the notation of expectation values will be dropped in the formulae, as will be specified below.

In contrast to the forward approach, one cannot specify that the source was switched on at $t = -\infty$, all parameters of the process can only be fixed at $t_0 = 0$. Instead, the stationary values can only be obtained by letting $t \to \infty$. This also means that the detection process will not start at $t = 0$, rather at $t - T$, and we will consider the random number of counts within the time interval $[t - T, t]$.

Accordingly, the following probability distributions are defined:

$$\mathcal{P}\{\mathbf{n}(t) = n, \mathbf{c}(t) = c, \mathbf{z}(t,T) = z | \, \mathbf{n}(0) = 1, \mathbf{c}(0) = 0, \mathbf{z}(0,T) = 0\} \equiv$$

$$p(n, c, z, T, t) \tag{78}$$

where $p(n, c, z, T, t)$ stands for the probability of having n neutrons and c precursors at time t in the system, induced by one initial neutron at $t = 0$, and that there were z detector counts between $t - T$ and t. Its probability generating function is defined as

$$g(x, y, v, T, t) = \sum_n \sum_c \sum_z x^n y^c v^z p(n, c, z, T, t). \tag{79}$$

Similarly, let

$$\mathcal{P}\{\mathbf{N}(t) = N, \mathbf{C}(t) = C, \mathbf{Z}(t,T) = Z | \, \mathbf{N}(0) = 0, \mathbf{C}(0) = 0, \mathbf{Z}(0,T) = 0\} \equiv$$

$$P(N, C, Z, T, t) \tag{80}$$

stand for the probability of having N neutrons and C precursors at time t in the system, and let there be Z detector counts between $t - T$ and t, induced by a source of intensity S, switched on at $t = 0$, given that there were no neutrons and delayed neutron precursors in the system at $t = 0$ and no detector counts have been registered up to time $t = 0$.

In the above, it was implicitly assumed that the mixed-type backward equation will be used, such that the time derivatives and subsequent integrals in the moment calculations with respect to t_0 of the proper backward equation were transferred to the final time t, after which the initial time t_0 was set to zero.

To expedite simplifications of the expressions, the following notations, used already in the foregoing, will be used:

$$\langle \mathbf{n}(t) \rangle \equiv n(t), \quad \langle \mathbf{z}(t,T) \rangle \equiv z(t,T), \tag{81}$$

$$\langle \mathbf{n}(t)(\mathbf{n}(t) - 1) \rangle \equiv m_{nn}(t),$$

$$\langle \mathbf{z}(t,T)(\mathbf{z}(t,T) - 1) \rangle \equiv m_{zz}(t,T) \tag{82}$$

for the moments of the single-particle-induced distributions, and

$$\langle \mathbf{N}(t) \rangle \equiv N(t), \quad \langle \mathbf{Z}(t,T) \rangle \equiv Z(t,T), \tag{83}$$

$$\langle \mathbf{N}(t)(\mathbf{N}(t) - 1) \rangle - \langle \mathbf{N}(t) \rangle^2 \equiv \mu_{NN}(t),$$

$$\langle \mathbf{Z}(t,T)(\mathbf{Z}(t,T) - 1) \rangle - \langle \mathbf{Z}(t,T) \rangle^2 \equiv \mu_{ZZ}(t,T) \tag{84}$$

for the moments of the source-induced distributions. The stationary values of the latter are denoted as

$$\lim_{t \to \infty} N(t) = N, \quad \lim_{t \to \infty} Z(t,T) = Z(T), \tag{85}$$

$$\lim_{t\to\infty}\mu_{NN}(t)=\mu_{NN},\quad \lim_{t\to\infty}\mu_{ZZ}(t,T)=\mu_{ZZ}(T). \tag{86}$$

Similar equations can be derived for the delayed neutron precursors, but these are not listed here since they do not appear in the derivation of the second factorial moment of the detector counts.

4.2.1. Distributions and Moments by One Single Starting Particle

With regards to the equations for the distributions started by one single particle, we need to account for the two possibilities that the process was started either by having one neutron, or one delayed neutron precursor in the system at time $t=0$. Accordingly, we shall derive two coupled equations for the following quantities. In addition to the already defined probability $p(n,c,z,T,t)$ of Equation (78), one needs also to define

$$\begin{gathered}\mathcal{P}\{\mathbf{n}(t)=n,\mathbf{c}(t)=c,\mathbf{z}(t,T)=z|\,\mathbf{n}(0)=0,\mathbf{c}(0)=1,\mathbf{z}(0,T)=0\}\equiv\\ w(n,c,z,T,t)\end{gathered} \tag{87}$$

as the probability that there are n neutrons and c precursors at time t in the system, induced by one initial precursor at $t=0$, and that there have been z detector counts between $t-T$ and t. The corresponding probability generating function is defined as

$$h(x,y,v,T,t)=\sum_n\sum_c\sum_z x^n y^c v^z w(n,c,z,T,t), \tag{88}$$

whereas the generating function $g(x,y,v,T,t)$ of $p(n,c,z,T,t)$ was defined in (79). The initial conditions for the above quantities read as

$$p(n,c,z,T,0)=\delta_{n,1}\delta_{c,0}\delta_{z,0};\qquad g(x,y,v,T,0)=x \tag{89}$$

and

$$w(n,c,z,T,0)=\delta_{n,0}\delta_{c,1}\delta_{z,0};\qquad h(x,y,v,T,0)=y. \tag{90}$$

The master equations for g and h can be obtained as follows. With the usual arguments one writes

$$\begin{aligned}p(n,c,z,T,t)&=(1-\lambda_a dt)p(n,c,z,T,t-dt)+\lambda_c\delta_{n,0}\delta_{c,0}\delta_{z,0}dt+\\ &\lambda_f dt\sum_{k,\ell}p_f(k,\ell)\sum_{\substack{n_1+n_2=n\\ c_1+c_2=c\\ z_1+z_2=z}}A_k(n_1,c_1,z_1,T,t)\;B_\ell(n_2,c_2,z_2,T,t)+\\ &\lambda_d dt\delta_{n,0}\delta_{c,0}\left[\Delta(t,T)\delta_{z,1}+\overline{\Delta}(t,T)\delta_{z,0}\right],\end{aligned} \tag{91}$$

where

$$A_k(n_1,c_1,z_1,T,t)=\sum_{\substack{n_{11}+\cdots+n_{1k}=n_1\\ c_{11}+\cdots+c_{1k}=c_1\\ z_{11}+\cdots+z_{1k}=z_1}}\prod_{j=1}^{k}p(n_{1j},c_{1j},z_{1j},T,t), \tag{92}$$

and

$$B_\ell(n_2,c_2,z_2,T,t)=\sum_{\substack{n_{21}+\cdots+n_{2k}=n_2\\ c_{21}+\cdots+c_{2k}=c_2\\ z_{21}+\cdots+z_{2k}=z_2}}\prod_{j=1}^{\ell}w(n_{2j},c_{2j},z_{2j},T,t). \tag{93}$$

The function $\Delta(t,T)$ is defined as

$$\Delta(t,T)=\begin{cases}1 & \text{for}\quad 0\le t\le T,\\ 0 & \text{otherwise}\end{cases} \tag{94}$$

and $\overline{\Delta}(t,T) = 1 - \Delta(t,T)$.

The terms on the right-hand side of (91) correspond to probabilities of events within the infinitesimal time interval $(0, dt)$ which are mutually exclusive in first order of dt. These are as follows: no collision, capture in the system, fission (leading to k prompt neutrons and ℓ precursors), and absorption in the detector, respectively. Since the temporal evolutions of the individual chains started by the fission neutrons through multiplication are independent from each other, the joint probability is a product of the individual probabilities, subject to the constraint that they together lead to n neutrons, c precursors, and z detector counts.

With similar arguments, for the cascade induced by a delayed neutron precursor one obtains

$$w(n,c,z,T,t) = (1-\lambda dt)w(n,c,z,T,t-dt) + \lambda dt p(n,c,z,T,t). \tag{95}$$

The notations in the above equations have their usual meaning. From (91) and (95), one obtains the following differential equations for the generating functions g and h of (79) and (88):

$$\begin{aligned}\frac{\partial g(x,y,v,T,t)}{\partial t} &= \lambda_f \sum_k \sum_\ell p_f(k,\ell) g^k(x,y,v,T,t)\; h^\ell(x,y,v,T,t) + \\ &\lambda_c - \lambda_a g(x,y,v,T,t) + \lambda_d\{(v-1)\Delta(t,T)+1\}\end{aligned} \tag{96}$$

and

$$\frac{\partial h(x,y,v,T,t)}{\partial t} = \lambda\{g(x,y,v,T,t) - h(x,y,v,T,t)\}. \tag{97}$$

By the definition (51) of the generating function $g_f(x,y)$ of the probability distribution $p_f(n,m)$ of the numbers of neutrons and delayed neutron precursors generated in fission, (96) can be written in the more concise form

$$\begin{aligned}\frac{\partial g(x,y,v,T,t)}{\partial t} &= \lambda_f\, g_f\big[g(x,y,v,T,t),\; h(x,y,v,T,t)\big] + \\ &\lambda_c - \lambda_a g(x,y,v,T,t) + \lambda_d\{(v-1)\Delta(t,T)+1\}\end{aligned} \tag{98}$$

where the generating functions $g(x,y,v,T,t)$ and $h(x,y,v,T,t)$ are arguments of $g_f(x,y)$.

Equation (97) can be solved explicitly. By accounting for the initial condition (90), one obtains

$$h(x,y,v,T,t) = \lambda \int_0^t e^{-\lambda(t-t')} g(x,y,v,T,t')dt' + ye^{-\lambda t}. \tag{99}$$

Using this in (96) to eliminate $h(x,y,v,T,t)$, one arrives at one single equation, from which all statistics can be derived, as

$$\begin{aligned}\frac{\partial g(x,y,v,T,t)}{\partial t} &= \lambda_f\, g_f\left[g(x,y,v,T,t),\; \lambda \int_0^t e^{-\lambda(t-t')} g(x,y,v,T,t')\; dt' + y\, e^{-\lambda t}\right] + \\ &\lambda_c - \lambda_a g(x,y,v,T,t) + \lambda_d\{(v-1)\Delta(t,T)+1\}.\end{aligned} \tag{100}$$

An important difference can be noticed here compared to the forward equation. As (100) shows, in the backward approach, having eliminated $h(x,y,v,T,t)$ via (99), one obtains one single equation for $g(x,y,v,T,t)$. Unlike the forward Equation (52), this equation does not contain any derivatives with respect to the variables x, y, and v. Hence, for any moment, i.e., expectations of any order, one can derive one single equation, which can be solved separately from the other moment equations, i.e., no knowledge of the other moments is needed for the solution. As will be seen soon, the only technical difficulty of the solution is the calculation of certain nested integrals.

In the forward approach, on the other hand, although there is also one single master equation as a starting point, the equation contains derivatives with respect to x and y. Because of this, for all moments of all orders, except the first moment of the detector count,

a coupled system of differential equations has to be solved. The order of the coupled system increases with the order of the moments. This, in general, poses more difficulties in the solution than the evaluation of the nested integrals in the backward approach.

Although our purpose here is to compare only the differing structures of the forward and the backward approaches, without solving the equations, for the backward approach this is only possible by outlining the course of the solutions. The moments of the single-particle-induced distribution can be calculated as follows. For the first moment

$$n(t) = \left.\frac{\partial g(x,y,v,T,t)}{\partial x}\right|_{x=y=v=1}, \tag{101}$$

one obtains from (100) the equation

$$\frac{dn(t)}{dt} = \lambda_f \nu(1-\beta) n(t) + \nu\beta\lambda_f\lambda \int_0^t e^{-\lambda(t-t')} n(t')dt' -$$

$$\lambda_a n(t) + \delta(t). \tag{102}$$

Here, the moments $\nu \equiv \langle \nu_p \rangle + \langle \nu_d \rangle$ and β were introduced as

$$\langle \nu_p \rangle = \left.\frac{\partial g_f(x,y)}{\partial x}\right|_{x=y=1} = \sum_k \sum_\ell k\ p_f(k,\ell) \equiv \nu(1-\beta) \tag{103}$$

and

$$\langle \nu_d \rangle = \left.\frac{\partial g_f(x,y)}{\partial y}\right|_{x=y=1} = \sum_\ell \sum_\ell \ell\ p_f(k,\ell) \equiv \nu\beta. \tag{104}$$

Similarly to (43), in Equation (102) the initial condition (89) was added directly to the equation such that one has

$$\langle n(t) \rangle|_{t=-0} = 0. \tag{105}$$

This step was made to help realise later that the first moment $n(t)$ is the Green's function of all the higher moments. To this end, (102) is re-written by the introduction of the linear operator $\widehat{\mathbb{M}}(t)$ as

$$\frac{dn(t)}{dt} = \widehat{\mathbb{M}}(t)\, n(t) \ + \ \delta(t). \tag{106}$$

where $\widehat{\mathbb{M}}(t)$ is defined through (102).

For the second moment

$$m_{nn}(t) = \left.\frac{\partial^2 g(x,y,v,T,t)}{\partial x^2}\right|_{x=y=v=1} \tag{107}$$

one obtains the equation

$$\frac{dm_{nn}(t)}{dt} = \lambda_f \nu(1-\beta) m_{nn}(t) +$$

$$\nu\beta\lambda_f\lambda \int_0^t e^{-\lambda(t-t')} m_{nn}(t')dt' - \lambda_a m_{nn}(t) +$$

$$\lambda_f \left[\langle \nu_p(\nu_p - 1) \rangle n^2(t) + 2\langle \nu_p \rangle \langle \nu_d \rangle n(t) \lambda \int_0^t e^{-\lambda(t-t')} n(t')dt' \right], \tag{108}$$

where $\langle \nu_p(\nu_p - 1) \rangle$ and $\langle \nu_p \rangle \langle \nu_d \rangle$ were defined in (75) and (76) with the subsequent comments. Rearranging, this can be written as

$$\frac{dm_{nn}(t)}{dt} = \lambda_f \nu(1-\beta) m_{nn}(t) + \nu\beta\lambda_f\lambda \int_0^t e^{-\lambda(t-t')} m_{nn}(t')dt' - \lambda_a m_{nn}(t) + q_{nn}(t) \tag{109}$$

which, in terms of the integral operator $\widehat{\mathbb{M}}(t)$ introduced in (106), can be further shortened as

$$\frac{dm_{nn}(t)}{dt} = \widehat{\mathbb{M}}(t) m_{nn}(t) + q_{nn}(t) \tag{110}$$

with

$$q_{nn}(t) = \lambda_f \left[\langle \nu_p(\nu_p - 1) \rangle n^2(t) + 2\langle \nu_p \rangle \langle \nu_d \rangle n(t) \int_0^t e^{-\lambda(t-t')} n(t')dt' \right], \tag{111}$$

Once the equation for the first moment $n(t)$ is solved, $q_{nn}(t)$ is known, and constitutes the inhomogeneous part of (110). A comparison with (106) shows that this way $n(t)$ serves also as the Green's function of the equation for $m_{nn}(t)$ with the source term $q_{nn}(t)$. Since, as seen from (89), the initial condition for $m_{nn}(t)$ is

$$m_{nn}(0) = 0, \tag{112}$$

the solution for $m_{nn}(t)$ can be written as

$$m_{nn}(t) = \int_0^t n(t-t') q_{nn}(t')dt', \tag{113}$$

As will be seen below, to obtain μ_{NN} one does not need to evaluate (113), only to identify the term $q_{nn}(t)$.

The example above demonstrates the already mentioned fact that in the backward approach, a single explicit expression is obtained for moments of all orders, which then can be evaluated separately from all the other auto- and cross-moments. On the other hand, in the forward approach, as was seen earlier, a coupled differential equation system between various auto- and cross-moments is obtained for all higher-order moments.

The mean and the modified variance of the detector counts of the single-particle cascades can be calculated by taking

$$z(t,T) = \left. \frac{\partial g(x,y,v,T,t)}{\partial v} \right|_{x=y=v=1} \tag{114}$$

and

$$m_{zz}(t,T) = \left. \frac{\partial^2 g(x,y,v,T,t)}{\partial v^2} \right|_{x=y=v=1} \tag{115}$$

from (100), respectively. For the first moment, one obtains the equation

$$\begin{aligned} \frac{dz(t,T)}{dt} &= \lambda_f \nu(1-\beta)\, z(t,T) + \nu\beta\lambda_f\lambda \textstyle\int_0^t e^{-\lambda(t-t')} z(t',T)dt' - \\ &\quad \lambda_a z(t,T) + \lambda_d \Delta(t,T) \\ &= \widehat{\mathbb{M}}(t)\, z(t,T) + \lambda_d \Delta(t,T). \end{aligned} \tag{116}$$

The initial condition is $z(0,T) = 0$. Hence, similarly to the moments of the number of neutrons in the system, the solution of (116) can be written as

$$z(t,T) = \lambda_d \int_0^t n(t-t')\Delta(t',T)dt'. \tag{117}$$

The second moment can be calculated by applying (115) to (100) with the result

$$\begin{aligned}\frac{d\,m_{zz}(t,T)}{dt} &= \lambda_f \nu(1-\beta) m_{zz}(t,T) + \nu\beta\lambda_f \lambda \int_0^t e^{-\lambda(t-t')} m_{zz}(t',T)dt' - \\ &\lambda_a m_{zz}(t,T) + q_{zz}(t,T) \\ &= \widehat{\mathbb{M}}(t)\, m_{zz}(t,T) + q_{zz}(t,T),\end{aligned} \tag{118}$$

where the source term is given as

$$\begin{aligned}q_{zz}(t,T) &= \lambda_f \langle \nu_p(\nu_p - 1)\rangle z^2(t,T) + \\ 2\lambda_f \langle \nu_p\rangle \langle \nu_d\rangle z(t,T) \lambda \int_0^t e^{-\lambda(t-t')} z(t',T)dt'.\end{aligned} \tag{119}$$

The initial condition, similarly to that of m_{nn}, is

$$m_{zz}(0,T) = 0, \tag{120}$$

see (89). Hence, the solution can be written in the form

$$m_{zz}(t,T) = \int_0^t n(t-t') q_{zz}(t',T)dt'. \tag{121}$$

That is, to calculate $m_{zz}(t)$, it was sufficient to identify the source term q_{zz}.

4.2.2. Distributions by a Process Maintained by an Extraneous Source

The derivation of the master equation that connects the single-particle-induced distribution with that of the source-induced one can be made by considering the probabilities, in first order of dt, of the mutually exclusive events of no source emission or one source neutron emission within the initial time interval $[0, dt]$:

$$\begin{aligned}P(N,C,Z,T,t) &= (1 - S\,dt) P(N,C,Z,T,t-dt) + \\ S\,dt \sum_{\substack{N_1+n_2=N \\ C_1+c_2=C \\ Z_1+z_2=Z}} &P(N_1,C_1,Z_1,T,t)\; p(n_2,c_2,z_2,T,t).\end{aligned} \tag{122}$$

Introducing the probability generating function

$$G(x,y,v,T,t) = \sum_N \sum_C \sum_Z x^N y^C v^Z P(N,C,Z,T,t) \tag{123}$$

one obtains from (122) the following differential equation:

$$\frac{dG(x,y,v,T,t)}{dt} = S\, G(x,y,v,T,t)\{g(x,y,v,T,t) - 1\}. \tag{124}$$

Taking into account the initial conditions

$$g(x,y,v,T,0) = x \quad \text{and} \quad G(x,y,v,T,0) = 1,$$

the solution of (124) is obtained as

$$G(x,y,v,T,t) = \exp\left\{ S \int_0^t \left[g(x,y,v,T,t') - 1 \right] dt' \right\}. \tag{125}$$

It can also be concluded that the asymptotic value of the generating function, i.e., the limit

$$\lim_{t\to\infty} G(x,y,v,T,t) = \exp\left\{ S \int_0^\infty \left[g(x,y,v,T,t') - 1 \right] dt' \right\} \tag{126}$$

is finite if the system is subcritical.

The moments of the source-induced distribution can be easily derived from the single-particle-induced ones by calculating the derivatives. Here, we shall use the same notation as in (81)–(86). The stationary (asymptotic) expectation of the number of neutrons in the source-driven system is given as

$$N = S \int_0^\infty n(t) dt. \tag{127}$$

The stationary value of the modified second moment of the source-induced neutron number, μ_{NN}, is obtained as an integral over the second factorial moment of the single-particle-induced distribution $m_{nn}(t)$

$$\mu_{NN} = S \int_0^\infty m_{nn}(t) dt. \tag{128}$$

As was seen in the previous section, the second factorial moment $m_{nn}(t)$ can be represented as a convolution of the single-particle-induced first moment, $n(t)$, which is the basic Green's function of the process, and the source function $q_{nn}(t)$ as

$$m_{nn}(t) = \int_0^t n(t-t') q_{nn}(t') dt'. \tag{129}$$

Hence, by virtue of (127)–(129), one has

$$\mu_{NN} = N \int_0^\infty q_{nn}(t) dt. \tag{130}$$

In an analogous manner, for the first moment $Z(T)$ and the modified second factorial moment, μ_{ZZ}, of the number of detector counts one obtains

$$Z(T) = S \int_0^\infty z(t,T) dt \tag{131}$$

and

$$\mu_{ZZ}(T) = S \int_0^\infty m_{zz}(t) dt = N \int_0^\infty q_{zz}(t,T) dt, \tag{132}$$

where again we made use of the fact that the single-particle-induced factorial moment $m_{zz}(t,T)$ is given as a convolution integral of $n(t)$ with the source function $q_{zz}(t,T)$ as

$$m_{zz}(t,T) = \int_0^t n(t-t') q_{zz}(t',T) dt'. \tag{133}$$

It is thus seen that although both the forward and the backward approaches lead to the same result, the structure of the equations is increasingly different with the increasing of the model order. Moreover, when the process is driven by an extraneous source and contains more than one variable (particle type), the two approaches result in rather different equations already for the first moment.

5. Discussion

As was mentioned in the Introduction, the reason for the non-adjoint property of any process, whether deterministic or stochastic, is the fact that the process is not invariant for a time reversal; following the process backward in time leads to a behaviour which is not possible for the direct process. For the deterministic transport equation, the reason for the irreversibility is that in the direct approach, neutrons leave the system at the boundaries,

but do not enter it from the outside; and also that the energy degradation of the high energy fission neutrons happens in small steps, through elastic and inelastic scattering, whereas the energy increase takes place in one single step, when the already slowed down (thermalised) neutrons induce fission. In the deterministic case, the branching does not appear explicitly: one follows only the fate of one neutron at a time, and the increase in the number of neutrons in fission is only described as an increased weight of the same neutron, described by the factor c of Equation (14). In a one-speed model without space dependence, the equations are self-adjoint, and the forward and backward equations are identical for the first moment, as was seen for the basic branching process, Equation (37).

However, it was also seen that even for the basic branching process with one particle type, the forward and backward equations for the generating function were rather different. Moreover, the equations were already different for the second factorial moment, even if the difference was milder than for the generating function. This can be explained by the increasing level of irreversibility of the branching process with the increasing moment order. From the mathematical point of view, since the forward equation operates on the final coordinates, it can only handle infinitesimal volumes of the phase space or time intervals. In order to calculate the variance (second moment) of the total number of particles in a finite volume of the phase space with the forward approach, one needs to treat two-point densities, such as the covariance function

$$c(\boldsymbol{R}_1\,\boldsymbol{R}_2) = n(\boldsymbol{R}_1\,\boldsymbol{R}_2) - n(\boldsymbol{R}_1)\,n(\boldsymbol{R}_2), \tag{134}$$

where $n(\boldsymbol{R}_1\,\boldsymbol{R}_2)$ is the joint density of finding a neutron around the point $\boldsymbol{R}_1$ and another at the point $\boldsymbol{R}_2$ of the phase space simultaneously, whereas $n(\boldsymbol{R}_1)$ and $n(\boldsymbol{R}_2)$ are the one-point densities around $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, respectively. Physically, this means that in order to calculate the second moment, it is necessary to consider particle pairs and follow up the evolution of both particles simultaneously. That is, one needs to consider that in a branching (e.g., fission), at least two particles are born simultaneously. The backward equation, which operates on the initial coordinates, permits writing down equations for the probability distribution in a finite volume of the phase space at the terminal time, and hence it can handle the original branching process, which the forward form cannot, due to the lack of invariance of time reversal. Namely, in the time-reversed process, branching would require the disappearance of two particles simultaneously. In a physical picture, this would require that two neutrons (and corresponding fission products) should collide simultaneously with the proper energies to build a nucleus which emits just one neutron. Similarly, for the calculation of the third moment of the number of particles in a finite phase space with the forward equations, one needs to treat three-point densities, and hence, consider the cases when three neutrons are born in a fission process. It is easy to see that the time-reversed process is even more unlikely, since it would require that three neutrons and two fission fragments should collide simultaneously, leading to the emission of one single neutron.

In the basic branching process of Section 3, the evolution of the fission fragments was not followed up, hence, the time reversal only applies to the neutrons. However, the probability of even such a reversed event is vanishing. For the third moment, one needs to consider triplets of particles born simultaneously, and for the time-reversed process, this would require three neutrons and corresponding fission fragments collide simultaneously, which is even more unlikely than the same process with only two neutrons, and so on. The equations for the generating functions contain all possible branching events, and hence the difference between the forward and the backward forms is the largest, as was seen for the basic branching process treated in Section 3.

A process with several variables (particle types) and added processes (detection and extraneous source) increases the lack of invariance to time reversal further. Unlike in the basic branching process, for the neutron processes in reactors, treated in Section 4, also the fission fragments (delayed neutron precursors) are followed up. Hence, in that case, the time-reversed process would require not only that several neutrons, but also a delayed neutron precursor should collide simultaneously, which again is an even more unlikely

event to happen than the simultaneous collision of neutrons. The presence of the extraneous source incurs even larger differences between the two forms, because its inclusion into the backward equations is only possible by a two-step process.

Finally, it is worth mentioning that the lack of invariance against time reversal in neutron transport as a branching process, as described above, is a statistical phenomenon, unlike the more fundamental symmetry breaking phenomena in quantum field theories, such as the violation of time (T), charge (C), and/or parity (P) reversal, or their double combinations, such as CT or PT violation. In classical particle transport, the underlying physical processes (particle collisions) are reversible. There is no physical law which would forbid a neutron continuously gaining energy by colliding with energetic moving target nuclei all the way from thermal energies (meV) to the energy of fission neutrons (MeV). Such energy gain of neutrons takes place at thermal energies due to the thermal motion of the scattering nuclides, but it is unlikely that a neutron in the epithermal or fast region would collide with an even faster moving nuclide. It is the vanishingly low probability of such events which leads to the irreversibility of the process. It is exactly this probabilistic reason of the symmetry breaking which made it possible to order the processes according to the increasing order of the vanishing of the likelihood of invariance for time reversal, and match them with the increasing asymmetry between the corresponding forward and backward equations.

6. Conclusions

In this paper, the differences between the forward and the backward approaches to branching processes, and in particular for neutron fluctuations in nuclear reactors, were demonstrated. We argue that the increasing difference between the two forms with increasing order of the statistical moments of the process, as well as with the number of variables, can be related to the increasing order of zero likelihood of the time-reversed process, which was illustrated with some examples.

It has to be added that, although in this paper only neutron chains were considered as examples, the insight and the results presented here are equally applicable in other processes including branching, such as population dynamics, epidemiology (spread of diseases), and many other processes.

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References

1. Watson, H.W.; Galton, F. On the probability of the extinction of families. *J. Anthropol. Inst.* **1875**, *4*, 103. [CrossRef]
2. Harris, T. *The Theory of Branching Processes*; Prentice-Hall Inc.: Englewood Cliffs, NJ, USA, 1963.
3. Bell, G.I.; Glasstone, S. *Nuclear Reactor Theory*; Van Nostrand Reinhold Company: New York, NY, USA, 1970.
4. Pázsit, I. *Transport Theory and Stochastic Processes*; Chalmers Lecture Notes; Chalmers University of Technology: Göteborg, Sweden, 2023.
5. Pázsit, I.; Pál, L. *Neutron Fluctuations: A Treatise on the Physics of Branching Processes*; Elsevier Ltd.: London, UK; New York, NY, USA; Tokyo, Japan, 2008.
6. Williams, M.M.R. Some considerations on the forward and backward Boltzmann equations. *Ann. Nucl. Energy* **1978**, *5*, 149–150. [CrossRef]
7. Williams, M.M.R. The role of the Boltzmann transport equation in radiation damage calculations. *Prog. Nucl. Energy* **1979**, *3*, 1–65. [CrossRef]
8. Pázsit, I. Duality in Transport-Theory. *Ann. Nucl. Energy* **1987**, *14*, 25–41. [CrossRef]

9. Williams, M.M.R. The relationship between the backward and forward equations of stochastic neutron transport and associated matters. *Ann. Nucl. Energy* **2023**, *186*, 109746. [CrossRef]
10. Lewins, J. *Importance: The Adjoint Function*; Pergamon Press: Oxford, UK; New York, NY, USA, 1965.
11. Pázsit, I. A Simple Derivation of the Neutron Transport Equation. *Nucl. Sci. Eng.* **1992**, *112*, 369–374. [CrossRef]
12. Srinivasan, S.K. *Stochastic Theory and Cascade Processes*; Number 15 in Modern Analytic and Computational Methods in Science and Mathematics; Elsevier: New York, NY, USA; London, UK; Amsterdam, The Netherlands, 1969.
13. Pázsit, I.; Yamane, Y. The variance-to-mean ratio in subcritical systems driven by a spallation source. *Ann. Nucl. Energy* **1998**, *25*, 667–676. [CrossRef]
14. Pázsit, I.; Yamane, Y. The backward theory of Feynman- and Rossi-alpha methods with multiple emission sources. *Nucl. Sci. Eng.* **1999**, *133*, 269–281. [CrossRef]