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ORIGINAL PAPER

Nonlinear semigroups for nonlocal conservation laws

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Abstract
We investigate a class of nonlocal conservation laws in several space dimensions, where the continuum average of weighted nonlocal interactions are considered over a finite horizon. We establish well-posedness for a broad class of flux functions and initial data via semigroup theory in Banach spaces and, in particular, via the celebrated Crandall–Liggett Theorem. We also show that the unique mild solution satisfies a Kružkov-type nonlocal entropy inequality. Similarly to the local case, we demonstrate an efficient way of proving various desirable qualitative properties of the unique solution.

Keywords Nonlocal differential equation · Conservation law · Nonlinear semigroup

Mathematics Subject Classification 35F25 · 35Q49 · 45K05

1 Introduction

We study the semigroup theory of nonlocal conservation laws of the form

$$\begin{aligned} &\frac{\partial u}{\partial t} + \int_{\mathbb{R}^n} \sum_{i=1}^{k} \frac{\phi_i(u, \tau_{\beta_i(h)}u) - \phi_i(\tau_{-\beta_i(h)}u, u)}{||\beta_i(h)||_{\mathbb{R}^n}} \omega_i\big(\beta_i(h)\big)\mathrm{d}h = 0, \quad &\text{in } \mathbb{R}^n \times \mathbb{R}_+; \\ &u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $\tau_{\pm h}u(x,t) = u(x \pm h, t)$ denote a spatial shift of the conserved quantity $u(x,t)$ and the flux functions $\phi_i : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are assumed to be increasing with respect to their first arguments and decreasing with respect to their second arguments, and to have the property

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$\phi_i(0,0) = 0$. The number $1 \leq k \leq n$ denotes the number of subinteractions and the functions $\beta_i : \mathbb{R}^n \mapsto \mathbb{R}^n$ are assumed to be of the form

$$\beta_i(h) = \sum_{j \in B_i} h_j e_j, \qquad h = (h_1, h_2, \dots, h_n),$$

where the nonempty, pairwise disjoint sets $B_i \subset \{1, 2, \dots, n\}$ are such that $\bigcup_{i=1}^k B_i = \{1, 2, \dots, n\}$ and e_j denotes the jth unit vector in $\mathbb{R}^n$. The kernel functions $\omega_i \in \mathcal{L}^1(\mathbb{R}^n) \cap \mathcal{L}^\infty(\mathbb{R}^n)$ are assumed to be nonnegative with $||\omega_i(\beta_i(.))||_{\mathcal{L}^1(\mathbb{R}^n)} = 1$. We further assume that the support of the kernel functions are finite and are either

1. symmetric around the origin, in which case we further assume that the kernels are even, or
2. contained in $\mathbb{R}^n_+$ such that the closure contains the origin.

For example, in the context of nonlocal particle flows, the above cases allows us to differentiate between multidirectional and unidirectional flows.

Our main examples for the choice of k, β_i and ω_i are as follows.

1. If $k = 1$ and $\beta_1(h) = h$, then the conservation law (1) takes the form

$$\frac{\partial u}{\partial t} + \int_{\mathbb{R}^n} \frac{\phi_1(u, \tau_h u) - \phi_1(\tau_{-h} u, u)}{||h||_{\mathbb{R}^n}} \omega(h) \mathrm{d}h = 0. \tag{2}$$

   This case describes a natural multidirectional generalization of the one-dimensional unidirectional nonlocal pair-interaction model investigated in [20]. In fact, if $n = 1$ and $\operatorname{supp}(\omega) \subset \mathbb{R}_+$, the law (2) coincides with the latter.
2. If $k = n$ and $\beta_i(h) = h_i e_i$ and $\omega_i(h) = \prod_{j=1}^n \tilde{\omega}_j(h_j)$, where the kernel functions $\tilde{\omega}_j$ have analogous properties to that of ω_i in $\mathbb{R}$ with $\operatorname{supp}(\tilde{\omega}_j) = (-\delta_j, \delta_j)$ for $\delta_j > 0$, then the conservation law (1) takes the form

$$\frac{\partial u}{\partial t} + \sum_{i=1}^n \int_{-\delta_i}^{\delta_i} \frac{\phi_i(u, \tau_{h_i e_i} u) - \phi_i(\tau_{-h_i e_i} u, u)}{|h_i|} \tilde{\omega}_i(h_i) \mathrm{d}h_i = 0.$$

   Should the underlying model allow such considerations, this case corresponds to interactions that can be unfolded into subinteractions along the individual axes. A clear advantage of this example is the ease of numerical approximation of the integral as described in [20, Section 3.1]. If $n = 1$ and $\operatorname{supp}(\tilde{\omega}_1) = (0, \delta_1)$ instead, then again, we obtain the one-dimensional unidirectional nonlocal pair-interaction model of [20], as in the previous special case.

We say that the nonlocal flux functions ϕ_i are consistent with the local fluxes ψ_i if $\phi_i(a, a) = \psi_i(a)$ holds for all $a \in \mathbb{R}$. For consistent flux functions, if in addition, the weighting kernels are smooth with their support approaching zero, both special cases formally lead to the standard local conservation law

$$\frac{\partial u}{\partial t} + \sum_{i=1}^n \frac{\partial \psi_i(u)}{\partial x_i} = 0. \tag{3}$$

For the formal derivation of (1) we utilize the nonlocal vector calculus established in [19, 25]. Let $\boldsymbol{\nu}, \tilde{\boldsymbol{\nu}}, \boldsymbol{\alpha} : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^k$ be vector two-point functions defined by the coordinate functions

$$\begin{aligned}
\boldsymbol{\nu}_i(u)(x,y,t) &= \phi_i\Big(u(x,t), u\big(x+\beta_i(y-x),t\big)\Big),\\
\tilde{\boldsymbol{\nu}}_i(u)(x,y,t) &= \phi_i\Big(u\big(x+\beta_i(y-x),t\big), u(x,t)\Big),\\
\boldsymbol{\alpha}_i(x,y) &= \frac{\omega_i\big(\beta_i(y-x)\big)}{||\beta_i(y-x)||_{\mathbb{R}^n}}.
\end{aligned}$$

Then, the nonlocal point divergence is defined as

$$\mathcal{D}\big(\boldsymbol{\nu}(u), \tilde{\boldsymbol{\nu}}(u)\big)(x,t) = \int_{\mathbb{R}^n} \big(\boldsymbol{\nu}(u)(x,y,t) - \tilde{\boldsymbol{\nu}}(u)(x,y,t)\big)\cdot \boldsymbol{\alpha}(x,y)\mathrm{d}y$$

and repeated changes of variables in the integral gives

$$\begin{aligned}
\mathcal{D}\big(\boldsymbol{\nu}(u), \tilde{\boldsymbol{\nu}}(u)\big)(x,t) &= \int_{\mathbb{R}^n}\sum_{i=1}^{k}\frac{\phi_i(u,\tau_{\beta_i(h)}u)-\phi_i(\tau_{\beta_i(h)}u,u)}{||\beta_i(h)||_{\mathbb{R}^n}}\omega_i\big(\beta_i(h)\big)\mathrm{d}h\\
&= \int_{\mathbb{R}^n}\sum_{i=1}^{k}\frac{\phi_i(u,\tau_{\beta_i(h)}u)-\phi_i(\tau_{-\beta_i(h)}u,u)}{||\beta_i(h)||_{\mathbb{R}^n}}\omega_i\big(\beta_i(h)\big)\mathrm{d}h.
\end{aligned} \tag{4}$$

The theory of abstract balance laws thoroughly discussed in [19, Sect. 7] shows that in the absence of external sources a class of nonlocal balance laws are given by

$$\frac{\partial u}{\partial t}(x,t) + \mathcal{D}\big(\boldsymbol{\nu}(u), \tilde{\boldsymbol{\nu}}(u)\big)(x,t) = 0,$$

which, combined with (4), gives exactly the law (1).

Local conservation and balance laws have been widely used in aerodynamics and Eulerian gas dynamics [32], pedestrian flows [39], ribosome flows [37] and many other fields [41] for the past decades. In recent years nonlocality has been introduced in multiple forms. A particular method is considering a nonlocal velocity, often expressed as a spatial convolution. The resulting family of models found many applications, for example for modelling supply chains [27, 28, 40] and traffic flows [8, 24]. However, in some cases these models lack monotonicity of solutions and violate the maximum principle, two naturally imposed property of physical models. Certain convolution based models do satisfy these constraints, see for example [2, 29]. Another approach to spatial nonlocality is considering pointwise interactions weighted by an appropriate integral kernel [18], which has many applications in the field of peridynamics [3, 26, 34]. However, the nonlocal model of [18] failed to preserve monotonicity, hence the authors formalized the nonlocal pair-interaction model in [20], where they show that it satisfies the desired properties. A final advantage of this model is that it reduces to its local counterpart (3) as the nonlocal horizon vanishes [21], while some other nonlocal models do not have this property [9]. Because of these improvements the nonlocal pair-interaction model, similarly to its earlier version, has seen many applications in the field of peridynamics [1, 43], while similar integral terms can be seen in nonlocal formulations of other problems as well, for example of that of the Allen–Cahn equation [42].

It is well known that the solution of (1) (including the local case (3) as well) may develop spatial discontinuities (shock waves) over time, even if the initial data is smooth. Hence the Cauchy problem must be considered in a weak or generalized sense. However, there might be infinitely many weak solutions of (1) for given initial data. This fact lead to the development of additional constraints, such as the entropy condition, selecting the unique, physically relevant weak solution, which in this case is the so-called entropy solution.

The well-posedness of the local conservation law (3) is a thoroughly investigated problem, heavily influenced by the profound work of Kružkov [31]. Kružkov showed uniqueness via a priori estimates and existence using the vanishing viscosity method for bounded and measurable initial data and sufficiently smooth flux functions, thus achieving well-posedness. Existence of entropy solutions can often be proved by the convergence of an appropriate numerical scheme [15, 38] (the technique was first used to prove the existence of weak solutions [10, 17]). Another classical framework is nonlinear semigroup theory and, in particular, the celebrated Crandall–Liggett Theorem [13], which was first used to prove well-posedness by Crandall [11]. Many combinations of these approaches were developed, a notable example being the approximation of semigroups of contractions [35].

The well-posedness of the one-dimensional nonlocal Cauchy problem with $\beta_1(h) = h$ was investigated in [20], where Kružkov's method was applied to prove uniqueness and existence was proved by the convergence of an appropriate finite volume scheme. While this approach could be extended for multidimensional non-homogeneous Cauchy problems in some special cases (see our second example above), the method is difficult to apply in the generality of (1) if $k < n$. Instead, we will also work with the semigroup framework, which provides an elegant way of handling further problems like inhomogeneous conservation laws [5] or error control of finite volume methods [36]. Another particular advantage of semigroup theory is the ability to handle $\mathcal{L}^1(\mathbb{R}^n)$ initial data, while with the methods of [20] one can only show existence and uniqueness for $\mathcal{L}^1(\mathbb{R}^n) \cap \mathcal{L}^\infty(\mathbb{R}^n)$ initial data. The semigroup framework considers generalized solutions of abstract Cauchy problems, often called mild solutions. In general, a mild solution can coincide with a weak solution or an entropy solution or, in some cases, with neither; after proving well-posedness an additional investigation is necessary to determine this.

The main results of the paper are contained in Theorems 3.8 and 3.9 and Corollary 3.11. In Theorem 3.8, we give appropriate circumstances under which there exists an operator satisfying the assumptions of the Crandall-Liggett Theorem. In Theorem 3.9, we show that the unique mild solution of (1) satisfies a nonlocal Kružkov-type entropy inequality and has many other qualitative properties that are desirable from a physical point of view. In Corollary 3.11 we extend the well-posedness to conservation laws under Carathéodory forcing.

The outline of the paper is as follows. In Sect. 2, we introduce notations and the abstract framework. In Sect. 3, we give the necessary definitions and state our main results. Section 4 contains the proof of the main results. The main steps of the proofs are based on [11], however, there are significant nontrivial differences in the details. The difficulty in carrying out this construction is the absence of flux derivatives rendering the method of integration by parts and thus many simplifying steps inapplicable. Most of these complications can be solved by a formally similar technique obtained via changes of variables in the integrals; the technique is often called integration by parts for difference quotients, see, for example [22, page 295]. However, a significant step that cannot be resolved in such manner is the verification of the range condition. Crandall uses a perturbation results to establish this, namely [30, Theorem 3.2], but this approach does not seem to be applicable in the nonlocal setting. Instead, we use a fix-point based approach similar to that of [33, Chapter 4] and [14, Proposition IV.3]. Throughout the paper the arguments of the functions β_i and ω_i are omitted unless necessary and C is used as a generic constant that may take on different values at different occurrences.

2 Preliminaries

We give a brief introduction of the abstract setting based on [4, 11, 12].

2.1 Mild solutions of the abstract Cauchy problem

Let X be a real Banach space and A be a possibly multivalued operator in X and $J = [0, T] \subset \mathbb{R}$ and $f \in \mathcal{L}^1(J, X)$. Consider the quasi-autonomous Cauchy problem

$$\begin{aligned} &u' + Au \ni f(t), \qquad t \in J; \\ &u(0) = u_0 \end{aligned} \tag{5}$$

for $u_0 \in \overline{D(A)}$. We call $u \in \mathcal{C}(J, X)$ a mild solution of (5) if for every $\epsilon > 0$ there exists a partition $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_N$ of $[0, t_N]$ and sequences $\{z_1, z_2, \dots, z_N\}, \{f_1, f_2, \dots, f_N\}$ in X such that

$$\begin{aligned} &t_i - t_{i-1} < \epsilon, && i = 1, \dots, N \\ &T - \epsilon < t_N \leq T, \\ &\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} ||f(s) - f_i|| \, \mathrm{d}s < \epsilon, \\ &\frac{z_i - z_{i-1}}{t_i - t_{i-1}} + Az_i \ni f_i, && i = 1, \dots, N \end{aligned}$$

and $||z(t) - u(t)|| \leq \epsilon$ on $[0, t_N]$, where $z : [0, t_N] \mapsto X$ is defined by

$$z(t) = z_i \qquad \text{for } t_{i-1} \leq t < t_i, \ i = 1, 2, \dots, N.$$

The piecewise constant function z is called an ϵ-approximate solution of (5).

Let $F : J \times \overline{D(A)} \mapsto 2^X \backslash \emptyset$. A mild solution of the Cauchy problem

$$\begin{aligned} &u' \in -Au + F(t, u), \qquad t \in J; \\ &u(0) = u_0 \end{aligned}$$

is a function that is a mild solution of the quasi-autonomous problem

$$\begin{aligned} &u' + Au \ni f(t), \qquad t \in J; \\ &u(0) = u_0 \end{aligned}$$

with some $f \in \mathcal{L}^1(J, X)$ such that $f(t) \in F\big(t, u(t)\big)$ a.e.

2.2 Crandall–Liggett Theorem

Here we discuss a special case of the Crandall–Liggett Theorem as it appears in [11, p. 110]. Let X be a Banach space and A be a possibly multivalued operator in X. The operator A is called accretive if, for any $\lambda > 0$ and $x, y \in D(A)$, the inequality

$$||(x + \lambda u) - (y + \lambda v)|| \geq ||x - y||$$

holds, where $u \in Ax$ and $v \in Ay$. The operator A is called m-accretive if it is accretive and the operator $I + \lambda A$ is surjective for $\lambda > 0$; that is, we have

$$R(I + \lambda A) = \bigcup_{x \in D(A)} \bigcup_{v \in Ax} \{x + \lambda v\} = X. \tag{6}$$

Theorem 2.1 [Crandall-Liggett Theorem] *Let X be a Banach space and A be a possibly multivalued m-accretive operator in X. Then for $\epsilon > 0$ and $u_0 \in X$ the problem*

$$\begin{aligned} &\frac{1}{\epsilon}\big(u_\epsilon(t) - u_\epsilon(t-\epsilon)\big) + Au_\epsilon(t) \ni 0, \qquad && t \geq 0; \\ &u_\epsilon(0) = u_0, && t < 0 \end{aligned} \tag{7}$$

has a unique solution $u_\epsilon(t)$ on $[0,\infty)$. If $u_0 \in \overline{D(A)}$, then $\lim_{\epsilon\to 0} u_\epsilon(t)$ converges uniformly to the unique mild solution of (5) in bounded sets and $\big(S(t)\big)_{t\geq 0}$ defined by $S(t)u_0 = \lim_{\epsilon\to 0} u_\epsilon(t)$ is a semigroup of contractions on $\overline{D(A)}$; that is, we have

(i) $S(t) : \overline{D(A)} \mapsto \overline{D(A)}$ *for $t \geq 0$,*
(ii) $S(t)S(\tau) = S(t+\tau)$ *for $t, \tau \geq 0$,*
(iii) $||S(t)v - S(t)w|| \leq ||v - w||$ *for $t \geq 0$ and $v, w \in D(A)$,*
(iv) $S(0) = I$,
(v) *$S(t)v$ is continuous in the pair (t, v).*

3 Statement of new results

The abstract framework of operator semigroups and, in particular, the fundamental Crandall–Liggett Theorem utilizes the notion of mild solutions. Later we will show that the unique mild solution of the conservation law (1) also satisfies a Kružkov-type entropy inequality. For the exact formulation of this inequality let us define the function $\eta : \mathbb{R}^n \mapsto \mathbb{R}$ to be an entropy of (1) with entropy fluxes $q_i : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ given that it is continuously differentiable and the equality

$$\eta'(u) \int_{\mathbb{R}^n} \frac{\phi_i(u, \tau_{\beta_i} u) - \phi_i(\tau_{-\beta_i} u, u)}{||\beta_i||_{\mathbb{R}^n}} \omega_i \mathrm{d}h = \int_{\mathbb{R}^n} \frac{q_i(u, \tau_{\beta_i} u) - q_i(\tau_{-\beta_i} u, u)}{||\beta_i||_{\mathbb{R}^n}} \omega_i \mathrm{d}h \tag{8}$$

holds for all $i = 1, 2, \ldots, k$. Then if $u(t, x)$ is a $\mathcal{C}^1$ solution of (1) then it also satisfies

$$\frac{\partial \eta(u)}{\partial t} + \int_{\mathbb{R}^n} \sum_{i=1}^{k} \frac{q_i(u, \tau_{\beta_i} u) - q_i(\tau_{-\beta_i} u, u)}{||\beta_i||_{\mathbb{R}^n}} \omega_i \mathrm{d}h = 0.$$

In the case of an $\eta \in \mathcal{C}^2$ convex entropy standard vanishing viscosity arguments (using integration by parts for difference quotients) show that the inequality

$$\int_0^T \int_{\mathbb{R}^n} \eta(u) \frac{\partial f}{\partial t} \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{i=1}^{k} \frac{\tau_{\beta_i} f - f}{||\beta_i||_{\mathbb{R}^n}} q_i(u, \tau_{\beta_i} u) \omega_i \mathrm{d}h\mathrm{d}x\mathrm{d}t \geq 0$$

holds for any $T > 0$, nonnegative $f \in \mathcal{C}_0^\infty\big(\mathbb{R}^n \times (0, T)\big)$. Our goal is to utilize classical Kružkov-entropies of the form $\eta(u) := \eta(u, c) = |u - c|$, however, in this case, an explicit formula for q_i does not seem to reveal itself. Instead, during the vanishing viscosity derivation we rely on (8) to arrive at the following definition:

Definition 3.1 A function $u \in \mathcal{L}^1(\mathbb{R}^n \times (0,T)) \cap \mathcal{L}^\infty(\mathbb{R}^n \times (0,T))$ is an entropy solution of (1) if the inequality

$$0 \le \int_0^T \int_{\mathbb{R}^n} \Bigg(|u-c| \frac{\partial f}{\partial t} + \int_{\mathbb{R}^n} \sum_{i=1}^k \frac{\tau_{\beta_i} f \operatorname{sign}_0(\tau_{\beta_i} u - c) - f \operatorname{sign}_0(u-c)}{||\beta_i||_{\mathbb{R}^n}} \big(\phi_i(u, \tau_{\beta_i} u) - \phi_i(c,c)\big)\omega_i \mathrm{d}h \Bigg) \mathrm{d}x \mathrm{d}t$$

holds for any $T > 0$, nonnegative $f \in \mathcal{C}_0^\infty\big(\mathbb{R}^n \times (0,T)\big)$ and $c \in \mathbb{R}$.

Remark 3.2 Let the functions $\tilde{q}_i$ be given by,[1]

$$\begin{aligned} \tilde{q}_i(a,b,c) &= \phi_i(a \vee c, b \vee c) - \phi_i(a \wedge c, b \wedge c) \\ &= \phi_i\big(\max\{a,c\}, \max\{b,c\}\big) - \phi_i\big(\min\{a,c\}, \min\{b,c\}\big) \\ &= \frac{\operatorname{sign}_0(a-c) + \operatorname{sign}_0(b-c)}{2}\big(\phi_i(a,b) - \phi_i(c,c)\big) \\ &\quad + \frac{\operatorname{sign}_0(a-c) - \operatorname{sign}_0(b-c)}{2}\big(\phi_i(a,c) - \phi_i(c,b)\big), \end{aligned}$$

where

$$\operatorname{sign}_0(x) = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

For the sake of notational simplicity, let us omit the sum in this remark. The properties of ϕ_i after adding and subtracting $\phi_i(c,c)$ imply that

$$\begin{aligned} &\operatorname{sign}_0(u-c) \int_{\mathbb{R}^n} \frac{\phi_i(u, \tau_{\beta_i} u) - \phi_i(\tau_{-\beta_i} u, u)}{||\beta_i||_{\mathbb{R}^n}} \omega_i \mathrm{d}h \\ &\ge \int_{\mathbb{R}^n} \frac{\tilde{q}_i(u, \tau_{\beta_i} u, c) - \tilde{q}_i(\tau_{-\beta_i} u, u, c)}{||\beta_i||_{\mathbb{R}^n}} \omega_i \mathrm{d}h, \end{aligned}$$

and thus it seems reasonable to define entropy solutions using $\tilde{q}_i$ as entropy fluxes corresponding to the entropy $|u-c|$. But, in fact, using the product rule for difference quotients shows that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\tau_{\beta_i} f \operatorname{sign}_0(\tau_{\beta_i} u - c) - f \operatorname{sign}_0(u-c)}{||\beta_i||_{\mathbb{R}^n}} \big(\phi_i(u, \tau_{\beta_i} u) - \phi_i(c,c)\big)\omega_i \mathrm{d}h \mathrm{d}x \mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\tau_{\beta_i} f - f}{||\beta_i||_{\mathbb{R}^n}} \operatorname{sign}_0(\tau_{\beta_i} u - c)\big(\phi_i(u, \tau_{\beta_i} u) - \phi_i(c,c)\big)\omega_i \mathrm{d}h \mathrm{d}x \mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f \frac{\operatorname{sign}_0(\tau_{\beta_i} u - c) - \operatorname{sign}_0(u-c)}{||\beta_i||_{\mathbb{R}^n}} \big(\phi_i(u, \tau_{\beta_i} u) - \phi_i(c,c)\big)\omega_i \mathrm{d}h \mathrm{d}x \mathrm{d}t. \end{aligned}$$

Clearly

$$\begin{aligned} &\operatorname{sign}_0(\tau_{\beta_i} v - c)\big[\phi_i(v, \tau_{\beta_i} v) - \phi_i(c,c)\big] \\ &\quad \le \phi_i(v \vee c, \tau_{\beta_i} v \vee c) - \phi_i(v \wedge c, \tau_{\beta_i} v \wedge c) = \tilde{q}_i(v, \tau_{\beta_i} v, c) \end{aligned}$$

[1] As already noted by [23, Definition 2.2] the second line is not identical to the corresponding equation in [20, p. 2470], which is assumed to be a misprint. Here we gave a more straightforward formula.

and similarly

$$-\operatorname{sign}_0(v-c)\big[\phi_i(v,\tau_{\beta_i}v)-\phi_i(c,c)\big]\le -\tilde q_i(v,\tau_{\beta_i}v,c)$$

holds, thus

$$\big[\operatorname{sign}_0(\tau_{\beta_i}v-c)-\operatorname{sign}_0(v-c)\big]\big[\phi_i(v,\tau_{\beta_i}v)-\phi_i(c,c)\big]\le 0 \tag{9}$$

and finally

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\tau_{\beta_i}f\operatorname{sign}_0(\tau_{\beta_i}u-c)-f\operatorname{sign}_0(u-c)}{||\beta_i||_{\mathbb{R}^n}}\big(\phi_i(u,\tau_{\beta_i}u)-\phi_i(c,c)\big)\omega_i\,\mathrm{d}h\mathrm{d}x\mathrm{d}t\\ &\quad\le\int_0^T\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\tau_{\beta_i}f-f}{||\beta_i||_{\mathbb{R}^n}}\tilde q_i(u,\tau_{\beta_i}u,c)\omega_i\,\mathrm{d}h\mathrm{d}x\mathrm{d}t;\end{aligned}$$

that is, in some sense, the inequality in Definition 3.1 is *more precise* in selecting the physically relevant weak solution than the right-hand side of the above inequality. This precision turns out to be crucial in later steps; the operator defined in Definition 3.6 does not seem to be accretive with the functions $\tilde q_i$ which is an essential property to derive uniqueness of solutions via the Crandall–Liggett theorem.

Throughout the paper difference quotients will be denoted by

$$D^y f=\frac{\tau_y f-f}{||y||_{\mathbb{R}^n}},$$

where $y\in\mathbb{R}^n$ and the partial derivative of the ϕ_i functions with respect to their first and second argument will be denoted by $\phi'_{i,1}$ and $\phi'_{i,2}$, respectively. For open subsets Ω of $\mathbb{R}^n$ let $\mathcal{W}^{k,p}(\Omega)$ denote the Sobolev space of functions whose distributional derivatives of order at most k are in $\mathcal{L}^p(\Omega)$. The space $\mathcal{W}_0^{k,p}(\Omega)\subset\mathcal{W}^{k,p}(\Omega)$ denotes the set of functions vanishing at the boundary of Ω and $\mathcal{W}^{k,p}_{loc}(\Omega)$ denotes the set of locally integrable functions whose restriction to any pre-compact $Q\Subset\Omega$ lies in $\mathcal{W}^{k,p}(Q)$. We will use the standard notation $\mathcal{H}^k(\Omega):=\mathcal{W}^{k,2}(\Omega)$.

We rewrite the nonlocal conservation law (1) using the operator

$$Bu=\int_{\mathbb{R}^n}\sum_{i=1}^k\frac{\phi_i(u,\tau_{\beta_i}u)-\phi_i(\tau_{-\beta_i}u,u)}{||\beta_i||_{\mathbb{R}^n}}\omega_i\,\mathrm{d}h$$

as

$$\frac{\partial u}{\partial t}+Bu=0. \tag{10}$$

The following lemma shows that for continuously differentiable fluxes the operator B maps $\mathcal{W}^{1,p}(\mathbb{R}^n)$ to $\mathcal{L}^p(\mathbb{R}^n)$.

Lemma 3.3 *Let $\phi_i\in\mathcal{C}^1(\mathbb{R}\times\mathbb{R})$ have bounded partial derivatives. Then $v\in\mathcal{W}^{1,p}(\mathbb{R}^n)$ implies $Bv\in\mathcal{L}^p(\mathbb{R}^n)$ for all $1\le p<\infty$. In particular, there is a constant $C=C(p)>0$ such that $||Bv||_{\mathcal{L}^p(\mathbb{R}^n)}\le C\,||\nabla v||_{\mathcal{L}^p(\mathbb{R}^n)}$ for all $v\in\mathcal{W}^{1,p}(\mathbb{R}^n)$.*

Proof Let $|\phi'_{i,1}| \le K_{i,1}$ and $|\phi'_{i,2}| \le K_{i,2}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Setting $K_i = \max\{K_{i,1}, K_{i,2}\}$ we find that

$$\begin{aligned}
||Bv||^p_{\mathcal{L}^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \sum_{i=1}^k \frac{\phi_i(v, \tau_{\beta_i} v) - \phi_i(\tau_{-\beta_i} v, v)}{||\beta_i||_{\mathbb{R}^n}} \omega_i \mathrm{d}h \right|^p \mathrm{d}x \\
&\le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \sum_{i=1}^k K_i \frac{|v - \tau_{-\beta_i} v| + |\tau_{\beta_i} v - v|}{||\beta_i||_{\mathbb{R}^n}} \omega_i \mathrm{d}h \right)^p \mathrm{d}x \\
&\le k^{p-1} \sum_{i=1}^k K_i^p \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left(\left| D^{\beta_i} \tau_{-\beta_i} v \right| + \left| D^{\beta_i} v \right| \right) \omega_i \mathrm{d}h \right)^p \mathrm{d}x \qquad (11) \\
&\le k^{p-1} \sum_{i=1}^k K_i^p \, ||\omega_i||^p_{\mathcal{L}^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} \int_{\mathrm{supp}(\omega_i)} \left(\left| D^{\beta_i} \tau_{-\beta_i} v \right| + \left| D^{\beta_i} v \right| \right)^p \mathrm{d}h \mathrm{d}x \\
&\le 2^{p-1} k^{p-1} \sum_{i=1}^k K_i^p \, ||\omega_i||^p_{\mathcal{L}^q(\mathbb{R}^n)} \int_{\mathrm{supp}(\omega_i)} ||\nabla v||^p_{\mathcal{L}^p(\mathbb{R}^n)} \mathrm{d}h = C \, ||\nabla v||^p_{\mathcal{L}^p(\mathbb{R}^n)},
\end{aligned}$$

where we used the Lipschitz continuity of ϕ in the first inequality, Hölder's inequality in the third inequality and finally Fubini's theorem and [6, Proposition 9.3(iii)] in the fourth inequality.

The continuity of B is established by our next lemma.

Lemma 3.4 *Let the assumptions of Lemma 3.3 hold. Then B is continuous from $\mathcal{H}^1(\mathbb{R}^n)$ to $\mathcal{L}^2(\mathbb{R}^n)$.*

Proof Let $u, v \in \mathcal{H}^1(\mathbb{R}^n)$. Similar estimates as in the proof of Lemma 3.3 lead to

$$\begin{aligned}
&||Bu - Bv||^2_{\mathcal{L}^2(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \sum_{i=1}^k D^{\beta_i} \left[\phi_i(\tau_{-\beta_i} u, u) - \phi_i(\tau_{-\beta_i} v, v) \right] \omega_i \mathrm{d}h \right)^2 \mathrm{d}x \\
&\le C \sum_{i=1}^k \int_{\mathbb{R}^n} \left| \left| D^{\beta_i} \left[\phi_i(\tau_{-\beta_i} u, u) - \phi_i(\tau_{-\beta_i} v, v) \right] \right| \right|^2_{\mathcal{L}^2(\mathbb{R}^n)} \mathrm{d}h \\
&\le C \sum_{i=1}^k \int_{\mathbb{R}^n} \left| \left| \nabla \left[\phi_i(\tau_{-\beta_i} u, u) - \phi_i(\tau_{-\beta_i} v, v) \right] \right| \right|^2_{\mathcal{L}^2(\mathbb{R}^n)} \mathrm{d}h \\
&= C \sum_{i=1}^k \int_{\mathbb{R}^n} \left| \left| \phi'_{i,1}(\tau_{-\beta_i} u, u) \nabla \tau_{-\beta_i} u + \phi'_{i,2}(\tau_{-\beta_i} u, u) \nabla u \right. \right. \\
&\quad \left. \left. - \phi'_{i,1}(\tau_{-\beta_i} v, v) \nabla \tau_{-\beta_i} v - \phi'_{i,2}(\tau_{-\beta_i} v, v) \nabla v \right| \right|^2_{\mathcal{L}^2(\mathbb{R}^n)} \mathrm{d}h.
\end{aligned}$$

By introducing mixed terms we find that

$$
\begin{aligned}
&||Bu - Bv||^2_{\mathcal{L}^2(\mathbb{R}^n)} \\
&\leq C \sum_{i=1}^{k} \int_{\mathbb{R}^n} \Big(\big|\big| \big[\phi'_{i,1}(\tau_{-\beta_i} u, u) - \phi'_{i,1}(\tau_{-\beta_i} v, v)\big] \nabla \tau_{-\beta_i} u \big|\big|_{\mathcal{L}^2(\mathbb{R}^n)} \\
&+ \big|\big| \big[\phi'_{i,2}(\tau_{-\beta_i} u, u) - \phi'_{i,2}(\tau_{-\beta_i} v, v)\big] \nabla u \big|\big|_{\mathcal{L}^2(\mathbb{R}^n)} \\
&+ \big|\big| \phi'_{i,1}(\tau_{-\beta_i} v, v) \big|\big|^2_{\mathcal{L}^\infty(\mathbb{R}^n)} \big|\big| \nabla \tau_{-\beta_i}(u - v) \big|\big|^2_{\mathcal{L}^2(\mathbb{R}^n)} \\
&+ \big|\big| \phi'_{i,2}(\tau_{-\beta_i} v, v) \big|\big|^2_{\mathcal{L}^\infty(\mathbb{R}^n)} ||\nabla(u - v)||^2_{\mathcal{L}^2(\mathbb{R}^n)} \Big) \mathrm{d}h.
\end{aligned}
\tag{12}
$$

Let v converge to u in $\mathcal{H}^1(\mathbb{R}^n)$ through a sequence $\{u_n\} \subset \mathcal{H}^1(\mathbb{R}^n)$ and let $\{u_{n_k}\}$ be a subsequence of $\{u_n\}$. Since u_{n_k} also converges to u as $n_k \to \infty$, there exists a subsequence $\{u_{n_{k_l}}\}$ of $\{u_{n_k}\}$ such that $u_{n_{k_l}} \to u$ a.e. as $n_{k_l} \to \infty$. Let $|\phi'_{i,1}| \leq K_{i,1}$ and $|\phi'_{i,2}| \leq K_{i,2}$ and observe that

$$
\begin{aligned}
&\Big| \big[\phi'_{i,1}(\tau_{-\beta_i} u, u) - \phi'_{i,1}(\tau_{-\beta_i} u_{n_{k_l}}, u_{n_{k_l}})\big] \nabla \tau_{-\beta_i} u \Big| \leq 2K_{i,1} \big| \nabla \tau_{-\beta_i} u \big|, \\
&\Big| \big[\phi'_{i,2}(\tau_{-\beta_i} u, u) - \phi'_{i,2}(\tau_{-\beta_i} u_{n_{k_l}}, u_{n_{k_l}})\big] \nabla \tau_{-\beta_i} u \Big| \leq 2K_{i,2} \big| \nabla u \big|.
\end{aligned}
$$

Using the dominated convergence theorem and the continuity of ϕ_i we find that the first two terms in (12) converge to zero as $n_{k_l} \to \infty$. Similarly, since $\phi'_{i,1}$ and $\phi'_{i,2}$ are bounded and $u_{n_{k_l}} \to u$ in $\mathcal{H}^1(\mathbb{R}^n)$, the second two terms also converge to zero as $n_{k_l} \to \infty$. Since $\{u_{n_k}\}$ was arbitrary we conclude that each subsequence of the sequence $||Bu - Bu_n||^2_{\mathcal{L}^2(\mathbb{R})}$ has a convergent subsequence with limit zero; that is, the sequence itself converges to zero and the proof is complete.

Remark 3.5 In [23] the authors consider the case (in one dimension) when $\int_{\mathbb{R}^n} \frac{\omega_i(\beta_i)}{||\beta_i||_{\mathbb{R}^n}} < \infty$. In this case the above calculations can be modified to show that $B : \mathcal{L}^1(\mathbb{R}^n) \mapsto \mathcal{L}^1(\mathbb{R}^n)$ is Lipschitz continuous. Hence, standard contraction mapping principle shows existence and uniqueness without entropy conditions. However, in this special case the kernels ω_i assign small weight to close interactions and more weight as the interaction distance increases. As such, the model's applicability to physically relevant problems is reduced.

We will consider $X = \mathcal{L}^1(\mathbb{R}^n)$ and proceed by verifying the hypotheses of the Crandall-Liggett Theorem for an appropriate operator A in $\mathcal{L}^1(\mathbb{R}^n)$ that is, in some sense, the generalization of the B of (10). The operator A will be the closure of the operator A_0 defined as follows.

Definition 3.6 Let A_0 be the operator in $\mathcal{L}^1(\mathbb{R}^n)$ defined by: $v \in D(A_0)$ and $w \in A_0 v$ if

(i) $v, w \in \mathcal{L}^1(\mathbb{R}^n)$,
(ii) $\phi_i(v, \tau_{\beta_i(h)} v) \in \mathcal{L}^1(\mathbb{R}^n)$ for $h \in \mathrm{supp}(\omega_i)$ and $i = 1, 2, \ldots, k$,
(iii) the inequality

$$
\int_{\mathbb{R}^n} \mathrm{sign}_0(v - c) w f \,\mathrm{d}x + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{i=1}^{k} D^{\beta_i} \times \big[f\, \mathrm{sign}_0(v - c)\big] \big(\phi_i(v, \tau_{\beta_i} v) - \phi_i(c, c)\big) \omega_i \,\mathrm{d}h \mathrm{d}x \geq 0 \tag{13}
$$

holds for any nonnegative $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $c \in \mathbb{R}$.

As we will see later, the inequality in Definition 3.6(iii) ensures that if $u \in D(A_0)$ is a solution of the abstract Cauchy problem, then it satisfies the entropy inequality in Definition 3.1. Lemmata 4.1 and 4.2 show that under appropriate circumstances A_0 is single-valued and coincides with B, further substantiating our definition.

While the accretivity of A_0, and thus the accretivity of its closure A, can be established in a straightforward manner using a tool described in [11, Proposition 2.1] (see Proposition 4.6), the verification of the range condition (6) is more intricate. In fact, it requires the treatment of the stationary equation

$$u + Bu = g. \tag{14}$$

We define the generalized solutions of (14) in terms of A.

Definition 3.7 Let $g \in \mathcal{L}^1(\mathbb{R}^n)$. Then $u \in \mathcal{L}^1(\mathbb{R}^n)$ is a generalized solution of (14) if $u \in D(A)$ and $g \in (I + A)u$.

Our first main result is the following theorem.

Theorem 3.8 *Let $\phi_i \in \mathcal{W}^{1,\infty}_{loc}(\mathbb{R} \times \mathbb{R})$ and $g \in \mathcal{L}^1(\mathbb{R}^n)$. Then A satisfies the assumptions of the Crandall-Liggett Theorem on $\mathcal{L}^1(\mathbb{R}^n)$ and the unique generalized solution of (14) is given by $u = (I + A)^{-1}g$.*

Theorem 3.8 and the Crandall-Liggett Theorem show that a semigroup of contractions is determined by the operator A, whose various properties are listed in the next theorem.

Theorem 3.9 *Let the assumptions of Theorem 3.8 hold and S be the semigroup of contractions on $\overline{D(A)}$ obtained from A via the Crandall–Liggett Theorem on $\mathcal{L}^1(\mathbb{R}^n)$. Let $u, v \in \overline{D(A)} \cap \mathcal{L}^\infty(\mathbb{R}^n)$ and $t \geq 0$. Then*

(i) (integrability) $S(t)v \in \mathcal{L}^p(\mathbb{R}^n)$ *for $p \geq 1$, furthermore the estimate* $||S(t)v||_{\mathcal{L}^p(\mathbb{R}^n)} \leq ||v||^{\frac{1}{p}}_{\mathcal{L}^1(\mathbb{R}^n)} ||v||^{1-\frac{1}{p}}_{\mathcal{L}^\infty(\mathbb{R}^n)}$ *holds,*
(ii) (maximum principle) $-\left|\left|v^-\right|\right|_{\mathcal{L}^\infty(\mathbb{R}^n)} \leq S(t)v \leq \left|\left|v^+\right|\right|_{\mathcal{L}^\infty(\mathbb{R}^n)}$, *where* $v^- = \max\{0, -v\}$ *and* $v^+ = \max\{0, v\}$.
(iii) (monotonicity) $\left|\left|(S(t)u - S(t)v)^+\right|\right|_{\mathcal{L}^1(\mathbb{R}^n)} \leq \left|\left|(u - v)^+\right|\right|_{\mathcal{L}^1(\mathbb{R}^n)}$,
(iv) (equicontinuity) *if $y \in \mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} \left|S(t)v(x + y) - S(t)v(x)\right| \mathrm{d}x \leq \int_{\mathbb{R}^n} \left|v(x + y) - v(x)\right| \mathrm{d}x,$$

(v) (conservation of mass) $\int_{\mathbb{R}^n} S(t)v(x)\mathrm{d}x = \int_{\mathbb{R}^n} v(x)\mathrm{d}x$,
(vi) *$S(t)v$ satisfies the nonlocal entropy inequality in Definition 3.1.*

Remark 3.10 Note that the properties (iii)–(v) still hold if we only assume $u, v \in \overline{D(A)}$.

Corollary 3.11 *Let $g : [0, T] \times \overline{D(A)} \mapsto \mathcal{L}^1(\mathbb{R}^n)$ be strongly measurable with respect to t and locally Lipschitz with respect to u such that*

$$||g(t, u)||_{\mathcal{L}^1(\mathbb{R}^n)} \leq c(t)\big(1 + ||u||_{\mathcal{L}^1(\mathbb{R}^n)}\big)$$

holds for some $c \in \mathcal{L}^1([0, T])$. Then the Cauchy problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + \int_{\mathbb{R}^n} \sum_{i=1}^{k} \frac{\phi_i(u, \tau_{\beta_i} u) - \phi_i(\tau_{-\beta_i} u, u)}{||\beta_i||_{\mathbb{R}^n}} \omega_i dh = g(t, u), \quad && \text{in } \mathbb{R}^n \times (0, T]; \\ &u(x, 0) = u_0(x), && x \in \mathbb{R}^n \end{aligned}$$

has a unique mild solution for each $u_0 \in \overline{D(A)}$ that depends continuously on u_0; that is, the map $u_0(.) \to u(., t)$ is continuous in the Banach space $X = \mathcal{L}^1(\mathbb{R}^n)$.

Proof The statement follows directly from [5, Theorem 5.2].

4 Proofs of the main results

The following lemma shows that A_0 is single-valued for bounded functions.

Lemma 4.1 *Let A_0 be given by Definition 3.6 and $v \in D(A_0) \cap \mathcal{L}^\infty(\mathbb{R}^n)$. Then A_0 is single-valued and the equality*

$$\int_{\mathbb{R}^n} A_0 v f \,\mathrm{d}x = -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^{k} D^{\beta_i} f \phi_i(v, \tau_{\beta_i} v)\omega_i \,\mathrm{d}h\mathrm{d}x$$

holds for any nonnegative $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$.

Proof Let $w \in A_0 v$. Then by (13) for any nonnegative $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $c \in \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} wf \,\mathrm{d}x + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^{k} D^{\beta_i}\big[f \,\mathrm{sign}_0(v-c)\big]\big(\phi_i(v, \tau_{\beta_i} v) - \phi_i(c,c)\big)\omega_i \,\mathrm{d}h\mathrm{d}x \geq 0,$$

thus for $c = ||v||_{\mathcal{L}^\infty(\mathbb{R}^n)} + 1$, we have that

$$\int_{\mathbb{R}^n} wf \,\mathrm{d}x \leq -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^{k} D^{\beta_i} f \phi_i(v, \tau_{\beta_i} v)\omega_i \,\mathrm{d}h\mathrm{d}x.$$

Similarly, letting $c = -(||v||_{\mathcal{L}^\infty(\mathbb{R}^n)} + 1)$ yields

$$\int_{\mathbb{R}^n} wf \,\mathrm{d}x \geq -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^{k} D^{\beta_i} f \phi_i(v, \tau_{\beta_i} v)\omega_i \,\mathrm{d}h\mathrm{d}x,$$

showing that for any $w \in A_0 v$, the following equality holds

$$\int_{\mathbb{R}^n} wf \,\mathrm{d}x = -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^{k} D^{\beta_i} f \phi_i(v, \tau_{\beta_i} v)\omega_i \,\mathrm{d}h\mathrm{d}x.$$

To show that $A_0 v$ is single-valued, suppose that $w_1, w_2 \in A_0 v$. Then the equality $\int_{\mathbb{R}^n} w_1 f \,\mathrm{d}x = \int_{\mathbb{R}^n} w_2 f \,\mathrm{d}x$ holds for all nonnegative $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, thus $w_1 = w_2$ a.e.

The following lemma shows that A_0 extends B on $\mathcal{C}_0^1(\mathbb{R}^n)$.

Lemma 4.2 *Let $\phi_i \in C^1(\mathbb{R} \times \mathbb{R})$ have bounded partial derivatives and A_0 be given by Definition 3.6. Then $\mathcal{C}_0^1(\mathbb{R}^n) \subset D(A_0)$ and for any $v \in \mathcal{C}_0^1(\mathbb{R}^n)$, the equality $A_0 v = Bv$ holds.*

Proof The fact $v \in \mathcal{C}_0^1(\mathbb{R}^n)$ implies that $\phi_i(v, \tau_{\beta_i(h)} v) \in \mathcal{L}^1(\mathbb{R}^n)$ holds for all $h \in \mathrm{supp}(\omega_i)$ and $i = 1, 2, \ldots, k$. Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be nonnegative and $c \in \mathbb{R}$. Multiply Bv by $\mathrm{sign}_0(v -$

$c)f$ and integrate over $\mathbb{R}^n$ to find that

$$\int_{\mathbb{R}^n} \operatorname{sign}_0(v-c) f Bv \mathrm{d}x \\ = -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sum_{i=1}^{k} D^{\beta_i}[f \operatorname{sign}_0(v-c)]\big(\phi_i(v,\tau_{\beta_i}v)-\phi_i(c,c)\big)\omega_i \mathrm{d}h\mathrm{d}x; \tag{15}$$

that is, we have $v \in D(A_0)$ and $Bv \in A_0 v$. This, combined with Lemma 4.1 implies that $A_0 v = Bv$ a.e.

We will use an efficient tool of Crandall to prove accretivity, characterized by the following definition and the two subsequent lemmata.

Definition 4.3 [11, Definition 2.1] For $u : \mathbb{R}^n \mapsto \mathbb{R}$ measurable, let

$$\operatorname{sign}(u) := \big\{v : \mathbb{R}^n \mapsto \mathbb{R}\big||v| \leq 1 \text{ a.e. and } vu = |u| \text{ a.e.}\}.$$

Note that $\operatorname{sign}_0(u) \in \operatorname{sign}(u)$, thus $\operatorname{sign}(u)$ is always nonempty.

Lemma 4.4 [11, Lemma 2.1] *Let $u, v \in \mathcal{L}^1(\mathbb{R}^n)$ and $\alpha \in \operatorname{sign}(u)$. If $\int_{\mathbb{R}^n} \alpha v \mathrm{d}x \geq 0$, then $||u+\lambda v||_{\mathcal{L}^1(\mathbb{R}^n)} \geq ||u||_{\mathcal{L}^1(\mathbb{R}^n)}$ holds for $\lambda > 0$.*

Lemma 4.5 [11, Lemma 2.2] *Let $\{\beta_k\}$ be a sequence in $\mathcal{L}^1(\mathbb{R}^n)$ with $\lim \beta_k = \beta$ in $\mathcal{L}^1(\mathbb{R}^n)$. If $\alpha_k \in \operatorname{sign}(\beta_k)$, then there exists a subsequence $\{\alpha_{k_l}\}$ and function $\alpha \in \operatorname{sign}(\beta)$ such that $\{\alpha_{k_l}\}$ converges to α in the weak-star topology on $\mathcal{L}^\infty(\mathbb{R}^n)$.*

Proposition 4.6 *Let A_0 be given by Definition 3.6. Then A_0 is accretive in $\mathcal{L}^1(\mathbb{R}^n)$.*

Proof Let $v \in D(A_0)$ and $w \in A_0 v$ and choose $u \in \mathcal{L}^1(\mathbb{R}^n)$ such that Definition 3.6 (ii) holds. Set $c = u(y)$ and $f(x) = g(x, y)$ in (13), where $g \in \mathcal{C}_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is nonnegative. We introduce the notations $\Pi = \left(\mathbb{R}^n\right)^2$ and

$$D_1^{\beta_i} g(x,y) = \frac{g(x+\beta_i, y) - g(x,y)}{||\beta_i||_{\mathbb{R}^n}},$$
$$D_2^{\beta_i} g(x,y) = \frac{g(x, y+\beta_i) - g(x,y)}{||\beta_i||_{\mathbb{R}^n}}.$$

For the sake of readability we omit most arguments in this proof. Integrating over y yields

$$\int_{\Pi} \operatorname{sign}_0(v-u) w g \mathrm{d}x\mathrm{d}y \\ + \int_{\Pi}\int_{\mathbb{R}^n}\sum_{i=1}^{k} D_1^{\beta_i}\big[g \operatorname{sign}_0(v-u)\big]\Big(\phi_i(v,\tau_{\beta_i}v)-\phi_i(u,u)\Big)\omega_i \mathrm{d}h\mathrm{d}x\mathrm{d}y \geq 0. \tag{16}$$

Suppose that $u \in D(A_0)$ as well and let $z \in A_0 u$. Set $c = v(x)$ and $f(y) = g(x, y)$ in (13) and integrate over x to find that

$$\int_{\Pi} \operatorname{sign}_0(u-v) z g \mathrm{d}y\mathrm{d}x \\ + \int_{\Pi}\int_{\mathbb{R}^n}\sum_{i=1}^{k} D_2^{\beta_i}\big[g \operatorname{sign}_0(u-v)\big]\Big(\phi_i(u,\tau_{\beta_i}u)-\phi_i(v,v)\Big)\omega_i \mathrm{d}h\mathrm{d}y\mathrm{d}x \geq 0. \tag{17}$$

and adding the inequalities (16) and (17) yields

$$\int_{\Pi} \operatorname{sign}_0(v-u)(w-z)g\mathrm{d}x\mathrm{d}y + \int_{\Pi}\int_{\mathbb{R}^n}\sum_{i=1}^{k}\Big(D_1^{\beta_i}\big[g\operatorname{sign}_0(v-u)\big]\Big(\phi_i(v,\tau_{\beta_i}v)-\phi_i(u,u)\Big) + D_2^{\beta_i}\big[g\operatorname{sign}_0(u-v)\big]\Big(\phi_i(u,\tau_{\beta_i}u)-\phi_i(v,v)\Big)\Big)\omega_i\mathrm{d}h\mathrm{d}x\mathrm{d}y \geq 0. \tag{18}$$

Let $\delta \in \mathcal{C}_0^\infty(\mathbb{R})$ be nonnegative and even such that $||\delta||_{\mathcal{L}^1(\mathbb{R}^n)} = 1$ and

$$\lambda(x) = \prod_{i=1}^{n}\delta(x_i),$$
$$\lambda_\epsilon(x) = \frac{1}{\epsilon^n}\lambda\left(\frac{x}{\epsilon}\right)$$

for $\epsilon > 0$. Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ nonnegative and set

$$g(x,y) = f\left(\frac{x+y}{2}\right)\lambda_\epsilon\left(\frac{x-y}{2}\right).$$

Setting $2\xi = x+y$, $2\eta = x-y$ in (18) yields

$$\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\operatorname{sign}_0(v-u)(w-z)f\mathrm{d}\xi\right)\lambda_\epsilon(\eta)\mathrm{d}\eta + \int_{\Pi} J_f^\epsilon(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta \geq 0. \tag{19}$$

where

$$J_f^\epsilon(\xi,\eta) = \int_{\mathbb{R}^n}\sum_{i=1}^{k}\frac{\omega_i}{||\beta_i||_{\mathbb{R}^n}} \times\Big[\Big(\tau_{\frac{\beta_i}{2}}f\tau_{\frac{\beta_i}{2}}\lambda_\epsilon\operatorname{sign}_0(\tau_{\beta_i}v-u) - f\lambda_\epsilon\operatorname{sign}_0(v-u)\Big)\Big(\phi_i(v,\tau_{\beta_i}v)-\phi_i(u,u)\Big) + \Big(\tau_{\frac{\beta_i}{2}}f\tau_{-\frac{\beta_i}{2}}\lambda_\epsilon\operatorname{sign}_0(\tau_{\beta_i}u-v) - f\lambda_\epsilon\operatorname{sign}_0(u-v)\Big)\Big(\phi_i(u,\tau_{\beta_i}u)-\phi_i(v,v)\Big)\Big]\mathrm{d}h.$$

Denote the integral in parenthesis in the first term of (19) with $I_f(\eta)$. We want to let $\epsilon \to 0$. Since I_f is bounded and $||\lambda_\epsilon||_{\mathcal{L}^1(\mathbb{R}^n)} = 1$ we have that

$$\liminf_{\epsilon\to 0}\int_{\mathbb{R}^n} I_f(\eta)\lambda_\epsilon(\eta)\mathrm{d}\eta \leq \limsup_{||\eta||_{\mathbb{R}^n}\to 0} I_f(\eta).$$

A similar argument after a change of variables shows that

$$\liminf_{\epsilon\to 0}\int_{\Pi} J_f^\epsilon(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta \leq \limsup_{||\eta||_{\mathbb{R}^n}\to 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sum_{i=1}^{k}\frac{\omega_i}{||\beta_i||_{\mathbb{R}^n}}\Big(\tau_{\beta_i}fq_i^{(1)}(v,\tau_{\beta_i}v,\tau_{\beta_i}u) - fq_i^{(2)}(v,\tau_{\beta_i}v,u) + \tau_{\beta_i}fq_i^{(1)}(u,\tau_{\beta_i}u,\tau_{\beta_i}v) - fq_i^{(2)}(u,\tau_{\beta_i}u,v)\Big)\mathrm{d}h\mathrm{d}\xi,$$

where

$$q_i^{(1)}(a,b,c) = \operatorname{sign}_0(a-c)\big(\phi_i(a,b) - \phi_i(c,c)\big),$$
$$q_i^{(2)}(a,b,c) = \operatorname{sign}_0(b-c)\big(\phi_i(a,b) - \phi_i(c,c)\big).$$

Introducing mixed terms yields

$$\begin{aligned}
&\liminf_{\epsilon\to 0} \int_\Pi J_f^\epsilon(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta \\
&\leq \limsup_{||\eta||_{\mathbb{R}^n}\to 0} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sum_{i=1}^k \Big(f\Big(\big(q_i^{(1)} - q_i^{(2)}\big)(v,\tau_{\beta_i}v,u) + \big(q_i^{(1)} - q_i^{(2)}\big)(u,\tau_{\beta_i}u,v)\Big) \\
&\quad + (\tau_{\beta_i}f - f)q_i^{(1)}(v,\tau_{\beta_i}v,\tau_{\beta_i}u) + (\tau_{\beta_i}f - f)q_i^{(1)}(u,\tau_{\beta_i}u,\tau_{\beta_i}v)\Big)\frac{\omega_i}{||\beta_i||_{\mathbb{R}^n}}\mathrm{d}h\mathrm{d}\xi.
\end{aligned}$$

But then (9) shows that the first two terms are nonpositive, thus we conclude that

$$\begin{aligned}
\liminf_{\epsilon\to 0}\int_\Pi J_f^\epsilon(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta &\leq \limsup_{||\eta||_{\mathbb{R}^n}\to 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sum_{i=1}^k \frac{\omega_i}{||\beta_i||_{\mathbb{R}^n}} \\
&\quad\times\Big((\tau_{\beta_i}f - f)q_i^{(1)}(v,\tau_{\beta_i}v,\tau_{\beta_i}u) + (\tau_{\beta_i}f - f)q_i^{(1)}(u,\tau_{\beta_i}u,\tau_{\beta_i}v)\Big)\mathrm{d}h\mathrm{d}\xi \\
&=: \limsup_{||\eta||_{\mathbb{R}^n}\to 0} \tilde{J}_f(\eta).
\end{aligned}$$

Choose a sequence $\{\eta_k\} \subset \mathbb{R}^n$ such that $||\eta_k||_{\mathbb{R}^n} \to 0$ and $\lim_{k\to\infty} I_f(\eta_k) = \limsup_{||\eta||_{\mathbb{R}^n}} \tilde{I}_f(\eta)$ and $\lim_{k\to\infty}\tilde{J}_f(\eta_k)) = \limsup_{||\eta||_{\mathbb{R}^n}\to 0}\tilde{J}_f(\eta)$ (note that it might be necessary to choose two different sequences for I_f and $\tilde{J}_f$). Using Lemma 4.5 we assume (passing to subsequences if necessary) that the sequence

$$\alpha_k(\xi) = \operatorname{sign}_0\big(v(\xi+\eta_k) - u(\xi-\eta_k)\big)$$

converges weakly-star in $\mathcal{L}^\infty(\mathbb{R}^n)$ to $\alpha \in \operatorname{sign}\big(v(\xi) - u(\xi)\big)$. We similarly assume that the sign_0 sequences appearing in $\tilde{J}_f(\eta_k)$ converge weakly-star in $\mathcal{L}^\infty(\mathbb{R}^n)$ and we denote the limit as

$$\lim_{k\to\infty}\tilde{J}_f(\eta_k) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sum_{i=1}^k D^{\beta_i}f\big(\gamma_i(v,\tau_{\beta_i}v,\tau_{\beta_i}u) + \gamma_i(u,\tau_{\beta_i}u,\tau_{\beta_i}v)\big)\omega_i\mathrm{d}h\mathrm{d}\xi.$$

Then

$$\begin{aligned}
\lim_{k\to\infty}\big(I_f(\eta_k) + \tilde{J}_f(\eta_k)\big) &= \int_{\mathbb{R}^n}\alpha(w-z)f\mathrm{d}\xi \\
&\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sum_{i=1}^k D^{\beta_i}f\big(\gamma_i(v,\tau_{\beta_i}v,\tau_{\beta_i}u) + \gamma_i(u,\tau_{\beta_i}u,\tau_{\beta_i}v)\big)\omega_i\mathrm{d}h\mathrm{d}\xi \geq 0.
\end{aligned} \tag{20}$$

Let $\kappa \in \mathcal{C}_0^\infty(\mathbb{R})$ be nonnegative such that $\kappa(s) = 1$ for $|s| \leq 1$. Set $f_l(\xi) = \kappa\Big(\frac{||\xi||_{\mathbb{R}^n}}{l}\Big)$ and let $l \to \infty$. Since the difference quotient

$$D^{\beta_i}f_l(x) = \int_0^1 \nabla f_l(x+\beta_i s)\cdot\frac{\beta_i}{||\beta_i||_{\mathbb{R}^n}}\mathrm{d}s \tag{21}$$

is bounded and is zero for $x \in \mathbb{R}^n$ such that $||x \pm \beta_i||_{\mathbb{R}^n} \leq l$, the second integral in (20) converges to zero; that is, we conclude that

$$\int_{\mathbb{R}^n} \alpha(w - z)\mathrm{d}\xi \geq 0.$$

Lemma 4.4 shows that the inequality

$$||v - u + \lambda(w - z)||_{\mathcal{L}^1(\mathbb{R}^n)} \geq ||v - u||_{\mathcal{L}^1(\mathbb{R}^n)}$$

holds for $\lambda > 0$. Since $u, v \in D(A_0)$ were arbitrary we conclude that A_0 is indeed accretive.

Remark 4.7 One can observe that in the above proof we did not use the fact that the kernels ω_i have finite support.

The stationary Eq. (14) will be investigated through the regularized equation

$$u + \lambda Bu - \epsilon\Delta u = g, \tag{22}$$

where $\lambda, \epsilon > 0$. In [11, Proposition 2.2] the author shows existence of solutions using a special version of the perturbation result [30, Theorem 3.2] without further preparations. A key step of the proof is the fact that for $u \in \mathcal{L}^2(\mathbb{R}^n)$, the $\tilde{B}$ local version of the operator B (see (3)) has the property $\langle \tilde{B}u, u\rangle = 0$. However, this is no longer true in the nonlocal case, and thus we instead use a fix-point approach based on [33, Chapter 4] and [14, Proposition IV.3]. In order to do so, we first establish some a priori estimates on the solutions.

Lemma 4.8 *Let $\phi_i \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R})$ have bounded partial derivatives and let $u \in \mathcal{H}^1(\mathbb{R}^n) \cap \mathcal{H}^2_{loc}(\mathbb{R}^n)$ satisfy (22) for $g \in \mathcal{L}^1(\mathbb{R}^n) \cap \mathcal{L}^\infty(\mathbb{R}^n)$. Then we have $u \in \mathcal{L}^1(\mathbb{R}^n) \cap \mathcal{L}^\infty(\mathbb{R}^n)$ and*

$$||u||_{\mathcal{L}^1(\mathbb{R}^n)} \leq ||g||_{\mathcal{L}^1(\mathbb{R}^n)},$$
$$||u||_{\mathcal{L}^\infty(\mathbb{R}^n)} \leq ||g||_{\mathcal{L}^\infty(\mathbb{R}^n)}.$$

Proof We treat the case of $\mathcal{L}^1(\mathbb{R}^n)$ first. Define

$$\Phi_l(s) = \begin{cases} -s & \text{if } s \leq -\frac{1}{l}, \\ \frac{l}{2}s^2 + \frac{1}{2l} & \text{if } |s| \leq \frac{1}{l}, \\ s & \text{if } s \geq \frac{1}{l} \end{cases} \tag{23}$$

and let $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $0 \leq f \leq 1$. Multiplying (22) by $\Phi_l'(u)f$ and integrating over $\mathbb{R}^n$ gives

$$\int_{\mathbb{R}^n} \big(u\Phi_l'(u)f + \lambda Bu\Phi_l'(u)f - \epsilon\Delta u\Phi_l'(u)f\big)\mathrm{d}x = \int_{\mathbb{R}^n} g\Phi_l'(u)f\mathrm{d}x \leq ||g||_{\mathcal{L}^1(\mathbb{R}^n)}. \tag{24}$$

Since the sequence $\{u\Phi_l'(u)f\}$ is a nonnegative and pointwise non-decreasing sequence with $u\Phi_l'(u)f \to |u|f$ as $l \to \infty$, the monotone convergence theorem and the fact that $0 \leq \Phi_l'f \leq 1$ implies

$$\lim_{l\to\infty}\int_{\mathbb{R}^n} u\Phi_l'(u)f\mathrm{d}x = \int_{\mathbb{R}^n} uf\mathrm{d}x. \tag{25}$$

Since Φ_l' is monotone, and f is nonnegative we have that

$$\begin{aligned}\int_{\mathbb{R}^n} \Delta u\Phi_l'(u)f\mathrm{d}x &= -\int_{\mathbb{R}^n} \Phi_l''(u)|\nabla u|^2 f\mathrm{d}x - \int_{\mathbb{R}^n} \Phi_l'(u)\nabla u\nabla f\mathrm{d}x \\ &= -\int_{\mathbb{R}^n} \Phi_l''(u)|\nabla u|^2 f\mathrm{d}x + \int_{\mathbb{R}^n} \Phi_l(u)\Delta f\mathrm{d}x \leq \int_{\mathbb{R}^n} \Phi_l(u)\Delta f\mathrm{d}x.\end{aligned} \tag{26}$$

By letting $l \to \infty$ we conclude that

$$-\limsup_{l\to\infty} \int_{\mathbb{R}^n} \Delta u \Phi_l'(u) f \mathrm{d}x \geq -\int_{\mathbb{R}^n} u \Delta f \mathrm{d}x.$$

Finally, the sequence $\{Bu\Phi_l'(u)f\}$ converges pointwise to $Bu\,\mathrm{sign}_0(u)f$ as $l \to \infty$ and is dominated by $|Bu|f$. The fact that $|Bu|f$ is integrable follows from Sobolev's embedding of $\mathcal{H}^2$ into $\mathcal{W}^{1,1}$ on the support of f and Lemma 3.3. Thus, using the dominated convergence theorem yields

$$\lim_{l\to\infty} \int_{\mathbb{R}^n} Bu\Phi_l'(u) f \mathrm{d}x = \int_{\mathbb{R}^n} Bu\,\mathrm{sign}_0(u) f \mathrm{d}x.$$

Use the integration by parts formula for difference quotients to find that

$$\begin{aligned}\lim_{l\to\infty} \int_{\mathbb{R}^n} Bu\Phi_l'(u) f \mathrm{d}x = &- \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^k D^{\beta_i}\,\mathrm{sign}_0(u)\tau_{\beta_i} f \phi_i(u, \tau_{\beta_i}u)\omega_i \mathrm{d}h\mathrm{d}x\\ &- \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi_l'(u) D^{\beta_i} f \phi_i(u, \tau_{\beta_i}u)\omega_i \mathrm{d}h\mathrm{d}x,\end{aligned}$$

and apply inequality (9) with $c = 0$ to conclude that

$$\lim_{l\to\infty} \int_{\mathbb{R}^n} Bu\Phi_l'(u) f \mathrm{d}x \geq -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^k \mathrm{sign}_0(u) D^{\beta_i} f \phi_i(u, \tau_{\beta_i}u)\omega_i \mathrm{d}h\mathrm{d}x. \tag{27}$$

Substituting (25), (26) and (27) into (24) yields

$$\int_{\mathbb{R}^n} (uf - \epsilon u\Delta f)\mathrm{d}x - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^k \mathrm{sign}_0(u) D^{\beta_i} f \phi_i(u, \tau_{\beta_i}u)\omega_i \mathrm{d}h\mathrm{d}x \leq ||g||_{\mathcal{L}^1(\mathbb{R}^n)}.$$

Let $\kappa \in \mathcal{C}_0^\infty(\mathbb{R})$ nonnegative such that $\kappa(s) = 1$ for $|s| \leq 1$. Set $f_l(\xi) = \kappa\left(\frac{||\xi||_{\mathbb{R}^n}}{l}\right)$. Since the difference quotient $D^{\beta_i} f_l$ is bounded and is zero for $x \in \mathbb{R}^n$ such that $||x \pm \beta_i||_{\mathbb{R}^n} \leq l$ (see (21)), letting $l \to \infty$ yields

$$||u||_{\mathcal{L}^1(\mathbb{R}^n)} \leq ||g||_{\mathcal{L}^1(\mathbb{R}^n)}.$$

For the case of $\mathcal{L}^\infty(\mathbb{R}^n)$, let $M \in \mathbb{R}$ be such that $M \geq g^+$ a.e. Subtract M from (22), multiply by $\Phi_l'^+(u - M)$ and integrate over $\mathbb{R}^n$ to find that

$$\int_{\mathbb{R}^n} (u - M + \lambda Bu - \epsilon\Delta u)\Phi_l'^+(u - M)\mathrm{d}x = \int_{\mathbb{R}^n} (g - M)\Phi_l'^+(u - M)\mathrm{d}x \leq 0. \tag{28}$$

A similar argument as in (26) gives

$$\lim_{l\to\infty} \int_{\mathbb{R}^n} \Delta u \Phi_l'^+(u - M)\mathrm{d}x \leq 0, \tag{29}$$

as before. Again, integration by parts for difference quotients and the inequality (9) with $c = M$ (the reader may want to check that sign_0 and $\mathrm{sign}_0^\pm$ are interchangeable in (9)) imply that

$$\lim_{l\to\infty}\int_{\mathbb{R}^n} Bu\Phi_l'^{+}(u-M)\mathrm{d}x \\ = -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sum_{i=1}^{k} D^{\beta_i}\,\mathrm{sign}_0^{+}(u-M)\big[\phi_i(u,\tau_{\beta_i}u)-\phi_i(M,M)\big]\omega_i\mathrm{d}h\mathrm{d}x \geq 0. \tag{30}$$

Substituting (29) and (30) into (28) yields

$$\int_{\mathbb{R}^n}(u-M)\Phi_l'^{+}(u-M)\mathrm{d}x \leq 0,$$

which implies that $u \leq M$ a.e.

To establish an analogous lower bound, let M be such that $M \leq g^-$ a.e. Add M to (22), multiply by $\Phi_l'^{-}(u+M)$ and integrate over $\mathbb{R}^n$ to conclude that

$$\int_{\mathbb{R}^n}(u+M+\lambda Bu-\epsilon\Delta u)\Phi_l'^{-}(u+M)\mathrm{d}x = \int_{\mathbb{R}^n}(g+M)\Phi_l'(u+M)^{-}\mathrm{d}x \leq 0.$$

Similar estimates as before show that

$$\int_{\mathbb{R}^n}(u+M)\Phi_l'^{-}(u+M)\mathrm{d}x \leq 0,$$

which implies that $-M \leq u$ a.e. Setting $M = ||g||_{\mathcal{L}^\infty(\mathbb{R}^n)}$ concludes the proof.

Remark 4.9 The proof also shows that the maximum principle holds for Eq. (22); that is, any solution $u \in \mathcal{H}^1(\mathbb{R}^n)\cap\mathcal{H}^2_{loc}(\mathbb{R}^n)$ of (22) satisfies the inequalities $-\left|\left|g^-\right|\right|_{\mathcal{L}^\infty(\mathbb{R}^n)} \leq u \leq \left|\left|g^+\right|\right|_{\mathcal{L}^\infty(\mathbb{R}^n)}$ a.e.

Hölder's inequality immediately yields the following result.

Corollary 4.10 *Let the assumptions of Lemma 4.8 hold and let $g \in \mathcal{L}^1(\mathbb{R}^n)\cap\mathcal{L}^\infty(\mathbb{R}^n)$. Then $u \in \mathcal{L}^p(\mathbb{R}^n)$ for $p \geq 1$ with $||u||_{\mathcal{L}^p(\mathbb{R}^n)} \leq ||g||_{\mathcal{L}^1(\mathbb{R}^n)}^{\frac{1}{p}}\,||g||_{\mathcal{L}^\infty(\mathbb{R}^n)}^{1-\frac{1}{p}}$.*

The next result shows the uniqueness of solutions of (22) for $g \in \mathcal{L}^1(\mathbb{R}^n)$.

Lemma 4.11 *Let the assumptions of Lemma 4.8 hold and let $u, v \in \mathcal{H}^1(\mathbb{R}^n)\cap\mathcal{H}^2_{loc}(\mathbb{R}^n)$ satisfy*

$$u+\lambda Bu-\epsilon\Delta u = g_1,$$
$$v+\lambda Bv-\epsilon\Delta v = g_2.$$

If $g_1, g_2 \in \mathcal{L}^1(\mathbb{R}^n)$, then

$$\left|\left|(u-v)^+\right|\right|_{\mathcal{L}^1(\mathbb{R}^n)} \leq \left|\left|(g_1-g_2)^+\right|\right|_{\mathcal{L}^1(\mathbb{R}^n)}.$$

Proof The proof follows the proof of Lemma 4.8. Let $w = u - v$. Then w satisfies

$$w+\lambda(Bu-Bv)-\epsilon\Delta w = g_1-g_2. \tag{31}$$

Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $0 \leq f \leq 1$. Define Ψ_l by setting $\Psi_l' = \Phi_l'^{+}$ and $\Psi_l(0) = 0$. Multiply (31) by $\Psi_l'(w)f$ and integrate over $\mathbb{R}^n$ to find that

$$\int_{\mathbb{R}^n}\big(w+\lambda(Bu-Bv)-\epsilon\Delta w\big)\Psi_l'(w)f\mathrm{d}x \\ = \int_{\mathbb{R}^n}(g_1-g_2)\Psi_l'(w)f\mathrm{d}x \leq \left|\left|(g_1-g_2)^+\right|\right|_{\mathcal{L}^1(\mathbb{R}^n)} \tag{32}$$

holds, since $0 \leq \Psi_l' f \leq 1$. The facts that $\Psi_l(w) \in \mathcal{H}^1_{loc}(\mathbb{R}^n)$ and that both $\Psi_l'', f \geq 0$ imply that

$$\int_{\mathbb{R}^n} \Delta w \Psi_l'(w) f \mathrm{d}x \leq \int_{\mathbb{R}^n} \Psi_l(w) \Delta f \mathrm{d}x,$$

and thus

$$-\limsup_{l\to\infty} \int_{\mathbb{R}^n} \Delta w \Psi_l'(w) f \mathrm{d}x \geq -\int_{\mathbb{R}^n} w^+ \Delta f \mathrm{d}x. \tag{33}$$

as before. Integration by parts for difference quotients yields

$$\begin{aligned}
&\int_{\mathbb{R}^n} (Bu - Bv)\Psi_l'(w) f \mathrm{d}x \\
&\quad = -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^k D^{\beta_i} \Psi_l'(w) \tau_{\beta_i} f \big[\phi_i(u, \tau_{\beta_i} u) - \phi_i(v, \tau_{\beta_i} v)\big]\omega_i \mathrm{d}h\mathrm{d}x \\
&\qquad - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^k \Psi_l'(w) D^{\beta_i} f \big[\phi_i(u, \tau_{\beta_i} u) - \phi_i(v, \tau_{\beta_i} v)\big]\omega_i \mathrm{d}h\mathrm{d}x.
\end{aligned}$$

Letting $l \to \infty$ in the first integral and using a similar argument as in (9) we find that

$$-\lim_{l\to\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^k D^{\beta_i} \Psi_l'(w) \tau_{\beta_i} f \big[\phi_i(u, \tau_{\beta_i} u) - \phi_i(v, \tau_{\beta_i} v)\big]\omega_i \mathrm{d}h\mathrm{d}x \geq 0,$$

and thus, by the dominated convergence theorem,

$$\begin{aligned}
&\lim_{l\to\infty} \int_{\mathbb{R}^n} (Bu - Bv)\Psi_l'(w) f \mathrm{d}x \\
&\quad \geq -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^k \mathrm{sign}_0^+(w) D^{\beta_i} f \big[\phi_i(u, \tau_{\beta_i} u) - \phi_i(v, \tau_{\beta_i} v)\big]\omega_i \mathrm{d}h\mathrm{d}x.
\end{aligned} \tag{34}$$

Using (33) and (34) in (32) and letting $l \to \infty$ gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} w^+ f \mathrm{d}x - \lambda \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^k \mathrm{sign}_0^+(w) D^{\beta_i} f \big[\phi_i(u, \tau_{\beta_i} u) - \phi_i(v, \tau_{\beta_i} v)\big]\omega_i \mathrm{d}h\mathrm{d}x \\
&- \epsilon \int_{\mathbb{R}^n} w^+ \Delta f \mathrm{d}x \leq \big|\big|(g_1 - g_2)^+\big|\big|_{\mathcal{L}^1(\mathbb{R}^n)}.
\end{aligned}$$

By the same argument as before, let $\kappa \in \mathcal{C}_0^\infty(\mathbb{R})$ nonnegative such that $\kappa(s) = 1$ for $|s| \leq 1$. Set $f_l(\xi) = \kappa\left(\frac{||\xi||_{\mathbb{R}^n}}{l}\right)$. Since the difference quotient $D^{\beta_i} f_l$ is bounded and is zero for $x \in \mathbb{R}^n$ such that $||x \pm \beta_i||_{\mathbb{R}^n} \leq l$ (see (21)), letting $l \to \infty$ yields

$$\int_{\mathbb{R}^n} w^+ \mathrm{d}x = \big|\big|(u - v)^+\big|\big|_{\mathcal{L}^1(\mathbb{R}^n)} \leq \big|\big|(g_1 - g_2)^+\big|\big|_{\mathcal{L}^1(\mathbb{R}^n)}.$$

Corollary 4.12 *Let the assumptions of Lemma 4.8 hold and let $u, v \in \mathcal{H}^1(\mathbb{R}^n) \cap \mathcal{H}^2_{loc}(\mathbb{R}^n)$ satisfy*

$$\begin{aligned}
u + Bu - \epsilon\Delta u &= g_1 \\
v + Bv - \epsilon\Delta v &= g_2.
\end{aligned}$$

If $g_1, g_2 \in \mathcal{L}^1(\mathbb{R}^n)$*, then*

$$||u - v||_{\mathcal{L}^1(\mathbb{R}^n)} \leq ||g_1 - g_2||_{\mathcal{L}^1(\mathbb{R}^n)} .$$

Proof Notice that the equality

$$||a - b||_{\mathcal{L}^1(\mathbb{R}^n)} = \left|\left|(a - b)^+\right|\right|_{\mathcal{L}^1(\mathbb{R}^n)} + \left|\left|(b - a)^+\right|\right|_{\mathcal{L}^1(\mathbb{R}^n)}$$

holds for any $a, b \in \mathcal{L}^1(\mathbb{R}^n)$. Lemma 4.11 shows that

$$\left|\left|(u - v)^+\right|\right|_{\mathcal{L}^1(\mathbb{R}^n)} \leq \left|\left|(g_1 - g_2)^+\right|\right|_{\mathcal{L}^1(\mathbb{R}^n)} ,$$
$$\left|\left|(v - u)^+\right|\right|_{\mathcal{L}^1(\mathbb{R}^n)} \leq \left|\left|(g_2 - g_1)^+\right|\right|_{\mathcal{L}^1(\mathbb{R}^n)} .$$

Hence, the inequality $||u - v||_{\mathcal{L}^1(\mathbb{R}^n)} \leq ||g_1 - g_2||_{\mathcal{L}^1(\mathbb{R}^n)}$ holds as claimed.

The next result shows the existence of a unique generalized solution of (22) for $g \in \mathcal{L}^1(\mathbb{R}^n) \cap \mathcal{L}^\infty(\mathbb{R}^n)$ and plays an essential role in our developments. In order to do so we consider the problem on the ball $B_r \subset \mathbb{R}^n$ for $r > 0$ with zero Dirichlet boundary condition. Let $u^r \in \mathcal{H}_0^1(B_r) \cap \mathcal{H}^2(B_r) =: \mathcal{H}_0^2(B_r)$ satisfy

$$\begin{aligned} &u^r(x) + \lambda B u^r(x) - \epsilon \Delta u^r(x) = g(x), && x \in B_r; \\ &u^r(x) = 0, && x \in \partial B_r, \end{aligned} \tag{35}$$

where Δ denotes the Dirichlet–Laplacian Δ_D on $\mathcal{L}^2(B_r)$ with $D(\Delta_D) = \mathcal{H}_0^2(B_r)$. For the operator B to remain meaningful we use the $E : \mathcal{H}_0^1(B_r) \mapsto \mathcal{H}^1(\mathbb{R}^n)$ extension operator [22, Chapter 5.4] on u^r supplemented with the fact that $\text{supp}(Eu^r) = \text{supp}(u^r)$ and $||Eu^r||_{\mathcal{H}^1(\mathbb{R}^n)} = ||u^r||_{\mathcal{H}_0^1(B_r)}$ [7]. Then we use the restriction operator $R : \mathcal{L}^2(\mathbb{R}^n) \mapsto \mathcal{L}^2(B_r)$ on BEu^r to obtain the operator $RBE : \mathcal{H}_0^1(B_r) \mapsto \mathcal{L}^2(B_r)$. As in (35), we will denote Δ_D by Δ and RBE by B for brevity.

Remark 4.13 One can verify from the proof of Lemmata 3.3, 3.4, 4.8 and 4.11 and Corollaries 4.10 and 4.12 that they all hold for the Dirichlet problem too. Minor steps of the proofs have to be modified, for example, in the proof of Lemma 4.8, instead of multiplying by $\Phi_l'(u)$ and integrating over $\mathbb{R}^n$ we multiply by $\Phi_l'(Eu^r)$ and integrate over B_r. Then we can repeat the same estimates as before. Similar arguments should be used in the rest of the proofs as well.

Proposition 4.14 *Let the assumptions of Lemma 4.8 hold. Then for each* $g \in \mathcal{L}^1(\mathbb{R}^n) \cap \mathcal{L}^\infty(\mathbb{R}^n)$ *there is a unique solution* $u \in \mathcal{H}^1(\mathbb{R}^n) \cap \mathcal{H}_{loc}^2(\mathbb{R}^n)$ *of* (22).

Proof We consider the Dirichlet problem (35) first. Define the operator $T : \mathcal{H}_0^1(B_r) \mapsto \mathcal{H}_0^2(B_r)$ by $T = -(I - \epsilon\Delta)^{-1}\lambda B u + (I - \epsilon\Delta)^{-1} g$ and let

$$\mathcal{S} := \{u \in \mathcal{H}_0^1(B_r) : u = \eta T u,\ \eta \in [0, 1]\}.$$

Note that $\mathcal{H}_0^2(B_r)$ can be compactly embedded into $\mathcal{H}_0^1(B_r)$, which implies that T is continuous (see also Lemma 3.4) and compact and maps the Banach space $\mathcal{H}_0^1(B_r)$ into itself. Observe that $u \in \mathcal{S}$ implies in fact $u \in \mathcal{H}_0^2(B_r)$, and thus $u = \eta T u$ is equivalent to

$$u + \eta\lambda B u - \epsilon \Delta u = \eta g \tag{36}$$

on B_r a.e. Multiply by u and integrate over B_r to find that

$$\begin{aligned}||u||^2_{\mathcal{L}^2(B_r)} + \epsilon\,||\nabla u||^2_{\mathcal{L}^2(B_r)} &= \eta \int_{B_r} gu\mathrm{d}x - \eta\lambda \int_{B_r} Buu\mathrm{d}x\\
&\leq \eta\,||g||_{\mathcal{L}^2(B_r)}\,||u||_{\mathcal{L}^2(B_r)} + \eta\lambda\,||Bu||_{\mathcal{L}^2(B_r)}\,||u||_{\mathcal{L}^2(B_r)}\\
&\leq \frac{\eta}{2}\,||g||^2_{\mathcal{L}^2(B_r)} + \frac{\eta}{2}\,||u||^2_{\mathcal{L}^2(B_r)} + \eta\lambda\delta^2\,||Bu||^2_{\mathcal{L}^2(B_r)} + \frac{\eta\lambda}{\delta^2}\,||u||^2_{\mathcal{L}^2(B_r)}\\
&\leq \frac{1}{2}\,||g||^2_{\mathcal{L}^2(B_r)} + \frac{1}{2}\,||u||^2_{\mathcal{L}^2(B_r)} + \lambda\delta^2\,||Bu||^2_{\mathcal{L}^2(B_r)} + \frac{\lambda}{\delta^2}\,||u||^2_{\mathcal{L}^2(B_r)}\end{aligned}$$

for any $\delta > 0$. Using (11) and Corollary 4.10 (note that the right-hand side is ηg in (36) and g in (22)) we find that

$$||u||^2_{\mathcal{L}^2(B_r)} \leq \eta\,||g||_{\mathcal{L}^1(B_r)}\,||g||_{\mathcal{L}^\infty(B_r)} \leq ||g||_{\mathcal{L}^1(B_r)}\,||g||_{\mathcal{L}^\infty(B_r)} \tag{37}$$

and that

$$\begin{aligned}(\epsilon - C\lambda\delta^2)\,||\nabla u||^2_{\mathcal{L}^2(B_r)} &\leq \frac{1}{2}\,||g||^2_{\mathcal{L}^2(B_r)} + \left(\frac{1}{2} + \frac{\lambda}{\delta^2}\right)||u||^2_{\mathcal{L}^2(B_r)}\\
&\leq \left(1 + \frac{\lambda}{\delta^2}\right)||g||_{\mathcal{L}^1(B_r)}\,||g||_{\mathcal{L}^\infty(B_r)}\,.\end{aligned} \tag{38}$$

The inequalities (37) and (38) show that by choosing δ small enough $\mathcal{S}$ is bounded in $\mathcal{H}^1_0(B_r)$. Then Schaefer's fixed point theorem shows that T has a fixed point [16, Corollary 8.1] and, in fact, Lemma 4.11 ensures that the fixed point is unique on B_r.

Choose a sequence $\{r_m\} \subset \mathbb{R}$ such that $r_m \to \infty$ in an increasing fashion as $m \to \infty$ and let $u^{r_m} \in \mathcal{H}^2_0(B_{r_m})$ be the corresponding sequence of solutions. Then clearly $\{Eu^{r_m}\} \subset \mathcal{H}^2(\mathbb{R}^n)$ and by Lemma 4.8 we also have $||Eu^{r_m}||_{\mathcal{L}^\infty(B_{r_m})} \leq ||g||_{\mathcal{L}^\infty(B_{r_m})} \leq ||g||_{\mathcal{L}^\infty(\mathbb{R}^n)}$. For any $r < r'$ we have by Corollary 4.12 that

$$\left|\left|Eu^r - Eu^{r'}\right|\right|_{\mathcal{L}^1(\mathbb{R}^n)} \leq ||g||_{\mathcal{L}^1\left(B_{r'}\setminus B_r\right)}\,,$$

and thus the sequence is Cauchy and converges in $\mathcal{L}^1(\mathbb{R}^n)$ to some $u \in \mathcal{L}^1(\mathbb{R}^n) \cap \mathcal{L}^\infty(\mathbb{R}^n)$. Furthermore, elliptic regularity [22, Sect. 6.3.1] combined with inequalities (37) and (38) imply that $\{Eu^{r_m}\}$ is uniformly bounded with

$$\begin{aligned}\left|\left|Eu^{r_m}\right|\right|_{\mathcal{H}^2(\mathbb{R}^n)} &= \left|\left|u^{r_m}\right|\right|_{\mathcal{H}^2_0(B_{r_m})} \leq C\big(||g||_{\mathcal{L}^2(B_{r_m})} + \left|\left|Bu^{r_m}\right|\right|_{\mathcal{L}^2(B_{r_m})}\big)\\
&\leq C\big(||g||_{\mathcal{L}^2(B_{r_m})} + \left|\left|u^{r_m}\right|\right|_{\mathcal{H}^1_0(B_{r_m})}\big) \leq C\Big(||g||_{\mathcal{L}^2(\mathbb{R}^n)} + ||g||^{\frac{1}{2}}_{\mathcal{L}^1(\mathbb{R}^n)}\,||g||^{\frac{1}{2}}_{\mathcal{L}^\infty(\mathbb{R}^n)}\Big).\end{aligned} \tag{39}$$

Let us consider B_{r_0} for some $r_0 > 0$ and let $\{Eu^{r_{m_k}}\}$ be any subsequence, which is then bounded in $\mathcal{H}^2(B_{r_0})$ and thus by the compact embedding of $\mathcal{H}^2(B_{r_0})$ into $\mathcal{H}^1(B_{r_0})$ it has a subsequence $\{Eu^{r_{m_{k_l}}}\}$ that converges in $\mathcal{H}^1(B_{r_0})$ to u. Since any subsequence has a convergent sequence with the same limit the original sequence converges in $\mathcal{H}^1(B_{r_0})$ to u. By (39) $||u||_{\mathcal{H}^1(B_{r_0})} \leq C$ independently of r_0 showing that u is in fact in $\mathcal{H}^1(\mathbb{R}^n)$ and is a weak solution. Thus, by elliptic regularity $u \in \mathcal{H}^2_{loc}(B_{r_0})$ as well and since $r_0 > 0$ was arbitrary we conclude that $u \in \mathcal{H}^1(\mathbb{R}^n) \cap \mathcal{H}^2_{loc}(\mathbb{R}^n)$ is a strong solution solution and by Corollary 4.12 it is unique.

In our next result we take the limit $\epsilon \to 0$. This will not only allow us to consider flux functions in $\mathcal{W}^{1,\infty}_{loc}(\mathbb{R} \times \mathbb{R})$ but will show that the various properties established for the solutions of (22) hold for the generalized solutions of (14), which in turn will imply that they hold for the semigroup as well.

Proposition 4.15 *Let $\phi_i \in \mathcal{W}^{1,\infty}_{loc}(\mathbb{R}\times\mathbb{R})$ and A_0 be given by Definition 3.6. Then $\mathcal{L}^1(\mathbb{R}^n)\cap\mathcal{L}^\infty(\mathbb{R}^n)\subseteq R(I+\lambda A_0)$ for $\lambda>0$. Accordingly, let $T_\lambda:\mathcal{L}^1(\mathbb{R}^n)\cap\mathcal{L}^\infty(\mathbb{R}^n)\mapsto\mathcal{L}^1(\mathbb{R}^n)$ be the restriction of $(I+\lambda A_0)^{-1}$ to $\mathcal{L}^1(\mathbb{R}^n)\cap\mathcal{L}^\infty(\mathbb{R}^n)$. If $g_1, g_2\in\mathcal{L}^1(\mathbb{R}^n)\cap\mathcal{L}^\infty(\mathbb{R}^n)$, then*

- (i) $T_\lambda g_1\in\mathcal{L}^p(\mathbb{R}^n)$ *for* $p\geq 1$ *with* $||T_\lambda g_1||_{\mathcal{L}^p(\mathbb{R}^n)}\leq||g_1||^{\frac{1}{p}}_{\mathcal{L}^1(\mathbb{R}^n)}\leq||g_1||^{1-\frac{1}{p}}_{\mathcal{L}^\infty(\mathbb{R}^n)}$,
- (ii) $-\big|\big|g_1^-\big|\big|_{\mathcal{L}^\infty(\mathbb{R}^n)}\leq T_\lambda g_1\leq\big|\big|g_1^+\big|\big|_{\mathcal{L}^\infty(\mathbb{R}^n)}$,
- (iii) $\big|\big|(T_\lambda g_1-T_\lambda g_2)^+\big|\big|_{\mathcal{L}^1(\mathbb{R}^n)}\leq\big|\big|(g_1-g_2)^+\big|\big|_{\mathcal{L}^1(\mathbb{R}^n)}$,
- (iv) T_λ *commutes with translations,*
- (v) $\int_{\mathbb{R}^n}T_\lambda g_1\mathrm{d}x=\int_{\mathbb{R}^n}g_1\mathrm{d}x$.

Proof Let $\{\phi_i^m\}\subset\mathcal{C}^1(\mathbb{R}\times\mathbb{R})$ be a sequence such that each ϕ_i^m is bounded and have the property $\phi_i^m(0,0)=0$ and $\{\phi_i^m\}$ converges to ϕ_i uniformly on compact sets. Define

$$B_m u=\int_{\mathbb{R}^n}\sum_{i=1}^k\frac{\phi_i^m(u,\tau_{\beta_i}u)-\phi_i^m(\tau_{-\beta_i}u,u)}{||\beta_i||_{\mathbb{R}^n}}\omega_i\mathrm{d}h$$

and the operator $T_{\lambda,m}:\mathcal{L}^1(\mathbb{R}^n)\cap\mathcal{L}^\infty(\mathbb{R}^n)\mapsto\mathcal{L}^1(\mathbb{R}^n)\cap\mathcal{L}^\infty(\mathbb{R}^n)$ by $T_{\lambda,m}g=u$ if $u\in\mathcal{H}^1(\mathbb{R}^n)\cap\mathcal{H}^2_{loc}(\mathbb{R}^n)$ and

$$u+\lambda B_m u-\frac{1}{m}\Delta u=g. \tag{40}$$

Proposition 4.14, Lemmata 4.8 and 4.11, Remark 4.9, Corollaries 4.10 and 4.12 and the fact that $T_{\lambda,m}$ commutes with translations imply that $T_{\lambda,m}$ is well-defined and has the properties (i)-(iv). Let $g\in\mathcal{L}^1(\mathbb{R}^n)\cap\mathcal{L}^\infty(\mathbb{R}^n)$ and $u_m=T_{\lambda,m}g$. By Lemma 4.11 and the translation invariance of $T_{\lambda,m}$ we conclude that

$$\int_{\mathbb{R}^n}\big|u_m(x+y)-u_m(x)\big|\mathrm{d}x\leq\int_{\mathbb{R}^n}\big|g(x+y)-g(x)\big|\mathrm{d}x$$

for $y\in\mathbb{R}^n$. The above estimate and $||u_m||_{\mathcal{L}^1(\mathbb{R}^n)}\leq||g||_{\mathcal{L}^1(\mathbb{R}^n)}$, by the means of the Fréchet-Kolmogorov compactness theorem, imply that $\{u_m\}$ is precompact in $\mathcal{L}^1_{loc}(\mathbb{R}^n)$. Thus, there is a subsequence $\{u_{m_j}\}$ which converges a.e. in $\mathcal{L}^1_{loc}(\mathbb{R}^n)$ to a limit $u\in\mathcal{L}^1(\mathbb{R}^n)$. This convergence will be denoted as $u_{m_j}\twoheadrightarrow u$. Let $f\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ be nonnegative and Φ_l be given by (23). Multiply (40) by $\Phi_l'(u_m-c)f$ and integrate over $\mathbb{R}^n$ to find that

$$\int_{\mathbb{R}^n}\left(u_m+\lambda B_m u_m-\frac{1}{m}\Delta u_m\right)\Phi_l'(u_m-c)f\mathrm{d}x=\int_{\mathbb{R}^n}g\Phi_l'(u_m-c)f\mathrm{d}x.$$

Integration by parts gives

$$\int_{\mathbb{R}^n}\Bigg((u_m-g)\Phi_l'(u_m-c)f+\lambda B_m u_m\Phi_l'(u_m-c)f+\frac{1}{m}\big(\Phi_l''(u_m-c)|\nabla u_m|^2f-\Phi_l(u_m-c)\Delta f\big)\Bigg)\mathrm{d}x=0.$$

Note that both $\Phi_l'', f\geq 0$ implies that

$$\frac{1}{m}\int_{\mathbb{R}^n}\Phi_l''(u_m-c)|\nabla u_m|^2f\mathrm{d}x\geq 0$$

and $||u_m||_{\mathcal{L}^\infty(\mathbb{R}^n)} \le ||g||_{\mathcal{L}^\infty(\mathbb{R}^n)}$ implies that the integral

$$\int_{\mathbb{R}^n} \Phi_l(u_m - c)\Delta f \,\mathrm{d}x$$

is bounded. Letting $m \to \infty$ through the subsequence $\{m_j\}$ and using the convergences $u_{m_j} \twoheadrightarrow u$ and $\phi_i^m \to \phi_i$ uniformly on compact sets yields

$$\int_{\mathbb{R}^n} \big((u-g)\Phi_l'(u-c)f + \lambda Bu\Phi_l(u-c)f\big)\mathrm{d}x \le 0.$$

Letting $l \to \infty$ and using (15) gives

$$\int_{\mathbb{R}^n} \Bigg(\operatorname{sign}_0(u-c)(u-g)f \\ - \lambda \int_{\mathbb{R}^n} \sum_{i=1}^{k} D^{\beta_i}\big[f \operatorname{sign}_0(u-c)\big]\big(\phi_i(u, \tau_{\beta_i}u) - \phi_i(c,c)\big)\omega_i \mathrm{d}h \Bigg)\mathrm{d}x \le 0.$$

Since $||u||_{\mathcal{L}^\infty(\mathbb{R}^n)} \le ||g||_{\mathcal{L}^\infty(\mathbb{R}^n)}$ and $\phi_i \in \mathcal{W}^{1,\infty}_{loc}(\mathbb{R}\times\mathbb{R})$ we have $\phi_i(u, \tau_{\beta_i}u) \in \mathcal{L}^1(\mathbb{R}^n)$. Thus, we have $g \in (I + \lambda A_0)u$ by Definition 3.6 and, in fact, by Lemma 4.1 the equality

$$u + \lambda A_0 u = g \tag{41}$$

holds. The accretivity of A_0 shows that u is unique, hence $\lim_{m\to\infty} T_{\lambda,m}g = T_\lambda g$ holds with convergence in $\mathcal{L}^1_{loc}(\mathbb{R}^n)$. Properties (i)-(iv) are preserved under $\mathcal{L}^1_{loc}(\mathbb{R}^n)$ convergence. Choose $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ nonnegative, multiply (41) with f and integrate over $\mathbb{R}^n$ to find that

$$\int_{\mathbb{R}^n} uf \,\mathrm{d}x + \lambda \int_{\mathbb{R}^n} A_0 uf \,\mathrm{d}x \\ = \int_{\mathbb{R}^n} uf \,\mathrm{d}x - \lambda \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^{k} D^{\beta_i} f \phi_i(u, \tau_{\beta_i}u)\omega_i \mathrm{d}h\mathrm{d}x = \int_{\mathbb{R}^n} gf \,\mathrm{d}x$$

also holds by Lemma 4.1. Let $\kappa \in \mathcal{C}_0^\infty(\mathbb{R})$ be nonnegative such that $\kappa(s) = 1$ for $|s| \le 1$. Set $f_l(\xi) = \kappa\Big(\frac{||\xi||_{\mathbb{R}^n}}{l}\Big)$ and let $l \to \infty$. Using (21) we find that the integral

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^{k} D^{\beta_i} f_l \phi_i(u, \tau_{\beta_i}u)\omega_i \mathrm{d}h\mathrm{d}x$$

converges to zero as $l \to \infty$ and thus property (v) holds as well.

Remark 4.16 By Definition 3.6 it is clear that $\overline{D(A)} \subset \mathcal{L}^1(\mathbb{R}^n)$ and in some cases, in fact, the equality $\overline{D(A)} = \mathcal{L}^1(\mathbb{R}^n)$ holds, see Lemma 4.2. However, this remains to be shown under our general assumption that $\phi_i \in \mathcal{W}^{1,\infty}_{loc}(\mathbb{R}\times\mathbb{R})$.

Proof of Theorem 3.8 Since A_0 is accretive it follows that the closure A is also accretive. Let $g \in \mathcal{L}^1(\mathbb{R}^n)$ and $\{g_m\} \subset \mathcal{L}^1(\mathbb{R}^n) \cap \mathcal{L}^\infty(\mathbb{R}^n)$ be such that $g_m \to g$ in $\mathcal{L}^1(\mathbb{R}^n)$. Since T_λ is a contraction, the sequence $\{T_\lambda g_m\}$ is Cauchy. Let $\lambda w_m = (I - T_\lambda)g_m$, so $w_m \in A_0 T_\lambda g_m$ and the sequence $\{w_m\}$ is also Cauchy. If $T_\lambda g_m \to v$ and $w_m \to w$, then $w \in Av$ and $g = v + \lambda w \in (I + \lambda A)v$. This shows that A is m-accretive and the proof is complete.

Proof of Theorem 3.9 The solution $u_\epsilon(t)$ of (7) is given by

$$u_\epsilon(t) = (I + \epsilon A)^{-\left\lfloor \frac{t}{\epsilon} \right\rfloor - 1} u_0.$$

The uniform convergence $\lim_{\epsilon\to 0} u_\epsilon(t) = S(t)u_0$ for t in $\mathcal{L}^1(\mathbb{R}^n)$ shows that properties (i)-(v) hold for $S(t)$, since by Proposition 4.15 they hold for $T_\lambda = (I + \lambda A)^{-1}$.

For property (vi) let $u_0 \in \mathcal{L}^1(\mathbb{R}^n) \cap \mathcal{L}^\infty(\mathbb{R}^n)$ (note that by Lemma 4.1 the operator A_0 is single-valued in this case) and $u_\epsilon(x,t)$ satisfy

$$\begin{aligned} &\frac{1}{\epsilon}\big(u_\epsilon(x,t) - u_\epsilon(x,t-\epsilon)\big) + A_0 u_\epsilon(x,t) = 0, && (x,t) \in \mathbb{R}^n \times (0,T);\\ &u_\epsilon(x,0) = u_0(x), && x \in \mathbb{R}^n. \end{aligned}$$

The definition of A_0 implies that

$$\begin{aligned} &\int_{\mathbb{R}^n} \operatorname{sign}_0\big(u_\epsilon(x,t) - c\big) A_0 u_\epsilon(x,t) f \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^k D^{\beta_i}\big[f \operatorname{sign}_0(u-c)\big]\big(\phi_i(u_\epsilon, \tau_{\beta_i} u_\epsilon) - \phi_i(c,c)\big)\omega_i \,\mathrm{d}h\mathrm{d}x \ge 0 \end{aligned}$$

holds for any nonnegative $f \in \mathcal{C}_0^\infty\big(\mathbb{R}^n \times (0,T)\big)$ and any $c \in \mathbb{R}$. Notice that

$$A_0 u_\epsilon(x,t) = \frac{1}{\epsilon}\big(u_\epsilon(x,t-\epsilon) - u_\epsilon(x,t)\big)$$

and that

$$\begin{aligned} &\operatorname{sign}_0\big(u_\epsilon(x,t) - c\big)\big(u_\epsilon(x,t-\epsilon) - u_\epsilon(x,t)\big) = \operatorname{sign}_0\big(u_\epsilon(x,t) - c\big)\big(u_\epsilon(x,t-\epsilon) - c\big)\\ &\quad + \operatorname{sign}_0\big(u_\epsilon(x,t) - c\big)\big(u_\epsilon(x,t) - c\big) \le \big|u_\epsilon(x,t-\epsilon) - c\big| - \big|u_\epsilon(x,t) - c\big|. \end{aligned}$$

Using the above and integrating over $(0,T)$ yields

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} \frac{1}{\epsilon}\Big(\big|u_\epsilon(x,t-\epsilon) - c\big| - \big|u_\epsilon(x,t) - c\big|\Big) f(x,t)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sum_{i=1}^k D^{\beta_i}\big[f \operatorname{sign}_0(u-c)\big]\big(\phi_i(u_\epsilon, \tau_{\beta_i} u_\epsilon) - \phi_i(c,c)\big)\omega_i \,\mathrm{d}h\mathrm{d}x\mathrm{d}t \ge 0. \end{aligned} \tag{42}$$

Observe that

$$\begin{aligned} &\frac{1}{\epsilon}\int_0^T \int_{\mathbb{R}^n} \Big(\big|u_\epsilon(x,t-\epsilon) - c\big| - \big|u_\epsilon(x,t) - c\big|\Big) f(x,t)\mathrm{d}x\mathrm{d}t \\ &\quad = \frac{1}{\epsilon}\bigg(\int_0^\epsilon \int_{\mathbb{R}^n} \big|u_\epsilon(x,t-\epsilon) - c\big| f(x,t)\mathrm{d}x\mathrm{d}t - \int_{T-\epsilon}^T \int_{\mathbb{R}^n} \big|u_\epsilon(x,t) - c\big| f(x,t)\mathrm{d}x\mathrm{d}t\bigg) \\ &\quad + \int_\epsilon^{T-\epsilon} \int_{\mathbb{R}^n} \big|u_\epsilon(x,t) - c\big| \frac{1}{\epsilon}\big(f(x,t+\epsilon) - f(x,t)\big)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Since $f \in \mathcal{C}_0^\infty\big(\mathbb{R}^n \times (0,T)\big)$ the first two integrals after the equal sign vanish for ϵ small enough. The uniform convergence $\lim_{\epsilon\to 0} u_\epsilon(x,t) = S(t)u_0(x)$ in $\mathcal{L}^1(\mathbb{R}^n)$ implies that the third integral tends to

$$\int_0^T \int_{\mathbb{R}^n} \big|S(t)u_0(x) - c\big| \frac{\partial f}{\partial t}\mathrm{d}x\mathrm{d}t;$$

that is, by taking the limit $\epsilon \to 0$ in (42) the proof is complete.

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Declarations

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