Quasi-monotone convergence of plurisubharmonic functions

Citation for the original published paper (version of record):
Guedj, V., Trusiani, A. (2023). Quasi-monotone convergence of plurisubharmonic functions. Bulletin des Sciences Mathematiques, 188. http://dx.doi.org/10.1016/j.bulsci.2023.103341

Bull. Sci. math. 188 (2023) 103341

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

journal homepage: www.elsevier.com/locate/bulsci

Quasi-monotone convergence of plurisubharmonic functions

Vincent Guedj [a], Antonio Trusiani [b,*]

[a] *Institut de Mathématiques de Toulouse, Université de Toulouse, 118 route de Narbonne, 31400 Toulouse, France*
[b] *Mathematical Sciences, Chalmers University of Technology, Chalmers tvärgata 3, 41296 Göteborg, Sweden*

A R T I C L E   I N F O

Article history:
Received 17 March 2023
Available online 25 September 2023

MSC:
32W20
32U05
32Q15
35A23

Keywords:
Plurisubharmonic functions
Strong topology
Complex Monge-Ampère operator

A B S T R A C T

The complex Monge-Ampère operator has been defined for locally bounded plurisubharmonic functions by Bedford-Taylor in the 80's. This definition has been extended to compact complex manifolds, and to various classes of mildly unbounded quasi-plurisubharmonic functions by various authors. As this operator is not continuous for the L^1-topology, several stronger topologies have been introduced over the last decades to remedy this, while maintaining efficient compactness criteria. The purpose of this note is to show that these stronger topologies are essentially equivalent to the natural quasi-monotone topology that we introduce and study here.

* Corresponding author.
E-mail addresses: vincent.guedj@math.univ-toulouse.fr (V. Guedj), trusiani@chalmers.se (A. Trusiani).
URLs: https://www.math.univ-toulouse.fr/~guedj/ (V. Guedj), https://sites.google.com/view/antonio-trusiani/home (A. Trusiani).

https://doi.org/10.1016/j.bulsci.2023.103341

0. Introduction

In connection with the spectacular developments of Kähler geometry in the last decade (see [11,24,2,4,34,33,35]), several finite energy spaces of quasi-plurisubharmonic functions have been studied, each of which endowed with a strong topology, that ensures completeness of the space, good compactness criteria, and continuity of the complex Monge-Ampère operator (the latter being discontinuous for the weaker L^1-topology). In this note we introduce yet another strong convergence, the *quasi-monotone convergence*.

Let (X, ω) be a compact Kähler manifold. A function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is quasi-plurisubharmonic if it is locally the sum of smooth and a plurisubharmonic function. The function φ is called ω-psh if $\omega + dd^c\varphi \geq 0$ in the sense of currents. We let $PSH(X, \omega)$ denote the set of all ω-psh functions; it can be endowed with the L^1-topology, which is equivalent to the weak topology of distributions. If $\varphi_j \in PSH(X, \omega)$ converges in L^1 to $\varphi \in PSH(X, \omega)$, then $\varphi_j^+ := \left(\sup_{\ell \geq j} \varphi_\ell\right)^*$ is a decreasing sequence of ω-psh functions that pointwise decreases to φ (see [29, Proposition 8.4]). The quasi-monotone convergence requires a dual property:

Definition. We say that φ_j converges quasi-monotonically if $\varphi_j^- := P_\omega\left(\inf_{\ell \geq j} \varphi_\ell\right)$ is a sequence of ω-plurisubharmonic functions that increases to φ.

Here $P_\omega(h)$ denotes the ω-psh envelope of the function h: this is the largest ω-psh function that lies below h. The family of such functions is compact, but it may happen that it is empty in which case $P_\omega(h) \equiv -\infty$ (see Example 3.3).

The complex Monge-Ampère operator MA is well defined for bounded ω-psh functions, as follows from Bedford-Taylor theory [1,29]. At the heart of the theory lies the continuity property of MA along monotone sequences. Our convergence notion naturally extends this property, allowing for sequences φ_j that are bounded from above and below $u_j \leq \varphi_j \leq v_j$ by a sequence of ω-psh functions u_j (resp. v_j) which increases (resp. decreases) to φ [see Lemma 2.2].

The property that we require is somewhat dual to the property for the upper-envelope, which always holds. This is well illustrated by the case when the φ_j's are solutions of a complex Monge-Ampère equation (or of a complex Monge-Ampère flow), where this operation (envelope of infima) leaves the space of super-solutions invariant (see [25, Theorem C] and Examples 3.1 and 3.2).

Our main result compares all these strong topologies:

Theorem. Let (X, ω) be a compact Kähler manifold, and $(\varphi_j) \in PSH(X, \omega)^{\mathbb{N}}$ a sequence which converges in $L^1(X)$ to some $\varphi \in PSH(X, \omega)$. Fix $\psi \in \mathcal{E}^1(X, \omega)$.

(1) If φ_j converges quasi-monotonically then φ_j converges in capacity.
(2) If φ_j converges in capacity and $\varphi_j \geq \psi$ then φ_j converges in $(\mathcal{E}^1(X, \omega), d_1)$.

(3) If φ_j converges in $(\mathcal{E}^1(X,\omega), d_1)$, then φ_j converges to φ in capacity. Moreover up to extracting and relabelling, the convergence is quasi-monotone and there exists $\tilde{\psi} \in \mathcal{E}^1(X,\omega)$ such that $\varphi_j \geq \tilde{\psi}$.

A sequence (φ_j) converges in capacity to φ if for all $\delta > 0$,

$$\mathrm{Cap}_\omega\left(\{z \in X,\ |\varphi_j(x) - \varphi(x)| \geq \delta\}\right) \longrightarrow 0,$$

as $j \to +\infty$, where Cap_ω denotes the Monge-Ampère capacity introduced by Kolodziej [32] and further studied in [26,31,39,22].

The set $(\mathcal{E}^1(X,\omega), d_1)$ denotes the finite energy space introduced in [27], endowed with the metrizable strong topology considered in [2] and further studied in [15]. One can equally well consider other finite energy classes $(\mathcal{E}_\chi(X,\omega), d_\chi)$, endowed with the induced Mabuchi topology, and show an appropriate version of the above result. We refer the reader to Definitions 1.2 and 1.8, and to Theorem 2.5 for the precise statements.

All the previous notions of strong convergence therefore coincide –up to extracting and relabelling– when the sequence is uniformly bounded:

Corollary. Let (X,ω) be a compact Kähler manifold, and $(\varphi_j) \in PSH(X,\omega)^{\mathbb{N}}$ a uniformly bounded sequence which converges in $L^1(X)$ to some $\varphi \in PSH(X,\omega)$. Up to extracting and relabelling, the following properties are equivalent

(1) φ_j converges quasi-monotonically to φ;
(2) φ_j converges to φ in capacity;
(3) φ_j converges to φ in $(\mathcal{E}^1(X,\omega), d_1)$.

Example 3.3 shows that a sequence can converge in capacity but not quasi-monotonically. The sequence $\varphi_j = \varphi/j$ converges to zero both in capacity and quasi-monotonically, but not in energy if φ does not belong to $\mathcal{E}^1(X,\omega)$. Example 3.4 shows that extracting is necessary to ensure the quasimonotone convergence. Finally Example 3.5 provides an example where these strong convergences do not hold, while Example 3.6 compares the various types of convergence in energy.

Contents We recall basic facts in Section 1. The quasi-monotone convergence is introduced in Section 2, where we prove our main Theorem. We provide several explicit examples in Section 3 and explain how our observations extend to big cohomology classes with prescribed singularities, as well as to the local setting.

Acknowledgments

We thank E. Di Nezza, H. C. Lu and A. Zeriahi for useful comments. The authors are partially supported by the research project Hermetic (ANR-11-LABX-0040), the ANR projects Paraplui. The second author is supported by a grant from the Knut and Alice Wallenberg Foundation.

1. Capacity and energies

In the whole article we let X denote a compact Kähler manifold of complex dimension $n \geq 1$, and we fix ω denote a Kähler form on X.

1.1. Convergence in capacity

1.1.1. Quasi-plurisubharmonic functions

A function is quasi-plurisubharmonic if it is locally given as the sum of a smooth and a psh function. Quasi-psh functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ satisfying $\omega + dd^c\varphi \geq 0$ in the weak sense of currents are called ω-plurisubharmonic (ω-psh for short). Here $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{2i\pi}(\partial - \overline{\partial})$.

A $\mathcal{C}^2$-smooth function u has bounded Hessian, hence εu is ω-psh if $0 < \varepsilon$ is small enough. Note that constants functions are also ω-psh functions.

Definition 1.1. We let $\mathrm{PSH}(X, \omega)$ denote the set of all ω-plurisubharmonic functions which are not identically $-\infty$.

The set $\mathrm{PSH}(X, \omega)$ is a closed subset of $L^1(X)$, for the L^1-topology. Subsets of ω-psh functions enjoy strong compactness and integrability properties, we mention notably the following: for any fixed $r \geq 1$,

- $\mathrm{PSH}(X, \omega) \subset L^r(X)$; the induced L^r-topologies are all equivalent;
- $\mathrm{PSH}(X, \omega) \subset W^{1,q}(X) = \{u \in L^1(X),\ \nabla u \in L^q(X)\}$ for all $q < 2$; the induced $W^{1,q}$-topology is again equivalent to the L^1-topology;
- $\mathrm{PSH}_A(X, \omega) := \{u \in \mathrm{PSH}(X, \omega),\ -A \leq \sup_X u \leq 0\}$ is compact.

We refer the reader to [29] for further basic properties of ω-psh functions.

Let φ be a plurisubharmonic function in $\mathbb{C}^n$. If φ is smooth, its complex Monge-Ampère measure is defined by

$$MA(\varphi) = (dd^c\varphi)^n = c_n \det\left(\frac{\partial^2\varphi}{\partial z_j \partial \overline{z_k}}\right) dV_{eucl},$$

where dV_{eucl} is the euclidean volume form, and $c_n > 0$ is a normalizing constant.

When φ is less regular, one can approximate it from above by smooth plurisubharmonic functions $\varphi_{\varepsilon_j} = \varphi \star \chi_{\varepsilon_j}$ obtained by convolution with a standard family of mollifiers. If the measures $MA(\varphi_{\varepsilon_j})$ converge to a limit μ_φ, one sets $MA(\varphi) := \mu_\varphi$. This definition has been shown to be consistent by Bedford and Taylor [1] when φ is *locally bounded*: in this case they have shown that μ_φ is the limit of *any* decreasing sequence of plurisubharmonic approximants.

Bedford-Taylor's theory has been adapted to the compact setting (see [29,21]), and the definition of MA has been extended to mildly unbounded quasi-plurisubharmonic functions (see [10,7,13,29]). In the compact setting, one needs to replace plurisubharmonic functions by quasi-plurisubharmonic ones. One can approximate $\varphi \in PSH(X,\omega)$ by a decreasing sequence of smooth ω-psh functions φ_j [19] and set $V = \int_X \omega^n$ and

$$MA(\varphi) := \lim_{j\to+\infty} V^{-1}(\omega + dd^c\varphi_j)^n,$$

whenever the limit is well-defined and independent of the approximants.

The complex Monge-Ampère measure $MA(\varphi)$ is in particular well-defined for any ω-psh function φ which is *bounded*. It is also well-defined for unbounded ω-psh functions that have *finite energy* (see Section 1.2).

1.1.2. The Monge-Ampère capacity

Definition 1.2. Given $K \subset X$ a compact set, its Monge-Ampère capacity is

$$\mathrm{Cap}_\omega(K) := \sup\left\{ \int_K (\omega + dd^c u)^n,\ u \in PSH(X,\omega) \text{ with } 0 \le u \le 1 \right\}.$$

This notion has been introduced in [32] and further studied in [26]. We refer the reader to [29, Chapter 9] for its basic properties.

Definition 1.3. A sequence $(\varphi_j) \in PSH(X,\omega)^{\mathbb{N}}$ converges in capacity to $\varphi \in PSH(X,\omega)$ if for all $\delta > 0$,

$$\mathrm{Cap}_\omega\left(\{x \in X,\ |\varphi_j(x) - \varphi(x)| \ge \delta\}\right) \overset{j\to+\infty}{\longrightarrow} 0.$$

It is known that convergence in capacity implies convergence in L^1 (see [29, Lemma 4.24]). One can moreover reduce to uniformly bounded sequences:

Proposition 1.4. *A sequence $(\varphi_j) \in PSH(X,\omega)^{\mathbb{N}}$ converges in capacity to $\varphi \in PSH(X,\omega)$ if and only if for all $C > 0$, the sequence* $\max(\varphi_j, -C)$ *converges in capacity to* $\max(\varphi, -C)$.

Proof. We can assume without loss of generality that $\varphi, \varphi_j \le 0$. It follows from the Chern-Levine-Nirenberg inequalities [29, Corollary 9.5] that

$$\mathrm{Cap}_\omega\left(\{x \in X,\ \varphi_j(x) < -C\}\right) \le \frac{||\varphi_j||_{L^1} + nV}{C}$$

and similarly $\mathrm{Cap}_\omega\left(\{\varphi < -C\}\right) \le \frac{||\varphi||_{L^1}+nV}{C}$. Thus $\{x \in X,\ |\varphi_j(x) - \varphi(x)| \ge \delta\}$ and $\{x \in X,\ |\max(\varphi_j(x), -C) - \max(\varphi(x), -C)| \ge \delta\}$ differ by a set whose capacity is uniformly small in j, as C goes to $+\infty$. The conclusion follows. $\square$

For uniformly bounded sequences, the convergence in capacity implies the convergence of Monge-Ampère measures [38, Theorem 1]. More generally we have the following consequence of [29, Theorem 4.26]:

Proposition 1.5. *Let φ_j^ℓ be a uniformly bounded sequence of ω-psh functions which converge in capacity to $\varphi^\ell \in PSH(X,\omega)$, $0 \le \ell \le n$. For all continuous weight $\chi : \mathbb{R} \to \mathbb{R}$, the weighted measure $\chi(\varphi_j^0)(\omega+dd^c\varphi_j^1)\wedge\cdots\wedge(\omega+dd^c\varphi_j^n)$ weakly converges to the weighted measure $\chi(\varphi^0)(\omega + dd^c\varphi^1) \wedge \cdots \wedge (\omega + dd^c\varphi^n)$.*

We shall need the following estimate, which is an adaptation to the compact setting of a local observation of Blocki [6].

Proposition 1.6. *Let u, v, w be ω-psh functions such that $-1 \le u \le 0$ and $v \le w$. Then*

$$\int_X (w-v)^{n+1}(\omega+dd^cu)^n \le (n+1)! \sum_{j=0}^{n} \int_X (w-v)^{n+1-j}\,(\omega+dd^cv)^j \wedge \omega^{n-j}.$$

Proof. Set $\omega_u = \omega + dd^cu$ and $\omega_v = \omega + dd^cv$. Let S be a closed current of bidegree $(1,1)$. By induction it suffices to establish the following inequality,

$$\int_X (w-v)^{p+1}\omega_u \wedge S \le \int_X (w-v)^{p+1}\,\omega \wedge S + (p+1)\int_X (w-v)^p\,\omega_v \wedge S,$$

and apply it to $S = \omega_u^a \wedge \omega_v^b \wedge \omega^c$. This follows from Stokes theorem, observing that $-dd^c(w-v)^{p+1} \le (p+1)(w-v)^p\omega_v$, hence $\int_X (w-v)^{p+1}dd^cu \wedge S = \int_X u\, dd^c(w-v)^{p+1} \wedge S \le (p+1)\int (w-v)^p\omega_v \wedge S$. □

1.2. Finite energy topologies

1.2.1. Finite energy classes

Given $\varphi \in \mathrm{PSH}(X,\omega)$, we consider

$$\varphi_j := \max(\varphi, -j) \in \mathrm{PSH}(X,\omega) \cap L^\infty(X).$$

The measures $MA(\varphi_j)$ are well defined probability measures and the sequence $\mu_j := \mathbf{1}_{\{\varphi>-j\}}MA(\varphi_j)$ is increasing [27, p.445], with total mass bounded from above by 1. We consider

$$\mu_\varphi := \lim_{j\to+\infty} \mu_j,$$

which is a positive Borel measure on X, with total mass ≤ 1.

Definition 1.7. We set $\mathcal{E}(X,\omega) := \{\varphi \in \mathrm{PSH}(X,\omega) \mid \mu_\varphi(X) = 1\}$.
For $\varphi \in \mathcal{E}(X,\omega)$, we set $MA(\varphi) := \mu_\varphi$.

It is proved in [27] that the Monge-Ampère operator MA is well defined on the class $\mathcal{E}(X,\omega)$. One has a stratification

$$\mathcal{E}(X,\omega) = \bigcup_{\chi \in \mathcal{W}} \mathcal{E}_\chi(X,\omega),$$

where $\mathcal{W}$ denotes the set of all functions $\chi : \mathbb{R} \to \mathbb{R}$ such that χ is increasing and $\chi(-\infty) = -\infty$, and the finite energy class $\mathcal{E}_\chi(X,\omega)$ is defined as follows:

Definition 1.8. We set $\mathcal{E}_\chi(X,\omega) := \left\{\varphi \in \mathcal{E}(X,\omega) \mid \chi(-|\varphi|) \in L^1(\mathrm{MA}(\varphi))\right\}$.
When $\chi(t) = -(-t)^p$, $p > 0$, we set $\mathcal{E}^p(X,\omega) = \mathcal{E}_\chi(X,\omega)$.

The set $\mathcal{E}^1(X,\omega)$ can be characterized as the set of $\varphi \in \mathcal{E}(X,\omega)$ such that

$$E(\varphi) := \frac{1}{n+1} \sum_{j=0}^{n} \int_X \varphi(\omega + dd^c\varphi)^j \wedge \omega^{n-j} > -\infty.$$

Observe that $\bigcap_{\chi \in \mathcal{W}} \mathcal{E}_\chi(X,\omega) = PSH(X,\omega) \cap L^\infty(X)$, so that finite energy classes interpolate between $\mathcal{E}(X,\omega)$ and bounded ω-psh functions.

It follows from [27, Theorem C] that a probability measure μ is the Monge-Ampère measure of a potential in $\mathcal{E}^p(X,\omega)$ if and only if $\mathcal{E}^p(X,\omega) \subset L^p(X,\mu)$, while [27,20] ensures that μ does not charge pluripolar sets if and only if there exists a unique $\varphi \in \mathcal{E}(X,\omega)$ such that $\mu = (\omega + dd^c\varphi)^n$ with $\sup_X \varphi = 0$.

Example 1.9. Every bounded ω-psh function belongs to $\mathcal{E}(X,\omega)$. The class $\mathcal{E}(X,\omega)$ also contains many ω-psh functions which are unbounded:

- when X is a compact Riemann surface, $\mathcal{E}(X,\omega)$ is precisely the set of ω-sh functions whose Laplacian does not charge polar sets.
- if $\varphi \in \mathrm{PSH}(X,\omega)$ satisfies $\varphi \le -1$, then $\varphi_\varepsilon = -(-\varphi)^\varepsilon$ belongs to $\mathcal{E}(X,\omega)$ whenever $0 \le \varepsilon < 1$, and φ_ε belongs to $\mathcal{E}^p(X,\omega)$ if $\varepsilon < 1/(n+p)$.
- the functions in $\mathcal{E}(X,\omega)$ have relatively mild singularities; in particular they have zero Lelong number at every point.

1.2.2. Mabuchi geometry

The class $\mathcal{E}^1(X,\omega)$ has played a key role in recent applications of pluripotential theory to Kähler geometry (see e.g. [4]). Set

$$I(\varphi,\psi) = \int_X (\varphi - \psi)[MA(\psi) - MA(\varphi)] \ge 0.$$

This quantity is well defined for $\varphi, \psi \in \mathcal{E}^1(X, \omega)$ and satisfies a quasi-triangle inequality [2, Theorem 1.8], hence induces a distance d_I.

Definition 1.10. The strong topology on $\mathcal{E}^1(X, \omega)$ is the one induced by d_I.

This notion has been introduced in [3, Section 5.3], it implies convergence in capacity [3, Theorem 5.7]. A sequence $(\varphi_j) \in \mathcal{E}^1(X, \omega)^{\mathbb{N}}$ strongly converges to φ iff it converges in L^1 and $E(\varphi_j)$ converges to $E(\varphi)$. Moreover the metric space $(\mathcal{E}^1(X, \omega), d_I)$ is complete [2, Propositions 2.3 and 2.4].

Let $\mathcal{H} = \{\varphi \in \mathcal{C}^\infty(X, \mathbb{R}),\ \omega + dd^c\varphi > 0\}$ denote the set of smooth and strictly ω-psh functions (*Kähler potentials*). This set can be thought of as an infinite dimensional Riemannian manifold, whose tangent space at $\varphi \in \mathcal{H}$ can be identified with $\mathcal{C}^\infty(X, \mathbb{R})$. Following earlier work of Mabuchi, Darvas [15] has considered the following Finsler structure: for $f \in T_\varphi \mathcal{H}$ he sets $|f|_\varphi := \int_X |f| MA(\varphi)$. If $\gamma : [0, 1] \to \mathcal{H}$ is a Lipschitz path, one then defines

$$\ell(\gamma) = \int_0^1 \int_X |\gamma'(t)| MA(\gamma(t)) dt,$$

and given $\varphi, \psi \in \mathcal{H}$, one considers

$$d_1(\varphi, \psi) := \inf \{\ell(\gamma),\ \gamma : [0, 1] \to \mathcal{H} \text{ with } \gamma(0) = \varphi,\ \gamma(1) = \psi\},$$

where the infimum runs over all Lipschitz paths joining φ to ψ. The following summarizes some results of [15] that we shall need:

Theorem 1.11. *Fix $\varphi, \psi \in \mathcal{E}^1(X, \omega)$. The following properties hold:*

- *d_1 is a distance on $\mathcal{H}$ which is uniformly equivalent to d_I;*
- *d_1 uniquely extends to $\mathcal{E}^1(X, \omega)$, $(\mathcal{E}^1(X, \omega), d_1)$ is a geodesic metric space;*
- *$P_\omega(\min(\varphi, \psi)) \in \mathcal{E}^1(X, \omega)$ with $d_1(\varphi, P_\omega(\min(\varphi, \psi))) \leq d_1(\varphi, \psi)$;*
- *if $\psi \leq \varphi$, then $d_1(\varphi, \psi)$ is comparable to $\int_X (\varphi - \psi) MA(\psi)$.*

We refer to [15, Theorem 2, Theorem 3, Corollary 4.14] for more details.

Analogous strong topologies have been defined on the other energy classes, notably the classes $\mathcal{E}^p(X, \omega)$ (see [15,25,16]). If $\chi \in \mathcal{W}$ is *convex* with polynomial growth at infinity, the class $\mathcal{E}_\chi(X, \omega)$ can be equipped with a Finsler metric d_χ making it a complete geodesic metric space [15]. The Mabuchi distance d_χ is again comparable to a pluripotential quasi-distance,

$$C^{-1} d_\chi(u, v) \leq I_\chi(u, v) := \int_X |\chi(u - v)| (MA(u) + MA(v)) \leq C d_\chi(u, v),$$

for a constant $C = C(\chi) > 0$ and $u, v \in \mathcal{E}_\chi(X, \omega)$.

Concave weights χ correspond to *low energy classes*. These are the weights one needs to consider for the stratification of the class $\mathcal{E}(X, \omega)$. One can still consider I_χ and d_χ, but the distance d_χ is no longer induced by a Finsler metric as emphasized in [16, p2]. The following summarizes the results obtained by Darvas in [16] that we shall need:

Theorem 1.12. *Fix $\varphi, \psi \in \mathcal{E}_\chi(X, \omega)$. The following properties hold:*

- *d_χ is a distance on $\mathcal{H}$ which is uniformly equivalent to I_χ;*
- *d_χ uniquely extends to $\mathcal{E}_\chi(X, \omega)$, $(\mathcal{E}_\chi(X, \omega), d_\chi)$ is a complete metric space;*
- *$P_\omega(\min(\varphi, \psi)) \in \mathcal{E}_\chi(X, \omega)$ with $d_\chi(\varphi, P_\omega(\min(\varphi, \psi))) \leq d_\chi(\varphi, \psi)$;*
- *if $\psi \leq \varphi$, then $d_\chi(\varphi, \psi)$ is comparable to $\int_X \chi \circ (\varphi - \psi) MA(\psi)$.*

We refer to [15] for convex weights with polynomial growth, and to [16, Proposition 5.3, Theorem 5.7, Theorem 6.1] for concave weights.

2. Quasi-monotone convergence

2.1. Capacity vs quasi-monotonicity

Continuity of complex Monge-Ampère operators along monotone sequences lies at the heart of Bedford-Taylor theory [1]. Several extensions of this continuity property have been proposed over the last decades under various restricted types of convergence. The following notion seems to encompass many of the latter:

Definition 2.1. A sequence $\varphi_j \in PSH(X, \omega)$ converges quasi-monotonically to $\varphi \in PSH(X, \omega)$ if there exists an increasing (resp. decreasing) sequence u_j (resp. v_j) in $PSH(X, \omega)$ such that $u_j \leq \varphi_j \leq v_j$ for all j and $u_j, v_j \to \varphi$ in $L^1(X)$.

It follows easily from the definition that φ_j converges to φ in $L^1(X)$. Observe that $\psi_j^+ := \sup_{\ell \geq j} \varphi_\ell$ decreases to φ, while $\psi_j^- := \inf_{\ell \geq j} \varphi_\ell$ increases to φ. However none of these functions usually belongs to $PSH(X, \omega)$: ψ_j^+ satisfies the mean value inequalities but it is no longer u.s.c., while ψ_j^- is u.s.c. but does not satisfy the mean value inequalities. It follows from [29, Proposition 8.4] that

$$\varphi_j^+ := \left(\sup_{\ell \geq j} \varphi_\ell \right)^* \in PSH(X, \omega)$$

and decreases to φ pointwise. By duality we consider the sequence

$$\varphi_j^- := P_\omega \left(\inf_{\ell \geq j} \varphi_\ell \right).$$

The latter belongs to $PSH(X,\omega)$ as soon as it is not identically $-\infty$, which is the case if $\varphi_j \geq u_j$ with u_j increasing, since we then obtain $\varphi_j^- \geq u_j$. We thus obtain the following reformulation of the quasi-monotone convergence:

Lemma 2.2. *A sequence $\varphi_j \in PSH(X,\omega)$ converges quasi-monotonically to $\varphi \in PSH(X,\omega)$ if and only if $\varphi_j^- := P_\omega \left(\inf_{\ell \geq j} \varphi_\ell\right)$ increases to φ.*

It is a celebrated result of [1] that if φ_j^- converges to φ in L^1, the convergence holds pointwise off a pluripolar set.

Example 2.3. Assume $\varphi_j \in PSH(X,\omega)$ converges in $L^1(X)$ to $\varphi \in PSH(X,\omega)$, and assume there exists $\varphi \geq \psi \in PSH(X,\omega)$ and $\varepsilon_j \in \mathbb{R}^+$ decreasing to 0 such that $\varphi_j \geq (1-\varepsilon_j)\varphi + \varepsilon_j\psi$. Then $\varphi_j^- \geq (1-\varepsilon_j)\varphi + \varepsilon_j\psi$, hence φ_j converges to φ quasi-monotonically. In this case it has been observed in [25, Lemma 1.2] that φ_j converges to φ in capacity. This is a special case of Theorem 2.4 below.

It is well-known that monotone convergence implies convergence in capacity (see [29, Proposition 4.25]). We extend this here to quasi-monotone convergence.

Theorem 2.4. *If a sequence $\varphi_j \in PSH(X,\omega)$ converges quasi-monotonically, then it converges in capacity.*

Conversely if $\varphi_j \in PSH(X,\omega)$ is uniformly bounded and converges in capacity, then a subsequence converges quasi-monotonically.

The converse does not hold without a uniform lower bound on the φ_j's, as shown in Example 3.3 which provides a sequence (φ_j) which converges in capacity while $\varphi_j^- \equiv -\infty$ (even after extracting). Example 3.4 moreover shows that it is usually necessary to extract, in order to reach the quasi-monotone convergence.

Proof. We let the reader check that if φ_j converges quasi-monotonically, then for all $C > 0$, $\max(\varphi_j, -C)$ converges quasi-monotonically to $\max(\varphi, -C)$. Since the convergence in capacity of φ_j to φ is equivalent to the convergence in capacity of $\max(\varphi_j, -C)$ to $\max(\varphi, -C)$, we are reduced to the uniformly bounded case.

Assume first that φ_j converges quasi-monotonically to $\varphi \in PSH(X,\omega)$. Fix $u \in PSH(X,\omega)$ such that $-1 \leq u \leq 0$. It follows from Proposition 1.6 that for all $\delta > 0$,

$$\int_{\{|\varphi_j - \varphi| \geq \delta\}} (\omega + dd^c u)^n \leq \delta^{-(n+1)} \int_X (\varphi_j^+ - \varphi_j^-)^{n+1} (\omega + dd^c u)^n$$

$$\leq \delta^{-(n+1)} (n+1)! \sum_{j=0}^{n} \int_X (\varphi_j^+ - \varphi_j^-)^{n+1-j} (\omega + dd^c \varphi_j^-)^j \wedge \omega^{n-j}.$$

Each of the above integral converges to zero as $j \to +\infty$, as follows from Bedford-Taylor monotone convergence theorem. Thus φ_j converges in capacity.

Assume now that φ_j converges in capacity. Rescaling ω, we can assume that $-1 \leq \varphi_j \leq 0$. Let $\delta_j, \varepsilon_j > 0$ be sequences decreasing to zero. Observe that

$$\{|\varphi_j - \varphi_{j+1}| \geq \delta\} \subset \{|\varphi_j - \varphi| \geq \delta/2\} \cup \{|\varphi_{j+1} - \varphi| \geq \delta/2\}.$$

Extracting and relabelling, we can thus assume that

$$\mathrm{Cap}_\omega\left(\{|\varphi_j - \varphi_{j+1}| \geq \delta_j\}\right) \leq \varepsilon_j - \varepsilon_{j+1}.$$

Set $E_j = \{\varphi_{j+1} \leq \varphi_j - \delta_j\}$ and $F_j = \cup_{\ell \geq j} E_\ell$. The sequence $j \mapsto F_j$ is decreasing with $\mathrm{Cap}_\omega(F_j) \leq \varepsilon_j$. Fix $A_j \geq 1$ and consider

$$h_j := \left(\sup\{u \in PSH(X, \omega),\ u \leq -A_j \text{ on } F_j \text{ and } u \leq 0 \text{ on } X\}\right)^*.$$

This is a variant of the "relative extremal function" considered in [29, Definition 9.13]. Adapting [29, Section 9.3], one easily obtains the following:

- $h_j \in PSH(X, \omega)$ with $-A_j \leq h_j \leq 0$;
- $h_j = -A_j$ on $F_j \setminus P_j$, where P_j is a pluripolar set;
- $\mathrm{Cap}_\omega(F_j) \geq A_j^{-n-1} \int_X (-h_j)(\omega + dd^c h_j)^n$.

Thus $\int_X (-h_j)(\omega + dd^c h_j)^n \leq 1$ if we choose $A_j^{n+1}\varepsilon_j = 1$ and it follows from Stokes theorem that

$$\int_X (-h_j)\omega^n \leq \int_X (-h_j)(\omega + dd^c h_j)^n \leq 1.$$

We infer that $\sum_{\ell \geq 1} 2^{-\ell} h_\ell \in PSH(X, \omega)$, hence the sequence

$$H_j := \sum_{\ell \geq j} 2^{-\ell-1} h_\ell \in PSH(X, 2^{-j}\omega)$$

increases to zero as $j \to +\infty$.

We set $\psi_j := (1 - 2^{-j})\varphi_j + H_j - 2^{-j+1} \in PSH(X, \omega)$. Since $H_j \leq 0$ and $\varphi_j \geq -1$, we obtain $\psi_j \leq \varphi_j$. We choose $\delta_j = 2^{-j-1}/(1 - 2^{-j})$ and $A_j = 2^{j+1}$ and claim that $j \mapsto \psi_j$ increases to φ as $j \to +\infty$. Indeed

- either $x \in X \setminus F_j$, then $\varphi_{j+1}(x) \geq \varphi_j(x) - \delta_j$ hence

  $$\psi_{j+1} - \psi_j = (1 - 2^{-j})(\varphi_{j+1} - \varphi_j) + 2^{-j-1}\varphi_{j+1} - 2^{-j-1}h_j + 2^{-j} \geq 0$$

  using that $-h_j \geq 0$ and $\varphi_{j+1} \geq -1$;

- or $x \in F_j$ and we obtain $\psi_{j+1} - \psi_j \geq -1 - 2^{-j-1} h_j = 0$ if $x \notin P_j$.

Thus $\psi_{j+1} \geq \psi_j$ a.e. hence everywhere. Replacing ψ_j by $\max(\psi_j, -1)$, we moreover obtain an increasing sequence which is uniformly bounded. $\square$

2.2. Finite energy sequences

Following Theorem 2.4 we now prove our main Theorem, which extends several partial results previously obtained ([38, Theorem 2], [31, Theorem 2.1], [39, Theorem 1.2], [3, Theorem 5.7], [15, Corollary 5.7], [5, Propositions 2.6 and 6.4], [25, Proposition 1.9], [36, Proposition 5.7], [30, Theorem 1.2]).

Theorem 2.5. *Assume $\varphi_j \in PSH(X, \omega)$ converges in $L^1(X)$ and fix $\chi \in \mathcal{W}$ a weight which is either convex or concave with polynomial growth at $-\infty$.*

- *If φ_j converges in capacity and $\varphi_j \geq \psi$ for some $\psi \in \mathcal{E}_\chi(X, \omega)$, then φ_j converges in $(\mathcal{E}_\chi(X, \omega), d_\chi)$.*
- *If φ_j converges in $(\mathcal{E}_\chi(X, \omega), d_\chi)$, then φ_j converges in capacity. Moreover up to extracting and relabelling, φ_j converges quasi-monotonically and there exists $\psi \in \mathcal{E}_\chi(X, \omega)$ such that $\varphi_j \geq \psi$.*

For sequences that are not uniformly bounded, one cannot expect that quasi-monotone convergence is equivalent to convergence in energy. For instance if $\psi \in PSH(X, \omega)$ has some positive Lelong number and ε_j decreases to zero, then $\varphi_j = \varepsilon_j \psi$ converges quasi-monotonically to $\varphi = 0$, but not in energy. A finite energy lower bound turns out to be a necessary and sufficient condition.

Proof. We start with the first item. Assume first that the sequence (φ_j) is uniformly bounded. When $\chi(t) = t$, it follows from [2, Proposition 2.3] that $\varphi_j \to \varphi$ in $(\mathcal{E}^1(X, \omega), d_1)$ if and only if $\int_X (\varphi_j - \varphi)[MA(\varphi) - MA(\varphi_j)] \to 0$. The latter holds when φ_j converges in capacity, as follows from Proposition 1.5. More generally the convergence in $(\mathcal{E}_\chi(X, \omega), d_\chi)$ is equivalent to the following

$$\int_X |\chi(\varphi_j - \varphi)| \, [MA(\varphi_j) + MA(\varphi)] \to 0,$$

which is a consequence of [29, Theorem 4.26].

We now reduce the general case to the uniformly bounded one. Fix $\tilde{\chi}$ a weight such that $\int_X (-\tilde{\chi} \circ \psi) \, MA(\psi) < +\infty$ and $\tilde{\chi}(t)/\chi(t) \to +\infty$ as $t \to -\infty$ (see [29, Exercise 10.5]). Set $\varphi_j^C = \max(\varphi_j, -C)$ and $\varphi^C = \max(\varphi, -C)$, it follows from Theorem 1.12 that

$$d_\chi(\varphi_j, \varphi_j^C) \sim \int_X \chi \circ (\varphi_j^C - \varphi_j)\, MA(\varphi_j) \le \int_{\{\varphi_j < -C\}} (-\chi \circ \varphi_j)\, MA(\varphi_j)$$
$$\le \frac{\chi(-C)}{\tilde{\chi}(-C)} \int_X (-\tilde{\chi} \circ \varphi_j)\, MA(\varphi_j) \le \frac{M^n \chi(-C)}{\tilde{\chi}(-C)} \int_X (-\tilde{\chi} \circ \psi)\, MA(\psi),$$

where the last inequality follows from [27, Lemmas 2.3 and 3.5]. The conclusion thus follows from the uniformly bounded case and the triangle inequality.

It thus remains to show the second item. To check that φ_j converge in capacity, it suffices to show that any subsequence admits a subsubsequence that converges in capacity. Extracting and relabelling, we can assume that $d_\chi(\varphi_j, \varphi_{j+1}) \le 2^{-j}$. Set $\varphi_{j,k}^- := P_\omega(\min_{j \le \ell \le j+k} \varphi_\ell)$. A repeated use of Theorem 1.12 ensures that $\varphi_{j,k}^- \in \mathcal{E}_\chi(X, \omega)$ with

$$\begin{aligned} d_\chi(\varphi_j, \varphi_{j,k}^-) &\le d_\chi\left(\varphi_j, P_\omega\left(\min_{j+1 \le \ell \le j+k} \varphi_\ell\right)\right) \\ &\le d_\chi(\varphi_j, \varphi_{j+1}) + d_\chi\left(\varphi_{j+1}, P_\omega\left(\min_{j+1 \le \ell \le j+k} \varphi_\ell\right)\right) \\ &\le d_\chi(\varphi_j, \varphi_{j+1}) + d_\chi\left(\varphi_{j+1}, P_\omega\left(\min_{j+2 \le \ell \le j+k} \varphi_\ell\right)\right) \\ &\le \sum_{\ell=j}^{j+k-1} d_\chi(\varphi_\ell, \varphi_{\ell+1}) \le 2^{-j+1}. \end{aligned}$$

We infer that $k \mapsto \varphi_{j,k}^-$ decreases, as k increases to $+\infty$, to $\varphi_j^- \in \mathcal{E}_\chi(X, \omega)$ with $d_\chi(\varphi_j, \varphi_j^-) \le 2^{-j+1}$. It follows that φ_j converges to φ quasi-monotonically and for all j, $\varphi_j \ge \psi := \varphi_1^- \in \mathcal{E}_\chi(X, \omega)$. In particular φ_j converges to φ in capacity. □

3. Examples and remarks

3.1. Quasi-monotone convergence of Monge-Ampère potentials

Families of solutions to complex Monge-Ampère equations or flows provide natural examples of sequences which converge quasi-monotonically. We illustrate this here with two typical situations.

Example 3.1. Let μ be a non pluripolar probability measure. It follows from [27,20] that there exists a unique function $\varphi \in \mathcal{E}(X, \omega)$ such that

$$MA(\varphi) = e^\varphi \mu.$$

[25, Theorem C] and Choquet's lemma show that φ is the quasi-monotone limit of a sequence of functions $\varphi_j \in \mathcal{E}(X, \omega)$ such that $MA(\varphi_j) \le e^{\varphi_j} \mu$.

We now consider smoothing properties of the Kähler-Ricci flow [28,23].

Example 3.2. Assume that X is a Calabi-Yau manifold and fix $T_0 = \omega + dd^c\varphi_0$ a positive closed current with zero Lelong numbers which is cohomologous to ω. It has been shown in [28,23] that there exists a unique family of Kähler forms $(\omega_t)_{t>0}$ on X, which evolve along the Kähler-Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t),$$

and such that $\omega_t \to T_0$ as $t \to 0$. The (normalized) potentials $\varphi_t \in PSH(X,\omega)$ of $\omega_t = \omega + dd^c\varphi_t$ are solutions of a complex Monge-Ampère flow,

$$(\omega + dd^c\varphi_t)^n = e^{\partial_t\varphi_t + h}\omega^n,$$

and the convergence $\varphi_t \to \varphi_0$ at time zero is such that $\varphi_t \geq \varphi_0 - A(t - t\log t)$ for some constant $A \geq 0$ [28, Lemma 2.9]. It follows that

$$\varphi_t^- := P_\omega\left(\inf_{0<s\leq t}\varphi_s\right) \geq \varphi_0 - A(t - t\log t),$$

hence the convergence of φ_t towards φ_0 is quasi-monotone.

3.2. Intermediate convergences

We first provide examples of ω-psh functions φ_j such that φ_j^- is identically $-\infty$, while φ_j converges in capacity to some ω-psh function φ.

Example 3.3. Consider the Riemann sphere endowed with the Fubini-Study Kähler form $(X,\omega) = (\mathbb{P}^1, \omega_{FS})$. Then

$$\varphi_j[z] = \log|z_1 - \tau_j z_0| - \log|z| \longrightarrow \varphi[z] = \log|z_1| - \log|z|$$

if the τ_j's converge to 0, and $P_\omega(\min(\varphi_j, \varphi_{j+1})) \equiv -\infty$ because functions in $PSH(\mathbb{P}^1, \omega_{FS})$ can have at most one Lelong number of size 1. On the other hand the convergence of φ_j towards φ is uniform on compact subsets of $\mathbb{P}^1 \setminus [1:0]$, so the sequence converges in L^1 and in capacity.

More generally if $\psi \in PSH(\mathbb{P}^n, \omega_{FS})$ the set $E_1(\psi) = \{x \in \mathbb{P}^n,\ \nu(\psi, x) \geq 1\}$ has to be included in a hyperplane [12, Proposition 2.2], so one can cook up similar examples for which φ_j^- is identically $-\infty$.

We now provide an example of a uniformly bounded sequence (φ_j) of ω-psh functions which converge to 0 in capacity with $\varphi_j^- \equiv -1$. This shows that it is necessary to use extractions in Theorem 2.5.

Example 3.4. Using local charts, we construct for each point $a \in X$ a function $G_a \in PSH(X, \omega)$ which has a logarithmic singularity at point a.

Observe that $\max(G_a, -1) \equiv -1$ in a neighborhood V_a of a. We cover X by finitely many such neighborhoods $V^1_{a_{1,1}}, \dots V^1_{a_{s_1,1}}$. We similarly cover X by finitely many neighborhoods $V^2_{a_{1,2}}, \dots, V^2_{a_{s_2,2}}$ on which the function $\max(G_a/2, -1)$ is identically -1. The latter neighborhoods are smaller and we need more points.

We go on by induction, dividing at each step G_a by an extra factor 2, considering the family of functions $\max(G_a/2^n, -1)$ at step n. We then label the corresponding sequence of functions (φ_j), so that $\varphi_j = \max(G_a/2^n, -1) \in PSH(X, \omega)$ for some $a_j = a_{l,n}$ with $1 \leq \ell \leq s_n$ and $n = n_j \to +\infty$ as $j \to +\infty$. By construction we obtain $\varphi_j \geq -1$ and $\varphi_j^- \equiv -1$, since φ_j^- lies below each function from a fixed step. On the other hand $G_{a_j}/2^{n_j} \to 0$ in capacity, hence $\varphi_j \to 0$ in capacity.

It has been observed by Cegrell in [9] that the Monge-Ampère operator is not continuous for the L^1_{loc}-topology. The following is an explicit example of this phenomenon, adapted to the compact context.

Example 3.5. Assume $(X, \omega) = (\mathbb{P}^2, \omega_{FS})$ and consider

$$\varphi_j[z] = \frac{1}{j} \max\left(\log\left|z_0^j + z_1^j + z_2^j\right|, \log|z_0|\right) - \log|z|.$$

Observe that $\varphi_j \in PSH(X, \omega)$ is locally bounded outside the finite set

$$F_j = \{[z] \in \mathbb{P}^2,\ z_0 = 0 \ \&\ z_1^j + z_2^j = 0\}$$

which is included in the circle $S^1 = \{z_0 = 0 \ \&\ |z_1| = |z_2|\}$. Note also that φ_j converges in L^1 to $\varphi[z] = \max_{0 \leq j \leq 2} \log^+|z_j| - \log|z|$. The Monge-Ampère measures $MA(\varphi_j)$ are combination of Dirac masses at points of F_j and converge to the Haar measure on S^1, while $MA(\varphi)$ is the Haar measure on the torus

$$\mathbb{T}^2 = \{[z] \in \mathbb{P}^2,\ |z_0| = |z_1| = |z_2|\}.$$

Thus φ_j does not converge in capacity to φ.

We finally compare the various types of convergence in energy classes.

Example 3.6. Assume $(X, \omega) = (\mathbb{P}^1, \omega_{FS})$ and consider

$$\varphi_j[z] = \varepsilon_j \max(\log|z_1| - \log|z|, -C_j) \in PSH(X, \omega),$$

where $0 \leq \varepsilon_j \leq 1$ and $C_j \geq 0$. These examples are toric, so one can use the dictionary established in [14] to justify the following assertions:

- $\varphi_j \to 0$ in L^1 iff it does so in capacity/quasi-monotonically iff $\varepsilon_j \to 0$;
- $\varphi_j \to 0$ in $(\mathcal{E}_\chi(X,\omega), d_\chi)$ iff $\varepsilon_j \chi(-\varepsilon_j C_j) \to 0$.

By considering weights with arbitrarily slow growth, we conclude that $\varphi_j \to 0$ in some $(\mathcal{E}_\chi(X,\omega), d_\chi)$ as soon as $\varepsilon_j \to 0$, whatever the speed at which $C_j \nearrow +\infty$.

3.3. *Concluding remarks*

3.3.1. *Independence on ω*

Let $\tilde{\omega} = \omega + dd^c \rho$ be a Kähler form cohomologous to ω. Then $\varphi_j \in PSH(X,\omega)$ if and only if $\psi_j = \varphi_j - \rho \in PSH(X, \tilde{\omega})$. We let the reader check that φ_j converges quasi-monotonically if and only if so does ψ_j (and similarly for the other notions of strong convergence).

Assume now $\tilde{\omega}$ is an arbitrary Kähler form. We want to compare notions of strong convergence with respect to ω and with respect to $\tilde{\omega}$. We claim that these are essentially the same. Using $\omega + \tilde{\omega}$ as a third auxiliary form, we see that it suffices to treat the case when $\omega \leq \tilde{\omega}$. The following are left to the reader:

- $PSH(X,\omega) \subset PSH(X,\tilde{\omega})$ and if $\varphi \in \mathcal{E}_\chi(X,\omega)$ then $\varphi \in \mathcal{E}_\chi(X,\tilde{\omega})$;
- if $\varphi \in \mathcal{E}_\chi(X,\tilde{\omega})$ then $P_\omega(\varphi) \in \mathcal{E}_\chi(X,\omega)$;
- $\varphi_j \in PSH(X,\omega)$ converges to $\varphi \in PSH(X,\omega)$ with respect to Cap_ω if and only if it does so with respect to $Cap_{\tilde{\omega}}$;
- if $\varphi_j \in PSH(X,\omega)$ converges quasi-monotonically to $\varphi \in PSH(X,\omega)$, then the same property holds with respect to $\tilde{\omega}$.

Adapting Example 3.3, one can find $(\varphi_j) \in PSH(X,\tilde{\omega})^{\mathbb{N}}$ which converges quasi-monotonically to $\varphi \in PSH(X,\tilde{\omega})$ with $P_\omega(\inf_{\ell \geq j} \varphi_\ell) \equiv -\infty$. This converse however holds if we assume an appropriate lower bound on the sequence.

In particular a uniformly bounded sequence $\varphi_j \in PSH(X,\omega) \cap PSH(X,\tilde{\omega})$ converges quasi-monotonically w.r.t. ω if and only if it does so w.r.t. $\tilde{\omega}$.

3.3.2. *Big classes and prescribed singularities*

All notions introduced previously and all properties established so far can be adapted to the case when the reference form ω is no longer Kähler, but merely a smooth closed real $(1,1)$-form representing a *big cohomology class*. We refer the reader to [8] for basics of pluripotential theory in that context. One can also extend these results to the case of big classes with prescribed singularities, a theory that has been developed by Darvas-Di Nezza-Lu in [17,18] and further studied in [36,37].

3.3.3. *The local setting*

Let Ω be a pseudoconvex domain of $\mathbb{C}^n$. We let $PSH(\Omega)$ denote the set of plurisubharmonic functions in Ω.

Definition 3.7. A sequence (φ_j) of plurisubharmonic functions in Ω converges to $\varphi \in PSH(\Omega)$ quasi-monotonically if $\varphi_j^- := P_\Omega\left(\inf_{\ell \geq j} \varphi_\ell\right)$ increases to φ.

Here $P_\Omega(h)$ denotes the largest plurisubharmonic function in Ω lying below h. Adapting what we have done in the compact case, we can establish that

- quasi-monotone convergence implies convergence in capacity;
- a sequence can converge in capacity but not quasi-monotonically;
- both notions essentially coincide for uniformly bounded sequences.

We leave the details to the reader.

Declaration of competing interest

None declared.

Data availability

No data was used for the research described in the article.

References

[1] E. Bedford, B.A. Taylor, A new capacity for plurisubharmonic functions, Acta Math. 149 (1–2) (1982) 1–40.

[2] R.J. Berman, S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, Kähler-Einstein metrics and Kähler-Ricci flow on log Fano varieties, J. Reine Angew. Math. 751 (2019) 27–89.

[3] R.J. Berman, S. Boucksom, V. Guedj, A. Zeriahi, A variational approach to complex Monge-Ampère equations, Publ. Math. IHÉS 117 (2013) 179–245.

[4] R.J. Berman, S. Boucksom, M. Jonsson, A variational approach to the Yau-Tian-Donaldson conjecture, J. Am. Math. Soc. (3) (2021) 605–652.

[5] R.J. Berman, T. Darvas, H.C. Lu, Convexity of the extended K-energy and the large time behavior of the weak Calabi flow, Geom. Topol. 21 (5) (2017) 2945–2988.

[6] Z. Blocki, Estimates for the complex Monge-Ampère operator, Bull. Pol. Acad. Sci., Math. 41 (2) (1993) 151–157.

[7] Z. Blocki, The domain of definition of the complex Monge-Ampère operator, Am. J. Math. 128 (2) (2006) 519–530.

[8] S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi, Monge-Ampère equations in big cohomology classes, Acta Math. 205 (2) (2010) 199–262.

[9] U. Cegrell, Discontinuité de l'opérateur de Monge-Ampère complexe, C. R. Acad. Sci. Paris Sér. I Math. 296 (21) (1983) 869–871.

[10] U. Cegrell, The general definition of the complex Monge-Ampère operator, Ann. Inst. Fourier (Grenoble) 54 (1) (2004) 159–179.

[11] X.X. Chen, S. Donaldson, S. Sun, Kähler-Einstein metrics on Fano manifolds, I, II & III, J. Am. Math. Soc. 28 (2015) 183–197, 199–234 & 235–278.

[12] D. Coman, V. Guedj, Quasi-plurisubharmonic Green functions, J. Math. Pures Appl. 92 (2009) 521–562.

[13] D. Coman, V. Guedj, A. Zeriahi, Domains of definition of Monge-Ampère operators on compact Kähler manifolds, Math. Z. 259 (2) (2008) 393–418.

[14] D. Coman, V. Guedj, S. Sahin, A. Zeriahi, Toric pluripotential theory, Ann. Pol. Math. 123 (1) (2019) 215–242.

[15] T. Darvas, The Mabuchi geometry of finite energy classes, Adv. Math. 285 (2015) 182–219.
[16] T. Darvas, The Mabuchi geometry of low energy classes, Math. Ann. (2023), https://doi.org/10.1007/s00208-023-02648-0, in press.
[17] T. Darvas, E. Di Nezza, H.C. Lu, On the singularity type of full mass currents in big cohomology classes, Compos. Math. 154 (2) (2018) 380–409.
[18] T. Darvas, E. Di Nezza, H.C. Lu, The metric geometry of singularity types, J. Reine Angew. Math. 771 (2021) 137–170.
[19] J.-P. Demailly, Regularization of closed positive currents and intersection theory, J. Algebraic Geom. 1 (3) (1992) 361–409.
[20] S. Dinew, Uniqueness in $\mathcal{E}(X, \omega)$, J. Funct. Anal. 256 (7) (2009) 2113–2122.
[21] S. Dinew, Pluripotential theory on compact Hermitian manifolds, Ann. Fac. Sci. Toulouse Math. (6) 25 (1) (2016) 91–139.
[22] S. Dinew, P.H. Hiep, Convergence in capacity on compact Kähler manifolds, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 11 (4) (2012) 903–919.
[23] E. Di Nezza, H.C. Lu, Uniqueness and short time regularity of the weak Kähler-Ricci flow, Adv. Math. 305 (2017) 953–993.
[24] S. Donaldson, Some recent developments in Kähler geometry and exceptional holonomy, in: Proc. Int. Cong. of Math., Rio de Janeiro 2018, vol. I, World Sci. Publ., 2018, pp. 425–451.
[25] V. Guedj, H.C. Lu, A. Zeriahi, Plurisubharmonic envelopes and supersolutions, J. Differ. Geom. 113 (2) (2019) 273–313.
[26] V. Guedj, A. Zeriahi, Intrinsic capacities on compact Kähler manifolds, J. Geom. Anal. 15 (4) (2005) 607–639.
[27] V. Guedj, A. Zeriahi, The weighted Monge-Ampère energy of quasiplurisubharmonic functions, J. Funct. Anal. 250 (2) (2007) 442–482.
[28] V. Guedj, A. Zeriahi, Regularizing properties of the Kähler-Ricci flow, J. Reine Angew. Math. 729 (2017) 275–304.
[29] V. Guedj, A. Zeriahi, Degenerate Complex Monge-Ampère Equations, EMS Tracts in Mathematics, vol. 26, European Mathematical Society (EMS), Zürich, 2017.
[30] P. Gupta, A complete metric topology on relative low energy spaces, Math. Z. 303 (3) (2023) 56.
[31] P.H. Hiep, On the convergence in capacity on compact Kähler manifolds and its applications, Proc. Am. Math. Soc. 136 (6) (2008) 2007–2018.
[32] S. Kolodziej, The Monge-Ampère equation on compact Kähler manifolds, Indiana Univ. Math. J. 52 (3) (2003) 667–686.
[33] C. Li, G-uniform stability and Kähler-Einstein metrics on Fano varieties, Invent. Math. 227 (2) (2022) 661–744.
[34] C. Li, G. Tian, F. Wang, On the Yau-Tian-Donaldson conjecture for singular Fano varieties, Commun. Pure Appl. Math. (2020).
[35] Y. Liu, C. Xu, Z. Zhuang, Finite generation for valuations computing stability thresholds and applications to K-stability, Ann. Math. 196 (2) (2022) 507–566.
[36] A. Trusiani, L1 metric geometry of potentials with prescribed singularities on compact Kähler manifolds, J. Geom. Anal. 32 (2) (2022) 37.
[37] A. Trusiani, The strong topology of ω-plurisubharmonic functions, Anal. PDE 16 (2) (2023) 367–405.
[38] Y. Xing, Continuity of the complex Monge-Ampère operator, Proc. Am. Math. Soc. 124 (2) (1996) 457–467.
[39] Y. Xing, Continuity of the complex Monge-Ampère operator on compact Kähler manifolds, Math. Z. 263 (2) (2009) 331–344.