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Citation for the original published paper (version of record):
Magnusson, T., Raum, M. (2023). Scalar-valued depth two Eichler–Shimura integrals of cusp forms.
Transactions of the London Mathematical Society, 10(1): 156-174.
http://dx.doi.org/10.1112/tlm3.12055

Received: 1 September 2022 | Revised: 7 December 2022 | Accepted: 18 June 2023

DOI: 10.1112/tlm3.12055

RESEARCH ARTICLE

Transactions of the London Mathematical Society

Scalar-valued depth two Eichler–Shimura integrals of cusp forms

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Funding information
Vetenskapsrådet, Grant/Award Number: 2019-03551

Abstract

Given cusp forms f and g of integral weight $k \geqslant 2$, the depth two holomorphic iterated Eichler–Shimura integral $I_{f,g}$ is defined by $\int_\tau^{i\infty} f(z)(X-z)^{k-2} I_g(z;Y)\,\mathrm{d}z$, where I_g is the Eichler integral of g and X, Y are formal variables. We provide an explicit vector-valued modular form whose top components are given by $I_{f,g}$. We show that this vector-valued modular form gives rise to a *scalar-valued* iterated Eichler integral of depth two, denoted by $\mathcal{E}_{f,g}$, that can be seen as a higher depth generalization of the scalar-valued Eichler integral $\mathcal{E}_f$ of depth one. As an aside, our argument provides an alternative explanation of an orthogonality relation satisfied by period polynomials originally due to Paşol–Popa. We show that $\mathcal{E}_{f,g}$ can be expressed in terms of sums of products of components of vector-valued Eisenstein series with classical modular forms after multiplication with a suitable power of the discriminant modular form Δ. This allows for effective computation of $\mathcal{E}_{f,g}$.

MSC 2020

11F11 (primary), 11F30, 11F75 (secondary).

Iterated Eichler–Shimura integrals have received a lot of interest in recent years. For example, they have been studied extensively by Brown [2–4], especially in the context of iterated extensions of motives and multiple modular values. They have also been related to what are known as higher order modular forms by Diamantis [7]. Furthermore, they are closely related to the string theoretic notion of modular graph functions [5, 6, 8, 9].

In this paper, we examine iterated Eichler–Shimura integrals of depth two in more detail. We first recall the definition of usual Eichler integrals. There are two kinds of them, scalar-valued and polynomial-valued ones, and they are given as follows. Let $f \in \mathrm{S}_k$, $k \in \mathbb{Z}_{\geqslant 2}$, be a cusp form of level one, X a formal variable, and $\tau \in \mathbb{H}$ in the Poincaré upper half plane, then the polynomial-valued Eichler integral $I_f(\,\cdot\,;X)$ and the scalar-valued Eichler integral $\mathcal{E}_f$ are given by

$$I_f(\tau;X) = \int_\tau^{i\infty} f(z)\,(X-z)^{k-2}\,\mathrm{d}z \quad \text{and} \quad \mathcal{E}_f(\tau) = \int_\tau^{i\infty} f(z)\,(\tau-z)^{k-2}\,\mathrm{d}z. \tag{0.1}$$

Note that $I_f(\tau;\tau) = \mathcal{E}_f(\tau)$. If $f, g \in \mathrm{S}_k$ are cusp forms, the depth two Eichler–Shimura integral is given by

$$I_{f,g}(\tau;X,Y) = \int_\tau^{i\infty} f(z)\,(X-z)^{k-2}\,I_g(z;Y)\,\mathrm{d}z. \tag{0.2}$$

This Eichler–Shimura integral is well-understood and its definition can readily be generalized to arbitrary depths [14, 15]. We provide a scalar-valued analogue $\mathcal{E}_{f,g}$ to $I_{f,g}$. It is given by

$$\mathcal{E}_{f,g}(\tau) = \int_\tau^{i\infty} f(z)\,\mathcal{E}_g(z)\,\mathrm{d}z. \tag{0.3}$$

We are not aware of any previous occurrence of $\mathcal{E}_{f,g}$ in the literature, but remark that its definition generalizes the one of $\mathcal{E}_f$ in a straightforward way. The first main focus of this paper is to provide a connection between the geometrically motivated $I_{f,g}$ and its classical counterpart $\mathcal{E}_{f,g}$ paralleling the connection of I_f to $\mathcal{E}_f$. The second main focus of this paper is to provide a framework that enables the effective computation of $\mathcal{E}_{f,g}$ following the approach taken in [1].

We study $\mathcal{E}_{f,g}$ and $I_{f,g}$ using the language of vector-valued modular forms. The work of Brown [4] implies via arguments of Mertens–Raum [16] that $I_{f,g}$ is a component of a vector-valued modular form. However, they do not specify this vector-valued modular form explicitly. This is the purpose of our first theorem.

For an integer $\mathsf{d} \geqslant 0$, $\mathrm{sym}^{\mathsf{d}}(X)$ denotes the dth symmetric power of the standard representation, whose representation space is the space $\mathbb{C}[X]_{\mathsf{d}}$ of complex polynomials in X of degree at most d. We write $\mathrm{sym}^{\mathsf{d}}(X,Y)$ for the tensor product $\mathrm{sym}^{\mathsf{d}}(X) \otimes \mathrm{sym}^{\mathsf{d}}(Y)$, and $\mathbb{C}[X,Y]_{\mathsf{d}}$ for its representation space. There is a $\mathrm{sym}^{\mathsf{d}}(X)$-invariant pairing on $\mathbb{C}[X]_{\mathsf{d}} \times \mathbb{C}[Y]_{\mathsf{d}}$ denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$. For the precise definitions, see Section 1.

Echoing $I_{f,g}$ and $\mathcal{E}_{f,g}$, we consider the following two related representations:

$$\begin{aligned}
\widetilde{\rho}_{f,g} \,:\, \mathrm{SL}_2(\mathbb{Z}) &\longrightarrow \mathrm{GL}(\mathbb{C}[X,Y]_{k-2} \oplus \mathbb{C}[X]_{k-2} \oplus \mathbb{C}),\\
\gamma &\longmapsto \begin{pmatrix} \mathrm{sym}^{k-2}(X,Y)(\gamma) & \phi_{I_g}(\gamma;Y)\cdot \mathrm{sym}^{k-2}(X)(\gamma) & \widetilde{\psi}_{f,g}(\gamma) \\ 0 & \mathrm{sym}^{k-2}(X)(\gamma) & \phi_{I_f}(\gamma) \\ 0 & 0 & 1 \end{pmatrix}, \text{ and}
\end{aligned} \tag{0.4}$$

$$\begin{aligned}
\rho_{f,g} \,:\, \mathrm{SL}_2(\mathbb{Z}) &\longrightarrow \mathrm{GL}(\mathbb{C} \oplus \mathbb{C}[X]_{k-2} \oplus \mathbb{C}),\\
\gamma &\longmapsto \begin{pmatrix} 1 & -\phi^\vee_{I_g}(\gamma) & \psi_{f,g}(\gamma) \\ 0 & \mathrm{sym}^{k-2}(X)(\gamma) & \phi_{I_f}(\gamma) \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned} \tag{0.5}$$

where for a cusp form $h \in \mathrm{S}_k$ and a formal variable W, we set

$$\phi^{\vee}_{I_h}(\gamma)(v) = \left\langle \phi_{I_h}(\gamma^{-1}), v \right\rangle, v \in \mathbb{C}[X]_{k-2}, \quad \text{with} \quad \phi_{I_h}(\gamma; W) = \int_{\gamma(i\infty)}^{i\infty} h(z)\,(W-z)^{k-2}\,\mathrm{d}z$$

and

$$\widetilde{\psi}_{f,g}(\gamma) = \int_{\gamma(i\infty)}^{i\infty} f(z)\,(X-z)^{k-2}\,I_g(z;Y)\,\mathrm{d}z, \quad \psi_{f,g}(\gamma) = \int_{\gamma(i\infty)}^{i\infty} f(z)\,\mathcal{E}_g(z)\,\mathrm{d}z.$$

Recall that ϕ_{I_h} is a parabolic $\mathrm{sym}^{\mathrm{d}}(X)$-cocycle, see Section 1 and for example [12].

As mentioned previously, our first main theorem provides an explicit vector-valued modular form of type $\widetilde{\rho}_{f,g}$ with $I_{f,g}$ as a component, thus amending Brown's results [4]. We write $\mathrm{M}_k(\rho)$ for the space of weight k modular forms of type ρ, which is defined in Subsection 1.2.

Theorem A. *Let $k \geqslant 2$ be an even integer, and let $f, g \in \mathrm{S}_k$. Then the arithmetic type $\widetilde{\rho}_{f,g}$ given by (0.4) is well-defined, and we have that*

$$\begin{pmatrix} I_{f,g} \\ I_f \\ 1 \end{pmatrix} \in \mathrm{M}_0\big(\widetilde{\rho}_{f,g}\big).$$

The relation between $\widetilde{\rho}_{f,g}$ and $\rho_{f,g}$, and also between $I_{f,g}$ and $\mathcal{E}_{f,g}$, is provided by the contraction map $\pi : \mathrm{sym}^{\mathrm{d}}(X, Y) \to \mathbb{C}$, given by

$$\pi(p \otimes q) = \langle p, q \rangle, \quad p \in \mathbb{C}[X]_{\mathrm{d}}, \quad q \in \mathbb{C}[Y]_{\mathrm{d}}. \tag{0.6}$$

In particular, we provide the following theorem.

Theorem B. *Let $k \geqslant 2$ be an even integer, and let $f, g \in \mathrm{S}_k$. Then the arithmetic type $\rho_{f,g}$ is well-defined, and the contraction map (0.6) induces a pushforward morphism $\pi_* : \widetilde{\rho}_{f,g} \to \rho_{f,g}$ given by $\pi_*(p, q, z) = (\pi(p), q, z)$. Furthermore, we have that*

$$\pi_* \circ \begin{pmatrix} I_{f,g} \\ I_f \\ 1 \end{pmatrix} = \begin{pmatrix} \mathcal{E}_{f,g} \\ I_f \\ 1 \end{pmatrix} \in \mathrm{M}_0(\rho_{f,g}).$$

We next introduce a family of representations ρ_{ϕ_1,ϕ_2}. We construct its members from pairs of parabolic $\mathrm{sym}^{\mathrm{d}}(X)$-cocycles (ϕ_1, ϕ_2) satisfying a certain orthogonality relation, see Subsection 2.1 and specifically Theorem 2.2. Theorem B implies that $\rho_{\phi_{I_f}, -\phi_{I_g}}$ is well-defined and equals $\rho_{f,g}$.

Remark. Our characterization of pairs (ϕ_1, ϕ_2) for which ρ_{ϕ_1,ϕ_2} is well-defined in conjunction with our result that $\rho_{f,g}$ is well-defined implies that ϕ_{I_f} and ϕ_{I_g} satisfies the aforementioned orthogonality relation. This relation between cocycles has previously been obtained by Paşol–Popa [17] in the language of period polynomials. Theorem B thus gives an alternate explanation of Paşol–Popa's result in level 1. The case of general levels treated by Paşol–Popa can be incorporated into our setting by introducing induced representations.

In Section 3, we provide a framework based on vector-valued Eisenstein series that allows for effective computation of modular forms of type ρ_{ϕ_1,ϕ_2}, that is, of forms that transform like the representation ρ_{ϕ_1,ϕ_2} (see Subsection 2.1 for the definition). This builds upon the framework developed in [1], and in particular enables us to evaluate $\mathcal{E}_{f,g}$.

To prepare for the setup of this framework, we record that the representation ρ_{ϕ_1,ϕ_2} features a function $\psi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ that behaves similar to a 1-cocycle. Specifically, it satisfies

$$\psi(\gamma_1\gamma_2) = \psi(\gamma_1) + \psi(\gamma_2) + \left\langle \phi_2(\gamma^{-1}), \phi_1(\gamma) \right\rangle, \quad \gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z}),$$

$$\psi(S) = -\tfrac{1}{2}\left\langle \phi_2(S^{-1}), \phi_1(S) \right\rangle, \quad \psi(T) = 0.$$

We also remark that in the case of $\phi_1 = \phi_{I_f}$ and $\phi_2 = -\phi_{I_g}$, this function coincides with $\psi_{f,g}$.

Let k be an integer, and let (ϕ_1, ϕ_2) be a pair of parabolic cocycles for which ρ_{ϕ_1,ϕ_2} is well-defined. Then we define the vector-valued Eisenstein series of weight $k \in \mathbb{Z}$, $k > 2 + \mathsf{d}$ and type ρ_{ϕ_1,ϕ_2} by

$$E_k(\tau; \rho_{\phi_1,\phi_2}) = \left(E_k^{[2]}(\tau; \phi_1, \phi_2),\ E_k^{[1]}(\tau; \phi_1),\ E_k \right)^T, \tag{0.7}$$

where E_k is the classical Eisenstein series of weight k, and where

$$E_k^{[1]}(\tau; \phi_1) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \frac{\phi_1(\gamma^{-1})}{(c\tau + d)^k} \quad \text{and} \quad E_k^{[2]}(\tau; \phi_1, \phi_2) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \frac{\psi(\gamma^{-1})}{(c\tau + d)^k}.$$

The series $E_k^{[1]}(\cdot; \phi_1)$ is called the generalized second-order Eisenstein series of type $(\mathrm{sym}^{\mathsf{d}}(X), \mathbf{1})$ associated to $(\phi_1, 1)$ and was a subject of study in [1], in which it was shown that it converges absolutely and locally uniformly on $\mathbb{H}$ for $k > 2 + \mathsf{d}$, and where its Fourier series expansion was provided. As for $E_k^{[2]}(\tau; \phi_1, \phi_2)$, we show that it converges in the same region and provide its Fourier series expansion in Theorem 3.1.

To state our last main theorem, we write $\mathrm{M}_\bullet$ for the graded ring of modular forms, and given a representation ρ, $\mathrm{M}_\bullet(\rho)$ for the corresponding graded $\mathrm{M}_\bullet$-module of modular forms of type ρ. Furthermore, if M is a $\mathrm{M}_\bullet$-module, $I \subseteq M$ is a submodule, and $f \in \mathrm{M}_\bullet$, then the saturation of I at f is the $\mathrm{M}_\bullet$-module

$$(I : f^\infty) = \left\{ g \in M : \exists n \in \mathbb{Z}_{\geqslant 0}.\, f^n g \in I \right\}.$$

Recall also that given a parabolic $\mathrm{sym}^{\mathsf{d}}(X)$-cocycle ϕ, we have the representation $\mathbf{1} \boxplus_{\phi^\vee} \mathrm{sym}^{\mathsf{d}}(X) : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C} \oplus \mathbb{C}[X]_{\mathsf{d}})$ given by

$$(\mathbf{1} \boxplus_{\phi^\vee} \mathrm{sym}^{\mathsf{d}}(X))(\gamma) = \begin{pmatrix} 1 & \left\langle \phi(\gamma^{-1}),\ \cdot\ \right\rangle \\ 0 & \mathrm{sym}^{\mathsf{d}}(X)(\gamma) \end{pmatrix}.$$

We provide the following theorem.

Theorem C. *Let $\mathsf{d} \geqslant 0$ and $k_0 > 2 + \mathsf{d}$ be integers and let ϕ_1, ϕ_2 be parabolic $\mathrm{sym}^{\mathsf{d}}(X)$-cocycles making ρ_{ϕ_1,ϕ_2} well-defined (see Theorem 2.2). Let also*

$$\mathrm{E}_{\geqslant k_0}(\rho_{\phi_1,\phi_2}) = \mathrm{span}\, \mathrm{M}_\bullet \left\{ E_k(\tau; \rho_{\phi_1,\phi_2}) : k \geqslant k_0 \right\}.$$

Then

$$\mathrm{M}_\bullet(\rho_{\phi_1,\phi_2}) = \Big(\mathrm{E}_{\geqslant k_0}(\rho_{\phi_1,\phi_2}) + \iota\Big(\mathrm{M}_\bullet(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathrm{d}}(X))\Big) : \Delta^\infty\Big),$$

where $\iota(f,g) = (f,g,0)^T$.

Since $\mathrm{M}_\bullet(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathrm{d}}(X))$ is described in [1] as the saturation at Δ of generalized second-order Eisenstein series and classical modular forms, Theorem C indeed implies that modular forms of type ρ_{ϕ_1,ϕ_2} can be expressed in terms of sums of products of (vector-valued) Eisenstein series with classical modular forms, after multiplication with a suitable power of Δ.

1 | PRELIMINARIES

In this section, we define the notation we use throughout the paper, and revisit the basic theory of Eichler cohomology, extensions of arithmetic types, and vector-valued modular forms. For further details, we direct the reader to [19] and [16].

The special linear group of degree two over the integers, is given by

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} \,:\, ad - bc = 1 \right\}.$$

It is a fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The parabolic subgroup $\Gamma_\infty \subseteq \mathrm{SL}_2(\mathbb{Z})$ is given by $\Gamma_\infty = \langle T, -1 \rangle$. We also let $U = TS$.

The upper-half plane $\mathbb{H}$ is given by $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$, the Möbius action is given by

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}.$$

Given a $\mathbb{C}$-vector space V, a function $f : \mathbb{H} \to V$, an integer k, and an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we define a new function $f|_k\gamma : \mathbb{H} \to V$ by

$$\big(f|_k\gamma\big)(\tau) = (c\tau + d)^{-k} f(\gamma\tau).$$

We record that this gives rise to a right-action of $\mathrm{SL}_2(\mathbb{Z})$ on the space of holomorphic functions from $\mathbb{H}$ to V, which we call the slash-action.

Let V be a $\mathbb{C}$-vector space with a norm $\|\cdot\|$, and let $f : \mathbb{H} \to V$ be a function. If there exists a real number $a \in \mathbb{R}$ such that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ it holds uniformly in $\mathrm{Re}(\tau)$ that

$$\big\|\big(f|_k\gamma\big)(\tau)\big\| = O(\mathrm{Im}(\tau)^a) \text{ as } \mathrm{Im}(\tau) \to \infty, \tag{1.1}$$

we say that f has moderate growth. If instead we have that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ it holds that

$$\big\|\big(f|_k\gamma\big)(\tau)\big\| \to 0 \text{ as } \mathrm{Im}(\tau) \to \infty, \tag{1.2}$$

we say that f is cuspidal. Note that the conditions (1.1) and (1.2) are independent of the choice of norm on V.

Let k be an integer. Then if $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic function of moderate growth satisfying that

$$f\big|_k\gamma = f \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}),$$

we call f a (scalar-valued) modular form of weight k. If f is cuspidal, it is called a cusp form. The set of modular forms of weight k forms a $\mathbb{C}$-vector space denoted by M_k. The corresponding subspace of cusp forms is denoted by S_k. If $f \in \mathrm{M}_k$, we let the conjugate modular form f^c be given by

$$f^c(\tau) = \overline{f(-\overline{\tau})}.$$

Note that if f has the Fourier series expansion $f(\tau) = \sum_{n\geqslant 0} c_n e(n\tau)$, then f^c has the Fourier series expansion $f^c(\tau) = \sum_{n\geqslant 0} \overline{c_n} e(n\tau)$.

Finally, we denote the trivial representation of $\mathrm{SL}_2(\mathbb{Z})$ by $\mathbf{1}$, so that $V(\mathbf{1}) = \mathbb{C}$ and $\mathbf{1}(\gamma)z = z$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{C}$.

1.1 | Cohomology and extensions of arithmetic types

An arithmetic type is a finite-dimensional complex representation of a subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. In the present paper, we restrict our scope to arithmetic types of $\mathrm{SL}_2(\mathbb{Z})$.

If ρ and σ are arithmetic types, the $\mathbb{C}$-vector space of (ρ, σ)-cocycles is given by

$$\mathrm{Z}^1(\rho,\sigma) = \big\{ f : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Hom}(V(\rho), V(\sigma)) \ : \ f(\gamma_1\gamma_2) = \sigma(\gamma_1)f(\gamma_2) + f(\gamma_1)\rho(\gamma_2) \big\},$$

and the subspace of (ρ, σ)-coboundaries is given by

$$\mathrm{B}^1(\rho,\sigma) = \big\{ f : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Hom}(V(\rho), V(\sigma)) : \exists h \in \mathrm{Hom}(V(\rho), V(\sigma)).\, f(\gamma) = \sigma(\gamma)h - h\rho(\gamma) \big\}.$$

If a cocycle $\phi \in \mathrm{Z}^1(\rho, \sigma)$ vanishes on every element of Γ_∞, we call it parabolic. The space of all parabolic cocycles (or coboundaries) is denoted by $\mathrm{Z}^1_{\mathrm{pb}}(\rho, \sigma)$ (or $\mathrm{B}^1_{\mathrm{pb}}(\rho, \sigma)$). We will identify $\mathrm{Hom}(\mathbf{1}, V(\sigma))$ with $V(\sigma)$.

Given a parabolic cocycle $\phi \in \mathrm{Z}^1_{\mathrm{pb}}(\sigma, \rho)$, we define the arithmetic type $\rho \boxplus_\phi \sigma$ by $V(\rho \boxplus_\phi \sigma) = V(\rho) \oplus V(\sigma)$ and

$$(\rho \boxplus_\phi \sigma)(\gamma)(v, v') = \big(\rho(\gamma)v + \phi(\gamma)v', \sigma(\gamma)v'\big). \tag{1.3}$$

The first parabolic cohomology group is the quotient $\mathrm{H}^1_{\mathrm{pb}}(\rho, \sigma) = \mathrm{Z}^1_{\mathrm{pb}}(\rho, \sigma)/\mathrm{B}^1_{\mathrm{pb}}(\rho, \sigma)$. We record that $\mathrm{H}^1_{\mathrm{pb}}(\sigma, \rho)$ is isomorphic to the group of parabolic extension classes of σ by ρ, denoted by $\mathrm{Ext}_{\mathrm{pb}}(\sigma, \rho)$. In particular, the following map is a well-defined isomorphism of groups

$$\mathrm{Ext}_{\mathrm{pb}}(\sigma, \rho) \ni [0 \to \rho \to \rho \boxplus_\phi \sigma \to \sigma \to 0] \longmapsto \phi + \mathrm{B}^1_{\mathrm{pb}}(\sigma, \rho) \in \mathrm{H}^1_{\mathrm{pb}}(\sigma, \rho).$$

In particular, this means that given cocycles ϕ_1 and ϕ_2, it holds that $\rho \boxplus_{\phi_1} \sigma$ is isomorphic to $\rho \boxplus_{\phi_2} \sigma$ if and only if ϕ_1 and ϕ_2 are cohomologous.

Note that by the cocycle relations, an arbitrary cocycle $\phi \in \mathrm{Z}^1_{\mathrm{pb}}(\rho, \sigma)$ is fully determined by its value at S.

1.2 | Vector-valued modular forms

Let k be an integer and let ρ be an arithmetic type. Given a function $f : \mathbb{H} \to V(\rho)$ and an element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we define $f|_k\gamma : \mathbb{H} \to V(\rho)$ by

$$\left(f|_{k,\rho}\gamma\right)(\tau) = \rho(\gamma^{-1})\big(f|_k\gamma\big)(\tau).$$

A vector-valued modular form of type ρ and weight k, is a holomorphic function of moderate growth $f : \mathbb{H} \to V(\rho)$ satisfying that

$$f|_{k,\rho}\gamma = f \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

If a vector-valued modular form f of type ρ and weight k is cuspidal, we call it a cusp form. The space of vector-valued modular forms of weight k and type ρ is denoted by $\mathrm{M}_k(\rho)$. The corresponding subspace of cusp forms is denoted by $\mathrm{S}_k(\rho)$.

We remark that scalar-valued modular forms of weight k are the same as vector-valued modular forms of type $\mathbf{1}$. That is, we have the equalities $\mathrm{M}_k(\mathbf{1}) = \mathrm{M}_k$ and $\mathrm{S}_k(\mathbf{1}) = \mathrm{S}_k$.

For an arithmetic type ρ we let the graded module of modular forms of type ρ be given by

$$\mathrm{M}_\bullet(\rho) = \bigoplus_{k\in\mathbb{Z}} \mathrm{M}_k(\rho).$$

If M is an $\mathrm{M}_\bullet$-module, $I \subseteq M$ is a submodule of M, and $f \in \mathrm{M}_\bullet$ then we recall that the saturation of I at f is given by

$$(I : f^\infty) = \left\{g \in M : \exists n \in \mathbb{Z}_{\geqslant 0}.\, f^n g \in I\right\}.$$

1.3 | Symmetric powers

Let $\mathrm{d} \geqslant 0$ be an integer. Then we let $\mathbb{C}[X]_\mathrm{d}$ be the space of polynomials with coefficients in $\mathbb{C}$ of degree at most d. We define the arithmetic type $\mathrm{sym}^\mathrm{d}(X)$ by $V(\mathrm{sym}^\mathrm{d}(X)) = \mathbb{C}[X]_\mathrm{d}$, and

$$\mathrm{sym}^\mathrm{d}(X)(\gamma)p = p\big|_{-\mathrm{d}}\gamma^{-1} = (-cX+a)^\mathrm{d} p\left(\frac{dX-b}{-cX+a}\right), \quad \text{where } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } p \in \mathbb{C}[X]_\mathrm{d}.$$

We remark that $\mathrm{sym}^\mathrm{d}(X)$ is a model of the dth symmetric power of the standard representation of $\mathrm{SL}_2(\mathbb{Z})$, explaining the notation. The group ring $\mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]$ acts linearly on $\mathbb{C}[X]_\mathrm{d}$ by

$$\gamma.p = \mathrm{sym}^\mathrm{d}(X)(\gamma)p \text{ and } (c_1\gamma_1 + c_2\gamma_2)p = c_1(\gamma_1.p) + c_2(\gamma_2.p),$$

where $\gamma, \gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$ and $c_1, c_2 \in \mathbb{C}$. As mentioned in the introduction there exists a symmetric pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathbb{C}[X]_\mathrm{d} \times \mathbb{C}[Y]_\mathrm{d} \to \mathbb{C}$ given by

$$\langle p, q\rangle = \sum_{i=0}^{\mathrm{d}} (-1)^i \binom{\mathrm{d}}{i}^{-1} p_i q_{\mathrm{d}-i},$$

satisfying $\langle \gamma.p, \gamma.q\rangle = \langle p, q\rangle$ for any $p \in \mathbb{C}[X]_\mathrm{d}$, $q \in \mathbb{C}[Y]_\mathrm{d}$, and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. As $\langle\,\cdot\,,\,\cdot\,\rangle$ is invariant and bilinear, we have an equivariant contraction map $\pi : \mathbb{C}[X, Y]_\mathrm{d} \to \mathbb{C}$ given by $\pi(p \otimes q) =$

$\langle p, q\rangle$. There is also a related antisymmetric bilinear form $\langle\!\langle \cdot\, , \cdot \rangle\!\rangle : \mathbb{C}[X]_{\mathsf{d}}^2 \to \mathbb{C}$ given by

$$\langle\!\langle p, q \rangle\!\rangle = \left\langle T^{-1}.p - T.p,\ q \right\rangle.$$

If $p = \sum p_i X^i \in \mathbb{C}[X]_{\mathsf{d}}$, then we let $\overline{p} = \sum \overline{p_i} X^i \in \mathbb{C}[X]_{\mathsf{d}}$. Note that the action of $\mathrm{SL}_2(\mathbb{Z})$ commutes with conjugation, so that $\gamma.\overline{p} = \overline{\gamma.p}$.

Henceforth, we will identify $\mathbb{C}[X]_{\mathsf{d}} \otimes \mathbb{C}[Y]_{\mathsf{d}}$ with $\mathbb{C}[X, Y]_{\mathsf{d}}$; the space of polynomials in X and Y of degree at most d in X and Y. Furthermore, we will use the shorthand notation $\mathrm{sym}^{\mathsf{d}}(X, Y) := \mathrm{sym}^{\mathsf{d}}(X) \otimes \mathrm{sym}^{\mathsf{d}}(Y)$. This coincides with the definition we used in the introduction.

1.4 | The Eichler–Shimura isomorphism

We shall now briefly describe the Eichler–Shimura isomorphism between scalar-valued cusp forms of weight $k \geqslant 2$ and $\mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{k-2}(X))$. This exposition follows [12]. It holds for arbitrary integers $\mathsf{d} \geqslant 0$, but the case relevant in this paper is $\mathsf{d} = k - 2$.

The space of parabolic $(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$-cocycles can be completely described as follows:

$$\mathrm{Z}^1_{\mathrm{pb}}\big(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X)\big) = \big\{\phi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}[X]_{\mathsf{d}} : \phi(-I) = \phi(T) = 0 \text{ and}$$
$$(1 + S).\phi(S) = (1 + U + U^2).\phi(S) = 0\big\},$$

Note that $\mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))$ is closed under complex conjugation. The space $\mathbb{C}[X]_{\mathsf{d}}$ splits up into its "even part" and "odd part". That is, we have $\mathbb{C}[X]_{\mathsf{d}} = \mathbb{C}[X]_{\mathsf{d}}^+ \oplus \mathbb{C}[X]_{\mathsf{d}}^-$ with

$$\mathbb{C}[X]_{\mathsf{d}}^+ = \big\{ \textstyle\sum_{i=0}^{\mathsf{d}} p_i X^i \in \mathbb{C}[X]_{\mathsf{d}} \,:\, p_{2j+1} = 0 \text{ for } 0 \leqslant 2j + 1 \leqslant \mathsf{d}\big\}$$
$$\mathbb{C}[X]_{\mathsf{d}}^- = \big\{ \textstyle\sum_{i=0}^{\mathsf{d}} p_i X^i \in \mathbb{C}[X]_{\mathsf{d}} \,:\, p_{2j} = 0 \text{ for } 0 \leqslant 2j \leqslant \mathsf{d}\big\}.$$

We further write

$$W_{\mathsf{d}} = \mathrm{Z}^1_{\mathrm{pb}}\big(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X)\big)(S), \quad W_{\mathsf{d}}^+ = W_{\mathsf{d}} \cap \mathbb{C}[X]_{\mathsf{d}}^+, \quad \text{and} \quad W_{\mathsf{d}}^- = W_{\mathsf{d}} \cap \mathbb{C}[X]_{\mathsf{d}}^-.$$

Let $f \in \mathrm{S}_k$ be a cusp form of weight $k \in \mathbb{Z}_{\geqslant 2}$. Then we define the polynomial-valued Eichler integral of f as

$$I_f(\tau; X) = \int_\tau^{i\infty} f(z)\,(X - \tau)^{k-2}\ \mathrm{d}z, \quad \text{where } \tau \in \mathbb{H}.$$

If the variable is understood from context, it is omitted from the notation. Note that as f vanishes at the cusp, $I_f(\tau)$ is well-defined. We also see that for $\tau \in \mathbb{H}$ we have that $I_f(\tau) \in \mathbb{C}[X]_{\mathsf{d}}$. Given $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we let

$$\phi_{I_f}(\gamma) = I_f\big|_{0,\mathrm{sym}^{k-2}(X)}(1 - \gamma^{-1}) = \int_{\gamma(i\infty)}^{i\infty} f(z)\,(X - \tau)^{k-2}\ \mathrm{d}z.$$

We have that $\phi_{I_f} \in \mathrm{Z}^1(\mathbf{1}, \mathrm{sym}^{k-2}(X))$ by construction, and as elements of Γ_∞ stabilize the cusp we also have that $\phi_{I_f} \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{k-2}(X))$. We now set $r_f = \phi_{I_f}(S) \in W_{k-2}$ and

$$r_{f,n} = \int_0^\infty f(it)t^n \,\mathrm{d}t = \Gamma(n+1)(2\pi)^{-n-1}\,\mathrm{L}(f, n+1) \quad \text{for } 0 \leqslant n \leqslant k-2,$$

where $\mathrm{L}(f, \,\cdot\,)$ denotes the Hecke L-function associated to f. By the binomial theorem, we see that

$$r_f = \sum_{n=0}^{k-2} i^{-n+1} \binom{k-2}{n} r_n(f) X^{k-2-n}.$$

Hence, we define $r_f^+ \in W_{k-2}^+$ and $r_f^- \in W_{k-2}^-$ by

$$r_f^+ = \sum_{\substack{0 \leqslant n \leqslant k-2 \\ 2 \mid n}} (-1)^{n/2} \binom{k-2}{n} r_{f,n} X^{k-2-n} \quad \text{and} \quad r_f^- = \sum_{\substack{0 \leqslant n \leqslant k-2 \\ 2 \nmid n}} (-1)^{(n-1)/2} \binom{k-2}{n} r_{f,n} X^{k-2-n},$$

so that $r = r^- + ir^+$. We can now provide the Eichler–Shimura isomorphism.

Theorem (Eichler–Shimura). *It holds that the maps*

$$\mathrm{S}_k \ni f \mapsto r_f^- \in W_{k-2}^- \text{ and}$$

$$\mathrm{S}_k \oplus \mathbb{C} \ni (f, z) \mapsto r_f^+ + z(X^{k-2} - 1) \in W_{k-2}^+$$

are isomorphisms of $\mathbb{C}$*-vector spaces.*

For proofs, see, for example, [18] or [13]. We now state an important result related to the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Recall first that the Petersson inner product $(\cdot, \cdot) : \mathrm{M}_k \times \mathrm{S}_k \to \mathbb{C}$ is given by

$$(f, g) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(x+iy)\overline{g(x+iy)}y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

We have the following theorem, due to Haberland and Paşol–Popa.

Theorem (Haberland [10] and Paşol–Popa [17]). *Let $k \geqslant 2$ be an even integer, and let $f, g \in \mathrm{S}_k$. Then it holds that*

$$\langle\!\langle r_f, \overline{r_g} \rangle\!\rangle = -6(2i)^{k-1}(f, g) \qquad \text{and} \qquad \langle\!\langle r_f, r_g \rangle\!\rangle = 0. \tag{1.4}$$

To contextualize the assumptions of Theorem 2.2 in the next section, we recall the following:

Remark 1.1. Let $\mathrm{d} \geqslant 0$ be an even integer and let $e = X^{\mathrm{d}} - 1 \in W_{\mathrm{d}}$. Then the relations $(1+S).e = 0$ and $(1+U+U^2).e = 0$ imply that $\langle\!\langle e, q \rangle\!\rangle = 0$ for all $q \in W_{\mathrm{d}}$. Conversely, the Eichler–Shimura isomorphisms imply that if an element $p \in W_{\mathrm{d}}$ satisfies that $\langle\!\langle p, q \rangle\!\rangle = 0$ for all $q \in W_{\mathrm{d}}$, then $p \in \mathbb{C}\{e\}$. This means that $(W_{\mathrm{d}}/\mathbb{C}\{e\}, \langle\!\langle \cdot, \cdot \rangle\!\rangle)$ is a nondegenerate symplectic vector space, and by applying the Eichler–Shimura isomorphisms to a basis of orthonormalized Hecke eigenforms one obtains an explicit isomorphism of $(W_{\mathrm{d}}/\mathbb{C}\{e\}, \langle\langle\cdot,\cdot\rangle\rangle)$ with the standard complex symplectic vector space, given by

$$\left(\mathbb{C}^{2D}, (x, y) \mapsto x^T \begin{pmatrix} 0 & -1_D \\ 1_D & 0 \end{pmatrix} y\right),$$

where $D = \dim(\mathrm{S}_{\mathrm{d}+2})$, and 1_D denotes the $D \times D$ identity matrix.

2 | FROM EXTENSIONS TO EICHLER–SHIMURA INTEGRALS

In this section, we show that polynomial- and scalar-valued depth two Eichler–Shimura integrals can be regarded as components of vector-valued modular forms of type $\widetilde{\rho}_{f,g}$ and $\rho_{f,g}$, respectively.

2.1 | The extension ρ_{ϕ_1,ϕ_2}

Let $\mathrm{d} \geqslant 0$ be an even integer. For $\phi \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathrm{d}}(X))$, we let

$$\phi^\vee(\gamma)(v) = \left\langle \phi(\gamma^{-1}),\, v \right\rangle. \tag{2.1}$$

The invariance of the pairing implies that $\phi^\vee \in \mathrm{Z}^1_{\mathrm{pb}}(\mathrm{sym}^{\mathrm{d}}(X), \mathbf{1})$, and that (2.1) defines an isomorphism of $\mathbb{C}$-vector spaces from $\mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathrm{d}}(X))$ to $\mathrm{Z}^1_{\mathrm{pb}}(\mathrm{sym}^{\mathrm{d}}(X), \mathbf{1})$.

Given parabolic cocycles $\phi_1, \phi_2 \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathrm{d}}(X))$ and a function $\psi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$, we let $\rho_{\phi_1,\phi_2,\psi} : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C} \oplus \mathbb{C}[X]_{\mathrm{d}} \oplus \mathbb{C})$ be given by

$$\rho_{\phi_1,\phi_2,\psi}(\gamma) = \begin{pmatrix} 1 & \phi_2^\vee(\gamma) & \psi(\gamma) \\ 0 & \mathrm{sym}^{\mathrm{d}}(X)(\gamma) & \phi_1(\gamma) \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.2}$$

We have the following proposition.

Proposition 2.1. *Let $\phi_1, \phi_2 \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathrm{d}}(X))$, and let $\psi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ be a function. Then the following are equivalent.*

(i) *$\rho_{\phi_1,\phi_2,\psi}$ is a representation.*
(ii) *For all $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$, it holds that $\psi(\gamma_1\gamma_2) = \psi(\gamma_1) + \psi(\gamma_2) + \phi_2^\vee(\gamma_1)\phi_1(\gamma_2)$.*
(iii) *$(\phi_2^\vee, \psi) \in \mathrm{Z}^1_{\mathrm{pb}}(\mathrm{sym}^{\mathrm{d}}(X) \boxplus_{\phi_1} \mathbf{1}, \mathbf{1})$.*
(iv) *$(\psi, \phi_1)^T \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathrm{d}}(X))$.*

Furthermore, if any of these conditions hold, then

$$\rho_{\phi_1,\phi_2,\psi} = \mathbf{1} \boxplus_{(\phi_2^\vee,\psi)} \left(\mathrm{sym}^{\mathrm{d}}(X) \boxplus_{\phi_1} \mathbf{1}\right) = \left(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathrm{d}}(X)\right) \boxplus_{\binom{\psi}{\phi_1}} \mathbf{1}.$$

Proof. Let $\rho = \rho_{\phi_1,\phi_2,\psi}$ and let $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$. Then using the cocycle relations, and the fact that $\mathrm{sym}^{\mathrm{d}}(X)$ is a representation, we find that

$$\rho(\gamma_1)\rho(\gamma_2) = \begin{pmatrix} 1 & \phi_2^\vee(\gamma_1\gamma_2) & \psi(\gamma_2) + \phi_2^\vee(\gamma_1)\phi_1(\gamma_2) + \psi(\gamma_1) \\ 0 & \mathrm{sym}^{\mathrm{d}}(X)(\gamma_1\gamma_2) & \phi_1(\gamma_1\gamma_2) \\ 0 & 0 & 1 \end{pmatrix}.$$

We have that ρ is a representation if and only if it is a homomorphism and therefore we see that (i) is equivalent to (ii).

Continuing, we have that $(\phi_2^\vee, \psi) \in \mathrm{Z}^1_{\mathrm{pb}}(\mathrm{sym}^{\mathrm{d}}(X) \boxplus_{\phi_1} \mathbf{1}, \mathbf{1})$ if and only if

$$\big(\phi_2^\vee(\gamma_2), \psi(\gamma_2)\big) + \big(\phi_2^\vee(\gamma_1), \psi(\gamma_1)\big)\Big(\mathrm{sym}^{\mathrm{d}}(X) \boxplus_{\phi_1} \mathbf{1}\Big)(\gamma_2) = \big(\phi_2^\vee(\gamma_1\gamma_2), \psi(\gamma_1\gamma_2)\big).$$

On the other hand, we have that

$$\begin{aligned} &\big(\phi_2^\vee(\gamma_2), \psi(\gamma_2)\big) + \big(\phi_2^\vee(\gamma_1), \psi(\gamma_1)\big)\Big(\mathrm{sym}^{\mathrm{d}}(X) \boxplus_{\phi_1} \mathbf{1}\Big)(\gamma_2) \\ &\quad = \big(\phi_2^\vee(\gamma_2) + \phi_2^\vee(\gamma_1)\mathrm{sym}^{\mathrm{d}}(X)(\gamma_2),\ \psi(\gamma_2) + \phi_2^\vee(\gamma_1)\phi_1(\gamma_2) + \psi(\gamma_1)\big) \\ &\quad = \big(\phi_2^\vee(\gamma_1\gamma_2),\ \psi(\gamma_1) + \psi(\gamma_2) + \phi_2^\vee(\gamma_1)\phi_2(\gamma_2)\big). \end{aligned}$$

Hence, it is clear that (iii) is equivalent to (ii). In the same way, we find that (iv) is equivalent to (ii), which finishes the proof. □

For a fixed pair of cocycles (ϕ_1, ϕ_2), there is at most one function ψ satisfying the conditions of Proposition 2.1.

Theorem 2.2. *Let* $\mathrm{d} \geqslant 0$ *be an even integer and let* $\phi_1, \phi_2 \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathrm{d}}(X))$*. Then there exists a function* ψ *satisfying the conditions of Proposition 2.1 if and only if* $\langle\!\langle \phi_1(S), \phi_2(S) \rangle\!\rangle = 0$*. Furthermore, if such a function exists, it is unique and is given by*

$$\psi(S) = -\frac{1}{2}\phi_2^\vee(S)\phi_1(S).$$

Proof. Let $\psi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ be a function and let $\rho = \rho_{\phi_1,\phi_2,\psi}$. Recall that $\mathrm{SL}_2(\mathbb{Z})$ has the presentation $\langle S, T : S^4 = (ST)^6 = 1 \rangle$. Hence, we have that ρ is a representation if and only if

$$\rho(S)^4 = 1 \text{ and } (\rho(S)\rho(T))^6 = 1.$$

However, we find that

$$\rho(S)^2 = \begin{pmatrix} 1 & 0 & 2\psi(S) + \phi_2^\vee(S)\phi_1(S) \\ 0 & 1_{\mathrm{d}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and}$$

$$(\rho(S)\rho(T))^3 = \begin{pmatrix} 1 & 0 & 3\psi(S) + \phi_2^\vee((ST)^2)\phi_1(S) + \phi_2^\vee(ST)\phi_1(S) \\ 0 & 1_{\mathrm{d}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where 1_{d} is the $\mathrm{d} \times \mathrm{d}$ identity matrix. Hence, ρ is a representation if and only if

$$2\psi(S) + \phi_2^\vee(S)\phi_1(S) = 0 \text{ and}$$
$$3\psi(S) + \phi_2^\vee((ST)^2)\phi_1(S) + \phi_2^\vee(ST)\phi_1(S) = 0.$$

We have that $\phi_2^\vee((ST)^2) = \phi_2^\vee(ST) + \phi_2^\vee(ST)ST$, that $\phi_2^\vee(ST) = \phi_2^\vee(S)T$, and that $T\phi_1(S) = \phi_1(TS)$. With $U = TS$, this yields that the above is equivalent to

$$\psi(S) = -\tfrac{1}{2}\phi_2^\vee(S)\phi_1(S) \text{ and}$$
$$\phi_2^\vee(S)(2\phi_1(U) + 2\phi_1(U^2) - 3\phi_1(S)) = 0.$$

However, applying the identity $(1 + U + U^2).\phi_1(S) = 0$ to expand $2\phi_1(S)$ and then $(1 + S).\phi_1(S) = 0$ to simplify the expression, we obtain that

$$\begin{aligned} 2\big(\phi_1(U) + \phi_1(U^2)\big) - 3\phi_1(S) &= 2\big(\phi_1(U) + \phi_1(U^2) + U.\phi_1(S) + U^2.\phi_1(S)\big) - \phi_1(S) \\ &= 2\big(T.\phi_1(S) + TST.\phi_1(S) + U^2.\phi_1(S)\big) - \phi_1(S) = 2T.\phi_1(S) - \phi_1(S), \end{aligned}$$

so that $\phi_2^\vee(S)(2\phi_1(U) + 2\phi_1(U^2) - 3\phi_1(S)) = \langle 2T^{-1}.\phi_2(S) - \phi_2(S),\ \phi_1(S)\rangle$. To finish the proof we have to identify the right-hand side with $\langle\langle \phi_2(S), \phi_1(S)\rangle\rangle = \langle (T^{-1} - T).\phi_2(S),\ \phi_1(S)\rangle$. To this end, note that for any element $H \in \mathbb{C}[\mathrm{SL}_2(\mathbb{Z})]$ the expression $\langle H.\phi_2(S), \phi_1(S)\rangle$ only depends on the image of H in the double quotient $(1 + S)\backslash\mathbb{C}[\mathrm{SL}_2(\mathbb{Z})/\pm I]/(1 + S)$. Denoting equality in this quotient by $\equiv$, the result follows from $(1 + U + U^2).\phi_1(S) = 0$ and

$$\begin{aligned} T^{-1} - 1 + T &= SU^{-1} - 1 + US^{-1} = 1 + (SU^{-1} - 1)(1 - US^{-1}) \\ &\equiv 1 + (-U^{-1} - 1)(1 + U) = -U^{-1} - 1 - U. \end{aligned}$$ □

If a pair of cocycles $(\phi_1, \phi_2) \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathrm{d}}(X))^2$ satisfies that $\langle\langle \phi_1, \phi_2\rangle\rangle = 0$, we call it admissible. If (ϕ_1, ϕ_2) is admissible, then we omit ψ from the notation and write $\rho_{\phi_1,\phi_2} = \rho_{\phi_1,\phi_2,\psi}$.

2.2 | Depth two Eichler–Shimura integrals

In this section, we describe depth two Eichler–Shimura integrals as components of vector-valued modular forms of the types $\widetilde{\rho}_{f,g}$ and $\rho_{f,g}$, where $f, g \in \mathrm{S}_k$, $k \in \mathbb{Z}_{\geqslant 2}$; defined in the introduction, see (0.4) and (0.5).

Recall that for $f, g \in \mathrm{S}_k$, $k \in \mathbb{Z}_{\geqslant 2}$, and indeterminates X and Y we have the depth two polynomial-valued Eichler–Shimura integral $I_{f,g}(\,\cdot\,; X, Y)$ and the depth two scalar-valued Eichler–Shimura integral $\mathcal{E}_{f,g}$, given by

$$I_{f,g}(\tau; X, Y) = \int_\tau^{i\infty} f(z)\,(X - z)^{k-2}\, I_g(z; Y)\; \mathrm{d}z \quad \text{and} \quad \mathcal{E}_{f,g}(\tau) = \int_\tau^{i\infty} f(z)\,\mathcal{E}_g(z)\; \mathrm{d}z, \quad \tau \in \mathbb{H},$$

where $I_g(\tau; Y) = \int_\tau^{i\infty} g(z)(Y - z)^{k-2}\, \mathrm{d}z$ and $\mathcal{E}_g(\tau) = I_g(\tau; \tau)$ are the polynomial-valued Eichler integral and the scalar-valued Eichler integral.

Recall that for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have that $\widetilde{\psi}_{f,g}(\gamma; X, Y) = \int_{\gamma(i\infty)}^{i\infty} f(z)\,(X-z)^{k-2}\, I_g(z;Y)\, \mathrm{d}z$. Let also

$$\phi_{I_g} \cdot \mathrm{sym}^{k-2}(X) \,:\, \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Hom}(\mathbb{C}[X]_{k-2}, \mathbb{C}[X,Y]_{k-2}) \quad \text{be given by}$$

$$\Big(\phi_{I_g} \cdot \mathrm{sym}^{k-2}(X)\Big)(\gamma)p = \phi_{I_g}(\gamma) \cdot \mathrm{sym}^{k-2}(X)(\gamma)p.$$

With this notation, we have that

$$\widetilde{\rho}_{f,g} = \mathrm{sym}^{k-2}(X,Y) \boxplus_{(\phi_{I_g} \cdot \mathrm{sym}^{k-2}(X), \psi_{f,g})} \Big(\mathrm{sym}^{k-2}(X) \boxplus_{\phi_{I_f}} \mathbf{1}\Big). \tag{2.3}$$

We have that $(I_f, 1)^T \in \mathrm{M}_0(\mathrm{sym}^{k-2}(X) \boxplus_{\phi_{I_f}} \mathbf{1})$, see [1], and for the depth two polynomial-valued Eichler–Shimura integral we have the following theorem.

Theorem 2.3. *Let $k \geqslant 2$ be an integer, and let $f, g \in \mathrm{S}_k$. Then the arithmetic type $\widetilde{\rho}_{f,g}$ given by (0.4) or (2.3) is well-defined, and we have that*

$$\begin{pmatrix} I_{f,g} \\ I_f \\ 1 \end{pmatrix} \in \mathrm{M}_0(\widetilde{\rho}_{f,g}).$$

Proof. For convenience, we use the shorthand notation $|$ for $|_{0,\mathrm{sym}^{k-2}(X,Y)}$, ϕ for $\phi_{I_g} \cdot \mathrm{sym}^{k-2}(X)$, and ρ for $\widetilde{\rho}_{f,g}$. Through direct calculation, we see that $\phi \in \mathrm{Z}^1_{\mathrm{pb}}(\mathrm{sym}^{k-2}(X), \mathrm{sym}^{k-2}(X,Y))$. By a standard change of variables (see [12]), we also obtain that

$$\phi_{f,g}(\gamma,\tau) := r\big(I_{f,g}|(1-\gamma^{-1})\big)(\tau) = \widetilde{\psi}_{f,g}(\gamma) + \phi(\gamma) I_f(\gamma^{-1}\tau), \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.4}$$

To finish the proof of the transformation behavior, we record that

$$\rho(\gamma^{-1})\begin{pmatrix} I_{f,g}(\gamma\tau) \\ I_f(\gamma\tau) \\ 1 \end{pmatrix} - \begin{pmatrix} I_{f,g}(\tau) \\ I_f(\tau) \\ 1 \end{pmatrix} = \begin{pmatrix} I_{f,g}|(\gamma-1) + \phi(\gamma^{-1}) I_f(\gamma\tau) + \widetilde{\psi}_{f,g}(\gamma^{-1}) \\ 0 \\ 0 \end{pmatrix} = 0.$$

Let now $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$. Then (2.4) implies that $\phi_{f,g}(\gamma_1\gamma_2, \tau) = \widetilde{\psi}_{f,g}(\gamma_1\gamma_2) + \phi(\gamma_1\gamma_2) I_f(\gamma_2^{-1}\gamma_1^{-1}\tau)$. On the other hand, we have that

$$\begin{aligned}\phi_{f,g}(\gamma_1\gamma_2,\tau) &= \big(I_{f,g}|(1-\gamma_2^{-1})|\gamma_1^{-1}\big)(\tau) + \big(I_{f,g}|(1-\gamma_1^{-1})\big)(\tau) \\ &= \phi(\gamma_1\gamma_2) I_f(\gamma_2^{-1}\gamma_1^{-1}\tau) + \phi(\gamma_1)\big(I_f(\gamma_1^{-1}\tau) - \gamma_2.I_f(\gamma_2^{-1}\gamma_1^{-1}\tau)\big) + \gamma_1.\widetilde{\psi}_{f,g}(\gamma_2) + \widetilde{\psi}_{f,g}(\gamma_1).\end{aligned}$$

However, $(I_f|_{0,\mathrm{sym}^{k-2}(X)}(1-\gamma^{-1}))(\tau)$ is independent of τ, and thus we have that

$$\phi_{I_f}(\gamma_2) = (I_f|_{0,\mathrm{sym}^{k-2}(X)}(1-\gamma_2^{-1}))(\gamma_1^{-1}\tau) = I_f(\gamma_1^{-1}\tau) - \gamma_2.I_f(\gamma_2^{-1}\gamma_1^{-1}\tau).$$

We thus obtain the identity

$$\widetilde{\psi}_{f,g}(\gamma_1\gamma_2) = \gamma_1.\widetilde{\psi}_{f,g}(\gamma_2) + \widetilde{\psi}_{f,g}(\gamma_1) + \phi(\gamma_1)\phi_{I_f}(\gamma_2). \tag{2.5}$$

This leads immediately to the fact that $(\phi, \widetilde{\psi}_{f,g}) \in \mathrm{Z}^1_{\mathrm{pb}}(\mathrm{sym}^{k-2}(X) \boxplus_{\phi_{I_f}} \mathbf{1}, \mathrm{sym}^{k-2}(X,Y))$, and hence ρ is well-defined. ☐

An analogous statement holds for $\mathcal{E}_{f,g}$.

Theorem 2.4. *Let $k \geqslant 2$ be an integer, and let $f, g \in \mathrm{S}_k$. Then the arithmetic type $\rho_{f,g}$ given by (0.5) is well-defined, and we have that*

$$\begin{pmatrix} \mathcal{E}_{f,g} \\ I_f \\ 1 \end{pmatrix} \in \mathrm{M}_0(\rho_{f,g}).$$

Proof. The argument is parallel to the one for Theorem 2.3. ☐

Combining Theorems 2.3 and 2.4 with the contraction map $\pi : \mathrm{sym}^{k-2}(X,Y) \to \mathbf{1}$, we obtain our next theorem.

Theorem 2.5. *Let $k \geqslant 2$ be an even integer, and let $f, g \in \mathrm{S}_k$. Then we have that*

$$\pi_* \circ \begin{pmatrix} I_{f,g} \\ I_f \\ 1 \end{pmatrix} = \begin{pmatrix} \mathcal{E}_{f,g} \\ I_f \\ 1 \end{pmatrix},$$

where $\pi_* : \widetilde{\rho}_{f,g} \to \rho_{f,g}$ is the pushforward along the map (0.6), given by $\pi_*(p,q,z) = (\pi(p), q, z)$.

Proof. Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $p \in \mathbb{C}[X]_{k-2}$, then

$$\pi\Big(\phi_{I_g}(\gamma) \cdot \mathrm{sym}^{k-2}(X)(\gamma)p\Big) = \big\langle \phi_{I_g}(\gamma), \gamma.p \big\rangle = \big\langle \gamma^{-1}.\phi_{I_g}(\gamma), p \big\rangle = -\big\langle \phi_{I_g}(\gamma^{-1}), p \big\rangle,$$

and therefore $\pi \circ (\phi_{I_g} \cdot \mathrm{sym}^{k-2}(X)) = -\phi^\vee_{I_g}$. That is, π_* maps $\widetilde{\rho}_{f,g}$ to $\rho_{f,g}$ as claimed.

We consider the difference

$$\pi_* \circ \begin{pmatrix} I_{f,g} \\ I_f \\ 1 \end{pmatrix} - \begin{pmatrix} \mathcal{E}_{f,g} \\ I_f \\ 1 \end{pmatrix} = \begin{pmatrix} \pi \circ I_{f,g} - \mathcal{E}_{f,g} \\ 0 \\ 0 \end{pmatrix} \in \mathrm{M}_0(\rho_{f,g}).$$

As the two bottom components vanish, we conclude that $\pi \circ I_{f,g} - \mathcal{E}_{f,g} \in \mathrm{M}_0$. As f and g are cusp forms, the zeroth Fourier coefficient of $\pi \circ I_{f,g} - \mathcal{E}_{f,g}$ vanishes, and we obtain the equality stated in the theorem. ☐

Remark 2.6. Theorem 2.4 implies that the pair of cocycles $(\phi_{I_f}, -\phi_{I_g})$ yields a representation, for which we see that $\rho_{f,g} = \rho_{\phi_{I_f}, -\phi_{I_g}}$. Combining this with the orthogonality relation in Theorem 2.2 we obtain an alternate proof of Paşol–Popa's identity in level 1, as mentioned in the introduction.

3 | EISENSTEIN SERIES AND SATURATION

Let k and $\mathrm{d} \geqslant 0$ be integers, and let $(\phi_1, \phi_2) \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathrm{d}}(X))^2$ be admissible. In this section, we provide the Eisenstein series of type ρ_{ϕ_1,ϕ_2} and weight k, converging absolutely and locally uniformly on $\mathbb{H}$ for $k > 2 + \mathrm{d}$. We also prove a more detailed version of Theorem C.

3.1 | Eisenstein series

Let $\mathrm{d} \geqslant 0$ and $k > 2 + \mathrm{d}$ be even integers, and let $(\phi_1, \phi_2) \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathrm{d}}(X))^2$ be an admissible pair of parabolic cocycles. Then the weight k Eisenstein series of type ρ_{ϕ_1,ϕ_2} is given by the series

$$E_k(\tau; \phi_1, \phi_2) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)\Big|_{k, \rho_{\phi_1,\phi_2}} \gamma = \begin{pmatrix} \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \psi(\gamma^{-1})(c\tau + d)^{-k} \\ E_k^{[1]}(\tau; \phi_1) \\ E_k \end{pmatrix},$$

where ψ is given as in Theorem 2.2, $E_k(\tau) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} (c\tau + d)^{-k}$ and $E_k^{[1]}(\tau; \phi_1)$ is the weight k generalized second-order Eisenstein series of type $(\mathrm{sym}^{\mathrm{d}}(X), \mathbf{1})$ associated to ϕ_1, given by

$$E_k^{[1]}(\tau; \phi_1) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \frac{\phi_1(\gamma^{-1})}{(c\tau + d)^k}.$$

By [1, Lemma 3.10], we have that $E_k^{[1]}(\tau; \phi_1)$ converges absolutely and locally uniformly for $k > 2 + \mathrm{d}$. Its Fourier series expansion is given in Theorem 3.8 of the same paper. We write

$$E_k^{[2]}(\tau; \phi_1, \phi_2) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \frac{\psi(\gamma^{-1})}{(c\tau + d)^k}.$$

Note that as ϕ_1 and ϕ_2 are parabolic, we have that

$$\psi(\pm T^m \gamma T^n) = \psi(T^m \gamma T^n) = \psi(\gamma T^n) = \psi(\gamma), \tag{3.1}$$

for any $m, n \in \mathbb{Z}$, so that ψ descends to a function on the double quotient $\Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) / \Gamma_\infty$.

We now have the following theorem.

Theorem 3.1. *Let $\mathrm{d} \geqslant 0$ and $k > 2 + \mathrm{d}$ be even integers, let $(\phi_1, \phi_2) \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathrm{d}}(X))^2$ be an admissible pair, and let ψ be given as in Theorem 2.2. Then $E_k^{[2]}(\tau; \phi_1, \phi_2)$ converges absolutely and locally uniformly on $\mathbb{H}$ and has the following Fourier series expansion*

$$E_k^{[2]}(\tau; \phi_1, \phi_2) = \sum_{n \geqslant 1} e(n\tau) \frac{(-2\pi i)^k}{(k-1)!} \sum_{\substack{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) / \Gamma_\infty \\ [\gamma] \neq 1}} \frac{n^{k-1} e(nd/c) \psi(\gamma^{-1})}{c^k}.$$

Proof. Reorganizing the defining series for $E_k^{[2]}(\tau;\phi_1,\phi_2)$, we obtain

$$\sum_{[\gamma]\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} \psi(\gamma^{-1})(c\tau+d)^{-k} = \sum_{[\gamma]\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty} \sum_{m\in\mathbb{Z}} \psi(T^{-m}\gamma^{-1})(c(\tau+m)+d)^{-k}.$$

However, by (3.1) we have that $\psi(T^{-m}\gamma^{-1}) = \psi(\gamma^{-1})$ and thus we obtain the Fourier series expansion by applying Lipschitz' summation formula.

As for convergence, we note that it is enough to show that $|\psi(\gamma^{-1})| \ll_\epsilon |c|^{\mathrm{d}+\epsilon}$ where $[\gamma]\in \Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty$ and $[\gamma]\neq 1$.

To obtain this bound, we use a bijection between $\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty$ and continued fractions. We first have a bijection $s_1 : \Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty \to \mathbb{Q}\cap[0,1)\cup\{\infty\}$ given by $s_1([\gamma]) = d'/c' + \lceil -d'/c' \rceil$ where $(c',d') = \mathrm{sgn}(c,d)\cdot(c,d)$. Let now S be given by

$$S = \big\{(0),()\big\} \cup \big\{(0,\alpha_1,\dots,\alpha_l) \ : \ l\geqslant 1, \alpha_l\geqslant 2, \forall 1\leqslant j<l.\, \alpha_j\geqslant 1\big\}.$$

Then we have a bijection $s_2 : S\to \mathbb{Q}\cap[0,1)\cup\{\infty\}$, given by $s_2(()) = \infty$ and $s_2(0,\alpha_1,\dots,\alpha_l) = [0;\alpha_1,\dots,\alpha_l]$, see [11]. For convenience, we set $\alpha_0 = 0$.

Let $1\neq[\gamma]\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma_\infty$ be arbitrary and let $\alpha = s_2^{-1}(s_1([\gamma]))$. We then have that

$$\begin{aligned}\psi(\gamma^{-1}) &= \psi\Big(T^{-\alpha_0}S^{-1}\cdots T^{(-1)^{l+1}\alpha_l}S^{-1}\Big)\\ &= \psi\Big(S^{2(l+1)}T^{-\alpha_0}S\cdots T^{(-1)^{l+1}\alpha_l}S\Big) = \psi\Big(T^{-\alpha_0}S\cdots T^{(-1)^{l+1}\alpha_l}S\Big).\end{aligned}$$

Let $\beta_i = (-1)^{l-i+1}\alpha_{l-i}$, $\delta_{-1} = I$, and $\delta_i = T^{\beta_i}S\delta_{i-1}$, so that $[\delta_l] = [\gamma^{-1}]$. We then have that

$$\psi(\delta_i) - \psi(\delta_{i-1}) = \psi(S) + \phi_2^\vee(S)\phi_1(\delta_{i-1}).$$

We thus find that

$$\psi(\delta_l) = \sum_{i=0}^{l}(\psi(\delta_i)-\psi(\delta_{i-1})) = \sum_{i=0}^{l}\phi_2^\vee(S)\Big(\phi_1(\delta_{i-1}) - \tfrac{1}{2}\phi_1(S)\Big).$$

Using the bound $\sum_{j=0}^{\mathrm{d}}\binom{d}{j}^{-1} \leqslant 2+4/\mathrm{d}$, which holds for $\mathrm{d}\geqslant 1$, we find that

$$\Big|\phi_2^\vee(S)\Big(\phi_1(\delta_{i-1}) - \tfrac{1}{2}\phi_1(S)\Big)\Big| \leqslant 4\|\phi_2(S)\|_1 \big\|\phi_1(\delta_{i-1}) - \tfrac{1}{2}\phi_1(S)\big\|_1.$$

Hence, we obtain the bound

$$|\psi(\delta_l)| \leqslant 4\|\phi_2(S)\|_1\left(\frac{l+1}{2}\|\phi_1(S)\|_1 + \sum_{i=0}^{l}\|\phi_1(\delta_{i-1})\|_1\right).$$

Lemma 3.10 in [1] tells us that for $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$ with $|d|<|c|$ we have that $\|\phi_1(\gamma^{-1})\|_1 \leqslant C_{\mathrm{d}}|c|^{\mathrm{d}}$, for a constant $C_{\mathrm{d}}\in\mathbb{R}_{>0}$. In our case, we have that

$$|d(\delta_i^{-1})/c(\delta_i^{-1})| = [\alpha_{l-i};\alpha_{l-i-1},\dots,\alpha_l] > 1 \qquad \text{for } i<l,$$

and thus we can apply the bound to $(\delta_i^{-1}S)^{-1}$. We have that $S^{-1}.\phi_1(\delta_i) = \phi_1(S^{-1}.\delta_i) - \phi_1(S)$ and $\|S^{-1}.v\|_1 = \|v\|_1$, and thus we obtain

$$\sum_{i=0}^{l} \|\phi_1(\delta_{i-1})\|_1 \leqslant (l+1)\|\phi_1(S)\|_1 + \sum_{i=0}^{l} \|\phi_1(S^{-1}\delta_{i-1})\|_1 \leqslant (l+1)\|\phi_1(S)\|_1 + C_{\mathsf{d}} \sum_{i=0}^{l} |c(\delta_{i-1}^{-1}S)|^{\mathsf{d}}$$

$$= (l+1)\|\phi_1(S)\|_1 + C_{\mathsf{d}} \sum_{i=0}^{l} |d(\delta_{i-1}^{-1})|^{\mathsf{d}}.$$

However, $|d(\delta_{i-1}^{-1})|$ is increasing in i, whence we obtain that

$$\sum_{i=0}^{l} |d(\delta_{i-1}^{-1})|^{\mathsf{d}} \leqslant (l+1)|d(\delta_{l-1}^{-1})| = (l+1)|c(\gamma)|^{\mathsf{d}}.$$

In conclusion, we obtain that $|\psi(\gamma^{-1})| \leqslant 4\|\phi_1(S)\|_1(l+1)(\frac{3}{2} + C_{\mathsf{d}}|c|^{\mathsf{d}})$. As α corresponds to a continued fraction, we have that $l+1 \ll \log(|c|)$ and thus $|\psi(\gamma^{-1})| \ll_\epsilon |c|^{\mathsf{d}+\epsilon}$ as desired. □

3.2 | Saturation

Let $\mathsf{d} \geqslant 0$ and $k \geqslant 2$ be even integers, let $0 \leqslant j \leqslant \mathsf{d}$ be an integer, and let $(\phi_1, \phi_2) \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))^2$ be admissible.

In [1], the authors of this paper and Ahlbäck introduced the jth Eisenstein series of type $\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathsf{d}}(X)$ and weight k, given by

$$E_k\Big(\tau; \mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathsf{d}}(X), j\Big) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \big(0\$X-\tau)^j \big)\Big|_{k, \mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathsf{d}}(X)} \gamma, \quad \tau \in \mathbb{H}.$$

Let $k_0 > 2 + \mathsf{d}$ be an integer. Then the $\mathrm{M}_\bullet$-module of Eisenstein series of type $\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathsf{d}}(X)$ is given by

$$\mathrm{E}_{\geqslant k_0}\Big(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathsf{d}}(X)\Big) = \mathrm{span}\, \mathrm{M}_\bullet \Big\{E_k\Big(\tau; \mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathsf{d}}(X), j\Big) \;:\; k \geqslant k_0, 0 \leqslant j \leqslant \mathsf{d}\Big\}.$$

Theorem 4.3 of [1] implies that for $\mathsf{d} \geqslant 0$ and $k_0 > 2 + \mathsf{d}$ it holds that

$$\Big(\mathrm{E}_{\geqslant k_0}\Big(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathsf{d}}(X)\Big) + \iota(\mathrm{M}_\bullet) : \Delta^\infty\Big) = \mathrm{M}_\bullet\Big(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathsf{d}}(X)\Big), \tag{3.2}$$

where $\iota(f) = (f, 0)^T$. In Remark 4.4 of the same paper, we mention that the proof can be generalized to higher depths by induction. Following this approach, we obtain our last theorem.

Theorem 3.2. *Let* $\mathsf{d} \geqslant 0$ *be an even integer and* $k_0 > 2 + \mathsf{d}$ *be an integer, and* $(\phi_1, \phi_2) \in \mathrm{Z}^1_{\mathrm{pb}}(\mathbf{1}, \mathrm{sym}^{\mathsf{d}}(X))^2$ *be admissible. Then it holds that*

$$\mathrm{M}_\bullet(\rho_{\phi_1,\phi_2}) = \Big(\mathrm{E}_{\geqslant k_0}(\rho_{\phi_1,\phi_2}) + \iota_1\Big(\mathrm{E}_{\geqslant k_0}(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathsf{d}}(X))\Big) + \iota_2(\mathrm{M}_\bullet) : \Delta^\infty\Big),$$

where $\iota_1(f, g) = (f, g, 0)^T$ *and* $\iota_2(f) = (f, 0, 0)^T$.

Proof. Consider the following diagram

$$\begin{tikzcd}
\mathrm{M}_\bullet(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathrm{d}}(X)) \arrow[r, hook, "\mu_1"] \arrow[d, "="] & (\mathrm{E}_{\geq k_0}(\rho_{\phi_1,\phi_2}) + \mu_1(\mathrm{M}_\bullet(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathrm{d}}(X))) :\, \Delta^\infty) \arrow[r, "\nu_1"] \arrow[d, hook, "f \mapsto f"] & (\mathrm{E}_{\geq k_0} :\, \Delta^\infty) \arrow[d, "="] \\
\mathrm{M}_\bullet(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathrm{d}}(X)) \arrow[r, hook, "\mu_2"] & \mathrm{M}_\bullet(\rho_{\phi_1,\phi_2}) \arrow[r, "\nu_2"] & \mathrm{M}_\bullet
\end{tikzcd}$$

where $\mu_i(f, g) = (f, g, 0)^T$ and $\nu_i(f, g, h) = h$ for $i \in \{1, 2\}$. It is clear that the rows are exact and that the diagram commutes. We have that $\mathrm{im}(\nu_1) \subseteq \mathrm{im}(\nu_2)$ and thus the map $\theta : \mathrm{coker}(\nu_1) \to \mathrm{coker}(\nu_2)$ given by $\theta(h + \mathrm{im}(\nu_1)) = h + \mathrm{im}(\nu_2)$ is well-defined. As $E_k = \nu_1(E_k(\cdot; \phi_1, \phi_2))$, it is also injective. The Four Lemma now implies that the vertical map in the middle column is surjective and thus an equality. Finally, we conclude the proof by applying (3.2) and observing that

$$\begin{aligned}
&\left(\mathrm{E}_{\geqslant k_0}(\rho_{\phi_1,\phi_2}) + \mu_1\Big((\mathrm{E}_{\geqslant k_0}(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathrm{d}}(X)) + \iota(\mathrm{M}_\bullet) :\, \Delta^\infty)\Big) :\, \Delta^\infty\right) \\
&= \left(\mathrm{E}_{\geqslant k_0}(\rho_{\phi_1,\phi_2}) + \iota_1\Big(\mathrm{E}_{\geqslant k_0}(\mathbf{1} \boxplus_{\phi_2^\vee} \mathrm{sym}^{\mathrm{d}}(X))\Big) + \iota_2(\mathrm{M}_\bullet) :\, \Delta^\infty\right).
\end{aligned}$$

□

ACKNOWLEDGMENTS

The author was partially supported by Vetenskapsrådet Grant 2019-03551.

JOURNAL INFORMATION

The *Transactions of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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