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ON THE ASYMPTOTIC BEHAVIOUR OF SUPEREXPONENTIAL LÉVY PROCESSES

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Abstract. We study tail probabilities of superexponential infinite divisible distributions as well as tail probabilities of suprema of Lévy processes with superexponential marginal distributions over compact intervals.

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1. Introduction

By a Lévy process we mean a stochastically continuous process $\xi = \{\xi(t)\}_{t\geq 0}$ starting at $\xi(0) = 0$, that has stationary and independent increments. Writing $\kappa(x) = x/(1 \vee |x|)$ for $x \in \mathbb{R}$, the finite dimensional distributions of a Lévy process are fully determined by its so called characteristic triplet (ν, m, s^2) through the relation

$$\mathbf{E}\{\mathrm{e}^{i\theta\xi(t)}\} = \exp\left\{it\theta m + t\int_{\mathbb{R}} \left(\mathrm{e}^{i\theta x} - 1 - i\theta\kappa(x)\right) \mathrm{d}\nu(x) - \frac{t\theta^2 s^2}{2}\right\} \quad \text{for } \theta \in \mathbb{R} \text{ and } t \geq 0. \tag{1.1}$$

Here $m \in \mathbb{R}$ and $s^2 \geq 0$ are constants while ν is the so called Lévy measure on $\mathbb{R}$ that satisfies $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|^2)\, \mathrm{d}\nu(x) < \infty$.

We call a Lévy process ξ superexponential if $\mathbf{E}\{\mathrm{e}^{\alpha\xi(1)}\} < \infty$ for $\alpha \geq 0$. It follows from Sato [29], Theorem 25.17, that ξ is superexponential if and only if it has a well-defined Laplace transform

$$\phi_t(\lambda) = \mathbf{E}\{\mathrm{e}^{-\lambda\xi(t)}\} = \phi_1(\lambda)^t < \infty \quad \text{for } \lambda \leq 0 \text{ and } t > 0. \tag{1.2}$$

Also, according to Sato [29], Theorem 25.17, (1.2) is equivalent to

$$\int_{|x|>1} \mathrm{e}^{-\lambda x} \mathrm{d}\nu(x) < \infty \quad \text{for } \lambda \leq 0.$$

Hence, in order for a process to be superexponential its Lévy measure has to decay more that exponentially fast approaching $+\infty$.

Keywords and phrases: Extreme value theory, infinitely divisible distributions, Lévy processes, superexponential distributions.

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In Theorems 2.2 and 2.12 of Section 2 we find the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ for large u in closed form. Besides being of interest on their own in, *e.g.*, extreme value theory, these findings are also crucially used for the proof of our main result (1.5) below.

As infinitely divisible distributions are built up as sums of independent identically distributed increments of Lévy processes, not surprisingly, it turns out that investigation of the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ for large u is directly linked to how big are increments of Lévy processes over infinitesimal time intervals. So as a biproduct of our results on tails for superexponential infinitely divisible distributions, under appropriate technical conditions, for a superexponential Lévy process ξ, in Theorems 2.11 and 2.12 below, we show that there exist functions $q, w : (0,\infty) \to (0,\infty)$ with $q(u), w(u) \to 0$ as $u \to \infty$ such that

$$\frac{\xi(aq(u))}{w(u)} \xrightarrow{\mathrm{d}} \zeta(a) \quad \text{as } u \to \infty \text{ for } a > 0 \tag{1.3}$$

for some random variables $\{\zeta(a)\}_{a>0}$. This condition controls the size of increments over small time intervals of the Lévy process. Further, in Theorems 2.2 and 2.12 below, we show that the limit

$$L(t,x) = \lim_{u\to\infty} \frac{\mathbf{P}\{\xi(h - q(u)t) > u + xw(u)\}}{\mathbf{P}\{\xi(h) > u\}} \quad \text{exists for } t \geq 0 \text{ and } x \in \mathbb{R}. \tag{1.4}$$

This condition is a version of belongingness to the so called Type I domain of attraction of extremes, see, *e.g.*, [22], Chapter 1. The verification of these two results use Tauberian techniques developed for this purpose.

In Section 3 we do some preparatory investigations of the behaviour of suprema of Lévy processes.

In Section 4, under a technical conditions on the superexponential Lévy process ξ, we establish the existence of a constant $H \geq 1$ such that

$$\lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\} = H \quad \text{for } h > 0. \tag{1.5}$$

This completes our findings in [3] where the tail behaviour of suprema of so called subexponential and exponential Lévy processes were studied. The constant H given by equation 4.8 below cannot be calculated explicitly in general. This is similar to what is the situation with the famous Pickands' constant H_α from extreme value theory of stationary Gaussian processes, the definition of which somewhat resembles that of H (when done as in Albin and Choi [4]). However, in some setups it can be seen that, for example, $H = 1$ from a certain degeneracy of weak limits ingredients. See the examples in Section 5 for more on this.

As we know the behaviour of $\mathbf{P}\{\xi(h) > u\}$ for large u in more or less closed form from Section 2, this means that the asymptotic behaviour of $\mathbf{P}\{\sup_{t\in[0,h]} \xi(t) > u\}$ is fully understood.

In the concluding Section 5 we consider six examples of usage of (1.5):

1. Brownian motion with drift for which $H = 2$ in (1.5).
2. Merton's jump diffusion for which $H = 1$ in (1.5).
3. Rapidly decreasing tempered stable Lévy processes for which $H = 1$ in (1.5).
4. Totally skewed to the left α-stable Lévy processes with $\alpha \in (1,2)$ for which $H > 1$ in (1.5).
5. Totally skewed to the left 1-stable Lévy processes for which $H = 1$ in (1.5).
6. An unnamed superexponential Lévy process defined by Linnik and Ostrovskiĭ [23] for which $H = 1$ in (1.5).

At a first reading some readers might want to skip Section 4 and go directly to the examples in Section 5.

Extreme value theory origins in the search for nondegenerate limit laws for $(\max_{1\leq i\leq n} X_i - a_n)/b_n$ as $n \to \infty$ when $\{X_i\}_{i=1}^\infty$ are iid. random variables and $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are suitable normalizing sequences. The solution to this problem is the so called extremal types theorem from around the middle of the previous century, see, *e.g.*, [22], Chapter 1. After this extensions were considered to stationary and other sequences $\{X_i\}_{i=1}^\infty$, see, *e.g.*, [22], Chapter 3–6. Subsequently interest moved to maxima of continuous time random processes, see, *e.g.*, [9] and [22], Chapters 7–13. However, arguably, there exists no really up to date comprehensive coverage of this latter topic. As for applications of extreme value theory, there are some material on this in [22], Chapters 14–15. Basically, in many applications of random processes, one of the most interesting problems is that of extreme values. This can be in risk analysis in mathematical finance or for construction applications, etc. And it is there extreme value theory provides the theoretical framework.

We now state our main result on extremes of superexponential Lévy processes. See Albin and Sundén [3] on bibliographic information for results on this type. In addition, a few relevant references not mentioned there are Albin [1] and Braverman [11–13]. In particular Braverman have studied related problems in many papers but with completely different techniques than ours. For exemple, [13] is an extensive treatment of light tailed Lévy processes made up of a finite variation process with (possibly) an added Brownian motion component.

Recall that the right end-point $\sup\{x\in\mathbb{R}:\mathbf{P}\{\xi(t)>x\}>0\}$ of a Lévy process ξ is infinite for some $t>0$ if and only if

$$\sup\left\{x\in\mathbb{R}:\mathbf{P}\{\xi(t)>x\}>0\right\}=\infty\quad\text{for each }\ t>0 \tag{1.6}$$

(see *e.g.*, Sato [29], Thm. 24.7). By inspection of, *e.g.*, Sato [29], Definition 11.9 and Theorem 24.7, (1.6) is same thing as

$$\int_{-1}^{0}(-x)\mathrm{d}\nu(x)=\infty\quad\text{or}\quad\nu((0,\infty))>0\quad\text{or}\quad s^2>0. \tag{1.7}$$

Theorem 1.1. *Let ξ be a separable superexponential Lévy process with infinite upper end-point (1.6). Assume that there exist functions $w>0$ and $q>0$ with w continuous and random variables $\{\zeta(a)\}_{a>0}$ such that (1.3) and (1.4) hold with $L(0,x)=\mathrm{e}^{-x}$. Further, assume that $\zeta(a)$ is continuously distributed for $a>0$, or that $L(t,\cdot)$ is a continuous function for $t>0$. If*

$$\lim_{T\to\infty}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\left\{\sup_{t\in[0,h-Tq(u)]}\xi(t)>u\right\}=0, \tag{1.8}$$

then the limit (1.5) exists with value $H\in[1,\infty)$.

The proof of Theorem 1.1 constitutes Section 4.

Sufficient conditions on the characteristic triplet for the conditions (1.3) and (1.4) to hold are established in Theorems 2.2, 2.11 and 2.12 together with Propositions 2.8, 2.9 and 2.10 below.

The constant H in (1.5) is a rather complicated functional of the quantities ζ and L, see the proof of Theorem 1.1 in Section 4 for more information. It seems, in general, that H cannot be calculated in closed form. However, as we will see below, in some cases we encounter H really can be calculated. Also, in other cases, qualitative information such as whether $H>1$ or not can be established.

Equation 1.8 means that the appropriate time scale of the Lévy process $\{\xi(t)\}_{t\in[0,h]}$ when it takes a large value u (always close to the terminal point h) is $q(u)$. This can be said to govern the whole analysis of the probability for such large values. Some conditions to check 1.8 are given in Section 3, especially in Proposition 3.4.

2. Tail probabilities of superexponential processes

We start with the notion of Type I domain of attraction of extremes:

Definition 2.1. A random variable X belongs to the *Type I domain of attraction* of extremes, with *auxiliary function* $w(u) > 0$, if

$$\lim_{u\to\infty} \frac{\mathbf{P}\{X > u + xw(u)\}}{\mathbf{P}\{X > u\}} = \mathrm{e}^{-x} \quad \text{for } x \in \mathbb{R}.$$

The auxiliary function in Definition 2.1 satisfies $\lim_{u\to\infty} w(u)/u = 0$ and can be chosen to be continuous (see *e.g.*, Bingham, Goldie and Teugels [10], Lem. 3.10.1 and Cor 3.10.9). Further, $\tilde{w}$ is another auxiliary for X function if and only if $\lim_{u\to\infty} \tilde{w}(u)/w(u) = 1$.

Feigin and Yashchin [17], Theorems 2 and 3, give a scheme to deduce the asymptotics of the right tail of a probability distribution function from the left tail of its Laplace transform. The usefulness of this to establish Type I attraction was noted in a particular case by Davis and Resnick, [14], Section 3, see also Rootzén [26, 27]. Balkema, Klüppelberg and Resnick [5–7] and Balkema, Klüppelberg and Stadtmüller [8] characterized convergence of the Esscher transforms (exponential families), which are the key ingredient of proofs in this area. But they impose conditions on densities that we are not comfortable with. And it is not that convergence which is our goal, but to find the actual tail behaviour and to show Type I attraction. In fact, we deal with random variables, the distribution of which depends on how far out we are in the tail (an "external parameter"). This makes the existing literature non-applicable anyway.

For a Lévy process ξ with Laplace transform (1.2), we introduce the following notation:

$$\begin{cases} \mu(\lambda) &= -\dfrac{\phi_1'(\lambda)}{\phi_1(\lambda)} = \displaystyle\int_{\mathbb{R}} \big(x\mathrm{e}^{-\lambda x} - \kappa(x)\big)\,\mathrm{d}\nu(x) + m - \lambda s^2 & \text{for } \lambda \le 0, \\ \sigma(\lambda)^2 &= \quad -\mu'(\lambda) \quad = \qquad \displaystyle\int_{\mathbb{R}} x^2 \mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x) + s^2 & \text{for } \lambda \le 0, \\ \mu^{\leftarrow}(u) &= \quad \inf\{\lambda \in \mathbb{R} : \mu(\lambda) \le u\} & \text{for } u > 0 \text{ large enough.} \end{cases} \tag{2.1}$$

The so called Esscher transform of $\xi(t)$ is defined to be a random variable $X_{t,\lambda}$ having probability distribution

$$dF_{X_{t,\lambda}}(x) = \frac{\mathrm{e}^{-\lambda x} dF_{\xi(t)}(x)}{\phi_t(\lambda)}, \tag{2.2}$$

where $F_{\xi(t)}$ denotes the cumulative probability distribution function of $\xi(t)$. It is easy to see that $t\mu(\lambda)$ and $t\sigma(\lambda)^2$ are the mean and variance of $X_{t,\lambda}$, respectively.

2.1. The asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ as $u \to \infty$.

The following Theorem 2.2 is a development of a scheme of Feigin and Yashchin [17], and Davis and Resnick [14], with additional input from Albin [2], to establish Type I attraction for infinitely divisible probability distributions. The sufficient conditions of Theorem 2.2 are rather involved but may be verified from properties of the characteristic triple by means of Propositions 2.8, 2.9 and 2.10 below.

Theorem 2.2. *Let ξ be a superexponential Lévy process with characteristic triplet (ν, m, s^2) and infinite upper end-point (1.6). With the notation (2.1), assume that*

$$\lim_{\lambda\to-\infty} \lambda^2 \sigma(\lambda)^2 = \infty, \tag{2.3}$$

$$\lim_{\lambda\to-\infty} \int_{|x|>\varepsilon\sigma(\lambda)} \frac{x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x) = 0 \quad \textit{for } \varepsilon > 0 \tag{2.4}$$

and

$$\lim_{K\to\infty}\limsup_{\lambda\to-\infty}\int_{|\theta|>K}\exp\left\{-t\left[\int_{\mathbb{R}}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)+\frac{\theta^2 s^2}{2\sigma(\lambda)^2}\right]\right\}\mathrm{d}\theta=0 \tag{2.5}$$

for t in a neighborhood of $h>0$. Further assume that the following limit exists

$$\lim_{\lambda\to-\infty}\frac{\lambda\mu(\lambda)}{\lambda\mu(\lambda)+\ln(\phi_1(\lambda))}=L \tag{2.6}$$

With the notation

$$w(u)=-\frac{1}{\mu^{\leftarrow}(u/h)}\qquad\text{and}\qquad q(u)=\frac{1}{\ln(\phi_1(\mu^{\leftarrow}(u/h)))}, \tag{2.7}$$

we have $\lim_{u\to\infty} q(u)/w(u)=0$,

$$\mathbf{P}\{\xi(h)>u\}\sim\frac{\mathrm{e}^{u\mu^{\leftarrow}(u/h)}\phi_1(\mu^{\leftarrow}(u/h))^h}{\sqrt{2\pi h}\,\sigma(\mu^{\leftarrow}(u/h))(-\mu^{\leftarrow}(u/h))}\quad\text{as } u\to\infty \tag{2.8}$$

as well as

$$\lim_{u\to\infty}\frac{\mathbf{P}\{\xi(h-q(u)t)>u+xw(u)\}}{\mathbf{P}\{\xi(h)>u\}}=\mathrm{e}^{-t-x}\quad\text{for } x\in\mathbb{R}\text{ and } t\geq 0. \tag{2.9}$$

Proof. Let

$$Q(\lambda)\equiv\frac{1}{\ln(\phi_1(\lambda))+\lambda\mu(\lambda)}=\left(\int_{\mathbb{R}}\left(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\right)\mathrm{d}\nu(x)-m\lambda+\frac{\lambda^2 s^2}{2}+\lambda\mu(\lambda)\right)^{-1}$$

and let $Z_{t,\lambda}$ be the Esscher transform of $\xi(h-Q(\lambda)t)$ given by

$$dF_{Z_{t,\lambda}}(x)=\frac{\mathrm{e}^{-\lambda x}dF_{\xi(h-Q(\lambda)t)}(x)}{\phi_{h-Q(\lambda)t}(\lambda)}\quad\text{for } x\in\mathbb{R}\text{ and } t>0,$$

for $\lambda\leq 0$ sufficiently small [recall (2.2)].

Our first aim is to establish asymptotic normality of a normalized Esscher transform $(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)$. Here $\mu_{t,\lambda}$ is the expected value of $Z_{t,\lambda}$ and $\sigma(\lambda)$ is the standard deviation of $Z_{0,\lambda}$ for $h=1$, respectively. From (1.7) we readily see that the function μ satisfies

$$\mu(\lambda)=\int_{-\infty}^{0}\left(\mathrm{e}^{-\lambda x}x-\kappa(x)\right)\mathrm{d}\nu(x)+\int_{0}^{\infty}\left(\mathrm{e}^{-\lambda x}x-\kappa(x)\right)\mathrm{d}\nu(x)+m+(-\lambda)s^2\to\infty \tag{2.10}$$

as $\lambda\to-\infty$ [note that all terms on the right of the equality in (2.10) are non-negative]. Further, observe that $Q(\lambda)$ satisfies

$$Q(\lambda)>0\quad\text{for }\lambda\text{ sufficiently small, with}\quad\lim_{\lambda\to-\infty}Q(\lambda)=0: \tag{2.11}$$

This follows readily when $\nu((0,\infty))>0$ or $s^2=0$ [recall (1.7)] using that

$$\int_{-1}^{0}\big(\mathrm{e}^{-\lambda x}-1+\lambda x\big)\,\mathrm{d}\nu(x)=o(\lambda^2) \tag{2.12}$$

as $\lambda\to-\infty$. This in turn is so since (integrating by parts)

$$\begin{aligned}&\int_{-1}^{0}\big(\mathrm{e}^{-\lambda x}-1+\lambda x\big)\mathrm{d}\nu(x)\\ &\quad=\bigg[\frac{\mathrm{e}^{-\lambda x}-1+\lambda x}{x^2}\int_{-1}^{x}y^2\mathrm{d}\nu(y)\bigg]_{-1}^{0}-\lambda^2\int_{0}^{-\lambda}\frac{x\,\mathrm{e}^{-x}+x+2\mathrm{e}^{-x}-2}{x^3}\bigg(\int_{-1}^{x/\lambda}y^2\mathrm{d}\nu(y)\bigg)\mathrm{d}x\\ &\quad\sim\lambda^2\int_{-1}^{0}y^2\mathrm{d}\nu(y)\bigg(\frac{1}{2}-\int_{0}^{-\lambda}\frac{x\,\mathrm{e}^{-x}+x+2\mathrm{e}^{-x}-2}{x^3}\mathrm{d}x\bigg),\end{aligned} \tag{2.13}$$

where the inner integral on the right-hand side converges to $1/2$ as $\lambda\to-\infty$, so that the whole expression under consideration is $o(\lambda^2)$ as $\lambda\to-\infty$, as required. If instead $\nu((0,\infty))=s^2=0$, then (2.11) holds since (1.7) ensures that

$$\lim_{\lambda\to-\infty}\frac{1}{|\lambda|}\int_{-\infty}^{0}\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,\mathrm{d}\nu(x)\geq\lim_{\lambda\to-\infty}\frac{1}{2}\int_{-\infty}^{2/\lambda}-\kappa(x)\mathrm{d}\nu(x)=\infty.$$

As a final preparation we observe that

$$\lim_{\lambda\to-\infty}\int_{\mathbb{R}}\bigg(\frac{\theta x}{\sigma(\lambda)}-\sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\bigg)\,\mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x)=0\quad\text{for }\ \theta\in\mathbb{R}. \tag{2.14}$$

This is so because (2.4) gives

$$\limsup_{\lambda\to-\infty}\int_{|x|>\varepsilon\sigma(\lambda)}\bigg|\frac{\theta x}{\sigma(\lambda)}-\sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\bigg|\mathrm{e}^{-\lambda x}\,d\nu(x)\leq\limsup_{\lambda\to-\infty}\int_{|x|>\varepsilon\sigma(\lambda)}\frac{2|\theta|\,x^2}{\varepsilon\,\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x)=0$$

for $\varepsilon>0$, while by Taylor expansion, given any $\delta>0$ and for $\varepsilon=\varepsilon(\theta)>0$ sufficiently small,

$$\limsup_{\lambda\to-\infty}\int_{|x|\leq\varepsilon\sigma(\lambda)}\bigg|\frac{\theta x}{\sigma(\lambda)}-\sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\bigg|\mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x)\leq\limsup_{\lambda\to-\infty}\int_{|x|\leq\varepsilon\sigma(\lambda)}\frac{\delta\theta^2x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x)\leq\delta\theta^2.$$

Notice that, writing

$$\big(\mathrm{d}\nu_{t,\lambda}(x),m_{t,\lambda},s_{t,\lambda}^2\big)=(h-Q(\lambda)t)\bigg(\mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x),\,m-\int_{\mathbb{R}}\kappa(x)\,(1-\mathrm{e}^{-\lambda x})\,\mathrm{d}\nu(x)-\lambda s^2,\,s^2\bigg),$$

the random variable $Z_{t,\lambda}$ has characteristic function

$$\mathbf{E}\big\{\mathrm{e}^{i\theta Z_{t,\lambda}}\big\}=\exp\bigg\{i\theta m_{t,\lambda}+\int_{\mathbb{R}}\big(\mathrm{e}^{i\theta x}-1-i\theta\kappa(x)\big)\,\mathrm{d}\nu_{t,\lambda}(x)-\frac{\theta^2s_{t,\lambda}^2}{2}\bigg\}$$

for $\theta \in \mathbb{R}$ and $t > 0$, for $\lambda \le 0$ sufficiently small. Hence the random variable $Z_{t,\lambda}$ is infinitely divisible with characteristic triplet $(\nu_{t,\lambda}, m_{t,\lambda}, s_{t,\lambda}^2)$. Observing that

$$\mathbf{E}\{Z_{t,\lambda}\} = (h - Q(\lambda)t)\mu(\lambda) \equiv \mu_{t,\lambda}$$

it follows that

$$\mathbf{E}\{\mathrm{e}^{i\theta Z_{t,\lambda}}\} = \exp\bigg\{ i\theta\mu_{t,\lambda} + \int_{\mathbb{R}} (\mathrm{e}^{i\theta x} - 1 - i\theta x)\mathrm{d}\nu_{t,\lambda}(x) - \frac{\theta^2 s_{t,\lambda}^2}{2} \bigg\}$$

(see *e.g.* Sato [29], p. 39). Hence the characteristic function $g_{t,\lambda}$ of $(Z_{t,\lambda} - \mu_{t,\lambda})/\sigma(\lambda)$ is given by

$$\begin{aligned} g_{t,\lambda}(\theta) = \bigg(\exp\bigg\{ &- \int_{\mathbb{R}} \bigg(1 - \cos\Big(\frac{\theta x}{\sigma(\lambda)} \Big) \bigg) \mathrm{e}^{-\lambda x} \mathrm{d}\nu(x) \\ &- i \int_{\mathbb{R}} \bigg(\frac{\theta x}{\sigma(\lambda)} - \sin\Big(\frac{\theta x}{\sigma(\lambda)} \Big) \bigg) \mathrm{e}^{-\lambda x} \mathrm{d}\nu(x) - \frac{\theta^2 s^2}{2\sigma(\lambda)^2} \bigg\} \bigg)^{h - Q(\lambda)t}. \end{aligned}$$

Here (2.4) and (2.14) together with (2.11) and a Taylor expansion readily give $\lim_{\lambda\to-\infty} g_{t,\lambda}(\theta) = \mathrm{e}^{-h\theta^2/2}$ for $\theta \in \mathbb{R}$ and $t > 0$. Since $1 - \cos(x) \ge x^2/4$ for $|x| \le 1$ we further have

$$\begin{aligned} \int_{\mathbb{R}} |g_{t,\lambda}(\theta)|\,\mathrm{d}\theta &= \int_{\mathbb{R}} \exp\bigg\{ -(h - Q(\lambda)t) \int_{\mathbb{R}} \bigg(1 - \cos\Big(\frac{\theta x}{\sigma(\lambda)} \Big) \bigg) \mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x) \bigg\} \mathrm{d}\theta \\ &\le \int_{|\theta| > K} \exp\bigg\{ -(h - Q(\lambda)t) \int_{\mathbb{R}} \bigg(1 - \cos\Big(\frac{\theta x}{\sigma(\lambda)} \Big) \bigg) \mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x) \bigg\} \mathrm{d}\theta \\ &\quad + \int_{|\theta| \le K} \exp\bigg\{ -(h - Q(\lambda)t)\, \frac{\theta^2}{4\sigma(\lambda)^2} \int_{|x| \le \sigma(\lambda)/K} x^2 \mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x) \bigg\} \mathrm{d}\theta. \end{aligned}$$

Here the first term on the right-hand side can be made arbitrarily small as $\lambda \to -\infty$ and $K \to \infty$ (on that order) using (2.5). For the second term on the right-hand side, (2.4) and (2.11) show that there exists a constant $\delta = \delta(K) \in (0, 1)$ such that

$$\begin{aligned} &\int_{|\theta| \le K} \exp\bigg\{ -(h - Q(\lambda)t)\, \frac{\theta^2}{4\sigma(\lambda)^2} \int_{|x| \le \sigma(\lambda)/K} x^2 \mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x) \bigg\} \mathrm{d}\theta \\ &\quad \le \int_{|\theta| \le K} \exp\bigg\{ -h(1 - \delta)\, \frac{\theta^2}{4} \bigg\} \mathrm{d}\theta \quad \text{for } \lambda \text{ small enough.} \end{aligned}$$

The integrability of $|g_{t,\lambda}|$ established in the previous paragraph together with the Riemann-Lebesgue lemma show that $(Z_{t,\lambda} - \mu_{t,\lambda})/\sigma(\lambda)$ has a well-defined continuous probability density function $f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}$ for λ small enough. Furthermore, using (2.5) again, we readily see that

$$\begin{aligned} &\limsup_{\lambda\to-\infty} \sup_{x\in\mathbb{R}} \bigg| f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(x) - \frac{1}{\sqrt{2\pi h}}\, \mathrm{e}^{-x^2/(2h)} \bigg| \\ &\le \limsup_{K\to\infty} \limsup_{\lambda\to-\infty} \bigg(\int_{|\theta| \le K} \big| g_{t,\lambda}(\theta) - \mathrm{e}^{-h\theta^2/2} \big|\,\mathrm{d}\theta + \int_{|\theta| > K} \big(|g_{t,\lambda}(\theta)| + \mathrm{e}^{-h\theta^2/2} \big)\,\mathrm{d}\theta \bigg) \\ &= 0. \end{aligned} \tag{2.15}$$

Observing that

$$f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(x) = \frac{\mathrm{e}^{-\lambda(\sigma(\lambda)x+\mu_{t,\lambda})} f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda}+\sigma(\lambda)x)\sigma(\lambda)}{\phi_{h-Q(\lambda)t}(\lambda)}$$

for $x \in \mathbb{R}$ and $t \geq 0$. Hence (2.10) together with (2.3) and (2.15) show that

$$\begin{aligned} f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda}+x/\lambda) &= \mathrm{e}^x \frac{f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(x/(\lambda\sigma(\lambda)))\mathrm{e}^{\lambda\mu_{t,\lambda}}\phi_{h-Q(\lambda)t}(\lambda)}{\sigma(\lambda)} \\ &\sim \mathrm{e}^x \frac{\mathrm{e}^{\lambda\mu_{t,\lambda}}\phi_1(\lambda)^{h-Q(\lambda)t}}{\sqrt{2\pi h}\,\sigma(\lambda)} \\ &\sim \mathrm{e}^{x-t} \frac{\mathrm{e}^{h\lambda\mu(\lambda)}\phi_1(\lambda)^h}{\sqrt{2\pi h}\,\sigma(\lambda)} \quad \text{as } \lambda \to -\infty. \end{aligned} \tag{2.16}$$

We are now prepared to establish (2.9): By the asymptotics (2.16) of $f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda}+x/\lambda)$ together with application of (2.3) and (2.15), we get

$$\begin{aligned} \lim_{\lambda\to-\infty} \frac{\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda} - y/\lambda\}}{-\lambda\, f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda}-x/\lambda)} &= \mathrm{e}^x \lim_{\lambda\to-\infty} \int_y^\infty \frac{f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda}-z/\lambda)}{f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda})}\,\mathrm{d}z \\ &= \mathrm{e}^x \lim_{\lambda\to-\infty} \int_y^\infty \mathrm{e}^{-z} \frac{f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(-z/(\lambda\sigma(\lambda)))}{f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(0)}\,\mathrm{d}z \\ &= \mathrm{e}^{x-y} \quad \text{for } x,y \in \mathbb{R}. \end{aligned} \tag{2.17}$$

Observing that

$$\frac{-\lambda\, f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda}+(Lt-y)/\lambda)}{\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda}+Lt/\lambda\}} \quad y \geq 0, \tag{2.18}$$

is a probability density function, (2.17) and the theorem of Scheffé [30] show that

$$\begin{aligned} &\lim_{\lambda\to-\infty} \frac{\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda}+(Lt-x)/\lambda\}}{\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda}+Lt/\lambda\}} \\ &\qquad = \lim_{\lambda\to-\infty} \int_x^\infty \frac{-\lambda\, f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda}+(Lt-y)/\lambda)}{\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda}+Lt/\lambda\}}\,\mathrm{d}y \\ &\qquad = \int_x^\infty \mathrm{e}^{-y}\mathrm{d}y = \mathrm{e}^{-x} \quad \text{for } x \geq 0. \end{aligned} \tag{2.19}$$

Using this in turn, together with (2.16) and (2.17), we readily obtain

$$\lim_{\lambda\to-\infty} \frac{\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda}+(Lt-x)/\lambda\}}{\mathbf{P}\{\xi(h) > \mu_{t,\lambda}+Lt/\lambda\}} = \mathrm{e}^{-x-t} \quad \text{for } x,t \geq 0.$$

As (2.6) shows that, given any $\varepsilon > 0$,

$$\mu_{t,\lambda} + \frac{Lt+\varepsilon}{\lambda} \leq h\mu(\lambda) \leq \mu_{t,\lambda} + \frac{Lt-\varepsilon}{\lambda} \quad \text{for } \lambda \text{ small enough},$$

we may now conclude that

$$\begin{aligned}
&\limsup_{\lambda\to-\infty}\frac{\mathbf{P}\{\xi(h-Q(\lambda)t)>h\mu(\lambda)-x/\lambda\}}{\mathbf{P}\{\xi(h)>h\mu(\lambda)\}}\\
&\le\limsup_{\lambda\to-\infty}\frac{\mathbf{P}\{\xi(h-Q(\lambda)t)>\mu_{t,\lambda}+(Lt+\varepsilon-x)/\lambda\}}{\mathbf{P}\{\xi(h)>\mu_{t,\lambda}+(Lt-\varepsilon)/\lambda\}}\\
&=\limsup_{\lambda\to-\infty}\frac{\mathbf{P}\{\xi(h-Q(\lambda)t)>\mu_{t,\lambda}+(Lt+\varepsilon-x)/\lambda\}}{\mathbf{P}\{\xi(h)>\mu_{t,\lambda}+Lt/\lambda\}}\,\frac{\mathbf{P}\{\xi(h)>\mu_{t,\lambda}+Lt/\lambda\}}{\mathbf{P}\{\xi(h)>\mu_{t,\lambda}+(Lt-\varepsilon)/\lambda\}}\\
&=\mathrm{e}^{2\varepsilon-x-t}\\
&\to\mathrm{e}^{-x-t}\quad\text{as }\ \varepsilon\downarrow 0.
\end{aligned}$$

Treating the corresponding $\liminf$ in an entirely similar fashion, it follows that

$$\lim_{\lambda\to-\infty}\frac{\mathbf{P}\{\xi(h-Q(\lambda)t)>h\mu(\lambda)-x/\lambda\}}{\mathbf{P}\{\xi(h)>h\mu(\lambda)\}}=\mathrm{e}^{-x-t}\quad\text{for }\ x,t\ge 0. \tag{2.20}$$

As μ is continuous and eventually strictly decreasing [by (2.3)], with $\mu(\lambda)\to\infty$ if and only if $\lambda\to-\infty$, we may substitute $\lambda=\mu^{\leftarrow}(u)$ in (2.20), to obtain

$$\lim_{u\to\infty}\frac{\mathbf{P}\{\xi(h-q(hu)t)>hu+xw(hu)\}}{\mathbf{P}\{\xi(h)>hu\}}=\mathrm{e}^{-x-t}\quad\text{for }\ x,t\ge 0. \tag{2.21}$$

From (2.21) it is a simple matter to establish (2.9) in full generality with $x\in\mathbb{R}$ rather for $x\ge 0$ only. Further, the asymptotics (2.8) follow from inspection of (2.16) and (2.17). Finally, by inspection of (2.7), the limit $\lim_{u\to\infty}q(u)/w(u)=0$ holds if $\lim_{\lambda\to-\infty}\lambda/\ln(\phi_1(\lambda))=0$. However, this latter limit holds by the arguments we use to establish (2.11). This finishes the proof of all claims of the theorem. ☐

Remark 2.3. For $\hat{h}\in(0,h)$ it is possible, with extra work, to prove a version of Theorem 2.2 where (2.9) holds uniformly (in an obvious sense) for $t\in[0,(h-\hat{h})/q(u)]$. As we do not need this extension, we do not elaborate on it.

To check all the technical conditions of Theorem 2.2 we provide Propositions 2.8, 2.9 and 2.10 below, the proofs of which involve the following concepts of regular variation at 0:

Definition 2.4. A monotone function $f:[x_0,0)\to(0,\infty)$ is *regularly varying* as $x\uparrow 0$ with index $\alpha\in\mathbb{R}$, denoted $f\in\mathcal{R}_{0^-}(\alpha)$, if

$$\lim_{x\uparrow 0}\frac{f(yx)}{f(x)}=y^{\alpha}\quad\text{for }\ y>0.$$

Definition 2.5. A monotone function $f:[x_0,0)\to(0,\infty)$ is *O-regularly varying* as $x\uparrow 0$, with Matuszewska indices $-\infty<\alpha\le\beta<\infty$, denoted $f\in\mathcal{OR}_{0^-}(\alpha,\beta)$, if for some constant $x_0<0$ and for each $\varepsilon>0$, there exists a constant $C\ge 1$, such that

$$\frac{y^{\beta+\varepsilon}}{C}\le\frac{f(yx)}{f(x)}\le Cy^{\alpha-\varepsilon}\quad\text{for }\ x\in[x_0,0)\ \text{ and }\ y\in(0,1],$$

where α and β are the largest and smallest numbers, respectively, such that these two inequalities hold.

By Potter's theorem (see *e.g.* Bingham, Goldie and Teugels [10], Thm. 1.5.6), we have $\mathcal{R}_{0^-}(\alpha) \subseteq \mathcal{OR}_{0^-}(\alpha,\alpha)$ for $\alpha \in \mathbb{R}$.

The next lemma which is used in the proof of Proposition 2.10 is a version at 0 of the Stieltjes' version of Karamata's theorem for one-sided indices at ∞ (see *e.g.* Bingham, Goldie and Teugels [10], Sect. 2.6.2):

Lemma 2.6. *For $U \in \mathcal{OR}_{0^-}(\alpha,\beta)$ nondecreasing with $-2 < \alpha \leq \beta < 0$, we have*

$$0 < \liminf_{x\uparrow 0} \frac{1}{x^2U(x)} \int_x^0 y^2\, dU(y) \leq \limsup_{x\uparrow 0} \frac{1}{x^2U(x)} \int_x^0 y^2\, dU(y) < \infty. \tag{2.22}$$

Proof. We have $\lim_{x\uparrow 0} x^2 U(x) = 0$ because

$$\limsup_{x\uparrow 0} x^2 \frac{U(x)}{U(x_0)} \leq \limsup_{x\uparrow 0} \frac{Cx^{2+\alpha-\varepsilon}}{x_0^{\beta+\varepsilon}} = 0 \quad \text{for } \varepsilon > 0 \text{ small enough.}$$

From this in turn we get the upper bound noticing that

$$\int_x^0 \frac{y^2\, dU(y)}{x^2U(x)} = 2\int_x^0 \frac{(-y)U(y)}{x^2U(x)}\,\mathrm{d}y - 1 = 2\int_0^1 \frac{zU(zx)}{U(x)}\,\mathrm{d}z - 1 \leq 2\int_0^1 Cz^{\alpha+1-\varepsilon}\mathrm{d}z - 1$$

where the right-hand side is finite for $\varepsilon > 0$ small enough. Further, as we have

$$\limsup_{z\downarrow 0} z\int_z^1 \frac{U(yx)}{U(x)}\,\mathrm{d}y \leq \limsup_{z\downarrow 0} z\int_z^1 Cy^{\alpha-\varepsilon}\mathrm{d}y = \limsup_{z\downarrow 0} \frac{C(z - z^{\alpha+2-\varepsilon})}{\alpha+1-\varepsilon} = 0$$

for $x \in [x_0, 0)$ and $\varepsilon > 0$ small enough, Fatou's Lemma gives

$$\liminf_{x\uparrow 0} \int_x^0 \frac{y^2\, dU(y)}{x^2U(x)} = 2\liminf_{x\uparrow 0} \int_0^1 \frac{zU(zx)}{U(x)}\mathrm{d}z - 1 \geq 2\int_0^1 \left(\int_z^1 \liminf_{x\uparrow 0} \frac{U(yx)}{U(x)}\mathrm{d}y\right)\mathrm{d}z - 1. \tag{2.23}$$

Since $U(yx)/U(x) \geq 1$ is a nondecreasing function of $y \in (0,1)$, the $\liminf$ on the left in (2.22) can be 0 only if $\liminf_{x\uparrow 0} U(yx)/U(x) = 1$ for $y \in (0,1)$, as otherwise the right-hand side of (2.23) is strictly greater than $2\int_0^1(\int_z^1 \mathrm{d}y)\mathrm{d}z - 1 = 0$. And so the $\liminf$ on the left in (2.22) must be strictly greater than 0, because

$$\liminf_{x\uparrow 0} \frac{U(yx)}{U(x)} \geq \frac{y^{\beta+\varepsilon}}{C} > 1 \quad \text{for } \varepsilon, y > 0 \text{ small enough.}$$

$\square$

Our second lemma, which is also used in the proof of Proposition 2.10, is a version at 0 of the de Haan-Stadtmüller theorem (see *e.g.* Bingham, Goldie and Teugels [10], Thm. 2.10.2):

Lemma 2.7. *For $U \in \mathcal{OR}_{0^-}(\alpha,\beta)$ non-increasing with $0<\alpha\le\beta<\infty$ and $\int_{-\infty}^0 e^{-\lambda x}\, d(-U)(x)$, $\int_{-\infty}^0 e^{-\lambda x}\, U(x)\,\mathrm{d}x<\infty$ for λ small enough (i.e., $|\lambda|$ large enough), we have*

$$0<\liminf_{\lambda\to-\infty}\int_{-\infty}^0 \frac{e^{-\lambda x}\, d(-U)(x)}{U(1/\lambda)} \le \limsup_{\lambda\to-\infty}\int_{-\infty}^0 \frac{e^{-\lambda x}\, d(-U)(x)}{U(1/\lambda)}<\infty.$$

Proof. We have

$$\begin{aligned}\int_{-\infty}^0 \frac{\mathrm{e}^{-\lambda x}\, d(-U)(x)}{U(1/\lambda)} &= \left[-\frac{\mathrm{e}^{-\lambda x}U(x)}{U(1/\lambda)}\right]_{-\infty}^0 - \lambda\int_{-\infty}^0 \frac{\mathrm{e}^{-\lambda x}U(x)}{U(1/\lambda)}\,\mathrm{d}x\\ &= 0+\int_{-\lambda x_0}^0 \frac{\mathrm{e}^{y}U(-y/\lambda)}{U(1/\lambda)}\,\mathrm{d}y-\lambda\int_{-\infty}^{x_0}\frac{\mathrm{e}^{-\lambda x}U(x)}{U(1/\lambda)}\,\mathrm{d}x.\end{aligned}$$

Here Definition 2.5 gives that lower and upper limits (as $\lambda\to-\infty$) of the first integral on the right-hand side are strictly positive and finite. Further

$$0<-\lambda\int_{-\infty}^{x_0}\frac{\mathrm{e}^{-\lambda x}U(x)}{U(1/\lambda)}\,\mathrm{d}x \le \frac{(-\lambda)\,\mathrm{e}^{-\lambda x_0/2}}{U(1/\lambda)}\int_{-\infty}^{x_0}\mathrm{e}^{-\lambda x/2}U(x)\,\mathrm{d}x,$$

the upper limit of which must be zero as $U(x_0/(-\lambda))\ge C^{-1}U(x_0)(-\lambda)^{-\varepsilon-\beta}$ for λ small enough, so that $U(1/\lambda)\ge C^{-1}U(x_0)(x_0\lambda)^{-\varepsilon-\beta}$. ☐

As have been mentioned already, the following three propositions are key results for verifying the conditions of Theorem 2.2:

Proposition 2.8. *For a superexponential Lévy process ξ with characteristic triplet (ν,m,s^2) and infinite upper end-point (1.6), we have the following implications:*

1. *If $s^2>0$, then (2.3) and (2.5) hold.*
2. *If $s^2>0$ and $\nu((0,\infty))=0$, then (2.3)–(2.6) hold.*

Proof. Statement 1 of the proposition is quite immediate.

To prove Statement 2, notice that

$$\limsup_{\lambda\to-\infty}\int_{|x|>\varepsilon\sigma(\lambda)}\frac{x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x)\le\limsup_{\lambda\to-\infty}\frac{1}{s^2}\int_{-\infty}^0 x^2\mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x)=0 \tag{2.24}$$

when $s^2>0$ and $\nu((0,\infty))=0$, so that (2.4) holds. In view of Statement 1 it thus remains to prove (2.6). To that end it is sufficient to show that the limit

$$\lim_{\lambda\to-\infty}\frac{\ln(\phi_1(\lambda))}{\lambda\mu(\lambda)}=\lim_{\lambda\to-\infty}\frac{\int_{\mathbb{R}}\left(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\right)\mathrm{d}\nu(x)+m\lambda+\lambda^2s^2/2}{\int_{\mathbb{R}}\left(\lambda x\mathrm{e}^{-\lambda x}-\lambda\kappa(x)\right)\mathrm{d}\nu(x)-m\lambda-\lambda^2s^2}\equiv\tilde{L} \tag{2.25}$$

exists and is not equal to -1. As it is obvious that

$$\int_{-\infty}^{-1}\left(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\right)\mathrm{d}\nu(x)=O(\lambda)\quad\text{and}\quad\int_{-\infty}^{-1}\left(\lambda x\mathrm{e}^{-\lambda x}-\lambda\kappa(x)\right)\mathrm{d}\nu(x)=O(\lambda)$$

as $\lambda \to -\infty$, (2.25) with $\tilde{L} = -1/2$ will in turn follow provided that we prove that

$$\int_{-1}^{0} \left(\mathrm{e}^{-\lambda x} - 1 + \lambda x\right) \mathrm{d}\nu(x) = o(\lambda^2) \quad \text{and} \quad \int_{-1}^{0} \left(\lambda x \mathrm{e}^{-\lambda x} - \lambda x\right) d\nu(x) = o(\lambda^2)$$

as $\lambda \to -\infty$. The first of these asymptotic relations is established in (2.13). The second asymptotic relation follows in a similar fashion noticing that, by integration by parts,

$$\begin{aligned}\int_{-1}^{0} \left(\lambda x \mathrm{e}^{-\lambda x} - \lambda x\right) \mathrm{d}\nu(x) &= -\lambda^2 \int_{0}^{-\lambda} \frac{x\mathrm{e}^{-x} + \mathrm{e}^{-x} - 1}{x^2} \left(\int_{-1}^{x/\lambda} y^2 \mathrm{d}\nu(y)\right) \mathrm{d}x \\ &\sim \lambda^2 \int_{-1}^{0} y^2 \mathrm{d}\nu(y) \left(-1 - \int_{0}^{-\lambda} \frac{x\mathrm{e}^{-x} + \mathrm{e}^{-x} - 1}{x^2} \mathrm{d}x\right),\end{aligned}$$

where the inner integral on the right-hand side converges to -1 as $\lambda \to -\infty$, so that the whole expression under consideration is $o(\lambda^2)$ as $\lambda \to -\infty$, as required. □

Proposition 2.9. *For a superexponential Lévy process ξ with characteristic triplet (ν, m, s^2) and infinite upper end-point (1.6), we have the following implications:*

1. *If $\nu((0,\infty)) > 0$, then (2.3) and (2.6) hold.*
2. *Equations (2.3), (2.4) and (2.6) hold if $\nu((0,\infty)) > 0$ and there exists a non-decreasing function g such that*

$$\lim_{x\to\infty} \frac{g(x)}{\ln(x)} = \infty \quad \textit{and} \quad \int_{1}^{\infty} \exp\{g(x)x\} \mathrm{d}\nu(x) < \infty. \tag{2.26}$$

3. *Equations (2.3)–(2.6) hold if $x_0 \equiv \sup\{x : \nu((x,\infty)) > 0\} \in (0,\infty)$ and ν is absolutely continuous with a version of $\mathrm{d}\nu(x)/\mathrm{d}x$ that is bounded, strictly positive for $x \in (x_1, x_2)$ for some $0 < x_1 < x_2 \le x_0$ and satisfies*

$$\frac{\mathrm{d}\nu(x)}{\mathrm{d}x} \sim C x^{-1-\rho} \quad \textit{as } x \downarrow 0^+ \quad \textit{for some constants } C > 0 \textit{ and } \rho \in (0,2). \tag{2.27}$$

4. *Equations (2.3)–(2.6) hold if ν is absolutely continuous with $\sup\{x : \nu((x,\infty)) > 0\} = \infty$, if ν satisfies (2.26) and (2.27), and if ν has a version of $\mathrm{d}\nu(x)/\mathrm{d}x$ that is ultimately decreasing.*

Proof. To prove Statement 1, notice that (2.12) readily gives (2.3). Further, by inspection of the proof of Proposition 2.8(2), (2.6) holds with $\tilde{L} = L = 0$ if

$$\lim_{\lambda\to-\infty} \int_{0}^{\infty} \left(\mathrm{e}^{-\lambda x} - 1 + \lambda\kappa(x)\right) \mathrm{d}\nu(x) \Big/ \left(\int_{0}^{\infty} \left(\lambda\kappa(x) - \lambda x \mathrm{e}^{-\lambda x}\right) \mathrm{d}\nu(x)\right) = 0$$

and

$$\lim_{\lambda\to-\infty} \frac{1}{\lambda^2} \int_{0}^{\infty} \left(\mathrm{e}^{-\lambda x} - 1 + \lambda\kappa(x)\right) \mathrm{d}\nu(x) = \lim_{\lambda\to-\infty} \frac{1}{\lambda^2} \int_{0}^{\infty} \left(\lambda\kappa(x) - \lambda x \mathrm{e}^{-\lambda x}\right) \mathrm{d}\nu(x) = \infty.$$

However, both these requirements are quite obvious consequences of the fact that

$$\int_{0}^{1} \left(\lambda\kappa(x) - \lambda x \mathrm{e}^{-\lambda x}\right) \mathrm{d}\nu(x) \ge \int_{0}^{1} \left(\mathrm{e}^{-\lambda x} - 1 + \lambda\kappa(x)\right) \mathrm{d}\nu(x) \ge 0.$$

To prove Statement 2, notice that by (2.26), the function $G(x)=g(\sqrt{x})$ is non-decreasing with

$$\lim_{x\to\infty}\frac{G(x)}{\ln(x)}=\infty \quad \text{and} \quad \int_1^\infty \exp\left\{G(x^2)x\right\}\mathrm{d}\nu(x)<\infty. \tag{2.28}$$

As we must have $\nu((\underline{x},\infty))>0$ for some $\underline{x}>0$, (2.28) gives that

$$\liminf_{\lambda\to-\infty}\frac{G(\varepsilon^2\sigma(\lambda)^2)}{-\lambda}\geq\liminf_{\lambda\to-\infty}\frac{1}{-\lambda}\,G\bigg(\varepsilon^2\int_{\underline{x}}^\infty x^2\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)\bigg)\geq\liminf_{\lambda\to-\infty}\frac{G\big(\varepsilon^2\underline{x}^2\nu((\underline{x},\infty))\mathrm{e}^{-\lambda\underline{x}}\big)}{-\lambda}=\infty$$

for $\varepsilon>0$. From this in turn we readily obtain, making use of (2.28) again [see also (2.24)],

$$\begin{aligned}&\limsup_{\lambda\to-\infty}\int_{|x|>\varepsilon\sigma(\lambda)}\frac{x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)\\&\leq\limsup_{\lambda\to-\infty}\frac{1}{\sigma(\lambda)^2}\int_{-\infty}^0 x^2\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)+\Big(\sup_{x<0}x^2\mathrm{e}^x\Big)\bigg(\limsup_{\lambda\to-\infty}\frac{1}{\lambda^2\sigma(\lambda)^2}\int_{|x|>\varepsilon\sigma(\lambda)}\mathrm{e}^{-2\lambda x}\mathrm{d}\nu(x)\bigg)\\&\leq 0+\Big(\sup_{x<0}x^2\mathrm{e}^x\Big)\bigg(\int_1^\infty\exp\left\{G(x^2)x\right\}\mathrm{d}\nu(x)\bigg)\bigg(\limsup_{\lambda\to-\infty}\sup_{x>\varepsilon\sigma(\lambda)}\exp\left\{-2\lambda x-G(x^2)x\right\}\bigg)\\&=0\quad\text{for }\ \varepsilon>0.\end{aligned}$$

Hence (2.4) holds. The statement now follows from Statement 1.

To prove Statement 3, notice that Statement 2 shows that (2.3), (2.4) and (2.6) hold. Using the elementary inequality $1-\cos(x)\geq x^2/4$ for $|x|\leq 1$ we further get

$$\begin{aligned}\limsup_{\lambda\to-\infty}&\int_{K<|\theta|\leq\sigma(\lambda)/x_0}\exp\bigg\{-t\int_{\mathbb{R}}\bigg(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\bigg)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)\bigg\}\mathrm{d}\theta\\&\leq\limsup_{\lambda\to-\infty}\int_{|\theta|>K}\exp\bigg\{-\int_0^{x_0}\frac{t\theta^2x^2}{4\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)\bigg\}\mathrm{d}\theta\\&=\int_{|\theta|>K}\exp\bigg\{-\frac{t\theta^2}{4}\bigg\}\mathrm{d}\theta\\&\to 0\quad\text{as }\ K\to\infty.\end{aligned}\tag{2.29}$$

Further, using (2.27) to find a $\delta\in(0,1\wedge x_0)$ such that $\mathrm{d}\nu(x)/\mathrm{d}x\geq\frac{1}{2}Cx^{-1-\rho}$ for $x\in(0,\delta)$, we get in a similar fashion

$$\begin{aligned}\limsup_{\lambda\to-\infty}&\int_{\sigma(\lambda)/x_0<|\theta|<\sigma(\lambda)\sqrt{-\lambda}}\exp\bigg\{-t\int_{\mathbb{R}}\bigg(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\bigg)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)\bigg\}\mathrm{d}\theta\\&\leq\limsup_{\lambda\to-\infty}\int_{\sigma(\lambda)/x_0<|\theta|<\sigma(\lambda)\sqrt{-\lambda}}\exp\bigg\{-\int_{(\delta\sigma(\lambda)/|\theta|)/2}^{\delta\sigma(\lambda)/|\theta|}\frac{Ct\theta^2x^{1-\rho}}{8\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}\mathrm{d}x\bigg\}\mathrm{d}\theta\\&\leq\limsup_{\lambda\to-\infty}\int_{|\theta|>\sigma(\lambda)/x_0}\exp\bigg\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{8(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\,\mathrm{e}^{\delta\sqrt{-\lambda}/2}\bigg\}\mathrm{d}\theta\\&=0,\end{aligned}\tag{2.30}$$

where we made use of the simple fact that

$$\limsup_{\lambda\to-\infty} \sigma(\lambda)^2 \mathrm{e}^{\lambda x_0} = 0 \tag{2.31}$$

to get the last equality. Finally, we have, based in part on a slight modification of (2.30), and noticing the quick oscillations of the cosine function,

$$\begin{aligned}
&\limsup_{\lambda\to-\infty} \int_{|\theta|>\sigma(\lambda)\sqrt{-\lambda}} \exp\Big\{-t\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)\Big\}\mathrm{d}\theta \\
&\le \limsup_{\lambda\to-\infty} \int_{|\theta|>\sigma(\lambda)\sqrt{-\lambda}} \exp\Big\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho} - \frac{t}{2}\int_{x_1}^{x_2}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)\Big\}\mathrm{d}\theta \\
&\le \limsup_{\lambda\to-\infty} \int_{|\theta|>\sigma(\lambda)\sqrt{-\lambda}} \exp\Big\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho} - t\,\frac{x_2-x_1}{4}\inf_{x\in(x_1,x_2)}\frac{\mathrm{d}\nu(x)}{\mathrm{d}x}\,\mathrm{e}^{-\lambda x_1}\Big\}\mathrm{d}\theta \\
&= 0,
\end{aligned} \tag{2.32}$$

using (2.31) at the end again. Putting (2.29), (2.30) and (2.32) together we arrive at (2.5).

To prove Statement 4, notice that Statement 2 shows that (2.3), (2.4) and (2.6) hold. As

$$2\ln(y) - \lambda y - yg(y) - \big(2\ln(x) - \lambda x - xg(x)\big) \le (y-x)(2-\lambda-g(x)) \quad \text{for } 1\le x\le y$$

by the first part of (2.26), we can further find a function $x_0(\lambda)$ such that $\lim_{\lambda\to-\infty} x_0(\lambda)=\infty$,

$$\lim_{\lambda\to-\infty} \exp\{\lambda\varepsilon\}x_0(\lambda) = 0 \quad \text{for } \varepsilon>0, \tag{2.33}$$

and $2\ln(x)-\lambda x - xg(x)$ is non-increasing for $x\ge x_0(\lambda)$, so that, by the second part of (2.26),

$$\int_{x_0(\lambda)}^\infty x^2\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x) \le \mathrm{e}^{2\ln(x_0(\lambda))-\lambda x_0(\lambda)-x_0(\lambda)g(x_0(\lambda))}\int_1^\infty \mathrm{e}^{xg(x)}\mathrm{d}\nu(x) \to 0 \quad \text{as } \lambda\to-\infty. \tag{2.34}$$

Now by (2.34), the argument for (2.29) in the proof of Statement 3 carries over to show that

$$\lim_{K\to\infty}\limsup_{\lambda\to-\infty}\int_{K<|\theta|\le\sigma(\lambda)/x_0(\lambda)} \exp\Big\{-t\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)\Big\}\mathrm{d}\theta = 0. \tag{2.35}$$

Notice that (2.34) also gives

$$\sigma(\lambda)^2 \sim \int_0^{x_0(\lambda)} x^2\mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x) \le x_0(\lambda)^2\mathrm{e}^{-\lambda x_0(\lambda)}\int_0^\infty (1\wedge x^2)\,\mathrm{d}\nu(x) \quad \text{as } \lambda\to-\infty,$$

from which we readily conclude that (2.33) implies

$$\lim_{\lambda\to-\infty}\frac{\exp\{-\lambda\varepsilon\}}{x_0(\lambda)^\rho\ln(\sigma(\lambda))} = \infty \quad \text{for } \varepsilon>0. \tag{2.36}$$

By (2.36) in turn, there exists a function $f(\lambda)>0$ with $\lim_{\lambda\to-\infty} f(\lambda)=0$ such that

$$\lim_{\lambda\to-\infty}\frac{\exp\{-\lambda f(\lambda)\}}{x_0(\lambda)^\rho\ln(\sigma(\lambda))} = \infty. \tag{2.37}$$

Selecting $\delta \in (0,1)$ such that $\mathrm{d}\nu(x)/\mathrm{d}x \geq \frac{1}{2}Cx^{-1-\rho}$ for $x \in (0,\delta)$, the analogue of (2.30) in the proof of Statement 3 now becomes

$$\begin{aligned}
&\limsup_{\lambda\to-\infty} \int_{\sigma(\lambda)/x_0(\lambda)<|\theta|<\delta\sigma(\lambda)/(2f(\lambda))} \exp\Big\{-t\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)\Big\}\mathrm{d}\theta \\
&\leq \limsup_{\lambda\to-\infty} \int_{\sigma(\lambda)/x_0(\lambda)<|\theta|<\delta\sigma(\lambda)/(2f(\lambda))} \exp\Big\{-\int_{(\delta\sigma(\lambda)/|\theta|)/2}^{\delta\sigma(\lambda)/|\theta|} \frac{Ct\theta^2 x^{1-\rho}}{8\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}\mathrm{d}x\Big\}\mathrm{d}\theta \\
&\leq \limsup_{\lambda\to-\infty} \int_{|\theta|>\sigma(\lambda)/x_0(\lambda)} \exp\Big\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{8(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\,\mathrm{e}^{-\lambda f(\lambda)}\Big\}\mathrm{d}\theta \\
&= \limsup_{\lambda\to-\infty} \frac{2}{\rho}\Big(\frac{Ct(1-2^{\rho-2})}{8(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\,\mathrm{e}^{-\lambda f(\lambda)}\Big)^{-1/\rho}\Gamma\Big(\frac{1}{\rho},\Big(\frac{\sigma(\lambda)}{x_0(\lambda)}\Big)^\rho\frac{Ct(1-2^{\rho-2})}{8(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\,\mathrm{e}^{-\lambda f(\lambda)}\Big) \\
&= \limsup_{\lambda\to-\infty} \frac{2}{\rho}\Big(\frac{\sigma(\lambda)}{x_0(\lambda)}\Big)^{1-\rho}\Big(\frac{Ct(1-2^{\rho-2})}{8(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\,\mathrm{e}^{-\lambda f(\lambda)}\Big)^{-1}\exp\Big\{-\Big(\frac{\sigma(\lambda)}{x_0(\lambda)}\Big)^\rho\frac{Ct(1-2^{\rho-2})}{8(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\,\mathrm{e}^{-\lambda f(\lambda)}\Big\} \\
&= 0,
\end{aligned} \tag{2.38}$$

by well-known asymptotics for the incomplete Gamma function $\Gamma(1/\rho,\cdot)$, and provided that

$$\lim_{\lambda\to-\infty} \frac{\sigma(\lambda)x_0(\lambda)^{\rho-1}}{\exp\{-\lambda f(\lambda)\}}\,\exp\Big\{-\frac{t\exp\{-\lambda f(\lambda)\}}{x_0(\lambda)^\rho}\Big\} = 0 \quad \text{for } t>0,$$

the latter fact which in turn holds when (2.37) does. Finally, the analogue of (2.32) becomes

$$\begin{aligned}
&\limsup_{\lambda\to-\infty} \int_{|\theta|>\delta\sigma(\lambda)/(2f(\lambda))} \exp\Big\{-t\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x)\Big\}\mathrm{d}\theta \\
&\leq \limsup_{\lambda\to-\infty} \int_{|\theta|>\delta\sigma(\lambda)/(2f(\lambda))} \exp\Big\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho} - \frac{t}{2}\int_1^{x_0(\lambda)}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x)\Big\}\mathrm{d}\theta \\
&\leq \limsup_{\lambda\to-\infty} \int_{|\theta|>\delta\sigma(\lambda)/(2f(\lambda))} \exp\Big\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho} - \frac{t}{4}\int_1^{x_0(\lambda)}\mathrm{e}^{-\lambda x-1}\,\mathrm{d}\nu(x)\Big\}\mathrm{d}\theta \\
&\leq \limsup_{\lambda\to-\infty} \int_{|\theta|>\delta\sigma(\lambda)/(2f(\lambda))} \exp\Big\{-\frac{Ct(1-2^{\rho-2})|\theta|^\rho}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho} - \frac{t\,\mathrm{e}^{-1}\sigma(\lambda)^2}{4\,x_0(\lambda)^2}\Big\}\mathrm{d}\theta \\
&= \limsup_{\lambda\to-\infty} \frac{2}{\rho}\Big(\frac{Ct(1-2^{\rho-2})}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\Big)^{-1/\rho}\Gamma\Big(\frac{1}{\rho},\Big(\frac{\delta\sigma(\lambda)}{2f(\lambda)}\Big)^\rho\frac{Ct(1-2^{\rho-2})}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\Big)\exp\Big\{-\frac{t\,\mathrm{e}^{-1}\sigma(\lambda)^2}{4\,x_0(\lambda)^2}\Big\} \\
&= \limsup_{\lambda\to-\infty} \frac{2}{\rho}\Big(\frac{\delta\sigma(\lambda)}{2f(\lambda)}\Big)^{1-\rho}\Big(\frac{Ct(1-2^{\rho-2})}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho}\Big)^{-1}\exp\Big\{-\Big(\frac{\delta\sigma(\lambda)}{2f(\lambda)}\Big)^\rho\frac{Ct(1-2^{\rho-2})}{16(2-\rho)\delta^{\rho-2}\sigma(\lambda)^\rho} - \frac{t\,\mathrm{e}^{-1}\sigma(\lambda)^2}{4\,x_0(\lambda)^2}\Big\} \\
&= 0,
\end{aligned} \tag{2.39}$$

by the already cited properties of the incomplete Gamma function, and provided that

$$\lim_{\lambda\to-\infty} \sigma(\lambda)f(\lambda)^{\rho-1}\exp\Big\{-t\,\frac{\sigma(\lambda)^2}{x_0(\lambda)^2}\Big\} = 0 \quad \text{for } t>0,$$

the latter fact which in turn holds provided that

$$\lim_{\lambda\to-\infty}\frac{\sigma(\lambda)^2}{x_0(\lambda)^2\ln(\sigma(\lambda))}=\infty, \tag{2.40}$$

because (2.37) readily gives that $f(\lambda)^{-1}=o(\sigma(\lambda))$ as $\lambda\to-\infty$. However, it is also readily seen that (2.36) implies (2.40), using that $\sigma(\lambda)/x_0(\lambda)\to\infty$ by (2.33), and that $\sigma(\lambda)^{2-\rho}\mathrm{e}^{\lambda\varepsilon}\to\infty$. Putting (2.35)–(2.39) together we now finally arrive at (2.5), which in turn completes the proof of Statement 4. □

Proposition 2.10. *For a superexponential Lévy process ξ with characteristic triplet (ν, m, s^2) and infinite upper end-point (1.6), we have the following implications:*

1. *Equation (2.3) holds if*

$$\nu((-\infty,\cdot))\in\mathcal{OR}_{0^-}(\alpha,\beta)\quad \textit{for some constants}\ \ -2<\alpha\le\beta<0. \tag{2.41}$$

2. *If $\nu((0,\infty))=0$ and (2.3) holds, then (2.4) holds.*
3. *If $\nu((0,\infty))=0$ and (2.41) holds, then (2.3)–(2.5) hold.*
4. *If ξ is selfdecomposable, then (2.3) and (2.4) hold.*
5. *If $\nu((0,\infty))=0$ and $\mathrm{d}\nu(x)=k(x)\mathrm{d}x/|x|^2$ for $x<0$ where $k>0$ is non-decreasing, then (2.3)–(2.5) hold.*
6. *Equations (2.3)–(2.6) hold if $\nu((0,\infty))=0$ and*

$$\nu((-\infty,\cdot))\in\mathcal{R}_{0^-}(\alpha)\quad \textit{for some constant}\ \ -2<\alpha<-1. \tag{2.42}$$

Proof. To prove Statement 1, notice that Lemma 2.6 gives

$$0<\frac{1}{C_1}\le\liminf_{x\uparrow0}\int_x^0\frac{y^2\mathrm{d}\nu(y)}{x^2\nu((-\infty,x))}\le\limsup_{x\uparrow0}\int_x^0\frac{y^2\mathrm{d}\nu(y)}{x^2\nu((-\infty,x))}\le C_1<\infty \tag{2.43}$$

for some constant $C_1\ge1$. As this also shows that $\int_\cdot^0 y^2\mathrm{d}\nu(y)$ belongs to $\mathcal{OR}_{0^-}(\alpha+2,\beta+2)$, Lemma 2.7 now in turn gives

$$\begin{aligned}0<\frac{1}{C_2}&\le\liminf_{\lambda\to-\infty}\bigg(\int_{1/\lambda}^0 y^2\mathrm{d}\nu(y)\bigg)^{-1}\int_{-\infty}^0\mathrm{e}^{-\lambda x}d\bigg(-\int_x^0 y^2\mathrm{d}\nu(y)\bigg)\\&\le\limsup_{\lambda\to-\infty}\bigg(\int_{1/\lambda}^0 y^2\mathrm{d}\nu(y)\bigg)^{-1}\int_{-\infty}^0\mathrm{e}^{-\lambda x}\mathrm{d}\bigg(-\int_x^0 y^2\mathrm{d}\nu(y)\bigg)\le C_2<\infty\end{aligned}$$

for some constant $C_2\ge1$. And so we get (2.3) in the following manner [recall (1.7)]:

$$\begin{aligned}\liminf_{\lambda\to-\infty}\lambda^2\sigma(\lambda)^2&\ge\liminf_{\lambda\to-\infty}\lambda^2\int_{-\infty}^0\mathrm{e}^{-\lambda x}\mathrm{d}\bigg(-\int_x^0 y^2\,\mathrm{d}\nu(y)\bigg)\\&\ge\frac{1}{C_2}\liminf_{\lambda\to-\infty}\lambda^2\int_{1/\lambda}^0 y^2\,\mathrm{d}\nu(y)\\&\ge\frac{1}{C_1C_2}\liminf_{\lambda\to-\infty}\nu((-\infty,1/\lambda))\\&=\infty\end{aligned} \tag{2.44}$$

To prove Statement 2, using that $-\varepsilon\sigma(\lambda) < 1/\lambda$ for λ small enough, we get (2.4) in the following manner:

$$\begin{aligned}
&\limsup_{\lambda\to-\infty} \int_{-\infty}^{-\varepsilon\sigma(\lambda)} \frac{x^2}{\sigma(\lambda)^2}\, \mathrm{e}^{-\lambda x}\, \mathrm{d}\nu(x) \\
&\quad \le \Big(\sup_{x<0} x^2 \mathrm{e}^{x/2}\Big) \limsup_{\lambda\to-\infty} \mathrm{e}^{\varepsilon\lambda\sigma(\lambda)/2}\, \nu((-\infty,-\varepsilon\sigma(\lambda))) \Big/ \Big(\frac{1}{\mathrm{e}} \int_{1/\lambda}^{0} x^2\, \mathrm{d}\nu(x)\Big) \\
&\quad = 0 \quad \text{for } \varepsilon > 0.
\end{aligned}$$

To prove Statement 3, in view of Statements 1 and 2, it is enough to prove that (2.5) holds. Note that, since $\nu((0,\infty)) = 0$, the arguments that were use to establish (2.44) carry over to show that

$$\frac{1}{C_1C_2} \le \liminf_{\lambda\to-\infty} \frac{\nu((-\infty,1/\lambda))}{\lambda^2\sigma(\lambda)^2} \le \limsup_{\lambda\to-\infty} \frac{\nu((-\infty,1/\lambda))}{\lambda^2\sigma(\lambda)^2} \le C_1C_2. \tag{2.45}$$

Further, using the inequality $1-\cos(x) \ge x^2/4$ for $|x| \le 1$ we have by (2.43) and (2.45)

$$\begin{aligned}
\int_{\mathbb{R}} \Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\mathrm{e}^{-\lambda x}\, \mathrm{d}\nu(x) &\ge \frac{\theta^2}{4\,\mathrm{e}\,\sigma(\lambda)^2} \int_{\max\{-\sigma(\lambda)/|\theta|,1/\lambda\}}^{0} x^2\, \mathrm{d}\nu(x) \\
&\ge \frac{1}{8C_1\mathrm{e}} \min\Big\{\nu((-\infty,-\sigma(\lambda)/|\theta|)), \frac{\nu((-\infty,1/\lambda))\,\theta^2}{\lambda^2\sigma(\lambda)^2}\Big\} \\
&\ge \frac{1}{8C_1\mathrm{e}} \min\Big\{\frac{|\theta|^{-\beta-\varepsilon}\nu((-\infty,-\sigma(\lambda)))}{C}, \frac{\theta^2}{2C_1C_2}\Big\}
\end{aligned}$$

for $|\theta| > 1$ and λ small enough. As the fact that $\lim_{\lambda\to-\infty}\sigma(\lambda) = 0$ implies that $\lim_{\lambda\to-\infty}\nu((-\infty, -\sigma(\lambda))) = \infty$ [recall (1.7)], it follows that (2.5) holds.

To prove Statement 4, by Proposition 2.8(1) and Proposition 2.9(1) we may assume that $\nu((0,\infty)) = 0$ and $s^2 = 0$. It is enough to prove (2.3), as Statement 2 then gives (2.4). Recall that selfdecomposability means that $\mathrm{d}\nu(x) = k(x)/|x|$ where $k > 0$ is non-decreasing (see *e.g.* Sato, [29], Cor. 15.11). From (1.7) we get in addition that $\lim_{x\uparrow 0} k(x) = \infty$. And so we get (2.3) as follows:

$$\begin{aligned}
\liminf_{\lambda\to-\infty} \lambda^2\sigma(\lambda)^2 &\ge \liminf_{\lambda\to-\infty} \int_{1/\lambda}^{0} \lambda^2 x^2 \mathrm{e}^{-\lambda x}\, \mathrm{d}\nu(x) \\
&\ge \frac{1}{\mathrm{e}} \liminf_{\lambda\to-\infty} \int_{1/\lambda}^{0} \lambda^2(-x)k(x)\, \mathrm{d}x \\
&\ge \frac{1}{\mathrm{e}} \liminf_{\lambda\to-\infty} \lambda^2(-0^-) \int_{1/\lambda}^{0^-} k(y)\, \mathrm{d}y + \frac{1}{\mathrm{e}} \liminf_{\lambda\to-\infty} \lambda^2 \int_{1/\lambda}^{0} \Big(\int_{1/\lambda}^{x} k(y)\, \mathrm{d}y\Big)\mathrm{d}x \\
&\ge \frac{1}{2\,\mathrm{e}} \liminf_{\lambda\to-\infty} k(1/\lambda) \\
&= \infty.
\end{aligned}$$

To prove Statement 5, by Proposition 2.8(1) we may assume that $s^2 = 0$. Further, ξ is selfdecomposable (see the proof of Statement 4), so that Statement 4 gives (2.4). Noticing that

$$\frac{\mathrm{d}}{\mathrm{d}x} - \int_{x}^{0} \frac{y^2}{x}\mathrm{e}^{-\lambda y}\, \mathrm{d}\nu(y) = \int_{x}^{0} \frac{k(y)}{x^2}\mathrm{e}^{-\lambda y}\, \mathrm{d}y + \frac{k(x)}{x}\mathrm{e}^{-\lambda x} = \int_{x}^{0} \frac{-y}{x^2} \frac{\mathrm{d}}{\mathrm{d}y}(\mathrm{e}^{-\lambda y}k(y))\, \mathrm{d}y \ge 0 \tag{2.46}$$

it is now an easy matter to finish off the proof: Using that $1-\cos(x)\ge x^2/4$ for $|x|\le 1$ we get (2.5), as (2.46) together with (2.4) give that

$$\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}\,\mathrm{d}\nu(x)\ge\int_{-\sigma(\lambda)/|\theta|}^{0}\frac{\theta^2x^2\mathrm{e}^{-\lambda x}}{4\sigma(\lambda)^2}\,\mathrm{d}\nu(x)\ge\int_{-\sigma(\lambda)}^{0}\frac{x^2|\theta|\,\mathrm{e}^{-\lambda x}}{4\sigma(\lambda)^2}\,\mathrm{d}\nu(x)\ge\frac{|\theta|}{8}$$

for λ small enough and $|\theta|\ge 1$.

To prove Statement 6, in view of Proposition 2.8(2) we may assume that $s^2=0$. By (2.52) below we have

$$\int_{-\infty}^{0}\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,\mathrm{d}\nu(x)\sim-\Gamma(1+\alpha)\,\nu((-\infty,1/\lambda))\quad\text{as }\ \lambda\to-\infty. \tag{2.47}$$

Moreover, by (2.51) below together with Feller's Tauberian theorem (see *e.g.* Bingham, Goldie and Teugels [10], Thm. 1.7.1′), we have

$$\begin{aligned}&\int_{-\infty}^{0}\big(\lambda x\,\mathrm{e}^{-\lambda x}-\lambda\kappa(x)\big)\,\mathrm{d}\nu(x)\\&=\int_{-1}^{0}\lambda x\,(\mathrm{e}^{-\lambda x}-1)\,\mathrm{d}\nu(x)+\int_{-\infty}^{-1}\big(\lambda x\,\mathrm{e}^{-\lambda x}+\lambda\big)\,\mathrm{d}\nu(x)\\&=\lambda(\mathrm{e}^{\lambda}-1)\nu((-\infty,-1))+\int_{-1}^{0}\big((\lambda^2x-\lambda)\,\mathrm{e}^{-\lambda x}+\lambda\,\nu((-\infty,x)\big)\,\mathrm{d}x+O(\lambda)\\&=\lambda^2\int_{-1}^{0}(\lambda x-2)\,\mathrm{e}^{-\lambda x}\mathrm{d}\Big(\int_{x}^{0}\Big(\int_{-1}^{y}\nu((-\infty,z))\,\mathrm{d}z\Big)\mathrm{d}y\Big)+O(\lambda)\\&\sim\lambda^3\,\frac{(1/\lambda)^3\Gamma(4+\alpha)\,\nu((-\infty,-1/\lambda))}{-(\alpha+1)\,(2+\alpha)}-2\lambda^2\,\frac{(1/\lambda)^2\Gamma(3+\alpha)\,\nu((-\infty,-1/\lambda))}{-(\alpha+1)\,(2+\alpha)}\\&=-\Gamma(2+\alpha)\,\nu((-\infty,-1/\lambda))\quad\text{as }\ \lambda\to-\infty,\end{aligned} \tag{2.48}$$

where $\alpha<-1$ ensures that $\lim_{\lambda\to-\infty}\nu((-\infty,1/\lambda))/(-\lambda)=\infty$. Putting (2.47) and (2.48) together we see that (2.25) holds with $\tilde{L}=1+\alpha$. □

2.2. Sufficient conditions for condition (1.3)

The next theorem gives sufficient conditions for condition (1.3) to hold in terms of the characteristic triplet:

Theorem 2.11. *Let ξ be a superexponential Lévy process with characteristic triplet (ν,m,s^2) and infinite upper end-point (1.6). With the notation (2.7) we have the following implications (with obvious notation):*

1. *If $\nu((0,\infty))>0$, then* $\dfrac{\xi(aq(u))}{w(u)}\xrightarrow{d}0$ *as $u\to\infty$ for $a>0$;*
2. *If $\nu((0,\infty))=0$ and $s^2>0$, then* $\dfrac{\xi(aq(u))}{w(u)}\xrightarrow{d}N(0,2a)$ *as $u\to\infty$ for $a>0$.*
3. *If $\nu((0,\infty))=0$ and $s^2=0$ and (2.42) holds, then*

$$\frac{\xi(aq(u))}{w(u)}\xrightarrow{d}S_{-\alpha}\Big((-a\cos(-\frac{\pi\alpha}{2}))^{-1/\alpha},-1,0\Big)\quad\textit{as }u\to\infty\textit{ for }a>0.$$

Proof. We have weak convergence $\xi(aq(u))/w(u) \xrightarrow{\mathrm{d}} X$ if and only if we have convergence of the Laplace transform

$$\begin{aligned}\lim_{u\to\infty}\mathbf{E}\big\{\mathrm{e}^{-t\xi(aq(u))/w(u)}\big\} &= \lim_{u\to\infty}\phi_1(t/w(u))^{aq(u)}\\ &= \lim_{u\to\infty}\exp\big\{aq(u)\ln(\phi_1(t/w(u)))\big\}\\ &= \lim_{\lambda\to-\infty}\exp\left\{\frac{a\ln(\phi_1(-t\lambda))}{\ln(\phi_1(\lambda))}\right\}\\ &= \lim_{\lambda\to-\infty}\exp\left\{a\,\frac{\int_{\mathbb{R}}\big(\mathrm{e}^{t\lambda x}-1-t\lambda\kappa(x)\big)\,\mathrm{d}\nu(x)+mt\lambda+(t\lambda s)^2/2}{\int_{\mathbb{R}}\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,\mathrm{d}\nu(x)-m\lambda+(\lambda s)^2/2}\right\}\\ &= \mathbf{E}\{\mathrm{e}^{-tX}\}\quad\text{for }\ t\in(-1,0)\end{aligned}\tag{2.49}$$

(see *e.g.* Hoffmann-Jørgensen [19], pp. 377–378).

To prove Statement 1, notice that by arguing as for the proof of (2.6) in Proposition 2.9(1), the limit in (2.49) is 1 when $\nu((0,\infty))>0$, which implies weak convergence to a degenerate random variable $X=0$.

To prove Statement 2, notice that by arguing as for the proof of (2.6) in Proposition 2.8(2), the limit in (2.49) is e^{at^2} when $\nu((0,\infty))=0$ and $s^2>0$, which implies weak convergence to a normal $\mathrm{N}(0,2a)$ distributed random variable X.

To prove Statement 3, assume that $\nu((0,\infty))=0$ and $s^2=0$. Notice that by Karamata's theorem (see *e.g.* Bingham, Goldie and Teugels [10], Sect. 1.5.6),

$$-\int_x^0 y\nu((-\infty,y))\,\mathrm{d}y \sim \frac{x^2\nu((-\infty,x))}{2+\alpha}\in\mathcal{R}_{0^-}(2+\alpha)\quad\text{as }\ x\uparrow 0.$$

Hence Feller's Tauberian theorem (see *e.g.* Bingham, Goldie and Teugels [10], Thm. 1.7.1′) gives

$$\begin{aligned}\int_{-\infty}^0\big(1-\mathrm{e}^{-\lambda x}(1+\lambda x)\big)\,\mathrm{d}\nu(x) &= \int_{-\infty}^0\lambda^2\mathrm{e}^{-\lambda x}\,\mathrm{d}\Big(-\int_x^0 y\nu((-\infty,y))\,\mathrm{d}y\Big)\\ &\sim \Gamma(2+\alpha)\,\nu((-\infty,1/\lambda))\quad\text{as }\ \lambda\to-\infty.\end{aligned}\tag{2.50}$$

Moreover, using Karamata's theorem again we get

$$\int_x^0\Big(\int_{-1}^y\nu((-\infty,z))\,\mathrm{d}z\Big)\mathrm{d}y\sim\frac{x^2\,\nu((-\infty,x))}{-(\alpha+1)\,(2+\alpha)}\in\mathcal{R}_{0^-}(2+\alpha)\quad\text{as }\ x\uparrow 0,\tag{2.51}$$

so that by Feller's Tauberian theorem

$$\begin{aligned}\int_{-\infty}^0\big(\mathrm{e}^{t\lambda x}-1-t\lambda\kappa(x)\big)\,\mathrm{d}\nu(x) &= o(1)+\int_{-1}^0\big(t\lambda-\mathrm{e}^{t\lambda x}t\lambda)\nu((-\infty,x)\big)\,\mathrm{d}x\\ &\sim (t\lambda)^2\int_{-1}^0\mathrm{e}^{t\lambda x}\,\mathrm{d}\Big(\int_x^0\Big(\int_{-1}^y\nu((-\infty,z))\,\mathrm{d}z\Big)\mathrm{d}y\Big)\\ &\sim\frac{\Gamma(2+\alpha)\,\nu((-\infty,-1/(t\lambda)))}{-(\alpha+1)}\\ &\sim -(-t)^{-\alpha}\,\Gamma(1+\alpha)\,\nu((-\infty,1/\lambda))\quad\text{as }\ \lambda\to-\infty\end{aligned}\tag{2.52}$$

for $t \in [-1,0)$. Since $\alpha < -1$ ensures that $\lim_{\lambda\to-\infty} \nu((-\infty,1/\lambda))/(-\lambda) = \infty$ it follows that the limit in (2.49) is $\mathrm{e}^{-a\,(-t)^{-\alpha}}$, which is the Laplace transform of the $-\alpha$-stable distribution in Statement 3 (see *e.g.* Samorodnitsky and Taqqu [28], Prop. 1.2.12). $\square$

By (1.6) we have $\alpha \le -1$ in (2.41) when $\nu((0,\infty)) = 0$ and $s^2 = 0$. But $\alpha = -1$ was not covered in Theorem 2.11 and turns out to behave differently than $\alpha < -1$:

Theorem 2.12. *Let ξ be a superexponential Lévy process with characteristic triplet $(\nu, m, 0)$ and infinite upper end-point (1.6). Assume that $\nu((0,\infty)) = 0$ and that $\nu((-\infty,\cdot)) \in \mathcal{R}_{0^-}(-1)$. Denoting*

$$w(u) = -\frac{1}{\mu^{\leftarrow}(u/h)} \qquad \textit{and} \qquad q(u) = \frac{1}{\ln(\phi_1(\mu^{\leftarrow}(u/h)))},$$

we have

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\xi(h-q(u)t) > u + xw(u)\}}{\mathbf{P}\{\xi(h) > u\}} = \begin{cases} \mathrm{e}^{-x} & \textit{for} \quad x \in \mathbb{R} \quad \textit{and} \quad t = 0 \\ 0 & \textit{for} \quad x \in \mathbb{R} \quad \textit{and} \quad t > 0 \end{cases} \tag{2.53}$$

and

$$\frac{\xi(aq(u))}{w(u)} \xrightarrow{d} a \quad \textit{as} \;\; u \to \infty \quad \textit{for} \;\; a > 0. \tag{2.54}$$

Further, (2.8) holds.

Proof. We still have (2.50) with $\alpha = -1$. However, by so called de Haan theory (see *e.g.* Bingham, Goldie and Teugels [10], Prop. 1.5.9a), (2.51) changes to

$$\int_{\cdot}^{0} \left(\int_{-1}^{y} \nu((-\infty,z))\mathrm{d}z \right) \mathrm{d}y \in \mathcal{R}_{0^-}(1) \tag{2.55}$$

with

$$\lim_{x\uparrow 0} \frac{1}{x^2 \nu((-\infty,x))} \int_x^0 \left(\int_{-1}^{y} \nu((-\infty,z))\,\mathrm{d}z \right) \mathrm{d}y = \infty. \tag{2.56}$$

And so by Feller's Tauberian theorem the corresponding modification of (2.52) becomes

$$\begin{aligned} \int_{-\infty}^{0} \big(\mathrm{e}^{-\lambda x} - 1 + \lambda\kappa(x)\big)\,\mathrm{d}\nu(x) &= o(1) + \lambda^2 \int_{-1}^{0} \mathrm{e}^{-\lambda x}\,\mathrm{d}\left(\int_x^0 \left(\int_{-1}^{y} \nu((-\infty,z))\,\mathrm{d}z \right) \mathrm{d}y \right) \\ &\sim \Gamma(2)\,\lambda^2 \int_{1/\lambda}^{0} \left(\int_{-1}^{y} \nu((-\infty,z))\,\mathrm{d}z \right) \quad \text{as} \;\; \lambda \to -\infty, \end{aligned} \tag{2.57}$$

where the right-hand side is regularly varying by (2.55). Since (1.7) shows that

$$\lim_{\lambda\to-\infty} \frac{1}{(-\lambda)} \int_{-\infty}^{0} (\mathrm{e}^{-\lambda x} - 1 + \lambda\kappa(x))\mathrm{d}\nu(x) = \infty,$$

we now readily obtain (2.54) in the following manner: For $t \in (-1,0)$ we have

$$\begin{aligned}
\lim_{u\to\infty} \mathbf{E}\big\{\mathrm{e}^{-t\xi(aq(u))/w(u)}\big\} &= \lim_{u\to\infty} \exp\left\{\frac{a\ln(\phi_1(-t\mu^{\leftarrow}(u/h)))}{\ln(\phi_1(\mu^{\leftarrow}(u/h)))}\right\}\\
&= \lim_{\lambda\to-\infty} \exp\left\{a\,\frac{\int_{-\infty}^0 \big(\mathrm{e}^{t\lambda x}-1-t\lambda\kappa(x)\big)\,\mathrm{d}\nu(x)+mt\lambda}{\int_{-\infty}^0 \big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,\mathrm{d}\nu(x)-m\lambda}\right\}\\
&= \lim_{\lambda\to-\infty} \exp\left\{a\,\frac{(-t)\int_{-\infty}^0 \big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,\mathrm{d}\nu(x)}{\int_{-\infty}^0 \big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,\mathrm{d}\nu(x)}\right\}\\
&= \mathrm{e}^{-at}.
\end{aligned}$$

Changing the definition of Q to $Q(\lambda) = 1/\ln(\phi_1(\lambda))$ in the proof of Theorem 2.2, that proof still goes through in essence. The only important change is that since

$$\lim_{\lambda\to-\infty} \frac{-\lambda\mu(\lambda)-\ln(\phi_1(\lambda))}{\ln(\phi_1(\lambda))} = 0$$

by (2.50) and (2.56)–(2.57) [recall that $\nu((-\infty,1/\lambda))/(-\lambda)\to\infty$], (2.16) changes to

$$f_{\xi(h-Q(\lambda)t)}(\mu_t(\lambda)+x/\lambda) \sim \mathrm{e}^{x-t\ln(\phi_1(\lambda))/(-\lambda\mu(\lambda)-\ln(\phi_1(\lambda)))}\,\frac{\mathrm{e}^{h\lambda\mu(\lambda)}\phi_1(\lambda)^h}{\sqrt{2\pi h}\,\sigma(\lambda)} \quad \text{as } \lambda\to-\infty.$$

This does not affect the validity of (2.17)–(2.19), while (2.20) and (2.21) change to

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\xi(h-q(hu)t) > hu - xw(hu)\}}{\mathbf{P}\{\xi(h) > hu\}} = \begin{cases} \mathrm{e}^{-x} & \text{for } x\in\mathbb{R} \text{ and } t=0,\\ 0 & \text{for } x\in\mathbb{R} \text{ and } t>0.\end{cases}$$

From this in turn it follows that (2.53) holds as claimed. The proof of (2.8) is as before. □

3. A general upper bound and consequences

We will study the probability $\mathbf{P}\{\sup_{t\in[0,h]}\xi(t) > u\}$ for a separable Lévy process ξ. As that probability coincide for all separable Lévy processes with the same finite dimensional distributions, it is enough to consider one specific such process: In proofs we can thus henceforth assume that ξ is càdlàg (right continuous with left limits).

The following simple general upper bound for the above mentioned probability will be an important tool for us:

Proposition 3.1. *For a separable Lévy process ξ we have*

$$\sup_{u\in\mathbb{R}} \frac{1}{\mathbf{P}\{\xi(h)>u-\varepsilon\}}\,\mathbf{P}\bigg\{\sup_{t\in[0,h]}\xi(t)>u\bigg\} \le \frac{1}{\inf_{t\in[0,h]}\mathbf{P}\{\xi(t)\ge-\varepsilon\}} \quad \textit{for } \varepsilon\ge 0.$$

Proof. Writing $T=\inf\{t>0:\xi(t)>u\}$ and $g(t)=\mathbf{P}\{\xi(t)\geq -\varepsilon\}$, we have

$$\begin{aligned}\mathbf{P}\{\xi(h)>u-\varepsilon\} &\geq \mathbf{E}\big\{\mathbf{P}\{T<h,\xi(h)-\xi(T)\geq -\varepsilon\,\big|\,T\}\big\}\\ &= \mathbf{E}\big\{\mathbf{1}_{\{T<h\}}\,\mathbf{P}\{\xi(h)-\xi(T)\geq -\varepsilon\,\big|\,T\}\big\}\\ &= \mathbf{E}\big\{\mathbf{1}_{\{T<h\}}\,g(h-T)\big\}\\ &\geq \mathbf{P}\Big\{\sup_{t\in[0,h)}\xi(t)>u\Big\}\inf_{t\in[0,h]}g(t)\\ &= \mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\inf_{t\in[0,h]}g(t). \qquad \square\end{aligned}$$

A simple version of the following corollary to Proposition 3.1 for symmetric processes appears already in Doob [15], p. 106:

Corollary 3.2. *For a separable Lévy process ξ such that*

$$\liminf_{t\downarrow 0}\mathbf{P}\{\xi(t)>0\}>0 \tag{3.1}$$

we have

$$\inf_{t\in[0,h]}\mathbf{P}\{\xi(t)\geq 0\}>0, \tag{3.2}$$

which in turn implies that

$$\sup_{u\in\mathbb{R}}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}<\infty.$$

Proof. By inspection of Proposition 3.1 it is enough to show that (3.1) implies (3.2). So assume that (3.1) holds and that (3.2) does not. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}\subseteq[0,h]$ such that

$$\mathbf{P}\{\xi(t_n)\geq 0\}\to\inf_{t\in[0,h]}\mathbf{P}\{\xi(t)\geq 0\}=0 \quad\text{as } n\to\infty.$$

Picking a convergent subsequence $\{t_n'\}_{n=1}^{\infty}\subseteq\{t_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty}t_n'=t_0$, we get

$$\mathbf{P}\{\xi(t_0)>0\}\leq\liminf_{n\to\infty}\mathbf{P}\{\xi(t_n')>0\}\leq\liminf_{n\to\infty}\mathbf{P}\{\xi(t_n')\geq 0\}=0 \tag{3.3}$$

by continuity in probability of ξ. Hence (3.1) implies that $t_0>0$. And so $\xi(t_0)$ is supported on $(-\infty,0]$ by (3.3), which contradicts the left condition in (3.1). $\square$

The next example addresses the difference between Proposition 3.1 and Corollary 3.2:

Example 3.3. Let $\{N(t)\}_{t\geq 0}$ be a unit rate Poisson process and $\{\eta_k\}_{k=1}^{\infty}$ independent Bernoulli distributed random variables satisfying $\mathbf{P}\{\eta_k=1\}=\mathbf{P}\{\eta_k=-1\}=\frac{1}{2}$. Rather spectacularly, Braverman [11], Section 4, shows that for the Lévy process $\xi(t)=\sum_{k=1}^{N(t)}\eta_k-t$, it holds that

$$1=\liminf_{u\to\infty}\frac{\mathbf{P}\big\{\sup_{t\in[0,h]}\xi(t)>u\big\}}{\mathbf{P}\{\xi(h)>u\}}<\limsup_{u\to\infty}\frac{\mathbf{P}\big\{\sup_{t\in[0,h]}\xi(t)>u\big\}}{\mathbf{P}\{\xi(h)>u\}}=\infty.$$

Hence neither Corollary 3.2 nor (3.1) holds for this process.

For an example of a Lévy process that does not satisfy (3.1) it is enough to consider $\xi(t)=N(t)-t$.

The following proposition can be very useful to verify that the condition (1.8) of Theorem 1.1 holds:

Proposition 3.4. *Let ξ be a separable Lévy process such that (3.1) or (3.2) holds and such that*

$$\lim_{u\to\infty}\frac{\mathbf{P}\{\xi(h-Tq(u))>u\}}{\mathbf{P}\{\xi(h)>u\}}\leq \mathrm{e}^{-T}\quad \textit{for } T\geq 0. \tag{3.4}$$

Then we have

$$\lim_{T\to\infty}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\left\{\sup_{t\in[0,h-Tq(u)]}\xi(t)>u\right\}=0.$$

Proof. According to Corollary 3.2 (3.1) implies (3.2). Further, Proposition 3.1 together with (3.2) and (3.4) give

$$\begin{aligned}
&\lim_{T\to\infty}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\left\{\sup_{t\in[0,h-Tq(u)]}\xi(t)>u\right\}\\
&\quad\leq \lim_{T\to\infty}\limsup_{u\to\infty}\frac{\mathbf{P}\{\xi(h-Tq(u))>u\}}{\mathbf{P}\{\xi(h)>u\}}\frac{1}{\inf_{t\in[0,h-Tq(u)]}\mathbf{P}\{\xi(t)\geq 0\}}\\
&\quad\leq \lim_{T\to\infty}\limsup_{u\to\infty}\frac{\mathbf{P}\{\xi(h-Tq(u))>u\}}{\mathbf{P}\{\xi(h)>u\}}\frac{1}{\inf_{t\in[0,h]}\mathbf{P}\{\xi(t\geq 0\}}\\
&\quad\leq \lim_{T\to\infty}\mathrm{e}^{-T}\frac{1}{\inf_{t\in[0,h]}\mathbf{P}\{\xi(t)\geq 0\}}\\
&\quad=0.
\end{aligned}$$

□

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Provided that $L(t,0)>0$ repeated use of (1.4) gives

$$\mathbf{P}\left\{\frac{\xi(h-q(u)t)-u}{w(u)}>x\,\middle|\,\xi(h-q(u)t)>u\right\}=\frac{\mathbf{P}\{\xi(h-q(u)t)>u+xw(u)\}}{\mathbf{P}\{\xi(h-q(u)t)>u\}}\to\frac{L(t,x)}{L(t,0)} \tag{4.1}$$

as $u\to\infty$ for $x>0$. Let $\{\zeta_i(a)\}_{i=1}^{\infty}$ be independent random variables distributed as $\zeta(a)$. Further, let $\eta_t\geq 0$ be a possibly infinite valued random variable that is independent of $\{\zeta_i(a)\}_{i=1}^{\infty}$, that has the possibly improper cumulative probability distribution function $1-L(t,x)/L(t,0)$ when $L(t,0)>0$ and that is infinite when $L(t,0)=0$.

By (1.3) and (4.1) we get

$$
\begin{aligned}
&\liminf_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t)>u\Big\} \\
&\quad\geq \lim_{T\to\infty} \limsup_{a\downarrow 0} \liminf_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\Big\{\max_{k=0,\dots,\lfloor T/a\rfloor} \xi(h-kaq(u))>u\Big\} \\
&\quad= \lim_{T\to\infty} \limsup_{a\downarrow 0} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0) \liminf_{u\to\infty} \mathbf{P}\Big\{\bigcap_{\ell=0}^{k-1} \{\xi(h-\ell aq(u)) \\
&\qquad\qquad - \xi(h-kaq(u)) + \xi(h-kaq(u)) - u \leq 0\} \,\Big|\, \xi(h-kaq(u))>u\Big\} \\
&\quad\geq \lim_{T\to\infty} \limsup_{a\downarrow 0} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0)\, \mathbf{P}\Big\{\bigcap_{\ell=0}^{k-1}\Big\{\sum_{i=0}^{k-\ell}\zeta_i(a)+\eta_{ka}<0\Big\}\Big\}. \\
&\quad= \lim_{T\to\infty} \limsup_{a\downarrow 0} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0)\, \mathbf{P}\Big\{\bigcap_{\ell=0}^{k-1}\Big\{\sum_{i=0}^{k-\ell}\zeta_i(a)+\eta_{ka}\leq 0\Big\}\Big\}.
\end{aligned}
\tag{4.2}
$$

Here the first inequality is due to discretization, while the equality follows from the inclusion-exclusion formula and the fact that

$$
\begin{aligned}
&\mathbf{P}\Big\{\max_{k=0,\dots,\lfloor T/a\rfloor} \xi(h-kaq(u))>u\Big\} \\
&\quad= \sum_{k=0}^{\lfloor T/a\rfloor} \mathbf{P}\Big\{\bigcap_{\ell=0}^{k-1}\{\xi(h-\ell aq(u))\leq u\} \,\Big|\, \xi(h-kaq(u))>u\Big\}\mathbf{P}\{\xi(h-kaq(u))>u\}.
\end{aligned}
$$

The last inequality follows from (1.3) and the reasoning on the first few lines of this proof by means of dividing by $w(u)$ in the featured intersected events and the last equality follows from the assumed continuity properties of $\zeta(a)$ or $L(t,\cdot)$. For an upper bound we make some preparations: The strong Markov property gives

$$
\begin{aligned}
&\mathbf{P}\Big\{\sup_{t\in[h-Tq(u),h]} \xi(t)>u+xw(u)\Big\} \\
&\leq \mathbf{P}\Big\{\max_{k=0,\dots,\lfloor T/a\rfloor} \xi(h-kaq(u))>u\Big\} \\
&\quad + \mathbf{P}\Big\{\sup_{t\in[h-Tq(u),h]} \xi(t)>u+xw(u)\Big\}\mathbf{P}\Big\{\inf_{t\in[0,aq(u)]} \xi(t)\leq -xw(u)\Big\} \quad \text{for } x>0,
\end{aligned}
\tag{4.3}
$$

so that by rearranging

$$
\begin{aligned}
&\mathbf{P}\Big\{\sup_{t\in[h-Tq(u),h]} \xi(t)>u+xw(u)\Big\}\,\mathbf{P}\Big\{\inf_{t\in[0,aq(u)]} \xi(t)>-xw(u)\Big\} \\
&\leq \mathbf{P}\Big\{\max_{k=0,\dots,\lfloor T/a\rfloor} \xi(h-kaq(u))>u\Big\} \quad \text{for } x>0.
\end{aligned}
\tag{4.4}
$$

Here we have $w(u)=o(u)$ since (1.4) implies that $\xi(h)$ belongs to the Type I domain of attraction of extremes, see Definition 2.1. By the continuity of w it hence follows that the functions u and $u+xw(u)$ range over the same values as $u \to \infty$ for any fixed $x > 0$. Hence we have

$$\limsup_{u\to\infty} g(u) = \limsup_{u\to\infty} g(u + xw(u)) \quad \text{for } x \in \mathbb{R} \text{ for any function } g. \tag{4.5}$$

From (1.3) together with basic theory of Lévy processes (see *e.g.* Sato [29], Thm. 8.7, together with Fristedt [18], p. 251), we have that $\{\xi(tq(u))/w(u)\}_{t\geq 0} \xrightarrow{\mathrm{d}} \{\zeta(t)\}_{t\geq 0}$ in the space of càdlàg functions equipped with the Skorohod J_1 topology, where $\{\zeta(t)\}_{t\geq 0}$ is a Lévy process. This gives that

$$\liminf_{a\downarrow 0} \liminf_{u\to\infty} \mathbf{P}\Big\{\inf_{t\in[0,aq(u)]} \xi(t) > -xw(u)\Big\} \geq \liminf_{a\downarrow 0} \mathbf{P}\Big\{\inf_{t\in[0,2a]} \zeta(t) > -x\Big\} = 1 \tag{4.6}$$

for $x > 0$. Using (4.3)–(4.6) together with (1.4) and (1.3), we get in the fashion of (4.2)

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t) > u\Big\} \\
&\quad = \lim_{T\to\infty} \limsup_{x\downarrow 0} \limsup_{u\to\infty} \frac{\mathrm{e}^x}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t) > u + xw(u)\Big\} \\
&\quad \leq \lim_{T\to\infty} \limsup_{x\downarrow 0} \liminf_{a\downarrow 0} \limsup_{u\to\infty} \bigg(\mathbf{P}\Big\{\inf_{t\in[0,aq(u)]} \xi(t) > -xw(u)\Big\}\bigg)^{-1} \\
&\qquad\qquad \times \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor T/a\rfloor} \xi(h - kaq(u)) > u\Big\} \\
&\qquad\quad + \lim_{T\to\infty} \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]} \xi(t) > u\Big\} \\
&\quad \leq \lim_{T\to\infty} \liminf_{a\downarrow 0} \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor T/a\rfloor} \xi(h - kaq(u)) > u\Big\} \\
&\quad \leq \lim_{T\to\infty} \liminf_{a\downarrow 0} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0)\, \mathbf{P}\bigg\{\bigcap_{\ell=0}^{k-1} \bigg\{\sum_{i=0}^{k-l} \zeta_i(a) + \eta_{ka} \leq 0\bigg\}\bigg\}.
\end{aligned} \tag{4.7}$$

Here the first equality is due to the Type I attraction, the first inequality is due to Boole's inequality (4.4), the second inequality is due to (4.6) and (1.8), while the last inequality follows from the same arguments as were used to establsih (4.2).

By (1.8) together with (4.2) and 4.7), the following three limits exist and coincide

$$\begin{aligned}
H &= \lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t) > u\Big\} \\
&= \lim_{T\to\infty} \limsup_{a\downarrow 0} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0)\, \mathbf{P}\bigg\{\bigcap_{\ell=0}^{k-1} \bigg\{\sum_{i=0}^{k-\ell} \zeta_i(a) + \eta_{ka} \leq 0\bigg\}\bigg\} \\
&= \lim_{T\to\infty} \liminf_{a\downarrow 0} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0)\, \mathbf{P}\bigg\{\bigcap_{\ell=0}^{k-1} \bigg\{\sum_{i=0}^{k-\ell} \zeta_i(a) + \eta_{ka} \leq 0\bigg\}\bigg\}.
\end{aligned} \tag{4.8}$$

As it is clear that $H \geq 1$ it only remains to show that $H < \infty$. However, this follows from applying (1.8) and (4.6) to the following version of (4.7), with $a > 0$ small enough and $T > 0$ large enough,

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t) > u\Big\} \\
&\leq \mathrm{e}^{x}\bigg(\mathbf{P}\Big\{\inf_{t\in[0,2a]} \zeta(t) > -\frac{x}{2}\Big\}\bigg)^{-1} \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor T/a\rfloor} \xi(h - kaq(u)) > u\Big\} \\
&\quad + \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]} \xi(t) > u\Big\} \\
&\leq \limsup_{x\downarrow 0}\bigg(\mathbf{P}\Big\{\inf_{t\in[0,2a]} \zeta(t) > -\frac{x}{2}\Big\}\bigg)^{-1} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0) \\
&\quad + \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]} \xi(t) > u\Big\} \quad \text{for } x > 0.
\end{aligned}$$

This concludes the proof of the full statement of Theorem 1.1. □

5. Examples

Example 5.1. Brownian motion with drift is a superexponential Lévy process ξ that has characteristic triplet $(0, m, s^2)$ for some constants $m \in \mathbb{R}$ and $s^2 > 0$.

By Proposition 2.8(2), (2.3)–(2.6) hold so that Theorem 2.2 gives (1.4) with $L(t,x) = \mathrm{e}^{-t-x}$ while Theorem 2.11(2) gives (1.3) with $\zeta(a)$ $\mathrm{N}(0, 2a)$ distributed. Further, as

$$\mu(\lambda) = m - \lambda\, s^2, \quad \sigma(\lambda) = s, \quad \mu^{\leftarrow}(u) = (m-u)/s^2 \quad \text{and} \quad \phi_1(\lambda) = \mathrm{e}^{-m\lambda + s^2\lambda^2/2},$$

(2.8) shows that

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{\mathrm{e}^{u\,(m-u/h)/s^2}\mathrm{e}^{mh(u/h-m)/s^2 + s^2h(u/h-m)^2/(2s^4)}}{\sqrt{2\pi h}\, s\,(u/h-m)/s^2} \sim \frac{s\sqrt{h}}{\sqrt{2\pi}\,u}\,\mathrm{e}^{-(u-mh)^2/(2s^2h)}$$

as $u \to \infty$, agreeing with what elementary considerations give using that $\xi(h)$ is $\mathrm{N}(mh, s^2h)$.

Notice that by Proposition 2.8(2) and Theorem 2.11(2), (1.3) and (1.4) hold with $\zeta(a)$ $\mathrm{N}(0, 2a)$ distributed and the functions w and q given by (2.7) for any Lévy process with characteristic triplet (ν, m, s^2) such that $\nu((0,\infty)) = 0$ and $s^2 > 0$. Only that now (2.8) changes as compared with above as soon as ν is not zero/absent.

Further, note that

$$\mathbf{P}\{\xi(t) > 0\} = \mathbf{P}\big\{mt + \sqrt{t}\,\mathrm{N}(0,s^2) > 0\big\} = \mathbf{P}\big\{m\sqrt{t} + \mathrm{N}(0,s^2) > 0\big\} \to \tfrac{1}{2}$$

as $t \downarrow 0$ so that (3.1) holds. Hence (1.8) holds by Proposition 3.4 so that (1.5) holds by Theorem 1.1.

We have that (1.3) and (1.4) hold with $\zeta(a)$ and $L(t,x)$ as above for any Lévy process with characteristic triplet (ν, m, s^2) such that $\nu((0,\infty)) = 0$ and $s^2 > 0$. Hence, if also (1.8) holds it follows that $H = 2$ in (1.5) by well-known properties of Brownian motion as the expression (4.8) for H depends only on $\zeta(a)$ and $L(t,x)$ and thus is same as for the triplet $(0, 0, s^2)$. For example, for a totally skewed to the left α-stable Lévy process η with $\alpha \in (0,2)$, see Examples 5.4 and 5.5 below, we have $\nu((0,\infty)) = 0$. Further, (1.1) together with [28] equation 1.1.6 shows that $\eta(t)/\sqrt{t} \stackrel{\mathrm{d}}{\to} 0$ as $t \downarrow 0$. And so (3.1) follows as before for the Lévy process $\xi + \eta$ when ξ is an independent Brownian motion with drift as before, as do then (1.5) with $H = 2$.

Example 5.2. The Merton jump-diffusion [24] is the superexponential Lévy process

$$\xi(t) = m\,t + s\,W(t) + \sum_{i=1}^{N(t)} Y_i \quad \text{for } \ t > 0,$$

where W is standard Brownian motion, $m \in \mathbb{R}$ and $s > 0$ are constants, N is a Poisson process with intensity $\gamma > 0$ and the Y_i:s independent $\mathrm{N}(0, \delta^2)$ distributed random variables.

The characteristic triplet is (ν, m, s^2) with

$$\mathrm{d}\nu/\mathrm{d}x = \frac{\gamma}{\sqrt{2\pi}\delta}\,\mathrm{e}^{-x^2/2\delta^2}.$$

Since $s^2 > 0$, (2.3) and (2.5) hold by Proposition 2.8(1), and we may take $g(x) = \sqrt{x}$ in Proposition 2.9(2) to see that (2.4) and (2.6) hold. Hence Theorem 2.2 gives (1.4) with $L(t, x) = \mathrm{e}^{-t-x}$ while we have (1.3) with $\zeta(a) = 0$ by Theorem 2.11(1). Of course, (2.8) holds but as

$$\mu(\lambda) = -\gamma\delta^2\lambda\,\mathrm{e}^{\delta^2\lambda^2/2} + m - \lambda s^2$$

does not allow an explicit closed form expression for $\mu^{\leftarrow}(u)$ the same applies to (2.8). We omit a discussion of the details.

As $\sum_{i=1}^{N(t)} Y_i/\sqrt{t} \xrightarrow{\mathrm{d}} 0$ as $t \downarrow 0$ we see that (3.1) holds as in Example 5.1. Hence (1.8) holds by Proposition 3.4 so that (1.5) holds by Theorem 1.1. Further, $\zeta(a) = 0$ gives $H = 1$ by inspection of (4.8).

Example 5.3. A rapidly decreasing tempered stable (RDTS) Lévy process [21] ξ has characteristic triplet $(\nu, m, 0)$ with

$$\frac{\mathrm{d}\nu}{\mathrm{d}x} = |x|^{-\alpha-1}\left(C_+\mathrm{e}^{-\lambda_+^2 x^2}\mathbf{1}_{\{x>0\}} + C_-\mathrm{e}^{-\lambda_-^2|x|^2}\mathbf{1}_{\{x<0\}}\right),$$

where $C_+, C_-, \lambda_+, \lambda_- > 0$, $\alpha \in (0, 2)$ and $\alpha \neq 1$. Choosing $g(x) = \frac{1}{2}\lambda_+^2 x$, we may employ Proposition 2.9(4) to show that (2.3)–(2.6) hold. Hence, Theorem 2.2 gives (1.4) with $L(t, x) = \mathrm{e}^{-t-x}$. Further, Theorem 2.11(1) shows that (1.3) holds with $\zeta(a) = 0$. Of course, (2.8) applies but again $\mu^{\leftarrow}(u)$ in (2.8) cannot be calculated explicitely in closed form.

By [21] Proposition 3.3 $\xi(t)$ has characteristic function

$$\mathbf{E}\{\mathrm{e}^{i\theta\xi(t)}\} = \exp\{it\theta\gamma + C_+t\,G(i\theta, \alpha, \lambda_+) + C_-t\,G(-i\theta, \alpha, \lambda_-)\} \quad \text{for } \theta \in \mathbb{R} \text{ and } t \geq 0, \tag{5.1}$$

for some constant $\gamma \in \mathbb{R}$ where

$$G(x, \alpha, \lambda) = 2^{-\alpha/2-1}\lambda^{\alpha}\Gamma(-\tfrac{\alpha}{2})\Big(\Phi\big(-\tfrac{\alpha}{2}, \tfrac{1}{2}, \tfrac{x^2}{2\,\lambda^2}\big) - 1\Big) + 2^{-\alpha/2-1/2}\lambda^{\alpha-1}x\,\Gamma(\tfrac{1-\alpha}{2})\Big(\Phi\big(\tfrac{1-\alpha}{2}, \tfrac{3}{2}, \tfrac{x^2}{2\,\lambda^2}\big) - 1\Big)$$

and Φ is the confluent hypergeometric function (sometimes denoted ${}_1F_1$).

We now restrict ourselves to the argubly most interesting case when $\alpha \in (1, 2)$, as $\alpha \in (0, 1)$ means that $\xi(t)$ is a finite variation process, *i.e.*, the difference between two subordinators (see, *e.g.*, [29] Thm. 21.5), although the latter case can also be studied with a technique similar to that we now employ: Using the fact that

$$\Phi(a, c, x) \sim \frac{\Gamma(c)}{\Gamma(c-a)}\,(-x)^{-a} \quad \text{as } \ x \to -\infty$$

(*e.g.*, [16] Sect. 6.13.1) together with a so called multiplication formula for the Gamma function (*e.g.*, [16] Sect. 1.2) we may readily conclude from (5.1) that

$$\xi(t)/t^{1/\alpha} \xrightarrow{\mathrm{d}} S_\alpha\Big(\big((C_+{+}C_-)[-\cos(\tfrac{\pi\alpha}{2})]\Gamma(-\alpha)\big)^{1/\alpha}, \frac{C_+{-}C_-}{C_+{+}C_-}, 0\Big) \quad \text{as } t\downarrow 0$$

(cf., *e.g.*, [28] Eq. 1.1.6). It follows that

$$\begin{aligned}\mathbf{P}\{\xi(t)>0\} &= \mathbf{P}\big\{\xi(t)/t^{1/\alpha}>0\big\}\\ &\to \mathbf{P}\Big\{S_\alpha\Big(\big((C_+{+}C_-)[-\cos(\tfrac{\pi\alpha}{2})]\Gamma(-\alpha)\big)^{1/\alpha}, \frac{C_+{-}C_-}{C_+{+}C_-}, 0\Big)>0\Big\}>0\end{aligned}$$

as $t\downarrow 0$. Hence (3.1) holds so that (1.8) holds by Proposition 3.4. And so (1.5) holds by Theorem 1.1. In addition we have $H=1$ for the same reason as in Example 5.2.

Example 5.4. Pick a constant $\alpha\in(1,2)$. A totally skewed to the left α-stable Lévy process ξ has characteristic triplet $(\nu,m,0)$ with

$$\frac{\mathrm{d}\nu(x)}{\mathrm{d}x} = \frac{Q}{(-x)^{\alpha+1}} \quad \text{for } x<0, \text{ for some constant } Q>0.$$

By Proposition 2.10(6), ξ satisfies (2.3)–(2.6) so that Theorem 2.2 shows that (1.4) holds with $L(t,x)=\mathrm{e}^{-t-x}$. Further, Theorem 2.11(3) shows that (1.3) holds with $\zeta(a)$ having a $S_\alpha((-a\cos(\frac{\pi\alpha}{2}))^{1/\alpha},-1,0)$ distribution.

Switching to the well established notation of [28] $\xi(1)$ is $S_\alpha(\sigma,-1,\eta)$ distributed for some $\sigma>0$ and $\eta\in\mathbb{R}$ meaning that ([28], Prop. 1.2.12)

$$\phi_1(\lambda) = \exp\Big\{\frac{\sigma^\alpha(-\lambda)^\alpha}{[-\cos(\frac{\pi\alpha}{2})]} - \eta\lambda\Big\} \quad \text{for } \lambda\le 0.$$

Hence we have

$$\begin{aligned}\mu(\lambda) &= \frac{\alpha\sigma^\alpha(-\lambda)^{\alpha-1}}{[-\cos(\frac{\pi\alpha}{2})]} + \eta\\ \sigma(\lambda)^2 &= \frac{\alpha(\alpha{-}1)\sigma^\alpha(-\lambda)^{\alpha-2}}{[-\cos(\frac{\pi\alpha}{2})]}\\ \mu^{\leftarrow}(u) &= -\Big(\frac{(u{-}\eta)[-\cos(\frac{\pi\alpha}{2})]}{\alpha\sigma^\alpha}\Big)^{1/(\alpha-1)}\end{aligned}$$

for $\lambda\le 0$ and large u. Inserting this in (2.8) straightforward calculations gives

$$\mathbf{P}\{\xi(h)>u\} \sim \frac{u^{-\frac{\alpha}{2(\alpha-1)}}}{\sqrt{2\pi(\alpha{-}1)}}\Big(\frac{\alpha h\sigma^\alpha}{[-\cos(\frac{\pi\alpha}{2})]}\Big)^{\frac{1}{2(\alpha-1)}} \exp\Big\{-\frac{\alpha{-}1}{\alpha}\Big(\frac{[-\cos(\frac{\pi\alpha}{2})]}{\alpha h\sigma^\alpha}\Big)^{\frac{1}{\alpha-1}}(u{-}h\eta)^{\frac{\alpha}{\alpha-1}}\Big\}$$

as $u\to\infty$. This result follows from, *e.g.*, Ibragimov and Linnik [20] Theorem 2.4.7. See also, *e.g.*, Minjheer [25] Lemma 2.1 and [28], equation 1.2.11. Note that the treatment of [20] is analytically exceptionally complicated and valid for the α-stable case only while our proof is just by insertion in (2.8).

By Albin [1], Theorem 1, (1.5) holds with $H > 1$ for $\eta = 0$. Here we use Theorem 1.1 to extend Albin's result to a general η without using difficult results from the literature about α-stable distributions, contrary to what did Albin.

By $1/\alpha$-self-similarity of $\xi(t) - \eta t$ (*e.g.*, [28] Eq. 1.1.6) we have

$$\liminf_{t\downarrow 0} \mathbf{P}\{\xi(t) > 0\} = \liminf_{t\downarrow 0} \mathbf{P}\big\{\xi(t)/t^{1/\alpha} > 0\big\} = \liminf_{t\downarrow 0} \mathbf{P}\big\{\xi(1) - \eta + \eta\, t^{1-1/\alpha} > 0\big\} > 0.$$

Hence (3.1) holds so that (1.8) holds by Proposition 3.4. And so Theorem 1.1 gives (1.5).

Example 5.5. A totally skewed to the left 1-stable Lévy process ξ has charactersitic triplet $(\nu, m, 0)$ with

$$\frac{\mathrm{d}\nu(x)}{\mathrm{d}x} = \frac{Q}{\pi(-x)^2} \quad \text{for } \ x<0, \ \text{ for some constants } \ Q>0 \ \text{ and } \ m\in\mathbb{R}.$$

By Theorem 2.12 (1.3) holds with $\zeta(a) = a$ while (1.4) holds with $L(0,x) = \mathrm{e}^{-x}$ and $L(t,x) = 0$ for $t > 0$.

Switching to the well established notation of [28] $\xi(1)$ is $S_1(\sigma, -1, \eta)$ distributed for some $\sigma > 0$ and $\eta \in \mathbb{R}$ meaning that (*e.g.*, [28], Prop. 1.2.12)

$$\phi_1(\lambda) = \exp\bigg\{\frac{2\sigma}{\pi}\,(-\lambda)\ln(-\lambda) - \eta\lambda\bigg\} \quad \text{for } \ \lambda \le 0,$$

Hence we have

$$\mu(\lambda) = \frac{2\sigma}{\pi}\big(\ln(-\lambda)+1\big) + \eta, \quad \sigma(\lambda)^2 = \frac{2\sigma}{\pi(-\lambda)} \quad \text{and} \quad \mu^{\leftarrow}(u) = \exp\bigg\{\frac{\pi(u-\eta)}{2\sigma} + 1\bigg\}$$

for $\lambda \le 0$ and large u. Inserting this in (2.8) straightforward calculations give

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{\sqrt{e}}{2\sqrt{h\sigma}} \exp\bigg\{-\frac{\pi\,(u-h\eta)}{4h\sigma} - \frac{2h\sigma}{\pi}\exp\Big(\frac{\pi\,(u-h\eta)}{2h\sigma} - 1\Big)\bigg\} \quad \text{as } \ u\to\infty.$$

This result fits with, *e.g.*, Ibragimov and Linnik [20] Theorem 2.4.4. See also *e.g.*, Minjheer [25] Lemma 2.1. The result does not fit with [28], equation 1.2.11, which seems to be because their formula comes from Zolotarev [31] which uses another parametrization of 1-stable distributions. Again, the treatment of [20] is analytically exceptionally complicated and valid for the 1-stable case only while our proof is just by insertion in (2.8).

By Albin [1], Theorem 2, (1.5) holds with $H = 1$ for $\eta = 0$. Here we use Theorem 1.1 to extend Albin's result to a general η without using difficult results from the literature about 1-stable distributions, contrary to what did Albin.

Further, equation 1.2.1 in [28] shows that

$$\mathbf{P}\{\xi(t) > 0\} = \mathbf{P}\{\xi(t)/t > 0\} = \mathbf{P}\big\{\xi(1) - \tfrac{2}{\pi}\sigma\ln(t) > 0\big\} \to 1 \quad \text{as } \ t\downarrow 0.$$

Hence (3.1) holds so that (1.8) holds by Proposition 3.4. And so (1.5) holds by Theorem 1.1. Finally an inspection of (4.8) shows that $H = 1$ when $\zeta(a) = a$.

The methodologies of Examples 5.4 and 5.5 readily carry over to, for example, the sum of two independent totally skewed stable Lévy processes with different stability indices.

Example 5.6. An unnamed superexponential Lévy process ξ is defined by Linnik and Ostrovskiĭ [23] pp. 52–53, see also Sato [29], Exercise 18.19, as having characteristics $(\nu, m, 0)$, where

$$\frac{\mathrm{d}\nu(x)}{\mathrm{d}x} = \frac{\mathrm{e}^{bx}}{|x|(1-\mathrm{e}^{ax})} \quad \text{for } x<0, \quad \text{for some constants } a,b>0.$$

For a suitable constant $c = c(a,b,m) > 0$ the corresponding Laplace transform is given by

$$\phi_1(\lambda) = \frac{\Gamma((b-\lambda)/a)c^{\lambda/a}}{\Gamma(b/a)} \quad \text{for } \lambda \le 0.$$

By Theorem 2.12, (1.3) holds with $\zeta(a') = a'$ while (1.4) holds with $L(0,x) = \mathrm{e}^{-x}$ and $L(t,x) = 0$ for $t>0$.

We finish by demonstrating how (2.8) gives the asymptotics of $\mathbf{P}\{\xi(h)>u\}$ as $u\to\infty$ in (1.5): Taking $a=1$ for simplicity and denoting the polygamma function ψ (see *e.g.* Erdélyi, Magnus, Oberhettinger and Tricomi [16], Sects. 1.16–1.17), we have

$$\begin{aligned}
\mu(\lambda) &= -\ln(c) + \psi(b-\lambda) = \ln((b-\lambda)/c) - \frac{1}{2(b-\lambda)} + O\Big(\frac{1}{\lambda^2}\Big) && \text{as } \lambda\to-\infty,\\
\sigma(\lambda)^2 &= \psi'(b-\lambda) = \frac{1}{b-\lambda} + O\Big(\frac{1}{\lambda^2}\Big) && \text{as } \lambda\to-\infty,\\
\mu^{\leftarrow}(x) &= b - \tfrac{1}{2} - c\mathrm{e}^{x} + O(\mathrm{e}^{-x}) && \text{as } x\to\infty.
\end{aligned}$$

Using this together with (2.8) and Stirling's formula, we get

$$\mathbf{P}\{\xi(h)>u\} \sim \frac{(2\pi)^{(h-1)/2}}{\sqrt{hc}\,\Gamma(b)^h}\exp\Big\{-(cu + hc\ln(c/\mathrm{e}))\mathrm{e}^{u/h} - \frac{u}{2h}\Big\} \quad \text{as } u\to\infty.$$

Picking a function $g(t)>0$ with $\lim_{t\downarrow 0} t\ln(1/g(t))/g(t) = 1$, Stirling's formula (see, *e.g.*, Erdélyi, Magnus, Oberhettinger and Tricomi [16], Eq. 1.18.2) gives

$$\phi_1\Big(\frac{\lambda}{g(t)}\Big)^t \sim \frac{1}{(2\pi)^{t/2}}\exp\Big\{t\Big(\frac{bg(t)-\lambda}{ag(t)} - \frac{1}{2}\Big)\ln\Big(\frac{bg(t)-\lambda}{ag(t)}\Big) - \frac{bt}{a} + \frac{\ln(c/\mathrm{e})\lambda t}{ag(t)}\Big\} \to \mathrm{e}^{-\lambda/a}$$

as $t\downarrow 0$ for $\lambda\le 0$, so that

$$\liminf_{t\downarrow 0}\mathbf{P}\{\xi(t)>0\} = \liminf_{t\downarrow 0}\mathbf{P}\{\xi(t)/g(t)>0\} \ge \mathbf{P}\{1/a>0\} = 1.$$

Hence (3.1) holds so that (1.8) holds by Proposition 3.4. And so Theorem 1.1 shows that (1.5) holds. In addition we have $H=1$ for the same reason as in Example 5.5.

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