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Weighted Bergman kernels for nearly holomorphic functions on bounded symmetric domains ☆

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A B S T R A C T

We identify the standard weighted Bergman kernels of spaces of nearly holomorphic functions, in the sense of Shimura, on bounded symmetric domains. This also yields a description of the analogous kernels for spaces of "invariantly-polyanalytic" functions — a generalization of the ordinary polyanalytic functions on the ball which seems to be the most appropriate one from the point of view of holomorphic invariance. In both cases, the kernels turn out to be given by certain spherical functions, or equivalently Heckman-Opdam hypergeometric functions, and a conjecture relating some of these to a Faraut-Koranyi hypergeometric function is formulated based on the study of low rank situations. Finally, analogous results are established also for compact Hermitian symmet-

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ric spaces, where explicit formulas in terms of multivariable Jacobi polynomials are given.

1. Introduction

Let Ω be an irreducible bounded symmetric domain in $\mathbf{C}^d$, $d \geq 1$, in its Harish-Chandra realization, and denote by p its genus and by $h(z,w)$ the associated Jordan triple determinant, which is a holomorphic polynomial in z and $\overline{w}$ on $\mathbf{C}^d$. The standard weighted Bergman spaces on Ω are the spaces

$$A_\nu(\Omega) \equiv A_\nu := L^2(\Omega, d\mu_\nu) \cap \mathcal{O}(\Omega) \tag{1}$$

of all holomorphic functions on Ω square-integrable with respect to the measure

$$d\mu_\nu(z) := h(z,z)^{\nu-p}\, dz, \tag{2}$$

where dz stands for the Lebesgue measure. It is well known that A_ν is nontrivial if and only if $\nu > p-1$, and in that case A_ν possesses a reproducing kernel — the *weighted Bergman kernel* — given by

$$K_\nu(z,w) = c_\nu h(z,w)^{-\nu}, \tag{3}$$

where $c_\nu = 1/\mu_\nu(\Omega)$ is a constant which can be evaluated explicitly.

For any $\phi \in \mathrm{Aut}(\Omega)$, the group of all biholomorphic self-maps of Ω, the Jordan triple determinant satisfies the transformation rule

$$h(\phi z, \phi w) = \frac{h(a,a)h(z,w)}{h(z,a)h(a,w)}, \qquad a = \phi^{-1}0. \tag{4}$$

(We will mostly write just ϕz instead of $\phi(z)$.) It follows that the Riemannian metric

$$ds^2 = -\sum_{j,k=1}^{d} \frac{\partial^2 \log h(z,z)}{\partial z_j \partial \overline{z}_k} \tag{5}$$

is invariant under $\mathrm{Aut}(\Omega)$. Recall now that, quite generally, for an arbitrary Kähler manifold Ω with Kähler metric $ds^2 = \sum_{j,k} g_{j\overline{k}}\, dz_j\, d\overline{z}_k$, the invariant Cauchy-Riemann operator $\overline{D}$, introduced by Peetre [23], is the map from functions into vector fields of type (1,0) defined by

$$\overline{D}f = (\overline{D}f)^j \frac{\partial}{\partial z_j}, \qquad (\overline{D}f)^j = g^{\overline{l}j}\overline{\partial}_l f,$$

where we have started to employ the Einstein summation convention, and also to write for brevity $\overline{\partial}_l := \partial/\partial\overline{z}_l$; namely, it is the $\overline{\partial}$ operator combined with the Riesz lemma identifying (0,1)-forms with vector fields of type (1,0). Here $g^{\overline{k}j}$ is the inverse matrix to $g_{j\overline{k}}$. One can iterate this construction and set, for $m = 1, 2, \dots,$

$$(\overline{D}^m f)^{k_m \dots k_1} = g^{\bar{l}_m k_m}\overline{\partial}_{l_m} \dots g^{\bar{l}_2 k_2}\overline{\partial}_{l_2} g^{\bar{l}_1 k_1}\overline{\partial}_{l_1} f.$$

It turns out that the tensor field $(\overline{D}^m f)^{k_m \dots k_1}$ is symmetric in the indices $k_1, \dots, k_m$ [23], and in fact coincides with the contravariant derivative $f^{/k_1 \dots k_m}$ with respect to the Hermitian connection [12]. The m-th Cauchy-Riemann space $\mathcal{N}^m$ [13], or the space of *nearly holomorphic functions of order* m, is, by definition, the kernel of $\overline{D}^m$:

$$\mathcal{N}^m(\Omega) \equiv \mathcal{N}^m := \{f \in C^\infty(\Omega) : \overline{D}^m f = 0 \text{ on } \Omega\}.$$

An alternative definition is due to Shimura [28]: $\mathcal{N}^m$ is the vector space of all functions on Ω that can (locally) be written as polynomials of degree $< m$ in the derivatives $\partial_j \Psi$, with holomorphic coefficients, where Ψ is a (local) potential for the Kähler metric, i.e. $g_{j\overline{k}} = \overline{\partial}_k \partial_j \Psi$. (This space does not depend on the choice of the local potential Ψ.) See e.g. Proposition 7 in [13] for a proof of the equivalence of these two definitions.

The above construction applies, in particular, to our bounded symmetric domain Ω with the invariant metric (5), possessing a global Kähler potential $\Psi(z) = -\log h(z, z)$. In analogy with (1), we can consider the *weighted Bergman spaces of nearly holomorphic functions*

$$\mathcal{N}^m_\nu := L^2(\Omega, d\mu_\nu) \cap \mathcal{N}^m. \tag{6}$$

Of course, if $m = 1$ then $\mathcal{N}^1 = \mathcal{O}(\Omega)$ and $\mathcal{N}^1_\nu = A_\nu$ for any ν.

For the simplest bounded symmetric domain $\Omega = \mathbf{B}^d$, the unit ball of $\mathbf{C}^d$, $d \geq 1$, the Jordan triple determinant is given simply by $h(z, w) = 1 - \langle z, w\rangle$, so that $\Psi_j = \frac{\overline{z}_j}{1-|z|^2}$. Nearly holomorphic functions of order m on $\mathbf{B}^d$ are thus precisely the polynomials of degree $\leq m - 1$ in $(1 - |z|^2)^{-1}\overline{z}$, with holomorphic coefficients. In other words,

$$\mathcal{N}^m(\mathbf{B}^d) = (1 - |z|^2)^{1-m}\mathcal{P}^m(\mathbf{B}^d), \tag{7}$$

where $\mathcal{P}^m(\mathbf{B}^d)$ consists, by definition, of all linear combinations, with holomorphic coefficients, of $(1 - |z|^2)^{m-1-|\alpha|}\overline{z}^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multiindex of length $|\alpha| := \alpha_1 + \dots + \alpha_d < m$; that is, by a simple check, $\mathcal{P}^m(\mathbf{B}^d)$ consists of all polynomials of degree $\leq m - 1$ in $\overline{z}$, with holomorphic coefficients. (Indeed, in one direction, $(1 - |z|^2)^{m-1-|\alpha|}\overline{z}^\alpha$ is clearly a polynomial in $\overline{z}$ of degree $m - 1$ with holomorphic coefficients; while in the other direction,

$$\overline{z}^\alpha = (|z|^2 + (1 - |z|^2))^{m-1-|\alpha|}\overline{z}^\alpha$$

$$= \sum_{j=0}^{m-1-|\alpha|} \binom{m-1-|\alpha|}{j} |z|^{2j}(1-|z|^2)^{m-1-|\alpha|-j}\overline{z}^\alpha$$

$$= \sum_{|\beta|<m-1-|\alpha|} \binom{m-1-|\alpha|}{\beta} z^\beta (1-|z|^2)^{m-1-|\alpha+\beta|}\overline{z}^{\alpha+\beta}$$

is a linear combination of $(1-|z|^2)^{m-1-|\gamma|}\overline{z}^\gamma$ with holomorphic coefficients.) The space $\mathcal{P}^m(\mathbf{B}^d)$ is thus nothing else than the well-known space of *m-analytic* functions on the ball, as studied by many authors. The reproducing kernel of the space

$$L^2(\mathbf{B}^d, (1-|z|^2)^s\, dz) \cap \mathcal{P}^q(\mathbf{B}^d), \qquad s > -1,$$

was recently found by the second author [32] to be

$$P^q_{s+d+1}(z,w) := \frac{\Gamma(q+s+d)}{\pi^d \Gamma(q+s)} \frac{(1-\langle w,z\rangle)^{q-1}}{(1-\langle z,w\rangle)^{q+s+d}} P^{(d,s)}_{q-1}(1-2|\phi_z w|^2), \tag{8}$$

where $P^{(d,s)}_n$ denotes the Jacobi polynomial of degree n with parameters d, s, and $\phi_z \in \mathrm{Aut}(\mathbf{B}^d)$ is the biholomorphic self-map of $\mathbf{B}^d$ interchanging z and the origin. Returning to our general bounded symmetric domain Ω, we are thus led to define, by analogy with (7), the space of *invariantly polyanalytic functions of order m on* Ω as

$$\mathcal{P}^m(\Omega) \equiv \mathcal{P}^m := h(z,z)^{m-1}\mathcal{N}^m, \tag{9}$$

and consider the corresponding weighted Bergman spaces

$$\mathcal{P}^m_\nu := L^2(\Omega, d\mu_\nu) \cap \mathcal{P}^m. \tag{10}$$

Our aim in this paper is to find the reproducing kernels N^m_ν and P^m_ν of the spaces $\mathcal{N}^m_\nu$ and $\mathcal{P}^m_\nu$, respectively, thus generalizing the formulas (3) (which corresponds to $m=1$) and (8) (which corresponds to $\Omega = \mathbf{B}^d$).

On an abstract level, the answer is given by the group representation theory, more specifically, by the Plancherel formula for certain representations of the identity connected component G of the automorphism group $\mathrm{Aut}(\Omega)$ of Ω. Namely, from the fact that $\overline{D}^m f$ is a tensor, it follows that the action of G by composition preserves the space $\mathcal{N}^m$; in other words,

$$f \longmapsto f \circ \phi^{-1}, \qquad f \in \mathcal{N}^m,\ \phi \in G,$$

is a representation of G on $\mathcal{N}^m$. In combination with the transformation rule for μ_ν,

$$d\mu_\nu(\phi z) = \left| \frac{h(a,a)^{\nu/2}}{h(z,a)^\nu} \right|^2 d\mu_\nu(z), \qquad a = \phi^{-1}0,\ \phi \in G, \tag{11}$$

which follows from (4), this implies that

$$f \longmapsto \frac{h(a,a)^{\nu/2}}{h(z,a)^{\nu}} f \circ \phi^{-1}, \qquad a = \phi 0,\ \phi \in G, \tag{12}$$

is a projective unitary representation of G on $\mathcal{N}_\nu^m$. It is now a result of the third author [35] that for each $m = 1, 2, \ldots$, $\mathcal{N}_\nu^m$ comes as an orthogonal direct sum of irreducible components which can be identified with certain so-called relative discrete series representations of G. Finally, a general Plancherel formula of Shimeno [27], applied to these representations, implies that the reproducing kernel at the origin of each of these irreducible components must be a constant multiple of $\phi_{\lambda,\ell}$, the spherical function of G with parameter ℓ (describing the representation, actually $\ell = \nu$) and weight λ (uniquely associated to each of the irreducible components). In this way, the reproducing kernel N_ν^m is thus expressed as a finite sum of terms involving spherical functions. (For the particular case of $\Omega = \mathbf{B}^d$, this expression was obtained in [33].)

Recoursing to the available theory of multivariable special functions (see e.g. Anker [1]), the spherical functions $\phi_{\lambda,\ell}$ can also be expressed as Heckman-Opdam hypergeometric functions, or, if one wishes, as multivariable Jacobi polynomials of Debiard [8] (and many other authors). For instance, the result for $\mathbf{B}^d$ from [33] just mentioned reads

$$N_{s+d+1}^m(z,w) = k_{s+d+1}^m(|\phi_w z|^2)$$

with

$$k_{s+d+1}^m(t) := \sum_{l=0}^{m-1} c_l(s) {}_2F_1\left(\begin{matrix} -l, l-s-1 \\ d \end{matrix} \middle| \frac{t}{t-1}\right), \tag{13}$$

where

$$c_l(s) = \frac{(s-2l+1)\Gamma(s+d+1-l)d}{\pi^d l! \Gamma(s-l+2)},$$

and ${}_2F_1$ is the ordinary (Gauss) hypergeometric function. On the other hand, from (8) one can express P_{s+d+1}^m and N_{s+d+1}^m in terms of a single Jacobi polynomial $P_{m-1}^{(d,s)}$. Comparing both expressions leads (after working out the details) to the equality

$$(1-t)^{q-1} \sum_{l=0}^{q-1} c_l(s+2q-2) {}_2F_1\left(\begin{matrix} -l, l-s-2q+1 \\ d \end{matrix} \middle| \frac{t}{t-1}\right) = \frac{\Gamma(q+s+d)}{\pi^d \Gamma(q+s)} P_{q-1}^{(d,s)}(1-2t). \tag{14}$$

It is amusing to prove this (valid) formula directly (cf. Lemma 33 below); note that

$$P_n^{(d,s)}(1-2t) = \binom{n+d}{d} {}_2F_1\left(\begin{matrix} -n, n+1+s+d \\ d+1 \end{matrix} \middle| t\right). \tag{15}$$

Performing explicit computer calculations for rank 2 and rank 3 bounded symmetric domains indicates that, analogously to the rank 1 situation just described, even for general bounded symmetric domains the kernels N_ν^m and P_ν^m can in some cases be expressed not only as a finite sum of terms involving Heckman-Opdam hypergeometric functions, but actually as a constant multiple of a single special function, namely a hypergeometric function of Faraut and Koranyi [14] with certain parameters. We offer a conjecture to this effect, together with some consequences that would follow; the latter include relations among the two kinds of hypergeometric functions, as well as a generalization of a theorem of Helgason [16, Theorem V.4.5] describing, in effect, the reproducing kernel for a certain space of radial functions on the complex projective space $\mathbf{C}P^d$.

The paper is organized as follows. Section 2 contains the necessary background material on bounded symmetric domains. Section 3 lists some elementary facts about the kernels N_ν^m and P_ν^m and discusses radial nearly-holomorphic functions, which are relevant for the sequel. The expressions for N_ν^m in terms of spherical functions and Heckman-Opdam hypergeometric functions are presented in Section 4. Section 5 describes the computations for particular bounded symmetric domains and the resulting conjectures mentioned above. The final section, Section 6, briefly treats also the dual case of compact Hermitian symmetric spaces.

2. Prerequisites on bounded symmetric domains

Throughout the rest of this paper, Ω will be an irreducible bounded symmetric domain in $\mathbf{C}^d$ in its Harish-Chandra realization (i.e. a Cartan domain). We denote by G the identity connected component of the group $\mathrm{Aut}(\Omega)$ of all biholomorphic self-maps of Ω, and by K the stabilizer in G of the origin $0 \in \Omega$. Then K consists precisely of the unitary maps on $\mathbf{C}^d$ that preserve Ω, and Ω is isomorphic to the coset space G/K. We further denote by r, a, b and p the rank, the characteristic multiplicities and the genus of Ω, respectively, so that

$$p = (r-1)a + b + 2, \qquad d = \frac{r(r-1)}{2}a + rb + r. \tag{16}$$

If $b = 0$, Ω is said to be of *tube type*.

Irreducible bounded symmetric domains were completely classified by E. Cartan. There are four infinite series of such domains plus two exceptional domains in $\mathbf{C}^{16}$ and $\mathbf{C}^{27}$. For future reference, we include a table with brief descriptions of these domains and with the corresponding values of r, a, b, p and d. The symbol $\mathbf{O}$ stands for the division algebra of octonions.

The unit balls $\mathbf{B}^d = I_{1d}$ are the only bounded symmetric domains of rank 1, and the only bounded symmetric domain with smooth boundary.

For $x \in \Omega$, ϕ_x will denote the (unique) geodesic symmetry which interchanges x and the origin, i.e.

Domain	Description	
I_{mn}	$Z \in \mathbf{C}^{m\times n}$: $\|Z\|_{\mathbf{C}^n\to\mathbf{C}^m} < 1$ $r=m,\ a=2,\ b=n-m,\ p=n+m,\ d=mn$	$n \geq m \geq 1$
II_n	$Z \in I_{nn},\ Z=Z^t$ $r=n,\ a=1,\ b=0,\ p=n+1,\ d=\frac{1}{2}n(n+1)$	$n \geq 2$
III_m	$Z \in I_{mm},\ Z=-Z^t$ $r=[\frac{m}{2}],\ a=4,\ b=2(m-2r),\ p=2m-2,\ d=\frac{1}{2}m(m-1)$	$m \geq 5$
IV_n	$Z \in \mathbf{C}^{n\times 1},\ \|Z^tZ\| < 1,\ 1+\|Z^tZ\|^2-2Z^*Z > 0$ $r=2,\ a=n-2,\ b=0,\ p=d=n$	$n \geq 5$
V	$Z \in \mathbf{O}^{1\times 2},\ \|Z\| < 1$ $r=2,\ a=6,\ b=4,\ p=12,\ d=16$	
VI	$Z \in \mathbf{O}^{3\times 3},\ Z=Z^*,\ \|Z\| < 1$ $r=3,\ a=8,\ b=0,\ p=18,\ d=27$	

$$\phi_x \circ \phi_x = \mathrm{id},\ \phi_x(0) = x,\ \phi_x(x) = 0, \tag{17}$$

and ϕ_x has only an isolated fixed-point. (In fact, ϕ_x has only one fixed point, namely the geodesic mid-point between 0 and x.) Note that from the definition of K it is immediate that any $\phi \in G$ is of the form $\phi = \phi_x k$, where $k \in K$ and $x \in \Omega$. (In fact $x = \phi(0)$.)

It is known that the ambient space $\mathbf{C}^d =: Z$ possesses a structure of *Jordan-Banach $*$-triple system* (or *JB*-triple* for short) for which Ω is the open unit ball. That is, there exists a Jordan triple product

$$\{\cdot,\cdot,\cdot\} : Z \times Z \times Z \to Z, \qquad x, y, z \mapsto \{x, y, z\},$$

(linear and symmetric in x, z and anti-linear in y) such that

$$\Omega = \{z \in Z : \|\{z, z, \cdot\}\| < 1\}.$$

Moreover, if one uses the notation, for $x, y \in Z$,

$$\begin{aligned} D(x,y) &= \{x, y, \cdot\} :\ Z \to Z, \\ Q(x) &= \{x, \cdot, x\} :\ Z \to Z, \end{aligned}$$

then for every $x \in \Omega$, $D(x, x)$ is Hermitian and has nonnegative spectrum, and $iD(x, x)$ is a triple derivation. The linear operator

$$B(x,y) = I - 2D(x,y) + Q(x)Q(y) \tag{18}$$

on Z is called the *Bergman operator*.

Two vectors $x, y \in Z$ are said to be *orthogonal* (in the Jordan-theoretic sense) if $D(x, y) = 0$, and a vector $v \in Z$ is called a *tripotent* if $\{v, v, v\} = v$. For any tripotent v, the ambient space admits the *Peirce decomposition*

$$Z = Z_0(v) \oplus Z_{1/2}(v) \oplus Z_1(v) \tag{19}$$

into the orthogonal components

$$Z_{j/2}(v) := \{z \in Z : D(v,v)z = \frac{j}{2}z\}.$$

(The orthogonality is only with respect to the inner product in $\mathbf{C}^d$, not in the triple-product (Jordan-theoretic) sense.) Each $Z_{j/2}(v)$ is a subtriple of Z, and $Z_1(v)$ is a JB*-algebra under the product $x \circ y = \{xvy\}$, with unit v and involution $z^* = \{vzv\}$. A tripotent v is called *minimal* if $\dim Z_1(v) = 1$. Any maximal set $e_1, \dots, e_r$ of pairwise orthogonal minimal tripotents is called a *Jordan frame*; its cardinality r is independent of the frame and equal to the rank r of Ω. For any Jordan frame $e_1, \dots, e_r$, we similarly as above have the *joint Peirce decomposition*

$$Z = \bigoplus_{0 \le i \le j \le r} Z_{ij} \tag{20}$$

with

$$Z_{ij} = \{z \in Z : D(e_k, e_k)z = \frac{\delta_{ik} + \delta_{jk}}{2} \ \forall k = 1, \dots, r\}. \tag{21}$$

Given any Jordan frame $e_1, \dots, e_r$ — which we choose and fix once and for all from now on — any $z \in Z$ has a *polar decomposition*

$$z = k(t_1 e_1 + \cdots + t_r e_r) \tag{22}$$

with $k \in K$ and $t_1 \ge t_2 \ge \cdots \ge t_r \ge 0$; the numbers $t_1, \dots, t_r$, called the *singular numbers* of z, are determined uniquely, but k need not be (it is if all the t_j are distinct). Further, $z \in \Omega$ if and only if $t_1 < 1$, $z \in \partial\Omega$ if and only if $t_1 = 1$, and z belongs to the Shilov boundary $\partial_e \Omega$ of Ω if and only if $t_1 = \cdots = t_r = 1$; that is, if and only if $z = ke$, where $e = e_1 + \cdots + e_r$ is a *maximal tripotent*.

Since the Jordan triple product is invariant under K (i.e. $\{kx, ky, kx\} = k\{x, y, z\}$ $\forall k \in K$), it is immediate from (21) that under the decomposition (20), the Bergman operator $B(z,z)$ with z as in (22) is given by

$$B(z,z)|_{Z_{ij}} = (1 - t_i^2)(1 - t_j^2) I|_{Z_{ij}} \tag{23}$$

(where $t_0 := 0$).

There exists a unique polynomial $h(x,y)$ on $\mathbf{C}^d \times \mathbf{C}^d$, holomorphic in x and anti-holomorphic in y, which is K-invariant, in the sense that

$$h(kx, ky) = h(x,y) \qquad \forall k \in K,$$

and satisfies

$$h(z,z)=\prod_{j=1}^{r}(1-t_j^2)\qquad \text{for } z \text{ as in (22).}$$

It is known that $h(x,y)$ is irreducible, of degree r in x as well as in $\overline{y}$, and $h(x,0)=h(0,x)=1\ \forall x\in\mathbf{C}^d$; also, $h(x,y)^p=\det B(x,y)$. Further, the measure

$$h(z,z)^{\nu-p}\,dz \tag{24}$$

is finite if and only if $\nu>p-1$, and the corresponding weighted Bergman kernel — i.e. the reproducing kernel of the space of all holomorphic functions on Ω square-integrable with respect to (24) — is equal to

$$K_\nu(x,y)=c_\nu h(x,y)^{-\nu} \tag{25}$$

where

$$c_\nu=\frac{\Gamma_\Omega(\nu)}{\pi^d\Gamma_\Omega(\nu-\frac{d}{r})}. \tag{26}$$

Here Γ_Ω is the *Gindikin-Koecher Gamma function*

$$\Gamma_\Omega(\nu):=\prod_{j=1}^{r}\Gamma\Big(\nu-\frac{j-1}{2}a\Big).$$

In the polar coordinates (22), the measures (24) assume the form

$$\int_\Omega f(z)\,h(z,z)^\nu\,dz=\quad c_\Omega\int_{[0,1]^r}\int_K f(k\sum_{j=1}^{r}\sqrt{t_j}e_j)\,dk\,d\mu_{b,\nu,a}(t), \tag{27}$$

where $d\mu_{b,\nu,a}$ is the *Selberg measure*

$$d\mu_{b,\nu,a}(t):=\prod_{j=1}^{r}(1-t_j)^{\nu-p}\prod_{j=1}^{r}t_j^b\prod_{1\le i<j\le r}|t_i-t_j|^a\,dt, \tag{28}$$

where $dt\equiv dt_1\dots dt_r$. Here dk is the normalized Haar measure on the (compact) group K, and

$$c_\Omega=\frac{\pi^d\Gamma(\frac{a}{2}+1)^r}{\Gamma_\Omega(\frac{ra}{2}+1)\Gamma_\Omega(\frac{d}{r})}. \tag{29}$$

Let $\mathbf{P}$ denote the vector space of all (holomorphic) polynomials on $\mathbf{C}^d$. We endow $\mathbf{P}$ with the *Fock* (or *Fischer*) inner product

$$\langle f, g\rangle_F := \pi^{-d} \int_{\mathbf{C}^d} f(z)\,\overline{g(z)}\,e^{-|z|^2}\,dz \\ = (f(\partial)g^*)(0) = (g^*(\partial)f)(0), \tag{30}$$

where

$$g^*(z) := \overline{g(\overline{z})}.$$

This makes $\mathbf{P}$ into a pre-Hilbert space, and the action

$$f \mapsto f \circ k^{-1}, \qquad k \in K,$$

is a unitary representation of K on $\mathbf{P}$. It is a deep result of W. Schmid [26] that this representation has a multiplicity-free decomposition into irreducibles

$$\mathbf{P} = \bigoplus_{\mathbf{m}} \mathbf{P_m}$$

where $\mathbf{m}$ ranges over all signatures, i.e. r-tuples $\mathbf{m} = (m_1, m_2, \dots, m_r) \in \mathbf{Z}^r$ satisfying $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$. Polynomials in $\mathbf{P_m}$ are homogeneous of degree $|\mathbf{m}| := m_1 + m_2 + \cdots + m_r$; in particular, $\mathbf{P}_{(0)}$ are the constants and $\mathbf{P}_{(1)}$ the linear polynomials. Any holomorphic function on Ω thus has a decomposition $f = \sum_{\mathbf{m}} f_{\mathbf{m}}$, $f_{\mathbf{m}} \in \mathbf{P_m}$, which refines the usual homogeneous expansion.

Since the spaces $\mathbf{P_m}$ are finite dimensional, they automatically possess a reproducing kernel: there exist polynomials $K_{\mathbf{m}}(x, y)$ on $\mathbf{C}^d \times \mathbf{C}^d$, holomorphic in x and $\overline{y}$, such that for each $f \in \mathbf{P_m}$ and $y \in \mathbf{C}^d$,

$$f(y) = \langle f, K_{\mathbf{m}}(\cdot, y)\rangle_F. \tag{31}$$

From the definition of the spaces $\mathbf{P_m}$ it also follows that the kernels $K_{\mathbf{m}}(x, y)$ are K-invariant.

It is a consequence of Schur's lemma from representation theory that for any K-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathbf{P}$, $\mathbf{P_m}$ and $\mathbf{P_n}$ are orthogonal if $\mathbf{m} \neq \mathbf{n}$, while on each $\mathbf{P_m}$, $\langle\cdot,\cdot\rangle$ is proportional to $\langle\cdot,\cdot\rangle_F$. In particular, for the inner product

$$\langle f, g\rangle_\nu := c_\nu \int_\Omega f(z)\,\overline{g(z)}\,d\mu_\nu(z) \qquad (\nu > p - 1),$$

(with c_ν as in (25)) we have, for any $f_{\mathbf{m}} \in \mathbf{P_m}$ and $g_{\mathbf{n}} \in \mathbf{P_n}$,

$$\langle f_{\mathbf{m}}, g_{\mathbf{n}}\rangle_\nu = \frac{\langle f_{\mathbf{m}}, g_{\mathbf{n}}\rangle_F}{(\nu)_{\mathbf{m}}} \tag{32}$$

(cf. [14]), where $(\nu)_{\mathbf{m}}$ is the *generalized Pochhammer symbol*

$$(\nu)_{\mathbf{m}} := (\nu)_{m_1}(\nu - \tfrac{a}{2})_{m_2} \dots (\nu - \tfrac{r-1}{2}a)_{m_r};$$

here

$$(\nu)_k := \nu(\nu+1)\dots(\nu+k-1) \qquad \Big(= \frac{\Gamma(\nu+k)}{\Gamma(\nu)} \text{ if } \nu \neq 0,-1,-2,\dots,\Big)$$

is the ordinary Pochhammer symbol.

A consequence of the relation (32) is the *Faraut-Koranyi formula*

$$h(x,y)^{-\nu} = \sum_{\mathbf{m}} (\nu)_{\mathbf{m}} K_{\mathbf{m}}(x,y) \tag{33}$$

relating the reproducing kernels K_ν from (25) and $K_{\mathbf{m}}$ from (31).

As already mentioned, the point $e = e_1 + \dots + e_r$ belongs to the Shilov boundary $\partial_e\Omega$ of Ω. The group K acts transitively on $\partial_e\Omega$, so that $\partial_e\Omega = \{ke, k \in K\} \simeq K/L$, where L is the stabilizer of e in K. Each Peter-Weyl space $\mathbf{P}_{\mathbf{m}}$ contains a unique L-invariant polynomial $\phi_{\mathbf{m}}$ satisfying the normalization condition $\phi_{\mathbf{m}}(e) = 1$. We will sometimes write just $\phi_{\mathbf{m}}(t_1,\dots,t_r)$ instead of $\phi_{\mathbf{m}}(t_1e_1 + \dots + t_re_r)$. These *spherical polynomials* $\phi_{\mathbf{m}}$ satisfy $\phi_{(0)} \equiv 1$,

$$\phi_{(m_1+1,m_2+1,\dots,m_r+1)}(t_1,\dots,t_r) = t_1 \cdots t_r \, \phi_{\mathbf{m}}(t_1,\dots,t_r), \tag{34}$$

and are related to the reproducing kernels $K_{\mathbf{m}}$ by the formula

$$K_{\mathbf{m}}(x,e) = \frac{d_{\mathbf{m}}}{(d/r)_{\mathbf{m}}} \phi_{\mathbf{m}}(x), \tag{35}$$

where $d_{\mathbf{m}} := \dim \mathbf{P}_{\mathbf{m}}$. It is known that the last dimension is given by the formula ([30], Lemmas 2.5 and 2.6)

$$d_{\mathbf{m}} = \frac{(d/r)_{\mathbf{m}}}{(q_\Omega)_{\mathbf{m}}} \pi_{\mathbf{m}}$$

where

$$q_\Omega := \frac{r-1}{2}a + 1 \tag{36}$$

and

$$\pi_{\mathbf{m}} := \prod_{1 \le i < j \le r} \frac{m_i - m_j + \frac{j-i}{2}a}{\frac{j-i}{2}a} \, \frac{(\frac{j-i+1}{2}a)_{m_i - m_j}}{(\frac{j-i-1}{2}a + 1)_{m_i - m_j}}. \tag{37}$$

Thus we may rewrite (35) as

$$K_{\mathbf{m}}(x,e)=\frac{\pi_{\mathbf{m}}}{(q_\Omega)_{\mathbf{m}}}\,\phi_{\mathbf{m}}(x). \tag{38}$$

Combining the last formula with the fact that [14, Lemma 3.2]

$$K_{\mathbf{m}}(\sum_j t_je_j,\sum_j s_je_j)=K_{\mathbf{m}}(\sum_j t_js_je_j,e),$$

we thus get

$$K_{\mathbf{m}}(k\sum_j t_je_j,k\sum_j t_je_j)=\frac{\pi_{\mathbf{m}}}{(q_\Omega)_{\mathbf{m}}}\,\phi_{\mathbf{m}}(t_1^2,\dots,t_r^2). \tag{39}$$

The polynomials $\phi_{\mathbf{m}}$ have also a combinatorial interpretation in terms of *Jack symmetric polynomials* $J_{\mathbf{m}}^{(\lambda)}$ with parameter λ (cf. [20], Section 10 of Chapter VI): namely,

$$\phi_{\mathbf{m}}(t_1,\dots,t_r)=j_{\mathbf{m}}^{-1}\,J_{\mathbf{m}}^{(2/a)}(t_1,\dots,t_r), \tag{40}$$

where

$$j_{\mathbf{m}}:=J_{\mathbf{m}}^{(2/a)}(\underbrace{1,\dots,1}_{r})=\Big(\frac{2}{a}\Big)^{|\mathbf{m}|}\Big(\frac{ra}{2}\Big)_{\mathbf{m}}. \tag{41}$$

We will usually suppress the superscripts $(2/a)$ in the sequel.

Recall that in any Jordan algebra J with unit v and product $x\circ y$ an element x is called *invertible* if it has a (necessarily unique) inverse $y=:x^{-1}$ satisfying $x\circ y=v$ and $x^2\circ y=x$. In the special case that the Jordan algebra arises as $J=Z_1(v)$ for a tripotent v of the JB*-triple Z then invertibility of $z\in J$ is equivalent to the invertibility of the operator $Q(z)$ on J and $z^{-1}=Q(z)^{-1}Q(v)z$. In particular, taking the inverse is a rational map on J that can be written (see e.g. [30, Chapter 4]) in exact (i.e. reduced) form as $z^{-1}=p(z)/N(z)$, where $p:J\to J$ is a polynomial which generalizes the matrix adjoint and $N:J\to\mathbf{C}$ is a polynomial called the *determinant polynomial*, or *Koecher norm*, of the Jordan algebra. In particular, fixing a Jordan frame $e_1,\dots,e_r$ of Z the above applies to the Jordan algebras $Z_1(e_1+\cdots+e_j)$, $1\le j\le r$; we denote the corresponding determinant polynomials by N_j and extend them to all of Z by defining $N_j(z):=N_j(P_1^{(j)}(z))$, where $P_1^{(j)}$ is the canonical projection of Z onto $Z_1(e_1+\cdots+e_j)$ given by the Peirce decomposition (19). For a signature $\mathbf{m}$, the *conical polynomial* $N^{\mathbf{m}}$ associated with $\mathbf{m}$ is

$$N^{\mathbf{m}}:=N_1^{m_1-m_2}N_2^{m_2-m_3}\cdots N_r^{m_r}. \tag{42}$$

In particular,

$$N^{\mathbf{m}}\Big(\sum_{j=1}^{r} t_j e_j\Big) = \prod_{j=1}^{r} t_j^{m_j}.$$

Each polynomial space $\mathbf{P}_{\mathbf{m}}$ is then spanned by $N^{\mathbf{m}} \circ k$, $k \in K$. In particular, the conical polynomials are related to the spherical polynomials by

$$\phi_{\mathbf{m}}(z) = \int_L N^{\mathbf{m}}(lz)\, dl,$$

where dl stands for the normalized Haar measure on L.

Standard references for the material in this section are [2], [19], [14], or [30].

3. Radial nearly holomorphic functions

The following relation between the nearly-holomorphic reproducing kernels N_ν^m and the invariantly-polyanalytic reproducing kernels P_ν^m is elementary.

Proposition 1. $P_\nu^m(z,w) = h(z,z)^{m-1} h(w,w)^{m-1} N^m_{\nu+2(m-1)}(z,w)$.

Proof. By their very definition (9), the mapping

$$T : f(z) \longmapsto h(z,z)^{m-1} f(z)$$

is a bijection of $\mathcal{N}^m$ onto $\mathcal{P}^m$. By (2), T clearly acts isometrically from $L^2(\Omega, d\mu_\nu)$ into $L^2(\Omega, d\mu_{\nu-2(m-1)})$, for any $\nu \in \mathbf{R}$. Thus if $\{e_j(z)\}_j$ is an orthonormal basis for $\mathcal{N}^m_{\nu+2(m-1)}$, then $\{h(z,z)^{m-1} e_j(z)\}_j$ will be an orthonormal basis for $\mathcal{P}_\nu^m$. Recalling the familiar formula for a reproducing kernel in terms of an orthonormal basis

$$K(z,w) = \sum_j e_j(z) \overline{e_j(w)}, \tag{43}$$

the assertion follows. $\square$

We also readily get a transformation formula for $N_\nu^m(z,w)$.

Proposition 2. *For any* $\phi \in \mathrm{Aut}(\Omega)$,

$$N_\nu^m(z,w) = \frac{h(a,a)^\nu}{h(z,a)^\nu h(a,w)^\nu} N_\nu^m(\phi z, \phi w), \qquad a := \phi^{-1} 0. \tag{44}$$

In particular,

$$N_\nu^m(z,w) = h(z,w)^{-\nu} N_\nu^m(\phi_w z, 0). \tag{45}$$

Proof. Since $\overline{D}^m f$ is a tensor and ϕ is just a coordinate change, the kernel $\mathcal{N}^m$ of $\overline{D}^m$ is automatically invariant under the composition $f \mapsto f \circ \phi$ with ϕ. As already observed in the Introduction, it therefore follows from the transformation formula (11) for the measure $d\mu_\nu$ (which formula is in turn a consequence of the transformation rule (4) for the Jordan triple determinant, in combination with the fact that the measure $d\mu_0$ is $\mathrm{Aut}(\Omega)$-invariant) that the operator (12) acts unitarily on $\mathcal{N}^m_\nu$. Employing again the formula (43), we thus obtain

$$N^m_\nu(z,w) = \frac{h(a,a)^\nu}{h(z,a)^\nu h(a,w)^\nu} N^m_\nu(\phi^{-1}z, \phi^{-1}w), \qquad a := \phi 0.$$

Replacing ϕ by ϕ^{-1} yields (44), and taking $\phi = \phi_w$ in (44) yields (45). $\square$

Corollary 3. $P^m_\nu(z,w) = h(w,z)^{m-1} h(z,w)^{1-m-\nu} P^m_\nu(\phi_w z, 0)$.

Proof. Combine the last two propositions. $\square$

We have thus reduced the identification of both N^m_ν and P^m_ν to finding the reproducing kernel $N^m_\nu(z,0)$ at zero. Note that by (44),

$$N^m_\nu(kz,0) = N^m_\nu(z,0) \qquad \forall k \in K,$$

where as before K is the stabilizer of the origin $0 \in \Omega$ in $G = \mathrm{Aut}(\Omega)_0$; in other words, $N^m_\nu(\cdot,0)$ is a *radial* function. We now proceed to identify the radial nearly holomorphic functions.

Recall that an element $z \in \mathbf{C}^d$ is called *quasi-invertible* with respect to another element $w \in \mathbf{C}^d$ if, by definition, the Bergman operator (18) is invertible on $\mathbf{C}^d$, and the *quasi-inverse* z^w is then defined as

$$z^w := B(z,w)^{-1}(z - Q(z)w).$$

Note that z^w is holomorphic in z and anti-holomorphic in w. Since $\det B(z,w) = h(z,w)^p$ does not vanish on $\Omega \times \Omega$, the quasi-inverse z^w is, in particular, defined for all $z, w \in \Omega$. It is now a result of [35, formula (3.2) and Proposition 3.1], that, first of all, $\overline{D} = B(z,z)\overline{\partial}$, and furthermore

$$\partial \Psi = \overline{z}^{\overline{z}} = B(\overline{z},\overline{z})^{-1}(\overline{z} - Q(\overline{z})\overline{z}), \tag{46}$$

where as before $\partial\Psi$ stands for the vector of derivatives $\partial_j \Psi$ of the Kähler potential $\Psi(z) = -\log h(z,z)$.

Proposition 4. *Radial functions in $\mathcal{N}^m$ consist precisely of functions of the form*

$$p(z, z^z), \tag{47}$$

where $p(z,w)$ is a polynomial in $z, \overline{w} \in \mathbf{C}^d$ of degree $< m$ in each argument which is K-invariant in the sense that

$$p(kz, kw) = p(z, w) \qquad \forall k \in K. \tag{48}$$

Consequently, the radial functions in $\mathcal{N}^m$ are the linear span of $K_{\mathbf{m}}(z, z^z)$, with $|\mathbf{m}| < m$.

Proof. By (46) and the very definition of nearly-holomorphic functions, any $f \in \mathcal{N}^m$ is of the form

$$f(z) = p(z, z^z),$$

with $p(z,w)$ holomorphic in $z \in \Omega$ and a polynomial of degree $< m$ in $\overline{w}$. Since elements of K are Jordan triple automorphisms, we have $kz^{kw} = k(z^w)$ for any $k \in K$, hence

$$f(kz) = p(kz, k(z^z))$$

with the same p. Thus f is radial if and only if

$$p(z, z^z) = p(kz, kz^{kz}) \qquad \forall z \in \Omega, \forall k \in K.$$

The last equality means that, for any fixed $k \in K$, the two holomorphic functions $p(z, \overline{w}^z)$ and $p(kz, k\overline{w}^{kz})$ of $z, w \in \Omega$ coincide on the anti-diagonal $z = \overline{w}$. By the well-known uniqueness principle [7, Proposition II.4.7], they must coincide for all z, w. Since, for each fixed $z \in \Omega$, the image of Ω under the (non-constant anti-holomorphic) map $w \mapsto \overline{w}^z$ is a (nonempty) open set and p is a polynomial in the second argument, actually $p(z, y) = p(kz, ky)$ for all $z \in \Omega$ and $y \in \mathbf{C}^d$, proving (48). Now it is well known basically from Schur's lemma [3, Proposition 2] that the functions p satisfying (48) are spanned by $K_{\mathbf{m}}(z, w)$, as $\mathbf{m}$ ranges over all signatures. As $K_{\mathbf{m}}(z, w)$ is homogeneous of degree $|\mathbf{m}|$ in both z and $\overline{w}$, the proposition follows. $\square$

Thanks to the last proposition, we can reduce the identification of $N_\nu^m(\cdot, 0)$ to that of the reproducing kernel at 0 of a certain space of symmetric polynomials on $\mathbf{R}^r$ (with r, as before, denoting the rank of Ω). First of all, denote by $\mathcal{R}_\nu^m$ the subspace of all radial functions in $\mathcal{N}_\nu^m$, and let $R_\nu^m(z, w)$ be its reproducing kernel. Then

$$N_\nu^m(\cdot, 0) = R_\nu^m(\cdot, 0). \tag{49}$$

Indeed, by the very definition of a reproducing kernel, $R_\nu^m(\cdot, 0)$ is the (unique) element of $\mathcal{R}_\nu^m$ which reproduces the value at 0 for all elements of $\mathcal{R}_\nu^m$. Now $N_\nu^m(\cdot, 0)$ reproduces the value at 0 even for all elements of $\mathcal{N}_\nu^m$, and belongs to $\mathcal{R}_\nu^m$ (being radial). So by uniqueness, (49) follows.

Secondly, the space $\mathcal{R}^m_\nu$ can be described explicitly as follows. For ease of notation, let us write for an r-tuple $t = (t_1, \dots, t_r) \in \mathbf{R}^r_+$,

$$t^b := \prod_{j=1}^r t_j^b, \qquad (1-t)^\nu := \prod_{j=1}^r (1-t_j)^\nu, \qquad \sqrt{t} = t^{1/2},$$
$$\frac{t}{1-t} := \Big(\frac{t_1}{1-t_1}, \dots, \frac{t_r}{1-t_r}\Big), \qquad dt := dt_1 \dots dt_r,$$

and so forth, and let $d\rho_{b,\nu,a}$ be the modified Selberg measure

$$d\rho_{b,\nu,a}(t) := c_\Omega \, t^b (1+t)^{-\nu} \prod_{1 \le i < j \le r} |t_i - t_j|^a \, dt. \tag{50}$$

Finally, if $e_1, \dots, e_r$ is a fixed Jordan frame, we will write just te for $t_1 e_1 + \dots + t_r e_r$. Let $\mathcal{S}^m$ be the vector space of all symmetric polynomials of degree $< m$ in r variables, denote

$$\mathcal{S}^m_\nu := \mathcal{S}^m \cap L^2(\mathbf{R}^r_+, d\rho_{b,\nu,a}),$$

and let $S^m_\nu(x, y)$ be the reproducing kernel of $\mathcal{S}^m_\nu$.

Proposition 5. *The mapping V from $\mathcal{S}^m$ into functions on Ω given by*

$$Vf(k\sqrt{t}e) := f\Big(\frac{t}{1-t}\Big), \qquad k \in K, \ t \in [0,1]^r, \tag{51}$$

is a bijection from $\mathcal{S}^m$ onto radial functions in $\mathcal{N}^m$. Furthermore, V sends $\mathcal{S}^m_\nu$ unitarily onto $\mathcal{R}^m_\nu$, and

$$R^m_\nu(\cdot, 0) = VS^m_\nu(\cdot, 0). \tag{52}$$

Proof. Let $z = k\sqrt{t}e$ be the polar decomposition of $z \in \Omega$. From the formula (23) for the action of $B(z, z)$ on the Peirce subspaces (and the similar formula for the action of $Q(z)$), one gets

$$z^z = k\frac{\sqrt{t}}{1-t}e.$$

Hence

$$K_{\mathbf{m}}(z, z^z) = K_{\mathbf{m}}(k\sqrt{t}e, k\tfrac{\sqrt{t}}{1-t}e) = K_{\mathbf{m}}(\tfrac{t}{1-t}e, e).$$

This is, as we have seen in Section 2, up to a constant factor just the Jack symmetric polynomial $J_{\mathbf{m}}(\frac{t}{1-t})$ in r variables evaluated at $\frac{t}{1-t}$. Since $J_{\mathbf{m}}$, $|\mathbf{m}| < m$, span all symmetric polynomials of degree $< m$, by the preceding proposition the radial functions in

$\mathcal{N}^m$ are precisely those of the form Vf, with V as in (51) and f a symmetric polynomial of degree $< m$. This proves the first assertion.

As for the second, we have by (27)

$$\|Vf\|^2_{L^2(d\mu_\nu)} = c_\Omega \int\limits_{[0,1]^r} \Big| f\Big(\frac{t}{1-t}\Big)\Big|^2 \, d\mu_{b,\nu,a}(t). \tag{53}$$

Making the change of variable $t_j = \frac{x_j}{1+x_j}$, $x \in \mathbf{R}^r_+$, we have

$$\begin{aligned} &\frac{t}{1-t} = x, \qquad dt = (1+x)^{-2}\, dx, \qquad t^b = x^b(1+x)^{-b}, \\ &(1-t)^{\nu-p} = (1+x)^{p-\nu}, \qquad t_i - t_j = \frac{x_i - x_j}{(1+x_i)(1+x_j)}, \end{aligned}$$

implying, by a direct computation using (16), that

$$c_\Omega \, d\mu_{b,\nu,a}(t) = d\rho_{b,\nu,a}(x). \tag{54}$$

By (53), the second claim follows.

Finally, (52) follows from the general formula (43) (applied to $\mathcal{S}^m_\nu$ and $\mathcal{R}^m_\nu$), together with the fact that under the above change of variables $x = \frac{t}{1-t}$, the point $t = 0$ corresponds to $x = 0$. $\square$

We summarize our findings so far as the main result of this section.

Theorem 6. *The reproducing kernels N^m_ν and P^m_ν of the spaces $\mathcal{N}^m_\nu$ and $\mathcal{P}^m_\nu$, respectively, are given by*

$$\begin{aligned} N^m_\nu(z,w) &= h(z,w)^{-\nu} N^m_\nu(\phi_z w, 0), \\ P^m_\nu(z,w) &= \frac{h(z,z)^{m-1} h(w,w)^{m-1}}{h(z,w)^{\nu+2m-2}} N^m_{\nu+2m-2}(\phi_w z, 0), \end{aligned}$$

where

$$N^m_\nu(k\sqrt{t}e, 0) = S^m_\nu(\tfrac{t}{1-t}, 0),$$

where S^m_ν is the reproducing kernel of the L^2 space of symmetric polynomials of degree $< m$ on $\mathbf{R}^r_+$ with respect to the measure (50).

Proof. Combine Propositions 1, 2, 4 and 5, and the formula (49). $\square$

We conclude this section by a simple observation concerning the nontriviality of the spaces $\mathcal{N}^m_\nu$ and $\mathcal{P}^m_\nu$.

Lemma 7. *A polynomial P belongs to $L^2(\mathbf{R}^r_+, d\rho_{b,\nu,a})$ if and only if its degree in each variable is less than $(\nu - p + 1)/2$.*

Proof. Let n_1 be the degree of $P(x)$ in the variable x_1; thus the leading term in the x_1 variable is $p_1(x')x_1^{n_1}$, where the polynomial p_1 in the $r-1$ variables $x' = (x_2, \dots, x_r)$ is not identically zero. The zero-set of p_1 is therefore a variety in $\mathbf{R}^{r-1}$ of codimension at least 1; we can therefore choose a closed ball Q (of positive finite radius) lying wholly in $\{y \in \mathbf{R}^{r-1} : y_j \neq y_k \text{ for all } j \neq k\}$ such that $|p_1| > 0$ on Q. Set $R := 1 + \sup\{\|y\| : y \in Q\}$. Then if $P \in L^2(\mathbf{R}^r_+, d\rho_{b,\nu,a})$, the integral

$$\int_R^\infty \int_Q |P(x_1, x')|^2 \, d\rho_{b,\nu,a}(x_1, x')$$

has to be finite. However, due to our choice of Q and R, the integrand is $\asymp x_1^{2n_1}$ (uniformly in x'), while the measure is $\asymp x_1^{b-\nu+(r-1)a} \, dx$ (uniformly in x'). Consequently, $x_1^{2n_1+(r-1)a+b-\nu}$ must be integrable at infinity, implying that $2n_1 + (r-1)a + b - \nu = 2n_1 + p - 2 - \nu < -1$, or $n_1 < (\nu - p + 1)/2$. Similarly, $n_j < (\nu - p + 1)/2$ for the degree n_j of $P(x)$ in the variable x_j, $j = 1, \dots, r$.

Conversely, let $P(x) = x_1^{n_1} \dots x_r^{n_r}$ with $n_j < (\nu - p + 1)/2$ for all j. Making again the change of variable $x = t/(1-t)$ shows that the L^2-norm of P with respect to $c_\Omega^{-1} d\rho_{b,\nu,a}$ equals

$$\int_{[0,1]^r} \prod_{j=1}^r \left(t_j^{2n_j+b} (1-t_j)^{\nu-p-2n_j} \right) \prod_{1 \le i < j \le r} |t_i - t_j|^a \, dt.$$

The second term in the integrand is bounded (by 1), while the first term yields just the product of single-variable integrals

$$\int_0^1 t_j^{2n_j+b} (1-t_j)^{\nu-p-2n_j} \, dt_j,$$

which are finite since $2n_j + b > -1$ and $\nu - p - 2n_j > -1$. □

Proposition 8.

(a) $\mathcal{N}_\nu^m \neq \{0\}$ *if and only if* $\nu > p - 1$, *and* $\mathcal{P}_\nu^m \neq \{0\}$ *if and only if* $\nu > p + 1 - 2m$.
(b) $\mathcal{N}_\nu^m \setminus \mathcal{N}_\nu^{m-1} \neq \{0\}$ *if and only if there exists a signature* $\mathbf{m}$ *with* $|\mathbf{m}| = m - 1$ *and* $m_1 < \frac{\nu - p + 1}{2}$.
(c) *In fact,* $K_{\mathbf{m}}(z, z^z) \in \mathcal{N}_\nu^m$ *if and only if* $|\mathbf{m}| < m$ *and* $m_1 < \frac{\nu - p + 1}{2}$.

Proof. (a) If $\mathcal{N}_\nu^m \neq \{0\}$ then its reproducing kernel is not identically zero; by the last theorem, this is equivalent, in turn, to $N_\nu^m(\cdot, 0) \not\equiv 0$ and $S_\nu^m(\cdot, 0) \not\equiv 0$. Thus $\mathcal{S}_\nu^m$ contains a nonzero polynomial $p(x)$ (even one that does not vanish at the origin). By the last lemma, necessarily $\nu - p + 1 > 0$, or $\nu > p - 1$.

Conversely, if $\nu > p - 1$, then $\mathcal{N}_\nu^1 = A_\nu$ is nontrivial (it contains all bounded holomorphic functions on Ω), hence so is $\mathcal{N}_\nu^m \supset \mathcal{N}_\nu^1$.

This settles the assertion for $\mathcal{N}_\nu^m$; the one for $\mathcal{P}_\nu^m$ then follows from Proposition 1.

(b) By the same argument as in the proof of part (a), $\mathcal{N}_\nu^m \setminus \mathcal{N}_\nu^{m-1} \neq \{0\}$ if and only if $\mathcal{S}_\nu^m$ contains a polynomial P whose total degree is $m - 1$ and whose degree in each variable is $< \frac{\nu-p+1}{2}$. If x^α, with α a multiindex, is any monomial in the top degree homogeneous component of P, then the nonincreasing rearrangement of α yields the desired signature $\mathbf{m}$.

(c) This follows in the same way as for part (b) from the fact that V maps $K_{\mathbf{m}}(z, z^z)$ into a (nonzero) constant multiple of the Jack polynomial $J_{\mathbf{m}}(x)$, and $J_{\mathbf{m}}(x)$ is equal to the symmetrization of (writing $x^{\mathbf{m}} := x_1^{m_1} \dots x_r^{m_r}$)

$$x^{\mathbf{m}} + \sum_{\mathbf{n}<\mathbf{m}} c_{\mathbf{mn}} x^{\mathbf{n}} \tag{55}$$

where the sum is over (some) signatures $\mathbf{n}$ smaller than $\mathbf{m}$ with respect to the lexicographic order; cf. Mcdonald [20, formula (10.13)]. $\square$

Corollary 9. $\mathcal{R}_\nu^m = \operatorname{span}\{K_{\mathbf{m}}(z, z^z) : |\mathbf{m}| < m,\ m_1 < \frac{\nu-p+1}{2}\}$.

In particular, if q denotes the nonnegative integer such that

$$q < \frac{\nu - p + 1}{2} \leq q + 1,$$

then

$$\mathcal{R}_\nu^m = \operatorname{span}\{K_{\mathbf{m}}(z, z^z) : |\mathbf{m}| < m\} \qquad \textit{if } m \leq q + 1, \tag{56}$$

$$\mathcal{R}_\nu^m = \operatorname{span}\{K_{\mathbf{m}}(z, z^z) : m_1 \leq q\} \qquad \textit{if } m \geq rq + 1. \tag{57}$$

This means that for $m \geq rq + 1$, $\mathcal{R}_\nu^m$ and, hence, $\mathcal{N}_\nu^m$ equals $\mathcal{N}_\nu^{rq+1}$ — i.e. the spaces $\mathcal{N}_\nu^m$ "stabilize" and stop growing with m (for fixed ν). Likewise, $N_\nu^m(z, w) = N_\nu^{rq+1}(z, w)$ for all $m \geq rq + 1$ if $2q - 1 < \nu - p \leq 2q + 1$.

Remark 10. We pause to note that while, clearly,

$$\mathcal{N}^1 \subset \mathcal{N}^2 \subset \mathcal{N}^2 \subset \dots,$$

no such inclusions hold for $\mathcal{P}^m$, except when the rank $r = 1$. More specifically, for $r > 1$, the function $\mathbf{1}$ (constant one) belongs to $\mathcal{P}^1$, but not to any $\mathcal{P}^m$, $m \geq 2$. Indeed, $\mathbf{1} \in \mathcal{P}^m \iff h(z, z)^{1-m} \in \mathcal{N}^m$, by (9); and by Proposition 4, the latter is equivalent to

$$(1 - t)^{1-m} = \sum_{|\mathbf{m}|<m} c_{\mathbf{m}} \phi_{\mathbf{m}}\left(\tfrac{t}{1-t}\right)$$

with some coefficients $c_{\mathbf{m}}$. Passing again to $x = \frac{t}{1-t}$, this translates into

$$(1+x)^{m-1} = \sum_{|\mathbf{m}|<m} c_{\mathbf{m}} \phi_{\mathbf{m}}(x). \tag{58}$$

But by the Faraut-Koranyi formula (33), the left-hand side equals

$$\sum_{\mathbf{n}} (1-m)_{\mathbf{n}} \frac{\pi_{\mathbf{n}}(-1)^{|\mathbf{n}|}}{(q_\Omega)_{\mathbf{n}}} \phi_{\mathbf{n}}(x).$$

Since the $\phi_{\mathbf{m}}$ are linearly independent, (58) can hold only if

$$(1-m)_{\mathbf{n}} = 0 \qquad \text{whenever } |\mathbf{n}| \geq m.$$

However, for $\mathbf{n} = (m-1, 1)$ one has $(1-m)_{\mathbf{n}} = (-1)^m (m-1)!(m-1+\frac{a}{2})$ which is nonzero. So $\mathbf{1} \notin \mathcal{P}^m$ for $m \geq 2$ if $r > 1$. $\square$

4. Spherical functions

Any $g \in G$ can be uniquely written in the form $g = k\phi_w$, with $w = g^{-1}0 \in \Omega$, $k \in K$ and ϕ_w the geodesic reflection (17) interchanging 0 and w. This yields the formula for the complex Jacobian

$$J_g(z) = \det k \cdot (-1)^d \frac{h(w,w)^{p/2}}{h(z,w)^p}, \tag{59}$$

which shows that the projective representation (12) is actually nothing else than

$$f \longmapsto J_{\phi^{-1}}^{\nu/p} \cdot f \circ \phi^{-1}, \qquad \phi \in G.$$

In order to make this not only projective but genuine representation if ν/p is not an integer, one needs to pass from G to its universal cover $\widetilde{G}$. The elements of $\widetilde{G}$ can be thought of as elements g of G together with a consistent choice of $\log J_g$. The operators

$$U_g^{(\nu)} : f \longmapsto J_{g^{-1}}^{\nu/p} \cdot f \circ g^{-1}, \qquad g \in \widetilde{G}, \tag{60}$$

then define a (honest, not only projective) unitary representation of $\widetilde{G}$ on $L^2(\Omega, d\mu_\nu)$; and one has $\Omega = \widetilde{G}/\widetilde{K}$, where $\widetilde{K}$, the preimage of K under the covering map $\widetilde{G} \to G$, is the universal cover of K and the stabilizer of $0 \in \Omega$ in $\widetilde{G}$. (Actually one has $\widetilde{K} \cong K \times \mathbf{R}$, but we will not need this fact.)

Using (60), one can identify $L^2(\Omega, d\mu_\nu)$ with a subspace of $L^2(\widetilde{G}/Z(\widetilde{G}))$, the L^2 space on the quotient of $\widetilde{G}$ modulo its center $Z(\widetilde{G})$ with respect to a suitably normalized Haar measure on $\widetilde{G}$. Namely, for $f \in L^2(\Omega, d\mu_\nu)$, the function $f^\#$ on $\widetilde{G}$ defined by

$$f^{\#}(g) := f(g0)J_g(0)^{-\nu/p}, \qquad g \in \widetilde{G}, \tag{61}$$

satisfies

$$(U_g^{(\nu)}f)^{\#} = f^{\#} \circ g^{-1} \tag{62}$$

(i.e. the map $f \mapsto f^{\#}$ intertwines the representation (60) with the left regular representation of $\widetilde{G}$ on $L^2(\widetilde{G})$) and

$$f^{\#}(gk) = f^{\#}(g)J_k^{-\nu/p}, \qquad g \in \widetilde{G},\ k \in \widetilde{K}. \tag{63}$$

(Note that $J_k \equiv J_k(0)$ is a constant function, so we will write just J_k instead of $J_k(0)$ or $J_k(z)$.) Furthermore, $f \mapsto f^{\#}$ is a unitary isomorphism of $L^2(\Omega, d\mu_\nu)$ onto the subspace $L^2(\widetilde{G}, \nu)$ of all functions in $L^2(\widetilde{G}/Z(\widetilde{G}))$ satisfying the transformation rule (63); the inverse of the map $f \mapsto f^{\#}$ is given by $F \mapsto F^\flat$, where

$$F^\flat(g0) := F(g)J_g(0)^{\nu/p} \tag{64}$$

(the right-hand side depends only on $g0$, thanks to (63)). See Proposition 2.1 in [11] for the proof of all these facts. (Note: there is a misprint in the first formula of Section 2 in [11], the τ_ν there should be $\tau_{-\nu}$.)

Using the above identification, one can view also $\mathcal{N}_\nu^m$ as a subspace of $L^2(\widetilde{G}, \nu)$ invariant under the left regular representation (62). Note that radial functions on Ω, i.e. those satisfying $f(kz) = f(z)$ for all $z \in \Omega$ and $k \in K$, correspond to functions $f^{\#}$ on $\widetilde{G}$ satisfying

$$f^{\#}(k'gk) = J_{k'}^{-\nu/p} f^{\#}(g)J_k^{-\nu/p}, \qquad g \in \widetilde{G},\ k, k' \in \widetilde{K}. \tag{65}$$

Such functions on $\widetilde{G}$ are called *ν-spherical*.

The representation theory for the space $L^2(\widetilde{G}, \nu)$ has been developed by Shimeno [27] (his notation $\tau_{-\ell}(k)$ corresponds to our $J_k^{-\nu/p}$). (Note that there is a $\widetilde{G}$-equivariant isomorphism between $L^2(\widetilde{G}, \nu)$ and $L^2(\widetilde{G}, -\nu)$, so we can always assume that $\nu > 0$ as we have started with.) Namely, let $\widetilde{G} = \widetilde{K}AN$ be the Iwasawa decomposition of $\widetilde{G}$ and let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{a}$ be the Lie algebras of G (and $\widetilde{G}$), K (and $\widetilde{K}$), AN and A, respectively, so that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{a} \subset \mathfrak{p}$ is a maximal Abelian subspace of $\mathfrak{p}$. For $g \in \widetilde{G}$ let $H(g)$ be the element in the Lie algebra $\mathfrak{a}$ of A uniquely determined by $g \in \widetilde{K} \exp H(g)N$. Similarly, let $\kappa(g) \in \widetilde{K}$ be uniquely determined by $g \in \kappa(g)AN$. For $\lambda \in \mathfrak{a}^{*\mathbf{C}}$, the complexification of the dual $\mathfrak{a}^*$ of $\mathfrak{a}$, one defines the *spherical function* $\phi_{\lambda,\nu}$ of type ν by

$$\phi_{\lambda,\nu}(g) := \int\limits_{\widetilde{K}/Z(\widetilde{G})} e^{-(\lambda+\rho)H(g^{-1}k)} J_{k^{-1}\kappa(g^{-1}k)}^{\nu/p}\, dk, \tag{66}$$

where $Z(\widetilde{G})$ denotes the center of $\widetilde{G}$, dk is the invariant measure on the quotient $\widetilde{K}/Z(\widetilde{G})$ with total mass 1, and $\rho \in \mathfrak{a}^*$ is the half-sum of positive roots (see (72) below). Then $\phi_{\lambda,\nu}$ is a ν-spherical function on $\widetilde{G}$, and one defines the spherical Fourier transform $\hat{f}$ of a ν-spherical function f on $\widetilde{G}$ by

$$\hat{f}(\lambda) := \int\limits_{\widetilde{G}/Z(\widetilde{G})} f(g)\phi_{-\lambda,-\nu}(g)\,dg, \qquad \lambda \in \mathfrak{a}^{*\mathbf{C}}. \tag{67}$$

This definition makes sense e.g. whenever f is compactly supported modulo $Z(\widetilde{G})$. The main result of [27] then states that there is an inversion formula

$$f(g) = \int\limits_{\bigcup_{j=0}^{r} D_{\nu,j}+i\mathfrak{a}^*_{\Theta_j}} \hat{f}(\lambda)\phi_{\lambda,\nu}(g)\,d\gamma(\lambda)$$

where $D_{\nu,j}$ and $\mathfrak{a}^*_{\Theta_j}$ are certain systems of hyperplanes in $\mathfrak{a}^*$ and $d\gamma$ is a certain measure on them; and there is also a corresponding Plancherel theorem. See Theorems 6.7 and 6.8 in [27] for the details. Both $D_{\nu,j}$ and $\mathfrak{a}^*_{\Theta_j}$ have codimension j in $\mathfrak{a}^*$; in particular, for $j = r$, $\mathfrak{a}^*_{\Theta_r} = \{0\}$ and

$$D_{\nu,r} = \{\lambda_{\mathbf{m}} : \mathbf{m} \text{ is a signature with } m_1 < \tfrac{\nu-p+1}{2}\} \tag{68}$$

where

$$\lambda_{\mathbf{m}} := \frac{1}{2}\sum_{j=1}^{r} \lambda_j \beta_j, \qquad \lambda_{r+1-j} = p - 1 - \nu - (j-1)a + 2m_j. \tag{69}$$

(Here $\beta_1, \ldots, \beta_r \in \mathfrak{a}^*$ are the long roots of the root system of Ω; see below.) Thus $D_{\nu,r}$ is a finite discrete set in $\mathfrak{a}^*$. Furthermore, the Plancherel measure $d\gamma$ on $D_{\nu,r}+i\mathfrak{a}^*_{\Theta_r} = D_{\nu,r}$ reduces just to a multiple $d_r(\lambda,\nu)\delta_\lambda$ of the Dirac mass at each $\lambda = \lambda_{\mathbf{m}}$. Here $d_r(\lambda,\nu)$ is given by an explicit expression involving Γ-functions; see (4.18), (4.19), (4.21) and (6.18) in [27]. Altogether, it thus follows that the space $\mathcal{A}^{\mathbf{m}}_\nu(\widetilde{G})$ spanned in $L^2(\widetilde{G})$ by $\widetilde{G}$-translates of $\phi_{\lambda_{\mathbf{m}},\nu}$, for each signature $\mathbf{m}$, is an irreducible direct summand of $L^2(\widetilde{G})$, and as $\mathbf{m}$ varies these summands are mutually orthogonal. Such summands are called *relative discrete series* representations of $\widetilde{G}$. Note that by (68), only signatures $\mathbf{m}$ with $m_1 < \frac{\nu-p+1}{2}$ occur (in particular, there are no relative discrete series representations if $\nu \leq p-1$).

Let

$$\mathcal{A}^{\mathbf{m}}_\nu(\Omega) := \{F^\flat : F \in \mathcal{A}^{\mathbf{m}}_\nu(\widetilde{G})\}$$

be the space of functions on Ω corresponding to $\mathcal{A}^{\mathbf{m}}_\nu(\widetilde{G})$ via (64). It is then, next, the central result of [35] that $\mathcal{A}^{\mathbf{m}}_\nu(\Omega)$ is actually a space of nearly holomorphic functions of order $|\mathbf{m}|$, and that, as $\mathbf{m}$ varies, these spaces exhaust all nearly holomorphic functions.

Namely, let $N^{\mathbf{m}}$ be the conical polynomial on Ω from (42); one can then form the composition $N^{\mathbf{m}}(\overline{z}^{\overline{z}})$ with the quasi-inverse (46), which gives a function on Ω. By Theorem 4.7 in [35], the space spanned by $U_g^{(\nu)} N^{\mathbf{m}}(\overline{z}^{\overline{z}})$, $g \in \widetilde{G}$, coincides with $\mathcal{A}_\nu^{\mathbf{m}}(\Omega)$.

Finally, arguing as in the beginning of Section 3 in [33], it follows from Shimeno's Plancherel formula that the reproducing kernel $A_\nu^{\mathbf{m}}(z,w)$ of the space $\mathcal{A}_\nu^{\mathbf{m}}(\Omega)$ is given for $w=0$ (i.e. at the origin) simply by the appropriate multiple of the spherical function:

$$A_\nu^{\mathbf{m}}(z,0) = d_r(\lambda_{\mathbf{m}},\nu)\phi^\flat_{\lambda_{\mathbf{m}},\nu}(z). \tag{70}$$

Summarizing the discussion so far, we have thus arrived at the following result.

Theorem 11. *The nearly-holomorphic reproducing kernel N_ν^m at the origin is given by*

$$N_\nu^{\mathbf{m}}(z,0) = \sum_{\substack{|\mathbf{m}|<m \\ m_1<\frac{\nu-p+1}{2}}} d_r(\lambda_{\mathbf{m}},\nu)\phi^\flat_{\lambda_{\mathbf{m}},\nu}(z), \tag{71}$$

with $d_r(\lambda,\nu)$, $\phi_{\lambda,\nu}$ and $\lambda_{\mathbf{m}}$ as above.

We pause to note that actually $d_r(\lambda,\nu) = \|\phi_{\lambda,\nu}\|^{-2}_{L^2(\widetilde{G})}$, cf. Remark 6.9 in [27].

We conclude by recalling the relation between the spherical functions $\phi_{\lambda,\nu}$ and the hypergeometric functions of Heckman and Opdam [15, Part I]. With our notation $\mathfrak{g}$ for the Lie algebra of G, let Σ be the restricted root system of the pair $(\mathfrak{g},\mathfrak{a})$. Then Σ is a root system of type BC, i.e. has the form

$$\Sigma = \{\pm\tfrac{1}{2}\beta_j, \pm\beta_j, \pm\tfrac{1}{2}(\beta_j \pm \beta_k),\ 1 \le j,k \le r,\ j \ne k\},$$

where $\{\beta_j\}_j$ is a certain basis of $\mathfrak{a}^*$. Here the *short roots* $\pm\frac{1}{2}\beta_j$ have multiplicity $m_S = 2b$, the *long roots* $\pm\beta_j$ have multiplicity $m_L = 1$, and the *middle roots* $\frac{1}{2}(\pm\beta_j \pm \beta_k)$ have multiplicity $m_M = a$; if $b=0$, then the short roots are actually absent (and if $r=1$, then the middle roots are actually absent). The positive roots are $\frac{1}{2}\beta_j$, β_j, $j=1,\dots,r$, and $\frac{1}{2}(\beta_j \pm \beta_k)$, $1 \le k < j \le r$; the *half-sum of positive roots* is thus given by

$$\rho = \sum_{j=1}^r \frac{b+1+(j-1)a}{2}\beta_j. \tag{72}$$

Let now $F(\lambda,\mathbf{k}_\nu,\cdot)$ be the Heckman-Opdam hypergeometric function with parameter $\lambda \in \mathfrak{a}^{*\mathbf{C}}$ corresponding to the root system 2Σ with multiplicities $\mathbf{k}_\nu$ given by

$$\mathbf{k}_{\nu,S} = \frac{m_S}{2} - \nu = b-\nu, \qquad \mathbf{k}_{\nu,L} = \frac{m_L}{2} + \nu = \frac{1}{2} + \nu, \qquad \mathbf{k}_{\nu,M} = \frac{m_M}{2} = \frac{a}{2}, \tag{73}$$

for the "doubles" of the short, long and middle roots of Σ, respectively. Then for any $g \in A \subset \widetilde{G}$ we have

$$\phi_{\lambda,\nu}(g) = h(g0, g0)^{-\nu/2} F(\lambda, \mathbf{k}_\nu, g). \tag{74}$$

See [15, Theorem 5.2.2] (cf. also Remark 3.8 in [27]). Note further that by [15, (4.4.10)] (cf. also Remark 5.12 in [27]), if λ is a dominant weight (i.e. $\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbf{N}$ for all positive roots α of Σ), then

$$F(\lambda, \rho_\nu, \mathbf{k}_\nu, \cdot) = c(\lambda + \rho_\nu, \mathbf{k}_\nu) P(\lambda, \mathbf{k}_\nu, \cdot), \tag{75}$$

where ρ_ν is given by (72) with b replaced by $b+\nu$, and $P(\lambda, \mathbf{k}_\nu, \cdot)$ are the multivariable Jacobi polynomials (cf. Debiard [8], [9]). Here $c(\lambda, \mathbf{k})$ is the generalized c-function of Harish-Chandra with line bundle parameter ν (cf. (3.15) in [27] or (3.4.3) in [15]).

Combining (71), (74) and (72) with Theorem 6, we can express also the invariantly-polyanalytic reproducing kernels P^m_ν and the reproducing kernels S^m_ν of the spaces $\mathcal{S}^m_\nu$ of symmetric polynomials on $\mathbf{R}^r_+$ in terms of spherical functions, or Heckman-Opdam hypergeometric functions, or multivariable Jacobi polynomials. We omit the details.

Example 12. For $\Omega = \mathbf{B}^d$, the unit ball of $\mathbf{C}^d$, we have $r = 1$, $b = d-1$, a is not defined, $p = d+1$ and $h(z, w) = 1 - \langle z, w\rangle$. The elements of the group $G = SU(1, n)$ can be identified with $(n+1) \times (n+1)$ complex matrices $\left(\begin{smallmatrix} A & B \\ C^t & D\end{smallmatrix}\right)$, with $A \in \mathbf{C}$, $B, C \in \mathbf{C}^{1\times n}$ and $D \in \mathbf{C}^{n\times n}$, acting by $z \mapsto (Az + B)(Cz + D)^{-1}$, with $z \in \mathbf{B}^d$ written as row vector. The Lie algebra $\mathfrak{a}$ equals $\mathbf{R}H$, where

$$H = \begin{pmatrix} 1 & 0^{1\times(n-1)} & 0 \\ 0^{(n-1)\times 1} & 0^{(n-1)\times(n-1)} & 0^{(n-1)\times 1} \\ 0 & 0^{1\times(n-1)} & 1 \end{pmatrix},$$

and defining $\beta \in \mathfrak{a}^*$ by $\beta(H) = 2$ the root system is given by $\Sigma = \{\pm\frac{1}{2}\beta, \pm\beta\}$. We have

$$\exp(tH) = \begin{pmatrix} \cosh t & 0^{1\times(n-1)} & \sinh t \\ 0^{(n-1)\times 1} & I^{(n-1)\times(n-1)} & 0^{(n-1)\times 1} \\ \sinh t & 0^{1\times(n-1)} & \cosh t \end{pmatrix},$$

hence $|\exp(tH)0| = |\tanh t|$ and

$$\cosh t = h(\exp(tH)0, \exp(tH)0)^{-1/2}. \tag{76}$$

The spherical functions $\phi_{\lambda,\nu}$ are given by ([27, (8.2) and (8.3)])

$$\begin{aligned}\phi_{\lambda,\nu}(\exp tH) &= (\cosh t)^{-\nu}{}_2F_1\left(\begin{matrix}\frac{d-\nu+\lambda}{2}, \frac{d-\nu-\lambda}{2} \\ d\end{matrix}\,\middle|\, -\sinh^2 t\right) \\ &= (\cosh t)^{\nu}{}_2F_1\left(\begin{matrix}\frac{d+\nu+\lambda}{2}, \frac{d+\nu-\lambda}{2} \\ d\end{matrix}\,\middle|\, -\sinh^2 t\right),\end{aligned} \tag{77}$$

that is, by (64) and (76), and since $J_{\exp tH}(0) = \cosh^{-p} t$,

$$\begin{aligned}\phi^\flat_{\lambda,\nu}(z) &= (1-|z|^2)^\nu {}_2F_1\Big(\begin{matrix}\frac{d-\nu+\lambda}{2}, \frac{d-\nu-\lambda}{2}\\ d\end{matrix}\Big|\frac{|z|^2}{|z|^2-1}\Big)\\ &= {}_2F_1\Big(\begin{matrix}\frac{d+\nu+\lambda}{2}, \frac{d+\nu-\lambda}{2}\\ d\end{matrix}\Big|\frac{|z|^2}{|z|^2-1}\Big),\end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function and on the right-hand sides, we write just λ for $\lambda(H)$. Using the standard transformation formula for ${}_2F_1$ [4, §2.1 (22)]

$$ {}_2F_1\Big(\begin{matrix}a, b\\ c\end{matrix}\Big|z\Big) = (1-z)^{-a} {}_2F_1\Big(\begin{matrix}a, c-b\\ c\end{matrix}\Big|\frac{z}{z-1}\Big) = (1-z)^{-b} {}_2F_1\Big(\begin{matrix}c-a, b\\ c\end{matrix}\Big|\frac{z}{z-1}\Big), \tag{78}$$

this can also be written as

$$\begin{aligned}\phi^\flat_{\lambda,\nu}(z) &= (1-|z|^2)^{\frac{d+\lambda+\nu}{2}} {}_2F_1\Big(\begin{matrix}\frac{d+\lambda+\nu}{2}, \frac{d+\lambda-\nu}{2}\\ d\end{matrix}\Big||z|^2\Big)\\ &= (1-|z|^2)^{\frac{d-\lambda+\nu}{2}} {}_2F_1\Big(\begin{matrix}\frac{d-\lambda+\nu}{2}, \frac{d-\lambda-\nu}{2}\\ d\end{matrix}\Big||z|^2\Big).\end{aligned}$$

The elements $\lambda_{\mathbf{m}} \in \mathfrak{a}^*$, $\mathbf{m} = (m_1)$, are given by $\lambda_{\mathbf{m}} = \lambda_1 \frac{\beta}{2}$ with $\lambda_1 = 2m_1 + d - \nu$, $m_1 \in \mathbf{Z}$, $0 \leq m_1 < \frac{\nu-d}{2}$. The corresponding space $\mathcal{A}^{\mathbf{m}}_\nu(\mathbf{B}^d)$ is spanned by G-translates of the function $\overline{z}_1^{m_1}(1-|z|^2)^{-m_1}$ under the action (12) [35, Section 5] and coincides with the orthogonal complement $\mathcal{N}^{m_1+1}_\nu \ominus \mathcal{N}^{m_1}_\nu$ [33, pp. 116-117]. The reproducing kernel of $\mathcal{A}^{\mathbf{m}}_\nu(\mathbf{B}^d)$ at the origin is given by (71) with

$$d_1(\lambda_{\mathbf{m}}, \nu) = \frac{(\nu-d-2m_1)\pi^{-d}\Gamma(d+m_1)\Gamma(\nu-m_1)}{m_1!\Gamma(d)\Gamma(\nu-d+1-m_1)},$$

in complete agreement with [33, Section 3, bottom of p. 116]. (Note that there is a misprint in the formula (1.5) in [33]: the $\Gamma(\alpha+1+d)$ in the numerator should be $\Gamma(\alpha+1+d-l)$. Also the labeling of spherical functions is different there: our $\phi_{\lambda,\nu}$ corresponds to $\phi_{i\lambda,\nu}$ in [33].)

By [22, p. 90], the Heckman-Opdam hypergeometric function for root system BC_1 is given by

$$F(\lambda, \mathbf{k}, \exp tH) = {}_2F_1\Big(\begin{matrix}\frac{d+\nu+\lambda}{2}, \frac{d+\nu-\lambda}{2}\\ d\end{matrix}\Big| -\sinh^2 t\Big),$$

in full accordance with (74) and (77) in view of (76).

Finally, for rank one (75) reduces just to (15), and thus for $m-1 < \frac{\nu-p+1}{2}$

$$\begin{aligned}N^m_\nu(z,0) &= \sum_{m_1=0}^{m-1} d_1(\lambda_{\mathbf{m}}, \nu) {}_2F_1\Big(\begin{matrix}-m_1, d-\nu+m_1\\ d\end{matrix}\Big|\frac{|z|^2}{|z|^2-1}\Big)\\ &= \sum_{m_1=0}^{m-1} d_1(\lambda_{\mathbf{m}}, \nu)(1-|z|^2)^{-m_1} {}_2F_1\Big(\begin{matrix}-m_1, \nu-m_1\\ d\end{matrix}\Big||z|^2\Big) \qquad \text{by (78)}\end{aligned}$$

$$= \sum_{m_1=0}^{m-1} d_1(\lambda_{\mathbf{m}}, \nu) \frac{(1-|z|^2)^{-m_1}}{\binom{m_1+d-1}{d-1}} P_{m_1}^{(d-1,\nu-d-2m_1)}(1-2|z|^2),$$

recovering (13) and its reformulation in terms of Jacobi polynomials.

From Theorem 6 we see that the reproducing kernel at the origin for the subspace of all polynomials of degree $< m$ in $L^2(\mathbf{R}_+, c_{\mathbf{B}^d} t^{d-1}(1+t)^{-\nu}\, dt)$ equals

$$S_\nu^m(x,0) = \sum_{0 \le m_1 < \min(m, \frac{\nu-d}{2})} d_1(\lambda_{(m_1)}, \nu)\ {}_2F_1\left(\begin{matrix} -m_1, d-\nu+m_1 \\ d \end{matrix} \middle| -x \right), \tag{79}$$

the summands being actually mutually orthogonal. □

5. Faraut-Koranyi hypergeometric functions

With Theorem 6 in mind, let us return to the reproducing kernels $S_\nu^m(x,y)$ of the subspaces $\mathcal{S}_\nu^m$ of symmetric polynomials of degree $< m$ in $L^2(\mathbf{R}_+^r, d\rho_{b,\nu,a})$. By Propositions 4 and 5, the functions

$$K_{\mathbf{m}}(xe, e), \qquad |\mathbf{m}| < m,\ m_1 < \tfrac{\nu-p+1}{2}, \tag{80}$$

span $\mathcal{S}_\nu^m$. The following easy fact — capturing, in effect, the standard Gram-Schmidt orthogonalization process — describes how to extract the reproducing kernel from an arbitrary basis.

Proposition 13. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space of functions with (not necessarily orthogonal) basis $\{f_j\}$. Denote by $\mathbf{U}(x)$ the column vector $(f_j(x))_j$. Then the reproducing kernel of $\mathcal{H}$ is given by*

$$K_{\mathcal{H}}(x,y) = \mathbf{U}(y)^* \mathbf{G}^{-1} \mathbf{U}(x), \tag{81}$$

where $\mathbf{G} = (\langle f_j, f_k \rangle_{\mathcal{H}})_{j,k=1}^{\dim \mathcal{H}}$ is the Grammian matrix of the basis $\{f_j\}_j$.

Proof. Let $\{e_l\}_l$ be an orthonormal basis of $\mathcal{H}$ and let $e_l = \sum_j c_{lj} f_j$ be the expressions of e_l as linear combinations of the f_j. Let C denote the matrix $(c_{lj})_{l,j=1}^{\dim \mathcal{H}}$. From

$$\delta_{lm} = \langle e_l, e_m \rangle_{\mathcal{H}} = \sum_{j,k} c_{lj} \overline{c_{mk}} \langle f_j, f_k \rangle_{\mathcal{H}} = (C\mathbf{G}C^*)_{lm}$$

we see that $C\mathbf{G}C^*$ is the identity matrix; that is, $\mathbf{G} = (C^*C)^{-1}$. Hence by (43)

$$\begin{aligned} K_{\mathcal{H}}(x,y) &= \sum_l e_l(x)\overline{e_l(y)} = \sum_{l,j,k} c_{lj} f_j(x) \overline{c_{lk} f_k(y)} \\ &= \mathbf{U}(y) C^* C \mathbf{U}(x) = \mathbf{U}(y) \mathbf{G}^{-1} \mathbf{U}(x), \end{aligned}$$

proving the claim. □

For the basis (80), the Grammian matrix $\mathbf{G}$ is in principle easy to compute explicitly for low values of m and r. For instance, for $r = 2$, starting from

$$(1-t)^n = h(te,e)^n = \sum_{\mathbf{m}:m_1 \le n} (-n)_{\mathbf{m}} K_{\mathbf{m}}(te,e), \qquad n = 0,1,2,\dots,$$

one recursively reads off $(-n)_{(n,m_2)} K_{(n,m_2)}(te,e)$ as the homogeneous component of degree $n + m_2$ in $(1-t)^n$:

$$K_{(0)} = \mathbf{1}, \qquad K_{(1)} = t_1 + t_2, \qquad K_{(1,1)} = \frac{2t_1t_2}{a+2},$$
$$K_{(2,0)} = \frac{t_1^2 + t_2^2}{2} + \frac{at_1t_2}{a+2}, \qquad K_{(2,1)} = \frac{2t_1t_2(t_1+t_2)}{a+4},$$
$$K_{(2,2)} = \frac{2t_1^2t_2^2}{(a+2)(a+4)}, \qquad \dots .$$

(For brevity, we have omitted the arguments (te, e).) This reduces the computation of $\mathbf{G}$ to evaluation of the integrals

$$\int_0^\infty \int_0^\infty x_1^{q_1} x_2^{q_2} (1+x_1)^{-\nu} (1+x_2)^{-\nu} |x_1 - x_2|^a \, dx_1 \, dx_2. \tag{82}$$

For a an even nonnegative integer, the last integral can be evaluated by expanding $(x_1 - x_2)^a$ via the binomial theorem and integrating term by term using the standard formula

$$\int_0^\infty \frac{x^q}{(1+x)^\nu} \, dx = \frac{\Gamma(q+1)\Gamma(\nu-q-1)}{\Gamma(\nu)} \equiv B(q+1, \nu-q-1), \qquad -1 < q < \nu - 1,$$

for the Beta integral. The outcome is that (82) equals

$$\sum_{j=0}^{a} \frac{(-a)_j}{j!} B(q_1+j+1, \nu-q_1-a-1) B(q_2+a-j+1, \nu-q_2-a-1), \qquad a \in 2\mathbf{N}.$$

Taking for $\mathbf{U}(x)$ the column vector $(K_{\mathbf{m}}(xe,e))_{|\mathbf{m}|<m, m_1<(\nu-p+1)/2}$, one can then use (81) to obtain a formula for $S_\nu^m(x,0)$.

For $a \notin 2\mathbf{N}$, a possible way of evaluating (82) is first making the change of variable $x = \frac{t}{1-t}$, which transforms (82) into

$$\begin{aligned}&\int_0^1\int_0^1 t_1^{q_1}t_2^{q_2}(1-t_1)^{\nu-a-2-q_1}(1-t_2)^{\nu-a-2-q_2}|t_1-t_2|^a\,dt_1\,dt_2\\&\qquad=\int_0^1\int_0^1(1-t_1)^{q_1}(1-t_2)^{q_2}t_1^{\nu-a-2-q_1}t_2^{\nu-a-2-q_2}|t_1-t_2|^a\,dt_1\,dt_2.\end{aligned}\tag{83}$$

Introducing temporarily the notation

$$I(\alpha,\beta,\gamma,\delta):=\int_0^1\int_0^{t_1}(1-t_1)^\alpha(1-t_2)^\beta t_1^\gamma t_2^\delta|t_1-t_2|^a\,dt_2\,dt_1,$$

(83) thus equals

$$I(q_1,q_2,\nu-a-2-q_1,\nu-a-2-q_2)+I(q_2,q_1,\nu-a-2-q_2,\nu-a-2-q_1).$$

Now making the change of variable $t_2=yt_1$ yields

$$\begin{aligned}I(\alpha,\beta,\gamma,\delta)&=\int_0^1\int_0^1(1-t_1)^\alpha(1-yt_1)^\beta t_1^\gamma(yt_1)^\delta t_1^a(1-y)^a\,t_1\,dy\,dt_1\\&=\sum_{j=0}^\infty\frac{(-\beta)_j}{j!}\int_0^1\int_0^1(1-t_1)^\alpha(yt_1)^j t_1^\gamma(yt_1)^\delta t_1^a(1-y)^a\,t_1\,dy\,dt_1\\&=\sum_{j=0}^\infty\frac{(-\beta)_j}{j!}B(\alpha+1,j+\gamma+\delta+a+2)B(a+1,j+\delta+1).\end{aligned}\tag{84}$$

For $\beta\in\mathbf{N}$ — which is our case in (83) — the series terminates, and one thus has an expression for (82), albeit the formula is a bit more unwieldy than the one from the previous paragraph.

Similarly, for rank 3, recall that

$$K_{\mathbf{m}}(te,e)=\frac{\pi_{\mathbf{m}}}{(q_\Omega)_{\mathbf{m}}}\phi_{\mathbf{m}}(te),$$

with $\pi_{\mathbf{m}}$ and q_Ω given by (37) and (36), respectively, and $\phi_{\mathbf{m}}$ the spherical polynomial corresponding to the signature $\mathbf{m}$. Using the formula (34), one can again successively read off $K_{\mathbf{m}}(te,e)$ as the term of the appropriate homogeneity degree in

$$(1-t)^n=h(te,e)^n=\sum_{\mathbf{m}:m_1\le n}(-n)_{\mathbf{m}}\frac{\pi_{\mathbf{m}}}{(q_\Omega)_{\mathbf{m}}}\phi_{\mathbf{m}}(te),\qquad n=0,1,2,\dots.$$

This yields (omitting again the argument te)

$$\phi_{(0)} = \mathbf{1}, \qquad \phi_{(1)} = \frac{t_1+t_2+t_3}{3}, \qquad \phi_{(1,1)} = \frac{t_1t_2+t_1t_3+t_2t_3}{3},$$

$$\phi_{(1,1,1)} = t_1t_2t_3, \qquad \phi_{(2)} = \frac{(a+2)(t_1^2+t_2^2+t_3^2)}{3(3a+2)} + \frac{2a(t_1t_2+t_1t_3+t_2t_3)}{3(3a+2)},$$

$$\phi_{(2,1)} = \frac{(a+1)(t_1^2t_2+t_1^2t_3+t_2^2t_1+t_2^2t_3+t_3^2t_1+t_3^2t_2)}{3(3a+2)} + \frac{3at_1t_2t_3}{3(3a+2)},$$

$$\phi_{(2,1,1)} = \frac{(t_1+t_2+t_3)t_1t_2t_3}{3},$$

$$\phi_{(2,2)} = \frac{(a+2)(t_1^2t_2^2+t_1^2t_3^2+t_2^2t_3^2)}{3(3a+2)} + \frac{2a(t_1+t_2+t_3)t_1t_2t_3}{3(3a+2)},$$

$$\phi_{(2,2,1)} = \frac{(t_1t_2+t_1t_3+t_2t_3)t_1t_2t_3}{3}, \qquad \phi_{(2,2,2)} = t_1^2t_2^2t_3^2, \qquad \dots .$$

This once more reduces the computation of $\mathbf{G}$ to the evaluation of the three-variable analogue of (82), which for $a \in 2\mathbf{N}$ is again by the same "bare hands" method seen to be equal to

$$\begin{aligned}\sum_{j,k,l=0}^{a} \frac{(-a)_j(-a)_k(-a)_l}{j!k!l!} & B(q_1+1+j+k, \nu-2a-1-q_1) \\ & \times B(q_2+1+a-j+l, \nu-2a-1-q_2) \\ & \times B(q_3+1+2a-k-l, \nu-2a-1-q_3), \qquad a \in 2\mathbf{N}.\end{aligned}$$

For $a \notin 2\mathbf{N}$, one can again proceed as for (84), but the outcome is quite cumbersome.

Carrying out all these calculations leads to the following conjecture.

Recall that for $\alpha, \beta, \gamma \in \mathbf{C}$, the *Faraut-Koranyi hypergeometric function* on Ω with parameters α, β, γ is defined by [14]

$$ {}_2\mathcal{F}_1^{\Omega}\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}\middle| z\right) := \sum_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}} K_{\mathbf{m}}(z,z). \tag{85}$$

Here γ is assumed to be such that $(\gamma)_{\mathbf{m}} \neq 0\ \forall \mathbf{m}$. Alternatively, one sometimes views these just as symmetric functions on $\mathbf{R}_+^r$ [31]:

$$ {}_2\mathcal{F}_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}\middle| t\right) := \sum_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}(\beta)_{\mathbf{m}}}{(\gamma)_{\mathbf{m}}} K_{\mathbf{m}}(te,e), \tag{86}$$

the two variants being related simply by ${}_2\mathcal{F}_1^{\Omega}(z) = {}_2\mathcal{F}_1(t)$ for $z = k\sqrt{t}e$.

Conjecture 14. *Assume that $m \geq rq+1$ where q is the nonnegative integer such that $q < \frac{\nu-p+1}{2} \leq q+1$. Then the reproducing kernel S_ν^m on $\mathbf{R}_+^r$ at the origin is given by*

$$S_\nu^m(x,0) = c_\nu^q {}_2\mathcal{F}_1\left(\begin{matrix}-q, -b-\nu+2p+q-2r \\ p\end{matrix}\middle| -x\right), \tag{87}$$

where

$$c_\nu^q = \frac{\Gamma_\Omega(\nu-p-q+2r+b)\Gamma_\Omega(q_\Omega)\Gamma_\Omega(p+q)}{\pi^d\Gamma_\Omega(\nu-p-q+2r-q_\Omega)\Gamma_\Omega(q_\Omega+q)\Gamma_\Omega(p)}. \tag{88}$$

The last conjecture holds for $r=1$, by (8), (15), Theorem 6 and (78). It has also been verified by computer for

$$\begin{aligned}
&r=2,\ q\in\{0,1,2\},\ a\in\{1,2,3,4,5,6,7,8\},\ b\in\{0,1,2,3\},\ \nu \text{ arbitrary},\\
&r=2,\ q=3,\ a\in\{1,2,3,4\},\ b\in\{0,1,2,3\},\ \nu \text{ arbitrary},\\
&r=3,\ q\in\{0,1,2\},\ a\in\{2,4\},\ b\in\{0,1,2,3\},\ \nu \text{ arbitrary},\\
&r=3,\ q\in\{0,1,2\},\ a=8,\ b=0,\ \nu \text{ arbitrary},
\end{aligned}$$

and a couple more values of a, b and ν for $r\in\{2,3\}$ and $q\in\{0,1,2\}$. (Note that the above values of r,a,b include, in particular, both exceptional bounded symmetric domains of dimensions 16 and 27.)

Note that the hypothesis of the conjecture, that is,

$$q-1<\frac{\nu-p-1}{2}\le q\le\frac{m-1}{r}, \tag{89}$$

corresponds precisely to the case (57) of the "stabilized" kernels from Corollary 9. Without this hypothesis, the conjecture fails, as the following example shows.

Example 15. Let $r=2$, $m=2$ and $\nu-p>1$ (note that this corresponds to the case (56) in Corollary 9). The space $\mathcal{S}_\nu^m$ is thus spanned by $K_{(0)}(xe,e)=\mathbf{1}$ and $K_{(1)}(xe,e)=x_1+x_2$. Performing the calculations outlined above yields

$$\tfrac{1}{C}S_\nu^m(x,0)=K_{(0)}+\frac{(b-\nu+a+3)(2b-2\nu+a+4)}{a^2+(7+4b-2\nu)a+(4b^2-4b\nu+16b-6\nu+14)}K_{(1)}$$

with some constant C. (We have omitted the arguments (xe,e) at $K_{(0)}$ and $K_{(1)}$.) Plainly, the right-hand side is not of the form ${}_2\mathcal{F}_1$. $\square$

In the remaining case from Corollary 9 (i.e. $q+1<m<rq+1$), the kernels can be expected to be even more "ugly" than in the last example.

By the results of the preceding sections, the validity of the conjecture would have the following consequences.

Corollary 16. (Subject to Conjecture 14) *Assume that $m\ge rq+1$ where q is the non-negative integer such that $q<\frac{\nu-p+1}{2}\le q+1$. Then the nearly-holomorphic reproducing kernel N_ν^m at the origin is given by*

$$N_\nu^m(k\sqrt{t}e,0)=c_\nu^q\,{}_2\mathcal{F}_1\left(\begin{matrix}-q,-b-\nu+2p+q-2r\\ p\end{matrix}\middle|\frac{t}{t-1}\right),$$

or

$$N_\nu^m(z,0) = c_\nu^q h(z,z)^{-q}{}_2\mathcal{F}_1^\Omega\left(\begin{matrix}-q, b+\nu-p-q+2r\\ p\end{matrix}\,\middle|\, z\right).$$

Proof. By Theorem 6,

$$N_\nu^m(k\sqrt{t}e,0) = S_\nu^m(\tfrac{t}{1-t},0),$$

and (87) gives the first formula. The second formula then follows from the Kummer relation (a counterpart of (78) for the ordinary ${}_2F_1$)

$${}_2\mathcal{F}_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\,\middle|\, t\right) = (1-t)^{-\alpha}{}_2\mathcal{F}_1\left(\begin{matrix}\alpha,\gamma-\beta\\ \gamma\end{matrix}\,\middle|\, \frac{t}{t-1}\right), \tag{90}$$

see [31, formula (35)]. □

Theorem 17. (Subject to Conjecture 14) *Assume that $m \geq rq+1$ where q is the nonnegative integer such that $q < \frac{\nu-p+1}{2} \leq q+1$. Then with the notation (73), (69) and (85),*

$$\begin{aligned}\sum_{|\mathbf{m}|:\, m_1\leq q} d_r(\lambda_{\mathbf{m}},\nu)F(\lambda_{\mathbf{m}},\mathbf{k}_\nu,g) = \\ h(z,z)^{-q}c_\nu^q{}_2\mathcal{F}_1^\Omega\left(\begin{matrix}-q, b+\nu-p-q+2r\\ p\end{matrix}\,\middle|\, z\right)\end{aligned} \tag{91}$$

for $z = g0$ with $g \in A$.

Proof. Since $J_g(0) = g(g0,g0)^{p/2}$ for $g \in A$, (64) and (74) yield

$$\phi^\flat_{\lambda,\nu}(g0) = F(\lambda,\mathbf{k}_\nu,g0).$$

Thus by (71), the left-hand side of (91) equals $N_\nu^m(g0,0)$. By Corollary 16, the latter is precisely the right-hand side of (91). □

Note that for $r=1$, (91) recovers the formula (14) from the Introduction.

Remark 18. By Theorem 4.2 of Beerends and Opdam [6], $F(\lambda,\mathbf{k}_\nu,\cdot)$ for the special value

$$\lambda = -\alpha\sum_j \beta_j + \rho_\nu, \qquad \alpha \in \mathbf{C},$$

can be expressed in terms of

$${}_2\mathcal{F}_1\left(\begin{matrix}\alpha, d/r+\nu-\alpha\\ d/r\end{matrix}\,\middle|\, \cdot\right);$$

however the ${}_2\mathcal{F}_1$ in (91) does not seem reducible to this form. □

Using again Theorem 6, the conjecture also implies a formula for the invariantly-polyanalytic kernel $S_\nu^m(x,0)$.

Corollary 19. (Subject to Conjecture 14) *Assume that $m \geq rq+1$ where q is the non-negative integer such that $q < \frac{\nu-p-1}{2} + m \leq q+1$. Then the reproducing kernel P_ν^m at the origin is given by*

$$P_\nu^m(z,0) = c_{\nu+2m-2}^m h(z,z)^{m-1-q}{}_2\mathcal{F}_1\left(\begin{matrix}-q, b+\nu+2m-2-p-q+2r\\ p\end{matrix}\middle| z\right).$$

Proof. By Theorem 6, $P_\nu^m(z,0) = h(z,z)^{m-1}N_{\nu+2m-2}^m(z,0)$, and the claim follows by Corollary 16. □

Example 20. Continuing our example of $\Omega = \mathbf{B}^d$ from the previous section, for rank 1 the Faraut-Koranyi hypergeometric function coincides with the ordinary Gauss hypergeometric function

$${}_2\mathcal{F}_1^{\mathbf{B}^d}\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\middle| z\right) = {}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\middle| |z|^2\right).$$

By (79) and (87) we therefore get, for $0 \leq q < \frac{\nu-d}{2} \leq q+1 \leq m$,

$$\sum_{j=0}^q d_1(\lambda_{(j)},\nu){}_2F_1\left(\begin{matrix}-j, d-\nu+j\\ d\end{matrix}\middle| -x\right) = c_\nu^q{}_2F_1\left(\begin{matrix}-q, d+q+1-\nu\\ d+1\end{matrix}\middle| -x\right).$$

This is, of course, just (14) in disguise. □

Remark 21. The formula (79) actually shows that $d_1(\lambda_{(j)},\nu){}_2F_1\left(\begin{matrix}-j, d-\nu+j\\ d\end{matrix}\middle| -x\right)$ is the reproducing kernel of the orthogonal complement $\mathcal{S}_\nu^j \ominus \mathcal{S}_\nu^{j-1}$, $0 \leq j < \frac{\nu-d}{2}$ (with $\mathcal{S}_\nu^{-1} := \{0\}$). Theorem V.4.5 in Helgason [16] identifies the last ${}_2F_1$ as the spherical function for the compact dual $SU(d+1)/SU(d) = \mathbf{C}P^d$ of $\mathbf{B}^d$. □

By the reproducing property, Conjecture 14 is equivalent to

$$\begin{aligned}\int_{\mathbf{R}_+^r} K_{\mathbf{m}}(xe,e)\,{}_2\mathcal{F}_1\left(\begin{matrix}-q, -b-\nu+2p+q-2r\\ p\end{matrix}\middle| -x\right) d\rho_{b,\nu,a}(x)\\ = \frac{1}{c_\nu^q}\delta_{\mathbf{m},(0)}, \qquad \forall \mathbf{m} \text{ with } m_1 \leq q,\end{aligned} \tag{92}$$

where q is the nonnegative integer such that $q < \frac{\nu-p+1}{2} \leq q+1$ and $m \geq rq+1$. Taking in particular $m = (0)$ yields

$$\frac{1}{c_\nu^q} = \int_{\mathbf{R}_+^r} {}_2\mathcal{F}_1\Big(\begin{matrix}-q, -b-\nu+2p+q-2r\\ p\end{matrix}\Big| -x\Big)\, d\rho_{b,\nu,a}(x)$$

(subject to the validity of Conjecture 14). The last integral can be evaluated explicitly.

Proposition 22.

$$\int_{\mathbf{R}_+^r} {}_2\mathcal{F}_1\Big(\begin{matrix}-q, -b-\nu+2p+q-2r\\ p\end{matrix}\Big| -x\Big)\, d\rho_{b,\nu,a}(x) = \frac{1}{c_{\nu-q}}\, {}_3\mathcal{F}_2^\Omega\Big(\begin{matrix}-q, b+\nu-p-q+2r, d/r\\ p, \nu-q\end{matrix}\Big| e\Big),$$

where the Faraut-Koranyi function ${}_3\mathcal{F}_2^\Omega$ is defined analogously as in (85).

Proof. Making again the change of variable $x = \frac{t}{1-t}$, we get from (54) and (86)

$$\begin{aligned}
&\int_{\mathbf{R}_+^r} {}_2\mathcal{F}_1\Big(\begin{matrix}-q, -b-\nu+2p+q-2r\\ p\end{matrix}\Big| -x\Big)\, d\rho_{b,\nu,a}(x)\\
&= c_\Omega \int_{[0,1]^r} {}_2\mathcal{F}_1\Big(\begin{matrix}-q, -b-\nu+2p+q-2r\\ p\end{matrix}\Big| \frac{t}{t-1}\Big)\, d\mu_{b,\nu,a}(t)\\
&= c_\Omega \int_{[0,1]^r} (1-t)^{-q}{}_2\mathcal{F}_1\Big(\begin{matrix}-q, b+\nu-p-q+2r\\ p\end{matrix}\Big| t\Big)\, d\mu_{b,\nu,a}(t)\\
&= c_\Omega \int_{[0,1]^r} {}_2\mathcal{F}_1\Big(\begin{matrix}-q, b+\nu-p-q+2r\\ p\end{matrix}\Big| t\Big)\, d\mu_{b,\nu-q,a}(t)\\
&= c_\Omega \sum_{|\mathbf{m}|<m} \frac{(-q)_{\mathbf{m}}(b+\nu-p-q+2r)_{\mathbf{m}}}{(p)_{\mathbf{m}}} \int_{[0,1]^r} K_{\mathbf{m}}(te, e)\, d\mu_{b,\nu-q,a}(t).
\end{aligned}$$

If $\{\psi_j\}_{j=1}^{d_{\mathbf{m}}}$ is an orthonormal basis of $\mathbf{P}_{\mathbf{m}}$ with respect to the Fock norm, the last integral equals, by (43),

$$\begin{aligned}
\int_{[0,1]^r} K_{\mathbf{m}}(\sqrt{t}e, \sqrt{t}e)\, d\mu_{b,\nu-q,a}(t) &= \int_K \int_{[0,1]^r} K_{\mathbf{m}}(k\sqrt{t}e, k\sqrt{t}e)\, d\mu_{b,\nu-q,a}(t)\, dk\\
&= \frac{1}{c_\Omega} \int_\Omega K_{\mathbf{m}}(z,z)\, d\mu_{\nu-q}(z) \qquad \text{by (27)}\\
&= \frac{1}{c_\Omega} \int_\Omega \sum_j |\psi_j(z)|^2\, d\mu_{\nu-q}(z)
\end{aligned}$$

$$= \frac{1}{c_\Omega} \sum_j \|\psi_j\|^2_{\nu-q}$$

$$= \frac{1}{c_\Omega} \sum_j \frac{\|\psi_j\|^2_F}{(\nu-q)_{\mathbf{m}} c_{\nu-q}} \qquad \text{by (32)}$$

$$= \frac{d_{\mathbf{m}}}{c_\Omega (\nu-q)_{\mathbf{m}} c_{\nu-q}}.$$

Consequently,

$$\begin{aligned}
&\int_{\mathbf{R}^r_+} {}_2\mathcal{F}_1\left(\begin{matrix} -q, -b-\nu+2p+q-2r \\ p \end{matrix} \middle| -x \right) d\rho_{b,\nu,a}(x) \\
&= \sum_{|\mathbf{m}|<m} \frac{(-q)_{\mathbf{m}} (b+\nu-p-q+2r)_{\mathbf{m}}}{(p)_{\mathbf{m}}} \frac{d_{\mathbf{m}}}{(\nu-q)_{\mathbf{m}} c_{\nu-q}} \\
&= \sum_{|\mathbf{m}|<m} \frac{(-q)_{\mathbf{m}} (b+\nu-p-q+2r)_{\mathbf{m}}}{(p)_{\mathbf{m}}} \frac{(d/r)_{\mathbf{m}}}{(\nu-q)_{\mathbf{m}} c_{\nu-q}} K_{\mathbf{m}}(e,e) \\
&= \frac{1}{c_{\nu-q}} \, {}_3\mathcal{F}_2^\Omega \left(\begin{matrix} -q, b+\nu-p-q+2r, d/r \\ p, \nu-q \end{matrix} \middle| e \right),
\end{aligned}$$

as claimed. Here the second equality is due to (35). $\square$

The formula (88) thus gives a conjectured value for this ${}_3\mathcal{F}_2$ function.

For rank 1, we have $b+\nu-p-q+2r = \nu-q$, so the ${}_3F_2$ becomes ${}_2F_1$ and (88) follows by the standard formula for ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| 1 \right)$.

6. Compact Hermitian symmetric spaces

We now consider also the compact duals of Hermitian symmetric spaces $\hat{\Omega}$, the simplest examples of these being the complex projective space $\mathbf{C}P^d$ as the compact dual of the unit ball $\mathbf{B}^d$ (including, in particular, the Riemann sphere $\mathbf{C}P^1$ as the compact dual of the unit disc). Most results are obtained by formally replacing ν by $-\nu$, $h(z,z)$ by $h(z,-z)$, and $\Omega \subset \mathbf{C}^d$ by the open chart $\mathbf{C}^d \subset \hat{\Omega}$. We shall be rather brief.

The symmetric space $\Omega = G/K$ has its compact dual $\hat{\Omega} = \hat{G}/K$ where $\hat{G}$ is a simply connected compact Lie group with Lie algebra $\hat{\mathfrak{g}} = \mathfrak{k} + i\mathfrak{p}$. There is a dense open subset of $\hat{\Omega}$ that is biholomorphic to $\mathbf{C}^d$, and we shall simply identify this local chart with $\mathbf{C}^d$ throughout. The stabilizer subgroup K of the origin in $\hat{G}$ is the same as in the bounded case. For $x \in \hat{\Omega}$, there is again a unique geodesic symmetry $\hat{\phi}_x \in \hat{G}$ which interchanges x and the origin, i.e. $\hat{\phi}_x \circ \hat{\phi}_x = \mathrm{id}$, $\hat{\phi}_x(0) = x$, $\hat{\phi}_x(x) = 0$, and $\hat{\phi}_x$ has only isolated fixed points. Any $g \in \hat{G}$ can be uniquely written in the form $g = \hat{\phi}_x k$ with $k \in K$ and $x = g0 \in \hat{\Omega}$. The measure

$$d\hat{\mu}_\nu(z) := h(z,-z)^{-\nu-p}\,dz$$

on $\mathbf{C}^d \subset \hat{\Omega}$ is finite if and only if $\nu > -1$, and one can again consider the spaces

$$\hat{A}_\nu := L^2(\hat{\Omega}, d\hat{\mu}_\nu) \cap \mathcal{O}(\mathbf{C}^d).$$

The elements of $\hat{A}_\nu$ extend to holomorphic sections on all of $\hat{\Omega}$ if and only if ν is an *integer*, which we will assume from now on throughout the rest of this section. In that case,

$$\hat{A}_\nu = \bigoplus_{\mathbf{m}:m_1\le\nu} \mathbf{P}_\mathbf{m},$$

and $\hat{A}_\nu$ possesses a reproducing kernel, given by

$$\hat{K}_\nu(z,w) = \hat{c}_\nu\; h(z,-w)^\nu, \qquad z,w \in \mathbf{C}^d \subset \hat{\Omega},\ \nu \in \mathbf{N},$$

where

$$\hat{c}_\nu = \frac{\Gamma_\Omega(\nu+p)}{\pi^d \Gamma_\Omega(\nu+p-\frac{d}{r})}.$$

(Here, as before, p, r, a and b denote the genus, the rank, and the characteristic multiplicities of $\hat{\Omega}$, which are all the same as for Ω.) From the transformation rule

$$h(\hat{\phi}z, -\hat{\phi}w) = \frac{h(a,-a)h(z,-w)}{h(z,-a)h(a,-w)}, \qquad a = \hat{\phi}^{-1}0, \qquad z,w \in \mathbf{C}^d,\ \hat{\phi} \in \hat{G},$$

it again follows that the measure $d\hat{\mu}_0$ is $\hat{G}$-invariant and that $\hat{\Psi}(z) := -\log h(z,-z)$ is the Kähler potential for a $\hat{G}$-invariant Riemannian metric on $\hat{\Omega}$. We thus again have the associated Cauchy-Riemann operator $\overline{D}$, and the corresponding spaces $\hat{\mathcal{N}}^m := \operatorname{Ker} \overline{D}^m$ of nearly holomorphic functions on $\hat{\Omega}$ of order m, as well as their Bergman-type subspaces

$$\hat{\mathcal{N}}^m_\nu := L^2(\hat{\Omega}, d\hat{\mu}_\nu) \cap \operatorname{Ker} \overline{D}^m.$$

One can also proceed to define the invariantly polyanalytic functions $\hat{\mathcal{P}}^m$ and their Bergman-type subspaces $\hat{\mathcal{P}}^m_\nu$ as in the bounded case.

In the polar coordinates (22), the measures $d\hat{\mu}_\nu$ assume the form

$$\int_{\hat{\Omega}} f(z)\,d\hat{\mu}_\nu(z) = \quad c_\Omega \int_{\mathbf{R}^r_+} \int_K f(k\sqrt{t}e)\,dk\,d\hat{\mu}_{b,\nu,a}(t), \tag{93}$$

where

$$d\hat{\mu}_{b,\nu,a}(t) := t^b(1+t)^{-\nu-p} \prod_{1\le i<j\le r} |t_i - t_j|^a \, dt \tag{94}$$

and c_Ω is given by (29).

By the above transformation rule for $h(z,-w)$, it again also follows that

$$f \longmapsto \frac{h(a,-a)^{-\nu/2}}{h(z,-a)^{-\nu}} f \circ \hat{\phi}^{-1}, \qquad a = \hat{\phi}0,\ \hat{\phi} \in \hat{G},\ \nu \in \mathbf{N},$$

is a projective unitary representation of $\hat{G}$ on $\hat{\mathcal{N}}^m_\nu$. Let $\hat{\mathcal{S}}^m$ be the vector space of all symmetric polynomials of degree $< m$ in r variables, denote

$$\hat{\mathcal{S}}^m_\nu := \hat{\mathcal{S}}^m \cap L^2([0,1]^r, d\hat{\rho}_{b,\nu,a}),$$

where

$$d\hat{\rho}_{b,\nu,a}(t) := c_\Omega \, t^b(1-t)^\nu \prod_{1\le i<j\le r} |t_i - t_j|^a \, dt,$$

and let $\hat{S}^m_\nu(x,y)$ be the reproducing kernel of $\hat{\mathcal{S}}^m_\nu$. Proceeding as in Section 3 above, we then obtain the following analogue of Theorem 6.

Theorem 23.

(a) *For any $\hat{\phi} \in \hat{G}$, the reproducing kernel $\hat{N}^m_\nu$ of $\hat{\mathcal{N}}^m_\nu$ satisfies*

$$\hat{N}^m_\nu(z,w) = \frac{h(z,-a)^\nu h(a,-w)^\nu}{h(a,-a)^\nu} \hat{N}^m_\nu(\hat{\phi}z, \hat{\phi}w), \qquad a := \hat{\phi}^{-1}0;$$

in particular,

$$\hat{N}^m_\nu(z,w) = h(z,-w)^\nu \hat{N}^m_\nu(\hat{\phi}_w z, 0).$$

(b) *Radial functions in $\hat{\mathcal{N}}^m$ consist precisely of functions of the form*

$$p(z,(-z)^z),$$

where $p(z,w)$ is a polynomial in $z, \overline{w} \in \mathbf{C}^d$ of degree $< m$ in each argument which is K-invariant in the sense of (48).
Consequently, the radial functions in $\hat{\mathcal{N}}^m$ coincide with the linear span of $K_{\mathbf{m}}(z, (-z)^z)$, $|\mathbf{m}| < m$.

(c) *The mapping $\hat{V}$ from $\hat{\mathcal{S}}^m$ into functions on $\hat{\Omega}$ given by*

$$\hat{V} f(k\sqrt{x}e) := f\Big(\frac{x}{1+x}\Big), \qquad k \in K,\ x \in \mathbf{R}^r_+,$$

is a bijection from $\hat{\mathcal{S}}^m$ *onto radial functions in* $\hat{\mathcal{N}}^m$*. Furthermore,* $\hat{V}$ *sends* $\hat{\mathcal{S}}^m_\nu$ *unitarily onto the subspace* $\hat{\mathcal{R}}^m_\nu$ *of radial function in* $\hat{\mathcal{N}}^m_\nu$*, and*

$$\hat{N}^m_\nu(\cdot, 0) = V\hat{S}^m_\nu(\cdot, 0).$$

Proof. The proof is the same as for Propositions 2, 4 and 5, hence omitted. □

Unlike the bounded case, for the compact dual we can give an explicit formula for the kernel $\hat{N}^m_\nu$ in terms of multivariable Jacobi polynomials $P^{(\alpha,\beta,a/2)}_{\mathbf{m}}$ (also called Heckman-Opdam polynomials; see [15, Section 1.3]). Recall from [10, Section 4.b] that $P^{(\alpha,\beta,a/2)}_{\mathbf{m}}(t)$ are symmetric polynomials on $\mathbf{R}^r$ such that

(i) $P^{(\alpha,\beta,a/2)}_{\mathbf{m}}(t)$ is the symmetrization of

$$t^{\mathbf{m}} + \sum_{\mathbf{n}<\mathbf{m}} c_{\mathbf{m}\mathbf{n}} t^{\mathbf{n}} \tag{95}$$

where the sum is over (some) signatures $\mathbf{n}$ smaller than $\mathbf{m}$ with respect to the lexicographic order; and

(ii) $P^{(\alpha,\beta,a/2)}_{\mathbf{m}}(t)$, $|\mathbf{m}| \geq 0$, are orthogonal on $[-1, +1]^r$ with respect to the measure

$$(1-t)^\alpha (1+t)^\beta \prod_{1\leq i<j\leq r} |t_i - t_j|^a \, dt.$$

By change of variable, it follows that $P^{(\alpha,\beta,a/2)}_{\mathbf{m}}(1-2t)$ are orthogonal on $[0, 1]^r$ with respect to the measure $t^\alpha(1-t)^\beta \prod_{1\leq i<j\leq r} |t_i - t_j|^a \, dt$, with the norm-square on $[-1, +1]^r$ given by $2^{r(r-1)a/2+r\alpha+r\beta+r}$ times the norm-square on $[0, 1]^r$. Setting in particular $\alpha = b$, $\beta = \nu$ we get an orthogonal basis for symmetric polynomials with respect to $d\hat{\rho}_{b,\nu,a}$ on $[0, 1]^r$. By (43), we thus arrive at the following theorem.

Theorem 24. *For* $\nu \in \mathbf{N}$*, the reproducing kernel* $\hat{S}^m_\nu$ *at the origin is given by*

$$\hat{S}^m_\nu(t, 0) = \sum_{|\mathbf{m}|<m} \frac{2^{d+r\nu} P^{(b,\nu,a/2)}_{\mathbf{m}}(1)}{\|P^{(b,\nu,a/2)}_{\mathbf{m}}\|^2} P^{(b,\nu,a/2)}_{\mathbf{m}}(1-2t). \tag{96}$$

Here the norm-square is understood on $[-1, +1]^r$*.*

We remark that explicit formulas both for $P^{(b,\nu,a/2)}_{\mathbf{m}}(1)$ and for $\|P^{(b,\nu,a/2)}_{\mathbf{m}}\|^2$ are available, see Theorems 3.5.5 and 3.6.6 in [15].

Example 25. For rank $r = 1$, $P^{(b,\nu,a/2)}_{\mathbf{m}}$ are up to a constant factor just the ordinary Jacobi polynomials $P^{(b,\nu)}_n$ of degree n on $[-1, +1]$:

$$P_{(n)}^{(b,\nu,a/2)}(t) = \frac{2^n}{\binom{2n+b+\nu}{n}} P_n^{(b,\nu)}(t).$$

From the formulas [5, Section 10.8]

$$P_n^{(b,\nu)}(1) = \binom{n+b}{n}, \qquad \|P_n^{(b,\nu)}\|^2 = \frac{2^{b+\nu+1}\Gamma(n+b+1)\Gamma(n+\nu+1)}{n!(2n+b+\nu+1)\Gamma(n+b+\nu+1)},$$

we therefore get

$$\hat{S}_\nu^m(t,0) = \sum_{j=0}^{m-1} \frac{\Gamma(j+\nu+1)}{(d-1)!j!^2(d+2j+\nu)\Gamma(d+j+\nu)} P_j^{(d-1,\nu)}(1-2t). \quad \square$$

Using Theorem 23, we can also obtain from (96) a formula for the nearly-holomorphic reproducing kernel $\hat{N}_\nu^m(z,w)$ on $\hat{\Omega}$ in terms of multivariable Jacobi polynomials.

Corollary 26. *For $\nu \in \mathbf{N}$, the nearly-holomorphic reproducing kernel $\hat{N}_\nu^m$ is given by $\hat{N}_\nu^m(z,w) = h(z,-w)^\nu \hat{N}_\nu^m(\hat{\phi}_w z, 0)$, where*

$$\hat{N}_\nu^m(k\sqrt{x}e,0) = \sum_{|\mathbf{m}|<m} \frac{2^{d+r\nu} P_{\mathbf{m}}^{(b,\nu,a/2)}(1)}{\|P_{\mathbf{m}}^{(b,\nu,a/2)}\|^2} P_{\mathbf{m}}^{(b,\nu,a/2)}\Big(\frac{1-x}{1+x}\Big).$$

Proof. Straightforward from Theorem 23 and (96). $\square$

One can also get the invariantly-polyanalytic kernels $\hat{P}_\nu^m$. We leave the details (which are utterly routine) to the interested reader.

Remark 27. Note that in (96) there is no restriction on m_1 in the sum, in contrast to Corollary 9 or (71); the reason being, of course, that $d\hat{\rho}_{b,\nu,a}$ is a finite measure on $[0,1]^r$ for $\nu \in \mathbf{N}$, so that the corresponding L^2 space contains all polynomials. Still, proceeding as in Section 5, one can get the following analogue of Conjecture 14 for the compact case. For $q, \nu \in \mathbf{N}$, let $\mathcal{Q}_\nu^q$ be the subspace of $L^2([0,1]^r, d\hat{\rho}_{b,\nu,a})$ spanned by $\{K_{\mathbf{m}}(te,e) : m_1 \le q\}$, and let Q_ν^q be its reproducing kernel. Then it seems that

$$Q_\nu^q(t,0) = \hat{c}_\nu^q \; {}_2\mathcal{F}_1\Big(\begin{matrix}-q, \nu+p+q \\ p\end{matrix}\Big| t\Big), \tag{97}$$

where

$$\hat{c}_\nu^q = \frac{\Gamma_\Omega(p+q)\Gamma_\Omega(p+q+\nu)\Gamma_\Omega(q_\Omega)}{\pi^d \Gamma_\Omega(p)\Gamma_\Omega(q+q_\Omega)\Gamma_\Omega(q+q_\Omega+\nu)}.$$

This has been checked for the same set of values of r, q, a, b as for Conjecture 14.

Note that in view of (55) and (95), the Jacobi polynomials $P_{\mathbf{m}}^{(b,\nu,a/2)}(1-2t)$ with $m_1 \le q$ form an orthogonal basis for $\mathcal{Q}_\nu^q$, thus again by (43)

$$Q_\nu^q(t,0) = \sum_{|\mathbf{m}|:\, m_1 \le q} \frac{2^{d+r\nu} P_{\mathbf{m}}^{(b,\nu,a/2)}(1)}{\|P_{\mathbf{m}}^{(b,\nu,a/2)}\|^2} \, P_{\mathbf{m}}^{(b,\nu,a/2)}(1-2t).$$

Hence (97) gives a conjectured value for this sum. $\square$

We conclude this section by deriving the counterpart of Section 4, i.e. the representation theory of for the L^2 spaces of sections of line bundles — especially the results of [35] — for the compact case. We give a representation theoretic proof of Corollary 26.

We follow the presentation as in [19]. We consider the holomorphic line bundle $\mathcal{L}$ over $\hat{G}/K$,

$$\hat{G} \times_{K,\tau} \mathbf{C} \to \hat{\Omega} = \hat{G}/K, \tag{98}$$

where $\tau(k) = (\det \mathrm{Ad}(k)|_{\mathfrak{p}^+})^{1/p}$, $k \in K$. This is the holomorphic line bundle such that $\mathcal{L}^p = \mathcal{K}^{-1}$ and it generates the Picard group of $\hat{\Omega}$; see [19, 7.1-7.11]. Here $\mathcal{K}^{-1}$ is the dual of the canonical line bundle. Let $\tau_\nu = \tau^\nu$ for any fixed integer ν, where as before we assume that $\nu \ge 0$.

Let $L^2(\hat{\Omega};\nu)$ be the L^2-space of sections of the line bundle $\mathcal{L}^\nu$. We normalize the measure so that the realization of sections $f \in L^2(\hat{\Omega};\nu)$ as functions on $L^2(\hat{G})$ is an isometry. More precisely $L^2(\hat{\Omega};\nu)$ consists of $f \in L^2(\hat{G})$ such that

$$\tau_\nu(k) f(gk) = f(g), \qquad k \in K,$$

and

$$\|f\|_\nu^2 = \int_{\hat{G}} |f(g)|^2 \, dg < \infty,$$

where dg is the Haar measure on $\hat{G}$ normalized so that $\int_{\hat{G}} dg = 1$.

The space $V := \mathbf{C}^d$ can be realized as an open subset in $\hat{\Omega}$ and we shall realize the space $L^2(\hat{\Omega};\nu)$ as point-wise functions on V. Under our assumption $\nu \ge 0$ the space of holomorphic sections of the line bundle (98) is non-zero, and there exists a global frame $e_\nu(z)$ with point-wise norm

$$\|e_\nu(z)\|_z^2 = h(z,-z)^{-\nu}.$$

Then a section $f \in L^2(\hat{\Omega};\nu)$ will be written as $f = f(z)e_\nu(z)$ for a point-wise function on V such that

$$\|f\|_\nu^2 = \hat{c}_\nu \int_V |f(z)|^2 h(z,-z)^{-\nu} d\mu_0(z), \qquad f = f(z)e_\nu(z),$$

where

$$d\mu_0(z) = \frac{dz}{h(z, -z)^p}$$

is the $\hat{G}$-invariant (Kähler) measure on $\hat{\Omega}$. To avoid confusion we write $L^2(V, \nu)$ for the space of L^2-functions $f(z)$ with the above norm. As an L^2-space and unitary representation of $\hat{G}$, $L^2(\hat{\Omega}; \nu) = L^2(V, \nu)$ via this identification.

Let $\mathfrak{g}^{\mathbf{C}} = \mathfrak{p}^+ + \mathfrak{k}^{\mathbf{C}} + \mathfrak{p}^-$ be the Harish-Chandra decomposition of $\mathfrak{g}^{\mathbf{C}}$. We use the same complex structure for Ω as for $\hat{\Omega}$, so that $\mathfrak{p}^+ = T_0^{(1,0)}(\hat{\Omega}) \equiv V$ is the holomorphic tangent space at $0 \in V \subset \hat{\Omega}$. Let $\mathfrak{t} \subset \mathfrak{k}^{\mathbf{C}}$ be a Cartan subalgebra, and let $\gamma_1 > \cdots > \gamma_r$ be the Harish-Chandra strongly orthogonal roots so that γ_1 is the highest root for $\mathfrak{p}^+$ as representation of $\mathfrak{k}^{\mathbf{C}}$. In particular γ_1 is the highest root of $\mathfrak{g}^{\mathbf{C}}$ as representation of $\mathfrak{g}^{\mathbf{C}}$. Let $\mathfrak{t}^-$ be the span of the co-roots of $\gamma_1, \cdots, \gamma_r$ and let $\mathfrak{t} = \mathfrak{t}^- + \mathfrak{t}^+$ with $\gamma_1, \cdots, \gamma_r$ vanishing on $\mathfrak{t}^+$. The root space decomposition of $\mathfrak{g}^{\mathbf{C}} = \mathfrak{p}^+ + \mathfrak{k}^{\mathbf{C}} + \mathfrak{p}^-$ is refined as $\mathfrak{g}^{\mathbf{C}} = (\mathfrak{p}^+ + \mathfrak{k}^+) + \mathfrak{t} + (\mathfrak{k}^- + \mathfrak{p}^-)$ with $\mathfrak{k}^- + \mathfrak{p}^-$ the space of negative root vectors, and $\mathfrak{k}^+ + \mathfrak{k}^- \subset [\mathfrak{k}^{\mathbf{C}}, \mathfrak{k}^{\mathbf{C}}]$.

The L^2-space $L^2(\hat{\Omega}, \nu)$ is decomposed as

$$L^2(\hat{\Omega}, \nu) = \sum_{\mathbf{m}} V_{\nu,\mathbf{m}} \tag{99}$$

where each $V_{\nu,\mathbf{m}}$ is of highest weight whose restriction on $\mathfrak{t}^-$ is

$$\frac{\nu}{2} + m_1\gamma_1 + \cdots + m_r\gamma_r, \qquad \frac{\nu}{2} := \frac{1}{2}\nu(\gamma_1 + \cdots + \gamma_r),$$

where m_j are nonnegative integers subject to the condition

$$m_1 \geq \cdots \geq m_r \geq 0. \tag{100}$$

When $\Omega = G/K$ is not of tube type this does not define completely the highest weights and it requires some extra specifications; however the highest weights of these representations that appear in $L^2(\hat{\Omega}, \nu)$ are uniquely determined by the condition above, see [25], [27], [34].

Recall the τ_ν-spherical functions on $\hat{G}$

$$\tau(k_1)^\nu \tau(k_2)^\nu f(k_1 g k_2) = f(g), \qquad g \in \hat{G},\ k_1, k_2 \in K.$$

As functions on $\hat{G}$ each space $V_{\nu,\mathbf{m}}$ contains a unique τ_ν-spherical function $\Psi_{\nu,\mathbf{m}}$ normalized by $\Psi_{\nu,\mathbf{m}}(e) = 1$. We set

$$\phi_{\nu,\mathbf{m}}(z) = J_g(0)^{-\frac{\nu}{p}} \Psi_{\nu,\mathbf{m}}(g), \qquad g \cdot 0 = z,$$

as a trivialization of the τ_ν-spherical function $\Psi_{\nu,\mathbf{m}}$. In particular $\phi_{\nu,\mathbf{m}}(z)$ is now both left and K-invariant, and thus can be realized as a left K-invariant function on $V \subset \hat{\Omega}$, $\phi_{\nu,\mathbf{m}}(kz) = \phi_{\nu,\mathbf{m}}(z)$, $\phi_{\nu,\mathbf{m}}(0) = 1$, and

$$\phi_{\nu,\mathbf{m}}(z) = h(z,-z)^{-\frac{\nu}{2}} \Psi_{\nu,\mathbf{m}}(\hat{\phi}_z).$$

In the notation above $\phi_{\mathbf{m}}(z)$ is the coefficient of the section $\Psi_{\nu,\mathbf{m}}$ with respect to the frame $e_\nu(z)$. The orthogonality relations for $\phi_{\nu,\mathbf{m}}$ read now

$$\hat{c}_\nu \int\limits_V \phi_{\nu,\mathbf{m}}(z)\overline{\phi_{\nu,\mathbf{m}'}(z)} h(z,-z)^{-\nu} d\mu_0(z)$$
$$= \hat{c}_\nu c_\Omega 2^r \int\limits_{\mathbf{R}_+^r} \phi_{\nu,\mathbf{m}}(x)\overline{\phi_{\nu,\mathbf{m}'}(x)} \prod_{j=1}^r (1+x_j^2)^{\nu-p} \prod_{1\le j<k\le r} (x_j^2 - x_k^2)^a \prod_{j=1}^r x_j^{2b+1}\, dx_j$$
$$= \frac{1}{d_{\nu,\mathbf{m}}} \delta_{\mathbf{m},\mathbf{m}'},$$

where $d_{\nu,\mathbf{m}} = \dim V_{\nu,\mathbf{m}}$ is the dimension of $V_{\nu,\mathbf{m}}$ (which can be computed using the Weyl dimension formula). These are the Jacobi polynomials of Heckman and Opdam. (The functions $\Psi_{\nu,\mathbf{m}}$ are the spherical functions $\phi_{\lambda,\nu}$ studied by Shimeno for specific discrete values of the parameter λ; see [27, Remark 5.12].)

In particular, for $\mathbf{m} = (0)$, $V_{\nu,(0)}$ is the Bergman space of holomorphic sections of the line bundle defined by ν in $L^2(\hat{\Omega},\nu)$. It can be realized as the space of holomorphic polynomials of degree $\le \nu$ and has reproducing kernel $\hat{c}_\nu h(z,-w)^\nu$. The corresponding Heckman-Opdam polynomial is the constant function $\phi_{\nu,(0)}(z) = 1$.

We equip $\hat{\Omega}$ with the $\hat{G}$-invariant (Kähler) metric and let $\overline{D}$ be the associated invariant Cauchy-Riemann operator. We describe the decomposition (99) using the kernels of $\overline{D}^m$. We shall need some results on the vanishing properties of Shimura operators on the spaces $V_{\nu,\mathbf{m}}$ obtained in [24]. First we recall the Shimura operators using our present formulation. Recall from Section 2 the Hua-Schmid decomposition

$$\otimes^m V = \sum_{|\mathbf{m}|=m} S^{\mathbf{m}} V$$

of the symmetric tensor product $\otimes^m V$ under K. Let $P_{\mathbf{m}}$ be the corresponding projection. It is a general fact that $\bar{D}^m : C^\infty(G,K;\tau_\nu) \to C^\infty(G,K;\tau_\nu \otimes \otimes^m V)$, where as before V is identified as the holomorphic tangent space $T_0^{(1,0)}(\hat{\Omega})$ of $\hat{\Omega}$ at 0, and $C^\infty(G,K;\tau_\nu \otimes \otimes^m V)$ is the space of smooth sections of the line bundle $\mathcal{L}^\nu \otimes \otimes^m T^{(1,0)}$ realized as functions on $\hat{G}$ transforming under K as

$$\tau_\nu(k) \otimes^m \mathrm{Ad}(k) f(gk) = f(g), \qquad g \in \hat{G}.$$

The Shimura operators are defined by

$$L_{\mathbf{m}} = (\bar{D}^{|\mathbf{m}|})^* P_{\mathbf{m}} \bar{D}^{|\mathbf{m}|}.$$

We have then

$$(\bar{D}^{m+1})^* \bar{D}^{m+1} = \sum_{|\mathbf{m}|=m+1} L_{\mathbf{m}}.$$

Theorem 28. *The kernel* $\operatorname{Ker} \bar{D}^{m+1}$ *in* $L^2(\hat{\Omega}, \nu)$ *is precisely the direct sum*

$$\operatorname{Ker} \bar{D}^{m+1} = \sum_{|\mathbf{m}|\le m}^{\oplus} V_{\nu,\mathbf{m}}.$$

In particular the reproducing kernel at the origin for the space of nearly holomorphic sections of order $m+1$ *in* $L^2(\hat{\Omega}, \nu)$ *is given by*

$$\hat{N}_\nu^{m+1}(z, 0) = \sum_{|\mathbf{m}|\le m} d_{\mathbf{m}} \phi_{\nu,\mathbf{m}}(z).$$

Proof. The operator $L_{\mathbf{m}}$ acts on each irreducible component $V_{\nu,\mathbf{n}}$ in (99) as a non-negative scalar multiple of the identity, by Schur's lemma, and their eigenvalues are shown in [24] to be given by Okounkov polynomials. More precisely, the eigenvalue of $L_{\mathbf{m}}$ on $V_{\nu,\mathbf{n}}$ is a symmetric polynomial $\widetilde{L}_{\mathbf{m}}(\frac{\nu}{2} + \mathbf{n} + \rho)$ of $\frac{\nu}{2} + \mathbf{n} + \rho$, where ρ is the half-sum of positive roots of $\mathfrak{t}$ in $\mathfrak{g}^{\mathbf{C}}$. (One may also take $\frac{\nu}{2}$ into the definition of ρ as above.) It follows from [24, Theorem 5.1] that $\widetilde{L}_{\mathbf{m}}(\frac{\nu}{2} + \mathbf{n} + \rho) = 0$ unless $\mathbf{m} \subseteq \mathbf{n}$ (i.e. $m_j \le n_j$ for all $j = 1, \ldots, r$). This implies that

$$\sum_{|\mathbf{n}|\le m}^{\oplus} V_{\nu,\mathbf{n}} \subseteq \operatorname{Ker}(\bar{D}^{m+1})^* \bar{D}^{m+1} = \operatorname{Ker} \bar{D}^{m+1}. \tag{101}$$

Now we prove the reverse inclusion, namely that if $|\mathbf{n}| > m$ then $\bar{D}^{m+1}$ on $V_{\nu,\mathbf{n}}$ is non-zero. Suppose to the contrary that $\bar{D}^{m+1} : V_{\nu,\mathbf{n}} \to 0$. We use the formulation as in [24, Section 3.4] for the realization of $V_{\nu,\mathbf{n}}$ to compute the action of $\bar{D}^{m+1}$. As a unitary representation $(V_{\nu,\mathbf{n}}, \hat{G}, \pi_{\mathbf{n}})$ of $\hat{G}$, the space $V_{\nu,\mathbf{n}}$ contains a unique non-zero vector v_ν such that

$$\pi_{\mathbf{n}}(k) v_\nu = \tau_\nu(k) v_\nu$$

where τ_ν is the one-dimensional representation defined as above. Moreover both representations $\tau_{-\nu}$ and τ_ν appear in $V_{\nu,\mathbf{n}}$. As functions on G, the space $V_{\nu,\mathbf{n}} \subset L^2(\hat{\Omega}, \nu) \subset L^2(\hat{G})$ is obtained as

$$v \in V_{\nu,\mathbf{n}} \mapsto f_v(g) = \langle \pi_{\mathbf{n}}(g^{-1}) v, v_{-\nu} \rangle, \qquad f_v \in V_{\nu,\mathbf{n}} \subset L^2(\hat{G}),$$

where with some abuse of notation we have used the same notation $V_{\nu,\mathbf{n}}$ both as $\hat{G}$-representation and as a space of functions. The assumption $\bar{D}^{m+1} : V_{\nu,\mathbf{n}} \to 0$ implies in particular that $\bar{D}^{m+1} f_{v_{-\nu}} = 0$, and its evaluation at $g = e$ implies further that

$$\pi_{\mathbf{n}}(X)v_{-\nu} = 0$$

for all $X \in S^{m+1}(\mathfrak{p}^-)$. Let $X = X_1 Y$ where $X_1 \in \mathfrak{p}^-$ is an arbitrary negative root vector and $Y \in S^{\mathbf{m}}(\mathfrak{p}^-)$ is a $\mathfrak{k}^+$-*lowest weight vector* in $S^{\mathbf{m}}(\mathfrak{p}^-)$ with lowest weight $-(m_1\gamma_1 + \cdots + m_r\gamma_r)$ with $m_1 \geq \cdots \geq m_r \geq 0$. (A construction of all lowest weight vectors is found in [29] but we shall not need the explicit form.) We have then $\pi_{\mathbf{n}}(X_1)\pi_{\mathbf{n}}(Y)v_{-\nu} = 0$. Since $v_{-\nu}$ defines a one-dimensional representation of $\mathfrak{k}^{\mathbb{C}}$ we have always $\pi_{\mathbf{n}}(X)\pi_{\mathbf{n}}(Y)v_{-\nu} = 0$, for $X \in \mathfrak{k}^- \subset [\mathfrak{k}^{\mathbf{C}}, \mathfrak{k}^{\mathbf{C}}]$. In other words, $\pi_{\mathbf{n}}(Y)v_{-\nu}$ is a lowest weight vector for the $\mathfrak{g}^{\mathbb{C}}$-representation unless it vanishes. However by the Hua-Schmid decomposition the element $\pi_{\mathbf{n}}(Y)v_{-\nu}$ has lowest weight $-\frac{\nu}{2} - (m_1\gamma_1 + \cdots + m_r\gamma_r)$, $m_1 + \cdots + m_r = m < |\mathbf{n}|$. But the space $V_{\nu,\mathbf{n}}$ has lowest weight $-\frac{\nu}{2} - (n_1\gamma_1 + \cdots + n_r\gamma_r)$ and thus $\pi_{\mathbf{n}}(Y)v_{-\nu} = 0$. Acting by $k \in K$ we find $\pi_{\mathbf{n}}(\mathrm{Ad}(k)Y)\pi_{\mathbf{n}}(k)v_{-\nu} = 0$. Again $v_{-\nu}$ defines a one-dimensional representation of K, $\pi_{\mathbf{n}}(k)v_{-\nu} = \tau_{-\nu}(k)v_{-\nu}$ with scalar character $\tau_{-\nu}(k)$. Thus $\pi_{\mathbf{n}}(\mathrm{Ad}(k)Y)v_{-\nu} = 0$. Furthermore $\{\mathrm{Ad}(k)Y, k \in K\}$ generates the irreducible the representation $S^{\mathbf{m}}(\mathfrak{p}^-)$ so we get $\pi_{\mathbf{n}}(X)v_{-\nu} = 0$ for all $X \in S^{\mathbf{m}}(\mathfrak{p}^-)$, and further $\pi_{\mathbf{n}}(X)v_{-\nu} = 0$ for all $X \in S^m(\mathfrak{p}^-)$. Continuing this procedure we get that $v_{-\nu} = 0$, a contradiction. This proves our claim on $\mathrm{Ker}\, \bar{D}^{m+1}$ and then on the reproducing kernel. □

Remark 29. As noted in [27, Remark 5.12] the spherical functions $\phi_{\nu,\mathbf{m}}(z)$ here are precisely the Heckman-Opdam polynomials in Corollary 26 under proper coordinate change. Thus Theorem 28 is just an abstract restatement and a different proof of the expansion in Corollary 26 (with $m+1$ replacing m for notational convenience) with interpretation of the coefficients using the dimension $d_{\nu,m}$. □

Remark 30. The subspace $V_{\nu,\mathbf{m}}$ can also be described using, as in Section 3, the quasi-inverse $\partial \log h(z, -z)$. In the local coordinates $z \in V \subset \hat{\Omega}$ the space $V_{\nu,\mathbf{m}}$ consists of functions

$$f(z) = \otimes^m (\partial \log h(z, -z))(F(z))$$

where F is a holomorphic section of the bundle $\mathcal{L}^\nu \otimes \otimes^m T^{(1,0)}$ in the highest weight representation above. □

Remark 31. It follows from the proof above that for any $\mathbf{n}$ there exists an $\mathbf{n}'$, $|\mathbf{n}'| \leq |\mathbf{n}|$ such that the eigenvalue $\widetilde{L}_{\mathbf{n}'}(\frac{\nu}{2} + \mathbf{n} + \rho)$ of the Shimura operator on $V_{\nu,\mathbf{n}}$ is nonvanishing, $\widetilde{L}_{\mathbf{n}'}(\frac{\nu}{2} + \mathbf{n} + \rho) \neq 0$. This might be a known fact or can be proved by using Koornwinder's formula (see [18], [21] and [24, Theorem 5.5]) for $\widetilde{L}_{\mathbf{n}'}$, which in turn can give a different proof of the reverse inclusion of (101). □

Example 32. Let us again make everything more specific for the rank one case, i.e. when $\hat{G}/K = \mathbf{C}P^d$ is the complex projective space. In this case it is more convenient to use the realization of $\mathbf{C}P^d$ as $\mathbf{C}P^d = U(d+1)/U(d) \times U(1)$. We choose the Cartan subalgebra of $U(d+1)$ as diagonal matrices identified as $\mathbf{R}^{d+1}$, with the Harish-Chandra root $\beta=(1,0,\cdots,0,-1)$. The highest weights above are now $(\nu+m,0,\cdots,-m)$. The sections of the line bundle with parameter ν on $\mathbf{C}P^d=U(d+1)/U(d)\times U(1)$ can be realized as functions on the sphere $S^{2d+1}=U(d+1)/U(d)$ and the representation space with the highest weight $(\nu+m,0,\cdots,-m)$ is the space of $(p,q)=(\nu+m,m)$-spherical harmonic polynomials. We write $\phi_{\nu,(m)}=\phi_{\nu,m}$.

When $\nu=0$, i.e. the spherical case, the highest weight is of the form $\mathbf{m}=m\beta$ with spherical polynomial

$$\phi_{0,m}(\exp(H))={}_2F_1\Big(\begin{matrix}-m,d+m\\ d\end{matrix}\Big|\sin^2\frac{\beta(H)}{2}\Big);$$

see [16, Theorem V.4.5] and Remark 21 above. For general $\nu\geq 0$,

$$\phi_{\nu,m}(\exp(tH))={}_2F_1\Big(\begin{matrix}d+m+\nu,-m\\ d\end{matrix}\Big|\sin^2 t\Big).$$

See [24], [17].

By the Schur orthogonality we have

$$\langle\phi_{\nu,m},\phi_{\nu,m'}\rangle=\frac{1}{d_{\nu,m}}\delta_{m,m'},$$

where $d_{\nu,m}$ is the dimension of the representation space $V_{\nu,m}$. Here the inner product is given by

$$\begin{aligned}\langle\phi,\psi\rangle&=\hat{c}_\nu\int_0^{\frac{\pi}{2}}\phi(\sin^2 t)\overline{\psi(\sin^2 t)}\sin^{2\nu+1}(2t)\sin^{2(d-1)-2\nu}(t)dt\\ &=\hat{c}_\nu\int_0^{\frac{\pi}{2}}\phi(\sin^2 t)\overline{\psi(\sin^2 t)}\sin^{2\nu}(2t)\sin^{2(d-1)-2\nu}(t)d\sin^2 t\\ &=\hat{c}_\nu\int_0^1\phi(x)\overline{\psi(x)}(1-x)^\nu x^{d-1}dx.\end{aligned}$$

The τ_ν-spherical function above is

$$\phi_{\nu,m}(x)={}_2F_1\Big(\begin{matrix}-m,m+d+\nu\\ d\end{matrix}\Big|x\Big).$$

The dimension of the representation space $V_{\nu,m}$ can be easily found using the Weyl dimension formula and equals

$$d_{\nu,m} = \frac{(2m+\nu+d)(m+\nu+1)_{d-1}(m+1)_{d-1}}{d!(d-1)!}.$$

In particular,

$$d_{\nu,0} = \frac{(\nu+d)(\nu+1)_{d-1}}{d!} = \frac{(\nu+1)_d}{d!} = \binom{\nu+d}{d}$$

which is precisely the dimension of the space of polynomials $\mathcal{P}_{\le\nu}(\mathbf{C}^d)$ on $\mathbf{C}^d$ of degree $\le \nu$ realized as the holomorphic sections in $L^2(\hat{\Omega}, \nu)$.

So we are computing the sum

$$\sum_{m\le n} d_{\nu,m}\phi_{\nu,m}(x) = \sum_{m\le n} \frac{(2m+\nu+d)(m+\nu+1)_{d-1}(m+1)_{d-1}}{d!(d-1)!}\; {}_2F_1\left(\begin{matrix}-m, m+d+\nu\\ d\end{matrix}\,\middle|\, x\right).$$

To carry out the summation we use the following elementary observation.

Lemma 33. *Let $d\mu(x)$ be a finite Borel measure on $\mathbf{R}_+$ such that all polynomials are dense in $L^2(\mathbf{R}_+, d\mu)$. Let $\{p_m\}_{m=0}^\infty$ be the orthonormal basis obtained from the Gram-Schmidt orthogonalization of the polynomials $\{x^m\}_{m=0}^\infty$. Then the reproducing kernel $\sum_{m=0}^n p_m(x)p_m(0)$ evaluated at 0 is*

$$\sum_{m=0}^{n} p_m(x)p_m(0) = A_n q_n(x)$$

for some constant A_n, where $\{q_n(x)\}_{n=0}^\infty$ is the orthonormal basis obtained from $\{x^n\}_{n=0}^\infty$ for the space $L^2(\mathbf{R}_+, d\tilde{\mu})$, where $d\tilde{\mu} = x\, d\mu(x)$.

Proof. Write $P_n(x) = \sum_{m=0}^n p_m(x)p_m(0)$. We prove that $P_n(x)$ is orthogonal to all polynomials x^m, $0 \le m \le n-1$, in the space $L^2(\mathbf{R}_+, d\tilde{\mu})$. Indeed the inner product of x^n and P_m in $L^2(\mathbf{R}_+, d\tilde{\mu})$ is

$$\int_0^\infty x^m P_n(x) x\, d\mu(x) = \int_0^\infty x^{m+1} P_n(x)\, d\mu(x) = x^{m+1}|_{x=0} = 0,$$

since $P_n(x)$ is the reproducing kernel at 0 in $L^2(\mathbf{R}_+, d\mu)$ for the polynomials of degree $\le n$ and $0 < m+1 \le n$. Thus P_n is proportional to q_n. This proves the lemma. □

Theorem 34. *The reproducing kernel $\hat{N}_\nu^n(z, 0)$ at the origin for the space $\hat{N}_\nu^n(\mathbf{C}P^d)$, under the local trivialization above using the local frame e_ν on $\mathbf{C}^d \subset \mathbf{C}P^d$, is*

$$\hat{N}_\nu^n(z,0) = \sum_{m\le n} d_{\nu,m}\phi_{\nu,m}(x) = A_n \; {}_2F_1\left(\begin{matrix}-n, n+d+\nu+2\\ d+1\end{matrix}\Big| x\right), \quad x = \frac{|z|^2}{1+|z|^2},$$

where the positive constant A_n is given by (103) below.

Proof. We use Lemma 33. The polynomials $\{\phi_{\nu,m}(x)\}$ form an orthogonal basis for the space $L^2((0,1), d\mu(x))$, $d\mu(x) = (1-x)^\nu x^{d-1}\, dx$, and they are the same orthogonal basis as obtained from the Gram-Schmidt process from the measure $d\mu(x)$. The orthogonal basis for the measure $d\tilde{\mu}(x) = x\, d\mu(x) = (1-x)^\nu x^{d+1}$ is ${}_2F_1\left(\begin{matrix}-m, m+d+\nu+2\\ d+1\end{matrix}\Big| x\right)$. Thus

$$\begin{aligned}\sum_{m\le n} d_{\nu,m}\phi_{\nu,m}(x) &= \sum_{m\le n} d_{\nu,m}\phi_{\nu,m}(x) \\ &= A_n \; {}_2F_1\left(\begin{matrix}-n, n+d+\nu+2\\ d+1\end{matrix}\Big| x\right)\end{aligned} \tag{102}$$

for some constant A_n. To find A_n, we view (102) as an identity of two polynomials of $x \in \mathbf{R}$. The leading coefficients of x^n in (102) are

$$d_{n,\nu}\frac{(-n)_n(n+d+\nu)_n}{(d)_n n!} = A_n \frac{(-n)_n(n+d+\nu+2)_n}{(d+1)_n n!}.$$

Thus

$$\begin{aligned}A_n &= d_{n,\nu}\frac{(n+d+\nu)_n(d+1)_n}{(n+d+\nu+2)_n(d)_n} = d_{n,\nu}\frac{(n+d+\nu)_n(d+n)}{(n+d+\nu+2)_n d} \\ &= \frac{(2n+\nu+d)(n+\nu+1)_{d-1}(n+1)_{d-1}}{d!(d-1)!}\frac{(n+d+\nu)_n(d+n)}{(n+d+\nu+2)_n d} \\ &= \frac{(n+\nu+1)_{d+1}(n+1)_{n+d-1}}{(2n+d+\nu+1)d!^2}. \quad \square\end{aligned} \tag{103}$$

Data availability

No data were used for the research described in the article.

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