Thesis for the degree of Licentiate of Philosophy

Geometric discretization for incompressible magnetohydrodynamics on the sphere

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Göteborg, Sweden 2023

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Typeset with LaTeX
Printed by Chalmers Reproservice
Göteborg, Sweden 2023

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Abstract

Many physical processes are modelled by partial differential equations (PDE), and their efficient discretization is still a challenging problem and an actively developing field. An important class of models arising in mathematical physics represents PDEs formulated in terms of a Lie–Poisson structure on the dual of infinite-dimensional Lie algebras, such as the Lie algebra of vector fields. They are usually referred to as *Euler–Arnold systems*. A natural approach to discretizing such PDEs is to develop numerical schemes that preserve the underlying Lie–Poisson structure. In the present thesis, an important class of such equations is addressed, namely equations of incompressible magnetohydrodynamics (MHD) on the sphere. The thesis comprises two papers.

In the first paper, a spatio-temporal discretization of MHD on the sphere is developed. This numerical scheme fully preserves the underlying Lie–Poisson structure. The discretization is performed into two steps. First, space discretization based on geometric quantization provides a finite-dimensional Lie–Poisson system on the dual of a semidirect product Lie algebra. Second, structure preserving time integrator is developed. It exactly preserves all the Casimirs and nearly preserves the Hamiltonian function in the sense of backward error analysis.

In the second paper, the developed structure preserving integrator is applied to Hazeltine's model of 2D turbulence in magnetized plasma. Simulations reveal formation of large-scale coherent structures in the long time behaviour, which indicates the presence of an inverse energy cascade.

Keywords: magnetohydrodynamics, Lie–Poisson structure, magnetic extension, Casimirs, Hamiltonian dynamics, symplectic Runge-Kutta integrators.

Contents

List of publications

The following papers are included in this thesis:

Paper 1: K. Modin, M. Roop, Spatio-temporal Lie–Poisson discretization for incompressible magnetohydrodynamics on the sphere. arXiv:2311.16045

Paper 2: K. Modin, M. Roop, Long time simulation of two-dimensional turbulence in magnetized ideal fluids. Preprint.

Author contribution:

Paper 1: I have developed most of the theoretical framework, derived the methods, modified the code `quflow`, and performed numerical simulations, prepared the draft of the manuscript.

Paper 2: I have performed numerical simulations, formulated and analyzed obtained results.

Acknowledgements

First, I would like to thank my supervisor Klas Modin for continuous support, guidance, and attention to the work. I am also grateful to Darryl D. Holm for pointing my attention to the problem that has become central for this thesis. Further, I would like to thank all my colleagues and friends. Last but not least, I am enormously grateful to my family.

Michael Roop
Göteborg, 2023

1 Introduction

The incompressible Euler equations is one of the most important mathematical model used to investigate the motion of fluids on different domains (with or without a boundary). It consists of two conservation laws, the momentum and mass conservation laws, accompanied with the incompressibility condition, which is constancy of the density of the fluid. The latter results in vanishing divergence of the velocity field of the fluid.

Numerous works have been devoted to understanding the properties of solutions to the Euler equations, both by analytical and numerical methods. Perhaps one of the most fundamental observations was made by V.I. Arnold in 1960s, who discovered their geodesic nature [2, 4]. Namely, the incompressible Euler equations constitute a geodesic flow on the group of volume preserving diffeomorphisms of the underlying manifold (for instance, the sphere), with respect to the right-invariant L_2 metric. Later, the same formalism has been shown to cover a large variety of equations of mathematical physics. Among them are inviscid Burger's equation, barotropic and fully compressible Euler's equations, magnetohydrodynamics equations, linear and non-linear Schrödinger equations, and many other [17]. Arnold's discovery gave rise to a new field in mathematics, *geometric fluid mechanics*, that opened up new insights in such fundamental problems as stability criteria for solutions to the Euler equations, global existence of solutions, turbulence of the Earth's atmosphere.

In the present thesis, we address one example of Euler–Arnold systems, the system of self-consistent magnetohydrodynamics equations. This model has important applications in astrophysics, physics of plasma, and geophysics [13, 10, 24, 11, 12]. The model describes charged incompressible fluids that, on the one hand, transport the magnetic field, and on the other hand experience influence from the magnetic field. This leads to an extension of the incompressible Euler model by adding the dynamics of the magnetic field and by including the Lorentz force in the momentum conservation law. Geometrically, the MHD system is a Lie–Poisson flow on the dual of the *magnetic extension* of the Lie algebra of volume preserving diffeomorphisms group.

The long time behaviour of solutions to the Euler equations is a prominent problem in mathematical fluid dynamics. Such applied questions as understanding the weather patterns on planets, formation of large scale coherent structures in atmospheric motions, are directly related to this problem. As there is no possibility to create a laboratory for experiments with the atmosphere, the way is to utilize computational facilities. The natural step then is to develop efficient numerical algorithms to simulate the equations of hydrodynamics.

Spatio-temporal discretization of hydrodynamic equations has always been a challenging problem, especially when it comes to long time simulations. The interpretation of the Euler equations as geodesic equations on the group of volume preserving diffeomorphisms gave significant contribution not only to the development of theoretical tools for investigating fluids' motion, but also paved a way for constructing efficient numerical methods allowing for long time simulation of fluids. Indeed, the Hamiltonian interpretation of Arnold's observation suggests that the Euler equations constitute a Lie–Poisson flow on the dual of the Lie algebra consisting of divergence-free vector fields, which is a Poisson reduction of Hamilton's equations on the cotangent bundle of the group of volume preserving diffeomorphisms. This means that the system admits a lot of (in fact, infinitely many) conservation laws, *Casimirs*. Preservation of Casimirs is known to be vital in long time simulations [1]. Indeed, conservation of Casimirs restricts the set of possible states that can be reached from a given initial state, thus determining the qualitative long time behaviour. Therefore, in order to capture that behaviour, one should use methods that preserve the underlying Lie–Poisson geometry, and, in particular, Casimirs.

The goal is achieved in two steps. First, one needs to discretize the equations in space. The main tool here is the theory of *Berezin–Toeplitz quantization* developed in the works [5, 6, 14, 15, 16]. The main idea is that the infinite-dimensional Poisson algebra of smooth functions is replaced with its finite-dimensional analogue, the Lie algebra of skew-hermitian matrices with the Lie bracket given by the matrix commutator. This makes it possible to introduce a finite dimensional approximation of the Euler equations — the flow on skew-hermitian matrices, known as the Euler–Zeitlin model [26]. Later, Zeitlin extended this approach to incompressible magnetohydrodynamics on the flat torus [27]. One crucial benefit of this approach is that the spatially quantized Euler's equations constitute a Lie–Poisson flow on $\mathfrak{su}(N)^*$, exactly as the continuous equations represent a Lie–Poisson flow on the dual of divergence-free vector fields.

In the present thesis, we extend this approach to incompressible magnetohydrodynamics on the sphere. The resulting quantized MHD system constitutes a finite-dimensional Lie–Poisson flow on the dual of the semidirect product Lie algebra $\mathfrak{f} = \mathfrak{su}(N) \ltimes \mathfrak{su}(N)^*$, which is usually referred to as the *magnetic extension* of $\mathfrak{su}(N)$, and is a quantized counterpart of the magnetic extension of the Lie algebra of divergence-free vector fields.

The second step is to discretize the matrix flow in time in such a way that the quantized analogues of Casimirs are exactly preserved. Such an integrator, the *isospectral symplectic Runge-Kutta method*, has been developed in the works [22, 25] for a large class of *isospectral flows*, including the Euler-Zeitlin model for incompressible Euler's equations. The main mechanism that allows for constructing such methods is the *discrete Poisson reduction*, motivated by the above mentioned observation that Lie–Poisson flows on $\mathfrak{g}^*$ can be treated as Poisson reduced Hamiltonian flows on T^*G. Under certain conditions, symplectic integrators for Hamilton's equations on T^*G descend to a Lie–Poisson integrator on $\mathfrak{g}^*$.

In the present thesis, we use a similar approach to develop structure preserving Lie–Poisson integrators for Lie–Poisson systems on the dual of the Lie algebra of the form $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{g}^*$, where $\mathfrak{g}$ is a J-quadratic Lie algebra. Such Lie algebras include all the classical Lie algebras. This extends the *isospectral symplectic Runge-Kutta integrators* (IsoSRK) developed by Modin and Viviani [22, 25].

Further, the developed structure preserving discretization is used to investigate the long time behaviour of magnetized fluids. It is worth mentioning that for the Euler equations there are a lot of results known, in particular those obtained in the works [22, 21, 8]. Namely, the long time behaviour of the incompressible Euler equations typically settles on a quasi-periodic motion of 2, 3, or 4 blobs for generic initial conditions. The amount of blobs is determined by the value of the total angular momentum normalized by the square root of the enstrophy Casimir, while their motion is closely related to point vortex dynamics.

However, much less is known about the long time behaviour of MHD. In particular, there are no systematic studies of the long time behaviour for Hazeltine's model of turbulence, which generalizes conventional models, such as reduced MHD and Charney-Hasegawa-Mima (CHM) equation. In the present thesis, we attempt to fill in this gap, and we reveal large scale coherent structures formation for Hazeltine's model of turbulence in magnetic fluids. These numerical results indicate the presence of an inverse energy cascade. Two regimes are considered: weak and generic magnetic field. In the case of weak magnetic field, the behaviour is close to the incompressible Euler dynamics. In the case of generic magnetic field, the fluid vorticity develops into formation of vortex condensates, and the magnetic field reaches the state of a magnetic dipole through intermediate mixing. At the same time, the MHD model does not behave similarly, and, in particular, there is no evidence of an inverse energy cascade. A theoretical explanation of this difference is one of the future directions for investigation.

2 Hamiltonian and Lie–Poisson systems

Classical mechanics is one of the first attempts to formulate empirical observations of the macroscopic world in terms of mathematical equations, and goes back to Newton, Lagrange, Euler, and Hamilton. Later, with the development of differential geometry, this classical field got a new breath. Indeed, the modern language of differential geometry formulates classical mechanics in an invariant and coordinate-free way, thus deepening the understanding of underlying fundamental structures and transforming this field into a beautiful and elegant science. The exposition here mainly follows [3, 4, 18].

2.1 Preliminaries from differential geometry

We start with some preliminary notions from Riemannian and symplectic geometry, and Hamiltonian mechanics. As the MHD equations will be the main subject of the present thesis, we shall provide (in a concise manner) background material that will be used to show the Hamiltonian structure of magnetohydrodynamics, which is the main goal of this chapter.

2.1.1 Riemannian structures and connections

Let M be a real smooth manifold of dimension $\dim(M) = n$, $C^\infty(M)$ be the space of smooth functions on M, $\mathcal{D}(M)$ be the module of vector fields on M, and $\Omega^1(M)$ be the module of differential 1-forms on M.

Definition 1. A smooth manifold M is called *Riemannian*, if it is equipped with a smoothly varying field of scalar products:

$$g_x \colon T_xM \times T_xM \to \mathbb{R}, \quad (X, Y) \mapsto g_x(X, Y), \quad x \in M.$$

A smooth manifold M with a given Riemannian structure g is denoted by (M, g).

The next important construction that we need is a *connection*. The notion of a connection on a smooth manifold M naturally appears when it comes to the definition of an acceleration in mechanics. Let $Y \in \mathcal{D}(M)$ be a vector field on M that can be thought of as a velocity of a particle, and $x(t)$ be a path on M. Then, to find an acceleration of a particle, one needs to compare vectors at different points of the curve $x(t)$, which is problematic, since they are elements of different vector spaces. To this end, let us equip the manifold M with linear isomorphisms $\lambda(t)\colon T_{x(t)}M \to T_{x(0)}M$ between the tangent spaces. This way of identification of tangent spaces is called a *connection*. Then, taking images $Y(t) = \lambda(t)\left(Y_{x(t)}\right) \in T_{x(0)}M$ of vectors $Y(t) \in T_{x(t)}M$, we get the velocity of variation of the vector field Y along the path $x(t)$:

$$\lim_{t\to 0} \frac{Y(t) - Y(0)}{t} = \left.\frac{\mathrm{d}Y(t)}{\mathrm{d}t}\right|_{t=0} \in T_{x(0)}M. \tag{2.1}$$

Let $x(t)$ be the trajectory of another vector field X on the manifold M. Then, the derivatives (2.1) at various points of M give us a vector field $\nabla_X Y$ on M, and thus come to the notion of a *covariant derivative*.

Definition 2. A covariant derivative is a map

$$\nabla_X\colon \mathcal{D}(M) \to \mathcal{D}(M), \quad X \in \mathcal{D}(M),$$

that satisfies the conditions

1. $\nabla_{X_1+X_2} = \nabla_{X_1} + \nabla_{X_2}$
2. $\nabla_{fX} = f\nabla_X$, $f \in C^\infty(M)$,
3. $\nabla_X(Y_1 + Y_2) = \nabla_X(Y_1) + \nabla_X(Y_2)$
4. $\nabla_X(fY) = X(f)Y + f\nabla_X(Y)$,

where $X_i, Y_i, X, Y \in \mathcal{D}(M)$, $f \in C^\infty(M)$.

In other words, the operator ∇ is $C^\infty(M)$-linear with respect to its first argument and is a derivation with respect to the second one. Any connection is determined by its covariant derivative.

The action of the covariant derivative on differential 1-forms is given by the following expression:

$$\langle \nabla_X \alpha, Y\rangle = X\langle \alpha, Y\rangle - \langle \alpha, \nabla_X Y\rangle,$$

where $\alpha \in \Omega^1(M)$, $X, Y \in \mathcal{D}(M)$, and brackets $\langle\cdot,\cdot\rangle$ stand for the natural pairing between 1-forms and vector fields.

By means of the Leibniz rule one can expand the action of the covariant derivative ∇_X to tensor fields of higher ranks. In particular, if g is a metric tensor on a smooth manifold M, then the action of the covariant derivative $\nabla_X(g)$ is given by the formula:

$$(\nabla_Z g)(X, Y) = Z(g(X, Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y),$$

where $X, Y, Z \in \mathcal{D}(M)$.

For a connection on a tangent bundle one can define a vector field

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

where $X, Y \in \mathcal{D}(M)$. The map

$$T\colon \mathcal{D}(M) \times \mathcal{D}(M) \to \mathcal{D}(M)$$

is called *the torsion tensor* of a given connection.

Definition 3. A connection on a tangent bundle is called *symmetric*, if its torsion tensor is trivial.

It is well known that there is a unique symmetric connection on a Riemannian manifold, which is also metric, that is $\nabla_X g = 0$ for any $X \in \mathcal{D}(M)$, and it is called *the Levi–Civita connection.*

Further on, for the scalar product $g(X, Y)$ of vector fields X, Y in terms of the metric g we will use the notation $\langle X, Y\rangle_g$, as well as $X \cdot Y$. For the Lie derivative along some vector field X, we will use the notation L_X. The Lie derivative L_X shows an infinitesimal change of a tensor field along the flow generated by the vector field X. Also, we will need some properties of the Levi–Civita connection.

Lemma 1. *Let ∇ be the Levi–Civita connection, associated with the metric g, and let $u, v, w \in \mathcal{D}(M)$. Then,*

$$\langle w, \nabla(v \cdot u)\rangle_g = \langle \nabla_w v, u\rangle_g + \langle v, \nabla_w u\rangle_g. \tag{2.2}$$

Proof. Since ∇ is the Levi-Civita connection, $\nabla_w g = 0$:

$$w(g(v, u)) = g(\nabla_w v, u) + g(v, \nabla_w u) = \langle \nabla_w v, u\rangle_g + \langle v, \nabla_w u\rangle_g.$$

Using the definition of the gradient $(\mathrm{d}f)(w) = \langle w, \nabla f\rangle_g$ and putting $f = g(v, u) = v \cdot u$, one can write down the left hand side as

$$w(g(v, u)) = (\mathrm{d}(v \cdot u))(w) = \langle w, \nabla(v \cdot u)\rangle_g.$$

□

Corollary 1. *Putting $w = v$ in* (2.2), *we get*

$$\langle v, \nabla_v u\rangle_g = \langle v, \nabla(v \cdot u)\rangle_g - \langle \nabla_v v, u\rangle_g. \tag{2.3}$$

Corollary 2. *Putting $u = v$ in* (2.2), *we get*

$$\langle w, \nabla|v|^2\rangle_g = \langle \nabla_w v, v\rangle_g + \langle v, \nabla_w v\rangle_g \Leftrightarrow \langle v, \nabla_w v\rangle_g = \frac{1}{2}\langle w, \nabla|v|^2\rangle_g. \tag{2.4}$$

Using the metric g one can define the *flat operator*, $\flat\colon \mathcal{D}(M) \to \Omega^1(M)$, $X^\flat(\cdot) = g(X,\cdot)$, and its inverse $\sharp\colon \Omega^1(M) \to \mathcal{D}(M)$, called *sharp operator*.

Lemma 2. *Let ∇ be the Levi-Civita connection, and $v \in \mathcal{D}(M)$. Then,*

$$(\nabla_v v)^\flat = L_v v^\flat - \frac{1}{2}\mathrm{d}|v|^2.$$

Proof. Let $Y \in \mathcal{D}(M)$ is an arbitrary vector field on M. We need to prove that

$$J = (\nabla_v v)^\flat(Y) - (L_v v^\flat)(Y) + \frac{1}{2}(\mathrm{d}|v|^2)(Y) = 0.$$

Let us use the formula of the action of the Lie derivative on 1-forms:

$$(L_v v^\flat)(Y) = v(v^\flat(Y)) - v^\flat([v,Y]) = v(g(v,Y)) - g(v,[v,Y]).$$

Therefore,

$$J = g(\nabla_v v, Y) - v(g(v,Y)) + g(v,[v,Y]) + \frac{1}{2}(\mathrm{d}|v|^2)(Y) \tag{2.5}$$

Taking into account that the torsion of ∇ is trivial, that is

$$\nabla_v Y - \nabla_Y v = [v,Y],$$

we reduce (2.5) to the form

$$J = g(\nabla_v v, Y) - v(g(v,Y)) + g(v,\nabla_v Y) - g(v,\nabla_Y v) + \frac{1}{2}Y(g(v,v)).$$

Since $\nabla g = 0$, then $v(g(v,Y)) = g(\nabla_v v, Y) + g(v, \nabla_v Y)$, and

$$J = -g(v,\nabla_Y v) + \frac{1}{2}Y(g(v,v)).$$

Finally, using the property of the Levi-Civita connection (2.4), $J = 0$. □

2.1.2 Hamiltonian mechanics

Let M again be a real smooth manifold of even dimension $\dim(M) = 2n$, and let Ω be a non-degenerate closed 2-form on M. Then, the pair (M,Ω) is called a *symplectic manifold*.

Theorem 1 (Darboux)**.** *Let (M,Ω) be a symplectic manifold. Then, in the neighborhood of $z \in M$, there exist local coordinates $(q_1,\dots,q_n,p_1,\dots,p_n)$, called canonical coordinates, such that*

$$\Omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

This means that all symplectic manifolds locally look similar (there are no local invariants).

Let (M_1, Ω_1) and (M_2, Ω_2) be two symplectic manifolds, and let $\varphi\colon M_1 \to M_2$ be a C^∞-map. Then φ is called a *symplectic map*, or a *symplectomorphism*, if φ is a diffeomorphism, if $\varphi^*(\Omega_2) = \Omega_1$.

Definition 4. Let (M, Ω) be a symplectic manifold. A vector field $X \in \mathcal{D}(M)$ is called *Hamiltonian*, if there is a function $H\colon M \to \mathbb{R}$, such that

$$\iota_X \Omega = \mathrm{d}H,$$

where $\iota\colon \mathcal{D}(M) \times \Omega^k(M) \to \Omega^{k-1}(M)$ is the contraction of a vector field and a differential form, that is (for $k = 2$) $(\iota_X \Omega)\,(Y) = \Omega(X, Y)$ for any $Y \in \mathcal{D}(M)$.

A Hamiltonian vector field with Hamiltonian H is denoted by X_H.

To compute the flow generated by the vector field X_H, one needs to solve the system

$$\dot{z} = X_H(z),$$

that in canonical coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ can be written as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{2.6}$$

The equations (2.6) are called *canonical*, or *Hamiltonian*.

Proposition 1. *Let X_H be a Hamiltonian vector field with the Hamiltonian H. Then,*

$$L_{X_H} \Omega = 0.$$

Proof. Using the Cartan's formula $L_X = \iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X$, we get

$$L_{X_H} \Omega = \iota_{X_H} \mathrm{d}\Omega + \mathrm{d}\,(\iota_{X_H} \Omega) = \mathrm{d}\,(\mathrm{d}H) = 0.$$

□

This property of Hamiltonian vector fields means that Hamiltonian flows are symplectic.

In canonical coordinates, a Hamiltonian vector field can be written as

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right).$$

Let F, G be two smooth functions on a symplectic manifold (M, Ω). Then, one can define a *Poisson bracket* of functions F and G by

$$\{F, G\}\,(z) = \Omega(X_F(z), X_G(z)), \quad z \in M.$$

Proposition 2. *Let φ_t be the flow of a Hamiltonian vector field X_H. Then,*

$$\varphi_t^* \{F, G\} = \{\varphi_t^* F, \varphi_t^* G\} \tag{2.7}$$

for all $F, G \in C^\infty(M)$.

Differentiating the expression (2.7) by t at $t = 0$, we get

$$X_H(\{F, G\}) = \{X_H(F), G\} + \{F, X_H(G)\} . \tag{2.8}$$

Further, from the definition of the Poisson bracket, we have

$$\{F, G\} = (\iota_{X_F}\Omega)\,(X_G) = (\mathrm{d}F)(X_G) = X_G(F),$$

which implies that

$$\{\{F, G\}, H\} = X_H(\{F, G\}) .$$

Using (2.8), we get

$$\{\{F, G\}, H\} = \{X_H(F), G\} + \{F, X_H(G)\} = \{\{F, H\}, G\} + \{F, \{G, H\}\} ,$$

which is the Jacobi identity.

Theorem 2. *The Poisson bracket $\{\cdot, \cdot\}$ has the following properties:*

- *skew-symmetry*
$$\{F, G\} = -\{G, F\} ,$$
- *Leibniz rule*
$$\{F, GH\} = \{F, G\} H + G \{F, H\} ,$$
- *Jacobi identity*
$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0.$$

Remark 1. The properties in Theorem 2 can serve as a definition of the Poisson bracket. Indeed, a set of smooth functions $C^\infty(M)$ can be endowed with a bilinear, skew-symmetric operation $\{\cdot, \cdot\}$ satisfying the Jacobi identity. The pair $(M, \{\cdot, \cdot\})$ then becomes a *Poisson manifold*.

Remark 2. A Poisson manifold is a more general object than a symplectic manifold. Indeed, any smooth manifold, not necessarily even-dimensional, can be endowed with a Poisson bracket. At the same time, any symplectic manifold is also a Poisson manifold, where the Poisson structure is induced by the symplectic form.

In canonical coordinates, a Poisson bracket can be written as

$$\{F, H\} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} \right).$$

Finally, in terms of the Poisson bracket, the canonical equations (2.6) take the form

$$\dot{F} = \{F, H\},$$

where F is any canonical coordinate function.

Among all Hamiltonian systems, a special place is occupied by *integrable* Hamiltonian systems, i.e., such Hamiltonian systems that can be solved explicitly. The following important theorem gives the integrability conditions for Hamiltonian systems, as well as provides a constructive method of integrating them:

Theorem 3 (Liouville, Arnold)**.** *Let $H = F_1, F_2 \ldots, F_n$ be independent functions on a symplectic manifold (M, Ω) in involution, i.e.*

$$\{F_i, F_j\} = 0, \quad i, j = 1, \ldots, n,$$

then the trajectories of the Hamiltonian system (2.6) *lie on an invariant n-dimensional manifold*

$$M_I = \{F_1(p, q) = I_1, \ldots, F_n(p, q) = I_n\} \subset (M, \Omega),$$

where $I_i \in \mathbb{R}$.

There exist canonical coordinates $(I_1, \ldots, I_n, \varphi_1, \ldots, \varphi_n)$, such that the symplectic form Ω has its canonical form

$$\Omega = \sum_{i=1}^{n} \mathrm{d}I_i \wedge \mathrm{d}\varphi_i,$$

and Hamilton's equations take their simplest form

$$\dot{I} = 0, \quad \dot{\varphi} = \omega(I).$$

If, in addition, M_I is compact and connected, then it is diffeomorphic to the n-torus T^n.

The canonical coordinates $(I_1, \ldots, I_n, \varphi_1, \ldots, \varphi_n)$ are called *action-angle* variables. Note that for Hamiltonian systems it is enough to have only n independent integrals in involution to find a solution, contrary to general type system of ODEs, for which $2n$ integrals would be needed.

2.2 Lie–Poisson systems

One of the most fundamental examples of a Poisson structure is the *Lie–Poisson bracket* on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$. Hamilton's equations written in terms of a Lie–Poisson structure are called *Lie–Poisson* systems. Through the *momentum map*, they are closely related to the Hamiltonian equations on T^*G, where G is a Lie group (not necessarily finite-dimensional) for the corresponding Lie algebra $\mathfrak{g}$. The process of passing from Hamilton's equations on T^*G to the Lie–Poisson equations on $\mathfrak{g}^*$ is called *Lie–Poisson reduction*, and the inverse process is called *Lie–Poisson reconstruction*. In this section, we address the main properties of such systems.

2.2.1 Adjoint and coadjoint representation

Let G be a Lie group, and $\mathfrak{g}$ be its Lie algebra.

Definition 5. The map

$$A_g\colon G \to G, \quad A_g\colon h \mapsto ghg^{-1}$$

for $g, h \in G$ is called an *inner automorphism*.

From now on, we will use the notation $F_*|_x\colon T_xG \to T_{F(x)}G$ for the derivative of a map $F\colon G \to G$.

Definition 6. The differential of the inner automorphism at the unit element e of the group G

$$\mathrm{Ad}_g\colon \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{Ad}_g(u) = ((A_g)_*|_e)\, u, \quad u \in \mathfrak{g},$$

is called the *group adjoint operator*.

The property $\mathrm{Ad}_{gh} = \mathrm{Ad}_g\mathrm{Ad}_h$ implies that adjoint operators form a representation of the group G in its Lie algebra $\mathfrak{g}$, called the *adjoint representation*.

By differentiating Ad_g at the group unit element e, on gets the *adjoint representation of the Lie algebra* $\mathfrak{g}$:

$$\mathrm{ad} = \mathrm{Ad}_{*e}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad \mathrm{ad}_v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{g(t)},$$

where $g(t)$ is a curve on G passing through the unit element $e \in G$ with the tangent vector $v \in T_eG = \mathfrak{g}$.

The following formula allows to express the ad operator in terms of the Lie bracket $[\cdot,\cdot]$ on $\mathfrak{g}$:

$$\mathrm{ad}_v(w) = [v, w], \quad v, w \in \mathfrak{g}.$$

Using the pairing $\langle\cdot,\cdot\rangle$ between $\mathfrak{g}$ and $\mathfrak{g}^*$, one can define the *coadjoint operator*:

$$\mathrm{Ad}_g^*\colon \mathfrak{g}^* \to \mathfrak{g}^*, \quad \mathrm{Ad}_g^*\colon \alpha \mapsto \mathrm{Ad}_g^*(\alpha), \quad \langle \mathrm{Ad}_g^*(\alpha), v\rangle = \langle \alpha, \mathrm{Ad}_g v\rangle,$$

for any $v \in \mathfrak{g}$, $\alpha \in \mathfrak{g}^*$, $g \in G$.

As $\mathrm{Ad}_{gh}^* = \mathrm{Ad}_h^*\mathrm{Ad}_g^*$, operators Ad_g^* form an *antirepresentation* of the group G in its coalgebra $\mathfrak{g}^*$.

Definition 7. The set of all points $\mathrm{Ad}_g^*\alpha$, $g \in G$, is called the *coadjoint orbit* of α:

$$\mathcal{O}_\alpha = \left\{\mathrm{Ad}_g^*\alpha \mid g \in G\right\}.$$

The dual operator for ad, the operator of the *coadjoint representation* is

$$\mathrm{ad}_v^*\colon \mathfrak{g}^* \to \mathfrak{g}^*, \quad \langle \mathrm{ad}_v^*\alpha, w\rangle = \langle \alpha, \mathrm{ad}_v w\rangle,$$

for any $v, w \in \mathfrak{g}$, $\alpha \in \mathfrak{g}^*$.

Example 1. Now we will illustrate the constructions introduced above for the group $F = G \ltimes \mathfrak{g}^*$, which is called the *magnetic extension* of G. This group will be central in the context of incompressible magnetohydrodynamics.

We define the group $F = G \ltimes \mathfrak{g}^*$ as a set of pairs

$$F = \{(\phi, a) \mid \phi \in G,\ a \in \mathfrak{g}^*\}$$

with the group multiplication

$$(\phi, a)\cdot(\psi, b) = (\phi\psi, \mathrm{Ad}_\psi^* a + b),$$

and $(e, 0)$ being the unit element.

It can easily be verified that the inverse element is $(\phi, a)^{-1} = (\phi^{-1}, -\mathrm{Ad}_{\phi^{-1}}^* a)$. Then, the formula for the inner automorphism is

$$\begin{aligned} A_{(\phi,a)}(\psi, b) = (\phi, a)\cdot(\psi, b)\cdot(\phi, a)^{-1} = (\phi, a)\cdot(\psi, b)\cdot(\phi^{-1}, -\mathrm{Ad}_{\phi^{-1}}^* a) = \\ = (\phi\psi\phi^{-1}, \mathrm{Ad}_{\psi\phi^{-1}}^* a + \mathrm{Ad}_{\phi^{-1}}^* b - \mathrm{Ad}_{\phi^{-1}}^* a). \end{aligned} \tag{2.9}$$

The Lie algebra of the group F is $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{g}^*$. Let us look for the adjoint operator on $\mathfrak{f}$ in the form $\mathrm{ad}_{(v,\tilde a)}(w, \tilde b) = (\xi, \eta)$, where $v, w, \xi \in \mathfrak{g}$, $\tilde a, \tilde b, \eta \in \mathfrak{g}^*$. Then, if the group elements (ϕ, a) and (ψ, b) are generated in the neighborhood of the unit element as

$$\begin{aligned} &\psi = e + tw + o(t), \quad b = t\tilde b + o(t), \quad w \in \mathfrak{g}, \quad \tilde b \in \mathfrak{g}^*, \\ &\phi = e + sv + o(s), \quad a = s\tilde a + o(s), \quad v \in \mathfrak{g}, \quad \tilde a \in \mathfrak{g}^*, \end{aligned}$$

differentiating the first component in (2.9), we get

$$\xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (\phi\psi\phi^{-1}) = vw - wv = [v, w].$$

Further, for the second component we get

$$\mathrm{Ad}^*_{\psi\phi^{-1}}a + \mathrm{Ad}^*_{\phi^{-1}}b - \mathrm{Ad}^*_{\phi^{-1}}a = s\mathrm{Ad}^*_{e-sv+tw}\tilde{a} + t\mathrm{Ad}^*_{e-sv}\tilde{b} - s\mathrm{Ad}^*_{e-sv}\tilde{a}.$$

Differentiating above expression, we get

$$\eta = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left(\mathrm{Ad}^*_{\psi\phi^{-1}}a + \mathrm{Ad}^*_{\phi^{-1}}b - \mathrm{Ad}^*_{\phi^{-1}}a\right) = \mathrm{ad}^*_w\tilde{a} - \mathrm{ad}^*_v\tilde{b}.$$

Finally, we can write (we remove tildes, as it is not essential):

$$\mathrm{ad}_{(v,a)}(w,b) = \left([v,w], \mathrm{ad}^*_w a - \mathrm{ad}^*_v b\right).$$

Let us now obtain the expression for the coadjoint operator. To that end, we need to specify what the dual $\mathfrak{f}^*$ is. We will identify the dual $\mathfrak{f}^*$ with $\mathfrak{f}$ itself,

$$\mathfrak{f}^* = \left\{(\xi, a) \mid \xi \in \mathfrak{g},\, a \in \mathfrak{g}^*\right\},$$

with the pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ defined as

$$\langle\!\langle(\xi,a),(\eta,b)\rangle\!\rangle = \langle b,\xi\rangle + \langle a,\eta\rangle, \quad (\xi,a)\in\mathfrak{f}^*, \quad (\eta,b)\in\mathfrak{f},$$

where $\langle\cdot,\cdot\rangle$ is the standard pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Using the definition of the coadjoint operator, we get

$$\begin{aligned}
&\langle\!\langle \mathrm{ad}^*_{(v,a)}(\xi,\eta),(w,b)\rangle\!\rangle = \langle\!\langle(\xi,\eta),\mathrm{ad}_{(v,a)}(w,b)\rangle\!\rangle = \langle\!\langle(\xi,\eta),([v,w],\mathrm{ad}^*_w a - \mathrm{ad}^*_v b)\rangle\!\rangle = \\
&= \langle\eta,[v,w]\rangle + \langle \mathrm{ad}^*_w a - \mathrm{ad}^*_v b, \xi\rangle = \langle\eta,\mathrm{ad}_v w\rangle + \langle a,\mathrm{ad}_w\xi\rangle - \langle b,\mathrm{ad}_v\xi\rangle = \\
&= \langle \mathrm{ad}^*_v\eta - \mathrm{ad}^*_\xi a, w\rangle + \langle b,\mathrm{ad}_\xi v\rangle = \langle \mathrm{ad}^*_v\eta - \mathrm{ad}^*_\xi a, w\rangle + \langle b,\mathrm{ad}_\xi v\rangle = \\
&= \langle\!\langle\left(\mathrm{ad}_\xi v, \mathrm{ad}^*_v\eta - \mathrm{ad}^*_\xi a\right),(w,b)\rangle\!\rangle.
\end{aligned}$$

Finally, we arrive at the following formula for the coadjoint operator:

$$\mathrm{ad}^*_{(v,a)}(w,b) = \left([w,v], \mathrm{ad}^*_v b - \mathrm{ad}^*_w a\right).$$

2.2.2 Momentum maps and Lie–Poisson reduction

It is well known that the dynamics of a mechanical system with symmetries can be reduced to the dynamics on a manifold of smaller dimension, obtained as a quotient manifold by the symmetry group action. In the context of geometric mechanics, this observation can be formalized via the notion of a *momentum map*.

Definition of the momentum map

Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold and let G be a Lie group acting on it:

$$\Phi\colon G\times M\to M, \quad (g,z)\mapsto\Phi_g(z)\in M \tag{2.10}$$

for any $z \in M$ and $g \in G$.

Let us also assume that the action Φ is canonical, i.e.

$$\Phi_g^* \{F_1, F_2\} = \left\{\Phi_g^* F_1, \Phi_g^* F_2\right\} \tag{2.11}$$

for any $F_1, F_2 \in C^\infty(M)$.

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G. Then, the action (2.10) is infinitesimally generated by the vector field $\xi_M \in \mathcal{D}(M)$ induced by an element $\xi \in \mathfrak{g}$:

$$T_z M \ni \xi_M(z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi)(z). \tag{2.12}$$

We observe that

$$\Phi_{g^{-1}}^*(\xi_M) = \left(\mathrm{Ad}_g \xi\right)_M . \tag{2.13}$$

Differentiating (2.13) at the neighborhood of the unit element, $g(t) = e + t\eta + o(t^2)$, we get an infinitesimal formulation of (2.13):

$$[\xi_M, \eta_M] = -[\xi, \eta]_M, \tag{2.14}$$

which implies that the map $\xi \mapsto \xi_M$ is a Lie algebra *antihomomorphism*.

Condition (2.11) implies that

$$\xi_M\left(\{F_1, F_2\}\right) = \{\xi_M(F_1), F_2\} + \{F_1, \xi_M(F_2)\}, \tag{2.15}$$

that, however, does not mean that ξ_M is necessarily Hamiltonian. We will require that ξ_M is globally Hamiltonian, i.e.

$$\xi_M = X_{J(\xi)}$$

for some $J(\xi) \in C^\infty(M)$.

Infinitesimal formulation of a canonical action (2.11) yields that we have a *canonical Lie algebra action* $\mathfrak{g} \ni \xi \mapsto \xi_M \in \mathcal{D}(M)$, where ξ_M satisfies (2.15).

Definition 8. Let a Lie algebra $\mathfrak{g}$ act canonically on a Poisson manifold M. Suppose there is a linear map $J\colon \mathfrak{g} \to C^\infty(M)$, such that

$$X_{J(\xi)} = \xi_M$$

for all $\xi \in \mathfrak{g}$. The map $\mu\colon M \to \mathfrak{g}^*$ defined by

$$\langle \mu(z), \xi \rangle = J(\xi)(z)$$

for all $\xi \in \mathfrak{g}$ and $z \in M$, is called a *momentum map*.

It is important to specify the construction of a momentum map for a subalgebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra, and assume that the action of $\mathfrak{g}$ is canonical on M. Then, $\mathfrak{h}$ also acts canonically on M. Assume also that $\mu_{\mathfrak{g}}$ be the momentum map associated to the action of $\mathfrak{g}$. Then, action of $\mathfrak{h}$ also admits a momentum map $\mu_{\mathfrak{h}}\colon M \to \mathfrak{h}^*$ defined by

$$\mu_{\mathfrak{h}}(z) = \mu_{\mathfrak{g}}(z)\big|_{\mathfrak{h}}\,, \quad z \in M. \tag{2.16}$$

To show that, let us take $\eta \in \mathfrak{h}$. Then, since the $\mathfrak{g}$ action admits a momentum map, and since also $\eta \in \mathfrak{g}$, we have $\eta_M = X_{J_{\mathfrak{g}}(\eta)}$. Therefore, putting $J_{\mathfrak{h}}(\eta) = J_{\mathfrak{g}}(\eta)$ for all $\eta \in \mathfrak{h}$ we define the induced $\mathfrak{h}$-momentum map. This is equivalent to

$$\langle \mu_{\mathfrak{h}}(z), \eta \rangle = \langle \mu_{\mathfrak{g}}(z), \eta \rangle$$

for all $z \in M$, $\eta \in \mathfrak{g}$, which proves (2.16).

Thus in order to get a momentum map for a subalgebra action, one should compute the momentum map for an ambient algebra, and then project to the subalgebra.

Example 2. We will illustrate the concept of a momentum map and how to compute it again on the semidirect product group example, but at this time we will take the magnetic extension of one of the matrix Lie groups $\mathrm{SU}(N)$, so that the group (previously denoted by G) is $F = \mathrm{SU}(N) \ltimes \mathfrak{su}(N)^*$. This group plays an important role in quantized MHD dynamics (see Paper I for more details).

Following the previous notation, we will specify the manifold M to be the total space of the cotangent bundle of F, i.e. $M = T^*F = T^*\left(\mathrm{SU}(N) \ltimes \mathfrak{su}(N)^*\right)$. First, we clarify what T^*F is:

$$T^*F = \left\{(Q, m, P, \alpha) \mid Q \in \mathrm{SU}(N), P \in T^*_Q(\mathrm{SU}(N)), m \in \mathfrak{su}(N)^*, \alpha \in \mathfrak{su}(N)\right\}.$$

Consider the left action of F on T^*F:

$$(G, u) \cdot (Q, m, P, \alpha) = (GQ, \mathrm{Ad}^*_Q u + m, (G^{-1})^\dagger P, \alpha) \tag{2.17}$$

for $(G, u) \in F$.

In order to find the corresponding momentum map (associated in this case to the left action) $\mu\colon T^*F \to \mathfrak{f}^*$, let us consider the infinitesimal action of F on T^*F. To that end, let (G, u) be close to the unit element:

$$\begin{aligned} G &= I + t\xi + o(t^2), \quad \xi \in \mathfrak{su}(N), \\ u &= 0 + t\eta + o(t^2), \quad \eta \in \mathfrak{su}(N)^*, \end{aligned}$$

thus, an infinitesimal generator is $(\xi, \eta) \in \mathfrak{su}(N) \ltimes \mathfrak{su}(N)^*$. Then, the infinitesimal left action will be

$$(G, u) \cdot (Q, m, P, \alpha) = \left(Q + t\xi Q + o(t^2), t\mathrm{Ad}^*_Q \eta + m + o(t^2), P - t\xi^\dagger P + o(t^2), \alpha\right).$$

Differentiating above expression by t at $t = 0$, we get a vector field on $M = T^*F$:

$$\xi_{T^*F} = \left(\xi Q, \mathrm{Ad}^*_Q \eta, -\xi^\dagger P, 0\right) = \left(\frac{\partial J}{\partial P}, \frac{\partial J}{\partial \alpha}, -\frac{\partial J}{\partial Q}, -\frac{\partial J}{\partial m}\right).$$

Solving this system of equations for the Hamiltonian J, we find

$$J = \operatorname{tr}\left(P^\dagger \xi Q\right) + \operatorname{tr}\left(\alpha^\dagger Q^\dagger \eta (Q^{-1})^\dagger\right) = \langle \mu, (\xi, \eta) \rangle,$$

and finally

$$\mu\colon T^*F \to \mathfrak{f}^*, \quad \mu(Q, m, P, \alpha) = \left(PQ^\dagger, Q\alpha Q^{-1}\right),$$

where we also used the pairing between $\mathfrak{su}(N)$ and $\mathfrak{su}(N)^*$ via the Frobenius inner product:

$$\langle A, B \rangle = \operatorname{tr}(A^\dagger B), \quad A \in \mathfrak{su}(N)^*,\, B \in \mathfrak{su}(N). \tag{2.18}$$

Note that we have not used so far that we work on a subalgebra $\mathfrak{su}(N) \subset \mathfrak{gl}(N, \mathbb{C})$. In particular, formula (2.18) does not guarantee that $PQ^\dagger \in \mathfrak{su}(N)$. As was discussed above, to get the final formula for the momentum map, we need to use an appropriate projector for (2.18) for the first component. That is not necessary for the second component, as it is an element of $\mathfrak{su}(N)$ already. The only simplification me can make is to put $Q^{-1} = Q^\dagger$. Finally, the formula for the momentum map takes the form:

$$\mu(Q, m, P, \alpha) = \left(\frac{PQ^\dagger - QP^\dagger}{2},\, Q\alpha Q^\dagger\right). \tag{2.19}$$

Lie–Poisson structure on $\mathfrak{g}^*$ and Lie–Poisson reduction

Especially important is the case when the manifold M coincides with the total space of the cotangent bundle of the group G acting on it (exactly as in the previous example), $M = T^*G$.

First, one can consider the two natural actions of G on T^*G, left and right, that are the cotangent lifts of left or right action of G on itself. Then, identifying the set of smooth functions $C^\infty(\mathfrak{g}^*)$ with the set of left (right) invariant functions on T^*G, one can obtain the Lie–Poisson bracket on $\mathfrak{g}^*$:

$$\{F, G\}_\pm (m) = \pm \left\langle m, \left[\frac{\delta F}{\delta m}, \frac{\delta G}{\delta m}\right]\right\rangle, \tag{2.20}$$

where $m \in \mathfrak{g}^*$, $F, G \in C^\infty(\mathfrak{g}^*)$, and the variational derivative $\delta F/\delta m \in \mathfrak{g}$ is defined as

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} F(m + \varepsilon w) = \left\langle w, \frac{\delta F}{\delta m}\right\rangle, \quad w \in \mathfrak{g}^*.$$

The sign $\pm$ in (2.20) is chosen to be $+$ for right-invariant functions, and $-$ for left-invariant.

The system of equations

$$\dot{F}(m(t)) = \{H, F\}_\pm (m(t)) \tag{2.21}$$

is called a *Lie–Poisson system*. The Hamiltonian function $H \in C^\infty(\mathfrak{g}^*)$ is the conserved quantity, along with *Casimir functions*, i.e. functions $C \in C^\infty(\mathfrak{g}^*)$, such that $\{C,\cdot\}_\pm = 0$. Casimir functions are conserved quantities of (2.21) for any choice of the Hamiltonian function H and play an important role in structure preserving numerical integration of (2.21), as their preservation by the numerical method is crucial for the long time dynamics.

Let us obtain a coadjoint representation of (2.21). For simplicity, we will choose the sign $+$ and omit it. One the one hand, using the chain rule, we get

$$\dot{F}(m(t)) = \left\langle \dot{m}, \frac{\delta F}{\delta m} \right\rangle.$$

On the other hand,

$$\{H,F\} = \left\langle m, \left[\frac{\delta H}{\delta m}, \frac{\delta F}{\delta m}\right] \right\rangle = \left\langle m, \mathrm{ad}_{\frac{\delta H}{\delta m}} \frac{\delta F}{\delta m} \right\rangle = \left\langle \mathrm{ad}^*_{\frac{\delta H}{\delta m}} m, \frac{\delta F}{\delta m} \right\rangle,$$

which implies that

$$\dot{m} = \mathrm{ad}^*_{\frac{\delta H}{\delta m}} m. \tag{2.22}$$

Integrating (2.22) in time, we get

$$\dot{m}(t) = \mathrm{Ad}^*_{\exp\left(\int_0^t \frac{\delta H}{\delta m}(m(s))ds\right)} m(0),$$

which drives us to a conclusion that the Lie–Poisson system (2.22) evolves on the coadjoint G-orbit of $m(0) \in \mathfrak{g}^*$, where the Casimir functions are constant. However, it is not always that Casimir functions completely define the coadjoint orbit.

A remarkable observation is that codajoint orbits can be endowed with the symplectic structure.

Theorem 4. *Let G be a Lie group and let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. Let also $X, Y \in \mathfrak{g}$. Then, there exist a symplectic form ω on $\mathcal{O}$ defined as*

$$\omega(\mathrm{ad}^*_X m, \mathrm{ad}^*_Y m)(m) = \langle m, [X,Y] \rangle \tag{2.23}$$

for all $m \in \mathfrak{g}^*$.

The symplectic form (2.23) is called *Kirillov-Kostant-Souriau form.* It is worth noting that (2.23) uses the identification $T_m\mathfrak{g}^* \simeq \mathfrak{g}^*$ for all $m \in \mathfrak{g}^*$.

Finally, we conclude that the dual $\mathfrak{g}^*$ is foliated by coadjoint orbits, each of which is a symplectic manifold, *symplectic leaf.*

Let us now establish relations between the Hamiltonian dynamics on T^*G and Lie–Poisson dynamics on $\mathfrak{g}^*$. To this end, we will assume that the Hamiltonian function H on T^*G is left (right) invariant. Let also μ_R and μ_L be the momentum maps for the right and left action of G on T^*G respectively. Then, the *Lie–Poisson reduction theorem* says that the momentum map μ_R (μ_L) reduces the Hamiltonian dynamics with the left (right) invariant Hamiltonian on T^*G to the Lie–Poisson dynamics (2.21) with the Hamiltonian H^- (H^+) satisfying $H = H^- \circ \mu_R$ ($H = H^+ \circ \mu_L$). If F_t is a flow of X_H on T^*G, then the corresponding flows $F_t^\pm$ of $X_{H^\pm}$ are related to F_t as

$$\mu_R\left(F_t\left(\alpha_g\right)\right) = F_t^-\left(\mu_R\left(\alpha_g\right)\right)$$
$$\mu_L\left(F_t\left(\alpha_g\right)\right) = F_t^+\left(\mu_L\left(\alpha_g\right)\right),$$

where $\alpha_g \in T_g^*G$.

Inversely, having a Lie–Poisson system (2.22), one can lift the Hamiltonian H by means of left (right) momentum map, thus obtaining right (left) invariant Hamiltonian $\tilde{H} = H \circ \mu_R$ (or $\tilde{H} = H \circ \mu_R$). Then, equations (2.22) become canonical equations on T^*G with respect to H. This process is called *Lie–Poisson reconstruction.*

Poisson property of the momentum map

Now we approach perhaps one of the most important property of the momentum map, which is the Poisson property.

Let us return to (2.14). Using that both maps $\xi \mapsto \xi_M$ and $H \mapsto X_H$ are Lie algebra antihomomorphisms, we obtain

$$X_{J([\xi,\eta])} = [\xi,\eta]_P = -[\xi_P,\eta_P] = -[X_{J(\xi)}, X_{J(\eta)}] = X_{\{J(\xi),J(\eta)\}}, \tag{2.24}$$

for all $\xi, \eta \in \mathfrak{g}$.

Note that (2.24) does not necessarily imply that

$$J([\xi,\eta]) = \{J(\xi), J(\eta)\} \tag{2.25}$$

Momentum maps that satisfy (2.25) are called *infinitesimally equivariant.*

Theorem 5. *Let $\mu\colon M \to \mathfrak{g}^*$ be an infinitesimally equivariant momentum map for a left Hamiltonian action on a Poisson manifold M, then, μ is a Poisson map:*

$$\mu^*\left\{F_1, F_2\right\}_+ = \left\{\mu^* F_1, \mu^* F_2\right\},$$

that is

$$\left\{F_1, F_2\right\}_+ \circ \mu = \left\{F_1 \circ \mu, F_2 \circ \mu\right\},$$

for all $F_1, F_2 \in C^\infty(\mathfrak{g}^*)$.

To prove the Poisson property, we will need the following

Lemma 3. *Let $\mu\colon M \to \mathfrak{g}^*$ be an infinitesimally equivariant momentum map. Then,*

$$X_{F\circ\mu} = X_{J(\delta F/\delta m)} = \left(\frac{\delta F}{\delta m}\right)_M,$$

for any $F \in C^\infty(\mathfrak{g}^*)$.

Proof. Let us take any $H \in C^\infty(\mathfrak{g}^*)$, and let $C^\infty(M) \ni \mu^*(F) = F \circ \mu$ be the pullback of F. Then,

$$X_{\mu^*(F)}(H) = -X_H(\mu^*(F)) = -\langle \mathrm{d}(\mu^*(F)), X_H\rangle = -\langle \mu^*(\mathrm{d}F), X_H\rangle =$$
$$= -\langle \mathrm{d}F, \mu_*(X_H)\rangle = -\left\langle \mu_*(X_H), \frac{\delta F}{\delta m}\right\rangle,$$

where in the last equality we used that $\mu_*(X_H) \in \mathcal{D}(\mathfrak{g}^*)$ can be identified with $\mathfrak{g}^*$ element. Further,

$$X_{\mu^*(F)}(H) = -\left\langle \mu_*(X_H), \frac{\delta F}{\delta m}\right\rangle = -\left\langle \mathrm{d}\left(J\left(\frac{\delta F}{\delta m}\right)\right), X_H\right\rangle =$$
$$= -X_H\left(J\left(\frac{\delta F}{\delta m}\right)\right) = X_{J(\delta F/\delta m)}(H).$$

□

Now we have all necessary ingredients to prove the Poisson property.

Proof of Theorem 5. Let $z \in M$, and $m = \mu(z) \in \mathfrak{g}^*$. Then,

$$\mu^*\{F, H\}_+ = \{F, H\}_+(\mu(z)) = \left\langle \mu(z), \left[\frac{\delta F}{\delta m}, \frac{\delta H}{\delta m}\right]\right\rangle = J\left(\left[\frac{\delta F}{\delta m}, \frac{\delta H}{\delta m}\right]\right)(z) =$$
$$= \left\{J\left(\frac{\delta F}{\delta m}\right), J\left(\frac{\delta H}{\delta m}\right)\right\}(z) = X_{J(\delta H/\delta m)}\left(J\left(\frac{\delta F}{\delta m}\right)\right)(z) = X_{H\circ\mu}\left(J\left(\frac{\delta F}{\delta m}\right)\right) =$$
$$= -X_{J(\delta F/\delta m)}(H \circ \mu)(z) = -X_{F\circ\mu}(H \circ \mu)(z) = \{F \circ \mu, H \circ \mu\},$$

which finalizes the proof. □

2.2.3 Incompressible MHD as a Lie–Poisson system

Here, we use the abstract constructions introduced previously to show the Hamiltonian structure of incompressible magnetohydrodynamics. Namely, we show that the system of self-consistent incompressible magnetohydrodynamics is a Lie–Poisson system on the dual $\mathfrak{imh}^*$ of the semidirect product Lie algebra $\mathfrak{imh} = \mathcal{D}_\mu(M) \ltimes \big(\Omega^1(M)/\mathrm{d}\Omega^0(M)\big)$. Informally speaking, the first component is responsible for the velocity field, and the second one stands for the magnetic field.

The system reads

$$\begin{cases} \dot{v} + \nabla_v v = -\nabla p + \mathrm{curl} B \times B, \\ \dot{B} = L_v B, \\ \operatorname{div} B = 0, \\ \operatorname{div} v = 0. \end{cases} \tag{2.26}$$

Here, $B(t, x)$ is the divergence-free magnetic field, $v(t, x)$ is the divergence-free velocity field, $p(t, x)$ is a pressure function, L_v denotes the Lie derivative along the vector field $v(t,x)$, and $\nabla_v v$ is the covariant derivative of the vector field v along itself, $x \in M$.

We start with the configuration space for MHD, which is the *magnetic extension* of the Lie group of volume-preserving diffeomorphisms $\mathrm{Diff}_\mu(M)$:

$$\mathrm{IMH} = \mathrm{Diff}_\mu(M) \ltimes \left(\Omega^1(M)/\mathrm{d}\Omega^0(M)\right),$$

where the subscript μ stands for the Riemannian volume form on M for the given Riemannian metric g on M.

The corresponding Lie algebra of the Lie group IMH is

$$\mathfrak{imh} = \mathcal{D}_\mu(M) \ltimes \left(\Omega^1(M)/\mathrm{d}\Omega^0(M)\right),$$

and its dual is

$$\mathfrak{imh}^* = \mathcal{D}^*_\mu(M) \ltimes \mathcal{D}_\mu(M) \simeq \mathcal{D}^*_\mu(M) \ltimes \Omega^2_{\mathrm{cl}}(M).$$

Magnetic fields $B \in \mathcal{D}_\mu(M)$ can be identified with closed differential 2-forms $\beta \in \Omega^2_{\mathrm{cl}}(M)$ by the following way:

$$\iota_B \mu = \beta.$$

In other words,

$$\mathfrak{imh}^* = \left\{(m, B) \mid m = v^\flat \otimes \mu,\ B \in \mathcal{D}_\mu(M)\right\}.$$

The pairing between $\mathfrak{imh}$ and $\mathfrak{imh}^*$ is given as follows:

$$\langle (v,\alpha), (m,B)\rangle = \int\limits_M (\iota_v v^\flat)\mu + \int\limits_M (\iota_B \alpha)\mu = \langle B, \alpha\rangle + \langle m, v\rangle.$$

The energy of a charged fluid is a sum of its kinetic energy and the energy of a magnetic field, and therefore the Hamiltonian of an incompressible charged fluid has the following form:

$$H = \frac{1}{2}\int\limits_M \left(|m|^2 + |B|^2\right)\mu.$$

The Lie–Poisson equations on $\mathfrak{imh}^*$ are

$$\dot{F} = \{H, F\},$$

where the expression for the Lie–Poisson bracket $\{\cdot,\cdot\}$ is given by (2.20) interpreted accordingly. Indeed, for any $(m, B) \in \mathfrak{imh}^*$

$$\{F, G\}(m, B) = \left\langle m, L_{\frac{\delta F}{\delta m}} \frac{\delta G}{\delta m}\right\rangle + \left\langle B, L_{\frac{\delta G}{\delta m}} \frac{\delta F}{\delta B} - L_{\frac{\delta F}{\delta m}} \frac{\delta G}{\delta B}\right\rangle.$$

Taking $F(m, B) = \langle m, u\rangle + \langle B, \theta\rangle$ for some $u \in \mathcal{D}_\mu(M)$ and $\theta \in \mathcal{D}^*_\mu(M) \simeq \Omega^1(M)/\mathrm{d}\Omega^0(M)$, we get

$$\dot{F} = \langle \dot{m}, u\rangle + \langle \dot{B}, \theta\rangle = \{H, F\} = \langle m, L_v u\rangle + \langle B, L_u \mathcal{B}\rangle - \langle B, L_v \theta\rangle, \tag{2.27}$$

where $\mathcal{B} = B^\flat \otimes \mu$. We will need a number of lemmas.

Lemma 4. $\langle m, L_v u\rangle = -\langle L_v m, u\rangle$.

Proof.

$$-\langle L_v m, u\rangle = -\int_M \iota_u(L_v m) = -\int_M \iota_u \Big((L_v v^\flat)\otimes\mu + v^\flat \otimes L_v\mu\Big)$$

Since $L_v\mu = (\mathrm{div} v)\mu = 0$, we get

$$\begin{aligned}-\langle L_v m, u\rangle &= -\int_M \Big(L_v v^\flat\Big)(u)\mu = -\int_M \left(v(g(v,u)) - g(v,[v,u])\right)\mu = \\ &= \int_M g(v, L_v u)\mu - \int_M v(g(v,u))\mu = \langle m, L_v u\rangle - \int_M v(g(u,v))\mu.\end{aligned}$$

Let us consider the last integral in more details.

$$\begin{aligned}\int_M v(g(u,v))\mu &= \int_M L_v(g(v,u))\mu = \int_M L_v(g(v,u)\mu) - \int_M g(v,u)L_v\mu = \\ &= \int_M \mathrm{div}(vg(v,u))\mu = \int_{\partial M}(v\cdot n)g(v,u)\mu = 0,\end{aligned}$$

since v is tangent to the boundary. □

Lemma 5. $\langle B, L_u \mathcal{B}\rangle = \langle L_B \mathcal{B}, u\rangle$.

Proof.

$$\langle B, L_u\mathcal{B}\rangle = \int_M \iota_B(L_u B^\flat)\mu = \int_M \left(B(g(B,u)) - g(B,[B,u])\right)\mu = -\langle \mathcal{B}, L_B u\rangle \tag{2.28}$$

by the same reasons as in the previous lemma.

$$\langle L_B\mathcal{B}, u\rangle = \int_M (L_B\mathcal{B})(u)\mu = \int_M B(B^\flat(u))\mu - \int_M B^\flat([B,u])\mu = -\langle\mathcal{B}, L_B u\rangle, \tag{2.29}$$

where we used the same ideas as in the previous lemma. Finally, from (2.28) and (2.29), we get $\langle B, L_u\mathcal{B}\rangle = \langle L_B\mathcal{B}, u\rangle$. □

Lemma 6. $\langle B, L_v\theta\rangle = -\langle L_v B, \theta\rangle$.

Proof.

$$\langle B, L_v\theta\rangle = \int_M \iota_B(L_v\theta)\mu = \int_M (L_v\theta)(B)\mu = \int_M (v(\theta(B)) - \theta([v,B]))\mu = -\langle L_v B, \theta\rangle.$$

□

Using results of these three lemmas, we can write down (2.27) as follows:

$$\langle \dot{m}, u\rangle + \langle \dot{B}, \theta\rangle = -\langle L_v m, u\rangle + \langle L_B \mathcal{B}, u\rangle + \langle L_v B, \theta\rangle,$$

and we conclude that incompressible MHD equations are

$$\begin{cases} \dot{m} = -L_v m + L_B \mathcal{B}, \\ \dot{B} = L_v B, \\ \mathrm{div} B = 0, \ \mathrm{div} v = 0. \end{cases} \tag{2.30}$$

It is important to get the first equation in terms of the velocity field v. To this end, let us take the first equation in (2.30) and apply the sharp operator $\sharp$:

$$(\dot{m} + L_v m - L_B \mathcal{B})^\sharp = \left(\dot{v}^\flat + L_v v^\flat - L_B B^\flat\right)^\sharp \otimes \mu.$$

Using the result of Lemma 2, we get

$$\left(L_v v^\flat\right)^\sharp = \nabla_v v + \nabla P_1$$

for some function $P_1(t, x)$.

One can verify in local coordinates that

$$\left(L_B B^\flat\right)^\sharp = \mathrm{curl} B \times B + \nabla |B|^2,$$

and finally we end up with

$$\dot{v} + \nabla_v v + \nabla P = \mathrm{curl} B \times B, \tag{2.31}$$

for some function $P(t, x)$, and (2.31) now coincides with the first equation in (2.26).

3 Numerical methods for Hamiltonian and Lie–Poisson systems

The basic idea that stands behind *geometric numerical integration* is to preserve the properties of an exact equation by a numerical method. For example, if some equation (or system of equations) has conservation laws, then the numerical method that is used to solve this equation approximately, must preserve those conservation laws. More generally, if there are geometric structures underlying a (system of) PDE, then the method must preserve these structures. The importance of preserving the geometrical properties is outlined in the first chapter of the book [9], where numerous examples that demonstrate a better reliability of geometric integrators are provided.

Here, we discuss some approaches to construction of geometric integrators for Hamiltonian and Lie–Poisson systems.

3.1 Symplectic integration of Hamiltonian systems

As was outlined previously, one of the main properties of Hamiltonian systems

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n, \tag{3.1}$$

is symplecticity.

Let $(p^{(k)}, q^{(k)})$ be the values of unknowns $p(t), q(t)$ at the discrete time moment t_k. If $\Phi_h \colon (p^{(k)}, q^{(k)}) \mapsto (p^{(k+1)}, q^{(k+1)})$, is a numerical scheme for (3.1) with h being the time stepping, then the condition for it to be symplectic can be expressed as

$$\mathrm{d}p^{(k)} \wedge \mathrm{d}q^{(k)} = \mathrm{d}p^{(k+1)} \wedge \mathrm{d}q^{(k+1)},$$

so the canonical symplectic form $\Omega = \mathrm{d}p \wedge \mathrm{d}q$ is invariant under Φ_h.

Given a Butcher tableau

$$\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\
c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
\hline
 & b_1 & b_2 & \cdots & b_s
\end{array}$$

with $c_i = \sum_{j=1}^{s} a_{ij}$, the corresponding s-stage *Runge-Kutta* method for an ODE (system of ODEs) $\dot{y} = f(t, y)$ is defined as

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i, \quad k_i = f\left(t_n + c_i h, y_n + h \sum_{j=1}^{s} a_{ij} k_j\right), \quad i = 1, \dots, s. \tag{3.2}$$

Theorem 6 ([23]). *If $b_i a_{ij} + b_j a_{ji} = b_i b_j$ for all $j = 1, \dots, s$, then the Runge-Kutta method* (3.2) *is symplectic.*

The simplest example of a symplectic Runge-Kutta method is the *implicit midpoint method*:

$$\begin{cases} q^{(k)} = \tilde{q} - \dfrac{h}{2} H_p(\tilde{p}, \tilde{q}), \\ p^{(k)} = \tilde{p} + \dfrac{h}{2} H_q(\tilde{p}, \tilde{q}), \end{cases} \qquad \begin{cases} q^{(k+1)} = \tilde{q} + \dfrac{h}{2} H_p(\tilde{p}, \tilde{q}), \\ p^{(k+1)} = \tilde{p} - \dfrac{h}{2} H_q(\tilde{p}, \tilde{q}). \end{cases} \tag{3.3}$$

3.2 Lie–Poisson integration of Lie–Poisson systems

For Lie–Poisson systems the problem of finding an integrator that respects the Lie–Poisson geometry (this means to be a symplectic map on coadjoint orbits) is more complicated. In particular, a symplectic Runge-Kutta scheme, in general, does not yield a Lie–Poisson integrator when directly applied to a Lie–Poisson system.

Existing approaches to constructing Lie–Poisson integrators include, for instance, *splitting methods* [19, 20]. They are used if the Hamiltonian can be decomposed into a sum of integrable Hamiltonians. The other approach is to use the constrained integrator RATTLE [7]. One lifts the equations on $\mathfrak{g}^*$ to T^*G and then solves a constrained Hamiltonian system. Most of the methods result in computationally expensive and complicated schemes, involving exponential maps and group to algebra maps.

In the framework of Lie–Poisson reduction described in the previous chapter, it is natural to develop a *discrete Lie–Poisson reduction*. The idea is to utilize the properties of the momentum map $\mu \colon T^*G \to \mathfrak{g}^*$, such as the Poisson property, and symplecticity of the Runge-Kutta scheme [22]. Indeed, having a discrete symplectic map provided by the symplectic Runge-Kutta integrator, that is also left (right) invariant, one reduces it to a Lie–Poisson integrator on $\mathfrak{g}^*$. As shown in [22], for the case when $\mathfrak{g}$ is reductive, and as shown in Paper I of this thesis, if $\mathfrak{g}$ has a semidirect product structure, the corresponding integrator on $\mathfrak{g}^*$ is

- formulated explicitly on $\mathfrak{g}^*$;
- does not require computation of expensive maps (matrix multiplication is the most expensive operation);
- applicable for any Hamiltonian function;
- has the Lie–Poisson properties intrinsically encoded.

4 Summary of included papers

4.1 Paper I

In Paper I, we develop a spatio-temporal discretization for MHD on the sphere that fully preserves the underlying Lie–Poisson geometry. This includes extension of the Euler–Zeitlin model to MHD resulting in a finite-dimensional Lie–Poisson system, and further discretization in time leading to a Lie–Poisson integrator for semidirect product Lie algebras.

First, we use the vorticity formulation of MHD (2.26) on the sphere S^2 in terms of four scalar fields, two vorticity fields ω and β for the velocity and magnetic fields respectively, and two stream functions ψ and θ:

Proposition 3. *The vorticity formulation for incompressible MHD equations* (2.26) *is*

$$\begin{cases} \dot{\omega} = \{\omega, \psi\} + \{\theta, \beta\}, & \omega = \Delta\psi, \\ \dot{\theta} = \{\theta, \psi\}, & \beta = \Delta\theta, \end{cases} \tag{4.1}$$

where $\{\cdot, \cdot\}$ *is a Poisson bracket on* S^2.

Then, based on *Berezin-Toeplitz quantization*, we provide a spatially discrete analogue of (4.1), which is a Lie–Poisson system on the dual of the semidirect product Lie algebra $\mathfrak{f}^* = \mathfrak{su}(N) \ltimes \mathfrak{su}(N)^*$:

$$\begin{cases} \dot{W} = [W, M_1] + [\Theta, M_2], \\ \dot{\Theta} = [\Theta, M_1], \end{cases} \tag{4.2}$$

where $W, \Theta \in \mathfrak{su}(N)$, $M_1 = \Delta_N^{-1} W$, $M_2 = \Delta_N \Theta$, and $\Delta_N \colon \mathfrak{su}(N) \to \mathfrak{su}(N)$ is the Hoppe–Yau Laplacian.

Proposition 4. *System* (4.2) *is a Lie–Poisson flow on the dual* $\mathfrak{f}^*$ *of the Lie algebra* $\mathfrak{f} = \mathfrak{su}(N) \ltimes \mathfrak{su}(N)^*$*:*

$$\dot{J} = \mathrm{ad}^*_M J,$$

where $J = (\Theta, W^\dagger) \in \mathfrak{f}^*$*,* $M = (M_1, M_2^\dagger) \in \mathfrak{f}$*, with the Hamiltonian*

$$H(W, \Theta) = \frac{1}{2}\left(\mathrm{tr}(W^\dagger M_1) + \mathrm{tr}(\Theta^\dagger M_2)\right). \tag{4.3}$$

The functions

$$C_f = \operatorname{tr}\left(f(\Theta)\right), \quad I_g = \operatorname{tr}(W g(\Theta)), \tag{4.4}$$

for arbitrary smooth functions f and g, are Casimirs for (4.2).

Further, we develop discrete Lie–Poisson reduction for semidirect products that can be summarized in the diagram Fig. 4.1.

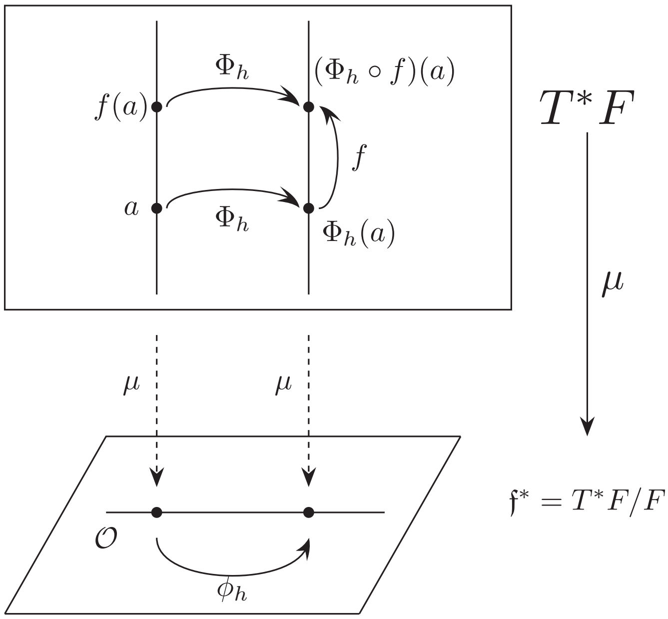


Fig. 4.1. Equivariance of a symplectic method $\Phi_h \colon T^*F \to T^*F$ with respect to the right action $f \colon T^*F \to T^*F$. Symplectic equivariant method $\Phi_h \colon T^*F \to T^*F$ descends to a Lie–Poisson method $\phi_h \colon \mathfrak{f}^* \to \mathfrak{f}^*$ on the coadjoint orbit $\mathcal{O} \subset \mathfrak{f}^*$.

Proposition 5. *The canonical equations on T^*F*

$$\begin{cases} \dot{Q} = -M_1 Q, \\ \dot{P} = M_1^\dagger P + 2M_2^\dagger Q \alpha^\dagger, \\ \dot{\alpha} = 0, \end{cases} \tag{4.5}$$

with right-invariant Hamiltonian $\tilde{H} = H \circ \mu$, where

$$M_1 = \Delta_N^{-1} W, \quad M_2 = \Delta_N \Theta, \quad H(W, \Theta) = \frac{1}{2}\left(\operatorname{tr}(W^\dagger M_1) + \operatorname{tr}(\Theta^\dagger M_2)\right),$$

are reduced to the Lie–Poisson system on $\mathfrak{f}^*$

$$\dot{W} = [W, M_1] + [\Theta, M_2], \quad \dot{\Theta} = [\Theta, M_1], \tag{4.6}$$

by means of the momentum map (2.19).

We use the implicit midpoint method (3.3) as a symplectic scheme on T^*F and prove that it descends to a Lie-Poisson integrator on $\mathfrak{g}^*$, and arrive at the main result.

Theorem 7. *The implicit midpoint method* (3.3) *for the Hamiltonian system* (4.5) *descends to a Lie–Poisson integrator* $\phi_h \colon \mathfrak{f}^* \to \mathfrak{f}^*$, $\phi_h \colon (W_n, \Theta_n) \mapsto (W_{n+1}, \Theta_{n+1})$ *for the Lie–Poisson flow* (4.2). *The method is given explicitly by the following formulas:*

$$
\begin{aligned}
\Theta_n &= \tilde{\Theta} - \frac{h}{2}[\tilde{\Theta}, \tilde{M}_1] - \frac{h^2}{4}\tilde{M}_1\tilde{\Theta}\tilde{M}_1,\\
\Theta_{n+1} &= \Theta_n + h[\tilde{\Theta}, \tilde{M}_1],\\
W_n &= \tilde{W} - \frac{h}{2}[\tilde{W}, \tilde{M}_1] - \frac{h}{2}[\tilde{\Theta}, \tilde{M}_2] - \frac{h^2}{4}\left(\tilde{M}_1\tilde{W}\tilde{M}_1 + \tilde{M}_2\tilde{\Theta}\tilde{M}_1 + \tilde{M}_1\tilde{\Theta}\tilde{M}_2\right),\\
W_{n+1} &= W_n + h[\tilde{W}, \tilde{M}_1] + h[\tilde{\Theta}, \tilde{M}_2],
\end{aligned}
\tag{4.7}
$$

where $\tilde{M}_1 = \Delta_N^{-1}(\tilde{W})$, $\tilde{M}_2 = \Delta_N(\tilde{\Theta})$.

The integrator (4.7) *preserves the Casimirs* (4.4)*:*

$$
\begin{aligned}
\mathrm{tr}(f(\Theta_n)) &= \mathrm{tr}(f(\Theta_{n+1})),\\
\mathrm{tr}(W_n g(\Theta_n)) &= \mathrm{tr}(W_{n+1} g(\Theta_{n+1})).
\end{aligned}
$$

Numerical simulations confirm that the method has all the properties indicated in Theorem 7. Variations of the Casimir functions shown in Fig. 4.2 indicate their exact preservation, as the magnitude 10^{-16} is the tolerance of the fixed point iterations. Also, we observe near preservation of the Hamiltonian function in Fig. 4.3.

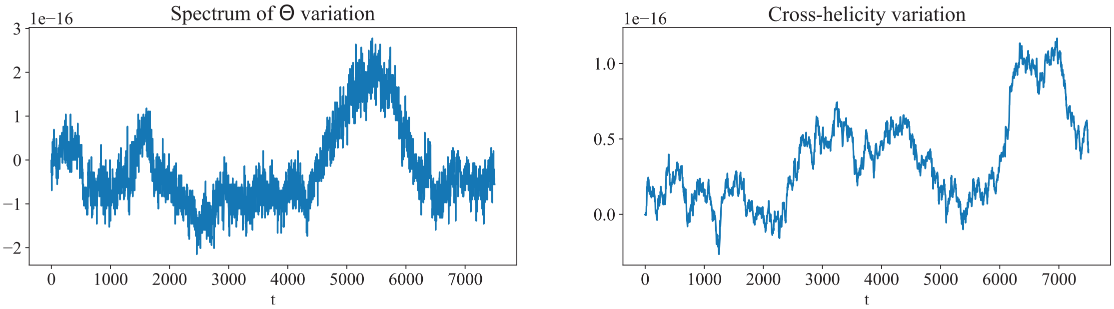


Fig. 4.2. Variation of the smallest eigenvalue of Θ (left), and cross-helicity $\mathrm{tr}(W\Theta)$ (right) for incompressible MHD equations. The order 10^{-16} of the magnitude of the variation indicates the exact preservation of the Casimirs.

4.2 Paper II

In Paper II, we apply the structure preserving discretization for MHD developed in Paper I to study the dynamics of magnetized fluids given by Hazeltine's model

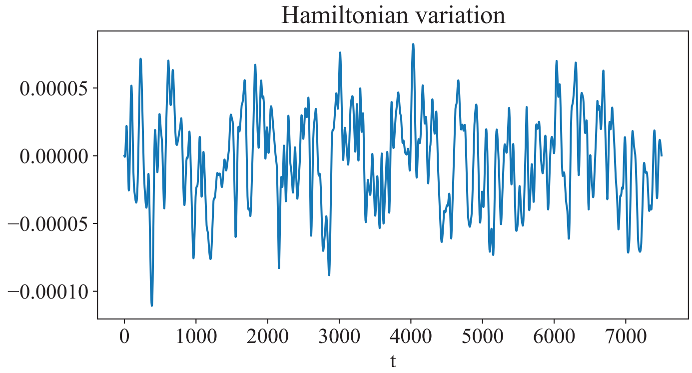


Fig. 4.3. Variation of the Hamiltonian for incompressible MHD equations. Absence of drift indicates nearly preservation of the Hamiltonian.

[13, 10, 11, 12]. Hazeltine's equations generalize the MHD system (4.1):

$$\begin{cases} \dot{\omega} = \left\{\omega, \Delta^{-1}\omega\right\} + \{\theta, \Delta\theta\}\,, \\ \dot{\theta} = \left\{\theta, \Delta^{-1}\omega\right\} - \alpha\,\{\theta, \chi\}\,, \\ \dot{\chi} = \left\{\chi, \Delta^{-1}\omega\right\} + \{\theta, \Delta\theta\}\,, \end{cases} \tag{4.8}$$

where ω and θ have the same meaning as before, χ is the normalized deviation of particle density from a constant equilibrium value, and α is a constant parameter.

The discretized analogue for (4.8) is

$$\begin{cases} \dot{\Psi} = [\Psi, M_1], \\ \dot{\Theta} = [\Theta, M_3], \\ \dot{\chi} = [\chi, M_3] + [\Theta, M_2], \end{cases} \tag{4.9}$$

where $M_3 = M_1 - \alpha\chi$, $\Psi = W - \chi$.

Proposition 6. *System* (4.9) *is a Lie–Poisson flow on the dual* $\mathfrak{f}^*$ *of the Lie algebra*

$$\mathfrak{f} = \mathfrak{su}(N) \oplus (\mathfrak{su}(N) \ltimes \mathfrak{su}(N)^*)$$

with Casimirs

- *Spectrum of* $\Psi = W - \chi$

$$\mathcal{E}_K = \mathrm{tr}(K(W - \chi)) \tag{4.10}$$

for any function K*,*

- *Spectrum of* Θ

$$\mathcal{C}_F = \mathrm{tr}(F(\Theta)) \tag{4.11}$$

for any function F*,*

- *Cross-helicity*

$$J = \mathrm{tr}(\chi \mathrm{G}(\Theta)), \tag{4.12}$$

for any function G,

and the Hamiltonian

$$H = \frac{1}{2}\mathrm{tr}\left(W M_1 + \Theta M_2 - \alpha\chi^2\right). \tag{4.13}$$

A structure preserving integrator for (4.9) is given by

$$\begin{aligned}
\Theta_n &= \tilde{\Theta} - \frac{h}{2}[\tilde{\Theta}, \tilde{M}_3] - \frac{h^2}{4}\tilde{M}_3\tilde{\Theta}\tilde{M}_3, \\
\Theta_{n+1} &= \Theta_n + h[\tilde{\Theta}, \tilde{M}_3], \\
\Psi_n &= \tilde{\Psi} - \frac{h}{2}[\tilde{\Psi}, \tilde{M}_1] - \frac{h^2}{4}\tilde{M}_1\tilde{\Psi}\tilde{M}_1, \\
\Psi_{n+1} &= \Psi_n + h[\tilde{\Psi}, \tilde{M}_1], \\
\chi_n &= \tilde{\chi} - \frac{h}{2}[\tilde{\chi}, \tilde{M}_3] - \frac{h}{2}[\tilde{\Theta}, \tilde{M}_2] - \frac{h^2}{4}\left(\tilde{M}_3\tilde{\chi}\tilde{M}_3 + \tilde{M}_2\tilde{\Theta}\tilde{M}_3 + \tilde{M}_3\tilde{\Theta}\tilde{M}_2\right), \\
\chi_{n+1} &= \chi_n + h[\tilde{\chi}, \tilde{M}_3] + h[\tilde{\Theta}, \tilde{M}_2],
\end{aligned} \tag{4.14}$$

where $\tilde{M}_1 = \Delta_N^{-1}(\tilde{W}) = \Delta_N^{-1}(\tilde{\Psi} + \tilde{\chi})$, $\tilde{M}_2 = \Delta_N(\tilde{\Theta})$, $\tilde{M}_3 = \tilde{M}_1 - \alpha\tilde{\chi}$.

Theorem 8. *The numerical scheme given by* (4.14) *is a Lie–Poisson integrator for* (4.9)*. It preserves the Casimirs exactly,*

$$\begin{aligned}
\mathrm{tr}(K(W_n - \chi_n)) &= \mathrm{tr}(K(W_{n+1} - \chi_{n+1})), \\
\mathrm{tr}(F(\Theta_n)) &= \mathrm{tr}(F(\Theta_{n+1})), \\
\mathrm{tr}(\chi_\mathrm{n}\mathrm{G}(\Theta_\mathrm{n})) &= \mathrm{tr}(\chi_\mathrm{n+1}\mathrm{G}(\Theta_\mathrm{n+1})),
\end{aligned}$$

and nearly preserves the Hamiltonian (4.13) *in the sense of backward error analysis.*

Simulations reveal the formation of large scale vortex condensates, see Fig. 4.4 and Fig 4.5, as well as the presence of an inverse energy cascade, which is shown in energy spectrum figures, see Fig. 4.6.

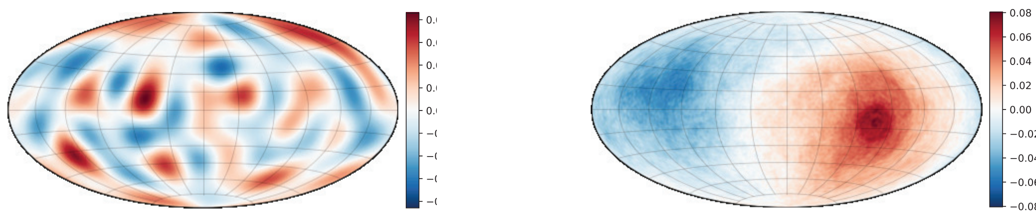


Fig. 4.4. Dynamics of Θ-field. Initial state and final state.

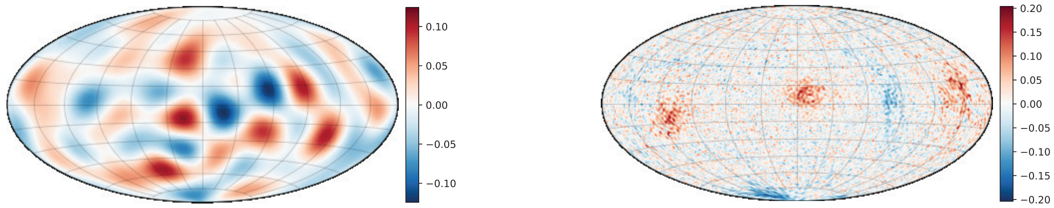
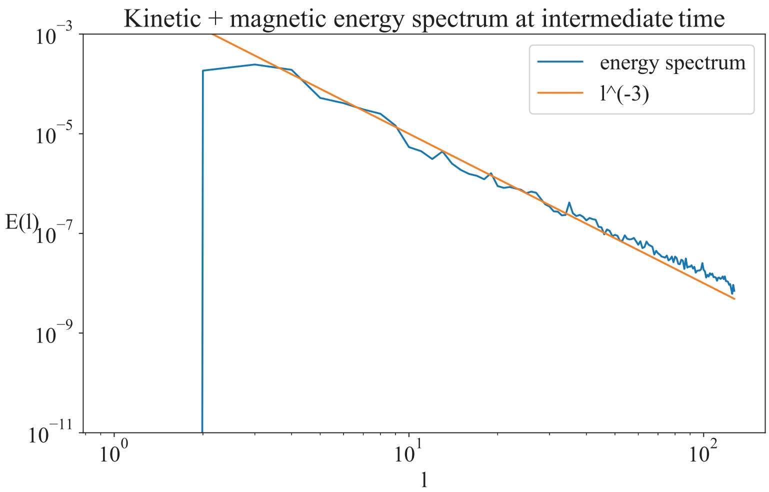
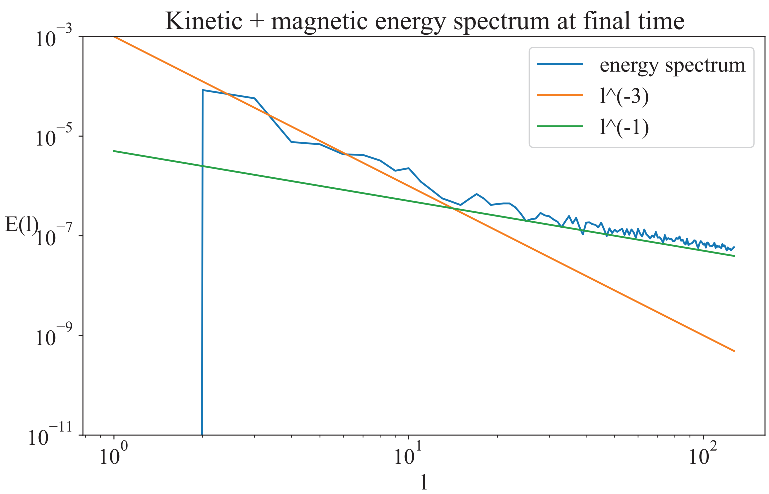


Fig. 4.5. Dynamics of Ψ-field. Initial state and final state.



(a)



(b)

Fig. 4.6. Kinetic and magnetic energy spectrum in the intermediate time (a) and final time (b).

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