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De-Higgsing in eleven-dimensional supergravity on the squashed S^7

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Abstract

In this paper we construct the subset of modes on S^7 that are relevant in the compactification of eleven-dimensional supergravity on a squashed S^7 when restricted to the sector that comprises singlets under the $Sp(1) \times Sp(2)$ isometry of the squashed sphere. Some of the properties of these modes, connected to the transition from the round S^7 to the squashed S^7, are analysed in detail. Special features of the Rarita–Schwinger operator, described in earlier work by Buchdahl, are explained and related to properties of the squashed S^7 operator spectrum obtained in previous work by the authors. We then discuss how the singlet modes give rise to supermultiplets in the left-squashed case, the phenomenon of de-Higgsing, and what happens to the AdS_4 fields in these supermultiplets under an orientation reversal ('skew-whiffing') of the squashed S^7. Finally, we consider the possible choices of boundary conditions that appear for some of these fields in AdS_4 in the case of the right-squashed non-supersymmetric compactification, and how these choices may affect the stability of the gravity theory.

Keywords: Kaluza–Klein, de-Higgsing, squashed S^7

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1. Introduction

This paper is submitted to the special issue of Journal of Physics A: *Fields, Gravity, Strings and Beyond: In Memory of Stanley Deser*. This is especially appropriate since it encounters issues involving the Buchdahl consistency problem for spin-$\frac{3}{2}$ fields interacting with gravity, which was a topic dear to Stanley's heart [1].

There has recently been a lot of effort put into deriving the complete spectrum of some non-supersymmetric AdS vacua of maximal supergravity theories in 10 and 11 dimensions. One reason for this endeavour is to challenge, or vindicate, conjectures related to the swampland project, in particular the AdS stability conjecture proposed by Ooguri and Vafa [2].

This issue becomes especially interesting when discussing vacua that are Breitenlohner–Freedman (BF) stable but which so far have not been proven unstable despite having no supersymmetry. In fact, for $D = 11$ supergravity, the squashed seven-sphere vacuum without supersymmetry, the so called right-squashed sphere [3, 4], is one of a very small set of such examples[4]. The squashed S^7 spectrum of fields and their $SO(2,3)$ irrep content has recently been derived in full detail [7–10][5] which makes it possible to study also the relation between the round S^7 compactification and its cousins with partially or totally broken supersymmetries, i.e. the left-squashed and right-squashed S^7 vacua with $\mathcal{N} = 1$ and $\mathcal{N} = 0$ supersymmetry, respectively. Some aspects of this relation for scalar and spin-$\frac{1}{2}$ modes were addressed in [12].

A number of comments on this relation were made in [7]. Here we discuss an additional aspect connected to, in particular, the Rarita–Schwinger (RS) operator and properties of its eigenvalue equation on non-Einstein versions of the squashed S^7. This is, in fact, a well-known issue which was studied some time ago by Buchdahl [13–15]. In the present context it becomes especially intriguing since for a large portion of the mode spectrum on S^7 can be traced continuously between the round and the Einstein-squashed vacuum solutions of $D = 11$ supergravity (see [12]), while Buchdahl's results indicate that this should not be possible in general.

Our results are based on an explicit construction of the entire set of singlet mode functions on the squashed S^7. Although they constitute only a small subset of the totality of modes in the complete spectrum, the singlet modes are of particular interest because they form a consistent truncation in their own right, and additionally, they have elegant geometrical interpretations, allowing their simple explicit construction. They also exhibit some of the most intriguing features in the relation between the round and the squashed S^7 vacua. The complete singlet sector involves in addition to the singlet RS mode relevant for the Buchdahl issue also its superpartners, the 'squashing mode,' which is the Lichnerowicz operator eigenmode first discussed by Page in [16], and two singlet modes of the first-order three-form operator $Q = *d$. Adding to these the scalar and spin-$\frac{1}{2}$ singlet modes they make up precisely two Wess–Zumino supermultiplets. The latter two singlet modes are also responsible for the $\mathcal{N} = 1$ supergravity multiplet. The properties of these modes on the non-supersymmetric right-squashed S^7 are also analysed in detail.

The paper is organised as follows. In section 2 we give some basic material needed in the following sections. Section 3 contains the construction of the singlet modes on S^7 and a discussion of some of their properties connected to the transition from the round S^7 compactification to the squashed one in particular the special features of the RS operator. Then, in section 4, we explain how the singlet modes give rise to supermultiplets in the left-squashed case and what happens to the AdS_4 fields in these supermultiplets under skew-whiffing. The possible

[4] Another case suggested recently is based on a deformed S-fold compactification [5, 6].

[5] The spectrum of the squashed seven-sphere was reproduced recently in [11] by entirely different methods.

choices of boundary conditions in the right-squashed non-supersymmetric case are identified and possible implications for the issue of stability are discussed. The final section contains a summary and some conclusions.

2. Preliminaries

We recall that the mass operators for the various towers in the spectrum of AdS_4 supergravity from compactification on M_7 are given by (see table 5 of [4][6])

$$\begin{aligned}
&2^+ && M^2 = \Delta_0\\
&\tfrac{3}{2}^{(1),(2)} && M = -i\not{D}_{1/2} + \tfrac{7}{2}m\\
&1^{-\,(1),(2)} && M^2 = \Delta_1 + 12m^2 \pm 6m\left(\Delta_1 + 4m^2\right)^{\frac{1}{2}}\\
&1^+ && M^2 = \Delta_2\\
&\tfrac{1}{2}^{(4),(1)} && M = -i\not{D}_{1/2} - \tfrac{9}{2}m\\
&\tfrac{1}{2}^{(3),(2)} && M = \tfrac{3}{2}m + i\not{D}_{3/2}\\
&0^{+\,(1),(3)} && M^2 = \Delta_0 + 44m^2 \pm 12m\left(\Delta_0 + 9m^2\right)^{\frac{1}{2}}\\
&0^{+\,(2)} && M^2 = \Delta_L - 4m^2\\
&0^{-\,(1),(2)} && M^2 = (Q+3m)^2 - m^2.
\end{aligned} \tag{2.1}$$

The internal space M_7 carries an Einstein metric, with Ricci tensor given by $R_{ab} = 6m^2\, g_{ab}$. The remaining notation here is as follows:

The entries in the left-hand column in (2.1) denote the spins of the four-dimensional fields in the Kaluza–Klein towers. The notation of the additional superscripts is explained in detail in [4]. The operators appearing in the right-hand column are defined as follows:

$$\begin{aligned}
\text{Scalar Laplacian}: &\ \Delta_0\,\phi = -\Box\phi,\\
\text{Vector Hodge-de Rham}: &\ \Delta_1\,V_a = -\Box\,V_a + R_a{}^b\,V_b,\\
\text{Lichnerowicz operator}: &\ \Delta_L\,h_{ab} = -\Box h_{ab} - 2R_{acbd}\,h^{cd} + 2R_{(a}{}^c\,h_{b)c},\\
\text{Two-form Hodge-de Rham}: &\ \Delta_2\,\omega_{ab} = -\Box\omega_{ab} - 2R_{acbd}\,\omega^{cd} - 2R^c{}_{[a}\,\omega_{b]c},\\
\text{Three-form operator}: &\ Q\,\omega_3 = *d\,\omega_3,\\
\text{Dirac operator}: &\ \not{D}_{1/2}\,\psi = \gamma^a\,D_a\,\psi,\\
\text{RS operator}: &\ \gamma^{abc}\nabla_b\psi_c.
\end{aligned} \tag{2.2}$$

For vectors, the transversality condition $\nabla_a V^a = 0$ is imposed and similarly for two-forms and three-forms. Lichnerowicz modes are required to be transverse and traceless and modes of the RS operator are required to be gamma-traceless, $\gamma^a\psi_a = 0$. As will be discussed later, consistency also then implies that $\nabla_a\psi^a = 0$ and that the metric be Einstein. The RS operator then reduces to the Dirac operator $\gamma^a\,\nabla_a$, but now acting on vector-spinors, which we denote by $\not{D}_{3/2}$.

[6] We have changed from the gamma matrix conventions in [4] to standard conventions where $\{\gamma_a,\gamma_b\} = +2\delta_{ab}$, by sending the Γ_a Dirac matrices in [4] to $-i\gamma_a$.

We shall be concerned with the spectrum of modes in the round and squashed S^7 vacua of $D=11$ supergravity. The family of squashed S^7 metrics, described as an $SU(2)$ bundle over S^4, is given by

$$ds^2 = d\mu^2 + \tfrac{1}{4}\sin^2\mu\,\Sigma_i^2 + \lambda^2\left(\sigma_i - A^i\right)^2. \tag{2.3}$$

Here $A^i = \cos^2\tfrac{1}{2}\mu\,\Sigma_i$, and σ_i and Σ_i are two sets of left-invariant one-forms on the group manifold $SU(2)$. They obey

$$d\sigma_i = -\tfrac{1}{2}\epsilon_{ijk}\,\sigma_j\wedge\sigma_k, \qquad d\Sigma_i = -\tfrac{1}{2}\epsilon_{ijk}\,\Sigma_j\wedge\Sigma_k. \tag{2.4}$$

When required, we use the vielbein basis e^a with

$$e^0 = d\mu, \qquad e^i = \tfrac{1}{2}\sin\mu\,\Sigma_i, \qquad e^{\hat{i}} = \lambda\nu_i = \lambda\left(\sigma_i - A^i\right), \tag{2.5}$$

where i ranges over the values 1, 2 and 3, and $\hat{i}=$ ranges over 4, 5 and 6, with $\hat{1}=4$, $\hat{2}=5$ and $\hat{3}=6$. The constant λ is the squashing parameter. The isometry group of the metric is $Sp(1)\times Sp(2)$ (which is locally the same as $SU(2)\times SO(5)$) for generic values of λ, enhancing to $SO(8)$ in the case $\lambda=1$, which corresponds to the round S^7 (of radius 2).

The metric (2.3) is Einstein when $\lambda=1$, obeying

$$R_{ab} = \frac{3}{2}g_{ab}, \tag{2.6}$$

and when $\lambda = 1\sqrt{5}$, with

$$R_{ab} = \frac{27}{10}g_{ab}. \tag{2.7}$$

For general values of λ, the Ricci scalar is

$$R = \frac{3\left(1+8\lambda^2-2\lambda^4\right)}{2\lambda^2}. \tag{2.8}$$

It will be convenient, when calculating the eigenvalues, to rescale the metric so that it has $R=42m^2$ for all values of the squashing parameter λ. This will imply that eigenvalues of a linear operator will need to be scaled by the factor

$$\frac{2\sqrt{7}\,\lambda m}{\sqrt{1+8\lambda^2-2\lambda^4}}. \tag{2.9}$$

Eigenvalues of a second-order operator will need to be scaled by the square of (2.9). Note that after the rescaling of the metric, the round and the squashed Einstein metrics will both depend on the parameter m^2 and obey

$$R_{ab} = 6m^2\,g_{ab}. \tag{2.10}$$

3. Singlet modes on the squashed S^7

In this section, we shall collect together a number of previously-known results for singlet modes of the relevant operators on the squashed S^7 whose eigenvalues govern the masses of the corresponding four-dimensional fields in the S^7 compactifications of $D=11$ supergravity. Some of our discussion here will also extend beyond that previously presented in the literature. The results we obtain here will then be used in the next section in order to study the mass spectrum in the singlet sector for both the left-squashed and the right-squashed S^7 vacua, and to relate these spectra to the spectrum in the round-sphere vacuum.

The singlet modes we shall consider in this section comprise the singlet mode of the scalar Laplacian $\Delta_0 = -\Box$ (this, of course, is just the trivial constant mode); a singlet in the spectrum of the Lichnerowicz operator Δ_L acting on transverse, traceless two-index symmetric tensors; two singlets in the spectrum of the first-order operator $Q = *d$ acting on three-forms; a singlet in the spectrum of the Dirac operator acting on spin-$\frac{1}{2}$ fields on the S^7; and a singlet in spectrum of the RS operator acting on transverse spin-$\frac{3}{2}$ fields on the S^7. In all, these comprise the entire set of singlet modes that are relevant for determining the spectrum of fields in the S^7 compactifications of $D = 11$ supergravity. Note that the Hodge–de Rham operators Δ_1 and Δ_2 acting on one-forms and two-forms have no $Sp(1) \times Sp(2)$ singlet modes in the round or squashed S^7 backgrounds.

3.1. Scalar singlet modes

The complete spectrum of the scalar Laplacian $\Delta_0 = -\Box$ on the whole family of squashed S^7 metrics was obtained in [12]. For our present purposes, it suffices to note that the only singlet in the spectrum of Δ_0 is the constant mode, with eigenvalue 0.

3.2. Lichnerowicz singlet modes

There is only one singlet in the spectrum of the Lichnerowicz operator acting on transverse, traceless symmetric tensor h_{ab} in the squashed S^7 metric, namely the so-called 'squashing mode' that corresponds to an infinitesimal variation of the squashing parameter while keeping the volume fixed. In the orthonormal frame defined in equation (2.5), it corresponds to a tensor whose non-vanishing components (up to an constant scale) are given by

$$h_{00} = 3, \qquad h_{ij} = 3\delta_{ij}, \qquad h_{\hat{i}\hat{j}} = -4\delta_{ij}. \tag{3.1}$$

It was first discussed in [16]. A straightforward calculation shows that in the background of the metric (2.3), it is an eigentensor satisfying $\Delta_L h_{ab} = \sigma_L h_{ab}$ with $\sigma_L = 7\lambda^2$. After making the rescaling by two powers of the factor in equation (2.9), so as to normalise to squashed metrics with $R = 42m^2$, we get the Lichnerowicz eigenvalue

$$\sigma_L = \frac{196\,\lambda^4\,m^2}{1 + 8\lambda^2 - 2\lambda^4}. \tag{3.2}$$

It should be noted that because the eigenfunctions of the Lichnerowicz operator are defined for all values of the squashing, we can in particular continuously follow them as we pass from the round S^7 to the Einstein-squashed S^7. For the particular case of the singlet mode (3.1), we see from equation (3.2) that we have

$$\begin{aligned} \lambda = 1: &\quad \sigma_L = 28m^2, \\ \lambda = \tfrac{1}{\sqrt{5}}: &\quad \sigma_L = \frac{28m^2}{9}. \end{aligned} \tag{3.3}$$

These values accord with those obtained in [16]. Note that the Lichnerowicz eigenvalue $\sigma_L = 28m^2$ on the round S^7 is in accordance with the known round-sphere results, where the lowest level of TT Lichnerowicz modes are in the **300** representation of $SO(8)$, and have eigenvalue $28m^2$ (see, for example, [4]). And indeed, as must be the case, the **300** of $SO(8)$ has a singlet in its decomposition under the $Sp(1) \times Sp(2)$ subgroup (the isometry group of the squashed S^7).

In fact one can straightforwardly see that this $Sp(1)\times Sp(2)$ singlet is the unique singlet in the entire spectrum of $SO(8)$ representations for TT eigentensors of the Lichnerowicz operator on the round sphere. This justifies the statement made above that the tensor (3.1) is the unique singlet TT Lichnerowicz mode in the squashed S^7 background[7].

3.3. Singlet modes of the $Q = *d$ operator on three-forms

Three-form modes of the operator $Q = *d$ exist for all values of the squashing parameter, and therefore we can again count the number of singlet modes by looking at the decomposition of the known $SO(8)$ representations for three-form modes on the round sphere, and counting the number of singlets in their decomposition under $Sp(1)\times Sp(2)$. There are in fact two singlets in total.

We can construct these singlet modes by considering three-forms in the squashed sphere metric (2.3) of the form

$$\omega = \alpha F^i \wedge \nu_i + \beta \nu_1 \wedge \nu_2 \wedge \nu_3, \tag{3.4}$$

where α and β are constants, $\nu_i \equiv \sigma_i - A^i$, and $F^i = dA^i + \frac{1}{2}\epsilon_{ijk}A^j \wedge A_k$. Solving for $Q\omega = \sigma\omega$, one finds

$$\alpha = \frac{\sigma\beta}{6\lambda^3}, \qquad \alpha(1-\sigma\lambda) = -\beta, \tag{3.5}$$

which can be solved for σ giving

$$\sigma_\pm = \frac{1}{2\lambda} \pm \frac{1}{2\lambda}\sqrt{1+24\lambda^4}. \tag{3.6}$$

Normalising the squashed sphere so that its Ricci scalar is $R = 42m^2$ for all λ implies we should scale these eigenvalues by the factor given in (2.9), and so the eigenvalues become

$$\sigma_+ = \frac{\sqrt{7}m\left[\sqrt{1+24\lambda^4}+1\right]}{\sqrt{1+8\lambda^2-2\lambda^4}}, \tag{3.7}$$

$$\sigma_- = -\frac{\sqrt{7}m\left[\sqrt{1+24\lambda^4}-1\right]}{\sqrt{1+8\lambda^2-2\lambda^4}}. \tag{3.8}$$

Thus for the σ_+ mode we have

$$\lambda = 1 : \sigma_+ = 6m,$$
$$\lambda = \frac{1}{\sqrt{5}} : \sigma_+ = 4m. \tag{3.9}$$

This three-form mode is the singlet in the decomposition of the 840 dimensional $(2,0,0,2)$ $SO(8)$ representation in the round-sphere vacuum.

For the the σ_- mode we have

$$\lambda = 1 : \quad \sigma_- = -4m,$$

[7] A crucial point in the argument is the fact, mentioned above, that every Lichnerowicz mode can be continuously followed from the round sphere to any of the squashed sphere family of metrics.

$$\lambda = \tfrac{1}{\sqrt{5}}: \quad \sigma_- = -\frac{2m}{3}. \tag{3.10}$$

This three-form mode is the singlet in the decomposition of the 35 dimensional $(0,0,2,0)$ $SO(8)$ representation.

Of course, since $Q = *d$ is a first-order operator its eigenvalues will reverse in sign if the orientation of the manifold is reversed.

3.4. Singlet modes of the Dirac operator

Turning now to the fermionic modes, we begin with the Dirac operator acting on spinor modes. Modes of the Dirac operator can be followed continuously from the round to the squashed sphere, and so the singlet Dirac modes on the squashed sphere can be counted by counting the number of singlets in the decomposition of the $SO(8)$ Dirac mode representations of the round sphere under the $Sp(1)\times Sp(2)$ subgroup. There is in fact a unique such singlet. These and the following features of the Dirac equation can be found in [12].

The $Sp(1)\times Sp(2)$ singlet Dirac mode can be constructed by first constructing the Killing spinor η in the Einstein-squashed S^7. Using the orthonormal frame in equation (2.5), and making a natural choice for the spin frame, η is a spinor with constant components, satisfying

$$\nabla_a \eta = -\frac{\mathrm{i}m}{2}\gamma_a \eta \tag{3.11}$$

in the $\lambda = \tfrac{1}{\sqrt{5}}$ squashed S^7 (we have rescaled the metric so that $R = 42m^2$, as discussed previously). A straightforward calculation shows that the same constant-component spinor η is an eigenspinor of the Dirac operator for arbitrary values of the squashing parameter, and its eigenvalue σ, defined by $\mathrm{i}\not{D}_{1/2}\,\eta = \sigma\,\eta$, is

$$\sigma = \frac{3\sqrt{7}\left(1+2\lambda^2\right) m}{2\sqrt{1+8\lambda^2-2\lambda^4}}. \tag{3.12}$$

Thus in particular, on the round S^7 and on the Einstein-squashed S^7 we have

$$\begin{aligned} \lambda = 1: &\quad \sigma = \frac{9m}{2}, \\ \lambda = \tfrac{1}{\sqrt{5}}: &\quad \sigma = \frac{7m}{2}. \end{aligned} \tag{3.13}$$

This Dirac mode is the singlet in the decomposition of the 56 dimensional $(1,0,0,1)$ $SO(8)$ representation of Dirac modes on the round S^7.

3.5. Singlet modes of the RS operator

Consider an RS mode obeying

$$\mathrm{i}\,\gamma^{abc}\nabla_b\psi_c = \sigma\,\psi^a. \tag{3.14}$$

Multiplying by γ_a, we obtain

$$5\mathrm{i}\,\nabla^a\psi_a = \left(5\mathrm{i}\,\gamma^b\nabla_b - \sigma\right)\left(\gamma^a\psi_a\right). \tag{3.15}$$

We should impose the gamma-traceless condition

$$\gamma^a \psi_a = 0, \tag{3.16}$$

so equation (3.15) then implies that

$$\nabla^a \psi_a = 0. \tag{3.17}$$

Now instead act on equation (3.14) with ∇_a. Using the identity

$$[\nabla_a, \nabla_b]\, \psi_c = \tfrac{1}{4} R_{abde}\, \gamma^{de}\, \psi_c + R_{cdab}\, \psi^d, \tag{3.18}$$

it follows after a little algebra that

$$\tfrac{\mathrm{i}}{2} R^{ab}\, \gamma_a\, \psi_b = \left(\sigma - \mathrm{i}\gamma^b \nabla_b\right) (\nabla^a \psi_a)\,. \tag{3.19}$$

If the metric is Einstein, this just gives the same conclusion that we saw before, namely that as well as the gamma-traceless condition (3.16) we shall also have the transversality condition (3.17). However, if the metric is not Einstein, we obtain the independent algebraic condition[8]

$$R_{ab}\, \gamma^a \psi^b = 0. \tag{3.20}$$

For our present example of the squashed S^7 we have in general, in vielbein components,

$$R_{\alpha\beta} = a\, \delta_{\alpha\beta}, \qquad R_{\hat{i}\hat{j}} = b\, \delta_{\hat{i}\hat{j}}, \tag{3.21}$$

where $\alpha = 0, 1, 2, 3$ and $\hat{i} = 4, 5, 6$. The constants a and b depend on the squashing parameter λ, and are unequal except when $\lambda^2 = 1$ or $\lambda^2 = \frac{1}{5}$. Thus (3.19), together with the original condition $\gamma^a\, \psi_a = 0$, implies that for general squashing we must have the two independent algebraic conditions

$$\gamma^\alpha\, \psi_\alpha = 0, \qquad \gamma^{\hat{i}}\, \psi_{\hat{i}} = 0, \tag{3.22}$$

whereas when the metric is Einstein, we have only the condition $\gamma^a\, \psi_a = 0$. Evidently, then, the RS eigenvalue problem is over-constrained if the metric is not Einstein.

Since we need to restrict attention to Einstein metrics when considering the modes of the RS operator, this means that we must consider only $\lambda = 1$ (the round S^7) or $\lambda = \frac{1}{\sqrt{5}}$ (the Einstein-squashed S^7). It is easily seen that the $SO(8)$ representations for the RS modes on the round sphere do not include any irreps whose decomposition under $Sp(1) \times Sp(2)$ includes any singlets. One might be tempted to conclude that there could therefore be no singlet RS modes in the Einstein-squashed S^7. However, unlike the situation for all the operators we discussed previously, here we have no continuous path by which we can deform the RS modes on the round sphere to the RS modes on the Einstein-squashed sphere. In fact, as we shall now show[9], there *does* exist a singlet RS mode in the Einstein-squashed S^7.

To see this, we first note that we can construct a class of RS modes in the Einstein-squashed S^7 as follows. Let h_{ab} be any transverse-traceless mode of the Lichnerowicz operator:

[8] This is the famous Buchdahl consistency condition [13–15].

[9] This fact was also demonstrated in [7] using a group theoretic construction of the irrep spectrum designed for coset spaces.

$$\Delta_L h_{ab} = \lambda_L h_{ab}, \qquad \nabla^a h_{ab} = 0, \quad h^a{}_a = 0. \tag{3.23}$$

We now consider a spin-$\frac{3}{2}$ field ψ_a, defined by

$$\psi_a = h_{ab}\gamma^b\eta + \alpha\,(\nabla_c h_{ab})\,\gamma^{cb}\eta, \tag{3.24}$$

where η is the singlet Killing spinor on the Einstein-squashed S^7, obeying equation (3.11), and α is some constant to be chosen later[10]. Given the Killing spinor equation (3.11) and the fact that h_{ab} is transverse and traceless, it is straightforward to see that ψ_a is divergence-free and gamma-traceless,

$$\nabla^a\psi_a = 0, \qquad \gamma^a\psi_a = 0. \tag{3.25}$$

Thus the RS eigenvalue equation (3.14) implies that ψ_a obeys the Dirac eigenvalue equation

$$\mathrm{i}\not{D}_{3/2}\,\psi_a = \sigma\,\psi_a, \tag{3.26}$$

where $\mathrm{i}\not{D}_{3/2}$ denotes the Dirac operator $\mathrm{i}\,\gamma^a\nabla_a$ when acting on vector-spinors.

Substituting the definition of ψ_a in equation (3.24) into equation (3.26) gives equations that determine the value that must be chosen for α, as well as determining the eigenvalue σ. After some calculation one finds that equation (3.14) implies

$$\begin{aligned}\mathrm{i}\left(-\alpha\lambda_L + 7\alpha m^2 + \frac{5\mathrm{i}\,m}{2}\right) h_{ab}\gamma^b\eta + \left(\mathrm{i} + \frac{3\alpha m}{2}\right)(\nabla_c h_{ab})\,\gamma^{cb}\eta \\ = \sigma\left[h_{ab}\gamma^b\eta + \alpha\,(\nabla_c h_{ab})\,\gamma^{cb}\eta\right].\end{aligned} \tag{3.27}$$

The two structures $h_{ab}\gamma^b\eta$ and $(\nabla_c h_{ab})\gamma^{cb}\eta$ are in general linearly independent, and so equating the coefficients for each implies

$$\sigma = -\mathrm{i}\,\alpha\lambda_L + 7\mathrm{i}\,\alpha m^2 - \frac{5m}{2} = \frac{\mathrm{i}}{\alpha} + \frac{3m}{2}. \tag{3.28}$$

Solving, this gives

$$\sigma_\pm = -\frac{m}{2} \pm \left(\lambda_L - 3m^2\right)^{1/2}, \tag{3.29}$$

$$\alpha_\pm = \frac{2\mathrm{i}m \pm \mathrm{i}\sqrt{\lambda_L - 3m^2}}{(\lambda_L - 7m^2)}. \tag{3.30}$$

Thus in general a Lichnerowicz mode h_{ab} with eigenvalue λ_L gives rise to two linearly-independent RS modes, with different eigenvalues σ_+ and σ_-, corresponding to the plus or minus sign choices.

If we now apply this construction to the Lichnerowicz squashing-mode of Page, given in equation (3.1), it turns out that for this particular case the two structures in the two terms in the definition of ψ_a in equation (3.24) are proportional to one another, with

[10] In the general case, the inclusion of both terms in equation (3.24) is necessary in order that ψ_a can be an eigenfunction of the RS operator. It will turn out, however, for the specific singlet mode of Δ_L that is of principle interest to us here, that the two structures in (3.24) are the same, up to a constant factor. See later.

$$(\nabla_c h_{ab})\gamma^{cb}\eta = -\frac{7\mathrm{i}m}{3}h_{ab}\gamma^b\eta. \tag{3.31}$$

Thus in this case the Lichnerowicz squashing mode gives rise to just one RS mode rather than two, and we may simply choose to write the RS mode in the form

$$\psi_a = h_{ab}\gamma^b\eta. \tag{3.32}$$

The RS eigenvalue then corresponds just to the minus sign choice in equation (3.29), implying, since $\lambda_L = \frac{28m^2}{9}$ for the Lichnerowicz singlet mode, that

$$\sigma = -\frac{m}{6}. \tag{3.33}$$

This is the eigenvalue of the singlet RS mode on the Einstein-squashed S^7.

As discussed above, because there is no way to make a continuous deformation between modes of the RS operator on the round S^7 and modes on the Einstein-squashed S^7, it does not necessarily have to be the case that the RS modes on the Einstein-squashed S^7 occur in irreps of $Sp(1)\times Sp(2)$ that are in 1–1 correspondence with the decomposition of the $SO(8)$ irreps of the round S^7 under the $Sp(1)\times Sp(2)$ subgroup. We have exhibited a concrete example of this phenomenon in the construction above. Namely, we have found that there exists a singlet RS mode on the Einstein-squashed S^7, while on the round S^7 none of the irreps of the RS modes decomposes to include a singlet under $Sp(1)\times Sp(2)$. This is very different from the situation for the singlet Dirac mode on the Einstein-squashed S^7 (i.e. the Killing spinor η), which can be traced back to the round S^7 as the singlet in the decomposition of the 56 Dirac modes in the $(1,0,0,1)$ of $SO(8)$ on the round sphere. Unlike the spinor η, which is responsible for the space-invader phenomenon in the squashed vacuum, the singlet RS mode on the Einstein-squashed S^7 is effectively a 'mode from nowhere,' which has no counterpart on the round sphere. This proves the correctness of this particular aspect of the spectrum construction in [7].

4. The singlets in the AdS_4 spectrum

In the previous section, we constructed explicitly the complete set of singlet modes of all the mass operators that arise in the analysis of the mass spectrum of the $\mathrm{AdS}_4\times M_7$ vacua of eleven-dimensional supergravity, where M_7 is either the round S^7 or the Einstein-squashed S^7. In this section, we shall show explicitly how these give rise to fields and, in the supersymmetric cases, supermultiplets, in the AdS_4 vacuum.

The four-dimensional fields form representations of the $SO(2,3)$ isometry group of the AdS_4 background metric. They are denoted by $D(E_0,s)$, where E_0 is the lowest energy eigenvalue and s is the total angular momentum quantum number of the lowest-energy state. The representation is unitary if $E_0 \geqslant s+1$ for $s\geqslant 1$, and if $E_0\geqslant s+\frac{1}{2}$ for $s=0$ or $s=\frac{1}{2}$.[11] We refer to [4, 18] for further details. In [4], the E_0 values of the AdS_4 representations are listed for each spin:

[11] Note that there exist also the so-called singleton representations $D(\frac{1}{2},0)$ and $D(1,\frac{1}{2})$ of $SO(2,3)$, which were discovered by Dirac [17]. These will feature in our later discussion.

$$
\begin{aligned}
s=0: &\qquad E_0=\tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{(M/m)^2+1}\\
s=\tfrac{1}{2}: &\qquad E_0=\tfrac{3}{2}\pm\tfrac{1}{2}|M/m|\\
s=1: &\qquad E_0=\tfrac{3}{2}+\tfrac{1}{2}\sqrt{(M/m)^2+1}\\
s=\tfrac{3}{2}: &\qquad E_0=\tfrac{3}{2}+\tfrac{1}{2}|M/m-2|\\
s=2: &\qquad E_0=\tfrac{3}{2}+\tfrac{1}{2}\sqrt{(M/m)^2+9}.
\end{aligned}
\tag{4.1}
$$

It should be noted that in general the eigenvalue spectra of the first-order mass operators (Dirac, RS and the three-form operator $Q=*d$) will be different if the orientation of the compactifying seven-manifold M_7 is reversed. Specifically, the orientation reversal will reverse the signs of all the eigenvalues of the first-order operators, while the eigenvalues of the second-order mass operators are unaffected. The round S^7 is an exception to the general rule; the eigenvalues of the first-order mass operators on the round S^7 occur always in pairs, with every positive eigenvalue having an equal and opposite negative-eigenvalue partner. By contrast, the eigenvalue spectra associated with first-order operators on a squashed S^7 are asymmetric between positive and negative. This can be seen in the results of the previous section, where the totality of singlet modes in the squashed S^7 background is presented, or by consulting the more general results of [8–10].

Because of the asymmetry of the squashed-sphere spectra under orientation reversal, there are two inequivalent squashed S^7 compactifications of eleven-dimensional supergravity, as was shown in [3, 4]. These are conventionally referred to as the left-squashed compactification and the right-squashed compactification. The left-squashed vacuum has $\mathcal{N}=1$ supersymmetry, while the right-squashed vacuum is non-supersymmetric, so $\mathcal{N}=0$.

For the left-squashed vacuum, we see from the results in section 3, from the mass operators in equation (2.1) and from equation (4.1) that singlets will give rise to the following Heidenreich-type supermultiplets: the massless supergravity supermultiplet with spin $(2,\frac{3}{2})$ and two Wess–Zumino multiplets with spin content $(0^+,0^-,\frac{1}{2})$, one of which is related to singlet modes for the operators $(\Delta_0,Q,i\not{D}_{1/2})$ and the second to the modes of the operators $(\Delta_L,Q,i\not{D}_{3/2})$.

Table 1 summarises the situation for the $Sp(1)\times Sp(2)$ singlet sectors of the round S^7 vacuum and the left-squashed S^7 vacuum. In particular, one can see where the singlet modes in the left-squashed vacuum have come from in the round-sphere vacuum. Two features in particular are noteworthy:

Firstly, the massless spin-$\frac{3}{2}$ field (the gravitino) of the left-squashed vacuum has not originated from one of the eight massless gravitini of the round-sphere vacuum; rather, it has come from the decomposition of the level $n=1$ $\mathbf{56}_c$ of massive gravitini in the round-sphere vacuum. This is the 'space-invaders' phenomenon that was first demonstrated in [3]. Of course, a corollary of this is that although the truncation of the full round-sphere spectrum to the singlets under the $Sp(1)\times Sp(2)$ subgroup of $SO(8)$ is a perfectly consistent one, the singlet modes do not fall into supermultiplets in the round-sphere vacuum, because all eight massless gravitini in the round-sphere vacuum are projected out in this singlet truncation.

Secondly, as we showed in [7] and have been further emphasising in this paper, because the massless gravitino in the left-squashed vacuum originated from a massive gravitino in the round-sphere vacuum, there must be an 'inverse Higgs', or de-Higgs, phenomenon, in which the helicity-$\frac{1}{2}$ states in the massive gravitino in the round-vacuum are ejected and become a genuine spin-$\frac{1}{2}$ mode in the left-squashed vacuum that was not seen as a distinct spin-$\frac{1}{2}$ mode in the round-sphere vacuum. This is precisely what is seen in the second row of fields listed

Table 1. The complete spectrum of $Sp(1)\times Sp(2)$ singlets, in the round and the left-squashed vacua. The $SO(8)$ reps in the fourth column contain an $Sp(1)\times Sp(2)$ singlet. In the squashed vacuum the $\mathcal{N}=1$ supermultiplets comprise the fields $(2, \frac{3}{2}^{(1)})$, $(0^{+(3)}, 0^{-(2)}, \frac{1}{2}^{(4)})$ and $(0^{+(2)}, 0^{-(1)}, \frac{1}{2}^{(2)})$. Note that the $\frac{1}{2}^{(2)}$ field occurs only in the squashed vacuum, since the Rarita–Schwinger operator $\not{D}_{3/2}$ has no $Sp(1)\times Sp(2)$ singlet modes on the round sphere. The boldface numbers denote irreducible representations of SO(8).

Field	Mass operator	Round	Level, $SO(8)$	Left-squashed
$\frac{3}{2}^{(1)}$	$M=-\mathrm{i}\not{D}_{1/2}+\frac{7m}{2}$	$M=-m$, $E_0=3$	$n=1$, $\mathbf{56}_c$	$M=0$, $E_0=\frac{5}{2}$
$\frac{1}{2}^{(2)}$	$M=\mathrm{i}\not{D}_{3/2}+\frac{3m}{2}$	—	—	$M=\frac{4m}{3}$, $E_0=\frac{13}{6}$
$\frac{1}{2}^{(4)}$	$M=-\mathrm{i}\not{D}_{1/2}-\frac{9m}{2}$	$M=-9m$, $E_0=6$	$n=3$, $\mathbf{56}_c$	$M=-8m$, $E_0=\frac{11}{2}$
2	$M^2=\Delta_0$	$M^2=0$, $E_0=3$	$n=0$, $\mathbf{1}$	$M^2=0$, $E_0=3$
$0^{+(3)}$	$M^2=(\sqrt{\Delta_0+9m^2}+6m)^2-m^2$	$M^2=80m^2$, $E_0=6$	$n=2$, $\mathbf{1}$	$M^2=80m^2$, $E_0=6$
$0^{-(2)}$	$M^2=(Q+3m)^2-m^2$	$M^2=80m^2$, $E_0=6$	$n=4$, $\mathbf{840}_s$	$M^2=48m^2$, $E_0=5$
$0^{+(2)}$	$M^2=\Delta_L-4m^2$	$M^2=24m^2$, $E_0=4$	$n=2$, $\mathbf{300}$	$M^2=-\frac{8m^2}{9}$, $E_0=\frac{5}{3}$
$0^{-(1)}$	$M^2=(Q+3m)^2-m^2$	$M^2=0$, $E_0=2$	$n=0$, $\mathbf{35}_c$	$M^2=\frac{40m^2}{9}$, $E_0=\frac{8}{3}$

Table 2. The modes in the multiplets are grouped according to the $\mathcal{N}=1$ supersymmetry of the left-squashed vacuum. The groupings for the right-squashed vacuum represent the same S^7 modes as in the left-squashed vacuum, now with sign-reversals for the $\mathrm{i}\not{D}_{1/2}$, $\mathrm{i}\not{D}_{3/2}$ and Q eigenvalues. The alternative possibilities for the E_0 values in the right-squashed vacuum represent the sign ambiguities for E_0 for spins 0 and $\frac{1}{2}$; these are discussed in the text.

Multiplet (L)	E_0 left-squashed	E_0 right-squashed
$s=\{2,\frac{3}{2}^{(1)}\}$	$(3,\frac{5}{2})$	$(3,4)$
$s=\{0^{+(3)},0^{-(2)},\frac{1}{2}^{(4)}\}$	$(6,5,\frac{11}{2})$	$(6,1,1)$ or $(6,1,2)$ or $(6,2,1)$ or $(6,2,2)$
$s=\{0^{+(2)},0^{-(1)},\frac{1}{2}^{(2)}\}$	$(\frac{5}{3},\frac{8}{3},\frac{13}{6})$	$(\frac{4}{3},\frac{10}{3},\frac{7}{3})$ or $(\frac{5}{3},\frac{10}{3},\frac{7}{3})$

in table 1, where the RS mass operator for the $\frac{1}{2}^{(2)}$ modes has a singlet mode on the Einstein-squashed S^7 but it has no singlet mode on the round S^7.

Table 2 contains a complete list of all $Sp(1)\times Sp(2)$ singlets fields in $SO(2,3)$ irreps $D(E_0,s)$, grouped into $\mathcal{N}=1$ supermultiplets in the supersymmetric left-squashed vacuum. This table also presents the field content of the non-supersymmetric right-squashed vacuum.

We end this section with a few comments on table 2 and then in particular on its relevance for the issue of stability of the right-squashed non-supersymmetric S^7 compactification. This theory is well-known to be BF stable and hence, in view of the AdS stability conjecture proposed in [2], it is of some interest to see if there are other decay modes possible. One such is connected to the appearance of marginal operators in the boundary theory [19] (see also, for instance, [20]). Whether or not the modes analysed in this paper lead to AdS fields whose dual operators on the boundary can be used to construct such marginal operators depends on the boundary conditions chosen for the fields in the third column of the second row of table 2. The fields in the third row cannot give rise to any such operators. Recall that if a scalar field has M^2 in the range $-m^2 \leqslant M^2 \leqslant 3m^2$ then both signs in $E_0 = \frac{3}{2} \pm \frac{1}{2}\sqrt{(M/m)^2+1}$ are compatible with unitarity, the upper sign corresponding to Dirichlet and the lower sign to Neumann boundary conditions. A similar situation arises for Dirac fields with a mass satisfying $|M| \leqslant m$ (m is assumed positive here).

Of the four options appearing in table 2 for the fields in the second row one can for the first case form marginal triple trace operators dual to three pseudo-scalar fields, P^3, or to two fermions and one pseudo-scalar, $P\bar{\lambda}\lambda$. Here we denote the fields, as well as their dual single trace operators on the boundary, as (S, P, λ). The second option of boundary conditions in table 2 allows only for a triple trace operator of the kind P^3 while the remaining two options cannot be used for this purpose.

In order to check if these operators can give rise to any instabilities one has, however, to compute $1/N$ corrections to the relevant beta-functions in the boundary theory but this is beyond the scope of this paper.

Needless to say it would be quite interesting if one could find arguments that would remove any of the four possible combinations of boundary conditions from the list on the second row in table 2. In fact, by adopting the point of view advocated in [7] there must be a fermionic singlet mode in the right-squashed case that has $E_0 = 1$ and therefore is identified as a singleton. This singleton could be created in another kind of de-Higgsing, in which a massive AdS_4 spin-$\frac{1}{2}$ fermion in the round-sphere vacuum splits up into a singleton and a spin-$\frac{1}{2}$ fermion with a different mass in the right-squashed vacuum. That this is possible is suggested by the state diagrams in the appendix of [21]. This singleton interpretation implies that only the first and the third cases are possible for the lowest-energies E_0 for the $\{0^{+(3)}, 0^{-(2)}, \frac{1}{2}^{(4)}\}$ fields in the right-squashed vacuum.

One further noteworthy feature of the singlet spectrum in the right-squashed S^7 vacuum is that one of the two eigenvalues of the three-form operator Q is now $-4m$, which implies that the associated pseudoscalar field in AdS_4 is massless. This accounts for the lowest-energy possibilities $E_0 = 1$ or $E_0 = 2$ for the $0^{-(2)}$ field in the right-squashed vacuum in table 2.

The rather intriguing corollary is then that while the first case does allow for marginal operators the third case does not. If the third case were to be established as the correct choice of boundary conditions the singlet sector of the theory analysed in this paper would not lead to any instabilities of the kind discussed here.

5. Conclusions

In this paper we considered the seven-sphere compactification of eleven-dimensional supergravity, and discussed the relation between the theories in AdS_4 resulting from using the round S^7 and its two squashed versions, the left-squashed S^7 that gives $\mathcal{N} = 1$ supersymmetry and the orientation-reversed right-squashed S^7, which gives no supersymmetry.

Our discussion relied on the explicit construction of all the generalised Fourier modes of the relevant operators on S^7 that are singlets under the squashed S^7 isometry group $Sp(1) \times Sp(2)$.

These modes were then analysed with respect to their properties under the continuous homogeneous deformation from the round to the Einstein-squashed S^7. While we showed that the singlet modes of the bosonic operators and the spin-$\frac{1}{2}$ Dirac operator can indeed be followed in this way, this is not the case for the RS singlet mode on the squashed S^7. We then related this to a special feature for higher-spin operators first discussed by Buchdahl [13–15]. This feature of the compactification is, however, in full accord with the squashed spectrum constructed in [7], which clearly shows that the RS singlet mode does not emanate from any mode in the round sphere operator spectrum.

In the AdS$_4$ theory coming from the left-squashed S^7 we may also identify the $SO(2,3)$ $\mathcal{N}=1$ supersymmetry multiplets associated with the singlet modes. These are the $\mathcal{N}=1$ supergravity multiplet and two Wess–Zumino multiplets. One of the latter is the so called 'Page multiplet,' which contains a scalar field that was shown by Page to be responsible for the squashing of the S^7. Its origin in the round sphere spectrum was also given. Curiously enough, as we explained here, the spin-$\frac{1}{2}$ field in this multiplet is associated with a RS mode in the Einstein-squashed S^7 that has no origin in the round sphere. Instead, from the point of view of the AdS$_4$ theory, this spin-$\frac{1}{2}$ field is ejected from a massive spin-$\frac{3}{2}$ field in the round sphere vacuum, when it becomes massless on the left-squashed sphere. In other words, this is a kind of inverse Higgsing, or de-Higgsing. This particular point was also emphasised in [7].

One thing that is lacking in most discussions, including ours, of the transition between the round and the squashed vacua, is a way of tracking the transition in a dynamical way. The two endpoints, corresponding to the round and the Einstein-squashed S^7 vacua, have been studied in their own right, and one can give a Higgs interpretation of the states in the squashed vacuum and their relation to the states in the round vacuum, as we have done, for example, in this paper. For some of the mass operators on the S^7, such as the scalar operator Δ_0 and the Dirac operator $\not{D}_{1/2}$, one can follow the evolution of the modes along the family of homogenous squashed spheres interpolating between the round and the Einstein-squashed limits, and this can give useful information about where the modes 'come from' in the transition. But the modes of the 'mass operators' on the intermediately-squashed spheres would presumably play no direct physical role if one were to construct a dynamically-evolving solution of eleven-dimensional supergravity that described the flow from the round to the squashed vacuum, since the actual intermediate metrics would be time-dependent and no longer of a simple $M_4 \times S^7$ direct-product form. Some substantial progress has been made recently in [22], where the domain-wall solution that interpolates between the two vacua was studied within the framework of exceptional field theory. This provides a method whereby in principle the quadratic couplings of all the Kaluza–Klein fluctuations in the domain-wall background could be computed. This opens up the possibility of tracking all the fields as the transition between the vacua proceeds, which would allow a more complete understanding of the Higgsing and de-Higgsing. This lies well beyond the scope of our present work, but it would be very interesting to pursue this line of investigation further.

Under an orientation-reversal of the squashed S^7, the $\mathcal{N}=1$ supersymmetry of the left-squashed vacuum is lost, which seems to introduce certain ambiguities concerning the boundary conditions to be imposed on the different scalar and fermionic fields. For some of the fields it is not clear how to discriminate between Dirichlet and Neumann boundary conditions. In the left-squashed case this issue is easily resolved by appealing to supersymmetry; the singlet fields must necessarily fall into $\mathcal{N}=1$ supermultiplets, but there is no such requirement in the right-squashed case. In fact, although the operator eigenvalues can be followed as a function of the squashing parameter this is probably not the case for the masses since their relation to the operators are not only theory dependent but also valid only in backgrounds that solve the field equations. This also presents a difficulty for following the change in E_0 as a function of the squashing parameter.

We have also discussed the relevance of our results for the issue of stability of the right-squashed non-supersymmetric S^7 compactification. This theory is BF stable but we found in the previous section that one can form marginal operators in the boundary theory. As is clear from table 2 above, this depends on the boundary conditions that are used for some of the singlet fields after skew-whiffing, where some options make it possible to construct marginal triple-trace operator. However, as mentioned at the end of the previous section, by adopting the reasoning of [7], there must exist a fermionic singleton in the right-squashed vacuum, and this excludes two of the four options on the second row of table 2. One of the two options that remain can then not lead to any marginal operators and would provide a theory that could still be completely stable.

In order to check if the marginal operators do indeed generate instabilities one has, however, to compute $1/N$ corrections to some beta-functions in the boundary theory, which is beyond the scope of the present paper. This would, however, be an interesting topic for a future investigation.

Data availability statement

No new data were created or analysed in this study.

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