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Research Article

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Galton–Watson Theta-Processes in a Varying Environment

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Abstract: We consider a special class of Galton–Watson theta-processes in a varying environment fully defined by four parameters, with two of them (θ, r) being fixed over time n, and the other two (a_n, c_n) characterizing the altering reproduction laws. We establish a sequence of transparent limit theorems for the theta-processes with possibly defective reproduction laws. These results may serve as a stepping stone towards incisive general results for the Galton–Watson processes in a varying environment.

Keywords: Branching Process, Varying Environment, Limit Theorem

MSC 2020: 60J80

1 Introduction

The basic version of the Galton–Watson process (GW-process) was conceived as a stochastic model of the population growth or extinction of a single species of individuals [3, 7]. The GW-process $\{Z_n\}_{n\geq 0}$ unfolds in the discrete time setting, with Z_n standing for the population size at the generation n under the assumption that each individual is replaced by a random number of offspring. It is assumed that the offspring numbers are independent random variables having the same distribution $\{p(j)\}_{j\geq 0}$.

By allowing the offspring number distribution $\{p_n(j)\}_{j\geq 0}$ to depend on the generation number *n*, we arrive at the GW-process in a varying environment [4]. This more flexible model is fully described by a sequence of probability generating functions

$$f_n(s) = \sum_{j\geq 0} p_n(j)s^j, \quad 0 \leq s \leq 1, \ n \geq 1$$

Introduce the composition of generating functions

$$F_n(s) = f_1 \circ \cdots \circ f_n(s), \quad 0 \le s \le 1, \ n \ge 1.$$

Given that the GW-process starts at time zero with a single individual, we get

$$E(s^{Z_n}) = F_n(s), \quad P(Z_n = 0) = F_n(0).$$

The state 0 of the GW-process is absorbing and the extinction probability for the modeled population is determined by

$$q = \lim F_n(0)$$

(here and throughout, all limits are taken as $n \to \infty$, unless otherwise specified). In the case of *proper* reproduction laws with $f_n(1) = 1$ for all $n \ge 1$, we get

$$E(Z_n) = F'_n(1) = f'_1(1) \cdots f'_n(1), \quad E(Z_n | Z_n > 0) = \frac{F'_n(1)}{1 - F_n(0)}$$

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In [5], the usual ternary classification of the GW-processes into supercritical, critical, and subcritical processes [1], was adapted to the framework of the varying environment. Given $0 < f'_n(1) < \infty$ for all *n*, it was shown that under a regularity condition (A) in [5], it makes sense to distinguish among four classes of the GW-processes in a varying environment: supercritical, asymptotically degenerate, critical, and subcritical processes. In a more recent paper [10] devoted to the Markov theta-branching processes in a varying environment, the quaternary classification of [5] was further refined into a quinary classification, which can be adapted to the discrete time setting as follows:

- *supercritical case:* q < 1 and $\lim E(Z_n) = \infty$,
- *asymptotically degenerate case:* q < 1 and $\liminf E(Z_n) < \infty$,
- *critical case:* q = 1 and $\lim E(Z_n | Z_n > 0) = \infty$,
- *strictly subcritical case:* q = 1 and a finite lim $E(Z_n | Z_n > 0)$ exists,
- *loosely subcritical case:* q = 1 and $\lim E(Z_n | Z_n > 0)$ does not exist.

Our paper is build upon the properties of a special parametric family of generating functions [9] leading to what will be called here the Galton–Watson theta-processes or GW^{θ} -processes. The remarkable property of the GW^{θ} -processes in a varying environment is that the generating functions $F_n(s)$ have explicit expressions presented in Section 2. An important feature of the GW^{θ} -processes is that they allow for defective reproduction laws. If the generating function $f_i(s)$ is *defective*, in that $f_i(1) < 1$, then $F_n(1) < 1$ for all $n \ge i$. In the defective case [6, 11], a single individual, with probability $1 - f_i(1)$ may force the entire GW-process to visit to an ancillary absorbing state Δ by the observation time n with probability

$$\mathbb{P}(Z_n = \Delta) = 1 - F_n(1).$$

In Sections 3 and 4, we state ten limit theorems for the GW^{θ} -processes in a varying environment. These results are illuminated in Section 5 by ten examples describing different growth and extinction patterns under environmental variation. The proofs are collected in Section 6.

2 Proper and Defective Reproduction Laws

Definition 1. Consider a sequence $(\theta, r, a_n, c_n)_{n \ge 1}$ satisfying one of the following sets of conditions:

- (a) $\theta \in (0, 1], r = 1$, and for $n \ge 1, 0 < a_n < \infty, c_n > 0, c_n \ge 1 a_n$,
- (b) $\theta \in (0, 1], r > 1$, and for $n \ge 1, 0 < a_n < 1, (1 a_n)r^{-\theta} \le c_n \le (1 a_n)(r 1)^{-\theta}$,
- (c) $\theta \in (-1, 0), r = 1$, and for $n \ge 1, 0 < a_n < 1, 0 < c_n \le 1 a_n$,
- (d) $\theta \in (-1,0), r > 1$, and for $n \ge 1, 0 < a_n < 1, (1-a_n)(r-1)^{-\theta} \le c_n \le (1-a_n)r^{-\theta}$,
- (e) $\theta = 0, r = 1$, and for $n \ge 1, 0 < a_n < 1, 0 \le c_n < 1$,
- (f) $\theta = 0, r > 1$, and for $n \ge 1, 0 < a_n < 1, 0 \le c_n \le 1$.

A GW^{θ}-process with parameters (θ , r, a_n , c_n)_{$n \ge 1$} is a GW-process in a varying environment characterized by a sequence of probability generating functions ($f_n(s)$)_{$n \ge 1$} defined by

$$f_n(s) = r - (a_n(r-s)^{-\theta} + c_n)^{-\frac{1}{\theta}}, \quad 0 \le s < r, \qquad f_n(r) = r,$$
(2.1)

for $\theta \neq 0$, and for $\theta = 0$, defined by

$$f_n(s) = r - (r - c_n)^{1 - a_n} (r - s)^{a_n}, \quad 0 \le s \le r.$$
(2.2)

Definition 1 is motivated by [9, Definitions 14.1 and 14.2], which also mentions a trivial case of $\theta = -1$ not included here. Observe that in the setting of varying environment, the key parameters $\theta \in (-1, 1]$ and $r \ge 1$ stay constant over time, while the parameters (a_n, c_n) may vary. The case $\theta = r = 1$ is the well studied case of the linearfractional reproduction law.

This section contains two key lemmas. Lemma 1 gives the explicit expressions for the generating functions $F_n(s)$ in terms of positive constants A_n , C_n , $D_n = D_n(r)$ defined by

$$A_0 = 1, \quad A_n = \prod_{i=1}^n a_i, \quad C_n = \sum_{i=1}^n A_{i-1}c_i, \quad D_n = \prod_{i=1}^n (r-c_i)^{A_{i-1}-A_i}$$

Lemmas 2 presents the asymptotic properties of the constants A_n , C_n , D_n leading to the limit theorems stated in Sections 3 and 4.

Lemma 1. Consider a GW^{θ}-process with parameters (θ , r, a_n , c_n). If $\theta \neq 0$, then

$$F_n(s) = r - (A_n(r-s)^{-\theta} + C_n)^{-\frac{1}{\theta}}, \quad 0 \le s < r, \qquad F_n(r) = r, \quad n \ge 1,$$

and if $\theta = 0$, then

$$F_n(s) = r - (r - s)^{A_n} D_n, \quad 0 \le s \le r, \ n \ge 1.$$

Here:

(a) for $\theta \in (0, 1], r = 1$,

$$0 < A_n < \infty, \quad C_n > 0, \quad C_n \ge 1 - A_n, \quad F_n(1) = 1, \quad F'_n(1) = A_n^{-\frac{1}{\theta}}, \quad n \ge 1,$$

(b) for $\theta \in (0, 1], r > 1$,

$$0 < A_n < 1, \quad (1 - A_n)r^{-\theta} \le C_n \le (1 - A_n)(r - 1)^{-\theta}, \quad F_n(1) \le 1, \quad n \ge 1,$$

with $F_n(1) = 1$ if and only if $c_k = (1 - a_k)(r - 1)^{-\theta}$, $1 \le k \le n$, implying $F'_n(1) = A_n$, (c) for $\theta \in (-1, 0)$, r = 1,

$$0 < A_n < \infty, \quad 0 < C_n \le 1 - A_n, \quad F_n(1) = 1 - C_n^{-\frac{1}{\theta}}, \quad n \ge 1,$$

(d) for $\theta \in (-1, 0), r > 1$,

$$0 < A_n < 1, \quad (1 - A_n)(r - 1)^{-\theta} \le C_n \le (1 - A_n)r^{-\theta}, \quad F_n(1) \le 1, \quad n \ge 1,$$

with $F_n(1) = 1$ if and only if $c_k = (1 - a_k)(r - 1)^{-\theta}$, $1 \le k \le n$, implying $F'_n(1) = A_n$, (e) for $\theta = 0, r = 1$,

$$0 < A_n < 1, \quad 0 < D_n \le 1, \quad F_n(1) = 1, \quad F'_n(1) = \infty, \quad n \ge 1,$$

(f) *for* $\theta = 0, r > 1$,

$$0 < A_n < 1, \quad (r-1)^{1-A_n} \le D_n \le r^{1-A_n}, \quad F_n(1) \le 1, \quad n \ge 1,$$

with
$$F_n(1) = 1$$
 if and only if $c_k = 1, 1 \le k \le n$, implying $F'_n(1) = A_n$

Lemma 2. Denote the limits $A = \lim A_n$, $C = \lim C_n$, $D = \lim D_n$, whenever they exist, whether finite or infinite. (a) If $\theta \in (0, 1]$, r = 1, then $C \in [1, \infty]$, and if $C < \infty$, then $A \in [0, \infty]$.

- (b) If $\theta \in (0, 1]$, r > 1, then $A \in [0, 1)$ and $(1 A)r^{-\theta} \le C \le (1 A)(r 1)^{-\theta}$.
- (c) If $\theta \in (-1, 0)$, r = 1, then $A \in [0, 1)$ and $0 < C \le 1 A$.
- (d) If $\theta \in (-1, 0)$, r > 1, then $A \in [0, 1)$ and $(1 A)(r 1)^{-\theta} \le C \le (1 A)r^{-\theta}$.
- (e) If $\theta = 0$, r = 1, then $A \in [0, 1)$ and $D = \prod_{n \ge 1} (1 c_n)^{A_{n-1} A_n}$ with $D \in [0, 1]$.
- (f) If $\theta = 0, r > 1$, then $A \in [0, 1)$ and $D = \prod_{n \ge 1} (r c_n)^{A_{n-1} A_n}$ with $(r 1)^{1 A} \le D \le r^{1 A}$.

3 Limit Theorems for the Proper GW^{θ} -Processes

Theorems 1–5 deal with the GW^{θ} -process in the case $\theta \in (0, 1]$, r = 1, when by Lemma 1,

$$E(Z_n) = A_n^{-\frac{1}{\theta}}, \quad P(Z_n > 0) = (A_n + C_n)^{-\frac{1}{\theta}}.$$

Putting $B_n = \frac{C_n}{A_n}$, we obtain

$$E(Z_n|Z_n > 0) = (1 + B_n)^{\frac{1}{\theta}}.$$

These five theorems fully cover the five regimes of reproduction in a varying environment and could be summarized as follows. Let $\theta \in (0, 1]$, r = 1,

- given $C < \infty$, the GW^{θ}-process is
 - supercritical if $A_n \rightarrow 0$, see Theorem 1,
 - − asymptotically degenerate if $A_n \rightarrow A \in (0, \infty)$, see Theorem 2,
 - − strictly subcritical if $A_n \rightarrow \infty$, see Theorem 4,

- given $C = \infty$, the GW^{θ}-process is
 - critical if $B_n \rightarrow \infty$, see Theorem 3,
 - − strictly subcritical if $B_n \rightarrow B \in [0, \infty)$, see Theorem 4,
 - loosely subcritical if the $\lim B_n$ does not exist, see Theorem 5.

This section also includes Theorem 6 addressing the proper case $\theta = 0$, r = 1. Notice that Theorem 6 deals with the case of infinite mean values, when the above mentioned quinary classification does not apply.

Theorem 1. Let $\theta \in (0, 1]$, r = 1, and $C < \infty$. If $A_n \to 0$, then $q = 1 - C^{-\frac{1}{\theta}}$ and $A_n^{\frac{1}{\theta}} Z_n$ almost surely converges to a random variable W such that

$$\mathrm{E}(e^{-\lambda W}) = 1 - (\lambda^{-\theta} + C)^{-\frac{1}{\theta}}, \quad \lambda \ge 0.$$

Theorem 2. Let $\theta \in (0, 1]$, r = 1, and $C < \infty$. If $A_n \rightarrow A \in (0, \infty)$, then

$$q = 1 - (A + C)^{-\frac{1}{\theta}}, \qquad \operatorname{E}(Z_n) \to A^{-\frac{1}{\theta}},$$

and Z_n almost surely converges to a random variable Z_∞ such that

$$E(Z_{\infty}) = A^{-\frac{1}{\theta}}, \quad E(s^{Z_{\infty}}) = 1 - (A(1-s)^{-\theta} + C)^{-\frac{1}{\theta}}, \quad 0 \le s \le 1.$$

Theorem 3. Let $\theta \in (0, 1]$, r = 1, and $C = \infty$. If $B_n \to \infty$, then q = 1,

$$\mathbb{P}(Z_n>0)\sim C_n^{-\frac{1}{\theta}},\quad \mathbb{E}(Z_n|Z_n>0)\sim B_n^{\frac{1}{\theta}},$$

and with $\lambda_n = \lambda B_n^{-\frac{1}{\theta}}$,

$$\mathbb{E}(e^{-\lambda_n Z_n}|Z_n>0) \to 1-(1+\lambda^{-\theta})^{-\frac{1}{\theta}}, \quad \lambda \ge 0$$

Theorem 4. Let $\theta \in (0, 1]$ and r = 1. If $A_n \to \infty$ and $B_n \to B \in [0, \infty)$, then q = 1,

$$\mathbb{P}(Z_n > 0) \sim (1+B)^{-\frac{1}{\theta}} A_n^{-\frac{1}{\theta}}, \quad \mathbb{E}(Z_n | Z_n > 0) \to (1+B)^{\frac{1}{\theta}},$$

and

$$E(s^{Z_n}|Z_n > 0) \to 1 - ((1+B)(1-s)^{-\theta} + B + B^2)^{-\frac{1}{\theta}}, \quad 0 \le s \le 1.$$

Theorem 5. Let $\theta \in (0, 1]$, r = 1, and assume that $\lim B_n$ does not exist. Then q = 1 and letting

$$B_{k_n} \to B \in [0,\infty]$$

along a subsequence $k_n \to \infty$, we get: (i) if $B = \infty$, then

$$\begin{split} \mathbb{P}(Z_{k_n} > 0) \sim C_{k_n}^{-\frac{1}{\theta}}, \quad \mathbb{E}(Z_{k_n} | Z_{k_n} > 0) \sim B_{k_n}^{\frac{1}{\theta}}, \\ \lambda B_n^{-\frac{1}{\theta}}, \quad & \\ \mathbb{E}(e^{-\lambda_{k_n} Z_{k_n}} | Z_{k_n} > 0) \to 1 - (1 + \lambda^{-\theta})^{-\frac{1}{\theta}}, \quad \lambda \ge 0, \end{split}$$

(ii) if $B \in [0, \infty)$, then $A_{k_n} \to \infty$,

and with $\lambda_n =$

$$P(Z_{k_n} > 0) \sim (1+B)^{-\frac{1}{\theta}} A_{k_n}^{-\frac{1}{\theta}}, \quad E(Z_{k_n} | Z_{k_n} > 0) \to (1+B)^{\frac{1}{\theta}},$$

and

$$E(s^{Z_{k_n}}|Z_{k_n} > 0) \to 1 - ((1+B)(1-s)^{-\theta} + B + B^2)^{-\frac{1}{\theta}}, \quad 0 \le s \le 1.$$

Theorem 6. Suppose $\theta = 0$ and r = 1. Then $P(Z_n > 0) = D_n$, so that q = 1 - D, with D given by Lemma 2 (e). Furthermore:

(i) if A = 0 and D = 0, then q = 1 and

$$\mathbb{P}(A_n \ln Z_n \le x | Z_n > 0) \to 1 - e^{-x}, \quad x \ge 0,$$

(ii) if A = 0 and D > 0, then q < 1 and

$$P(A_n \ln Z_n \le x) \to 1 - e^{-x} D, \quad x \ge 0$$

(iii) *if* $A \in (0, 1)$ *and* D = 0*, then* q = 1 *and*

$$E(s^{Z_n}|Z_n > 0) \to 1 - (1 - s)^A, \quad 0 \le s \le 1,$$

(iv) if $A \in (0, 1)$ and D > 0, then q < 1 and Z_n almost surely converges to a random variable Z_{∞} such that

$$E(Z_{\infty}) = \infty, \quad E(s^{Z_{\infty}}) = 1 - (1 - s)^A D, \quad 0 \le s \le 1.$$

Remarks

We make the following observations.

- (i) It is a straightforward exercise to check that the above mentioned regularity condition (A) in [5] is valid for the GW^{θ} -process in the case $\theta \in (0, 1]$, r = 1.
- (ii) The limiting distribution obtained in Theorem 3 coincides with that of [12] obtained for the critical GW-processes in a constant environment with a possibly infinite variance for the offspring number.
- (iii) Statement (ii) of Theorem 6 is of the Darling–Seneta-type limit theorem obtained in [2] for GW-processes with infinite mean.
- (iv) Part (iv) of Theorem 6 presents the pattern of limit behavior similar to the asymptotically degenerate regime in the case of infinite mean values. The conditions of Theorem 6 (iv) hold if and only if

$$\sum_{n\geq 1} (1-a_n) < \infty \tag{3.1}$$

and

$$\sum_{n\geq 1} (1-a_n) \ln \frac{1}{1-c_n} < \infty.$$
(3.2)

4 Limit Theorems for the Defective GW^{θ} -Process

In the defective case, there are two kinds of absorption times:

(i) τ_0 the absorption time of the GW^{θ}-process at 0,

(ii) τ_{Δ} the absorption time of the GW^{θ}-process at the state Δ .

Let $\tau = \min(\tau_0, \tau_{\Delta})$ be the absorption time of the GW^{θ} -process either at 0 or at the state Δ . Let us recall that $q = P(\tau_0 < \infty)$ and denote

$$q_{\Delta} = P(\tau_{\Delta} < \infty), \quad Q = P(\tau < \infty) = q + q_{\Delta}.$$

Clearly,

$$P(\tau \le n) = P(\tau_0 \le n) + P(\tau_\Delta \le n) = F_n(0) + 1 - F_n(1),$$

implying

$$P(\tau > n) = F_n(1) - F_n(0).$$

Furthermore,

$$\mathrm{E}(Z_n;\tau_{\Delta}>n)=F_n'(1),\qquad \mathrm{E}(s^{Z_n};\tau_{\Delta}>n)=F_n(s),\quad 0\leq s\leq 1,$$

so that

$$\mathsf{E}(Z_n|\tau > n) = \frac{F'_n(1)}{F_n(1) - F_n(0)}, \qquad \mathsf{E}(s^{Z_n}|\tau > n) = \frac{F_n(s) - F_n(0)}{F_n(1) - F_n(0)}, \quad 0 \le s \le 1.$$

Theorems 7–10 present the transparent asymptotical results on these absorption probabilities and the limit behavior of the GW^{θ} -process in the four defective cases. Corollaries of Theorems 7–9 deal with the proper subcases, where $\tau = \tau_0$. All three corollaries describe a strictly subcritical case, when A = 0, and an asymptotically degenerate case, when $A \in (0, 1)$.

Theorem 7. *Consider the case* $\theta \in (0, 1]$ *,* r > 1*. Then*

$$q = r - (Ar^{-\theta} + C)^{-\frac{1}{\theta}}, \quad q_{\Delta} = 1 - r + (A(r-1)^{-\theta} + C)^{-\frac{1}{\theta}}$$

where $A \in [0, 1)$ and $(1 - A)r^{-\theta} \le C \le (1 - A)(r - 1)^{-\theta}$. (i) If A = 0, then

$$q=1-q_{\Delta}=r-C^{-\frac{1}{\theta}}\in[0,1],$$

so that Q = 1. Furthermore,

$$\begin{split} A_n^{-1} \mathbf{P}(\tau > n) &\to ((r-1)^{-\theta} - r^{-\theta})\theta^{-1}C^{-\frac{1}{\theta}-1}, \\ \mathbf{E}(Z_n | \tau > n) &\to \frac{(r-1)^{-\theta-1}}{(r-1)^{-\theta} - r^{-\theta}}, \quad \mathbf{E}(s^{Z_n} | \tau > n) \to \frac{(r-s)^{-\theta} - r^{-\theta}}{(r-1)^{-\theta} - r^{-\theta}}, \quad 0 \le s \le 1 \end{split}$$

(ii) If $A \in (0, 1)$, then $Q \in [0, 1)$,

$$\mathcal{E}(Z_n; \tau_{\Delta} > n) \to A(A + C(r-1)^{\theta})^{-\frac{1}{\theta}-1}$$

and Z_n almost surely converges to a random variable Z_{∞} taking values in the set { Δ , 0, 1, 2, ... }, with

$$\begin{split} & \mathbb{P}(Z_{\infty} = \Delta) = 1 - r + (A(r-1)^{-\theta} + C)^{-\frac{1}{\theta}}, \\ & \mathbb{E}(s^{Z_{\infty}}; Z_{\infty} \neq \Delta) = r - (A(r-s)^{-\theta} + C)^{-\frac{1}{\theta}}, \quad 0 \le s \le 1. \end{split}$$

Corollary. *Consider the case* $\theta \in (0, 1]$ *,* r > 1 *assuming*

$$c_n = (1 - a_n)(r - 1)^{-\theta}, \quad n \ge 1,$$
(4.1)

so that $C = (1 - A)(r - 1)^{-\theta}$ implying $q_{\Delta} = 0$. (i) If A = 0, then q = 1 with

$$A_n^{-1} \mathbb{P}(Z_n > 0) \to ((r-1)^{-\theta} - r^{-\theta})\theta^{-1}(r-1)^{\theta+1}.$$

Furthermore,

$$\mathsf{E}(Z_n|Z_n > 0) \to \frac{(r-1)^{-\theta-1}}{\theta((r-1)^{-\theta} - r^{-\theta})}, \qquad \mathsf{E}(s^{Z_n}|Z_n > 0) \to \frac{(r-s)^{-\theta} - r^{-\theta}}{(r-1)^{-\theta} - r^{-\theta}}, \quad 0 \le s \le 1.$$

(ii) *If* $A \in (0, 1)$, *then*

$$q = 1 - r + (Ar^{-\theta} + C)^{-\frac{1}{\theta}}, \quad \mathcal{E}(Z_n) \to A$$

so that $q \in (0, 1)$, and Z_n almost surely converges to a random variable Z_{∞} such that

$$E(Z_{\infty}) = A,$$
 $E(s^{Z_{\infty}}) = r - (A(r-s)^{-\theta} + (1-A)(r-1)^{-\theta})^{-\frac{1}{\theta}}, \quad 0 \le s \le 1.$

Theorem 8. Consider the case $\theta \in (-1, 0)$, r > 1 and put $\alpha = -\frac{1}{\theta}$, so that $\alpha > 1$. Then

$$q = r - (Ar^{\frac{1}{\alpha}} + C)^{\alpha}, \quad q_{\Delta} = 1 - r + (A(r-1)^{\frac{1}{\alpha}} + C)^{\alpha}$$

where $A \in [0, 1)$ and $(1 - A)(r - 1)^{\frac{1}{\alpha}} \le C \le (1 - A)r^{\frac{1}{\alpha}}$. (i) If A = 0, then

$$q=1-q_{\Delta}=r-C^{\alpha}\in[0,1],$$

so that Q = 1. Furthermore,

$$\begin{split} &A_n^{-1} \mathbb{P}(\tau > n) \to \alpha C^{\alpha - 1}(r^{\frac{1}{\alpha}} - (r - 1)^{\frac{1}{\alpha}}), \\ &\mathbb{E}(Z_n | \tau > n) \to \frac{(r - 1)^{\frac{1}{\alpha} - 1}}{r^{\frac{1}{\alpha}} - (r - 1)^{\frac{1}{\alpha}}}, \quad \mathbb{E}(s^{Z_n} | \tau > n) \to \frac{r^{\frac{1}{\alpha}} - (r - s)^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}} - (r - 1)^{\frac{1}{\alpha}}}, \quad 0 \le s \le 1. \end{split}$$

(ii) If $A \in (0, 1)$, then $Q \in [0, 1)$,

$$\mathbb{E}(Z_n; \tau_{\Delta} > n) \to A(A + C(r-1)^{-\frac{1}{\alpha}})^{\alpha-1}$$

and Z_n almost surely converges to a random variable Z_{∞} taking values in the set { Δ , 0, 1, 2, ... }, with

$$\begin{split} & \mathbb{P}(Z_{\infty} = \Delta) = 1 - r + (A(r-1)^{\frac{1}{\alpha}} + C)^{\alpha}, \\ & \mathbb{E}(s^{Z_{\infty}}; Z_{\infty} \neq \Delta) = r - (A(r-s)^{\frac{1}{\alpha}} + C)^{\alpha}, \quad 0 \le s \le 1. \end{split}$$

Corollary. Consider the case $\theta \in (-1, 0)$, r > 1 assuming (4.1), so that $C = (1 - A)(r - 1)^{\frac{1}{a}}$ implying $q_{\Delta} = 0$. (i) If A = 0, then q = 1 with

$$A_n^{-1} \mathbb{P}(Z_n > 0) \to \alpha (r-1)^{1-\frac{1}{\alpha}} (r^{\frac{1}{\alpha}} - (r-1)^{\frac{1}{\alpha}}).$$

Furthermore,

$$\mathsf{E}(Z_n|Z_n > 0) \to \frac{(r-1)^{-\theta-1}}{\theta((r-1)^{-\theta} - r^{-\theta})}, \qquad \mathsf{E}(s^{Z_n}|Z_n > 0) \to \frac{(r-s)^{-\theta} - r^{-\theta}}{(r-1)^{-\theta} - r^{-\theta}}, \quad 0 \le s \le 1.$$

(ii) *If* $A \in (0, 1)$, *then*

$$q = 1 - r + (Ar^{\frac{1}{\alpha}} + (1 - A)(r - 1)^{\frac{1}{\alpha}})^{\alpha}, \quad \mathcal{E}(Z_n) \to A,$$

so that $q \in (0, 1)$, and Z_n almost surely converges to a random variable Z_∞ such that

$$E(Z_{\infty}) = A,$$
 $E(s^{Z_{\infty}}) = r - (A(r-s)^{\frac{1}{\alpha}} + (1-A)(r-1)^{\frac{1}{\alpha}})^{\alpha}, \quad 0 \le s \le 1.$

Theorem 9. Consider the case $\theta = 0$, r > 1 implying

$$q = r - r^A D$$
, $q_{\Delta} = 1 - r + (r - 1)^A D$, $Q = 1 - (r^A - (r - 1)^A) D$

where D is given by Lemma 2 (f).

(i) *If* A = 0, *then* Q = 1, *and*

$$\mathbf{P}(\tau > n) \sim (\ln r - \ln(r-1))A_n D_n.$$

Moreover,

$$E(Z_n|\tau > n) \to \frac{(r-1)^{-1}}{\ln r - \ln(r-1)}, \qquad P(s^{Z_n}|\tau > n) \to \frac{\ln r - \ln(r-s)}{\ln r - \ln(r-1)}, \quad 0 \le s \le 1.$$

(ii) If $A \in (0, 1)$, then Q < 1,

$$(r-1)^{1-A} \le D \le r^{1-A}, \quad \mathcal{E}(Z_n; \tau_\Delta > n) \to A(r-1)^{A-1}D,$$

and Z_n almost surely converges to a random variable Z_{∞} taking values in the set { Δ , 0, 1, 2, ...}, with

$$\mathbb{P}(Z_{\infty} = \Delta) = 1 - r + (r - 1)^{A}D, \qquad \mathbb{E}(s^{Z_{\infty}}; Z_{\infty} \neq \Delta) = r - (r - s)^{A}D, \quad 0 \le s \le 1.$$

Corollary. Given $\theta = 0$, r > 1, assume $c_n \equiv 1$. Then $D = (r - 1)^{1-A}$ implying $q_{\Delta} = 0$. (i) If A = 0, then q = 1, and

q

$$P(Z_n > 0) \sim (\ln r - \ln(r - 1))A_n D_n$$

Moreover,

$$\mathsf{E}(Z_n|Z_n>0)\to \frac{(r-1)^{-1}}{\ln r - \ln(r-1)}, \qquad \mathsf{P}(s^{Z_n}|Z_n>0)\to \frac{\ln r - \ln(r-s)}{\ln r - \ln(r-1)}, \quad 0\le s\le 1.$$

(ii) If $A \in (0, 1)$, then

$$= r - r^A (r-1)^{1-A}, \quad \operatorname{E}(Z_n) \to A$$

so that $q \in (0, 1)$, and Z_n almost surely converges to a proper random variable Z_{∞} , such that

$$E(Z_{\infty}) = A, \qquad E(s^{Z_{\infty}}) = r - (r - s)^A (r - 1)^{1 - A}, \quad 0 \le s \le 1.$$

Theorem 10. In the case $\theta \in (-1, 0)$, r = 1, put $\alpha = -\frac{1}{\theta}$, so that $\alpha > 1$. Then

$$q = 1 - (A + C)^{\alpha}, \quad q_{\Delta} = C^{\alpha}, \quad Q = 1 - (A + C)^{\alpha} + C^{\alpha},$$

where $A \in [0, 1)$ and $0 < C \le 1 - A$. (i) If A = 0, then $q = 1 - q_{\Delta} = 1 - C^{\alpha}$, Q = 1, and

$$A_n^{-1}\mathbb{P}(\tau > n) \to \alpha C^{\alpha - 1}.$$

Moreover,

$$E(s^{Z_n}|\tau > n) \to 1 - (1-s)^{\frac{1}{\alpha}}, \quad 0 \le s \le 1.$$

(ii) If $A \in (0, 1)$, then Q < 1,

$$\mathrm{E}(Z_n;\tau_{\Delta}>n)=\infty,$$

and Z_n almost surely converges to a random variable Z_{∞} taking values in the set { Δ , 0, 1, 2, ...}, with

$$\mathbb{P}(Z_{\infty} = \Delta) = C^{\alpha}, \qquad \mathbb{E}(s^{Z_{\infty}}; Z_{\infty} \neq \Delta) = 1 - (A(1-s)^{\frac{1}{\alpha}} + C)^{\alpha}, \quad 0 \le s \le 1.$$

Remarks

We make the following observations.

- (i) Theorem 7 (ii) should be compared to the more general [6, Theorem 1], which allows the limit Z_{∞} to take the value ∞ with a positive probability. The convergence results for the conditional expectation should be compared to the statements of [6, Theorems 3 and 4].
- (ii) The conditional convergence in distribution stated in Theorem 7 (i) should be compared to [11, Theorem 2a (k = 0)] in the more general setting under the assumption of constant environment.

5 Examples

The following ten examples illustrate each of the ten theorems of this paper. Observe that given

$$c_n = (1 - a_n)\sigma, \quad n \ge 1, \tag{5.1}$$

for some suitable positive constant σ , we get $C_n = (1 - A_n)\sigma$, $n \ge 1$. Similarly, if

$$c_n = (a_n - 1)\sigma, \quad n \ge 1, \tag{5.2}$$

for some suitable positive constant σ , then $C_n = (A_n - 1)\sigma$, $n \ge 1$.

Example 1. Suppose $\theta \in (0, 1]$, r = 1, and

$$a_n = \frac{n}{n+1}, \quad A_n = \frac{1}{n+1}, \quad n \ge 1.$$
 (5.3)

If (5.1) holds for some $\sigma \ge 1$, then by Theorem 1,

$$q = 1 - \sigma^{-\frac{1}{\theta}}, \quad n^{-\frac{1}{\theta}} \mathcal{E}(Z_n) \to 1,$$

and $n^{-\frac{1}{\theta}}Z_n \to W$ almost surely, with

$$\mathrm{E}(e^{-\lambda W}) = 1 - (\lambda^{-\theta} + \sigma)^{-\frac{1}{\theta}}, \quad \lambda \ge 0.$$

Example 2. Suppose $\theta \in (0, 1]$, r = 1, and

$$a_n = \frac{n(n+3)}{(n+1)(n+2)}, \quad A_n = \frac{n+3}{3(n+1)}, \quad n \ge 1.$$
 (5.4)

If (5.1) holds for some $\sigma \ge 1$, then by Theorem 2,

$$q = 1 - \left(\frac{3}{1+2\sigma}\right)^{\frac{1}{\theta}}, \quad \mathrm{E}(Z_n) \to 3^{\frac{1}{\theta}},$$

and $Z_n \rightarrow Z_\infty$ almost surely, with

$$E(Z_{\infty}) = 3^{\frac{1}{\theta}}, \qquad E(s^{Z_{\infty}}) = 1 - 3^{\frac{1}{\theta}}(2\sigma + (1-s)^{-\theta})^{-\frac{1}{\theta}}, \quad 0 \le s \le 1.$$

Example 3. Suppose $\theta \in (0, 1]$ and r = 1. Let

$$a_1 = \frac{1}{2},$$
 $a_{2n} = 4,$ $a_{2n+1} = \frac{1}{4},$
 $c_{2n-1} = 1,$ $c_{2n} = 2,$
 $A_{2n-1} = \frac{1}{2},$ $A_{2n} = 2,$ $n \ge 1$

Then $C = \infty$ and $B_n \to \infty$ implying the conditions of Theorem 3. Observe that for this example, $\lim A_n$ does not exist.

Example 4. Suppose $\theta \in (0, 1]$ and r = 1. Recall that Theorem 4 is the only one among Theorems 1–5 which may hold both with $C < \infty$ and $C = \infty$. For this reason, we present two examples (1) and (2) for each of these two situations:

(1) Let

$$a_n=\frac{n+1}{n},\quad c_n=\frac{1}{n^2(n+1)},\quad n\geq 1,$$

implying

$$A_n = n + 1, \quad C_n = \frac{n}{n+1}, \quad B_n = \frac{n}{(n+1)^2}, \quad n \ge 1$$

In this case, according to Theorem 4,

$$\mathbb{P}(Z_n > 0) \sim n^{-\frac{1}{\theta}}, \quad \mathbb{E}(Z_n | Z_n > 0) \to 1$$

and

$$\mathcal{E}(s^{Z_n}|Z_n>0)\to s, \quad 0\leq s\leq 1$$

(2) Let

$$a_n=\frac{n+1}{n}, \quad A_n=n+1, \quad n\geq 1,$$

and (5.2) hold for some σ > 0. Then

$$C_n = \sigma n, \quad B_n = \frac{\sigma n}{n+1}, \quad n \ge 1.$$

In this case, according to Theorem 4,

$$\mathbb{P}(Z_n > 0) \sim (1 + \sigma)^{-\frac{1}{\theta}} n^{-\frac{1}{\theta}}, \quad \mathbb{E}(Z_n | Z_n > 0) \rightarrow (1 + \sigma)^{\frac{1}{\theta}},$$

and

$$E(s^{Z_n}|Z_n > 0) \to 1 - ((1 + \sigma)(1 - s)^{-\theta} + \sigma + \sigma^2)^{-\frac{1}{\theta}}, \quad 0 \le s \le 1.$$

Example 5. Suppose $\theta \in (0, 1]$ and r = 1. Let

$$a_{n} = \begin{cases} n & \text{for } n = 2^{k} - 1, k \ge 1, \\ \frac{1}{n-1} & \text{for } n = 2^{k}, k \ge 1, \\ 1 & \text{otherwise,} \end{cases} \quad A_{n} = \begin{cases} n & \text{for } n = 2^{k} - 1, k \ge 1, \\ 1 & \text{otherwise.} \end{cases}$$

Taking

$$c_n = \begin{cases} 1 & \text{for } n = 2^k, k > 1, \\ \frac{1}{n^2} & \text{otherwise,} \end{cases}$$

we get

$$C_n = \sum_{k: \ 2 \le 2^k \le n} (2^k - 1 - 2^{-2k}) + \sum_{k=1}^n k^{-2}, \quad n \ge 1,$$

implying $C_{k_n} \sim 2^{n+1}$, provided $2^n - 1 \le k_n < 2^{n+1} - 1$. Thus, by Theorem 5, for $k_n = 2^n$, $\lambda_n = \lambda(2n)^{-\frac{1}{\theta}}$,

$$\mathbb{P}(Z_{k_n} > 0) \sim (2k_n)^{-\frac{1}{\theta}}, \qquad \mathbb{E}(e^{-\lambda_{k_n} Z_{k_n}} | Z_{k_n} > 0) \rightarrow 1 - (1 + \lambda^{-\theta})^{-\frac{1}{\theta}}, \quad \lambda \ge 0,$$

and on the other hand, for $k_n = 2^n - 1$,

$$\mathbb{P}(Z_{k_n} > 0) \sim (3k_n)^{-\frac{1}{\theta}}, \qquad \mathbb{E}(s^{Z_{k_n}} | Z_{k_n} > 0) \to 1 - (3(1-s)^{-\theta} + 6)^{-\frac{1}{\theta}}, \quad 0 \le s \le 1.$$

Example 6. Suppose $\theta = 0$, r = 1, and assume $c_n = 1 - e^{-n^{\sigma}}$, $-\infty < \sigma < \infty$, $n \ge 1$, yielding

$$D_n = \exp\left(-\sum_{i=1}^n i^{\sigma}(A_{i-1} - A_i)\right), \quad n \ge 1.$$

Notice that (5.3) implies A = 0 and

$$D_n = \exp\left(-\sum_{i=1}^n \frac{i^{\sigma-1}}{i+1}\right), \quad n \ge 1,$$

on the other hand, (5.4) implies $A = \frac{1}{3}$ and

$$D_n = \exp\left(-\sum_{i=1}^n \frac{2i^{\sigma-1}}{3(i+1)}\right), \quad n \ge 1.$$

(i) If (5.3) holds and $\sigma \ge 1$, then

$$A_n \sim n^{-1}, \quad D_n = \exp\left(-\sum_{i=1}^n \frac{i^{\sigma-1}}{i+1}\right) \to 0,$$

so that the conditions of Theorem 6 (i) are satisfied.

(ii) If (5.3) holds and σ < 1, then

$$A_n \sim n^{-1}, \quad D = \exp\left(-\sum_{i=1}^{\infty} \frac{i^{\sigma-1}}{i+1}\right),$$

so that the conditions of Theorem 6 (ii) are satisfied.

(iii) If (5.4) holds and $\sigma \ge$ 1, then

$$A = \frac{1}{3}, \quad D_n = \exp\left(-\sum_{i=1}^n \frac{2i^{\sigma-1}}{3(i+1)}\right) \to 0,$$

so that the conditions of Theorem 6 (iii) are satisfied.

(iv) If (5.4) holds and σ < 1, then

$$A = \frac{1}{3}, \quad D = \exp\left(-\sum_{i=1}^{\infty} \frac{2i^{\sigma-1}}{3(i+1)}\right),$$

so that the conditions of Theorem 6 (iv) are satisfied.

Example 7. Suppose $\theta \in (0, 1]$, r > 1 assuming (5.1) with $r^{-\theta} \le \sigma \le (r - 1)^{-\theta}$. (i) If (5.3), then the conditions of Theorem 7 (i) hold with $A_n \sim n^{-1}$ and $C = \sigma$. (ii) If (5.4), then the conditions of Theorem 7 (ii) hold with $A = \frac{1}{3}$ and $C = \frac{2\sigma}{3}$.

Example 8. Suppose $\theta \in (-1, 0)$, r > 1 assuming (5.1) with $r - 1 \le \sigma^{\alpha} \le r$, where $\alpha = -\frac{1}{\theta}$. (i) If (5.3), then the conditions of Theorem 8 (i) hold with $A_n \sim n^{-1}$ and $C = \sigma$. (ii) If (5.4), then the conditions of Theorem 8 (ii) hold with $A = \frac{1}{3}$ and $C = \frac{2\sigma}{3}$.

Example 9. Suppose $\theta = 0$ and r > 1 and assume

$$c_n = \sigma$$
, $0 \le \sigma \le 1$, $n \ge 1$

which implies

$$D_n = (r - \sigma)^{1 - A_n}, \quad n \ge 1.$$

(i) If (5.3), then by Theorem 9 (i), we get in particular,

$$\mathbb{P}(\tau > n) \sim \gamma n^{-1}, \quad \gamma = (r - \sigma) \ln \frac{r}{r - 1}.$$

(ii) If (5.3), then by Theorem 9 (ii), we get in particular,

$$q = r - r^{\frac{1}{3}}(r - \sigma)^{\frac{2}{3}}, \quad q_{\Delta} = 1 - r + (r - 1)^{\frac{1}{3}}(r - \sigma)^{\frac{2}{3}}, \quad Q = 1 - (r^{\frac{1}{3}} - (r - 1)^{\frac{1}{3}})(r - \sigma)^{\frac{2}{3}}.$$

Example 10. Suppose $\theta \in (-1, 0)$, r = 1. Put $\alpha = -\frac{1}{\theta}$ and assume (5.1) with $0 < \sigma \le 1$. (i) If (5.3), then by Theorem 10 (i), we get in particular, $q_{\Delta} = \sigma^{\alpha}$ and

$$\mathbb{P}(\tau > n) \sim \alpha \sigma^{\alpha - 1} n^{-1}.$$

(ii) If (5.4), then by Theorem 10 (ii), we get in particular, $Q = 1 - (\frac{1}{3} + \frac{2\sigma}{3})^{\alpha} + \frac{2\sigma}{3}^{\alpha}$.

6 Proofs

In this section we sketch the proofs of lemmas and theorems of this paper. The corollaries to Theorems 7–9 are easily obtained from the corresponding theorems.

Proof of Lemma 1

Relations (2.1) and (2.2) imply respectively

$$(r - f_k \circ f_{k+1}(s))^{-\theta} = a_k (r - f_{k+1}(s))^{-\theta} + c_k = a_k a_{k+1} (r - s)^{-\theta} + c_k + a_k c_{k+1},$$

and

$$r - f_k \circ f_{k+1}(s) = (r - c_k)^{1 - a_k} (r - f_{k+1}(s))^{a_k} = (r - c_k)^{1 - a_k} (r - c_{k+1})^{(1 - a_{k+1})a_k} (r - s)^{a_k a_{k+1}}$$

entailing the main claims of Lemma 1. Parts (a)–(f) follow from the respective restrictions (a)–(f) on (a_n, c_n) stated in Definition 1.

Proof of Lemma 2

(a) In the case $\theta \in (0, 1]$, r = 1, the claim follows from the existence of $\lim C_n$ and $\lim (A_n + C_n)$, which in turn, follows from monotonicity of the two sequences. To see that $A_n + C_n \le A_{n+1} + C_{n+1}$, it suffices to observe that

$$A_n - A_{n+1} = A_n(1 - a_{n+1}) \le A_n c_{n+1} = C_{n+1} - C_n.$$

The second part of Lemma 2 is a direct implication of the definition of C_n .

(b)–(f) The rest of the stated results follows immediately from the restrictions (b)–(f) imposed on (a_n, c_n) in Definition 1.

Church–Lindvall Condition for the GW^{θ} -Process

In [8] it was shown for the GW-processes in a varying environment that the almost surely convergence $Z_n \xrightarrow{a.s.} Z_{\infty}$ holds with $P(0 < Z_{\infty} < \infty) > 0$ if and only if the following condition holds:

i

$$\sum_{n\geq 1} (1 - p_n(1)) < \infty.$$
(6.1)

Relation (6.1) is equivalent to

$$\prod_{n\geq n_0} p_n(1) > 0 \tag{6.2}$$

for some $n_0 \ge 1$. For the GW^{θ}-process, the equality $p_n(1) = f'_n(0)$ implies

$$p_n(1) = a_n (a_n + c_n r^{\theta})^{-\frac{1}{\theta} - 1}$$
(6.3)

for $\theta \neq 0$, and for $\theta = 0$,

$$p_n(1) = a_n(1 - c_n r^{-1})^{1 - a_n}.$$
(6.4)

Lemma 3. In the case $\theta \in (0, 1]$ and r = 1, relation (6.1) holds if and only if

$$A_n \to A \in (0, \infty) \tag{6.5}$$

and

$$\sum_{n\geq 1}c_n<\infty.$$
(6.6)

Proof. In view of (6.3), we have

$$\prod_{i=1}^{n} p_i(1) = A_n G_n^{-\frac{1}{\theta}-1}, \quad G_n := \prod_{i=1}^{n} (a_i + c_i).$$

Since $a_n + c_n \ge 1$, we have

$$\lim G_n = G \in [1, \infty].$$

If $G = \infty$, then (6.2) is not valid, implying that (6.1) is equivalent to (6.5) plus $G < \infty$. It remains to verify that under (6.5), the inequality $G < \infty$ is equivalent to (6.6). Suppose (6.5) holds, and observe that in this case, $G < \infty$ is equivalent to

$$\prod_{n\geq 1}\left(1+\frac{c_n}{a_n}\right)<\infty,$$

which is true if and only if

$$\sum_{n\geq 1}\frac{c_n}{a_n}<\infty.$$

Since under (6.5), $a_n \rightarrow 1$, the latter condition is equivalent to (6.6).

Lemma 4. In the case $\theta = 0$ and r = 1, relation (6.1) holds if and only if $A \in (0, 1)$ and $D \in (0, 1)$.

Proof. In view of (6.4), we have

$$\prod_{n\geq 1} p_n(1) = A \prod_{n\geq 1} (1-c_n)^{1-a_n}$$

It remains to observe that given $A \in (0, 1)$ the relation $D \in (0, 1)$ is equivalent to

$$\prod_{n\geq 1}(1-c_n)^{1-a_n}>0.$$

Lemma 5. Assume that $\theta \neq 0$ and r > 1, and consider $\{\tilde{Z}_n\}$, a GW-process in a varying environment with the proper probability generating functions

$$\tilde{f}_n(s) = \frac{f_n(s)}{f_n(1)} = \frac{r - (a_n(r-s)^{-\theta} + c_n)^{-\frac{1}{\theta}}}{r - (a_n(r-1)^{-\theta} + c_n)^{-\frac{1}{\theta}}}.$$

Relation (3.1) implies

$$\sum_{n=1}^{\infty} (1 - \tilde{p}_n(1)) < \infty.$$

Proof. Assume $\theta \in (0, 1]$ and r > 1 together with (3.1). Then $A_n \to A \in (0, 1)$, $a_n \to 1$, and $c_n \to 0$. We have

$$\tilde{p}_n(1) = \tilde{f}'_n(0) = a_n h_n^{-\frac{1}{\theta}-1} k_n^{-1},$$

where

$$h_n = a_n + c_n r^{\theta}, \quad k_n = r - (a_n (r-1)^{-\theta} + c_n)^{-\theta}$$

are such that $h_n \ge 1$ and $k_n \in (0, 1]$. The statement follows from the representation

$$\prod_{n\geq 1}p_n(1)=AH^{-\frac{1}{\theta}-1}K^{-1},$$

where $H = \prod_{n \ge 1} h_n$ and $K = \prod_{n \ge 1} k_n$. It is easy to show that (3.1) and $(1 - a_n)r^{-\theta} \le c_n \le (1 - a_n)(r - 1)^{-\theta}$ yield

$$\sum_{n\geq 1}(h_n-1)\leq r^{\theta}\sum_{n\geq 1}c_n\leq r^{\theta}(r-1)^{-\theta}\sum_{n\geq 1}(1-a_n)<\infty,$$

implying $H \in [1, \infty)$. On the other hand, $K \in (0, 1]$, since

$$\sum_{n\geq 1}(1-k_n)<\infty,$$

which follows from

$$1-k_n \leq (r-1)(a_n+c_n(r-1)^{\theta})^{-\frac{1}{\theta}}-1) \leq r(1-(a_n+c_n(r-1)^{\theta})^{\frac{1}{\theta}}) \leq r\theta^{-1}(1-a_n).$$

In the other case, when (3.1) holds together with $\theta \in (-1, 0)$ and r > 1, the lemma is proven similarly.

Proof of Theorems 1–5

The proofs of these theorems are done using the usual for these kind of results arguments applied to the explicit expressions available for $F_n(s)$. In particular, the following standard formula is a starting point for computing the conditional limit distributions:

$$E(s^{Z_n}|Z_n > 0) = \frac{E(s^{Z_n}) - P(Z_n = 0)}{P(Z_n > 0)} = 1 - \frac{1 - F_n(s)}{1 - F_n(0)}.$$
(6.7)

Thus in the case $\theta \in (0, 1]$ and r > 1, Lemma 1 and (6.7) imply

$$\mathbf{E}(s^{Z_n}|Z_n>0) = 1 - \frac{((1-s)^{-\theta} + B_n)^{-\frac{1}{\theta}}}{(1+B_n)^{-\frac{1}{\theta}}} \to 1 - \frac{((1-s)^{-\theta} + B)^{-\frac{1}{\theta}}}{(1+B)^{-\frac{1}{\theta}}},$$

proving the main statement of Theorem 4. The almost sure convergence stated in Theorem 2 follows from Lemma 3 and the earlier cited criterium of [8].

Proof of Theorem 6

Suppose
$$\theta = 0$$
, $r = 1$, in which case $A \in [0, 1)$ and $D \in [0, 1]$.
(i) Suppose $A = D = 0$. In this case $q = 1 - D = 1$, and by (6.7) and Lemma 1,

$$E(s^{Z_n}|Z_n > 0) = 1 - (1 - s)^{A_n}.$$

Putting here $s_n = \exp(-\lambda e^{-\frac{\chi}{A_n}})$, we get as $n \to \infty$,

$$\mathsf{E}(s_n^{Z_n}|Z_n>0) = 1 - (1 - \exp(-\lambda e^{-\frac{\chi}{A_n}}))^{A_n} = 1 - \exp(A_n \ln(\lambda e^{-\frac{\chi}{A_n}}(1 + o(1)))) \to 1 - e^{-\chi}.$$

This implies a convergence in distribution

$$(Z_n e^{-\frac{X}{A_n}} | Z_n > 0) \stackrel{\mathrm{d}}{\to} W(X),$$

where the limit W(x) has a degenerate distribution with

$$P(W(x) \le w) = (1 - e^{-x}) \mathbb{1}_{\{0 \le w < \infty\}}.$$

In other words,

$$\mathbb{P}(Z_n \le w e^{\frac{X}{A_n}} | Z_n > 0) \to (1 - e^{-X}) \mathbb{1}_{\{0 \le w < \infty\}}.$$

After taking the logarithm of Z_n , we arrive at the statement of Theorem 6 (i).

(ii) Statement (ii) follows from Lemma 1 and relation (6.7) in a similar way as statement (i).

(iii) If $A \in (0, 1)$ and D = 0, then q = 1 and by relation (6.7) and Lemma 1,

$$E(s^{Z_n}|Z_n > 0) = 1 - (1 - s)^{A_n} \to 1 - (1 - s)^A.$$

(iv) Let A > 0 and D > 0. Since q = 1 - D, similarly to part (iii), we obtain

$$\mathcal{E}(s^{\mathbb{Z}_n}) \to 1 - (1-s)^A D.$$

By Lemma 4, the convergence in distribution $Z_n \xrightarrow{d} Z_{\infty}$ can be upgraded to the almost surely convergence $Z_n \xrightarrow{a.s.} Z_{\infty}$.

Proof of Theorems 7 and 8

In this section we prove only Theorem 7. Theorem 8 is proven similarly.

By Lemma 1,

$$F_n(0) = r - (A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}}, \quad F_n(1) = r - (A_n (r-1)^{-\theta} + C_n)^{-\frac{1}{\theta}}.$$

It follows that

$$\begin{split} \mathsf{P}(\tau > n) &= (A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}} - (A_n (r-1)^{-\theta} + C_n)^{-\frac{1}{\theta}}, \\ \mathsf{E}(Z_n | \tau > n) &= \frac{F'_n(1)}{F_n(1) - F_n(0)} = \frac{\theta^{-1} A_n (A_n + C_n (r-1)^{\theta})^{-\frac{1}{\theta}-1}}{(A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}} - (A_n (r-1)^{-\theta} + C_n)^{-\frac{1}{\theta}}}, \\ \mathsf{E}(s^{Z_n} | \tau > n) &= \frac{F_n(s) - F_n(0)}{F_n(1) - F_n(0)} = \frac{(A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}} - (A_n (r-s)^{-\theta} + C_n)^{-\frac{1}{\theta}}}{(A_n r^{-\theta} + C_n)^{-\frac{1}{\theta}} - (A_n (r-1)^{-\theta} + C_n)^{-\frac{1}{\theta}}}, \end{split}$$

(i) Assume that A = 0. Then the sequence of positive numbers

$$V_n = A_n^{-1}(C - C_n) = c_{n+1} + c_{n+2}a_{n+1} + c_{n+3}a_{n+2}a_{n+1} + \cdots$$

satisfies

$$r^{-\theta} \leq \liminf V_n \leq \limsup V_n \leq (r-1)^{-\theta}$$

For a given $x \in (0, \infty)$, put

$$W_n(x) = A_n^{-1} (C^{-\frac{1}{\theta}} - (A_n x + C_n)^{-\frac{1}{\theta}})$$

Since

$$W_n(x) = A_n^{-1}(C^{-\frac{1}{\theta}} - (A_n(x - V_n) + C)^{-\frac{1}{\theta}}) = \theta^{-1}C^{-\frac{1}{\theta}-1}(x - V_n + o(1)),$$

the representation

$$A_n^{-1} \mathbb{P}(\tau > n) = W_n((r-1)^{-\theta}) - W_n(r^{-\theta})$$

yields the first asymptotic result stated in part (i) of Theorem 7. The other two asymptotic results follow from the representations

$$E(Z_{n}|\tau > n) = \frac{\theta^{-1}(A_{n} + C_{n}(r-1)^{\theta})^{-\frac{1}{\theta}-1}}{W_{n}((r-1)^{-\theta}) - W_{n}(r^{-\theta})},$$

$$E(s^{Z_{n}}|\tau > n) = \frac{W_{n}((r-s)^{-\theta}) - W_{n}(r^{-\theta})}{W_{n}((r-1)^{-\theta}) - W_{n}(r^{-\theta})}.$$

(ii) The second claim follows from the equality

$$E(s^{Z_n}; \tau_{\Delta} > n) = r - (A_n(r-s)^{-\theta} + C_n)^{-\frac{1}{\theta}}.$$

Proof of Theorem 9

If $\theta = 0$ and r > 1, then by Lemma 1

$$P(Z_n = 0) = r - r^{A_n} D_n, \quad P(Z_n \neq \Delta) = r - (r - 1)^{A_n} D_n.$$

It follows that

$$q = r - r^A D$$
, $q_{\Delta} = 1 - r + (r - 1)^A D$, $Q = 1 - (r^A - (r - 1)^A) D$.

(i) If A = 0, then clearly

$$q=r-D, \quad q_{\Delta}=1-r+D, \quad Q=1,$$

and

$$\mathbb{P}(\tau > n) = (r^{A_n} - (r-1)^{A_n})D_n \sim (\ln r - \ln(r-1))A_nD_n.$$

Furthermore,

$$\begin{split} \mathsf{E}(Z_n|\tau>n) &= \frac{F_n'(1)}{F_n(1) - F_n(0)} = \frac{A_n(r-1)^{A_n-1}}{r^{A_n} - (r-1)^{A_n}} \to \frac{(r-1)^{-1}}{\ln r - \ln(r-1)},\\ \mathsf{P}(s^{Z_n}|\tau>n) &= \frac{F_n(s) - F_n(0)}{F_n(1) - F_n(0)} = \frac{r^{A_n} - (r-s)^{A_n}}{r^{A_n} - (r-1)^{A_n}} \to \frac{\ln r - \ln(r-s)}{\ln r - \ln(r-1)}. \end{split}$$

(ii) In the case A > 0, the main claim is obtained as

$$\mathrm{E}(s^{Z_n};\tau_{\Delta}>n)=r-(r-s)^{A_n}D_n\to r-(r-s)^AD.$$

Proof of Theorem 10

If $\theta \in (-1, 0)$ and r = 1, then by Lemmas 1 and 2,

$$F_n(0) = 1 - (A_n + C_n)^{\alpha}, \quad F_n(1) = 1 - C_n^{\alpha}$$

and

$$q = 1 - (A + C)^{\alpha}, \quad q_{\Delta} = C^{\alpha},$$

where $\alpha = -\frac{1}{\theta}$ and $0 < C \le 1 - A$.

(i) Suppose A = 0. Then the sequence of positive numbers $V_n = A_n^{-1}(C - C_n)$ satisfies

$$0 \leq \liminf V_n \leq \limsup V_n \leq 1$$

For a given $x \in (0, \infty)$, put

$$W_n(x) = A_n^{-1}((A_n x + C_n)^{\alpha} - C^{\alpha})$$

Since

$$W_n(x) = \alpha C^{\alpha-1}(x - V_n + o(1)),$$

the representation

$$A_n^{-1} \mathbf{P}(\tau > n) = W_n(1) - W_n(0)$$

yields the first asymptotic result stated in part (i) of Theorem 10. The other asymptotic result follows from the representation

$$\mathcal{E}(s^{Z_n}|\tau > n) = \frac{W_n(1) - W_n((1-s)^{\frac{1}{\alpha}})}{W_n(1) - W_n(0)}$$

(ii) Claim (ii) is derived as

$$\mathbb{P}(s^{Z_n}; \tau > n) = 1 - (A_n(1-s)^{\frac{1}{\alpha}} + C_n)^{\alpha} \to 1 - (A(1-s)^{\frac{1}{\alpha}} + C)^{\alpha}.$$

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