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# Mean-square exponential stabilization of coupled hyperbolic systems with random parameters

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**Abstract:**
In this paper, we consider a system of two coupled scalar-valued hyperbolic partial differential equations (PDEs) with random parameters. We formulate a stability condition under which the classical backstepping controller (designed for a nominal system whose parameters are constant) stabilizes the system. More precisely, we guarantee closed-loop mean-square exponential stability under random system parameter perturbations, provided the nominal parameters are sufficiently close to the stochastic ones on average. The proof is based on a Lyapunov analysis, the Lyapunov functional candidate describing the contraction of $L^2$-norm of the system states. An illustrative traffic flow regulation example shows the viability and importance of the proposed result.



*Keywords:* PDE, backstepping, stochastic parameters, exponential stability, Lyapunov function

## 1. INTRODUCTION

Hyperbolic PDEs are frequently used in the literature to model physical phenomena involving matter transportation across space and propagation. Among the numerous examples of conservation and balance laws that hyperbolic systems can model, we can cite traffic flow (Yu and Krstic, 2019), heat exchangers, fluids in open channels (Bastin and Coron, 2016), mechanical vibrations in drilling devices (Saldivar et al., 2016). This explains why the control and estimation of coupled hyperbolic PDEs is an active research topic, different types of control strategies having been developed in the literature (Litrico and Fromion, 2009; Woittennek et al., 2017; Bastin and Coron, 2016; Strecker and Aamo, 2017). Among them, we can emphasize the backstepping methodology (Krstic and Smyshlyaev, 2008) that enabled the design of explicit controllers first for scalar hyperbolic systems (Coron et al., 2013), then for general $n+m$ systems (Coron et al., 2017).

In all these contributions, the system parameters are assumed to be time-independent, and only a few results focus on hyperbolic systems with time-varying coefficients (Coron and Nguyen, 2021; Mokhtari and Ammar Khodja, 2022). However, when considering applications as freeway transportation systems, some parameters may be subject to abrupt changes due to external causes, e.g., the random flux at the entrance of the freeway (Colombo, 2003) or changes in drivers' behavior. This has motivated the stability analysis of switching hyperbolic systems (Amin et al., 2011) or Markov jump linear hyperbolic conservation laws (Zhang and Prieur, 2017). In this latter contribution, the authors considered stochastic velocities and showed mean-square exponential stability under appropriate conditions (balance between the dissipativity of the hyperbolic and the transition probability of the Markov process). The proof relied on a Lyapunov analysis.

In the present contribution, we consider a $2\times 2$ hyperbolic system where all parameters are stochastic. More precisely, they are modeled by independent Markov processes with a finite number of states (Kolmanovsky and Maizenberg, 2001). The objective is to guarantee the mean-square closed-loop stability. To do so, we consider a backstepping controller designed for a nominal system with constant coefficients (Vazquez et al., 2011). We show that the closed loop system is mean-square exponentially stable, provided the nominal parameters are sufficiently close to the stochastic ones on average. The proposed approach follows the methodology presented in (Kong and Bresch-Pietri, 2022a,b) for the case of linear systems with random input delays. It relies on a Lyapunov analysis of the closed-loop system. The exponential stability is shown using the so-called technique of probabilistic delay averaging (Kolmanovsky and Maizenberg, 2001).

Notations: We denote $L^2([0,1],\mathbb{R})$ the space of real-valued square-integrable functions defined on $[0,1]$ with the standard $L^2$ norm, i.e., for any $f\in L^2([0,1],\mathbb{R})$, we have $||f||_{L^2} = \left(\int_0^1 f^2(x)dx\right)^{\frac{1}{2}}$. $\mathbb{E}(x)$ denotes the expectation of a random variable $x$. For a random signal $x(t)$, the conditional expectation of $x(t)$ at the instant $t$ knowing that $x(s)=x_0$ at the instant $s\leq t$ is denoted $\mathbb{E}_{[s,x_0]}(x(t))$.


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## 2. PRELIMINARIES AND PROBLEM FORMULATION

### 2.1 Nominal system

Let us consider the following nominal $2 \times 2$ linear hyperbolic system

$$\partial_t u_{\mathrm{nom}}(t,x) + \lambda_0 \partial_x u_{\mathrm{nom}}(t,x) = \sigma_0^+ v_{\mathrm{nom}}(t,x), \quad (1)$$

$$\partial_t v_{\mathrm{nom}}(t,x) - \mu_0 \partial_x v_{\mathrm{nom}}(t,x) = \sigma_0^- u_{\mathrm{nom}}(t,x), \quad (2)$$

with the boundary conditions

$$u_{\mathrm{nom}}(t,0) = q_0 v_{\mathrm{nom}}(t,0), \quad (3)$$

$$v_{\mathrm{nom}}(t,1) = \rho_0 u_{\mathrm{nom}}(t,1) + U(t), \quad (4)$$

where $(u_{\mathrm{nom}}(t,x), v_{\mathrm{nom}}(t,x))^T$ is the state of the system, the different arguments evolving in $\{(t,x) \text{ s.t. } t > 0,\ x \in [0,1]\}$. The nominal velocities are denoted $\lambda_0 > 0$ and $\mu_0 > 0$ are constant. The in-domain couplings $\sigma_0^+$ and $\sigma_0^-$, and the boundary couplings $\rho_0$ and $q_0$ are also assumed to be constant. The function $U(t)$ corresponds to the actuation, which can be chosen as desired. We assume $q_0 \neq 0$. The initial conditions $u^0_{nom}$ and $v^0_{nom}$ are assumed to belong in $L^2([0,1],\mathbb{R})$. System (1)-(4) is the more general form for a one-dimensional $2 \times 2$ hyperbolic linear system (without integral or boundary terms) (Vazquez et al., 2011). For instance, it appears in the linearization of ARZ equations for traffic networks (Yu et al., 2020b; Espitia et al., 2022). System (1)-(4) may be unstable due to the in-domain couplings $\sigma^+$ and $\sigma^-$. The control input $U(t)$ may be designed to guarantee the exponential stability of the closed-loop system. We assume here that the boundary conditions are *dissipative* (i.e., $|\rho_0 q_0| < 1$) to guarantee the delay-robustness of the closed-loop system (Auriol and Di Meglio, 2019; Auriol, 2018). In this line of research, a stabilizing control law has been designed in (Coron et al., 2013; Vazquez et al., 2011; Auriol et al., 2018) using the backstepping approach. More precisely, we define the control law as

$$U(t) = \int_0^1 (\rho_0 K^{uu}(1,y) - K^{vu}(1,y)) u_{nom}(t,y) dy + \int_0^1 (\rho_0 K^{uv}(1,y) - K^{vv}(1,y)) v_{nom}(t,y) dy, \quad (5)$$

where the kernels $K^{\cdot\cdot}$ can be found in Vazquez et al. (2011). Then the closed-loop system (1)-(4) is well-posed and exponentially converges to zero in the sense of the $L^2$-norm (Coron et al., 2013) (Auriol et al., 2018).

### 2.2 Real system

We now consider the real stochastic $2 \times 2$ linear hyperbolic system

$$\partial_t u(t,x) + \lambda(t) \partial_x u(t,x) = \sigma^+(t) v(t,x), \quad (6)$$

$$\partial_t v(t,x) - \mu(t) \partial_x v(t,x) = \sigma^-(t) u(t,x), \quad (7)$$

with the boundary conditions

$$u(t,0) = q(t) v(t,0), \quad (8)$$

$$v(t,1) = \rho(t) u(t,1) + U(t), \quad (9)$$

where the state of the system is now $(u(t,x), v(t,x))^T$. The different parameters are now *random independent* variables. We denote $\mathfrak{S} = \{\lambda, \mu, \sigma^+, \sigma^-, q, \rho\}$ the set of random variables. Each random element $X$ of the set $\mathfrak{S}$ is a Markov process with the following properties.

- (P1) $X(t) \in \{X_i, i \in \{1, \dots, r_X\}\}$, $r_X \in \mathbb{N}$ with $\underline{X} \leq X_1 < \cdots < X_{r_X} \leq \bar{X}$.
- (P2) The transition probabilities $P^X_{ij}(t_1,t_2)$ qualify the probability to switch from $X_i$ at time $t_1$ to $X_j$ at time $t_2$ ($(i,j) \in \{1,\dots,r_X\}^2$, $0 \leq t_1 \leq t_2$). They satisfy
  (1) $P^X_{ij} : \mathbb{R}^2 \to [0,1]$ with $\sum_{j=1}^{r_X} P^X_{ij}(t_1,t_2) = 1$.
  (2) $P^X_{ij}$ is a differentiable function which, for $s < t$ follows the Kolmogorov equation
  $$\partial_t P^X_{ij}(s,t) = -c^X_j(t) P^X_{ij}(s,t) + \sum_{k=1}^{r_X} P^X_{ik}(s,t) \tau^X_{kj}(t),$$
  $$P^X_{ii}(s,s) = 1, \text{ and } P^X_{ij}(s,s) = 0 \text{ for } i \neq j, \quad (10)$$
  where $\tau_{ij}$ and $c^X_j = \sum_{k=1}^{r_X} \tau^X_{jk}$ are nonnegative-valued functions such that for any $t$, $\tau^X_{ii}(t) = 0$. Moreover, the functions $\tau^X_{ik}$ are upper bounded by a constant $\tau^\star_X$.
- (P3) The realizations of $X$ are right-continuous.

Moreover, we assume that for all $X \in \mathfrak{S}$, we have $\underline{X} \leq X_0 \leq \bar{X}$ (where $X_0$ is the nominal value given in the definition of system (1)-(4), e.g. $X_0 = \lambda_0$ if $X = \lambda$). We also assume that $\underline{\lambda} > 0$, $\underline{\mu} > 0$ and $|\bar{\rho}\bar{q}| < 1$. It is common to assume only a finite number of values in (P1) (Kolmanovsky and Maizenberg, 2001; Sadeghpour et al., 2019). Similarly, it is standard to assume Property (P3) for the modeling of continuous-time Markov chains. It is important to mention that the properties (P1) and (P3), along with the Markov property, guarantee that $P^X_{ij}$ satisfies the Kolmogorov Equation (10) for certain positive-valued functions $\tau^X_{ij}, c^X_j$ (Rausand and Hoyland, 2003; Ross, 2014). Thus Property (P2) only implies that the functions $\tau^X_{ij}$ are bounded, which is a mild modeling assumption. We emphasize that the parameter $\tau^X_{ij}\Delta t$ is approximately the probability of transition from $X_i$ to $X_j$ on the interval $[t, t+\Delta t)$. Moreover, $1 - c^X_j(t)\Delta t$ is the probability of staying at $X_j$ during this time interval.

For each $X \in \mathfrak{S}$, we denote by $T_X = \{X_1, \dots, X_{r_X}\}$ the set of possible realizations for the variable $X$. Let $\mathfrak{T} = T_\lambda \times T_\mu \times T_{\sigma^+} \times T_{\sigma^-} \times T_q \times T_\rho$ and for any $t$, let $\delta(t) \in \mathbb{R}^6$ be defined by

$$\delta(t) = (\lambda(t), \mu(t), \sigma^+(t), \sigma^-(t), q(t), \rho(t)). \quad (11)$$

Since all the variables are independent, $\delta$ is a Markov process whose transition probabilities can be deduced from those of $\mathfrak{S}$, and with a finite number of states $r = r_\lambda r_\mu r_{\sigma^+} r_{\sigma^-} r_q r_\rho$. Finally, we denote by $\mathfrak{R}$ the Cartesian product of the sets $\{1, \dots, r_X\}$, $X \in \mathfrak{S}$. An element $j \in \mathfrak{R}$ is a multi-index composed of 6 indices (integers): $(j_\lambda, j_\mu, j_{\sigma^+}, j_{\sigma^-}, j_q, j_\rho)$, and we will say that $\delta(t) = \delta_j$ if $X(t) = X_{j_X}$ for any $X \in \mathfrak{S}$. Finally, we denote $\tau_{j\ell} = \Pi_{X \in \mathfrak{S}} \tau^X_{j_X \ell_X}$, and similarly, $P_{ij} = \Pi_{X \in \mathfrak{S}} P_{i_X j_X}$.

We now prove the well-posedness of our stochastic system.

*Lemma 1.* For any initial condition $(u^0, v^0) \in L^2[0,1]$ and for any initial states $\delta(0)$ for the stochastic parameters, the closed-loop system (6)-(7) with the control law (5) admits a unique solution $(u,v)$ such that for any t,

$$\mathbb{E}_{[0,(u^0,v^0\delta(0))]}\{||(u(t,\cdot), v(t,\cdot)||_{L^2[0,1]}\} < \infty.$$

**Proof.** The proof can be easily adjusted from (Zhang and Prieur, 2017). □

### 2.3 Problem statement

The objective is this paper is to show that the nominal controller defined in equation (5) still stabilizes the stochastic system (6)-(9), provided the nominal parameters are sufficiently close to the stochastic ones on average. More precisely, we want to show the following sufficient condition for robust stabilization

*Theorem 2.* Consider the closed-loop system (6)-(9) with the control law (5). There exists a positive constant $\epsilon^\star$, such that if, for all time $t \geq 0$ and all $X \in \mathfrak{S}$,

$$\sum_{X\in\mathfrak{S}} \mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|) \leq \epsilon^\star, \tag{12}$$

then the closed loop system is *mean-square exponentially stable*, that is, there exists $\kappa > 0$ and $\gamma > 0$ such that

$$\mathbb{E}_{[0,(w(0),\delta(0))]}(w(t)) \leq \kappa \mathrm{e}^{-\gamma t} w(0), \tag{13}$$

where $w(t) = \int_0^1 u^2(t,x) + v^2(t,x)dx$.

Theorem 2 generalizes the deterministic robustness results stated in (Auriol and Di Meglio, 2020) to the case of stochastic parameters. The proof of Theorem 2 will be given in the next sections. It is inspired by (Kong and Bresch-Pietri, 2022b). First, the system is simplified using a backstepping transformation (Section 3). Then, the stability is shown using a Lyapunov analysis (Section 4).

## 3. BACKSTEPPING TRANSFORMATION

To prove Theorem 2, we first simplify the structure of the system using a backstepping transformation (Krstic and Smyshlyaev, 2008). Consider the integral change of coordinates

$$\alpha(t,x) = u(t,x) + \int_0^x K^{uu}(x,y)u(t,y) + K^{uv}(x,y)v(t,y)dy, \tag{14}$$

$$\beta(t,x) = v(t,x) + \int_0^x K^{vu}(x,y)u(t,y) + K^{vv}(x,y)v(t,y)dy, \tag{15}$$

where the kernels $K^{\cdot\cdot}$ are defined by Vazquez et al. (2011). The transformation (14)-(15) is a Volterra transformation and is invertible (Yoshida, 1960). In particular, there exist bounded functions $L^{\cdot\cdot}$ defined on the triangular domain $\mathcal{T}$ such that

$$u(t,x) = \alpha(t,x) + \int_0^x L^{\alpha\alpha}(x,y)\alpha(t,y) + L^{\alpha\beta}(x,y)\beta(t,y)dy, \tag{16}$$

$$v(t,x) = \beta(t,x) + \int_0^x L^{\beta\alpha}(x,y)\alpha(t,y) + L^{\beta\beta}(x,y)\beta(t,y)dy. \tag{17}$$

The kernels $L$ are explicitly defined in (Bou Saba et al., 2019). The states $(\alpha, \beta)$ and $(u, v)$ have equivalent $L^2$ norms, i.e. there exist $m > 0$ and $M > 0$ such that

$$m||(u,v)||_{L^2} \leq ||(\alpha,\beta)||_{L^2} \leq M||(u,v)||_{L^2} \tag{18}$$

Consider now $U(t)$ as defined by equation (5). Differentiating equations (16)-(15) with respect to time and space and integrating by parts, we can show that the variables $\alpha$ and $\beta$ are solutions to the following set of PDEs

$$\partial_t\alpha(t,x) + \lambda(t)\partial_x\alpha(t,x) = f_1(\delta(t))v(t,x) + f_2(\delta(t))\beta(t,0)$$
$$+ \int_0^x f_3(\delta(t),x,y)u(t,y) + f_4(\delta(t),x,y)v(t,y)dy \tag{19}$$

$$\partial_t\beta(t,x) - \mu(t)\partial_x\beta(t,x) = g_1(\delta(t))u(t,x) + g_2(\delta(t))\beta(t,0)$$
$$+ \int_0^x g_3(\delta(t),x,y)u(t,y) + g_4(\delta(t),x,y)v(t,y)dy, \tag{20}$$

with the boundary conditions

$$\alpha(t,0) = q(t)\beta(t,0), \tag{21}$$

$$\beta(t,1) = \rho(t)\alpha(t,1) + (\rho(t)-\rho_0)\int_0^1 K^{vu}(1,y)u(t,y)dy$$
$$+ (\rho(t)-\rho_0)\int_0^1 K^{vv}(1,y)v(t,y)dy \tag{22}$$

where the different functions are not given here due to space limitations but become *small* if the stochastic parameters are close enough to the nominal ones. More precisely, we have the following lemma.

*Lemma 3.* There exists a constant $M_0 > 0$ such that for any realization $\delta(t) = \delta_j$ $(j \in \mathfrak{R})$ of the stochastic variable $\delta$ and for any $(x,y) \in \mathcal{T}$, for all $i \in \{1,2,3,4\}$, we have

$$|f_i(\delta_j)| < M_0 \sum_{X\in\mathfrak{S}} |X_0 - X_j|, \tag{23}$$

$$|g_i(\delta_j)| < M_0 \sum_{X\in\mathfrak{S}} |X_0 - X_j|, \tag{24}$$

**Proof.** The proof is omitted due to the page limitation. □

Note that all the terms that depend on $(u,v)$ in the target system (19)-(22) could be expressed in terms of $(\alpha,\beta)$ using the inverse transformation (16)-(17). Thus, the target system (19)-(22) is simpler in the sense that it simplifies the robustness analysis that will be carried out in the next section. With this new set of coordinates, we can now analyze the exponential stability of the closed-loop system.

## 4. LYAPUNOV ANALYSIS

Let us denote the state of the target system (19)-(22) as $z(t,\cdot) = (\alpha(t,\cdot),\beta(t,\cdot)) \in (L^2([0,1],\mathbb{R}))^2$. As the solution to (19)-(22) is unique (due to the unicity of the solution of (6)-(9)), $(z,\mathfrak{S})$ defines a continuous-time Markov process. Define the infinitesimal generator $L$ (Kolmanovskii and Myshkis, 2013; Ross, 2014) acting on a functional $V : (L^2([0,1],\mathbb{R}))^2 \times \mathfrak{T} \to \mathbb{R}$ as

$$LV(z,\delta) = \limsup_{\Delta t\to 0^+} \frac{1}{\Delta t}$$
$$\times \mathbb{E}_{[t,(z,\delta)]}(V(z(t+\Delta t),\delta(t+\Delta t)) - V(z,\delta)). \tag{25}$$

We define $L_j$, the infinitesimal generator of the Markov process $(z,\delta)$ obtained from the system (19)-(22) by fixing $\delta(t) = \delta_j$ $(j \in \mathfrak{R})$ as

$$L_jV(z) = \frac{dV}{dz}(z,\delta_j)h_j(z) + \sum_{\ell\in\mathfrak{R}}(V_\ell(z) - V_j(z))\tau_{j\ell}(t), \tag{26}$$

where $V_\ell(z) = V(z,\delta_\ell)$, and $h_j$ is the operator corresponding to the dynamics of the target system (19)-(22) with the fixed value $\delta(t) = \delta_j$. To shorten the computations, we denote in the sequel $V(t), LV(t), V_j(t)$ and $L_jV(t)$ instead of (respectively) $V(z(t),\delta(t)), LV(z(t),\delta(t)), V(z(t),\delta_j)$ and $L_j(V(z(t)))$. From now, we consider that $\delta(t=0) = \delta_i$ for some $i \in \mathfrak{R}$.

### 4.1 Derivation of the Lyapunov function

Consider the following Lyapunov functional candidate

$$V(z,\delta) = \int_0^1 \frac{\mathrm{e}^{-\frac{\nu}{\lambda(t)}x}}{\lambda(t)}\alpha^2(t,x) + a\frac{\mathrm{e}^{\frac{\nu}{\mu(t)}x}}{\mu(t)}\beta^2(t,x)dx, \tag{27}$$

with $a, \nu > 0$. This functional explicitly depend on $\delta$ through the velocities $\lambda$ and $\mu$. Since the velocities $\lambda(t)$ and $\mu(t)$ are upper and lower bounded, the functional $V$ is equivalent to the $L^2$-norm of the state $(\alpha, \beta)$ (and consequently to the $L^2$-norm of the state $(u, v)$ due to (18)), i.e. there exists $k_u$ and $k_\ell$ such that

$$k_\ell ||(\alpha,\beta)||^2_{L^2} \leq V(t) \leq k_u ||(\alpha,\beta)||^2_{L^2} \tag{28}$$

We have the following lemma

*Lemma 4.* There exists $\eta > 0$, $M_1 > 0$ and $M_2 > 0$ such that the Lyapunov functional $V$ satisfies

$$\sum_{j=1}^{r} P_{ij}(0,t) L_j V(t) \leq -V(t)\big(\eta - M_2 k(t) - (M_1 + M_2 r\tau^\star) \sum_{X\in\mathfrak{S}} \mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|)\big), \tag{29}$$

where the function $k$ is defined by

$$k(t) = \sum_{X\in\mathfrak{S}} \sum_{j=1}^{r} |X_j - X_0| (\partial_t P_{ij}(0,t) + c_j P_{ij}(0,t)).$$

**Proof.** Consider that $\delta = \delta_j$. Let us compute the first term of equation (26) ($\frac{dV_j}{dz}(z) h_j(z)$). Applying integration by parts, we obtain

$$\frac{dV_j}{dz}(z)h_j(z) = -\nu V_j(t) + \int_0^1 \frac{2}{\lambda_{j_\lambda}} \mathrm{e}^{-\frac{\nu}{\lambda_{j_\lambda}}x} \alpha(t,x)[f_1(\delta_j) v(t,x) + f_2(\delta_j)\beta(t,0) + \int_0^x f_3(\delta_j,x,y)u(t,y) + f_4(\delta_j,x,y) v(t,y)dy] + \frac{2}{\mu_{j_\mu}} \mathrm{e}^{\frac{\nu}{\mu_{j_\mu}}x} \beta(t,x)[g_1(\delta_j)v(t,x) + g_2(\delta_j)\beta(t,0) + \int_0^x g_3(\delta_j,x,y)u(t,y) + g_4(\delta_j,x,y)v(t,y)dy]dx + (q^2_{j_q} - a)\beta^2(t,0) - \mathrm{e}^{-\frac{\nu}{\lambda_{j_\lambda}}} \alpha^2(t,1) + a\mathrm{e}^{\frac{\nu}{\mu_{j_\mu}}} (\rho_{j_\rho}\alpha(t,1) + (\rho_{j_\rho} - \rho_0) \int_0^1 K^{vu}(1,y)u(t,y) + K^{vv}(1,y)v(t,y)dy)^2.$$

Notice first that there exists $\eta > 0$ such that for all $j$, $-\nu V_j(t) \leq -\eta V(t)$. Applying Young's inequality and Lemma 3, we obtain

$$\frac{dV_j}{dz}(z)h_j(z) \leq -\eta V(t) + M_1 \sum_{X\in\mathfrak{S}} |X_0 - X_j| V(t) + (\epsilon_1 + q^2_{j_q} - a)\beta^2(t,0) + \alpha^2(t,1)(a\mathrm{e}^{\frac{\nu}{\mu_{j_\mu}}} \rho^2_{j_\rho}(1+\epsilon_2) - \mathrm{e}^{-\frac{\nu}{\lambda_{j_\lambda}}}),$$

where the constant $\epsilon_2 > 0$ can be chosen arbitrarily small and where the constant $M_1$ depends on $M_0$, $\epsilon_1$ and on the parameters of the system. Let us now choose $\nu, \epsilon_1$ and $\epsilon_2$ such that

$$\epsilon_1 + q^2_{j_q} - a < 0, \ a\mathrm{e}^{\frac{\nu}{\mu_{j_\mu}}} \rho^2_{j_\rho}(1+\epsilon_2) - \mathrm{e}^{-\frac{\nu}{\lambda_{j_\lambda}}} < 0.$$

These conditions are always feasible since $|\rho_{j_\rho} q_{j_q}| < 1$. Consequently, we obtain

$$\frac{dV_j}{dz}(z)h_j(z) \leq -\eta V(t) + M_1 \sum_{X\in\mathfrak{S}} |X_0 - X_j| V(t). \tag{30}$$

Consider now the second term of $L_j V(z)$ that appear in (26). Using the mean value theorem, we obtain

$$\sum_{\ell=1}^{r} (V_\ell(z) - V_j(z))\tau_{j\ell} \leq M_2 \sum_{\ell=1}^{r} \sum_{X\in\mathfrak{S}} \tau_{j\ell} |X_\ell - X_j| V(t),$$

where $M_2$ is defined by $M_2 = \frac{2\bar{\lambda}}{k_\ell}\frac{1}{\underline{\lambda}^2}(\frac{\nu}{\underline{\lambda}} + \bar{\lambda}) + \frac{2\bar{\mu}}{k_\ell}\frac{1}{\underline{\mu}^2}(\frac{\nu}{\underline{\mu}} + \bar{\mu})\mathrm{e}^{\frac{\nu}{\underline{\mu}}}$. Combining this inequality with equation (30), we get $L_j V(t) \leq -\eta V(t) + M_1 \sum_{X\in\mathfrak{S}} |X_0 - X_j| V(t) + M_2 \sum_{\ell=1}^{r} \sum_{X\in\mathfrak{S}} \tau_{j\ell} |X_\ell - X_j| V(t)$. We now compute the quantity $\bar{L} = \sum_{j=1}^{r} P_{ij}(0,t) L_j V(t)$. Since all the variables are independent, applying the triangular inequality and (10), we have

$$\bar{L} \leq -(\eta - (M_1 + M_2 r\tau^\star) \sum_{X\in\mathfrak{S}} \mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|))V(t) + M_2 \sum_{X\in\mathfrak{S}} \sum_{j=1}^{r} |X_j - X_0| (\partial_t P_{ij}(0,t) + c_j P_{ij}(0,t))V(t).$$

This concludes the proof of Lemma 4. □

### *4.2 Proof of Theorem 2*

We now have all the tools to prove Theorem 2. Let us denote $k_0(t) = \eta - (M_1 + M_2 r\tau^\star) \sum_{X\in\mathfrak{S}} \mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|) - M_2 k(t))$ and define the functional $Z(t)$ as $Z(t) = \exp(\int_0^t k_0(s)ds)V(t)$. Using the expression of $k(t)$ in Lemma 4, we have

$$\int_0^t k(s)ds \leq \sum_{X\in\mathfrak{S}} (\mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|) + c^\star \int_0^t \mathbb{E}_{[0,X(0)]}(|X_0 - X(s)|)ds),$$

where $c^\star = r\tau^\star$. Consequently, choosing $\epsilon^\star$ (defined in the statement of Theorem 2) as $\epsilon^\star = \frac{\eta}{2(2M_2 c^\star) + M_1}$, we obtain

$$\mathbb{E}_{[0,(z,\delta)(0))]}(Z(t)) \geq \mathbb{E}_{[0,(z,\delta)(0))]}(\mathrm{e}^{-M_2\epsilon^\star + \frac{\eta}{2}t} V(t)). \tag{31}$$

In the meantime, we have

$$\mathbb{E}_{[0,(z(0),\delta(0))]}(LZ(t)) = \mathrm{e}^{\int_0^t k_0(s)ds} \mathbb{E}_{[0,(z(0),\delta(0))]}(LV(t)).$$

Since $\mathbb{E}_{[0,(z(0),\delta(0))]}(LV(t)) = \mathbb{E}_{[0,(z(0),\delta(0))]}(\sum_{j=1}^{r} P_{ij}(0,t) L_j V(t))$, we obtain using equation (29), $\mathbb{E}_{[0,(z(0),\delta(0))]}(LZ(t)) \leq 0$. Therefore, according to Dynkin's formula (Dynkin, 2012, Theorem 5.1, p. 132), we obtain

$$\mathbb{E}_{[0,(z(0),\delta(0))]}(Z(t)) - Z(0) \leq 0. \tag{32}$$

Consequently, defining $\gamma = \frac{\eta}{2}$, and combining equations (31) and (32), we obtain

$$\mathbb{E}_{[0,(z(0),\delta(0))]}(V(t)) \leq V(0)\mathrm{e}^{M_2\epsilon^\star}\mathrm{e}^{-\gamma t}. \tag{33}$$

We can then easily conclude the proof of Theorem 2 using the fact that $V(t)$ is equivalent to $w(t)$.

## 5. NUMERICAL EXAMPLE: TRAFFIC CONGESTION CONTROL

In this section, we illustrate our theoretical results with the example of traffic congestion regulation under uncertain driver behavior. The Aw–Rascle–Zhang (ARZ) model (Aw and Rascle, 2000; Zhang, 2002), consisting of second-order nonlinear partial differential equations (PDEs), can describe the macroscopic traffic dynamics. More precisely, considering a freeway segment of length $L$, the traffic flux $q$ (number of vehicles per unit time which cross a given point) and the traffic speed $v$ verify (Yu and Krstic, 2019)

$$\partial_t q(t,x) + v\partial_x q(t,x) = \frac{q(\gamma p - v)}{v} \partial_x v(t,x)$$

$$+\frac{q(v_f - p - v)}{\tau v}, \quad (34)$$

$$\partial_t v(t,x) - (\gamma p - v)\partial_x q(t,x) = \frac{v_f - p - v}{\tau}, \quad (35)$$

where the traffic pressure is defined by $p(t,x) = \frac{v_f}{\rho_m^\gamma}(\frac{q}{v})^\gamma$. The parameter $\gamma$ represents the overall drivers' property, reflecting their change of driving behavior to the increase of density, $v_f$ is the maximum velocity, and $\tau$ is the relaxation time related to driving behavior (Dabiri and Kulcsar, 2022). In (Yu and Krstic, 2019), the system (34)-(35) is linearized around a steady-state $(q^\star, v^\star)$. The small deviations from the nominal profile are defined as $\tilde{q}(t,x) = q(t,x) - q^\star$ and $\tilde{v}(t,x) = v(t,x) - v^\star$. At the boundary, we consider a constant traffic flux $q^\star$ entering the domain (it can be obtained using mainline flux metering at the inlet): $q = q^\star$. At the other extremity, we have $v = \frac{v^\star}{q^\star}q$. Depending on the choice of parameters $(q^\star, v^\star)$ the traffic may be congested (Yu and Krstic, 2019). We therefore consider that we control the traffic upstream of the ramp metering (UORM). The objective is to stabilize the deviations $\tilde{q}(t,x)$ and $\tilde{v}(t,x)$. After linearization and a change of coordinates, these states verify equations (1)-(4) where

$$\lambda_0 = v^\star,\ \mu_0 = \gamma p^\star - v^\star,\ q = 1 - \frac{\gamma p^\star}{v^\star},$$
$$\rho = \exp(-\frac{L}{\tau v^\star}),\ \sigma_0^+ = 0,\ \sigma_0^-(x) = -\frac{1}{\tau}\exp(-\frac{x}{\tau v^\star}),$$

For this linearized system, the controller (5) can be used to guarantee exponential stability. To obtain a more realistic description of the drivers' behavior, we now consider that the variable $\gamma(t)$ is stochastic and that it follows a Markov process characterized by the properties (P1)-(P3). Note that this example cannot be seen as a direct application of Theorem 2 for several reasons: the parameter $\sigma_0^-$ is spatially varying, $\mu_0$ and $q$ both depend on $\gamma$ (therefore they are not independent), the change of coordinates required to rewrite the linearized ARZ equations in the Riemann coordinates depend on $\gamma$ and may add extra terms in the analysis. Despite these limitations, we chose to consider the proposed test case to emphasize possible extensions of our methodology. The length of the freeway segment is chosen as $L = 0.5$ km. The simulation time is $T = 600s$. The maximum speed limit is $v_m = 40$ ms$^{-1}$. The maximum density of the road is $\rho_m = 800$ vehicles per kilometer. The steady states are $(q^\star, v^\star) = (11640$ vehicles per hour$, 19.5$ km.h$^{-1})$. The relaxation time is $\tau = 60s$. We consider five different values for $\gamma$ $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (0.1, 0.48, 0.52, 0.53, 1.5)$. The initial transition probabilities are taken as $(0.02, 0.32, 0.32, 0.32, 0.02)$, which means that the values are initially concentrated between $\gamma_2, \gamma_3, \gamma_4 \in [0.48, 0.53]$. We choose the reference value $\gamma_0 = 0.5$. We operate the simulation with the transition rates $\tau_{ij}$ defined as

$$\tau_{ij}(t) = \begin{cases} 0 & \text{if } i = j \\ 2\kappa & \text{if } i \in \{1,5\} \\ 0.1\kappa & \text{if } i \in \{2,3,4\}, j \in \{1,5\} \\ \kappa(1 + 2\cos(\epsilon(i+5j)t)^2 & \text{if } i,j \in \{2,3,4\}, i \neq j \end{cases}$$

where $\kappa = 10$ and $\epsilon = 10^{-3}$. The dynamic of the state probabilities $P_j^\gamma(t) = \mathbb{P}(\gamma(t) = \gamma_j)$ is pictured in Figure 1, and reflects the choice of transition rates: the mass of the probability function is concentrated on the three states $\gamma_2, \gamma_3, \gamma_4$ close to the nominal value (which take turns being the most probable state), and a residual mass (close to zero) is put evenly on the two remaining states $\gamma_1$ and $\gamma_5$. We show in Figure 2 the Monte-Carlo simulation of $||(\tilde{q}, \tilde{v})||_{L^2}$ (100 trials) with the nominal control law (5) for the chosen stochastic parameter $\gamma(t)$. The solver is identical to the one used in (Yu et al., 2020a, 2022). As expected, since the nominal value $\gamma_0$ is close to the different realisations, we have the exponential mean-square stability of the closed-loop system. This illustrates the properties stated in Theorem 2.

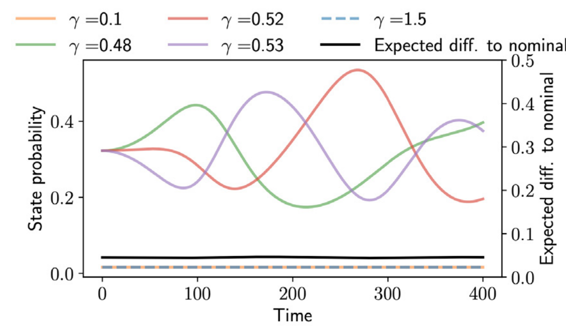


Fig. 1. Dynamic of the transition probabilities $P_1, P_2, P_3, P_4, P_5$ and expected difference to the nominal value.

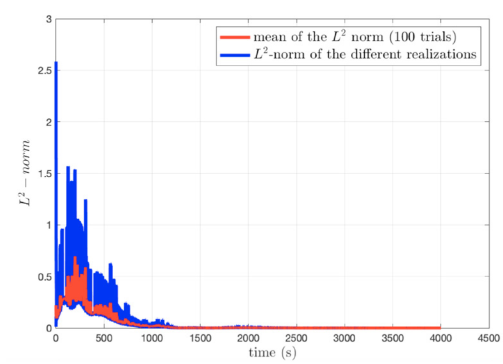


Fig. 2. Monte Carlo simulation of $||(\tilde{q}, \tilde{v})||_{L^2}$ (100 trials) with the nominal control law (5) for the chosen stochastic parameter $\gamma(t)$.

## 6. CONCLUSIONS

In this paper, we considered the stability of coupled and uncertain $2 \times 2$ hyperbolic system. Parametric uncertainties are modeled as independent Markov processes with a finite number of values. We showed that the classical backstepping controller designed with nominal constant parameters guarantees the exponential mean-square stability of the closed-loop system provided that the nominal values are close enough on average to the real ones. This result generalizes the deterministic robustness results stated in (Auriol and Di Meglio, 2020). The proposed methodology is adjusted from (Kong and Bresch-Pietri, 2022a) and is based on a Lyapunov analysis. Future works will focus on the generalization of our approach to non-independent stochastic parameters and to a larger class of random variables (that may not be described by Markov processes) and random fields.

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