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RESEARCH

The geometry of the equilibrium of forces and moments in shells

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Abstract We demonstrate that the internal forces and moments in a shell structure can be described by 4 vectors in 3 dimensional space which could be used to express the equilibrium of a shell graphically. The internal forces and moments described by 2 of the 4 vectors actively carry imposed loads and imposed couples, while the other 2 describe 'redundant' internal forces and moments that are purely in equilibrium with themselves. Nevertheless redundant stresses and moments are important in controlling internal stresses and satisfying boundary conditions. We show that forces and moments 'flow' across a shell in exactly the same way that fluid flows, with a combination of irrotational flow and incompressible flow. The aim is that these ideas can help in the initial design of those shell structures which have to rely on bending moments in addition to membrane action in supporting loads, and also in the interpretation of results from an analysis using the finite element method. We demonstrate the use of the technique for the design of an umbrella structure like those built by Amancio Williams.

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1 Introduction

In the design of shell structures the aim is as far possible to carry loads by membrane action, that is tensions and compressions in the tangent plane to the surface, as opposed to bending action involving moments and shear forces perpendicular to the surface. However, there are many situations in which bending action is unavoidable, and the aim of this paper is to examine how we can deliberately include bending action in the design process and actively decide how we want the moments to act. Having produced a proposed design, then conventional techniques, such as the finite element method can be used to confirm that the structure is indeed acting as we intended.

Some authors describe this approach as the 'force method' in which we concentrate upon finding a system of forces and moments in equilibrium with the applied loads. This system will in general include a set of 'redundant' forces and moments which carry no load, but the choice of redundancies is arbitrary and this applies to shell structures in the same way that it does for frameworks [29]. Having found such a system, including a choice of the redundancies, we then need to design the structure so that its stiffness is appropriate for producing the distribution of forces and moments which we desire.

In the equilibrium theory of shell structures there are 6 unknown components of internal force and 6 unknown components of internal moment. Many theories assume that 2 of the possible components of moment are zero, and 2 of the remaining components are equal, or equal and opposite, according to sign convention. However we shall not make these assumptions, at least to begin with.

We have 3 equations of equilibrium of force and 3 equations of equilibrium of moments. Thus we have a possible $2 \times 6 = 12$ unknowns and $2 \times 3 = 6$ equations, so that shell structures are 6 times statically indeterminate, unless we make further assumptions. A structure is potentially statically determinate if the number of equations equals the number of unknowns and we use the word 'potentially' since having the correct number of equations does not mean we have a solution, particularly if we have partial differential equations, as is the case with shells.

We will demonstrate that these statical indeterminacies or redundant forces and moments can be expressed by 2 redundancy vectors in 3 dimensional space, leaving 2 further vectors to describe the internal forces and moments that resist loads and loading couples. These vectors can be expressed graphically as surfaces in themselves, although interpretation of the geometry of these surfaces is by no means an easy task.

In addition we shall show that forces and moments 'flow' across a shell in exactly the same way that fluid flows, with a combination of irrotational flow and incompressible flow. Irrotational flow is flow with zero mean angular velocity.

The design of structures that contain statical indeterminacies is always difficult, firstly in making often arbitrary decisions as to how forces and moments should travel through the structure and secondly in ensuring that the structure does indeed function as predicted taking into account creep, foundation settlement etc. The ideas introduced in this paper are intended to help understand the structural behaviour of shell structures which rely on bending moments as part of their primary structural action.

2 Organization of the paper

In Sect. 3 we try and summarize the theoretical background to our work and we use this to inform our treatment of internal forces and internal moments in a shell or gridshell in Sect. 4.

In Sect. 5 we show how the internal forces and moments can be expressed in terms of the vectors to which we give the symbols $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$. We believe this to be the main contribution of the paper.

In Sect. 6 we introduce the analogy the flow of a fluid across a surface and show how this mirrors the flow of the Cartesian components of force and moment. As with all analogies this does not make it easier to solve the equations, but it does help in physical understanding, and is interesting in its own right.

In Sect. 7 we discuss the relationship between the specification of moment using $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ and its specification using the tensor $\mathbf{M}$ that is usually used.

In Sect. 8 we discuss the boundary conditions at a free edge, which is an important issue in the application of the theory. This leads on to Sect. 9 in which we discus a partially loaded shell and this gives further insights into the boundary conditions.

In Sect. 10 we show how the theory can be applied to the membrane theory, leading to 1 equation in 1 unknown. In Sect. 11 we apply the theory to a flat plate and here the theory could be used in plastic design of concrete slabs using the lower bound theory.

In Sect. 12 we introduce the 'moment surface' which best represents a shell and allow this to be different from the 'reference surface' on which we do our calculations. This is a generalisation of the concept of projecting a shell onto a plane.

Finally, in Sect. 13 we apply the the theory to the Monumento en homenaje a Amancio Williams, a shell structure which relies on both membrane action and bending action. We use a flat surface for the reference surface.

3 Theoretical background

In his paper [4] entitled *The theory of shell structures: Aims and methods* Professor Calladine writes

> The theory of shell structures is indeed a large subject. It has been actively studied for about 100 yr; and the literature is enormous.
> In some large subjects it is possible to discern a main stream of thought, perhaps associated with a succession of dominating personalities. In the field of shell structures there have been, and indeed still are, some dominant figures; but in general they appear to me to be more like the leaders of a collection of schismatic sects than a continuing chain of central authority. The sectarian nature of the subject springs partly from the fact that problems concerning shell structures crop up in many diverse types of engineering practice; and accordingly the stimulus and sponsorship for academic work has come, and still comes, from a wide range of sources.

and he goes on to write

> One of the underlying difficulties of the subject lies in the fact that the basic equations of elastic shells are of eighth order; and they can only be expressed properly in terms of coordinates which fit the curved surface.

Some books, for example Timoshenko and Woinowsky-Krieger [35] and Flügge [16, 17] use what might be termed 'engineering notation' together with numerous complicated diagrams, although in the bibliography for chapter 6 in the first edition of his book Wilhelm Flügge gives a list of papers using the tensor notation which 'establish sets of basic equations for shells of arbitrary shape'. However, he goes on to state that

> The papers of this group have the merit that they provide an extremely general formulation of the theory. However, it is a long way from such general formulations to a solution applicable to a concrete engineering problem.

This neatly summarises the difficulty.

The tensor notation with subscripts and superscripts, as used in Green and Zerna [18] avoids the need for the diagrams, but is difficult to apply in practice, as Flügge observed. The tensor notion is not difficult to learn, but it does take time.

Even though different books use different notations, they all essentially agree on the equations of equilibrium of force and equilibrium of moments for shell structures. There is more scope for variation in the assumed relationship between deformation and strain in a shell. The Kirchhoff-Love theory [27] of the bending of plates and shells is an extension of the Euler-Bernoulli beam theory in which plane sections are assumed to remain plane and perpendicular to the axis of an arch or the surface representing a shell. The Uflyand-Mindlin and Cosserat [19, 28] theories allow for the shear deformation of plates and shells in very much the same way as Timoshenko-Ehrenfest beam theory.

In this paper we are primarily interested in the equilibrium equations, which naturally leads to application of the lower bound theory of plasticity [14]. Professor Heyman's application of plasticity theory to masonry arches and vaults is particularly relevant [21, 22].

As curved surface structures the theory of shell structures bears a close relationship with the differential geometry of surfaces, and here again books adopt different notations to describe the same relationships. Struik [34] uses what might be termed 'elementary' notation, whereas Eisenhart [15] uses tensor notation. More modern differential geometry books use differential forms pioneered by Élie Cartan [8], and it is possible that this could be applied to shell structures.

The application of the finite element method to shell structures uses the elements to interpolate between a finite number of nodes. This is essentially the same problem as surface representation in computer aided geometric design and the introduction of isogeometric analysis [10] brings the two disciplines together.

The application of the finite element method to the design of shell structures involves the generation of a geometric model which is then analysed in the finite element program. The results are studied and the the model is modified to try and improve the behaviour. This might include some automated optimization process, but this is no means a simple process since increasing the strength of over-stressed elements increases their stiffness, attracting more force and moment.

It might be argued that the best structures have a relatively simple mode of structural action and the aim of this paper is to help designers to search for

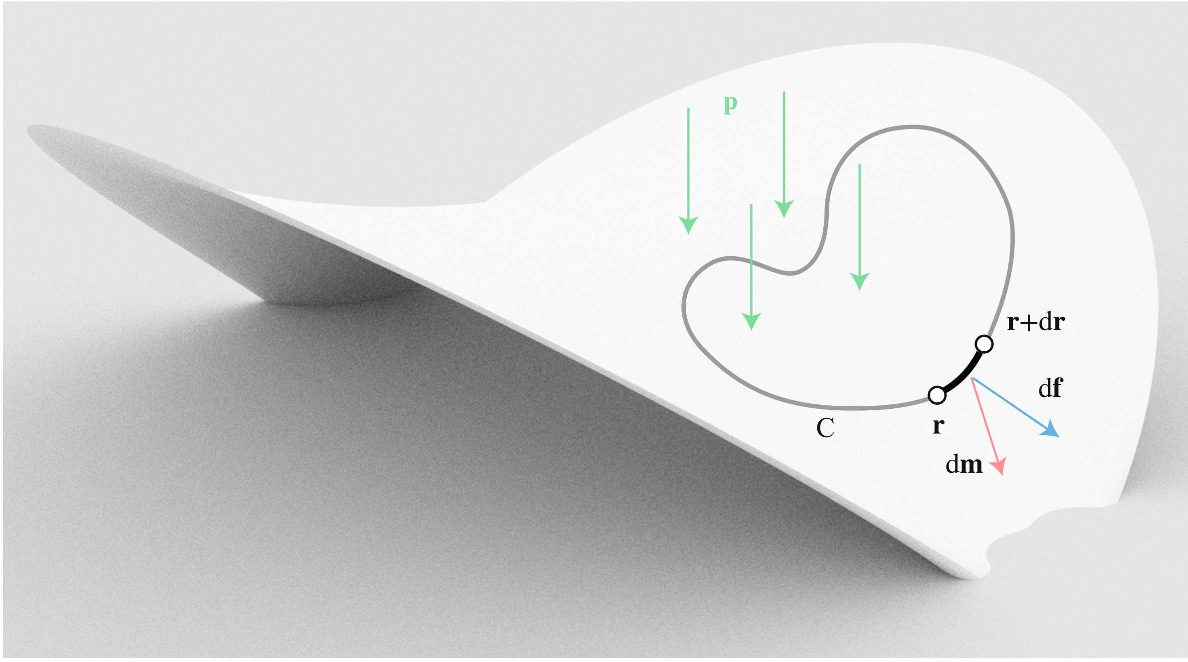

Fig. 1 A shell structure

such a structure and interpret the results from a finite element analysis.

However, it should always be remembered that if a shell can be designed to carry load primarily by membrane action, then it can be very thin and prone to buckling. Shells are very sensitive to imperfections and so a linear or eigenvalue buckling analysis may greatly overestimate their strength [31].

4 Internal forces and moments in a shell structure

In this section we define the tensor $\boldsymbol{\sigma}$ describing the internal forces within a shell structure and the tensor $\boldsymbol{\mu}$ describing the internal moments. In a gridshell the internal forces and moments are concentrated in individual members, but this in no way negates the use of $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$, it just means that they have high values where there is a member and zero between the members. If we want to represent the members by lines in space, then $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ tend to infinity, but the integral across the member remains finite.

Let us imagine making a cut through the shell structure shown in Fig. 1 using the fret saw shown in Fig. 2. The cut is closed so that we could remove a portion of shell. We can use a reference surface S to describe the saw blade, so that for each point on S there is a corresponding curve in space describing the saw blade. The saw blade will often be assumed to be straight, and its orientation is usually assumed to be perpendicular to the reference surface, although in some circumstances it might be more appropriate to make it vertical.

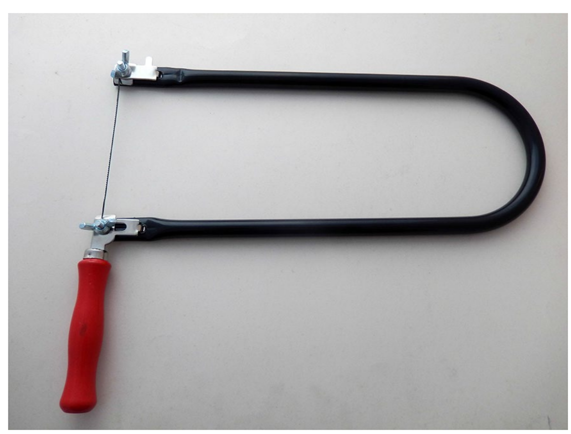

Fig. 2 A fret saw. Image: Gerd Fahrenhorst

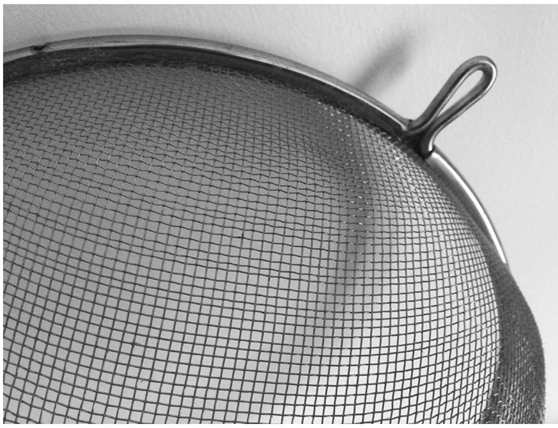

Fig. 3 A kitchen sieve

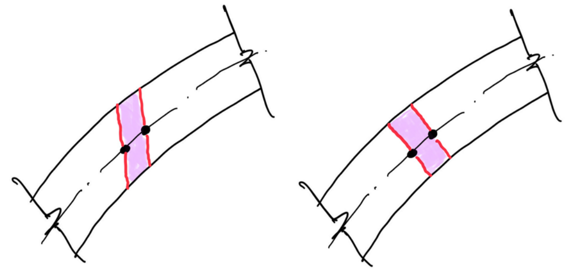

Fig. 4 Element cut with vertical blade on the left and blade normal to the reference surface on the right

In the case of a gridshell, such as the kitchen sieve in Fig. 3 we have a choice of either considering each wire separately or 'smoothing out' the effects of the wires to produce an equivalent continuous shell.

In this paper we are primarily concerned with equilibrium of a shell with a fixed geometry, and various elastic, plastic and creep deformations and prestressing actions will have occurred to achieve the internal forces and moments within the shell. Even if our imaginary cut was done with a straight blade, subsequent deformation may mean that the cut will no longer be a ruled surface with straight line generators.

The blade may or may not pass through the corresponding point on S, but it is usually simpler to assume that it does.

The closed curve C on S represents the entire cut necessary to remove a portion of shell. We make no assumption that the shell is thin - that assumption is only necessary when analysing the stiffness and strength properties of the shell.

The reference surface is arbitrary, and it would usually be chosen to 'best represent' the shell.

There will be an element of force $d\mathbf{f}$ and moment $d\mathbf{m}$ crossing the element of cut represented by the element of curve between the adjacent points $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ on the curve C. $d\mathbf{m}$ acts about the point $\mathbf{r}$, and since $d\mathbf{r} \to 0$ it makes no difference whether we take moments about $\mathbf{r}$ or $\mathbf{r} + d\mathbf{r}$, or even $\mathbf{r} + d\mathbf{r}/2$.

> However, it could be convenient to choose a simpler shape for the reference surface, such as a plane, a cylinder or a sphere. In Sect. 12 we discus the introduction of a second 'moment surface' and take moments about a point on this surface instead. For each point on S there is a corresponding point on the moment surface.

$d\mathbf{f}$ and $d\mathbf{m}$ will depend upon $d\mathbf{r}$, and also upon the orientation of the saw blade. Since $d\mathbf{r} \to 0$ the element of shell producing $d\mathbf{m}$ will be a narrow strip, and if the saw blade is straight and passes through $\mathbf{r}$ then $d\mathbf{m}$ will be about an axis perpendicular to the blade, that is about the horizontal if the blade is vertical or about axes tangential to the surface if the blade is normal to the surface - see Fig. 4. If the shell is thin compared to the radius of curvature of the reference surface, then the component of moment about an axis tangential to the surface is the same for both elements in Fig. 4, but the element of cut on the left will also produce a moment about the normal. The saw blade orientation clearly cannot influence the physical equilibrium of a shell, only our perceived values of moments and we shall see in Sect. 5 that the moment about the normal is largely a purely internal 'redundant' moment, although in the case of a sieve-like structure (Fig. 3) the moment about the normal due to the geodesic curvature of the members clearly affects the structural behaviour. These 'geodesic moments' about the normal might also be described as Cosserat moments [9].

We can use $d\mathbf{f}$ and $d\mathbf{m}$ to define the internal forces $\boldsymbol{\sigma}$ and internal moments $\boldsymbol{\mu}$ in the shell by

$$d\mathbf{f} = (d\mathbf{r} \times \mathbf{n}) \cdot \boldsymbol{\sigma} = d\mathbf{r} \cdot (\mathbf{n} \times \boldsymbol{\sigma}) = -d\mathbf{r} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{\sigma} \tag{1}$$

and

$$d\mathbf{m} = (d\mathbf{r} \times \mathbf{n}) \cdot \boldsymbol{\mu} = d\mathbf{r} \cdot (\mathbf{n} \times \boldsymbol{\mu}) = -d\mathbf{r} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{\mu} \tag{2}$$

in which $\mathbf{n}$ is the unit normal to the reference surface and $\boldsymbol{\epsilon}$ is the surface permutation tensor defined in (A3). In writing these equations we are effectively assuming that the shell is a continuum, although that does not preclude concentrations of force and moment in individual members. The equations apply whether we are in a loaded or unloaded region of shell.

Equations (1) and (2) are the *definitions* of $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ and therefore require no proof, except that it is implicitly assumed that a small element with constant $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ will be in equilibrium.

$\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are 2nd order tensors which are composed of the sum of tensor products of vectors. The tensor product $\mathbf{uv}$ of the vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{uv}\cdot\mathbf{w} = (\mathbf{uv})\cdot\mathbf{w} = \mathbf{u}(\mathbf{v}\cdot\mathbf{w}) \quad \text{and}$$
$$\mathbf{w}\cdot\mathbf{uv} = \mathbf{w}\cdot(\mathbf{uv}) = (\mathbf{w}\cdot\mathbf{u})\mathbf{v}$$

where $\mathbf{w}$ is any vector and the $\cdot$ represents the inner or scalar product. Thus $\mathbf{uv}\cdot\mathbf{w}$ is a vector in the direction of $\mathbf{u}$, but with a magnitude $(\mathbf{v}\cdot\mathbf{w})$ times that of $\mathbf{u}$. The tensor product is sometimes written $\mathbf{u}\otimes\mathbf{v}$ and is also called the dyadic product or the outer product. Scalars are 0th order tensors, vectors are 1st order tensors and by adding repeated tensor products of vectors we can produce tensors of any order.

Thus we can use tensor products of the Cartesian base vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ to write the stress tensor in 3D [36] as

$$\begin{aligned}\sigma_x\mathbf{ii} + \tau_{xy}\mathbf{ij} + \tau_{xz}\mathbf{ik}\\ + \tau_{yx}\mathbf{ji} + \sigma_y\mathbf{jj} + \tau_{yz}\mathbf{jk}\\ + \tau_{zx}\mathbf{ki} + \tau_{zy}\mathbf{kj} + \sigma_z\mathbf{kk}\end{aligned}$$

which has units force per unit area. However our $\boldsymbol{\sigma}$ in (1) is a force per unit length of cut and $\boldsymbol{\mu}$ in (2) is a moment per unit length of cut.

The tensor properties that we shall be using are summarized in Appendices A and B.

Since $d\mathbf{r}\times\mathbf{n}$ is perpendicular to $\mathbf{n}$ in (1) and (2) we can stipulate that

$$\mathbf{n}\cdot\boldsymbol{\sigma} = 0 \quad \text{and} \quad \mathbf{n}\cdot\boldsymbol{\mu} = 0 \tag{3}$$

without affecting the values of $d\mathbf{f}$ and $d\mathbf{m}$. In 3D a 2nd order tensor has $3\times 3 = 9$ components. However (3) means that $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ each only have 6 components, of which 4 are tangential to the surface and 2 are normal to the surface.

We shall see in Sect. 7 how the moment can be specified in a way which is more familiar to engineers.

We are attempting to write equations in using a coordinate free notation. However coordinates are useful for calculations and are, of course, used in formulating finite elements. In their chapter 10 Green and Zerna [18] use $n^{\alpha\beta}$ and q^{α} for the 6 components of $\boldsymbol{\sigma}$, in which case we can write

$$\boldsymbol{\sigma} = n^{\alpha\beta}\mathbf{a}_\alpha\mathbf{a}_\beta + q^\alpha\mathbf{a}_\alpha\mathbf{n}. \tag{4}$$

There is no other way of writing $\boldsymbol{\sigma}$ to satisfy (3), except, of course we could write

$$n^{\alpha\beta}\mathbf{a}_\alpha\mathbf{a}_\beta = n^{\alpha}_{\cdot\beta}\mathbf{a}_\alpha\mathbf{a}^\beta = n_{\alpha}^{\cdot\beta}\mathbf{a}^\alpha\mathbf{a}_\beta = n_{\alpha\beta}\mathbf{a}^\alpha\mathbf{a}^\beta$$

and

$$q^\alpha\mathbf{a}_\alpha = q_\alpha\mathbf{a}^\alpha$$

if we were so minded. Note that Green and Zerna's use of $n^{\alpha\beta}$ is in no way related to our use of $\mathbf{n}$ for the unit normal. They use $\mathbf{a}_3 = \mathbf{a}^3$ for the unit normal.

5 Solution of the equilibrium equations

In this section we show how the 12 components of the internal forces and moments in a shell structure, $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$, can be expressed in terms of the 12 components of the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ and substitute the vectors into the equilibrium equations.

The equilibrium equations are derived in Appendix C. The reason that they are relegated to an appendix is that they are well known, although perhaps not in the form we repeat here using the ∇ (pronounced nabla or del) notation [37]:

$$\nabla\cdot\boldsymbol{\sigma} + \mathbf{p} = 0 \tag{5}$$

for equilibrium of forces and

$$\nabla\cdot\boldsymbol{\mu} + \boldsymbol{\sigma} : \boldsymbol{\chi} + \mathbf{c} = 0 \tag{6}$$

for equilibrium of moments. $\mathbf{p}$ and $\mathbf{c}$ are the loading forces and loading couples per unit area of the reference surface. $\boldsymbol{\chi}$ is the Levi-Civita permutation pseudotensor in 3D space defined in (A2) and (B4). $\boldsymbol{\epsilon}$ is related to $\boldsymbol{\chi}$ by (A3). $\boldsymbol{\chi}$ is used in performing the vector product and here isolates the non-symmetric parts of $\boldsymbol{\sigma}$ which contribute to equilibrium of moments.

We could, of course, write these equations in component form, so that if we write $\boldsymbol{\sigma}$ in the form (4), then (5) becomes

$$\begin{aligned}&\left(n^{\alpha\beta}|_{\alpha} - b_{\alpha}^{\beta} q^{\alpha} + p^{\beta}\right)\mathbf{a}_{\beta}\\&\quad+\left(b_{\alpha\beta} n^{\alpha\beta} + q^{\alpha}|_{\alpha}\right)\mathbf{n} = 0\end{aligned} \tag{7}$$

which are equations (10.4.4) and (10.4.5) in Green and Zerna [18]. Writing (7) starting from (4) and (5) is a purely mechanical process using the methods in Appendices A and B. The inverse process of inferring (5) from (7) is not so simple.

In (5) and (6) $\mathbf{p}$ and $\mathbf{c}$ are usually assumed to be known, and $\mathbf{c}$ is almost invariably taken as zero. $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are the unknown internal forces and moments.

The meaning of the divergence $\nabla \cdot \boldsymbol{\sigma}$ and $\nabla \cdot \boldsymbol{\mu}$ is given in (B12). The divergence is the inner product of the gradient in (A4) with itself. The gradient and the divergence are physical entities which exist entirely separately from any coordinate system, and so as far as possible coordinates should be avoided in their definition.

Let us now consider internal forces of the form

$$\boldsymbol{\sigma} = \boldsymbol{\epsilon} \cdot \nabla\boldsymbol{\psi} + \nabla\boldsymbol{\beta} \tag{8}$$

and moments of the form

$$\begin{aligned}\boldsymbol{\mu} &= \boldsymbol{\epsilon} \cdot \nabla\boldsymbol{\phi} + \nabla\boldsymbol{\alpha}\\&\quad+ \boldsymbol{\psi} \cdot (\mathbf{nI} - \mathbf{In}) + \boldsymbol{\beta} \cdot (\mathbf{n}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{n})\end{aligned} \tag{9}$$

in which $\mathbf{I}$ is the unit tensor, defined in (A1) and H is the mean curvature of the reference surface in (B7). $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ are vectors which will in general have both tangential and normal components. These equations automatically satisfy (3).

Equations (8) and (9) are of a very similar form to those in the *Note Added in Proof* at the end of Carlson's paper on the Günther Cosserat stress functions [7] in 3 dimensional space in which he refers to equations given in a lecture given by Professor Schaefer in September 1965 in Augustów, Poland. In the case of an unloaded non-Cosserat material, the Günther stress functions reduce to the Beltrami stress functions. Schaefer's [12, 32] concept of a Krustenschale (crust shell) corresponds to discontinuities across the shell in the Günther Cosserat stress functions [11, 20] in 3 dimensional space. However, even though the derivation of the equilibrium equations for a shell starting from the 3 dimensional equations is of great interest, it is probably simpler to start with the reference surface, even though the resulting equations should be identical. In addition our definition of $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ apply regardless of how the physical stresses vary through the thickness of the shell or of how thick the shell is - it could be very thick. All we assume is that $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are the result of some integration, the details of which do not concern us.

The substitution (8) and (9) into the equilibrium equations is a pure mathematical process and is given in Appendix D. The equation of equilibrium of forces, becomes

$$\nabla^2\boldsymbol{\beta} + \mathbf{p} = 0 \tag{10}$$

where $\nabla^2\boldsymbol{\beta} = \nabla \cdot \nabla\boldsymbol{\beta}$ is the Laplacian of the vector $\boldsymbol{\beta}$ and the equation of equilibrium of moments becomes

$$\nabla^2\boldsymbol{\alpha} + 2H\boldsymbol{\epsilon} \cdot \boldsymbol{\beta} + \mathbf{c} = 0. \tag{11}$$

Thus we have 2 vector differential Eqs. () and (11) in the 2 unknown vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, assuming that $\mathbf{p}$, $\mathbf{c}$ and the geometry of the reference surface are known.

We can first solve (10) for $\boldsymbol{\beta}$ and then solve (11) for $\boldsymbol{\alpha}$. Both these equations are Poisson [30] equations. In these Poisson equations the Cartesian components of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are independent so that we have 6 completely separate Poisson equations to solve.

The vectors $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ do not appear in (10) or (11), so they describe a redundant system of internal forces and moments in equilibrium with zero applied loads and loading couples. The loads are resisted by the action of the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ which are statically determinate in the sense that we have the same number of differential equations as unknowns. Boundary conditions can be satisfied by a combination of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ which we can also use to apportion loads following Calladine's [5, 6] stretching action and bending action.

Since the only the gradients of $\boldsymbol{\phi}$ and $\boldsymbol{\alpha}$ appear in the expression for $\boldsymbol{\mu}$ and not at all in the expression for $\boldsymbol{\sigma}$ we can allow discontinuities in the value of $\boldsymbol{\phi}$ and $\boldsymbol{\alpha}$ without introducing a concentration of moment or force. In other words, we may go around a closed path and find that $\boldsymbol{\phi}$ and/or $\boldsymbol{\alpha}$ have a different value at the end of the path than at the beginning.

From Eqs. (1) and (2) together with (8) and (9) we have

$$d\mathbf{f} = d\boldsymbol{\psi} - d\mathbf{r} \cdot \boldsymbol{\epsilon} \cdot \nabla\boldsymbol{\beta} \tag{12}$$

and

$$d\mathbf{m} = d\boldsymbol{\phi} - d\mathbf{r} \cdot \boldsymbol{\epsilon} \cdot \begin{pmatrix} \nabla\boldsymbol{\alpha} + \boldsymbol{\psi} \cdot (\mathbf{n}\mathbf{I} - \mathbf{I}\mathbf{n}) \\ +\boldsymbol{\beta} \cdot (\mathbf{n}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{n}) \end{pmatrix} \tag{13}$$

for the elements of force and moment crossing an element of cut $d\mathbf{r}$. These equations show more clearly how the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ contribute to the force and moment crossing an element of cut.

It is perhaps worth noting, *but not pursuing* that we could dispense with $\boldsymbol{\alpha}$ and write

$$\boldsymbol{\sigma} = \boldsymbol{\epsilon} \cdot \nabla\boldsymbol{\psi} - \mathbf{c} \cdot \boldsymbol{\Xi} + \nabla\boldsymbol{\beta} - 2H(\mathbf{I} - \mathbf{n}\mathbf{n}) \cdot \boldsymbol{\beta}\mathbf{n}$$

and

$$\boldsymbol{\mu} = \boldsymbol{\epsilon} \cdot \nabla\boldsymbol{\phi} + \boldsymbol{\psi} \cdot (\mathbf{n}\mathbf{I} - \mathbf{I}\mathbf{n}) + \boldsymbol{\beta} \cdot (\mathbf{n}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{n})$$

in which case we automatically satisfy equilibrium of moments. $\boldsymbol{\Xi}$ is the 3$^{\text{rd}}$ order tensor defined in (B6). However the equations of equilibrium of force in the 3 Cartesian directions are now coupled via the term containing the mean curvature, H. Therefore we will not adopt this alternative. Note that the values of $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ and $\boldsymbol{\beta}$ will be different from those in (8) and (9).

6 An analogy from fluid mechanics

In this section we shall show how Eqs. (8) and (9) can be thought of as forces and moments 'flowing' across a shell structure, possibly concentrated in the members of a gridshell.

In the flow of fluids, such as air or water, in 2 dimensions it is usual to express the 2 components of fluid velocity as

$$\mathbf{v} = \boldsymbol{\epsilon} \cdot \nabla\psi + \nabla\beta \tag{14}$$

in which the scalar β is the velocity potential and the scalar ψ is the stream function which Lamb [26] attributes to Lagrange. If $\beta = 0$ the flow is incompressible and if $\psi = 0$, the flow is irrotational, that is the vorticity is zero. If

$$\mathbf{v} = \boldsymbol{\epsilon} \cdot \nabla\psi = \nabla\beta$$
$$\nabla^2\psi = 0$$
$$\nabla^2\beta = 0$$

then the flow is both incompressible and irrotational. $\boldsymbol{\epsilon} \cdot \nabla\psi = \nabla\beta$ are the Cauchy-Riemann equations leading to analytic functions of a complex variable - see §62 of Lamb [26]. This applies for flow in a plane, but also for flow over a curved surface, if we assume that the layer of fluid is thin and of constant thickness.

We have used the symbols β and ψ deliberately because then (14) and (8) are similar, except β and ψ are scalars, whereas $\boldsymbol{\psi}$ and $\boldsymbol{\beta}$ are vectors. Correspondingly $\mathbf{v}$ is a vector and $\boldsymbol{\sigma}$ a second order tensor.

We can imagine $\boldsymbol{\sigma}$ flowing across the reference surface. The 'irrotational stress' represented by $\boldsymbol{\beta}$ carries the load whereas the 'incompressible stress' corresponding to $\boldsymbol{\psi}$ carries no load, corresponding to the fact that incompressible flow does not pick up any 'rain' falling on the surface.

We can also say that the moment $\boldsymbol{\mu}$ in (9) flows across the surface with the 'irrotational moment' represented by $\boldsymbol{\alpha}$ and the 'incompressible moment' represented by $\boldsymbol{\phi}$. However, now we have additional moment flows represented by $\boldsymbol{\psi}$ and $\boldsymbol{\beta}$ in (9).

Finally 2 dimensional fluid flow can also be written as

$$\mathbf{v} = \xi\boldsymbol{\epsilon} \cdot \nabla\zeta$$

as an alternative to (14). ξ and ζ are scalars and the direction of flow is tangential to the contour lines of ζ and the magnitude

$$|\mathbf{v}| = \xi|\nabla\zeta|. \tag{15}$$

The disadvantage of this approach is that it is non-linear, but on the other hand for a known $\mathbf{v}$ it is easy to plot ζ by drawing streamlines parallel to the flow.

7 Specification of moment

In this section we shall examine the way in which we specify the internal moments $\boldsymbol{\mu}$. It is worth noting that if we set $\mathbf{V} = 0$ in (16)(below) then

$$\mu^1_{.2} = \sqrt{a}M^{11}$$
$$\mu^1_{.1} = -\sqrt{a}M^{12}$$
$$\mu^2_{.2} = \sqrt{a}M^{21}$$
$$\mu^2_{.1} = -\sqrt{a}M^{22}$$

where $\sqrt{a}$ is given in (B3). Thus $\mu^1_{.2}$ and $\mu^2_{.1}$ are the components of bending moments, whereas $\mu^1_{.1}$ and $\mu^2_{.2}$ are the components of twisting moments. This is explained by the fact that in Cartesian coordinates the component of the sagging moment M_{xx} for plate bending acts about the y axis in an anticlockwise direction.

The internal moment $\boldsymbol{\mu}$ contains 6 independent components. However (9) contains 4 vectors, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ making 12 components in all. The reason for doing this is that there is no unique way of finding the internal forces from the internal moments because the term $\boldsymbol{\sigma} : \boldsymbol{\chi}$ in (6) is not influenced by a contribution to $\boldsymbol{\sigma}$ corresponding to a symmetric surface tensor. However, by including $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ we can include all possibilities for $\boldsymbol{\sigma}$ in (8).

It is more usual to write the 6 components of moment as

$$\begin{aligned}\boldsymbol{\mu} &= -\mathbf{M}\cdot\boldsymbol{\epsilon} + \mathbf{V}\cdot(\mathbf{nI}-\mathbf{In})\\ &= -\mathbf{M}\cdot\boldsymbol{\epsilon} - \mathbf{Vn}\end{aligned} \tag{16}$$

where $\mathbf{M}$ is a 2nd order surface tensor, that is a tensor with only surface components and $\mathbf{V}$ is a surface vector which represents the Cosserat geodesic moments. It is usually assumed that $\mathbf{V}=0$ and that $\mathbf{M}$ is symmetric in the theory of plates and shells [18, 35]. If $\mathbf{M}$ is symmetric, then the trace $\mathrm{tr}\,(\boldsymbol{\mu}) = 0$.

Shell structures often have arches, cables or beams as boundaries. If we think of these elements as part of the shell, then they are simply concentrations of force and moment, which we can include in our formulation. Thus if we want a moment about an axis normal to the shell in an arch, it is simply a concentration of geodesic moment. If we want a torsional moment in an arch, it represents a concentration in both the symmetric and antisymmetric parts of $\mathbf{M}$. Thus to exclude geodesic moment and making $\mathbf{M}$ symmetric means that edge arches have to be considered to be separate structures.

If we equate (9) and (16) we obtain

$$\begin{aligned}&\boldsymbol{\epsilon}\cdot\nabla\boldsymbol{\phi} + \nabla\boldsymbol{\alpha} + \boldsymbol{\beta}\cdot(\mathbf{n}\boldsymbol{\epsilon}+\boldsymbol{\epsilon}\mathbf{n})\\ &\quad = -\mathbf{M}\cdot\boldsymbol{\epsilon} + (\mathbf{V}-\boldsymbol{\psi})\cdot(\mathbf{nI}-\mathbf{In})\end{aligned}$$

which means that

$$(\boldsymbol{\epsilon}\cdot\nabla\boldsymbol{\phi} + \nabla\boldsymbol{\alpha})\cdot\mathbf{n} + \boldsymbol{\beta}\cdot\boldsymbol{\epsilon} = -(\mathbf{V}-\boldsymbol{\psi})\cdot(\mathbf{I}-\mathbf{nn}) \tag{17}$$

and

$$\begin{aligned}&(\boldsymbol{\epsilon}\cdot\nabla\boldsymbol{\phi} + \nabla\boldsymbol{\alpha})\cdot(\mathbf{I}-\mathbf{nn}) + (\boldsymbol{\beta}\cdot\mathbf{n})\boldsymbol{\epsilon}\\ &\quad = -\mathbf{M}\cdot\boldsymbol{\epsilon} + ((\mathbf{V}-\boldsymbol{\psi})\cdot\mathbf{n})(\mathbf{I}-\mathbf{nn}).\end{aligned} \tag{18}$$

Thus, taking the trace of (18), and noting that $\mathrm{tr}\,(\mathbf{I}-\mathbf{nn}) = 3-1 = 2$,

$$-\boldsymbol{\epsilon} : \nabla\boldsymbol{\phi} + \nabla\cdot\boldsymbol{\alpha} = \mathbf{M} : \boldsymbol{\epsilon} + 2(\mathbf{V}-\boldsymbol{\psi})\cdot\mathbf{n}.$$

and so (17) gives

$$\begin{aligned}-(\mathbf{V}-\boldsymbol{\psi}) &= (\boldsymbol{\epsilon}\cdot\nabla\boldsymbol{\phi} + \nabla\boldsymbol{\alpha})\cdot\mathbf{n} + \boldsymbol{\beta}\cdot\boldsymbol{\epsilon}\\ &\quad - \frac{1}{2}(-\boldsymbol{\epsilon} : \nabla\boldsymbol{\phi} + \nabla\cdot\boldsymbol{\alpha} - \mathbf{M} : \boldsymbol{\epsilon})\mathbf{n}.\end{aligned}$$

Thus, remembering that $\mathbf{V}$ is a surface vector, we can ensure that $\mathbf{V}=0$ by setting

$$\boldsymbol{\psi}\cdot(\mathbf{I}-\mathbf{nn}) = (\boldsymbol{\epsilon}\cdot\nabla\boldsymbol{\phi} + \nabla\boldsymbol{\alpha})\cdot\mathbf{n} + \boldsymbol{\beta}\cdot\boldsymbol{\epsilon}$$

and we can ensure that $\mathbf{M}$ is symmetric by setting

$$\boldsymbol{\psi}\cdot\mathbf{n} = -\frac{1}{2}(-\boldsymbol{\epsilon} : \nabla\boldsymbol{\phi} + \nabla\cdot\boldsymbol{\alpha}). \tag{19}$$

Thus we can ensure that the geodesic moments represented by $\mathbf{V}$ are zero and that $\mathbf{M}$ is symmetric purely by adjusting $\boldsymbol{\psi}$, regardless of the values of $\boldsymbol{\phi}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.
However, this does not necessarily mean that we have satisfied the boundary conditions.

We can also write (18) as

$$\begin{aligned}&(\boldsymbol{\epsilon}\cdot\nabla\boldsymbol{\phi} + \nabla\boldsymbol{\alpha})\cdot\boldsymbol{\epsilon} - (\boldsymbol{\beta}\cdot\mathbf{n})(\mathbf{I}-\mathbf{nn})\\ &= \mathbf{M} + ((\mathbf{V}-\boldsymbol{\psi})\cdot\mathbf{n})\boldsymbol{\epsilon}\end{aligned}$$

so that

$$\begin{aligned}\frac{1}{2}\left(\mathbf{M}+\mathbf{M}^{\mathrm{T}}\right) &= \frac{1}{2}\boldsymbol{\epsilon}\cdot\left(\nabla\boldsymbol{\phi} + (\nabla\boldsymbol{\phi})^{\mathrm{T}}\right)\cdot\boldsymbol{\epsilon}\\ &\quad + \frac{1}{2}\left(\nabla\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon} + (\nabla\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon})^{\mathrm{T}}\right)\\ &\quad - (\boldsymbol{\beta}\cdot\mathbf{n})(\mathbf{I}-\mathbf{nn})\end{aligned}$$

which applies regardless of whether $\mathbf{V} = 0$ or $\mathbf{M}$ is symmetric. The superscript T is the transpose. The solution of this equation for $\boldsymbol{\phi}$, assuming $\mathbf{M}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are known, is the same as the problem of finding the velocity of a surface for a given rate of membrane strain [18]. To find the velocity we have to introduce another unknown, which is the mean angular velocity and first solve for that. This is equivalent to what we do in Sect. 10.

8 Boundary conditions at a free edge

Timoshenko and Woinowsky-Krieger [35] discuss the boundary conditions at a free edge of an elastic flat plate in §22 of chapter 4. They use the Kirchhoff-Love theory of plates leading to a non-homogeneous biharmonic equation, which only allows 2 boundary conditions at a free edge, whereas there are 2 components of moment and one of vertical shear force.

This problem can be overcome using virtual work, or better puissance virtuelle (virtual power) using velocities and angular velocities because the equation applies to increments of what may be large displacements or rotations. The puissance virtuelle of the twisting moments and vertical shear force are combined since the angular velocity of the boundary curve is not independent of its velocity.

These considerations apply equally to shells. However we do not need to go through this process of reducing the boundary conditions because we are not making the assumptions of the Kirchhoff-Love theory.

Equations (12) and (13) apply equally to an element $d\mathbf{r}$ along a boundary. Along a free boundary we have $d\mathbf{f} = 0$, which can be satisfied by an appropriate choice of $\boldsymbol{\psi}$ and $d\mathbf{m} = 0$, which can be satisfied by an appropriate choice of $\boldsymbol{\phi}$.

> We may have edge elements, such as arches or beams, which simply correspond to concentrations of stress and moment. If we imagine a 'ghost shell' attached to the boundary, then a step change in $\boldsymbol{\beta}$ between the real shell and the ghost shell gives a concentrated force in the edge element and a step change in $\boldsymbol{\alpha}$ gives a concentrated moment.

We also need to think about the boundary conditions that we use in solving the Poisson equations (10) and (11). In fact we can choose whatever we like, either the values of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to give Dirichlet problems, or their normal derivatives to give Neumann problems, or some mixture of the two. In heat flow a specified temperature corresponds to a Dirichlet boundary, whereas an insulator corresponds to a Neumann boundary. The reason that we can choose whatever we like is that we can then satisfy the physical boundary conditions using $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$.

Examination of (12) shows that it makes sense to use the Neumann boundary condition and set the normal derivative of $\boldsymbol{\beta}$ to zero. This corresponds to a 'force insulator' ensuring that no load flows across the free edge. Then we simply have $\boldsymbol{\psi} = \text{constant}$ along a free edge.

It also makes sense to set the normal derivative of $\boldsymbol{\alpha}$ to zero in (13), but we still have to find $\boldsymbol{\phi}$ because $\boldsymbol{\beta}$ and $\boldsymbol{\psi}$ will not be zero.

9 A partially loaded shell

Figure 5 shows a shell which is only loaded in the shaded areas and unloaded elsewhere. In the unloaded regions $\mathbf{p} = 0$ and $\mathbf{c} = 0$ so that (10) and (11) become

$$\nabla^2 \boldsymbol{\beta} = 0$$

and

$$\nabla^2 \boldsymbol{\alpha} + 2H\boldsymbol{\epsilon} \cdot \boldsymbol{\beta} = 0.$$

The discussion in Sect. 8 tells us that we can set $\boldsymbol{\alpha} = \text{constant}$ and $\boldsymbol{\beta} = \text{constant}$ in an area which is in direct contact with a free edge, *provided that we also apply the same conditions to the boundaries of the loaded areas, effectively surrounding the loaded areas with their own edge beams.*

Thus (12) and (13) become

$$d\mathbf{f} = d\boldsymbol{\psi}$$

so that

$$\mathbf{f} = \boldsymbol{\psi} \tag{20}$$

and

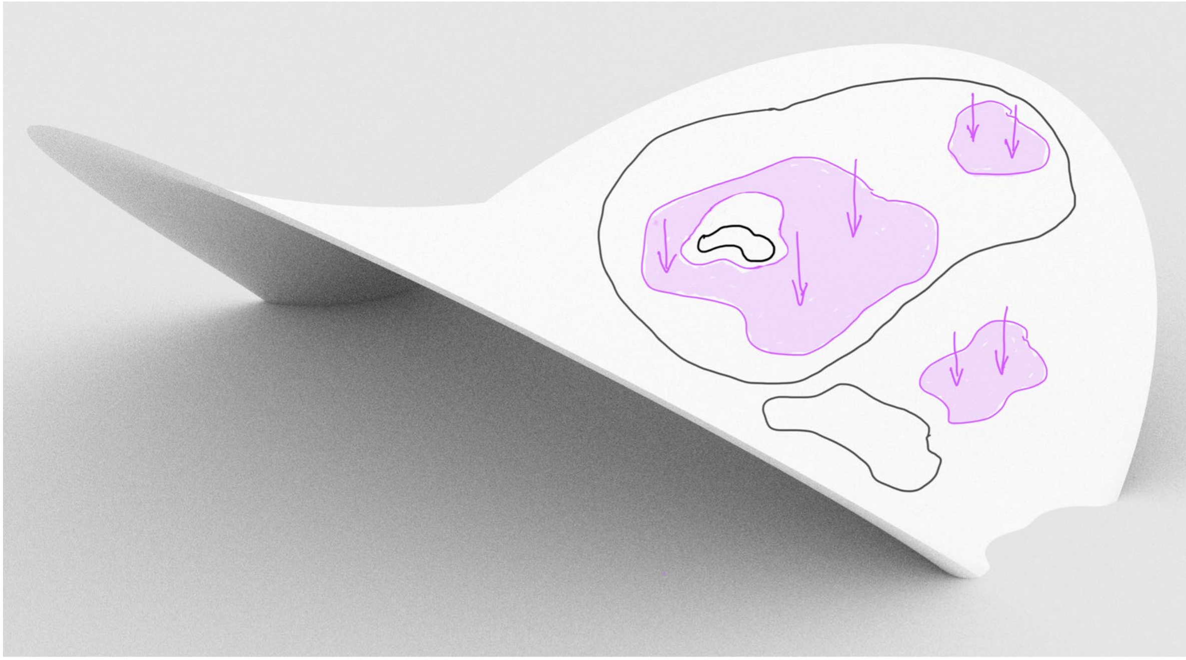

Fig. 5 Partially loaded shell, loaded areas are shown shaded and curve C_2 surrounds load, while C_1 and C_2 do not

$$\begin{aligned} d\mathbf{m} &= d\boldsymbol{\phi} - d\mathbf{r} \cdot \boldsymbol{\epsilon} \cdot (\boldsymbol{\psi} \cdot (\mathbf{nI} - \mathbf{In})) \\ &= d\boldsymbol{\phi} - d\mathbf{r} \cdot (\boldsymbol{\psi} \cdot (\mathbf{n}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{n})). \end{aligned} \tag{21}$$

The integral $\int d\mathbf{f} = 0$ for C_1 in Fig. 5 because it does not surround any load. However for C_2 the integral $\int d\mathbf{f}$ must be equal to minus the load contained within the curve. This means that we have a discontinuity in $\boldsymbol{\psi}$, but no discontinuity in $\nabla\boldsymbol{\psi}$. This discontinuity in $\boldsymbol{\psi}$ does not correspond to a real concentrated force. This might seem strange, but the reason is that we have to imagine some fictitious 'loading framework' which applies the loads and it is that that carries the force due to the discontinuity in $\boldsymbol{\psi}$.

A similar consideration applies to the moment, where now the discontinuity in $\boldsymbol{\phi}$ corresponds to the moment in the loading frame.

However, it may well be easier to instead include the unloaded areas in the solution of the Poisson equations for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, which avoids the discontinuities in $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, and also the need for the fictitious edge beams surrounding the loaded areas.

Equations equivalent to (20) and (21) were presented in a previous paper [1]. However the previous paper did not include $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and so require the discontinuities in $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ to transmit a load through an unloaded region.

10 Membrane stress

In this theory we discuss the membrane theory. As we stated in Sect. 1, the aim is always to ensure that shells act primarily by membrane action whenever possible.

The membrane theory of shells assumes that the applied loading couples $\mathbf{c}$ and internal moments $\boldsymbol{\mu}$ are both zero. It therefore follows from (6) that $\boldsymbol{\sigma} : \boldsymbol{\chi} = 0$ so that the internal force tensor is a symmetric surface tensor. Using (D3) (8) becomes

$$\nabla \cdot \boldsymbol{\psi}\mathbf{n} - \nabla\boldsymbol{\psi} \cdot \mathbf{n} + \nabla\boldsymbol{\beta} : \boldsymbol{\chi} = 0$$

so that

$$\begin{aligned} \nabla \cdot \boldsymbol{\psi} + \nabla\boldsymbol{\beta} : \boldsymbol{\epsilon} &= 0 \\ -\nabla\boldsymbol{\psi} \cdot \mathbf{n} + \boldsymbol{\epsilon} \cdot \nabla\boldsymbol{\beta} \cdot \mathbf{n} &= 0. \end{aligned}$$

We now need to solve these equations for $\boldsymbol{\psi}$, assuming that we have already solved the equilibrium equation for $\boldsymbol{\beta}$. In order to do this let us write $\boldsymbol{\psi}$ in terms of its tangential and normal components as

$$\boldsymbol{\psi} = \widetilde{\boldsymbol{\psi}} + \psi \mathbf{n} \tag{22}$$

where

$$\widetilde{\boldsymbol{\psi}} \cdot \mathbf{n} = 0.$$

Then

$$\begin{aligned} \nabla \cdot \boldsymbol{\psi} &= \nabla \cdot \widetilde{\boldsymbol{\psi}} + \nabla \cdot (\psi \mathbf{n}) \\ &= \nabla \cdot \widetilde{\boldsymbol{\psi}} - 2H\psi = -\nabla \boldsymbol{\beta} : \boldsymbol{\epsilon} \end{aligned} \tag{23}$$

where H is the mean curvature (B7), and

$$\begin{aligned} \nabla \boldsymbol{\psi} \cdot \mathbf{n} &= \nabla \widetilde{\boldsymbol{\psi}} \cdot \mathbf{n} + \nabla(\psi \mathbf{n}) \cdot \mathbf{n} \\ &= -\nabla \mathbf{n} \cdot \widetilde{\boldsymbol{\psi}} + \nabla \psi = \mathbf{b} \cdot \widetilde{\boldsymbol{\psi}} + \nabla \psi \\ &= \boldsymbol{\epsilon} \cdot \nabla \boldsymbol{\beta} \cdot \mathbf{n}. \end{aligned}$$

We can now use (B9) to write

$$\begin{aligned} K\widetilde{\boldsymbol{\psi}} &= \boldsymbol{\epsilon} \cdot \mathbf{b} \cdot \boldsymbol{\epsilon} \cdot (\nabla \psi - \boldsymbol{\epsilon} \cdot \nabla \boldsymbol{\beta} \cdot \mathbf{n}) \\ &= \boldsymbol{\epsilon} \cdot \mathbf{b} \cdot \boldsymbol{\epsilon} \cdot \nabla \psi + \boldsymbol{\epsilon} \cdot \mathbf{b} \cdot \nabla \boldsymbol{\beta} \cdot \mathbf{n}. \end{aligned}$$

where K is the Gaussian curvature.

We can now substitute for $\widetilde{\boldsymbol{\psi}}$ in (23) to give

$$\begin{aligned} &\nabla \cdot \left(\frac{1}{K} \boldsymbol{\epsilon} \cdot \mathbf{b} \cdot \boldsymbol{\epsilon} \cdot \nabla \psi \right) - 2H\psi \\ &= -\nabla \cdot \left(\frac{1}{K} \boldsymbol{\epsilon} \cdot \mathbf{b} \cdot \nabla \boldsymbol{\beta} \cdot \mathbf{n} \right) - \nabla \boldsymbol{\beta} : \boldsymbol{\epsilon} \end{aligned}$$

which is 1 equation in the 1 unknown ψ from which we can calculate $\boldsymbol{\sigma}$. This approach has to be modified when $K = 0$.

The differential equation is elliptic if $K > 0$ and hyperbolic if $K < 0$, and this explains why shells with positive or negative Gaussian curvature behave so differently. Shells made of developable surfaces with $K = 0$, such as cylindrical shells, require a parabolic differential equation to be solved.

An elliptic partial differential equation requires one condition all the way around the boundary and examples include Laplace's equation and Poisson's equation. In the membrane theory this means that a shell with positive Gaussian curvature requires a support all the way around, unless the shape of the shell is such that it is funicular for a particular load. The easiest way to find funicular shapes is to use hanging models in tension which are then inverted to form compression shells, as was done by Antoni Gaudí, Heinz Isler and Frei Otto [2].

A hyperbolic partial differential equation requires two boundary conditions around part of the boundary. An example includes the wave equation for a vibrating string in which the lateral displacement, y, is a function of x and time, t. We have to specify the initial shape and initial velocity along the boundary $t = 0$. Waves will travel along the string with a wave speed c which correspond to lines in (x, t) space. The equivalent to these lines for a membrane shell are the asymptotic lines which are in the directions of zero normal curvature [34]. For a shell with negative Gaussian curvature to be able to carry all loads by membrane action each asymptotic line must have one end attached to a boundary. If both ends are connected to a boundary, then the structure is statically indeterminate and can be prestressed. Of course, structures which are mechanisms like cable nets and tensegrity structures can also be prestressed, but only if they have the correct geometry [3].

Shells with zero Gaussian curvature share some of the properties of shells with positive Gaussian curvature and some of the properties of shells with negative Gaussian curvature. Shell whose Gaussian curvature change from positive to negative are more complicated.

In Sect. 12 we shall derive Pucher's equation which is an alternative way of solving the membrane equations for equilibrium. The form of the equation-elliptic, parabolic or hyperbolic is the same in both cases.

11 A flat plate

In this section we discuss the application of our theory to flat plates, which is the simplest special case of a surface structure containing bending moments.

A flat plate is simply a shell whose normal is in a constant direction given by the unit vector $\mathbf{k}$. The equilibrium Eqs. (5) and (6) are unchanged, as are the stresses and moments in (8) and (9), except for the substitution of $\mathbf{k}$ for $\mathbf{n}$, which we also have to do for $\boldsymbol{\epsilon}$ in (A3), which means that $\nabla \boldsymbol{\epsilon} = 0$. Clearly the mean curvature of a flat plate, H, is zero.

We shall use the words 'vertical' for the direction of $\mathbf{k}$ and 'horizontal' for directions perpendicular to $\mathbf{k}$, even though the flat plate may have a different orientation, as wall for example.

As the equivalent of (22) we will write

$$\boldsymbol{\psi} = \widetilde{\boldsymbol{\psi}} + \psi\mathbf{k}$$

where

$$\widetilde{\boldsymbol{\psi}} \cdot \mathbf{k} = 0.$$

We will assume that the loading couples $\mathbf{c} = 0$ and so (8) becomes

$$\boldsymbol{\sigma} \cdot (\mathbf{I} - \mathbf{kk}) = \boldsymbol{\epsilon} \cdot \nabla\widetilde{\boldsymbol{\psi}} + \nabla\widetilde{\boldsymbol{\beta}} \tag{24}$$

$$\mathbf{Z} = \boldsymbol{\sigma} \cdot \mathbf{k} = \boldsymbol{\epsilon} \cdot \nabla\psi + \nabla\beta \tag{25}$$

in which we have introduced the horizontal vector $\mathbf{Z}$ and set $H = 0$ so that (11) is satisfied by $\boldsymbol{\alpha} = 0$ for a flat plate. Equation (9) becomes

$$\boldsymbol{\mu} \cdot (\mathbf{I} - \mathbf{kk}) = \boldsymbol{\epsilon} \cdot \nabla\widetilde{\boldsymbol{\phi}} + \psi(\mathbf{I} - \mathbf{kk}) + \beta\boldsymbol{\epsilon} \tag{26}$$

$$\boldsymbol{\mu} \cdot \mathbf{k} = \boldsymbol{\epsilon} \cdot \nabla\phi - \widetilde{\boldsymbol{\psi}} + \widetilde{\boldsymbol{\beta}} \cdot \boldsymbol{\epsilon}. \tag{27}$$

If there is no horizontal loading on the plate, then $\widetilde{\boldsymbol{\beta}} = 0$, and if there are no geodesic moments, then $\boldsymbol{\mu} \cdot \mathbf{k} = 0$ and

$$\widetilde{\boldsymbol{\psi}} = \boldsymbol{\epsilon} \cdot \nabla\phi$$

and

$$\boldsymbol{\sigma} \cdot (\mathbf{I} - \mathbf{kk}) = \boldsymbol{\epsilon} \cdot \nabla(\boldsymbol{\epsilon} \cdot \nabla\phi) = -\boldsymbol{\epsilon} \cdot \nabla\nabla\phi \cdot \boldsymbol{\epsilon}.$$

so that ϕ is the Airy stress function [36].

We will now concentrate on vertical equilibrium of forces and moments about horizontal axes. The lower bound theorem of plasticity was applied to the design of slabs by Hillerborg [23, 24] and in applying the lower bound or 'safe' theorem we are only required to satisfy equilibrium and not violate the yield condition for the slab in bending. Johansen's yield line theory [25] is the equivalent upper bound method, but we are concentrating on equilibrium corresponding to the lower bound method.

Using (16) we can rewrite the moments about horizontal axes from (26) as

$$\mathbf{M} = \boldsymbol{\epsilon} \cdot \nabla\widetilde{\boldsymbol{\phi}} \cdot \boldsymbol{\epsilon} + \psi\boldsymbol{\epsilon} - \beta(\mathbf{I} - \mathbf{kk}). \tag{28}$$

The vertical shear force is described by $\mathbf{Z}$ in (25), which can also be written

$$\mathbf{Z} = -\nabla \cdot \mathbf{M}$$

and the load

$$\begin{aligned} p &= -\nabla \cdot \boldsymbol{\sigma} \cdot \mathbf{k} = -\nabla \cdot \mathbf{Z} \\ &= \nabla \cdot (\nabla \cdot \mathbf{M}) = -\nabla^2\beta. \end{aligned} \tag{29}$$

We can again treat $\mathbf{Z}$ as flow of a fluid in 2 dimensional fluid mechanics [26] as we did in Sect. 6. (29) is again Poisson's equation which we need to solve for β if the load per unit area p is known.

$\mathbf{M}$ and $\mathbf{Z}$ may be smoothly distributed over the plate or they may be concentrated in beams or ribs which might be curved in plan. There may be torsional moments in beams in which case $\mathbf{M}$ will not be symmetric, however if we do impose the condition that $\mathbf{M}$ is symmetric then (19) gives

$$\psi = \frac{1}{2}\boldsymbol{\epsilon} : \nabla\widetilde{\boldsymbol{\phi}}$$

so that

$$\begin{aligned} \mathbf{M} &= \boldsymbol{\epsilon} \cdot \nabla\widetilde{\boldsymbol{\phi}} \cdot \boldsymbol{\epsilon} + \frac{1}{2}\boldsymbol{\epsilon} : \nabla\widetilde{\boldsymbol{\phi}}\boldsymbol{\epsilon} - \beta(\mathbf{I} - \mathbf{kk}) \\ &= \frac{1}{2}\left(\nabla\widetilde{\boldsymbol{\phi}} + \left(\nabla\widetilde{\boldsymbol{\phi}}\right)^{\mathrm{T}}\right) \\ &\quad - \left(\nabla \cdot \widetilde{\boldsymbol{\phi}} + \beta\right)(\mathbf{I} - \mathbf{kk}) \end{aligned}$$

which is indeed symmetric. One could equally well write

$$\mathbf{M} = \frac{1}{2}\boldsymbol{\epsilon} \cdot \left(\nabla\widetilde{\boldsymbol{\phi}} + \left(\nabla\widetilde{\boldsymbol{\phi}}\right)^{\mathrm{T}}\right) \cdot \boldsymbol{\epsilon} - \beta(\mathbf{I} - \mathbf{kk}) \tag{30}$$

and these 2 forms are equivalent in light of (B14).

Let us now write the vector $\widetilde{\boldsymbol{\phi}}$ as

$$\widetilde{\boldsymbol{\phi}} = \nabla\lambda + \boldsymbol{\epsilon} \cdot \nabla\omega = \nabla\lambda - \nabla\omega \cdot \boldsymbol{\epsilon}$$

where λ and ω are scalars. Then

$$\begin{aligned} \nabla\widetilde{\boldsymbol{\phi}} &= \nabla\nabla\lambda - \nabla\nabla\omega \cdot \boldsymbol{\epsilon} \\ \nabla \cdot \widetilde{\boldsymbol{\phi}} &= \nabla^2\lambda \end{aligned}$$

and

$$\psi = -\frac{1}{2}\nabla^2\omega \tag{31}$$

so that (28) and (30) become

$$
\begin{aligned}
\mathbf{M} &= \nabla\nabla\lambda + \frac{1}{2}(\boldsymbol{\epsilon}\cdot\nabla\nabla\omega - \nabla\nabla\omega\cdot\boldsymbol{\epsilon}) \\
&\quad - \left(\nabla^2\lambda + \beta\right)(\mathbf{I} - \mathbf{kk}) \\
&= \boldsymbol{\epsilon}\cdot\nabla\nabla\lambda\cdot\boldsymbol{\epsilon} + \frac{1}{2}(\boldsymbol{\epsilon}\cdot\nabla\nabla\omega - \nabla\nabla\omega\cdot\boldsymbol{\epsilon}) \\
&\quad - \beta(\mathbf{I} - \mathbf{kk}) \\
&= \boldsymbol{\epsilon}\cdot\nabla\nabla\lambda\cdot\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\cdot\nabla\nabla\omega \\
&\quad - \frac{1}{2}\nabla^2\omega\boldsymbol{\epsilon} - \beta(\mathbf{I} - \mathbf{kk})
\end{aligned} \tag{32}
$$

and (25) becomes

$$
\mathbf{Z} = -\frac{1}{2}\boldsymbol{\epsilon}\cdot\nabla\left(\nabla^2\omega\right) + \nabla\beta.
$$

We can use the identity (B14) to compare (32) with the equivalent expression for an elastic plate [35], together with the equivalents of (25) and (29). For an elastic plate we have

$$
\begin{aligned}
\mathbf{M} &= D(\nabla\nabla u - \nu\boldsymbol{\epsilon}\cdot\nabla\nabla u\cdot\boldsymbol{\epsilon}) \\
&= D((1+\nu)\nabla\nabla u - \nu\,\mathrm{tr}\,(\mathbf{Q})(\mathbf{I} - \mathbf{kk})) \\
\mathbf{Z} &= -D\nabla\cdot(\nabla\nabla u) \\
p &= D\nabla^4 u
\end{aligned}
$$

where u is the deflection of the plate, which is assumed to be small, D is the bending stiffness and ν is Poisson's ratio. The discussion in Sect. 8 explains how the boundary conditions for a free edge can be satisfied when only u and its normal derivative that can be adjusted.

11.1 Principal directions and discontinuities

The only differential equation the we have is (29) for β in terms of the load per unit area p. In general loads will be distributed, but we can have line loads or point loads for which the load per unit area is infinite and are equivalent to line or point sources in the Poisson equation. The total load, including support reactions, will always be zero for overall equilibrium.

The scalar λ in (32) acts very much like the Airy stress function and a 'fold' in the λ surface corresponds to a concentrated moment acting about an axis perpendicular to the fold, and so the fold represents a beam. This corresponds to the fact that the principal directions of the curvature of the λ surface and the moment are interchanged.

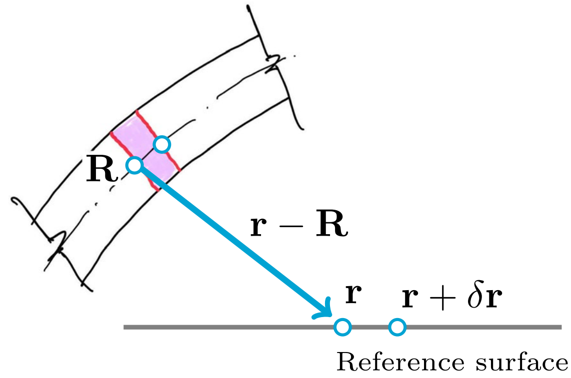

Fig. 6 Separate moment and reference surfaces

Let us imagine that the ω surface is defined locally by

$$
\omega = \kappa_x\frac{x^2}{2} + \kappa_y\frac{y^2}{2}
$$

so that the principal curvatures of the ω surface are κ_x in the x direction and κ_y in the y direction.

Then

$$
\frac{1}{2}(\boldsymbol{\epsilon}\cdot\nabla\nabla\omega - \nabla\nabla\omega\cdot\boldsymbol{\epsilon}) = \left(\kappa_y - \kappa_x\right)(\mathbf{ij} + \mathbf{ji})
$$

and so the principal values of that part of the moment due to ω are at 45° to those of the ω surface itself.

12 A separate moment surface

In Sect. 4 we discussed the introduction of 'moment surface' which is distinct from the 'reference surface'. The reference surface then becomes a map which we use to refer to points on the moment surface which better represents the shape of the shell. We can choose a simple shape for the reference surface upon which we do our calculations, typically a plane, provided that we take into account the change of lever arm when we take moments.

In Eqs. (1) and (2), which we repeat again here,

$$
d\mathbf{f} = (d\mathbf{r}\times\mathbf{n})\cdot\boldsymbol{\sigma} = d\mathbf{r}\cdot(\mathbf{n}\times\boldsymbol{\sigma}) = -d\mathbf{r}\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma} \tag{1}
$$

$$
d\mathbf{m} = (d\mathbf{r}\times\mathbf{n})\cdot\boldsymbol{\mu} = d\mathbf{r}\cdot(\mathbf{n}\times\boldsymbol{\mu}) = -d\mathbf{r}\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{\mu} \tag{2}
$$

the internal moment $d\mathbf{m}$ is acting about the point $\mathbf{r}$ on the reference surface, S, which is used to define the saw blade, but note that the saw blade does not necessarily pass through the point $\mathbf{r}$. We design the shell to resist the element of force $d\mathbf{f}$ and moment $d\mathbf{m}$ crossing that part of the shell corresponding to $d\mathbf{r}$. But now let us suppose that the actual shell is separated from $\mathbf{r}$ by some distance, which might be large.

The point $\mathbf{r}$ is an arbitrary point on the cut represented by the arbitrary curve C on the surface S. Thus $\mathbf{r}$ is an arbitrary point on S and in the following which shall write 'on $\mathbf{r}$' to mean 'on the surface S' to avoid having to introduce yet more symbols.

$\mathbf{R}$ is a point on the moment surface which better represents the shell as shown in Fig. 6. Each point on $\mathbf{R} = \mathbf{R}(\mathbf{r})$ corresponds to a point on the reference surface $\mathbf{r}$. In the figure $\mathbf{r}$ is on a flat surface, but as we stated above, it could be on some other shape.

If $\mathbf{r}$ is on a flat surface, then the simplest thing to do would be to have $\mathbf{R}$ vertically above $\mathbf{r}$, so that $\mathbf{R}$ is defined simply by its height above $\mathbf{r}$ as is done in §10.6 of Green & Zerna [18].

Let us suppose that $d\overset{\mathbf{R}}{\mathbf{m}}$ is the moment acting about the point $\mathbf{R}$ on the moment surface then

$$\begin{aligned} d\overset{\mathbf{R}}{\mathbf{m}} &= d\mathbf{m} + (\mathbf{r} - \mathbf{R}) \times d\mathbf{f} \\ &= (d\mathbf{r} \times \mathbf{n}) \cdot \boldsymbol{\mu} - (d\mathbf{r} \times \mathbf{n}) \cdot \boldsymbol{\sigma} \times (\mathbf{r} - \mathbf{R}) \\ &= (d\mathbf{r} \times \mathbf{n}) \cdot \boldsymbol{\Upsilon} \end{aligned}$$

where the tensor representing moments is

$$\begin{aligned} \boldsymbol{\Upsilon} &= \boldsymbol{\mu} - \boldsymbol{\sigma} \times (\mathbf{r} - \mathbf{R}) \\ &= \boldsymbol{\mu} - (\boldsymbol{\sigma}(\mathbf{r} - \mathbf{R})) : \boldsymbol{\chi} \end{aligned} \tag{33}$$

in which $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are given by (8) and (9).

Note that we use $(d\mathbf{r} \times \mathbf{n}) \cdot \boldsymbol{\Upsilon}$ and not $(d\mathbf{R} \times \mathbf{N}) \cdot \boldsymbol{\Upsilon}$, in which $\mathbf{N}$ is the unit normal to $\mathbf{R}$, and so $\boldsymbol{\Upsilon}$, which is an uppercase upsilon, represents the moment acting about a point on $\mathbf{R}$ per unit length on $\mathbf{r}$.

The loading couple $\mathbf{c}$ is about a point on $\mathbf{r}$. The corresponding loading couple $\mathbf{C}$ about a point on $\mathbf{R}$ is

$$\mathbf{C} = \mathbf{c} + (\mathbf{r} - \mathbf{R}) \times \mathbf{p} \tag{34}$$

where $\mathbf{p}$ is the loading force per unit area on $\mathbf{r}$. Note that even though $\mathbf{C}$ is applied on $\mathbf{R}$, it is an applied couple per unit area on $\mathbf{r}$.

We can define the gradient

$$d\boldsymbol{\Upsilon} = d\mathbf{r} \cdot \nabla\boldsymbol{\Upsilon}$$

which gives us the change in $\boldsymbol{\Upsilon}$ on $\mathbf{R}$ in terms of a displacement $d\mathbf{r}$ on $\mathbf{r}$.

The gradient of $\mathbf{R}$ tells us the displacement on $\mathbf{R}$ in terms of a displacement on $\mathbf{r}$,

$$d\mathbf{R} = d\mathbf{r} \cdot \nabla\mathbf{R}.$$

We can form the divergence of (33),

$$\begin{aligned} \nabla \cdot \boldsymbol{\Upsilon} &= \mathbf{a}^{\alpha} \cdot \boldsymbol{\Upsilon}_{,\alpha} \\ &= \nabla \cdot \boldsymbol{\mu} - \nabla \cdot \boldsymbol{\sigma} \times (\mathbf{r} - \mathbf{R}) \\ &\quad - \mathbf{a}^{\alpha} \cdot \boldsymbol{\sigma} \times \left(\mathbf{a}_{\alpha} - \mathbf{a}_{\alpha} \cdot \nabla\mathbf{R}\right) \\ &= \nabla \cdot \boldsymbol{\mu} + \mathbf{p} \times (\mathbf{r} - \mathbf{R}) \\ &\quad - \mathbf{a}^{\alpha} \cdot \boldsymbol{\sigma}\left(\left(\mathbf{a}_{\alpha} - \mathbf{a}_{\alpha} \cdot \nabla\mathbf{R}\right)\right) : \boldsymbol{\chi} \\ &= \nabla \cdot \boldsymbol{\mu} + \mathbf{c} - \mathbf{C} + \boldsymbol{\sigma} : \boldsymbol{\chi} + \left(\boldsymbol{\sigma}^{\mathrm{T}} \cdot \nabla\mathbf{R}\right) : \boldsymbol{\chi} \end{aligned}$$

in which we have used the equation of equilibrium of force, (5) and (34).

Thus the equation of equilibrium of moments, (6), which we repeat here,

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\sigma} : \boldsymbol{\chi} + \mathbf{c} = 0 \tag{6}$$

becomes

$$\nabla \cdot \boldsymbol{\Upsilon} + \left((\nabla\mathbf{R})^{\mathrm{T}} \cdot \boldsymbol{\sigma}\right) : \boldsymbol{\chi} + \mathbf{C} = 0.$$

We could at this stage examine what happens to $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ and $\boldsymbol{\beta}$ when we are considering moments referred to $\mathbf{R}$ in (8) and (9).

However, let us instead consider a much simpler case and derive Pucher's equation which is described in §113 of Timoshenko and Woinowsky-Krieger [35], although our derivation is quite different to theirs.

Let us take $\mathbf{r}$ to be a flat surface, in which case we can use the results in Sect. 11, and let us assume that a point on $\mathbf{R}$ is a height z immediately above the corresponding point on $\mathbf{r}$. Thus

$$\mathbf{R} - \mathbf{r} = z\mathbf{k} \tag{35}$$

where $\mathbf{k}$ is a vertical unit vector. Let us further assume for simplicity that there are no horizontal loads and so vertical loads follow the direction from $\mathbf{R}$ to $\mathbf{r}$ and therefore have no eccentricity.

Let us write $\widetilde{\boldsymbol{\beta}} = 0$ and $\widetilde{\boldsymbol{\psi}} = \boldsymbol{\epsilon} \cdot \nabla\phi$ in (24) and (25) to give

$$\boldsymbol{\sigma} = -\boldsymbol{\epsilon} \cdot \nabla\nabla\phi \cdot \boldsymbol{\epsilon} + (\boldsymbol{\epsilon} \cdot \nabla\psi + \nabla\beta)\mathbf{k} \tag{36}$$

and also in (26) and (27),

$$\boldsymbol{\mu} = \boldsymbol{\epsilon} \cdot \nabla\widetilde{\boldsymbol{\phi}} + \psi(\mathbf{I} - \mathbf{kk}) + \beta\boldsymbol{\epsilon} \tag{37}$$

so that there are no moments about a vertical axis.

Thus

$$\nabla \cdot \boldsymbol{\mu} = \nabla\psi + \nabla\beta \cdot \boldsymbol{\epsilon}$$

and

$$\begin{aligned}\boldsymbol{\sigma} : \boldsymbol{\chi} &= ((\boldsymbol{\epsilon} \cdot \nabla\psi + \nabla\beta)\mathbf{k}) : \boldsymbol{\chi} \\ &= -(\boldsymbol{\epsilon} \cdot \nabla\psi + \nabla\beta) \cdot \boldsymbol{\epsilon} = -\nabla \cdot \boldsymbol{\mu}\end{aligned}$$

so that equilibrium of moments is satisfied, as we would have expected. The vertical load per unit area is given by (29), $p = -\nabla^2\beta$.

Thus far we are considering a flat plate carrying the load p by bending moments and vertical shear forces. But now let us use (33) to find the moments in the surface $\mathbf{R}$. Substituting (36) and (37) into (33), and using (35),

$$\begin{aligned}\boldsymbol{\Upsilon} &= \boldsymbol{\epsilon} \cdot \nabla\widetilde{\boldsymbol{\phi}} + \psi(\mathbf{I} - \mathbf{kk}) + \beta\boldsymbol{\epsilon} \\ &\quad + ((-\boldsymbol{\epsilon} \cdot \nabla\nabla\phi \cdot \boldsymbol{\epsilon} + (\boldsymbol{\epsilon} \cdot \nabla\psi + \nabla\beta)\mathbf{k})z\mathbf{k}) : \boldsymbol{\chi} \\ &= \boldsymbol{\epsilon} \cdot \nabla\widetilde{\boldsymbol{\phi}} + \psi(\mathbf{I} - \mathbf{kk}) + \beta\boldsymbol{\epsilon} \\ &\quad - z(-\boldsymbol{\epsilon} \cdot \nabla\nabla\phi \cdot \boldsymbol{\epsilon}) \cdot \boldsymbol{\epsilon} \\ &= \boldsymbol{\epsilon} \cdot \nabla\widetilde{\boldsymbol{\phi}} + \psi(\mathbf{I} - \mathbf{kk}) + \beta\boldsymbol{\epsilon} - z\boldsymbol{\epsilon} \cdot \nabla\nabla\phi.\end{aligned}$$

Thus if we set $\boldsymbol{\Upsilon} = 0$ for pure membrane action, we have 4 equations from which we could find $\widetilde{\boldsymbol{\phi}}$, ψ and β for given z and ϕ. But it simpler to just write

$$\boldsymbol{\mu} = z\boldsymbol{\epsilon} \cdot \nabla\nabla\phi \tag{38}$$

so that

$$\nabla \cdot \boldsymbol{\mu} = \nabla z \cdot \boldsymbol{\epsilon} \cdot \nabla\nabla\phi$$

and therefore equilibrium of moments is satisfied by

$$\boldsymbol{\sigma} = -\boldsymbol{\epsilon} \cdot \nabla\nabla\phi \cdot \boldsymbol{\epsilon} - \nabla z \cdot \boldsymbol{\epsilon} \cdot \nabla\nabla\phi \cdot \boldsymbol{\epsilon}\mathbf{k}$$

since

$$(\boldsymbol{\epsilon}\mathbf{k}) : \boldsymbol{\chi} = \mathbf{I} - \mathbf{kk}.$$

Thus the load,

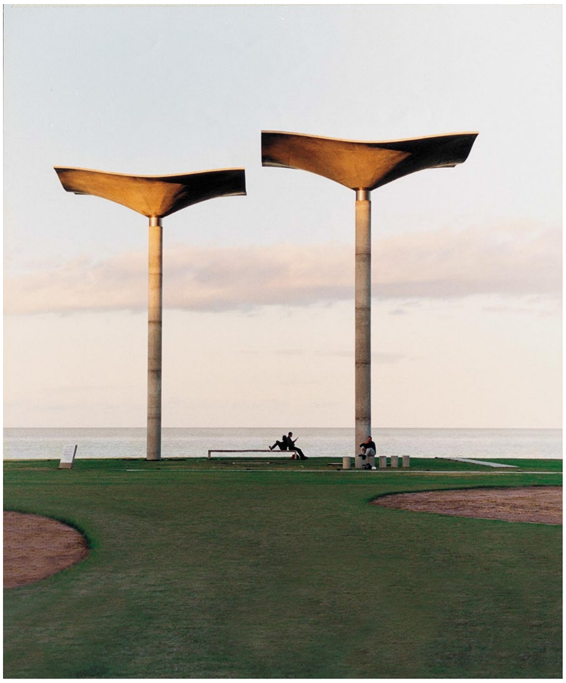

Fig. 7 Monumento en homenaje a Amancio Williams 1999, by Claudio Williams and Claudio Vekstein Photo: Fonds Amancio Williams, Centre Canadien d'Architecture, Don des enfants d'Amancio Williams

$$\begin{aligned}p\mathbf{k} &= -\nabla \cdot \boldsymbol{\sigma} = \nabla\nabla z : (\boldsymbol{\epsilon} \cdot \nabla\nabla\phi \cdot \boldsymbol{\epsilon})\mathbf{k} \\ &= -\epsilon^{\alpha\lambda}\epsilon^{\beta\mu} z|_{\alpha\beta}\phi|_{\lambda\mu}\mathbf{k}\end{aligned}$$

and in Cartesian coordinates we have the well known expression [35]

$$\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 z}{\partial y^2} + p = 0 \tag{39}$$

which is Pucher's equation.

13 An example

Let us imagine that we are designing a shell of a form similar to the Monumento en homenaje a Amancio Williams in Fig. 7, and that we expect it to act partly as a shell and partly as a plate in bending.

We can begin by considering a flat plate containing bending moments, vertical shear forces and horizontal membrane stresses. We can then project up onto the actual shell represented by $\mathbf{R}$ as described in Sect. 12.

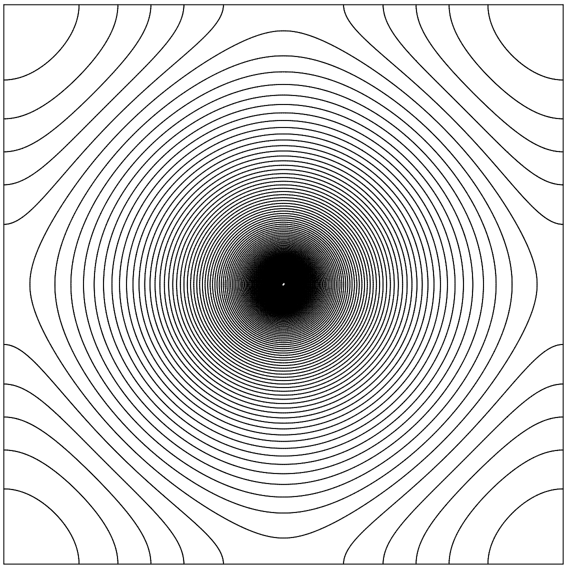

Fig. 8 Contour lines for β. The corresponding vector field is normal to the contour lines

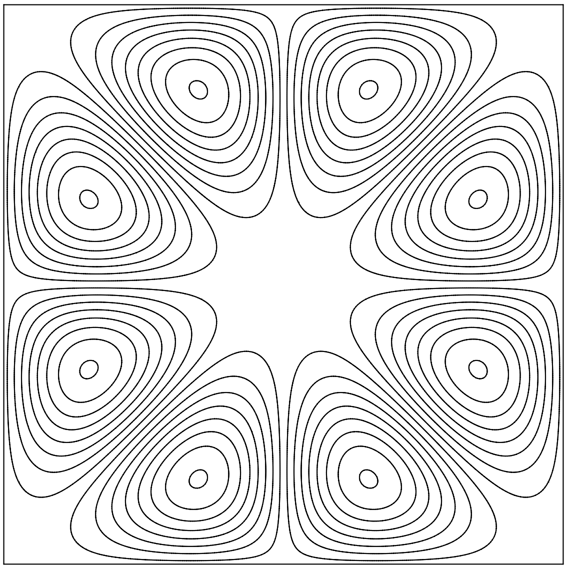

Fig. 9 Contour lines for ψ which we can consider as streamlines. The corresponding vector field is tangential to the streamlines

The vertical force will 'flow' towards the column in exactly the same way as that for an interior bay of a slab supported on a square grid of columns. We know that the load will not be uniform because the concrete thickness will vary, as does the slope of the shell, but let us for simplicity imagine that the load p is constant. The solution to the Poisson equation (29), $p = -\nabla^2\beta$ is

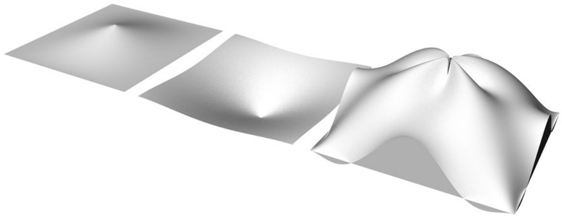

Fig. 10 ϕ left, z centre and $-p$ right

$$\beta = \frac{py^2}{2} - \frac{p}{2\pi}\sum_{n=-\infty}^{n=\infty} \Re\left\{ \log\left(\sin\frac{\pi(x+i(y-nL))}{L} \right) \right\}$$

in which L is the column spacing and the summation is obtained from rows of sinks [33] to produce the square grid. $\Re\{\}$ indicates the real part and $i = \sqrt{-1}$. The summation is stopped after a finite number of terms.

β is plotted in Fig. 8 and this distribution for β is unique, except for the addition of an arbitrary constant, for the given loading and symmetry conditions. β corresponds to an isotropic moment and the corresponding vertical shear force flows in a direction perpendicular to the contour lines of β. It can be seen that the shear force flows parallel to the edges of the shell, as we would expect. However, we still have to satisfy the condition of zero moment on the free edges.

Figure 9 shows a plot of ψ from (31), which was obtained from a possible candidate for ω in (32),

$$\omega = \sin\frac{2\pi x}{L}\sin\frac{2\pi y}{L}\sin\frac{2\pi(x+y)}{2L}\sin\frac{2\pi(x-y)}{2L}.$$

Unlike in Fig. 8 where the flow of shear force is perpendicular to the contour lines, in Fig. 9 the flow is parallel to the stream lines. The total shear force is given by the sum of the two flows. Because the flow is conserved in Fig. 9 this contribution to shear force is carrying no load.

The actual distribution of ω and hence ψ is arbitrary, but Fig. 9 has the correct symmetry.

We still have to achieve zero moments on the free edge. Symmetry tells us that we only have bending moments and not twisting moments to remove and we could do this simply by choosing α in (32) of the form

$$\alpha = f(x) + f(y)$$

where $f()$ is a Fourier series chosen to balance the moments due to β and ψ.

Finally we can use the technique in Sect. 12 to reduce the moments by shell action. Here we have choice of the shell shape, given by z and the Airy stress function ϕ. Figure 10 shows possible shapes for ϕ, z given by

$$\phi = -\text{constant} \times \sqrt{1 - \left(\frac{4x^2}{L^2} - 1\right)^2 \left(\frac{4y^2}{L^2} - 1\right)^2}$$

and

$$z = \text{constant} \times \sqrt{\frac{\left(x^2 + y^2\right)}{L^2} - \frac{\left(x^2 + y^2\right)^2}{L^4}}.$$

Fig. 10 also shows $-p$ obtained from (39). As expected membrane action cannot support the corners. However it can support load towards the middle of the edges since there can be tension parallel to the edge and there is curvature in the direction parallel to the edge. In combining the effects of membrane action (that is Calladine's stretching action) and bending action we can either think in terms of their contribution to load or in the contribution of membrane action in reducing moment, in effect using the $z\boldsymbol{\epsilon} \cdot \nabla\nabla\phi$ in (38) where $\nabla\nabla\phi$ is the force per unit width and z is the lever arm.

Thus there is tremendous freedom in controlling the moments in the shell. Having decided the moments that we would like to achieve, we can try adjusting the shell thickness, introducing ribs or prestressing to control the moments.

In this section we have concentrated upon moments acting about a horizontal axis, but if desired the geodesic component of moment can be removed by adjusting ψ in (9) without affecting equilibrium, although ψ does affect boundary conditions.

14 Conclusions

The design of shells which combine both bending and membrane actions is difficult, although their analysis using the finite element method presents no problem. The ideas in this paper enable the designer to better understand how bending and membrane actions can be combined, and also how to interpret the results of a finite element analysis in order to improve a design. As always, if a part of a structure is over stressed in bending, it might be better to make it thinner, rather than thicker, in order to send the moment somewhere else.

However there are real difficulties in applying the theory, and therefore ways of relating the theory to the finite element method should be investigated.

A finite element analysis will produce all the internal forces and moments in a shell structure, which will be concentrated in the members if it is a gridshell. The internal forces can be resolved into their Cartesian components,

$$\boldsymbol{\sigma}_x = \boldsymbol{\sigma} \cdot \mathbf{i}$$
$$\boldsymbol{\sigma}_y = \boldsymbol{\sigma} \cdot \mathbf{j}$$
$$\boldsymbol{\sigma}_z = \boldsymbol{\sigma} \cdot \mathbf{k}$$

and the internal moments can be resolved into their Cartesian components,

$$\boldsymbol{\mu}_x = \boldsymbol{\mu} \cdot \mathbf{i}$$
$$\boldsymbol{\mu}_y = \boldsymbol{\mu} \cdot \mathbf{j}$$
$$\boldsymbol{\mu}_z = \boldsymbol{\mu} \cdot \mathbf{k}.$$

$\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$, $\boldsymbol{\sigma}_z$, $\boldsymbol{\mu}_x$, $\boldsymbol{\mu}_y$ and $\boldsymbol{\mu}_z$ are not strictly vector fields since the depend on the directions of the Cartesian axes, but nevertheless they are tangent to the reference surface, even if we have chosen the reference surface of a curved shell to be a flat plane.

$\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$, $\boldsymbol{\sigma}_z$, $\boldsymbol{\mu}_x$, $\boldsymbol{\mu}_y$ and $\boldsymbol{\mu}_z$ could then be plotted each using the equivalent of ξ and ζ in (15).

Finally, our understanding of shell structures is partly mathematical and partly qualitative as part of our presence in the physical world and the ideas presented in this paper may be of more help in the latter. As Marcel Proust wrote in the introduction to his 1904 translation into French of John Ruskin's The Bible of Amiens,

> Mais une cathédrale n'est pas seulement une beauté à sentir. Si même ce n'est plus pour vous un enseignement à suivre, c'est du moins encore un livre à comprendre.

> But a cathedral is not only a thing of beauty to be felt. It may, for you, no longer be a lesson to be followed, but at least it is a book to understand.

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Availability of data and materials All information necessary to replicate the work in this paper is contained within the paper itself.

Declarations

Ethical approval Not applicable.

Appendix A: Some tensor properties

There are two tensors with special significance in 3 dimensions. Firstly there is the unit tensor $\mathbf{I}$ which is a 2$^{\text{nd}}$ order tensor with the property

$$\mathbf{v} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{v} = \mathbf{v} \tag{A1}$$

where $\mathbf{v}$ is any vector. In Cartesian coordinates

$$\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk}$$

which is equation (33) in Chapter V of Wilson [37].

Secondly, the Levi-Civita permutation pseudotensor $\boldsymbol{\chi}$ has the property

$$\mathbf{u} \times \mathbf{v} = -\mathbf{u} \cdot \boldsymbol{\chi} \cdot \mathbf{v}$$

in which $\mathbf{u} \times \mathbf{v}$ is the vector product of any two tensors. In Cartesian coordinates

$$\boldsymbol{\chi} = \mathbf{ijk} + \mathbf{jki} + \mathbf{kij} - \mathbf{kji} - \mathbf{ikj} - \mathbf{jik} \tag{A2}$$

and we can also use the double dot notation

$$(...\mathbf{uv}) : (\mathbf{xy}...) = ...(\mathbf{u} \cdot \mathbf{x})(\mathbf{v} \cdot \mathbf{y})...$$

to write

$$\mathbf{u} \times \mathbf{v} = (\mathbf{uv}) : \boldsymbol{\chi} = \mathbf{v} \cdot (\mathbf{u} \cdot \boldsymbol{\chi}) = \boldsymbol{\chi} : (\mathbf{uv}).$$

We shall also use the antisymmetric 2$^{\text{nd}}$ order pseudotensor known as the surface permutation tensor [18],

$$\boldsymbol{\epsilon} = \boldsymbol{\chi} \cdot \mathbf{n} = \mathbf{n} \cdot \boldsymbol{\chi} \tag{A3}$$

and note that while $\boldsymbol{\chi}$ is a constant, $\boldsymbol{\epsilon}$ is not since it varies with the orientation of the unit normal to the reference surface, $\mathbf{n}$.

We also need to be able to examine how some scalar, vector and higher order tensor quantity, $\mathbf{Q}$ varies over our reference surface. The gradient of $\mathbf{Q}$, which is written $\nabla\mathbf{Q}$ is defined such that

$$d\mathbf{Q} = d\mathbf{r} \cdot \nabla\mathbf{Q} \tag{A4}$$

and

$$\mathbf{n} \cdot \nabla\mathbf{Q} = 0$$

for any small displacement $d\mathbf{r}$ on the surface. It is important to write the $d\mathbf{r}$ in front of the $\nabla\mathbf{Q}$ because $d\mathbf{r} \cdot \nabla\mathbf{Q}$ will not in general be the same as $\nabla\mathbf{Q} \cdot d\mathbf{r}$. $\nabla\mathbf{Q}$ is a tensor of order one higher than $\mathbf{Q}$, so that the gradient of a scalar is a vector, and so on. The gradient is a real physical quantity and it exists independent of any coordinate system, and so it is preferable to avoid coordinates in its definition, although clearly they will be useful in doing calculations.

Appendix B: Tensors in curvilinear coordinates on a surface

If we have surface coordinates or parameters θ^1 and θ^2, replacing the u and v that are often used, then

$$\mathbf{a}_\alpha = \frac{\partial\mathbf{r}}{\partial\theta^\alpha} = \mathbf{r}_{,\alpha} \tag{B1}$$

and

$$\mathbf{a}^\alpha = \nabla\theta^\alpha \tag{B2}$$

are the covariant and contravariant base vectors on the surface respectively. Greek indices, are assumed to have the value 1 or 2, and in general we will use the notation in Green and Zerna [18], except we use $\mathbf{n}$ for the unit normal, whereas they use $\mathbf{a}_3 = \mathbf{a}^3$. Because surfaces are curved we have to use the curvilinear tensor notation, rather than Cartesian tensors.

We have

$$\begin{aligned}\mathbf{a}_\alpha \cdot \mathbf{n} &= 0\\ \mathbf{a}^\alpha \cdot \mathbf{n} &= 0\end{aligned}$$

and

$$d\theta^\alpha = d\mathbf{r} \cdot \nabla\theta^\alpha = \sum_{\beta=1}^{2} \left(\mathbf{a}_\beta d\theta^\beta\right) \cdot \mathbf{a}^\alpha$$

so that

$$\begin{aligned}\mathbf{a}_\beta \cdot \mathbf{a}^\alpha &= \delta^\alpha_\beta\\ &= 1 \text{ if } \alpha = \beta\\ &= 0 \text{ if } \alpha \neq \beta.\end{aligned}$$

δ^α_β is the Kronecker delta. The Einstein summation convention enables us to write

$$\sum_{\beta=1}^{2} \left(\mathbf{a}_\beta d\theta^\beta\right) = \mathbf{a}_\beta d\theta^\beta$$

in which the summation is implied whenever the same index is repeated as a subscript and as a superscript.

We can use the contravariant base vectors to write the gradient

$$\nabla\mathbf{Q} = \mathbf{a}^\alpha \frac{\partial \mathbf{Q}}{\partial\theta^\alpha} = \mathbf{a}^\alpha \mathbf{Q}_{,\alpha}$$

because then

$$d\mathbf{Q} = d\mathbf{r} \cdot \nabla\mathbf{Q} = d\theta^\beta \mathbf{a}_\beta \cdot \mathbf{a}^\alpha \mathbf{Q}_{,\alpha} = d\theta^\alpha \mathbf{Q}_{,\alpha}$$

as required.

The gradient of $\mathbf{r}$ itself is

$$\nabla\mathbf{r} = \mathbf{I} - \mathbf{nn}$$

which satisfies $d\mathbf{r} = d\mathbf{r} \cdot \nabla\mathbf{r}$ and $\mathbf{n} \cdot \nabla\mathbf{r} = 0$. Using the summation convention we can also write

$$\nabla\mathbf{r} = \mathbf{a}^\alpha \mathbf{a}_\alpha = a^{\alpha\beta} \mathbf{a}_\alpha \mathbf{a}_\beta = a_{\alpha\beta} \mathbf{a}^\alpha \mathbf{a}^\beta$$

in which

$$\begin{aligned}a^{\alpha\beta} &= \mathbf{a}^\alpha \cdot \mathbf{a}^\beta\\ a_{\alpha\beta} &= \mathbf{a}_\alpha \cdot \mathbf{a}_\beta.\end{aligned}$$

$a_{\alpha\beta}$ are known as the coefficients of the first fundamental form [15, 34] or the components of the metric tensor because

$$d\mathbf{r} \cdot d\mathbf{r} = a_{\alpha\beta} d\theta^\alpha d\theta^\beta.$$

$a^{\alpha\beta}$ and $a_{\alpha\beta}$ are also used for raising and lowering indices, so that for the vector

$$\begin{aligned}\mathbf{v} &= v^\alpha \mathbf{a}_\alpha + v\mathbf{n} = v_\alpha \mathbf{a}^\alpha + v\mathbf{n}\\ v &= \mathbf{v} \cdot \mathbf{n}\\ v_\alpha &= \mathbf{v} \cdot \mathbf{a}_\alpha = a_{\alpha\beta} v^\beta\\ v^\alpha &= \mathbf{v} \cdot \mathbf{a}^\alpha = a^{\alpha\beta} v_\beta.\end{aligned}$$

$\boldsymbol{\epsilon}$ in (A3) can be written in terms of its components,

$$\begin{aligned}\boldsymbol{\epsilon} &= \epsilon_{\lambda\mu} \mathbf{a}^\lambda \mathbf{a}^\mu\\ \epsilon_{\lambda\mu} &= -\epsilon_{\mu\lambda}\\ \epsilon_{11} &= 0\\ \epsilon_{22} &= 0\\ \epsilon_{12} &= -\epsilon_{21} = \sqrt{a}\\ a &= a_{11}a_{22} - a_{12}^2\end{aligned} \tag{B3}$$

which enable us to write

$$\boldsymbol{\chi} = \epsilon_{\lambda\mu}\left(\mathbf{a}^\lambda \mathbf{a}^\mu \mathbf{n} + \mathbf{n}\mathbf{a}^\lambda \mathbf{a}^\mu + \mathbf{a}^\mu \mathbf{n}\mathbf{a}^\lambda\right). \tag{B4}$$

Therefore

$$\begin{aligned}\boldsymbol{\epsilon} \cdot \boldsymbol{\chi} &= \epsilon^{\alpha\beta}\epsilon_{\lambda\mu} \mathbf{a}_\alpha \mathbf{a}_\beta \cdot \left(\mathbf{a}^\lambda \mathbf{a}^\mu \mathbf{n} + \mathbf{n}\mathbf{a}^\lambda \mathbf{a}^\mu + \mathbf{a}^\mu \mathbf{n}\mathbf{a}^\lambda\right)\\ &= -\mathbf{a}_\alpha \mathbf{a}^\alpha \mathbf{n} + \mathbf{a}_\alpha \mathbf{n}\mathbf{a}^\alpha\\ &= -(\mathbf{I} - \mathbf{nn})\mathbf{n} + \mathbf{a}_\alpha \mathbf{n}\mathbf{a}^\alpha\end{aligned} \tag{B5}$$

which is difficult to write without having recourse to base vectors. We also have

$$\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon} = \boldsymbol{\epsilon} \cdot \boldsymbol{\chi} \cdot \mathbf{n} = -(\mathbf{I} - \mathbf{nn})$$

and the 3rd order tensor

$$\boldsymbol{\Xi} = \frac{1}{2}\mathbf{n}\boldsymbol{\epsilon} - \boldsymbol{\epsilon}\mathbf{n} \tag{B6}$$

has the useful properties that

$$\boldsymbol{\Xi} : \boldsymbol{\chi} = \frac{1}{2} 2\mathbf{nn} - \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon} = \mathbf{I}$$

and

$$\mathbf{n} \cdot (\mathbf{v} \cdot \boldsymbol{\Xi}) = 0$$

where $\mathbf{v}$ is any vector.

The normal curvature tensor, or shape operator, $\mathbf{b}$ is minus the gradient of the normal,

$$\mathbf{b} = -\nabla \mathbf{n}$$

and since

$$\begin{aligned} \mathbf{n} \cdot \mathbf{n} &= 1 \\ \mathbf{n} \cdot \mathbf{a}_\alpha &= 0 \end{aligned}$$

we have

$$\begin{aligned} \mathbf{n}_{,\beta} \cdot \mathbf{n} &= 0 \\ \mathbf{n}_{,\beta} \cdot \mathbf{a}_\alpha &= -\mathbf{n} \cdot \mathbf{a}_{\alpha,\beta} = -\mathbf{n} \cdot \mathbf{r}_{,\alpha\beta} \end{aligned}$$

so that $\mathbf{b}$ is a symmetric surface tensor, that is a symmetric tensor with no normal components. The covariant components of $\mathbf{b}$ are known as the coefficients of the 2nd fundamental form [15, 34]. Because $\mathbf{b}$ is symmetric we can write $\mathbf{b} = b^\alpha_\beta \mathbf{a}_\alpha \mathbf{a}^\beta$ and not worry about the order of α and β. Otherwise we would have to write $b^\alpha_{\cdot\beta}$.

As a symmetric surface tensor $\mathbf{b}$ has 2 orthogonal principal directions in which the curvatures are equal to the principal curvatures. The mean of the principal curvatures is

$$H = \frac{1}{2} b^\alpha_\alpha = \frac{1}{2} \operatorname{tr}(\mathbf{b}) \tag{B7}$$

where $\operatorname{tr}(\mathbf{b})$ is the trace of $\mathbf{b}$ and the product of the principal curvatures is the Gaussian curvature,

$$\begin{aligned} K &= \epsilon^{12}\epsilon^{12}\left(b_{11}b_{22} - b^2_{12}\right) = b^1_1 b^2_2 - b^1_2 b^2_1 \\ &= \frac{1}{2}\left((\operatorname{tr}(\mathbf{b}))^2 - \mathbf{b} : \mathbf{b}\right) = -\frac{1}{2} \operatorname{tr}(\boldsymbol{\epsilon} \cdot \mathbf{b} \cdot \boldsymbol{\epsilon} \cdot \mathbf{b}). \end{aligned} \tag{B8}$$

We also have the useful result that

$$\boldsymbol{\epsilon} \cdot \mathbf{b} \cdot \boldsymbol{\epsilon} \cdot \mathbf{b} = -K(\mathbf{I} - \mathbf{nn}) \tag{B9}$$

which is consistent with (B8).

$\mathbf{I}$ and $\boldsymbol{\chi}$ are constant tensors and therefore their gradient is zero. However,

$$\nabla \boldsymbol{\epsilon} = \nabla \mathbf{n} \cdot \boldsymbol{\chi} = -\mathbf{b} \cdot \boldsymbol{\chi}. \tag{B10}$$

In addition to $\mathbf{b}$ the Christoffel symbols,

$$\Gamma^\eta_{\alpha\beta} = \mathbf{g}^\eta \cdot \mathbf{a}_{\alpha,\beta}$$

are useful in finding the gradient. Thus if we have the vector

$$\begin{aligned} \mathbf{v} &= v^\alpha \mathbf{a}_\alpha + v\mathbf{n} = v_\alpha \mathbf{a}^\alpha + v\mathbf{n} \\ v^\alpha &= \mathbf{a}^\alpha \cdot \mathbf{v} = a^{\alpha\beta} v_\beta \\ v_\alpha &= \mathbf{a}_\alpha \cdot \mathbf{v} = a_{\alpha\beta} v^\beta \\ v &= \mathbf{n} \cdot \mathbf{v} \end{aligned}$$

then the gradient

$$\nabla \mathbf{v} = \mathbf{a}^\beta\left(\left(v^\alpha|_\beta - v b^\beta_\alpha\right)\mathbf{a}_\alpha + \left(v^\alpha b_{\alpha\beta} + v|_\beta\right)\mathbf{n}\right) \tag{B11}$$

in which

$$v^\alpha|_\beta = v^\alpha_{,\beta} + v^\eta \Gamma^\alpha_{\eta\beta}.$$

$v^\alpha|_\beta$ is the covariant derivative [13, 18]. They are the components of a tensor, even though $v^\alpha_{,\beta}$ and $\Gamma^\alpha_{\eta\beta}$ are not because they do not obey the rules under a change of coordinates and are therefore, at least partly, a property of the coordinate system, rather than physical entities.

The covariant derivatives of $a_{\alpha\beta}$, $a^{\alpha\beta}$, $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ are all zero, even though none of these quantities is constant. This is due to the presence of the Christoffel symbols.

We can form the covariant derivative of higher order tensors, so that, for example,

$$Q^\alpha_{\cdot\beta}|_\lambda = Q^\alpha_{\cdot\beta,\lambda} + Q^\eta_{\cdot\beta}\Gamma^\alpha_{\eta\lambda} - Q^\alpha_{\cdot\eta}\Gamma^\eta_{\beta\lambda}.$$

Thus, if

$$\mathbf{Q} = Q^\alpha_{\cdot\beta}\mathbf{a}_\alpha \mathbf{a}^\beta \mathbf{n}$$

then

$$\begin{aligned} \nabla \mathbf{Q} &= Q^\alpha_{\cdot\beta}|_\lambda \mathbf{a}^\lambda \mathbf{a}_\alpha \mathbf{a}^\beta \mathbf{n} \\ &\quad + Q^\alpha_{\cdot\beta}\mathbf{a}^\lambda \begin{pmatrix} b_{\alpha\lambda}\mathbf{n}\mathbf{a}^\beta \mathbf{n} + b^\beta_\lambda \mathbf{a}_\alpha \mathbf{nn} \\ -b_{\lambda\mu}\mathbf{a}_\alpha \mathbf{a}^\beta \mathbf{a}^\mu \end{pmatrix}. \end{aligned}$$

The divergence,

$$\nabla \cdot \mathbf{Q} = \mathbf{I} : (\nabla \mathbf{Q}) = \mathbf{a}^\alpha \cdot \mathbf{Q}_{,\alpha} \tag{B12}$$

and it is a tensor of order 2 less than $\nabla \mathbf{Q}$ and therefore 1 less than $\mathbf{Q}$. For the vector in (B11),

$$\nabla \cdot \mathbf{v} = v^\alpha|_\alpha - v b^\alpha_\alpha$$

and from (B10),

$$\nabla \cdot \boldsymbol{\epsilon} = -\mathbf{b} : \boldsymbol{\chi} = 0.$$

The gradient of the gradient,

$$\nabla\nabla\mathbf{Q} = \mathbf{a}^\alpha\left(\mathbf{a}^\beta\mathbf{Q}_{,\beta}\right)_{,\alpha} = \mathbf{a}^\alpha\left(\mathbf{a}^\beta_{,\alpha}\mathbf{Q}_{,\beta} + \mathbf{a}^\beta\mathbf{Q}_{,\alpha\beta}\right)$$

and therefore,

$$\begin{aligned}\boldsymbol{\epsilon} : \nabla\nabla\mathbf{Q} &= \epsilon^{\lambda\mu}\left(\mathbf{a}_\mu \cdot \mathbf{a}^\beta_{,\lambda}\mathbf{Q}_{,\beta} + \mathbf{Q}_{,\lambda\mu}\right)\\ &= -\epsilon^{\lambda\mu}\mathbf{a}_{\mu,\lambda} \cdot \mathbf{a}^\beta\mathbf{Q}_{,\beta} = 0.\end{aligned}$$

In the special case when $\mathbf{Q}$ is the unit normal, $\mathbf{n}$, we have

$$\boldsymbol{\epsilon} : \nabla\nabla\mathbf{n} = -\boldsymbol{\epsilon} : \nabla\mathbf{b} = 0$$

which are the Peterson–Mainardi–Codazzi equations for the surface [15, 34]. There are 6 quantities a_{11}, $a_{12} = a_{21}$, a_{22}, b_{11}, $b_{12} = b_{21}$ and b_{22} which are functions of the surface coordinates θ^α, but which can be expressed in terms of the Cartesian (x, y, z) coordinates, which are also functions of θ^α. Thus there are 3 compatibility equations relating $a_{\alpha\beta}$ and $b_{\alpha\beta}$, 2 of which are the Peterson–Mainardi–Codazzi equations, and the 3rd is Gauss's *Theorema Egregium* [15, 34] which enables the Gaussian curvature to be calculated purely by measuring lengths on a surface. There does not seem to be a nice way of writing the Theorema Egregium without involving differentials of the Christoffel symbols, but the rate of change of Gaussian curvature can be written in terms of the rate of change of membrane strain over a surface.

Also,

$$\begin{aligned}\nabla(\boldsymbol{\epsilon} \cdot \nabla\mathbf{Q}) &= \mathbf{a}^\alpha(\mathbf{n} \cdot \boldsymbol{\chi} \cdot \nabla\mathbf{Q})_{,\alpha}\\ &= \mathbf{a}^\alpha\left(\mathbf{n}_{,\alpha} \cdot \boldsymbol{\chi} \cdot \nabla\mathbf{Q} + \boldsymbol{\epsilon} \cdot \left(\mathbf{a}_\alpha \cdot \nabla\nabla\mathbf{Q}\right)\right)\\ &= \mathbf{a}^\alpha\mathbf{n}\mathbf{a}_\alpha \cdot \mathbf{b} \cdot \boldsymbol{\epsilon} \cdot \nabla\mathbf{Q} - \boldsymbol{\epsilon} \cdot \nabla\nabla\mathbf{Q}\end{aligned}$$

and

$$\begin{aligned}\nabla \cdot (\boldsymbol{\epsilon} \cdot \nabla\mathbf{Q}) &= \mathbf{a}^\alpha \cdot \mathbf{n}\mathbf{a}_\alpha \cdot \mathbf{b} \cdot \boldsymbol{\epsilon} \cdot \nabla\mathbf{Q} + \boldsymbol{\epsilon} : \nabla\nabla\mathbf{Q}\\ &= 0.\end{aligned} \tag{B13}$$

(B13) is very useful in discussing the equilibrium of shells.

Finally, if $\mathbf{Q}$ is a 2nd order surface tensor, then

$$\begin{aligned}\mathbf{Q} - \boldsymbol{\epsilon} \cdot \mathbf{Q} \cdot \boldsymbol{\epsilon} &= \left(Q^{\alpha\beta} - \epsilon^{\alpha\lambda}\epsilon^{\mu\beta}Q_{\lambda\mu}\right)\mathbf{a}_\alpha\mathbf{a}_\beta\\ &= \left(Q^{\alpha\beta} - \epsilon^{\alpha\lambda}\epsilon_{\mu\eta}Q_{\lambda}^{\cdot\mu}a^{\eta\beta}\right)\mathbf{a}_\alpha\mathbf{a}_\beta\\ &= \left(Q^{\alpha\beta} - Q_{\lambda}^{\cdot\alpha}a^{\lambda\beta} + Q_{\lambda}^{\cdot\lambda}a^{\alpha\beta}\right)\mathbf{a}_\alpha\mathbf{a}_\beta\\ &= \mathbf{Q} - \mathbf{Q}^{\mathrm{T}} + \mathrm{tr}\,(\mathbf{Q})(\mathbf{I} - \mathbf{n}\mathbf{n})\end{aligned}$$

where $\mathrm{tr}\,(\mathbf{Q})$ is the trace of $\mathbf{Q}$ and $\mathbf{Q}^{\mathrm{T}}$ is its transpose. If $\mathbf{Q}$ is symmetric,

$$\mathbf{Q} - \boldsymbol{\epsilon} \cdot \mathbf{Q} \cdot \boldsymbol{\epsilon} = \mathrm{tr}\,(\mathbf{Q})(\mathbf{I} - \mathbf{n}\mathbf{n}). \tag{B14}$$

Appendix C: The equilibrium equations

Equilibrium of forces

Note that in this section we are effectively deriving the divergence theorem for our own purposes. If we were to assume prior knowledge of the divergence theorem on a surface, we could simply write down the equilibrium equations.

Consider the closed curve C in Fig. 1 representing a cut through the shell reference surface, and also through the shell. For convenience, let us introduce the second order tensor,

$$\mathbf{F} = \mathbf{n} \times \boldsymbol{\sigma} \tag{C1}$$

so that from (1) the total force crossing the cut acting on the portion of the shell within the cut is

$$\mathbf{f} = \oint_{\mathrm{C}} d\mathbf{r} \cdot \mathbf{F}.$$

Now let us imagine that we move the boundary slightly by an amount $\xi\boldsymbol{\rho}$ where $\boldsymbol{\rho}$ is a function of $\mathbf{r}$ and $\boldsymbol{\rho}$ is perpendicular to the surface normal. ξ is a scalar constant which will allow to tend to zero. For convenience we can in addition imagine that $\boldsymbol{\rho}$ is a function of $\mathbf{r}$ even on points on the surface not on C.

The increase in $\mathbf{f}$ due to moving the boundary is

$$\oint_C d(\mathbf{r}+\xi\boldsymbol{\rho})\cdot(\mathbf{F}+\xi\boldsymbol{\rho}\cdot\nabla\mathbf{F}) - \oint_C d\mathbf{r}\cdot\mathbf{F}$$
$$= \xi\oint_C d\mathbf{r}\cdot(\nabla\boldsymbol{\rho}\cdot\mathbf{F}+\boldsymbol{\rho}\cdot\nabla\mathbf{F})$$
$$= \xi\oint_C d\mathbf{r}\cdot(\nabla(\boldsymbol{\rho}\cdot\mathbf{F})+\boldsymbol{\rho}\cdot\nabla\mathbf{F})$$
$$-\xi\oint_C \boldsymbol{\rho}\cdot(d\mathbf{r}\cdot\nabla\mathbf{F})$$
$$= \xi\oint_C (d\mathbf{r}\boldsymbol{\rho}-\boldsymbol{\rho}d\mathbf{r}) : \nabla\mathbf{F}$$

in which we have used the fact that

$$\oint_C d\mathbf{r}\cdot\nabla(\boldsymbol{\rho}\cdot\mathbf{F}) = \oint_C d(\boldsymbol{\rho}\cdot\mathbf{F}) = 0.$$

If $\mathbf{p}$ is the load on the shell per unit area of reference surface, including the own weight of the structure, then the *change* in load on the area contained within C due to the movement of C is

$$-\xi\oint_C (d\mathbf{r}\times\boldsymbol{\rho})\cdot\mathbf{n}\mathbf{p} = \xi\oint_C (d\mathbf{r}\boldsymbol{\rho}) : \boldsymbol{\chi}\cdot\mathbf{n}\mathbf{p}$$
$$= \frac{1}{2}\xi\oint_C (d\mathbf{r}\boldsymbol{\rho}-\boldsymbol{\rho}d\mathbf{r}) : \boldsymbol{\epsilon}\mathbf{p}$$

in which the 3rd tensor $\boldsymbol{\chi}$ used for the vector product is defined in (A2) and (B4).

For equilibrium of forces the sum of these two forces must be zero and therefore

$$\oint_C (d\mathbf{r}\boldsymbol{\rho}-\boldsymbol{\rho}d\mathbf{r}) : \left(\nabla\mathbf{F}+\frac{1}{2}\boldsymbol{\epsilon}\mathbf{p}\right) = 0.$$

$(d\mathbf{r}\boldsymbol{\rho}-\boldsymbol{\rho}d\mathbf{r})$ is an antisymmetric surface tensor and therefore it is equal to a scalar times $\boldsymbol{\epsilon}$. Thus

$$\boldsymbol{\epsilon} : \left(\nabla\mathbf{F}+\frac{1}{2}\boldsymbol{\epsilon}\mathbf{p}\right) = 0$$

so that

$$\boldsymbol{\epsilon} : \nabla\mathbf{F}+\mathbf{p} = 0.$$

We can rewrite this equation using $\boldsymbol{\sigma}$ instead of $\mathbf{F}$ as follows using (C1),

$$\begin{aligned}\boldsymbol{\epsilon} : \nabla(\mathbf{n}\times\boldsymbol{\sigma}) &= -\boldsymbol{\epsilon} : \nabla(\mathbf{n}\cdot\boldsymbol{\chi}\cdot\boldsymbol{\sigma})\\ &= -\boldsymbol{\epsilon} : (\nabla\mathbf{n}\cdot\boldsymbol{\chi}\cdot\boldsymbol{\sigma})\\ &\quad -\boldsymbol{\epsilon} : \mathbf{a}^{\alpha}\left(\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}_{,\alpha}\right)\\ &= -\mathbf{a}^{\alpha}\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}_{,\alpha} = \mathbf{a}^{\alpha}\cdot\boldsymbol{\sigma}_{,\alpha}\\ &= \nabla\cdot\boldsymbol{\sigma}\end{aligned}$$

in which $\boldsymbol{\epsilon} : (\nabla\mathbf{n}\cdot\boldsymbol{\chi}\cdot\boldsymbol{\sigma}) = 0$ because of (3) and the fact that $\boldsymbol{\epsilon}$ and $\nabla\mathbf{n}$ contain no normal components, whereas all the parts of $\boldsymbol{\chi}$ contain the normal. Therefore

$$\nabla\cdot\boldsymbol{\sigma}+\mathbf{p} = 0$$

in which the load $\mathbf{p}$ is usually considered to be a known quantity. The equation number () corresponds to its number in Sect. 5.

Equilibrium of moments

Again for convenience let us introduce the second order tensor

$$\mathbf{G} = \mathbf{n}\times\boldsymbol{\mu} \tag{C2}$$

as we did in (C2) for $\boldsymbol{\sigma}$ and $\mathbf{F}$. Then using (1) and (2) the total moment about some arbitrary fixed point $\mathbf{Y}$ crossing the closed curve C is

$$\mathbf{m}_{\mathbf{Y}} = \oint_C d\mathbf{r}\cdot(\mathbf{G}-\mathbf{F}\times(\mathbf{r}-\mathbf{Y})).$$

Using the same working that we used above the increase in $\mathbf{m}_{\mathbf{Y}}$ due to moving the boundary by $\xi\boldsymbol{\rho}$ is

$$\xi\oint_C (d\mathbf{r}\boldsymbol{\rho}-\boldsymbol{\rho}d\mathbf{r}) : \nabla(\mathbf{G}-\mathbf{F}\times(\mathbf{r}-\mathbf{Y})) \tag{C3}$$

The change in the moment about $\mathbf{Y}$ due to the movement of C from the applied load and loading couples is

$$\begin{aligned}&-\xi\oint_C (d\mathbf{r}\times\boldsymbol{\rho})\cdot\mathbf{n}(\mathbf{c}-\mathbf{p}\times(\mathbf{r}-\mathbf{Y}))\\ &= \frac{1}{2}\xi\oint_C (d\mathbf{r}\boldsymbol{\rho}-\boldsymbol{\rho}d\mathbf{r}) : \boldsymbol{\epsilon}(\mathbf{c}-\mathbf{p}\times(\mathbf{r}-\mathbf{Y})).\end{aligned} \tag{C4}$$

If we add (C3) to (C4) we obtain the equation of equilibrium of moments,

$$\begin{aligned} 0 &= \boldsymbol{\epsilon} : \begin{pmatrix} \nabla(\mathbf{G} - \mathbf{F} \times (\mathbf{r} - \mathbf{Y})) \\ +\frac{1}{2}\boldsymbol{\epsilon}(\mathbf{c} - \mathbf{p} \times (\mathbf{r} - \mathbf{Y})) \end{pmatrix} \\ &= \boldsymbol{\epsilon} : \left(\nabla\mathbf{G} - \mathbf{a}^{\alpha}\mathbf{F} \times \mathbf{a}_{\alpha}\right) + \mathbf{c} \end{aligned}$$

in which the terms containing $\mathbf{p}$ and $\nabla\mathbf{F}$ have cancelled because of equilibrium of forces and we have used the base vectors (B1) and (B2).

Again we can use (C1) and (C2) to rewrite this equation in terms of $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$,

$$\begin{aligned} 0 &= \nabla \cdot \boldsymbol{\mu} + \boldsymbol{\epsilon} : \left(\mathbf{a}^{\alpha}(\boldsymbol{\epsilon} \cdot \boldsymbol{\sigma}) \times \mathbf{a}_{\alpha}\right) + \mathbf{c} \\ &= \nabla \cdot \boldsymbol{\mu} - \mathbf{a}^{\alpha} \cdot \boldsymbol{\sigma} \times \mathbf{a}_{\alpha} + \mathbf{c} \\ &= \nabla \cdot \boldsymbol{\mu} + \mathbf{a}^{\alpha} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\chi} \cdot \mathbf{a}_{\alpha} + \mathbf{c} \end{aligned}$$

so that

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\sigma} : \boldsymbol{\chi} + \mathbf{c} = 0 \tag{6}$$

in which the equation number (6) corresponds to its number in Sect. 5.

Appendix D: Substitution into the equilibrium equations

In order to substitute (8) and (9) into the equilibrium equations we have firstly that

$$\nabla \cdot (\boldsymbol{\epsilon} \cdot \nabla\boldsymbol{\psi}) = 0 \tag{D1}$$

$$\nabla \cdot (\boldsymbol{\epsilon} \cdot \nabla\boldsymbol{\phi}) = 0 \tag{D2}$$

from (B13).

For the equation of equilibrium of moments we will need the identity

$$\begin{aligned} &(\boldsymbol{\epsilon} \cdot \nabla\boldsymbol{\psi}) : \boldsymbol{\chi} \\ &= -\nabla\boldsymbol{\psi} : (\boldsymbol{\epsilon} \cdot \boldsymbol{\chi}) \\ &= -\nabla\boldsymbol{\psi} : \left(-(\mathbf{I} - \mathbf{nn})\mathbf{n} + \mathbf{a}_{\alpha}\mathbf{n}\mathbf{a}^{\alpha}\right) \\ &= \nabla \cdot \boldsymbol{\psi}\mathbf{n} - \nabla\boldsymbol{\psi} \cdot \mathbf{n} \\ &= -\nabla \cdot (\boldsymbol{\psi} \cdot (\mathbf{nI} - \mathbf{In})). \end{aligned} \tag{D3}$$

in which we have used (B5).

Finally,

$$\begin{aligned} &\nabla \cdot (\boldsymbol{\beta} \cdot (\mathbf{n}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{n})) \\ &= \nabla \cdot ((\mathbf{I} - \mathbf{nn}) \cdot (\boldsymbol{\beta} \cdot \boldsymbol{\chi})) \\ &= \mathbf{a}^{\alpha} \cdot \left(\left(-\mathbf{n}_{,\alpha}\mathbf{n} - \mathbf{n}\mathbf{n}_{,\alpha}\right) \cdot (\boldsymbol{\beta} \cdot \boldsymbol{\chi}) + \boldsymbol{\beta}_{,\alpha} \cdot \boldsymbol{\chi}\right) \\ &= 2H\mathbf{n} \cdot (\boldsymbol{\beta} \cdot \boldsymbol{\chi}) - \nabla\boldsymbol{\beta} : \boldsymbol{\chi} \\ &= 2H\boldsymbol{\epsilon} \cdot \boldsymbol{\beta} - \nabla\boldsymbol{\beta} : \boldsymbol{\chi} \end{aligned} \tag{D4}$$

where H is the mean curvature (B7).

Substituting (8) into (5) and using (D1) gives the equation of equilibrium of forces,

$$\nabla^2\boldsymbol{\beta} + \mathbf{p} = 0$$

The Laplacian of the vector $\boldsymbol{\beta}$ is $\nabla^2\boldsymbol{\beta} = \nabla \cdot \nabla\boldsymbol{\beta}$.

If we substitute (8) and (9) into (6) we obtain

$$\begin{aligned} &\nabla \cdot \begin{pmatrix} \boldsymbol{\epsilon} \cdot \nabla\boldsymbol{\phi} + \nabla\boldsymbol{\alpha} \\ +\boldsymbol{\psi} \cdot (\mathbf{nI} - \mathbf{In}) + \boldsymbol{\beta} \cdot (\mathbf{n}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{n}) \end{pmatrix} \\ &\quad + (\boldsymbol{\epsilon} \cdot \nabla\boldsymbol{\psi} + \nabla\boldsymbol{\beta}) : \boldsymbol{\chi} + \mathbf{c} = 0 \end{aligned}$$

and using (D2), (D3), (D4) we find that the equation of equilibrium of moments becomes

$$\nabla^2\boldsymbol{\alpha} + 2H\boldsymbol{\epsilon} \cdot \boldsymbol{\beta} + \mathbf{c} = 0. \tag{11}$$

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