Thesis for the degree of licentitate

Buchsbaum-Rim multiplicities and residue currents

Rahim Nkunzimana

Department of Mathematical Sciences

Division of Algebra and Geometry

Chalmers University of Technology and the University of Gothenburg

Göteborg, Sweden 2024

Buchsbaum-Rim multiplicities and residue currents
Rahim Nkunzimana

Department of Mathematical Sciences
Division of Algebra and Geometry
Chalmers University of Technology and the University of Gothenburg
SE-412 96 Göteborg
Sweden

Typeset with LaTeX
Printed by Chalmers Reproservice
Göteborg, Sweden 2024

BUCHSBAUM-RIM MULTIPLICITIES AND RESIDUE CURRENTS

Rahim Nkunzimana

Department of Mathematical Sciences
Division of Algebra and Geometry
Chalmers University of Technology and the University of Gothenburg

Abstract

Let f be a holomorphic $(r \times m)$-matrix defined near the origin in $\mathbb{C}^n$ and with full rank outside the origin. To the submodule N of $\mathcal{O}_0^r$ defined by the image of f there is a notion of multiplicity called the Buchsbaum-Rim multiplicity of N. This is the leading coefficient of a Hilbert polynomial of a certain graded algebra defined from N. In the special case when f is a row matrix, the image is given by the ideal in $\mathcal{O}_0$ defined by f and the Buchsbaum-Rim multiplicity coincides with the classical Hilbert-Samuel multiplicity.

In this thesis we represent the Buchsbaum-Rim multiplicity $\mathrm{e}_{\mathrm{BR}}(N)$ of N in terms of (residue) currents in the special case when the matrix f is block diagonal. More precisely, we prove that the point mass $\mathrm{e}_{\mathrm{BR}}(N)[0]$ factors into a product of a smooth form and a residue current associated to the so-called Buchsbaum-Rim complex of f. This generalises a result in [And05], where a similar factorisation is proven for row matrices and Hilbert-Samuel multiplicities. When f is block diagonal, the Buchsbaum-Rim multiplicity is given as a sum of so-called mixed multiplicities. By King's formula, these multiplicities can be expressed with mixed Monge-Ampère products. We show that the mixed Monge-Ampère products can be represented as the product of the smooth form and residue current defined from the Buchsbaum-Rim complex of f.

Keywords: Buchsbaum-Rim multiplicity, mixed multiplicity, Buchsbaum-Rim complex, residue current, holomorphic morphism, Mongè-Ampere

product, King's formula

Acknowledgements

This thesis would not exist without the extensive guidance of my supervisors Elizabeth Wulcan and Richard Lärkäng. For your persistent support and motivation, for sharing your knowledge and wisdom (in mathematics and in life), and for your patience, I am ever grateful. Thank you also to my friends and colleagues in the complex analysis group for creating a warm and welcoming environment, not only for me but for our many guests and visitors. It is a pleasure to know you all. I would not have gotten this far without my dear friends and peers at the department; to Rickard, Robin, Mattias, Ludvig, Douglas, Rolf, Jan, Julia, Petar, Rémi, and Antonio, to mention only a few, thank you! Finally, I am filled with gratitude to my friends and family for their endless love and support.

Rahim Nkunzimana
Göteborg, 2024

Chapter 1

Introduction

Let X be a neighbourhood of the origin $0 \in \mathbb{C}^n$ and consider a tuple $f = (f_1, \dots, f_m)$ of holomorphic functions defined on X such that $Z(f) := \{z \in X | f(z) = 0\} = \{0\}$. Let F be a trivial holomorphic bundle over X with rank m and fix a frame $e = (e_1, \dots, e_m)$ with dual frame $e^* = (e_1^*, \dots, e_m^*)$. We can then view f as a section of the dual bundle F^* by writing

$$f = \sum_{k=1}^{m} f_k e_k^*.$$

The morphism $f : F \to \mathcal{O}$, where $\mathcal{O}$ is the sheaf of holomorphic functions on X, then coincides with contraction δ_f with the section f, which is a map defined by

$$\delta_f e_k = f_k.$$

The contraction $\delta_f : F \to \mathcal{O}$ can be extended to a differential δ_f on the exterior algebra $H := \Lambda F$, cf. (2.30). This gives rise to the *Koszul complex* (H, φ) of f, which is the complex

$$0 \longrightarrow H_m := \Lambda^m F \xrightarrow{\varphi_m := \delta_f} \cdots \qquad \cdots \xrightarrow{\varphi_2 := \delta_f} H_1 := F \xrightarrow{\varphi_1 := \delta_f} H_0 := \mathcal{O}$$

From this data, Andersson constructs in [And04] a *residue current* $\widetilde{R}^f$ associated to f, which is a $(0,n)$-current with values in $\Lambda^n F$ and support on $Z(f)=\{0\}$. The coefficients of the residue current $\widetilde{R}^f$ are the so-called Bochner-Martinelli type currents, as constructed in [PTY00]. In particular, when $m=n$ and f defines a complete intersection, the current R^f is

$$\widetilde{R}^f = \bar{\partial}\frac{1}{f_n} \wedge \cdots \wedge \bar{\partial}\frac{1}{f_1} \otimes e_1 \wedge \cdots \wedge e_n$$

where $\bar{\partial}(1/f_n) \wedge \cdots \wedge \bar{\partial}(1/f_1)$ is the classical Coleff-Herrera product, constructed in [CH78].

The ideal $\mathcal{I} = (f_1, \dots, f_m)$, defined by f, is Artinian with support at the origin. There is a notion of multiplicity of the ideal, called the Hilbert-Samuel multiplicity $\mathrm{e}(\mathcal{I})$. A consequence of the classical King's formula is that the mass at the origin of the Monge-Ampère product $(dd^c \log|f|^2)^n$ (see Section 2.2) coincides with $e(\mathcal{I})$, i.e.

$$\int_{\{0\}} (dd^c \log|f|^2)^n = e(\mathcal{I}). \tag{1.1}$$

Therefore, this mass is sometimes taken as an analytic definition of the Hilbert-Samuel multiplicity.

The starting point of this thesis is [And05, Theorem 1.1] where Andersson proves the formula

$$\frac{1}{(2\pi i)^n n!} d\varphi \widetilde{R}^f = \mathrm{e}(\mathcal{I})[0], \tag{1.2}$$

where $d\varphi := d\varphi_1 \cdots d\varphi_n = (d\delta_f)^n$. Consider the special case when the dimension $n=1$ and F is a trivial line bundle with frame e, and $fe^* \in \mathcal{O}(F^*)$. Then $f = z^a g$ for some positive integer a and a nonvanishing holomorphic function g. The ideal $\mathcal{I}$ defined by f has multiplicity $\mathrm{e}(\mathcal{I}) = a$. The residue current is the $(0,1)$-current

$\widetilde{R}^f = \bar{\partial}(1/f) \otimes e$, cf. (2.17). In this case, (1.2) becomes[1]

$$\frac{1}{2\pi i}\bar{\partial}\frac{1}{f} \wedge df = a[0] \tag{1.3}$$

which can be viewed as a smooth version of the argument principle (see Remark 2.11).

We now consider the case when f is a holomorphic $(r \times m)$-matrix on X such that $Z(f) = \{0\}$, where $Z(f)$ is the set where f does not have full rank. We let E, Q be trivial holomorphic vector bundles over X of rank m and r, respectively. Let $e = (e_1, \dots, e_m)$ be a frame for E and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$ be a frame for Q, with dual frame e^* and ε^*, respectively. We can then identify f with a morphism $f : E \to Q$ by letting

$$f = \sum_{k=1}^{r} \sum_{\ell=1}^{m} f_{k\ell}\varepsilon_k \otimes e_\ell^*.$$

where $f_{k\ell}$ are the functions appearing in the matrix f. In a similar way as before, there is an associated complex (H, φ), called the *Buchsbaum-Rim complex* of f (see Section 3.1), such that f is the first map in the complex. When $r = 1$, so that f is just a row matrix, this complex coincides with the Koszul complex. We equip H with a trivial metric and connection. In [And06], Andersson constructs a residue current R^f (see Section 3.2) associated to the matrix f from this complex. In fact, R^f is a current with support in $Z(f) = \{0\}$ and values in $\mathrm{Hom}(H_0, H_n)$, i.e. we obtain a matrix of currents. We note also that $d\varphi R^f$ is a current with values in $H_0 \otimes H_0^*$. When $r = 1$, we have $R^f = \widetilde{R}^f$. We are interested in studying the current $\mathrm{tr}(d\varphi R^f)$ that generalises the left hand side of (1.2).

Since f has full rank outside the origin, the module $\mathcal{M} = \mathcal{O}(Q)/\operatorname{im} f$ is Artinian with support at the origin. For such a module there is a multiplicity called the Buchsbaum-Rim multiplicity $\mathrm{e}_{\mathrm{BR}}(\mathcal{M})$ (sometimes written $\mathrm{e}_{\mathrm{BR}}(\operatorname{im} f)$, see Section 2.1). When $r = 1$, then $\operatorname{im} f = \mathcal{I}$, $\mathcal{M} = \mathcal{O}/\mathcal{I}$, and it holds that $\mathrm{e}_{\mathrm{BR}}(\mathcal{I}) = \mathrm{e}(\mathcal{I})$.

[1] The order of df and $\bar{\partial}(1/f)$ is explained by a superstructure, see Section 2.4.

We now consider the special case when f is block diagonal with each block being a row matrix f_k, i.e. we can decompose E as a direct sum $E = \oplus_{k=1}^r E_k$ of trivial holomorphic subbundles E_k, and write

$$f = \sum_{k=1}^{r} f_k \otimes \varepsilon_k$$

where $f_k \in \mathcal{O}(E_k^*)$ (or rather, the trivial extension of this section to a section of E^*). We also assume that f_k satisfies $Z(f_k) = \{0\}$. Let $\mathcal{I}_k$ be the ideal defined by f_k and note first that the ideals $\mathcal{I}_k$ are Artinian. Further, it holds that $\mathcal{M} := \mathcal{O}(Q)/\operatorname{im} f \cong \oplus_{k=1}^r \mathcal{O}/\mathcal{I}_k$. Our main result is the following extension of (1.2).

Theorem 1.1. *Assume f is a block diagonal $(r \times m)$-matrix where each block is a tuple f_k such that $Z(f_k) = \{0\}$. Let R^f be the residue current associated to the Buchsbaum-Rim complex (H, φ) defined from f. Then it holds that*

$$\frac{1}{(2\pi i)^n n!} \operatorname{tr}(d\varphi R^f) = \mathrm{e}_{\mathrm{BR}}(\mathcal{M})[0], \tag{1.4}$$

where $d\varphi = d\varphi_1 \cdots d\varphi_n$ and $\mathcal{M} = \mathcal{O}(Q)/\operatorname{im} f$.

When f is a tuple of functions, Andersson's proof of (1.2) relies on the following factorisation

$$\mathbf{1}_{\{0\}}(dd^c \log |f|^2)^n = \frac{1}{(2\pi i)^n n!} d\varphi \widetilde{R}^f, \tag{1.5}$$

and our proof rests on the following generalisation.

Theorem 1.2. *Suppose we are in the situation of Theorem 1.1. Then it holds that*

$$\frac{1}{(2\pi i)^n n!} \operatorname{tr}(d\varphi R^f) = \sum_{\substack{\alpha \in \mathbb{N}^r \\ |\alpha| = n}} \mathbf{1}_{\{0\}} \left(dd^c \log |f|^2\right)^{\alpha}, \tag{1.6}$$

where

$$(dd^c \log|f|^2)^\alpha = (dd^c \log|f_1|^2)^{\alpha_1} \wedge \cdots \wedge (dd^c \log|f_r|^2)^{\alpha_r}.$$

For r Artinian ideals $\mathcal{I}_1, \ldots, \mathcal{I}_r$ supported at the origin there is a notion of multiplicity $\mathrm{e}_\alpha(\mathcal{I}_1, \ldots, \mathcal{I}_r)$ called the mixed multiplicity of type $\alpha \in \mathbb{N}^r$ (see Section 2.1). When $\mathcal{M} \cong \oplus_{k=1}^r \mathcal{O}/\mathcal{I}_k$, as in our situation, the Buchsbaum-Rim multiplicity $\mathrm{e}_{\mathrm{BR}}(\mathcal{M})$ is calculated from the mixed multiplicities as

$$\mathrm{e}_{\mathrm{BR}}(\mathcal{M}) = \sum_{|\alpha|=n} \mathrm{e}_\alpha(\mathcal{I}_1, \ldots, \mathcal{I}_r). \tag{1.7}$$

If f is a matrix as in our situation, then by polarising King's formula we obtain

$$\int_{\{0\}} (dd^c \log|f|^2)^\alpha = \mathrm{e}_\alpha(\mathcal{I}_1, \ldots, \mathcal{I}_r), \tag{1.8}$$

see Proposition 2.3. Thus, from (1.7)-(1.8) together with Theorem 1.2 we immediately obtain Theorem 1.1.

Chapter 2

Preliminaries

2.1 The Buchsbaum-Rim multiplicity

In this section we recall some basic facts and the definitions of the multiplicities that we consider. For a general reference, see e.g. [Rob98, Chapter 2].

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension n. Let $I \subseteq \mathfrak{m}$ be an $\mathfrak{m}$-primary ideal. Then A/I has finite length. Moreover, for $\ell \in \mathbb{N}$ large enough

$$\mathrm{length}(A/I^\ell)$$

is a polynomial in ℓ of degree n. The *Hilbert-Samuel multiplicity* $\mathrm{e}(I)$ is defined as the following normalisation of the leading term coefficient

$$\mathrm{e}(I) := n! \,\mathrm{coeff}(\ell^n, \mathrm{length}(A/I^\ell))$$

where $\ell \gg 1$. In fact, the multiplicity depends only on the *integral closure* $\bar{I}$ of the ideal I. An element $x \in A$ is *integral over* I precisely if there is a monic equation

$$x^m + a_1 x^{m-1} + \cdots + a_m = 0$$

with $a_k \in I^k$ and the ideal $\bar{I}$ consists precisely of all $x \in A$ that are integral over I. Note that $I \subseteq \bar{I}$. An ideal $J \subseteq I$ such that $I \subseteq \bar{J}$, i.e. such that all elements of I are integral over J, is said to be a *reduction* of the ideal I. If I, J are $\mathfrak{m}$-primary ideals such that J is a reduction of I, then $e(J) = e(I)$.

Suppose that $N \subseteq \mathfrak{m}F$ is a submodule of a free A-module F of rank r such that $M = F/N$ is of finite length. The symmetric algebra $S(F)$ can be identified with the polynomial ring $A[X_1, \ldots, X_r]$ as follows. Fix a basis $f_1, \ldots, f_r$ of A. Let $\varphi : S(F) \to A[X_1, \ldots, X_r]$ be the homomorphism $\varphi(f_k) := X_k$. The *Rees ring* $R(N)$ of N is the subring generated by $\varphi(N) \subseteq A[X_1, \ldots, X_r]$. Let $S_\ell(F)$ and $R_\ell(N)$ denote the submodules of $S(F)$ and $R(N)$, respectively, containing homogenoeus polynomials of degree ℓ. For large enough $\ell \in \mathbb{N}$

$$\operatorname{length}(S_\ell(F)/R_\ell(N))$$

is a polynomial in ℓ of degree $n + r - 1$. The Buchsbaum-Rim multiplicity $\mathrm{e}_{\mathrm{BR}}(M)$ is then defined as

$$\mathrm{e}_{\mathrm{BR}}(M) := (n + r - 1)! \operatorname{coeff}(\ell^{n+r-1}, \operatorname{length}(S_\ell(F)/R_\ell(N)) \tag{2.1}$$

where $\ell \gg 1$.

Let $I_1, \ldots, I_r \subseteq \mathfrak{m}$ be $\mathfrak{m}$-primary ideals. For any $\ell = (\ell_1, \ldots, \ell_r) \in \mathbb{N}^r$, it holds that

$$e(I_1^{\ell_1} \cdots I_r^{\ell_r})$$

is a homogeneous polynomial of degree n in $\ell_1, \ldots, \ell_r$. Let $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ be a multi-index with $|\alpha| = n$. The mixed multiplicity $e_\alpha(I_1, \ldots, I_r)$ of type α of the ideals $I_1, \ldots, I_r$ is defined as

$$\binom{n}{\alpha} \mathrm{e}_\alpha(I_1, \ldots, I_r) := \operatorname{coeff}\left(\ell_1^{\alpha_1} \cdots \ell_r^{\alpha_r}, \mathrm{e}(I_1^{\ell_1} \cdots I_r^{\ell_r})\right). \tag{2.2}$$

In fact, we can calculate the mixed multiplicity as the Hilbert-Samuel multiplicity of an ideal by the following proposition (see e.g. [Swa07, Lemma 2.5]).

Proposition 2.1. *Let $I_1, \dots, I_r \subseteq A$ be $\mathfrak{m}$-primary and $\alpha \in \mathbb{N}^r$ a multi-index with $|\alpha| = n$. Let J be the ideal generated by α_1 generic elements of I_1, α_2 generic elements of I_2, ..., α_r generic elements of I_r. Then*

$$e_\alpha(I_1, \dots, I_r) = e(J). \tag{2.3}$$

Note that, as a consequence, for any $\mathfrak{m}$-primary ideal I it holds that

$$e_\alpha(I, \dots, I) = e(I), \tag{2.4}$$

for any $\alpha \in \mathbb{N}^r$. This is because the ideal J constructed in Proposition 2.1 is a reduction of I.

Lemma 2.2 (Kirby-Rees, [KR96]). *Let $(A, \mathfrak{m})$ be a local Noetherian ring of dimension n. Let $I_1, \dots, I_r$ be $\mathfrak{m}$-primary ideals and let $M = \bigoplus_{k=1}^r A/I_k$, so that M is an A-module of finite length. Then the Buchsbaum-Rim multiplicity is given by*

$$\mathrm{e}_{\mathrm{BR}}(M) = \sum_{|\alpha|=n} \mathrm{e}_\alpha(I_1, \dots, I_r), \tag{2.5}$$

where e_α is the mixed multiplicity of type $\alpha \in \mathbb{N}^r$.

Let X be a neighbourhood of the origin $0 \in \mathbb{C}^n$ and consider a morphism $f : E \to Q$ of trivial holomorphic bundles over X. If $Z(f) = \{0\}$, where $Z(f)$ is the set where f is not surjective, then $\mathcal{M} := \mathcal{O}(Q)/\operatorname{im} f$ is an Artinian $\mathcal{O}_X$-module with support at the origin, i.e. $\mathcal{M}_z = 0$ if $z \neq 0$. It can thus be identified with the module $M := \mathcal{M}_0$ which is a module of finite length over the local Noetherian ring $(\mathcal{O}_0, \mathfrak{m}_0)$. We can therefore define the Buchsbaum-Rim multiplicity $\mathrm{e}_{\mathrm{BR}}(\mathcal{M})$ of $\mathcal{M}$ as $\mathrm{e}_{\mathrm{BR}}(\mathcal{M}) = \mathrm{e}_{\mathrm{BR}}(M)$. Similarily, when Q is the trivial line bundle, so that $f = (f_1, \dots, f_m)$, we can define the Hilbert-Samuel multiplicity of the Artinian ideal $\mathcal{I}$ that f defines, as $\mathrm{e}(\mathcal{I}) = \mathrm{e}(I)$, where $I = \mathcal{I}_0$. The main result we need from this section is that if f is block diagonal as in Theorem 1.1, so that

$\mathcal{M} \cong \oplus_{k=1}^{r} \mathcal{O}_X/\mathcal{I}_k$, then

$$\mathrm{e}_{\mathrm{BR}}(\mathcal{M}) = \sum_{|\alpha|=n} \mathrm{e}_\alpha(\mathcal{I}_1, \ldots, \mathcal{I}_r). \tag{2.6}$$

2.2 Monge-Ampère products

Throughout, let X be a neighbourhood of the origin $0 \in \mathbb{C}^n$. Let $\psi_1, \ldots, \psi_r$ be smooth plurisubharmonic (psh) functions on X which are locally bounded outside the origin. Then their mixed Monge-Ampère products (cf. [Dem12, Theorem III.4.5]) are the currents defined recursively as

$$dd^c\psi_k \wedge \cdots \wedge dd^c\psi_1 = dd^c\left(\psi_k dd^c\psi_{k-1} \wedge \cdots \wedge dd^c\psi_1\right) \tag{2.7}$$

for $1 \leqslant k \leqslant r$, and where d and

$$d^c := \frac{1}{4\pi i}(\partial - \bar{\partial})$$

are taken in the sense of currents. These are closed and positive currents, and in particular, this means that they are order 0 currents, i.e. they are currents with measure coefficients. If u_k^N is a sequence of psh functions decreasing to ψ_k, for each $k = 1, \ldots, r$, then the mixed Monge-Ampère product can be obtained as the limit (cf. [Dem12, Theorem III.4.5 & Proposition III.4.9])

$$dd^c\psi_r \wedge \cdots \wedge dd^c\psi_1 = \lim_{N\to\infty} dd^c u_r^N \wedge \cdots \wedge dd^c u_1^N. \tag{2.8}$$

In view of this, it is clear that the Monge-Ampère product is multilinear and symmetric in the factors ψ_k. Sometimes we will use the following multi-index notation. Suppose $\psi_1, \ldots, \psi_r$ are functions as above and that $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ is a multi-index. Then we define

$$(dd^c\psi)^\alpha := (dd^c\psi_1)^{\alpha_1} \wedge \cdots \wedge (dd^c\psi_r)^{\alpha_r}. \tag{2.9}$$

We will consider the typical case $\psi_k = \log|f_k|^2$ where f_k are tuples of holomorphic functions defined on a neighbourhood X of the origin $0 \in \mathbb{C}^n$, such that $Z(f_k) = \{0\}$. Then the ideal $\mathcal{I}_k$ defined by f_k is Artinian with support at the origin, for $k = 1, \dots, r$. The main result we need from this section is the following well-known consequence of polarising King's formula (1.1). We provide a proof for the convenience of the reader.

Proposition 2.3. *Let X be a neighbourhood of the origin $0 \in \mathbb{C}^n$. Suppose f_k are tuples of holomorphic functions on X such that $Z(f_k) = \{0\}$ and let $\mathcal{I}_k$ be the ideal defined by f_k, for $k = 1, \dots, r$. Then for a multi-index $\alpha \in \mathbb{N}^r$ such that $|\alpha| = n$, it holds that*

$$\int_{\{0\}} (dd^c \log|f|^2)^\alpha = \mathrm{e}_\alpha(\mathcal{I}_1, \dots, \mathcal{I}_r). \tag{2.10}$$

Note that the left hand side makes sense since the Monge-Ampère product has measure coefficients. As an immediate consequence of this proposition together with (2.6) we get the following.

Lemma 2.4. *Let f_k be tuples of holomorphic functions on a neighbourhood X of the origin $0 \in \mathbb{C}^n$ such that $Z(f_k) = \{0\}$ for $k = 1, \dots, r$. Then it holds that*

$$\mathbf{1}_{\{0\}} \sum_{|\alpha|=n} (dd^c \log|f|^2)^\alpha = \mathrm{e}_{\mathrm{BR}}(\mathcal{M})[0] \tag{2.11}$$

where $\mathcal{M} = \oplus_{k=1}^r \mathcal{O}/\mathcal{I}_k$ and $\mathcal{I}_k$ are the ideals defined by f_k.

Proof of Proposition 2.3

Throughout this section, we let

$$\mathscr{A} = \{\text{Artinian ideals } J \subseteq \mathcal{O}_X \text{ supported at the origin}\} \cup \{\mathcal{O}_X\}.$$

Then $\mathscr{A}$ is a commutative monoid with the product defined by multiplying ideals. For $J_1, \dots, J_n \in \mathscr{A}$ we define $m(J_1, \dots, J_n)$ as the number

$$m(J_1, \ldots, J_n) = \int_{\{0\}} dd^c \log|g_1|^2 \wedge \cdots \wedge dd^c \log|g_n|^2 \qquad (2.12)$$

where $g_k = (g_{k1}, \ldots, g_{km_k})$ are tuples generating J_k.

Proposition 2.5. *The function $m : \mathscr{A}^n \to \mathbb{R}$ is well-defined, symmetric, and multilinear.*

Proof. To prove that m is well-defined, we need to show that it is independent of the generators of the ideals. Let $J_1 = (u_1, \ldots, u_p) = (v_1, \ldots, v_q) \in \mathscr{A}$, and $J_k := (g_{k1}, \ldots, g_{km_k}) \in \mathscr{A}$, for $k = 2, \ldots, n$. Since the Monge-Ampère product is symmetric (see Section 2.2) it suffices to show that

$$\int_{\{0\}} dd^c \log|u|^2 \wedge dd^c \log|g_2|^2 \wedge \cdots \wedge dd^c \log|g_n|^2 = \int_{\{0\}} dd^c \log|v|^2 \wedge dd^c \log|g_2|^2 \wedge \cdots \wedge dd^c \log|g_n|^2. \qquad (2.13)$$

Now, note that $u = vA$ for some holomorphic matrix A with positive rank on X. Hence, we have

$$\log|u|^2 \leqslant \log|v|^2 + \varphi$$

where $\varphi = \log\|A\|_{op}^2$ is a locally bounded function. Similarily, since $v = uB$ for some B of positive rank, we get

$$\log|v|^2 \leqslant \log|u|^2 + \psi$$

for some locally bounded ψ. Thus,

$$\lim_{z\to 0} \frac{\log|u|^2}{\log|v|^2} = 1$$

and thus, (2.13) follows from the first comparison theorem (see [Dem12, Theorem III.7.1]). Hence, m is well-defined.

That m is symmetric follows immediately from the fact that the mixed Monge-Ampère product is symmetric.

It remains to show that m is multilinear. Since m is symmetric, it is enough to show

$$m(IJ, J_2, \dots, J_n) = m(I, J_2, \dots, J_n) + m(J, J_2, \dots, J_n). \tag{2.14}$$

Suppose $I = (u_1, \dots, u_p)$, $J = (v_1, \dots, v_q) \in \mathscr{A}$. Then IJ is generated by $h_{k\ell} := u_k v_\ell$, for $k = 1, \dots, p$ and $\ell = 1, \dots, q$. Let $h = (h_{11}, \dots, h_{1q}, \dots, h_{p1}, \dots, h_{pq})$. Then clearly $|h|^2 = |u|^2|v|^2$, and hence, $\log|h|^2 = \log|u|^2 + \log|v|^2$. Thus, (2.14), follows from the multilinearity of the mixed Monge-Ampère product (see Section 2.2). This finishes the proof. ■

Now, let $\gamma \in \mathbb{N}^n$ be the multi-index with $\gamma_k = 1$ for $k = 1, \dots, n$. For $J_1, \dots, J_n \in \mathscr{A}$ we define $e(J_1, \dots, J_n)$ as the number

$$e(J_1, \dots, J_n) = e_\gamma(J_1, \dots, J_n). \tag{2.15}$$

Proposition 2.6. *The function $e : \mathscr{A}^n \to \mathbb{R}$ is symmetric and multilinear.*

Proof. That e is symmetric is clear in view of (2.2).

Let $J_1, \dots, J_{n-1}, I, J \in \mathscr{A}$. Then by [Ree84, Lemma 2.5] we have linearity in the last factor

$$e(J_1, \dots, J_{n-1}, IJ) = e(J_1, \dots, J_{n-1}, I) + e(J_1, \dots, J_{n-1}, J).$$

Since e is symmetric, it follows that it is multilinear. ■

We want to show that $e = m$, and to do this we invoke the following.

Proposition 2.7. *Suppose $\psi_1, \psi_2 : \mathscr{A}^n \to \mathbb{R}$ are symmetric and multilinear such that for all $a \in \mathscr{A}$ we have $\psi_1(a, \dots, a) = \psi_2(a, \dots, a)$.*

Then $\psi_1 = \psi_2$.

This is immediate from the following elementary polarisation formula (written in multiplicative notation rather than the usual additive, since we are multiplying ideals).

Proposition 2.8. *Suppose* $\mathscr{A}$ *is a commutative monoid and let* $\psi : \mathscr{A}^n \to \mathbb{R}$ *be symmetric and multilinear. Define* $\Psi : \mathscr{A} \to \mathbb{R}$ *by* $\Psi(a) = \psi(a, \ldots, a)$. *Then it holds that*

$$\psi(a_1, \ldots, a_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^k \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \Psi(a_{i_1} \cdots a_{i_k}).$$

Proof of Proposition 2.3. The functions $e, m : \mathscr{A}^n \to \mathbb{R}$ are multilinear and symmetric. From (2.4) we have for any $I \in \mathscr{A}$ that $e(I, \ldots, I) = e(I)$. From King's formula, (1.1), we get $m(I, \ldots, I) = e(I)$, whence $m(I, \ldots, I) = e(I, \ldots, I)$ follows. Thus, from Proposition 2.7 we conclude that $e = m$.

Now, given ideals $I_1, \ldots, I_r \in \mathscr{A}$ and a multi-index $\alpha \in \mathbb{N}^r$ as in the formulation of Proposition 2.3, we define $J_1, \ldots, J_n \in \mathscr{A}$ as follows. Let

$$\begin{aligned} J_k &= I_1, \quad \text{for } k = 1, \ldots, \alpha_1 \\ J_k &= I_2, \quad \text{for } k = \alpha_1 + 1, \ldots, \alpha_2 \\ &\vdots \\ J_k &= I_r, \quad \text{for } k = \alpha_1 + \cdots + \alpha_{r-1} + 1, \ldots, \alpha_1 + \cdots + \alpha_r. \end{aligned}$$

Let $\gamma \in \mathbb{N}^n$ be the multi-index with $\gamma_k = 1$ for $k = 1, \ldots, n$. It follows from Proposition 2.1 that

$$e_\gamma(J_1, \ldots, J_n) = e_\alpha(I_1, \ldots, I_r).$$

As a consequence, $e(J_1, \dots, J_n) = e_\alpha(I_1, \dots, I_r)$, whence

$$\int_{\{0\}} (dd^c \log |f|^2)^\alpha = m(J_1, \dots, J_n) = \\ e(J_1, \dots, J_n) = e_\alpha(I_1, \dots, I_r), \quad (2.16)$$

which is precisely what we wanted to prove. ■

2.3 Residue currents

A function $\chi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is called a smooth approximand of the characteristic function $\chi_{[1,\infty)}$ of the interval $[1, \infty)$, denoted

$$\chi \sim \chi_{[1,\infty)},$$

if $\chi(t) \equiv 0$ for $t \ll 1$ and $\chi(t) \equiv 1$ for $t \gg 1$.

Let f be a holomorphic function on a manifold X such that $Z(f) := \{f = 0\}$ has positive codimension. Herrera and Lieberman proved in [HL71] that the limit

$$\lim_{\varepsilon \to 0} \int_{|f|^2 > \varepsilon} \frac{\xi}{f}$$

exists for test forms ξ and defines the *principal value current* $1/f$ of f. From the above limit, it follows that the current $\bar{\partial}(1/f)$ is supported at $Z(f)$, and such a current is called a *residue current*. Let s be a generically non-vanishing holomorphic section of a Hermitian vector bundle over X such that $Z(f) \subseteq Z(s)$. If $\chi \sim \chi_{[1,\infty)}$ then we can regularise these currents (see e.g. [AW18]) as

$$\frac{1}{f} = \frac{\chi(|s|^2/\varepsilon)}{f} \quad \text{and} \quad \bar{\partial}\frac{1}{f} = \frac{\bar{\partial}\chi(|s|^2/\varepsilon)}{f}. \quad (2.17)$$

There are several generalisations of this type of currents. For instance, we can define the principal value and residue of a generically non-vanishing holomorphic section f of a line bundle $L \to X$. Moreover, Coleff and Herrera introduced in [CH78] products of the form

$$\frac{1}{f_r}\cdots\frac{1}{f_{s+1}}\bar\partial\frac{1}{f_s}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}. \tag{2.18}$$

When $m = \operatorname{codim} Z(f)$, where f is the tuple $f = (f_1, \dots, f_m)$, then the *Coleff-Herrera product* $\bar\partial(1/f_m)\wedge\cdots\wedge\bar\partial(1/f_1)$ is anti-commutative and is supported on $Z(f)$.

Pseudomeromorphic currents

For details and a general reference of the material presented in this section and the next, see e.g. [AW18]. To get a coherent framework for a calculus of residue and principal value currents the sheaf $\mathcal{PM}$ of *pseudomeromorphic currents* on X was introduced in [AW10] and further developed in [AS12]. It consists of direct images under holomorphic mappings of products of test forms and currents on the form (2.18). Moreover, $\mathcal{PM}$ is closed under ∂, $\bar\partial$ and multiplication with smooth forms. Further, pseudomeromorphic currents satisfy the following dimension principle.

Proposition 2.9. *Suppose $\mu \in \mathcal{PM}$ has bidegree (p, q). If μ is supported on a subvariety $Z \subseteq X$ such that* $\operatorname{codim} Z > q$*, then $\mu = 0$.*

Furthermore, pseudomeromorphic currents admit natural restrictions to constructible subsets of X. In particular, if $V \subseteq X$ is a subvariety and s is a holomorphic section of a Hermitian bundle over X such that $V = \{s = 0\}$, then the restriction $\mu|_{X\setminus V}$ of μ to the open set $X \setminus V$ has an extension $\mathbf{1}_{X\setminus V}\mu$ to a pseudomeromorphic current on X. This current can be obtained as a limit of pseudomeromorphic currents

$$\mathbf{1}_{X\setminus V}\mu = \lim_{\varepsilon\to 0}\chi(|s|^2/\varepsilon)\mu \tag{2.19}$$

where $\chi \sim \chi_{[1,\infty)}$. In fact, the limit is independent of the choice of χ and s. It follows that $\mathbf{1}_V\mu := \mu - \mathbf{1}_{X\setminus V}\mu$ is a pseudomeromorphic current on X supported on V.

Almost semi-meromorphic currents

A *semi-meromorphic current* is a current of the form ω/f where f is a generically non-vanishing holomorphic section of a line bundle $L \to X$ and ω is a smooth form with values in L. An *almost semi-meromorphic* current a in X is a current of the form

$$a = \pi_* \left(\frac{\omega}{f} \right) \tag{2.20}$$

where $\pi : X' \to X$ is a modification and ω/f is semi-meromorphic. More generally, if E is a holomorphic bundle over X, we say that a current valued section a is almost semi-meromorphic if there is a modification π, a smooth form-valued section ω of $L \otimes \pi^* E$, and a holomorphic section f of a line bundle $L \to X$, such that $a = \pi_*(\omega/f)$. By definition, an almost semi-meromorphic current is a pseudomeromorphic on X. Hence, ∂a and $\bar{\partial} a \in \mathcal{PM}$ for any $a \in \mathrm{ASM}(X)$. In fact, we have the following (see e.g. [AW18, Proposition 4.16]).

Proposition 2.10. *Suppose $a \in \mathrm{ASM}(X)$ is smooth outside a subvariety $V \subseteq X$. Then $\partial a \in \mathrm{ASM}(X)$ and $\mathbf{1}_{X \setminus V} \bar{\partial} a \in \mathrm{ASM}(X)$.*

Let $\mathrm{ZSS}(a)$ denote the *Zariski singular support* of a, i.e. the smallest Zariski-closed set $V \subseteq X$ such that a is smooth outside V. Then the pseudomeromorphic current

$$r(a) := \mathbf{1}_{\mathrm{ZSS}(a)} \bar{\partial} a \tag{2.21}$$

is the *residue of a*. Note that the residue current $\bar{\partial}(1/f)$ considered above is precisely the residue of the almost semi-meromorphic current $1/f$. Almost semi-meromorphic currents have the *standard extension property* (SEP), which means that for $a \in \mathrm{ASM}(X)$ and any subvariety $V \subseteq X$ of positive codimension we have $\mathbf{1}_V a = 0$. Thus, if s is a section of a Hermitian bundle with $\mathrm{ZSS}(a) \subseteq \{s = 0\}$ and $\chi \sim \chi_{[1,\infty)}$, then we have the following regularisations of the residue of an almost semi-meromorphic a

$$r(a) = \lim_{\varepsilon \to 0} \bar\partial \chi(|s|^2/\varepsilon) \wedge a = \lim_{\varepsilon \to 0} d\chi(|s|^2/\varepsilon) \wedge a. \tag{2.22}$$

The set ASM(X) of almost semi-meromorphic currents in X in fact forms an algebra over the smooth forms $\mathscr{E}^\bullet$ on X. If $a, b \in$ ASM(X) are smooth outside a subvariety $V \subseteq X$, then there is a current $A \in$ ASM(X) that coincides with $a \wedge b$ outside ZSS(a) $\cup$ ZSS(b). By the SEP it then follows that $a \wedge b$ extends as an almost semi-meromorphic current in X. Note that in the special case when $\omega \in \mathscr{E}^\bullet$ and $a \in$ ASM(X) then

$$r(\omega \wedge a) = \omega \wedge r(a) \tag{2.23}$$

follows immediately from the SEP.

2.11 Remark. Suppose X is a neighbourhood of the origin $0 \in \mathbb{C}$. Suppose $f \in \mathcal{O}_X$ satisfies $Z(f) = \{0\}$ so that $f = z^a h$, for some non-vanishing $h \in \mathcal{O}_X$. Since the current $\bar\partial(1/f) \wedge df$ is a $(1,1)$-current supported on $Z(f) = \{0\}$ it acts on any smooth function g on X. Let $g \in \mathcal{O}_X$ and suppose $\chi \sim \chi_{[1,\infty)}$ such that $\chi(t) \equiv 1$ for $t \geqslant 1$. Then from (2.17) the action $\left\langle \bar\partial \frac{1}{f} \wedge df, g \right\rangle$ of the current $\bar\partial(1/f) \wedge df$ on g is obtained as the limit as $\varepsilon \to 0$ of

$$\int_X g \frac{\bar\partial \chi(|f|^2/\varepsilon) \wedge df}{f} = \int_{|f|^2 \leqslant \varepsilon} g \frac{\bar\partial \chi(|f|^2/\varepsilon) \wedge df}{f} =$$
$$\int_{|f|^2 = \varepsilon} g \frac{\chi(|f|^2/\varepsilon) df}{f} = \int_{|f|^2 = \varepsilon} \frac{g df}{f} = 2\pi i a g(0)$$

where we have applied Stokes' theorem in the second equality, and in the last equality we invoke the argument principle. Hence, we can view (1.3) as a smooth version of the argument principle, since it in fact holds for any smooth g. With this perspective, (1.2) is a generalisation of the argument principle to a tuple f with an isolated zero at the origin in arbitrary dimension n.

2.4 Superstructure

In the sequel, we work with currents and forms with values in graded holomorphic bundles. Endowing these bundles with a so called superstructure gives a coherent framework for how to manipulate these objects.

Suppose $H = \oplus_{k=0}^{N} H_k$ is a graded holomorphic bundle over a complex manifold X. We get an induced grading on the endomorphism bundle

$$\operatorname{End} H = \bigoplus_{\nu=-N}^{N} \left(\bigoplus_{\nu=k-\ell} \operatorname{Hom}(H_\ell, H_k) \right) =: \bigoplus_{\nu} \operatorname{End}_\nu H. \tag{2.24}$$

We get a superstructure by taking these gradings modulo 2 giving us a $\mathbb{Z}/2\mathbb{Z}$-grading

$$H = H_+ \oplus H_-, \quad \operatorname{End} H = \operatorname{End}_+ H \oplus \operatorname{End}_- H$$

where H_+, $\operatorname{End}_+ H$ denote the direct sum of the subbundles of even degrees and $H_-, \operatorname{End}_- H$ denote the direct sum of the subbundles of odd degrees of H and $\operatorname{End} H$, respectively.

Suppose the bundle H is equipped with a product $\otimes$ that respects the grading, so that the smooth sections $\mathscr{E}(H)$ of H is a graded algebra over the smooth functions $\mathscr{E}$ on X. Then, with the superstructure, we can extend this product to the smooth form-valued sections $\mathscr{E}^\bullet(H)$ of H, turning $\mathscr{E}^\bullet(H)$ into a graded algebra over smooth forms $\mathscr{E}^\bullet$ on X as follows. First, we give $\mathscr{E}^\bullet(H)$ a grading. If $\alpha = \omega \otimes \xi$, where ω is a homogeneous form and ξ is a homogeneous section of H, then we denote by $\deg \alpha$ the total degree of α

$$\deg \alpha := \deg \omega + \deg \xi.$$

Given a homogeneous form ω and a homogeneous form-valued section $\omega' \otimes \xi$, we turn $\mathscr{E}^\bullet(H)$ into a right-module over $\mathscr{E}^\bullet$ by

$$(\omega' \otimes \xi) \otimes \omega := (-1)^{\deg \xi \deg \omega} (\omega \wedge \omega') \otimes \xi.$$

Then, for homogeneous $\alpha = \omega \otimes \xi$ and $\beta = \omega' \otimes \xi'$, we define

$$\alpha \otimes \beta = (-1)^{\deg \xi \deg \omega'} \omega \wedge \omega' \otimes \xi \otimes \xi'$$

extending the product to a product on $\mathscr{E}^\bullet(H)$ that respects the grading.

Similarly, given homogeneous $\alpha = \omega \otimes \varphi$, with $\varphi \in \mathscr{E}^\bullet(\mathrm{End}_\nu H)$, and $\beta = \omega' \otimes \xi$, we define

$$\alpha(\beta) = (-1)^{\deg \varphi \deg \omega'} \omega \wedge \omega' \otimes \varphi(\xi). \tag{2.25}$$

Moreover, the form-valued sections $\mathscr{E}^\bullet(\mathrm{End}\, H)$ of the endomorphism bundle naturally has structure of a graded algebra over $\mathscr{E}$ under composition of maps, and we can extend this to a graded algebra over $\mathscr{E}^\bullet$ as above.

Let D_H be a connection on H. Then for form-valued sections $\alpha, \beta \in \mathscr{E}^\bullet(H)$ we have a Leibniz rule

$$D_H(\alpha \otimes \beta) = D_H\alpha \otimes \beta + (-1)^{\deg \alpha} \alpha \otimes D_H\beta. \tag{2.26}$$

We also get an induced connection D_{End} on $\mathrm{End}\, H$ which on form-valued endomorphisms $\alpha \in \mathscr{E}^\bullet(\mathrm{End}\, H)$ is defined by

$$D_{\mathrm{End}}\alpha = D_E \circ \alpha - (-1)^{\deg \alpha} \alpha \circ D_E. \tag{2.27}$$

This connection also satisfies Leibniz' rule, i.e. for $\alpha, \beta \in \mathscr{E}^\bullet(\mathrm{End}\, H)$, we have

$$D_{\mathrm{End}}(\alpha\beta) = D_{\mathrm{End}}\alpha\beta + (-1)^{\deg \alpha} \alpha D_{\mathrm{End}}\beta. \tag{2.28}$$

Finally, note that a form-valued section $\alpha \in \mathscr{E}^\bullet(H)$ defines an endomorphism

$$\alpha(\beta) := \alpha \otimes \beta \tag{2.29}$$

and in fact

$$D_{\mathrm{End}}\alpha(\beta) = D_E\alpha(\beta).$$

2.5 The Koszul complex and residue current

Let X be a neighbourhood of the origin $0 \in \mathbb{C}^n$ and let $f = (f_1, \dots, f_m)$ be a tuple of holomorphic functions defined on X such that $Z(f) := \{f = 0\} = \{0\}$. Let F be a trivial holomorphic rank m bundle over X and fix a frame $e = (e_1, \dots, e_m)$ with dual frame $e^* = (e_1^*, \dots, e_m^*)$. We view f as a section of the dual bundle F^*

$$f := \sum_{k=1}^{m} f_k e_k^*.$$

Let δ_f be the map given by contraction with f

$$\delta_f e_k := f_k.$$

Contraction with f extends to a map on the exterior algebra ΛF of F by defining

$$\delta_f(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_{k=1}^{r} (-1)^{k-1} f_{i_k} \widehat{e_{i_k}} \tag{2.30}$$

where the circumflex means that e_{i_k} has been omitted from the exterior product $e_{i_1} \wedge \cdots \wedge e_{i_r}$. Note that δ_f is anti-commutative, i.e. for homogeneous $\xi, \eta \in \mathscr{E}(\Lambda F)$

$$\delta_f(\xi \wedge \eta) = \delta_f(\xi) \wedge \eta + (-1)^{\deg \xi} \xi \wedge \delta_f(\eta). \tag{2.31}$$

As a result, δ_f defines a differential, $\delta_f^2 = 0$, on the exterior algebra. The Koszul complex associated to f is the complex

$$0 \longrightarrow \Lambda^m F \xrightarrow{\delta_f} \cdots \qquad \cdots \xrightarrow{\delta_f} \Lambda^2 F \xrightarrow{\delta_f} F \xrightarrow{\delta_f} \mathcal{O}.$$

We now recall Andersson's contruction in [And04] of the residue current $\widetilde{R}^f$, see also e.g. [AW18, Example 4.18]. First, we view the Koszul complex $H = \oplus_k H_k := \oplus_k \Lambda^k F$ as a graded holomorphic bundle with the product $\otimes$ being the usual exterior product $\wedge$, and we equip H with a superstructure as in Section 2.4. We equip H_1 with a trivial metric and connection d with respect to the frame $e_1, \ldots, e_m$ and take the induced metric and connection on H. Let τ be the section of H_1 of minimal norm such that $f(\tau) = 1$ outside the origin. In the given frame, we can then write

$$\tau = \frac{1}{|f|^2} \sum_{k=1}^{m} \overline{f_k} e_k. \tag{2.32}$$

Note that $\tau \in \mathscr{E}^{\bullet}_{X \setminus \{0\}}(H)$ is odd and $\bar{\partial}\tau$ is even. Moreover, one can show that τ extends across the origin as an almost semi-meromorphic current. Since $\mathrm{ASM}(X)$ is an algebra, we get from Proposition 2.10 that the section $v_n \in \mathscr{E}^{\bullet}_{X \setminus \{0\}}(H)$ defined by

$$v_n = \tau \wedge (\bar{\partial}\tau)^{n-1} \tag{2.33}$$

extends to an almost semi-meromorphic current V_n across the origin. The residue current $\widetilde{R}^f$ associated to f is then the residue of the almost semi-meromorphic current V_n

$$\widetilde{R}^f := r(V_n).$$

Let $\varphi_k = \delta_f$, $k = 1, \ldots, m$, be the morphisms appearing in the Koszul complex and

$$d\varphi := d\varphi_1 \cdots d\varphi_n = (d\delta_f)^n.$$

Then, as noted in the introduction, the residue current $\widetilde{R}^f$ satisfies the formula (1.2). Note that δ_f is an odd section of $\mathscr{E}^{\bullet}(\operatorname{End} H)$ and that from (2.27) we get

$$d\delta = \delta_{df} \tag{2.34}$$

where δ_{df} is contraction with the section $\sum_{k=1}^{m} df_k \otimes e_k^* \in \mathscr{E}^{\bullet}(H^*)$, so that δ_{df} is an even section of $\mathscr{E}^{\bullet}(\operatorname{End} H)$, cf. Section 2.4. For the sequel, we need the following factorisation of the Monge-Ampère product.

Proposition 2.12. *For any $\ell \geqslant 1$, we have*

$$\left(dd^c \log |f|^2\right)^{\ell} = \frac{1}{(2\pi i)^{\ell} \ell!} \delta_{df}^{\ell} \left(\left(\bar{\partial}\tau\right)^{\ell}\right). \tag{2.35}$$

outside the origin.

Proof. We give a proof by induction. Moreover, we get from (2.30) together with (2.25) that

$$\delta_{df}\left(\bar{\partial}\tau\right) = \bar{\partial}\left(\frac{1}{|f|^2}\sum_{k=1}^{m}\overline{f_k}df_k\right) = (2\pi i)dd^c\log|f|^2,$$

which proves the base case $\ell = 1$.

Suppose now that (2.35) holds for some $\ell \geqslant 1$. For $\ell + 1$ we have

$$\delta_{df}^{\ell+1}\left((\bar{\partial}\tau)^{\ell+1}\right) = \delta_{df}^{\ell}\left(\delta_{df}((\bar{\partial}\tau)^{\ell+1})\right) = (\ell+1)\delta_{df}^{\ell}\left(\delta_{df}(\bar{\partial}\tau)\wedge(\bar{\partial}\tau)^{\ell}\right) = (\ell+1)\delta_{df}(\bar{\partial}\tau)\wedge\delta_{df}^{\ell}(\bar{\partial}\tau)^{\ell} = (2\pi i)^{\ell+1}(\ell+1)!\left(dd^c\log|f|^2\right)^{\ell+1}, \quad (2.36)$$

where the third equality follows from the fact that $\delta_{df}(\bar{\partial}\tau)$ is a pure diffferential form, whence $\delta_{df}(\delta_{df}(\bar{\partial}\tau)) = 0$. By induction, this proves the result. ■

Chapter 3

The Buchsbaum-Rim residue current

In the given setting, we briefly recall Andersson's construction in [And06] of the residue current R^f associated to a holomorphic morphism $f : E \to Q$ of bundles E, Q over a manifold X. This residue is constructed from a complex (H, φ), the so-called *Buchsbaum-Rim complex* associated to f, consisting of holomorphic bundles over X.

3.1 The Buchsbaum-Rim complex

Let X be a neighbourhood of the origin $0 \in \mathbb{C}^n$ and let $f = (f_{k\ell})$ be a $(r \times m)$-matrix of holomorphic functions $f_{k\ell}$ on X such $Z(f) = \{0\}$, where $Z(f)$ is the set where f has sub-optimal rank. Let E, Q be trivial holomorphic bundles over X of rank m and r and with frames $e_1, \ldots, e_m$ and $\varepsilon_1, \ldots, \varepsilon_r$, respectively. We identify f with the holomorphic bundle morphism $f : E \to Q$ defined by

$$f = \sum_{k=1}^{r} \sum_{\ell=1}^{m} f_{k\ell} \varepsilon_k \otimes e_\ell.$$

We now define the Buchsbaum-Rim complex (H, φ) associated to f.

Let $H_0 := Q$, $H_1 := E$ and for $\nu \geqslant 2$

$$H_\nu := \Lambda^{r+\nu-1} H_1 \otimes S^{\nu-2}(H_0^*) \otimes \det H_0^*. \tag{3.1}$$

For $\nu \geqslant 2$, a section $\eta \in \mathscr{E}(H_\nu)$ can be written in the frame ε_k, with dual frame ε_k^*, as

$$\eta = \sum_{\substack{\alpha \in \mathbb{N}^r \\ |\alpha| = \nu - 2}} \eta_\alpha \otimes \varepsilon_\alpha^* \otimes \varepsilon^*$$

with $\eta_\alpha \in \mathscr{E}(\Lambda^{r+\nu-1} H_1)$ and where

$$\varepsilon_\alpha^* = \frac{1}{\alpha!} (\varepsilon_1^*)^{\alpha_1} \cdots (\varepsilon_r^*)^{\alpha_r}.$$

Write $f = \sum_{k=1} f_k \otimes \varepsilon_k$ where $f_k \in \mathcal{O}(H_1^*)$ correspond to the rows in f. Let δ_{f_k} be the contraction with f_k, which extends to the exterior algebra ΛH_1 of H_1, cf. (2.30). We can then view f as the morphism

$$f = \sum_{k=1}^{r} \delta_{f_k} \otimes \varepsilon_k : H_1 \to H_0. \tag{3.2}$$

which acts on sections $\eta \in \mathscr{E}(H_1)$ by

$$\sum_{k=1}^{r} \delta_{f_k}(\eta) \varepsilon_k.$$

Let ε_k^* be the dual frame of ε_k and define $\varepsilon^* = \varepsilon_1^* \wedge \cdots \wedge \varepsilon_r^*$. Define a morphism

$$\delta_F = \delta_{f_r} \cdots \delta_{f_1} \rho : H_2 \to H_1, \tag{3.3}$$

where $\rho : \det H_0^* \to \mathcal{O}_X$ is the morphism defined by $\varepsilon^* \mapsto 1$. Let $u \in \mathcal{O}(H_0)$ and write $u = \sum_{k=1}^{r} u_k \varepsilon_k$. Contraction $\delta_u : H_0^* \to \mathcal{O}$ with u extends to a map on the symmetric algebra $S(H_0^*)$

$$\delta_u(\varepsilon_{i_1}^* \cdots \varepsilon_{i_s}^*) := \sum_{k=1}^{s} u_{i_k} \widehat{\varepsilon_{i_k}^*} \tag{3.4}$$

where the circumflex means that $\varepsilon^*_{i_k}$ has been omitted from the symmetric product $\varepsilon^*_{i_1}\cdots\varepsilon^*_{i_r}$. Note that δ_u is commutative, i.e. for $v,w\in\mathcal{O}(S(H_0^*))$, we have

$$\delta_u(vw)=\delta_u(v)w+v\delta_u(w). \tag{3.5}$$

As a consequence,

$$\delta_u(v^k)=k\delta_u(v)v^{k-1}. \tag{3.6}$$

Finally, for $\nu\geqslant 3$, we define morphisms

$$\delta=\sum_{k=1}^{r}\delta_{f_k}\delta_{\varepsilon_k}:H_\nu\to H_{\nu-1} \tag{3.7}$$

which act on sections of H_ν as

$$\delta(\xi\otimes u\otimes\varepsilon^*)=\sum_{k=1}^{r}\delta_{f_k}(\xi)\otimes\delta_{\varepsilon_k}(u)\otimes\varepsilon^*,$$

where $\xi\in\mathscr{E}(\Lambda H_1)$ and $u\in\mathscr{E}(S(H_0^*))$. We note that $\delta^2=0$, $\delta_F\delta=0$ and $f\delta_F$, which follows from the fact that the δ_{f_k} are anti-commutative (2.31) while the δ_{ε_k} are commutative (3.5). Hence, we get a complex (H,φ) with

$$\varphi_1:=f,\quad \varphi_2:=\delta_F,\quad \varphi_\nu:=\delta:H_\nu\to H_{\nu-1},\quad \nu\geqslant 3 \tag{3.8}$$

This complex is the *Buchsbaum-Rim complex associated to f*.

We define an auxiliary graded holomorphic bundle

$$A=(\Lambda H_1)\otimes S(H_0)\otimes(\det H_0^*\oplus\mathcal{O}) \tag{3.9}$$

with the grading induced from letting

$$\deg(\Lambda^k H_1)=k,\quad \deg(S^k(H_0))=0,\quad \deg\mathcal{O}=0,\quad \deg(\det H_0^*)=-r+1.$$

(Note that the last one is non-standard.) We can define a product on

$\mathscr{E}(A)$. For $\xi, \xi' \in \mathscr{E}(\Lambda H_1)$, $u, v \in \mathscr{E}(S(H_0))$ and $a, b \in \mathscr{E}(\det H_0^*) \oplus \mathscr{E}$

$$(\xi \otimes u \otimes a) \otimes (\xi' \otimes v \otimes b) := (\xi \wedge \xi') \otimes (uv) \otimes (a \wedge b)$$

where $\wedge$ is the usual exterior product and the concatenation uv is the symmetric product in $S(H_0)$. Note that the product $\otimes$ respects the grading, so that $\mathscr{E}(A)$ is a graded algebra over $\mathscr{E}$. We equip A with a superstructure and extend the product to $\mathscr{E}^\bullet(A)$ as in Section 2.4.

As a subbundle of A

$$H := \bigoplus_{\nu \in \mathbb{N}} H_\nu$$

inherits a grading with $\deg H_\nu = \nu$, as expected, and H is further equipped with the superstructure inherited from A. We also equip H_0 and H_1 with trivial metrics and connections with respect to the frames $e_1, \ldots, e_m$ and $\varepsilon_1, \ldots, \varepsilon_r$ of H_1 and H_0, respectively. The Buchsbaum-Rim complex inherits a trivial metric and connection d. The morphisms f, δ_F and δ extend to maps on form-valued sections of H (cf. (2.25)). Note that with the superstructure all of these maps are odd endomorphisms. As before, we get an even endomorphism

$$d\delta_{f_k} = \delta_{df_k}, \tag{3.10}$$

cf. (2.34). Note also that δ_{ε_k} is even and that $d\delta_{\varepsilon_k} = 0$. Moreover, $\deg \rho = r - 1$ and $d\rho = 0$. Finally, in $\mathscr{E}^\bullet(\operatorname{End} H)$ we have the following commutation rules

$$\delta \circ \bar\partial = -\bar\partial \circ \delta, \quad \delta_F \circ \bar\partial = -\bar\partial \circ \delta_F \quad \delta_{f_k}\delta_{f_\ell} = -\delta_{f_\ell}\delta_{f_k}$$
$$\delta_{f_k}\delta_{\varepsilon_\ell} = \delta_{\varepsilon_\ell}\delta_{f_k}, \quad \rho\delta_{f_k} = (-1)^{r-1}\delta_{f_k}\rho, \quad \rho\delta_{\varepsilon_k} = \delta_{\varepsilon_k}\rho. \tag{3.11}$$

3.2 The Buchsbaum-Rim residue current

Let σ_f be the minimal inverse of $f : H_1 \to H_0$, i.e the section of $H_1 \otimes H_0^*$ such that if we write

$$\sigma_f = \sum_{k=1}^{r} \sigma_k \delta_{\varepsilon_k^*}$$

then $\sigma_k \in \mathscr{E}_{X\setminus\{0\}}(H_1)$ are the sections of minimal norms such that outside the origin

$$f_k(\sigma_\ell) = \delta_{k\ell}. \tag{3.12}$$

We also define the section $\widetilde{\sigma} \in \mathscr{E}_{X\setminus\{0\}}(\wedge^r E)$ as

$$\widetilde{\sigma} = \sigma_1 \wedge \cdots \wedge \sigma_r.$$

Then the section $\tau := \widetilde{\sigma} \otimes \varepsilon^* \in \mathscr{E}_{X\setminus\{0\}}(\wedge^r E \otimes \det Q^*)$ induces a morphism

$$\tau(\xi) := \tau \otimes \xi : H_1 \to H_2.$$

Note that $\delta_F(\tau) = 1$. Finally, for $\nu \geqslant 2$, the section σ of $H_1 \otimes H_0^*$ defined by

$$\sigma = \sum_{k=1}^{r} \sigma_k \otimes \varepsilon_k^*$$

induces morphisms

$$\sigma(\xi \otimes u \otimes \varepsilon^*) := \sigma \otimes (\xi \otimes u \otimes \varepsilon^*) : H_\nu \to H_{\nu+1}.$$

Note that σ_f, τ and σ are odd sections of the auxiliary bundle A and they define odd endomorphisms of H. Moreover, σ_f, τ and σ (which a priori are defined only outside the origin) extend as almost semi-meromorphic currents across the origin, see [And06, Lemma 4.1].

Outside the origin we define a form-valued section u_n of $\mathrm{Hom}(H_0, H_n)$ (cf. (3.1)) by

$$u_n = (\bar{\partial}\sigma)^{\otimes(n-2)} \otimes \tau \otimes \bar{\partial}\sigma_f.$$

In the frame ε_k we can write

$$u_n = \sum_{k=1}^{r} \sum_{\substack{\alpha\in\mathbb{N}^r \\ |\alpha|=\nu-2}} \widetilde{\sigma} \wedge \bar{\partial}\sigma_k \wedge (\bar{\partial}\sigma)^\alpha \otimes \varepsilon_\alpha^* \otimes \varepsilon^* \otimes \delta_{\varepsilon_k^*}, \tag{3.13}$$

where

$$(\bar{\partial}\sigma)^\alpha = (\bar{\partial}\sigma_1)^{\alpha_1} \wedge \cdots \wedge (\bar{\partial}\sigma_r)^{\alpha_r}. \tag{3.14}$$

Note that we have used that $\bar{\partial}\sigma_\ell$ is even, so that we can place the σ-terms in any order. The form-valued section u_n is of bi-degree $(0, n-1)$ and it is smooth outside the origin. In fact, since $\mathrm{ASM}(X)$ is an algebra, we get from Proposition 2.10 that u_n extends to an almost semi-meromorphic current U_n across the origin. The *(Buchsbaum-Rim) residue current R^f associated to the matrix f* is then the residue of this almost semi-meromorphic current

$$R^f := r(U_n)$$

and R^f is a $(0,n)$-current supported at the origin and with values in $\mathrm{Hom}(H_0, H_n)$.

Chapter 4

Proofs

Suppose now that we are in the setting of Theorem 1.1, i.e. f is a block diagonal matrix of holomorphic functions on X where the blocks are tuples f_k satisfying $Z(f_k) = \{0\}$. We get a decomposition $E = \oplus_{k=1}^r E_k$ of trivial holomorphic subbundles $E_k \subseteq E$ such that

$$f = \sum_{k=1}^{r} f_k \otimes \varepsilon_k$$

with $f_k \in \mathcal{O}(E_k^*)$. In this case the sections σ_k defined in (3.12) are precisely the minimal inverses of the tuples f_k (cf. (2.32)), since the sections f_k of E^* take values in different subbundles E_k^* of E^*. Let (H, φ) be the Buchsbaum-Rim complex of f and let $d\varphi$ be the (smooth) form-valued section

$$d\varphi = d\varphi_1 \cdots d\varphi_n \tag{4.1}$$

of $\operatorname{Hom}(H_n, H_0)$.

First, as noted in the introduction, Theorem 1.1 follows from Theorem 1.2 together with Lemma 2.4.

To prove Theorem 1.2, we analyse the differential form $\operatorname{tr}(d\varphi u_n)$, where u_n is the form-valued section of $\operatorname{Hom}(H_n, H_0)$ defined (outside the origin) from the Buchsbaum-Rim complex, cf. (3.13). Since $d\varphi$ is smooth, and u_n extends to an almost semi-meromorphic current U_n across the origin, it holds that $\operatorname{tr}(d\varphi u_n)$ extends to the almost semi-meromorphic current $\operatorname{tr}(d\varphi U_n)$ across the origin. Moreover, (cf. (2.23))

$$r(\operatorname{tr}(d\varphi U_n)) = \operatorname{tr}(d\varphi R^f), \tag{4.2}$$

where $R^f = r(U_n)$ is the residue associated to f, see Section 3.2. We study the form $\operatorname{tr}(d\varphi u_n)$ and calculate the residue of its almost semi-meromorphic current extension.

Proposition 4.1. *Outside the origin, it holds that*

$$\operatorname{tr}(d\varphi u_n) = (n-1)! \sum_{k=1}^{r} \sum_{\substack{\alpha \in \mathbb{N}^r \\ |\alpha| = n-1}} (\alpha_k + 1) \delta_{df_k}(\sigma_k) \wedge \left(\delta_{df}\left(\bar\partial\sigma\right)\right)^{\alpha}, \tag{4.3}$$

where $\left(\delta_{df}\left(\bar\partial\sigma\right)\right)^{\alpha} = \left(\delta_{df_1}(\bar\partial\sigma_1)\right)^{\alpha_1} \wedge \cdots \wedge \left(\delta_{df_r}(\bar\partial\sigma_r)\right)^{\alpha_r}$.

Proof. Recall, from (3.8) and (4.1), that

$$d\varphi = df\, d\delta_F (d\delta)^{n-2}.$$

We begin by calculating the differentials of the morphisms separately (see (3.2), (3.3), and (3.7) for definitions of the maps). Throughout the proof, we use (3.10) and the commutation rules (3.11) freely.

First,

$$df = \sum_{k=1}^{r} \delta_{df_k} \otimes \varepsilon_k.$$

Next, note that $d\rho = 0$. Hence, from Leibniz' rule (2.28), we get

$$d\delta_F = \sum_{\ell=1}^{r} (-1)^{r-\ell} \delta_{f_r} \cdots \delta_{df_\ell} \cdots \delta_{f_1} \rho.$$

Lastly, $d\delta = \sum_k \delta_{df_k} \delta_{\varepsilon_k}$, since $d\delta_{\varepsilon_k} = 0$. Hence the multinomial theorem implies that

$$(d\delta)^{n-2} = \sum_{|\beta|=n-2} \binom{n-2}{\beta} \delta_{df}^{\beta} \delta_{\varepsilon}^{\beta},$$

where $\delta_{df}^{\beta} = \delta_{df_1}^{\beta_1} \cdots \delta_{df_r}^{\beta_r}$ and $\delta_{\varepsilon}^{\beta} = \delta_{\varepsilon_1}^{\beta_1} \cdots \delta_{\varepsilon_r}^{\beta_r}$. Taking all of this together, we see that

$$d\varphi = \sum_{k,\ell=1}^{r} \sum_{|\beta|=n-2} (-1)^{r-\ell} \binom{n-2}{\beta} \delta_{df_\ell} \delta_{df_k} \delta_{df}^{\beta} \delta_{f_r} \cdots \widehat{\delta_{f_\ell}} \cdots \delta_{f_1} \rho \delta_{\varepsilon}^{\beta} \otimes \varepsilon_k.$$

After expanding u_n as in (3.13), we find that

$$\sum_{|\beta|=n-2} (-1)^{r-\ell} \binom{n-2}{\beta} \delta_{df_\ell} \delta_{df_k} \delta_{df}^{\beta} \delta_{f_r} \cdots \widehat{\delta_{f_\ell}} \cdots \delta_{f_1} (\widetilde{\sigma} \wedge \bar{\partial}\sigma_k \wedge (\bar{\partial}\sigma)^{\beta}),$$

is the coefficient of $\varepsilon_k \otimes \delta_{\varepsilon_k^*}$ in $d\varphi(u_n)$. Now, since δ_{f_m} is holomorphic, it follows from (2.27) together with (3.12), that $\delta_{f_m}(\bar{\partial}\sigma_p) = 0$, for any m and p. Hence,

$$\delta_{f_r} \cdots \widehat{\delta_{f_\ell}} \cdots \delta_{f_1} \left(\widetilde{\sigma} \wedge \bar{\partial}\sigma_k \wedge (\bar{\partial}\sigma)^{\beta}\right) = (-1)^{r-\ell} \sigma_\ell \wedge \bar{\partial}\sigma_k \wedge (\bar{\partial}\sigma)^{\beta}.$$

As a consequence, we see that

$$\begin{aligned} \operatorname{tr}(d\varphi u_n) &= \sum_{k,\ell=1}^{r} \sum_{|\beta|=n-2} \binom{n-2}{\beta} \delta_{df_\ell} \delta_{df_k} \delta_{df}^{\beta} \left(\sigma_\ell \wedge \bar{\partial}\sigma_k \wedge \left(\bar{\partial}\sigma\right)^{\beta}\right) \\ &= \sum_{\ell=1}^{r} \sum_{|\alpha|=n-1} \binom{n-1}{\alpha} \delta_{df_\ell} \delta_{df}^{\alpha} \left(\sigma_\ell \wedge \left(\bar{\partial}\sigma\right)^{\alpha}\right). \end{aligned}$$

Finally, it follows from (2.36) that $\delta_{df_k}^{\alpha_k}((\bar{\partial}\sigma_k)^{\alpha_k}) = \alpha_k!(\delta_{df_k}(\bar{\partial}\sigma_k))^{\alpha_k}$. Similarly, we get

$$\delta_{df_k}^{\alpha_k+1} \left(\sigma_k \wedge \left(\bar{\partial}\sigma_k\right)^{\alpha_k}\right) = (\alpha_k+1)! \delta_{df_k}(\sigma_k) \wedge (\delta_{df_k}(\bar{\partial}\sigma_k))^{\alpha_k}. \tag{4.4}$$

Moreover, since the f_k take values in different subbundles E_k, it holds that $\delta_{df_k}(\bar{\partial}\sigma_\ell) = 0$ whenever $k \neq \ell$. Thus,

$$\delta_{df_k}\delta_{df}^{\alpha}\left(\sigma_k \wedge \left(\bar{\partial}\sigma\right)^{\alpha}\right) = \alpha!(\alpha_k + 1)\delta_{df_k}(\sigma_k) \wedge \left(\delta_{df}\left(\bar{\partial}\sigma\right)\right)^{\alpha}$$

and the result follows. ■

Note that the terms on the right hand side of (4.3) are almost semi-meromorphic. Indeed, this follows from Proposition 2.10 since each σ_k is almost semi-meromorphic (see (2.32)) and $\mathrm{ASM}(X)$ forms an algebra. In fact, we have the following computation of the residue of such a term.

Lemma 4.2. *Suppose $\alpha \in \mathbb{N}^r$ is a multi-index with $|\alpha| = n - 1$. Then*

$$r(\delta_{df_k}(\sigma_k)(\delta_f(\bar{\partial}\sigma))^{\alpha}) = (2\pi i)^n \mathbf{1}_{\{0\}} dd^c \log|f_k|^2 \wedge (dd^c \log|f|^2)^{\alpha} \quad (4.5)$$

where

$$(dd^c \log|f|^2)^{\alpha} = (dd^c \log|f_1|^2)^{\alpha_1} \wedge \cdots \wedge (dd^c \log|f_r|^2)^{\alpha_r}. \quad (4.6)$$

To prove this lemma we need the following result.

Proposition 4.3. *Let $g = (g_1, \ldots, g_p)$ and $h = (h_1, \ldots, h_q)$ be tuples of holomorphic functions in a neighbourhood X of the origin $0 \in \mathbb{C}^n$ such that the ideal $(g_1, \ldots, g_p) \subseteq \mathfrak{m}$ and the ideal $(h_1, \ldots, h_q)$ is $\mathfrak{m}$-primary, where $\mathfrak{m} = (z_1, \ldots, z_n) \subseteq \mathcal{O}_X$ is the maximal ideal at the origin $0 \in \mathbb{C}^n$. Then there is a positive integer N_0 such that for any integer $N \geqslant N_0$, the inequality $|g|^2 \geqslant e^{-N}/2$ implies $|h|^2 \geqslant e^{-N^2}$.*

Proof. First note that since $(h_1, \ldots, h_q)$ is $\mathfrak{m}$-primary, there is a positive integer a such that $\mathfrak{m}^a \subseteq (h)$. Now, from the inclusions $(g) \subseteq \mathfrak{m}$ and $\mathfrak{m}^a \subseteq (h)$ we get the inequalities $|g| \leqslant A|z|$ and $|z|^a \leqslant B|h|$ for some positive constants A, B. Thus, there is a positive constant C such that $|h| \geqslant C|g|^a$.

Suppose now that $|g|^2 \geqslant e^{-N}/2$. Then we have

$$|h|^2 \geqslant C^2|g|^{2a} \geqslant C^2\frac{e^{-aN}}{2^a}.$$

Hence, we can get an inequality $|h|^2 \geqslant e^{-N^2}$ by ensuring that $C^2\frac{e^{-aN}}{2^a} \geqslant e^{-N^2}$. This inequality can then be rewritten as

$$N^2 \geqslant aN + a\log 2 - 2\log C$$

and we can take N_0 to be the smallest positive integer such that this inequality holds. This proves the proposition. ■

Proof of Lemma 4.2. We compare the regularisation (2.8) of the Monge-Ampère product with the regularisation (2.22) of the residue. Without loss of generality, we can assume $k = r$, since the Monge-Ampère product is commutative (cf. (2.8)).

Write $\psi_\ell := \log|f_\ell|^2$. We regularise the current

$$\mathbf{1}_{\{0\}}\bar{\partial}\partial\psi_r \wedge \big(\bar{\partial}\partial\psi\big)^\alpha$$

as follows. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth, convex, increasing function such that $\rho(t)$ is constant for $t \leqslant -\log 2$ and $\rho(t) = t$ for $t \geqslant 0$. Given a positive integer M, define $\rho_M(t) = \rho(t+M) - M$. For $\ell = 1,\dots,r$, we define $u_\ell^M = \rho_M \circ \psi_\ell$ and note that u_ℓ^M is a sequence of plurisubharmonic functions decreasing to ψ_ℓ. Then, by (2.8), we get

$$T := \bar{\partial}\partial\psi_r \wedge \big(\bar{\partial}\partial\psi\big)^\alpha = \lim_{N\to\infty} \bar{\partial}\partial u_r^N \wedge (\bar{\partial}\partial u^{N^2})^\alpha. \tag{4.7}$$

Let $\chi = \rho \circ \log$, and observe that $\chi \sim \chi_{[1,\infty)}$. Define

$$\chi_{\ell,M}(z) = \chi(|f_\ell(z)|^2/e^{-M}) \tag{4.8}$$

and note that $\partial u_\ell^M = \chi_{\ell,M}\partial\psi_\ell$, whence

$$(\bar{\partial}\partial u_\ell^M)^{\alpha_\ell} = \alpha_\ell\chi_{\ell,M}^{\alpha_\ell-1}\bar{\partial}\chi_{\ell,M} \wedge \partial\psi_\ell \wedge (\bar{\partial}\partial\psi_\ell)^{\alpha_\ell-1} + \chi_{\ell,M}^{\alpha_m} \wedge (\bar{\partial}\partial\psi_\ell)^{\alpha_\ell}.$$

It follows that in the right-hand side of (4.7), there appear products with factors $\chi_{r,N}$, $\bar\partial\chi_{r,N}$ and χ_{ℓ,N^2}, $\bar\partial\chi_{\ell,N^2}$ for $\ell = 1,\dots,r$. By construction $\chi(t) = 0$ when $t \leqslant 1/2$ and $\chi(t) = 1$ when $t \geqslant 1$. We therefore see that $\chi_{\ell,M}(z) = 0$ when $|f_\ell|^2 \leqslant e^{-M}/2$ and $\chi_{\ell,M} = 1$ when $|f_\ell|^2 \geqslant e^{-M}$, for $\ell = 1,\dots,r$. From Proposition 4.3 we get that there is a positive integer N_0 such that if $N \geqslant N_0$, then the inequality $|f_r|^2 \geqslant e^{-N}/2$ implies the inequality $|f_\ell|^2 \geqslant e^{-N^2}$, for all $\ell = 1,\dots,r$. Thus, for $\ell = 1,\dots,r$, we see that $\chi_{\ell,N^2} = 1$ on the support of $\chi_{r,N}$, for any $N \geqslant N_0$. As a consequence, for $N \geqslant N_0$ and $\ell = 1,\dots,r$ it holds that

$$\chi_{r,N}\chi_{\ell,N^2} = \chi_{r,N}, \quad \bar\partial\chi_{r,N}\chi_{\ell,N^2} = \bar\partial\chi_{r,N},$$
$$\chi_{r,N}\bar\partial\chi_{\ell,N^2} = 0, \quad \bar\partial\chi_{r,N} \wedge \bar\partial\chi_{\ell,N^2} = 0.$$

Thus,

$$T = \lim_{N\to\infty} \bar\partial\partial u_r^N \wedge (\bar\partial\partial u^{N^2})^\alpha = \lim_{N\to\infty} \bar\partial\chi_{r,N} \wedge \partial\psi_r \wedge \big(\bar\partial\partial\psi\big)^\alpha + \lim_{N\to\infty} \chi_{r,N}\bar\partial\partial\psi_r \wedge \big(\bar\partial\partial\psi\big)^\alpha =: A + B.$$

A calculation shows that $\partial\psi_\ell = \delta_{df_\ell}(\sigma_\ell)$ and $\bar\partial\partial\psi_\ell = \delta_{df_\ell}(\bar\partial\sigma_\ell)$ and hence, by (2.22) (cf. (4.8)), we recognise the current

$$A = \lim_{N\to\infty} \bar\partial\chi_{r,N} \wedge \partial\psi_r \wedge \big(\bar\partial\partial\psi\big)^\alpha = \lim_{N\to\infty} \bar\partial\chi_{r,N} \wedge \delta_{df_r}(\sigma_r) \wedge \big(\delta_{df}(\bar\partial\sigma)\big)^\alpha$$

as the residue of the almost semi-meromorphic current $\delta_{df_r}(\sigma_r)\wedge\big(\delta_{df}(\bar\partial\sigma)\big)^\alpha$, which is supported precisely at the origin. The current B is the restriction $\mathbf{1}_{X\setminus\{0\}}B'$ of the order 0 current (cf. Section 2.2)

$$B' = \bar\partial\partial\psi_r \wedge \big(\bar\partial\partial\psi\big)^\alpha$$

whence $\mathbf{1}_{\{0\}}B = 0$. Finally, this means that

$$(2\pi i)^n \mathbf{1}_{\{0\}} dd^c \log|f_r|^2 \wedge (dd^c \log|f|^2)^\alpha = \mathbf{1}_{\{0\}} T = A = r(\delta_{df_r}(\sigma_r)(\delta_f(\bar\partial\sigma))^\alpha)$$

which proves the results. ∎

Proof of Theorem 1.2. From Proposition 4.1 together with Lemma 4.2 we get that

$$r(\mathrm{tr}(d\varphi u_n)) = (2\pi i)^n \mathbf{1}_{\{0\}} \left(\sum_{k=1}^{r} \sum_{|\alpha|=n-1} (n-1)!(\alpha_k+1) dd^c \log|f_k|^2 \wedge (dd^c \log|f|^2)^\alpha \right). \tag{4.9}$$

Let $v_k \in \mathbb{N}^r$ be the unit vector with a 1 in the k:th position. We rewrite the sum in (4.9)

$$\sum_{k=1}^{r} \sum_{|\alpha|=n-1} \binom{n-1}{\alpha} (\alpha+v_k)!(dd^c \log|f|^2)^{\alpha+v_k} = \sum_{|\beta|=n} \sum_{k=1}^{r} \binom{n-1}{\beta-v_k} \beta!(dd^c \log|f|^2)^\beta. \tag{4.10}$$

The sum over k is then calculated to

$$\sum_{k=1}^{r} \binom{n-1}{\beta-v_k} \beta! = \binom{n}{\beta} \beta! = n! \tag{4.11}$$

as the coefficient of $(dd^c \log|f|^2)^\beta$. Finally, we see that the right hand side of (4.9) can be written

$$(2\pi i)^n n! \sum_{|\beta|=n} \mathbf{1}_{\{0\}} (dd^c \log|f|^2)^\beta \tag{4.12}$$

which is precisely what we wanted to prove. ■

4.4 Example. Suppose now that the tuples f_k coincide, i.e. (with slight abuse of notation) there is a tuple $g = (g_1, \ldots, g_s)$ of holomorphic functions such that $f_k = g$ for each $k = 1, \ldots, r$. This means that $\sigma_k = \tau$, where τ is the minimal inverse of g, see (2.32). In this special case, we get Theorem 1.1 as a consequence of Andersson's result (1.2) and we do not need to invoke Theorem 1.2.

Indeed, from Proposition 4.1 and a similar calculation as in (4.10)-(4.11), we get

$$\operatorname{tr}(d\varphi u_n) = (n-1)! \sum_{k=1}^{r} \sum_{|\alpha|=n-1} (\alpha_k + 1) \delta_{dg}(\tau) \wedge \left(\delta_{dg}(\bar\partial\tau)\right)^{n-1} = \left(\sum_{|\alpha|=n} 1\right) n! \delta_{dg}(\tau) \wedge \left(\delta_{dg}(\bar\partial\tau)\right)^{n-1} = \binom{n+r-1}{r-1} n! \delta_{dg}(\tau) \wedge \left(\delta_{dg}(\bar\partial\tau)\right)^{n-1}.$$

Now, since

$$n! \delta_{dg}(\tau) \wedge \left(\delta_{dg}(\bar\partial\tau)\right)^{n-1} = \delta_{dg}^n \left(\tau \wedge (\bar\partial\tau)^{n-1}\right)$$

(cf. (4.4)) we get from (2.23) together with (1.2) that

$$r(\operatorname{tr}(d\varphi u_n)) = \binom{n+r-1}{r-1} \delta_{dg}^n r\left(\tau \wedge (\bar\partial\tau)^{n-1}\right) = \binom{n+r-1}{r-1} \delta_{dg}^n \widetilde{R}^g = \binom{n+r-1}{r-1} \mathrm{e}(\mathcal{I})[0].$$

Finally, by (2.4) and (2.6), the module $\mathcal{M} = \oplus_{k=1}^r \mathcal{O}/\mathcal{I}$ defined from f has multiplicity

$$\mathrm{e}_{\mathrm{BR}}(\mathcal{M}) = \binom{n+r-1}{r-1} \mathrm{e}(\mathcal{I}).$$

Hence, we see that in this special case, Theorem 1.1

$$\frac{1}{(2\pi i)^n n!}\,\mathrm{tr}(d\varphi R^f) = \mathrm{e}_{\mathrm{BR}}(\mathcal{M})[0]$$

follows directly from (1.2).

Bibliography

[And04] Mats Andersson. “Residue currents and ideals of holomorphic functions”. *Bull. Sci. Math.* 128.6 (2004), 481–512.

[And05] Mats Andersson. “Residues of holomorphic sections and Lelong currents”. *Ark. Mat.* 43.2 (2005), 201–219.

[And06] Mats Andersson. “Residue currents of holomorphic morphisms”. *J. Reine Angew. Math.* 596 (2006), 215–234.

[AS12] Mats Andersson and Håkan Samuelsson. “A Dolbeault-Grothendieck lemma on complex spaces via Koppelman formulas”. *Invent. Math.* 190.2 (2012), 261–297.

[AW10] Mats Andersson and Elizabeth Wulcan. “Decomposition of residue currents”. *J. Reine Angew. Math.* 638 (2010), 103–118.

[AW18] Mats Andersson and Elizabeth Wulcan. “Direct images of semi-meromorphic currents”. *Ann. Inst. Fourier (Grenoble)* 68.2 (2018), 875–900.

[CH78] Nicolas R. Coleff and Miguel E. Herrera. *Les Courants Résiduels Associés à une Forme Méromorphe.* Berlin, Heidelberg: Springer Berlin Heidelberg, 1978.

[Dem12] Jean-Pierre Demailly. *Complex analytic and differential geometry.* 2012.

[HL71] M. Herrera and D. Lieberman. “Residues and principal values on complex spaces”. *Math. Ann.* 194 (1971), 259–294.

[KR96] D. Kirby and D. Rees. “Multiplicities in graded rings. II. Integral equivalence and the Buchsbaum-Rim multiplicity”. *Math. Proc. Cambridge Philos. Soc.* 119.3 (1996), 425–445.

[PTY00] Mikael Passare, August Tsikh, and Alain Yger. “Residue currents of the Bochner-Martinelli type”. *Publ. Mat.* 44.1 (2000), 85–117.

[Ree84] D. Rees. “Generalizations of reductions and mixed multiplicities”. *J. London Math. Soc. (2)* 29.3 (1984), 397–414.

[Rob98] Paul C. Roberts. *Multiplicities and Chern classes in local algebra.* Vol. 133. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1998, xii+303.

[Swa07] Irena Swanson. “Multigraded Hilbert functions and mixed multiplicities”. In: *Syzygies and Hilbert functions.* Vol. 254. Lect. Notes Pure Appl. Math. Chapman & Hall/CRC, Boca Raton, FL, 2007, 267–280.