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RESEARCH ARTICLE

Evaluation of $\zeta(2,\ldots,2,4,2,\ldots,2)$ and period polynomial relations

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Abstract
In studying the depth filtration on multiple zeta values, difficulties quickly arise due to a disparity between it and the coradical filtration [9]. In particular, there are additional relations in the depth graded algebra coming from period polynomials of cusp forms for $\mathrm{SL}_2(\mathbb{Z})$. In contrast, a simple combinatorial filtration, the block filtration [13, 28] is known to agree with the coradical filtration, and so there is no similar defect in the associated graded. However, via an explicit evaluation of $\zeta(2,\ldots,2,4,2,\ldots,2)$ as a polynomial in double zeta values, we derive these period polynomial relations as a consequence of an intrinsic symmetry of block graded multiple zeta values in block degree 2. In deriving this evaluation, we find a Galois descent of certain alternating double zeta values to classical double zeta values, which we then apply to give an evaluation of the multiple t values [22] $t(2\ell, 2k)$ in terms of classical double zeta values.

Contents

1. Introduction

For any tuple $(k_1, k_2, \ldots, k_r)$ of positive integers with $k_r \geq 2$, we may define a multivariable analogue of the Riemann zeta values, called a multiple zeta value (MZV) of weight $k_1 + \cdots + k_r$ and depth r, by

$$\zeta(k_1, k_2, \ldots, k_r) := \sum_{0<n_1<n_2<\cdots<n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}.$$

These numbers arise naturally in many areas of mathematics and mathematical physics, including in connection to associators [30, 34], Feynman amplitudes [3], and as periods of mixed Tate motives [5].

Unlike single zeta values, multiple zeta values have a rich algebraic structure, the study of which goes back to Euler 15. Many families of relations, such as the associator relations [30], the double shuffle relations [35], and the confluence relations [22], are conjectured to exhaust all relations among MZVs. However, this is incredibly challenging and encompasses still-open questions such as the transcendence of $\zeta(2k+1)$.

One approach to make this more manageable is to consider instead motivic multiple zeta values. Via their connection to mixed Tate motives, MZVs may be lifted to formal, algebraic objects, only satisfying relations coming from the geometry of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ [5]. In this setting, much more is known: the ring $\mathcal{H}$ of motivic MZVs are known to be graded by weight, with weight graded dimensions d_n given by

$$\sum_{n\geq 0} d_n x^n = \frac{1}{1-x^2-x^3}.$$

Motivic multiple zeta values have an explicit basis [5], given by the Hoffman zeta values

$$\{\zeta^{\mathfrak{m}}(k_1, \ldots, k_r) \mid k_1, \ldots, k_r \in \{2, 3\}\}.$$

However, the question of providing a complete set of relations remains an open problem. One approach to describing all (motivic) relations among MZVs is to consider the associated graded algebra with respect to the depth filtration

$$\mathcal{D}_n\mathcal{H} = \langle \zeta^{\mathfrak{m}}(k_1, \ldots, k_r) \mid r \leq n \rangle_{\mathbb{Q}}.$$

Relations in $\mathrm{gr}^{\mathcal{D}}_{\bullet}\mathcal{H}$ are much simpler, with the stuffle product reducing to a simple shuffle product. However, this introduces additional relations [17],

$$14\zeta^{\mathfrak{m}}(3,9) + 75\zeta^{\mathfrak{m}}(5,7) + 84\zeta^{\mathfrak{m}}(7,5) \equiv 0 \pmod{\text{lower depth}}$$

and the associated Lie algebra of relations is no longer free [27]

$$\{\sigma_3, \sigma_9\} - 3\{\sigma_5, \sigma_7\} \equiv 0 \pmod{\text{terms of higher depth}}.$$

In particular, there are a family of such quadratic relations, arising from period polynomials of cusp forms [17]. Both the relations among multiple zeta values and among elements of the motivic Lie algebra are commonly referred to as the *period polynomial relations*. It is conjectured that these relations determine all additional relations among depth graded multiple zeta values – that is to say, the the associated Lie algebra of relations is the quotient of a free Lie algebra by the idea generated by these quadratic relations.

Conjecture 1.1 (Broadhurst-Kreimer, [3]). *The generating series for the dimension of the depth graded multiple zeta values is given by*

$$BK(x,y) := \frac{1}{1-O(x)y+S(x)y^2-S(x)y^4},$$

where

$$O(x) := \frac{x^3}{1-x^2},$$

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and

$$S(x) = \frac{x^{12}}{(1-x^4)(1-x^6)}$$

is the generating function for the space of cusp forms of weight n for the full modular group. That is to say, that the number of linearly independent depth graded multiple zeta values of weight n and depth d is given by the coefficient of $x^n y^d$ of $BK(x,y)$, and that these dimensions are determined by the relations coming from cusp forms.

However, a proof of this remains out of sight, and these additional relations make using the depth graded Lie algebra to conclude statements about ungraded MZVs challenging.

An alternative approach, first explored in [28, 29] and based on results in [11, 13], is to consider the so-called block filtration. This filtration provides a simple description of the coradical filtration associated to the motivic coaction in terms of a combinatorial degree function. Specifically

$$\mathcal{B}_n\mathcal{H} = \langle \zeta^{\mathfrak{m}}(w) \mid \deg_{\mathcal{B}}(w) \leq n\rangle_{\mathbb{Q}},$$

where $\deg_{\mathcal{B}}(w)$ counts the number of subsequences e_ie_i in e_0we_1. In [29], we see that in the associated graded algebra with respect to the block filtration, there are no additional relations, and furthermore that a complete set of relations can be given in low block degree. One might then ask how the period polynomial relations manifest in this setting.

Lemma 1.2. *The depth filtration is a subfiltration of the block filtration:*

$$\mathcal{D}_n\mathcal{H} \subset \mathcal{B}_n\mathcal{H}.$$

Proof. First note that the depth filtration is motivic:

$$\Delta\mathcal{D}_n\mathcal{H} \subset \sum_{i+j=n} \mathcal{D}_i\mathcal{A} \otimes \mathcal{D}_j\mathcal{H}.$$

As such, since the block filtration is equal to the coradical filtration, it suffices to show that $\mathcal{D}_1\mathcal{H} \subset \mathcal{B}_1\mathcal{H}$. This is an immediate consequence of Lemma 3.2 [5]. □

As depth is a subfiltration of the block filtration, it is clear that we should be able to express double zeta values in terms of block degree 2 zeta values, and hence that all block graded relations among them, modulo products, should be determined by relations describing $\mathfrak{bg}$, the associated Lie algebra of relations among block graded MZVs. However, Lemmas 1.2 and 2.8 imply that, in block degree 2 and even weight, relations among multiple zeta values modulo terms of lower block degree are genuine relations modulo products. Thus, the period polynomial relations, modulo products, should arise as a consequence of the relations among block graded MZVs introduced in [29].

And indeed, this seems to be the case. The following is a consequence of Lemma 4.1 and allows us to show that relations among double zeta values of even weight are encoded by certain explicit polynomials in $\mathbb{Q}[x_1, x_2, x_3]$.

Corollary 1.3. *Modulo products, the following holds for any* $0 \leq 2a \leq n$,

$$\begin{aligned}\sum_{i=a}^{n-a} \zeta^{\mathrm{I}}(\{2\}^i, 4, \{2\}^{n-i}) &= 4(-1)^{n+1} \sum_{i=a}^{n-a} \zeta^{\mathrm{I}}(2i+3, 2n-2i+1) \\ &= 4(-1)^n \zeta^{\mathrm{I}}(2a+1, 2n-2a+3).\end{aligned}$$

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Proof. Letting $n \coloneqq a + b$ in Lemma 4.1, we have

$$\zeta^{\mathrm{I}}(\{2\}^a, 4, \{2\}^b) = 4(-1)^n \Bigg[-\zeta^{\mathrm{I}}(2a+2, 2b+2) - \zeta^{\mathrm{I}}(2a+3, 2b+1) \\ + \sum_{j=1}^{2n+3} 2^{j-4-2n} \left(\binom{2n+3-j}{2b+1} - \binom{2n+3-j}{2a+1} \right) \zeta^{\mathrm{I}}(j, 2n+4-j) \Bigg].$$

Noting that both $\zeta^{\mathrm{I}}(2a+2, 2b+2)$ and

$$\binom{2n+3-j}{2b+1} - \binom{2n+3-j}{2a+1}$$

are antisymmetric in a and b, the result immediately follows. □

From this, it is possible to deduce that the dimension of double zeta values of weight $2n + 2$ modulo products is bounded above by $\lfloor \frac{n}{3} \rfloor$, which is precisely the dimension predicted by Conjecture 1.1. As the modulo products version of Conjecture 1.1 is known to hold in depth 2 [36], we must have that all period polynomial relations can be written in terms of the block relations, defined in Section 3, and thus Proposition 3.5 holds.

Proposition 3.5. *All relations among double zeta values of weight* $2n+2$ *modulo products are determined by* (Relation 1) *and* (Relation 3) *via Corollary 1.3.*

Indeed, using a computer, one can easily write the period polynomial relations as linear combinations of relations coming from the dihedral symmetry of Section 8 of [29]. A more explicit connection is given in Proposition 3.7.

The structure of this paper is as follows. We first will briefly remind readers of the motivic formalism, and in particular, the use of the motivic coaction to deduce relations. We then describe the block filtration and review several of the results of [29]. In particular, we will introduce the block dihedral symmetry and the necessary framework to discuss it.

In Section 3, we then apply these results, along with a number of new evaluations to conclude that the period polynomial relations are a consequence of this block dihedral symmetry in block degree 2. The remainder of the paper is then dedicated to the necessary technical results needed for this section. Specifically, an evaluation of $\zeta(\{2\}^a, 4, \{2\}^b)$in terms of double zeta values[1], and a computation of the dimension of the space cut out by the block dihedral symmetry. The latter is a straightforward argument from representation theory, while the former is a trek through the world of MZV relations and machinery, including: motivic cobracket calculations; multiple Euler sums (also called alternating MZVs) and the octagon-relation-induced dihedral symmetries thereof [18, 19]; multiple zeta star values and the stuffle-antipode [31, Equation 2.4],[24, Proposition 1]; Zhao's generalised 2-1 theorem [41] (and the first author's block-decomposition description thereof [12]); (alternating) multiple zeta-half values [39]; the explicit depth-parity theory for depth 3 alternating MZVs [21, 33]; and a vital (and serendipitously unearthed) generalised doubling relation [42, Section 14.2.5]. We divide these results between Section 4 and an appendix, according to how central they are to the main results of the article.

We end the main body of the paper with a short corollary of Proposition A.3 in relation to a variation of multiple zeta values, called multiple t values [23]

$$t(k_1, \ldots, k_d) \coloneqq \sum_{0<n_1<\cdots<n_d} \frac{1}{(2n_1-1)^{k_1} \cdots (2n_d-1)^{k_d}}.$$

[1]Computer readable versions of the full evaluations from Theorems A.6 and A.7 in Appendix A are included in the supplementary materials, as plain text files in `Mathematica` syntax and in `pari/gp` syntax.

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In particular, we provide an evaluation of $t(2\ell, 2k)$, when the arguments are even, via classical double zeta values, improving upon the results of [32, Theorem 1] by giving an explicit formula for the Galois descent in this case.

2. The motivic Lie algebra and block graded multiple zeta values

An essential observation in the study of multiple zeta values is that they may be lifted to motivic periods – algebraic objects satisfying only relations coming from geometry. Because of this, motivic multiple zeta values (mMZVs) are much simpler to study. They are known to be graded by weight, and they come equipped with a coaction that encodes all motivic relations. We may consider their graded analogues with respect to a number of filtrations, or consider the associated Lie coalgebra of mMZVs modulo products, whose relations are encoded in a free Lie algebra. The theory of motivic periods is substantial [7], so we give only an essential overview here and refer the reader to [6] for more details.

2.1. Motivic multiple zeta values

The formal definition of mMZVs relies on the Tannakian formalism for the category of mixed Tate motives over $\operatorname{Spec}\mathbb{Z}$, and is intimately related to the motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ [5]. In brief, letting $\mathrm{MT}(\mathbb{Z})$ denote the category of mixed Tate motives, and denoting by

$$\omega_B, \omega_{dR} : \mathrm{MT}(\mathbb{Z}) \to \mathrm{Vec}_{\mathbb{Q}}$$

the Betti and de Rham realisation functors, the ring of motivic periods of $\mathrm{MT}(\mathbb{Z})$ is the ring of functions on the scheme of tensor isomorphisms

$$\mathcal{P}^{\mathfrak{m}}_{\mathrm{MT}(\mathbb{Z})} := \mathcal{O}(\mathrm{Isom}^{\otimes}_{\mathrm{MT}(\mathbb{Z})}(\omega_{dR}, \omega_B)).$$

The results of Brown [5] tell us that this is isomorphic to $\mathcal{H}[\mathbb{L}^{-1}]$, where $\mathcal{H}$ will be the algebra of motivic multiple zeta values, and $\mathbb{L}$ is a motivic analogue of $2\pi\mathrm{i}$.

However, for our purposes, the reader need only keep in mind the following properties.

Properties 2.1. The algebra $\mathcal{H}$ of motivic multiple zeta values is the $\mathbb{Q}$-algebra spanned by symbols

$$\mathrm{I}^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1}) \text{ where } a_i \in \{0, 1\},$$

called motivic multiple zeta values or motivic iterated integrals, satisfying the following properties:

i) (Equal boundaries) $\mathrm{I}^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_0) = \delta_{n,0}$,
ii) (Reversal of paths) $\mathrm{I}^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1}) = (-1)^n \mathrm{I}^{\mathfrak{m}}(a_{n+1}; a_n, \ldots, a_1; a_0)$,
iii) (Functoriality) $\mathrm{I}^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1}) = \mathrm{I}^{\mathfrak{m}}(1 - a_0; 1 - a_1, \ldots, 1 - a_n; 1 - a_{n+1})$
iv) (Shuffle product) For $1 < r < n$, denote by $\mathrm{Sh}_{r,n-r}$ the set of permutations σ on n satisfying

$$\sigma(1) < \sigma(2) < \cdots < \sigma(r) \text{ and } \sigma(r+1) < \cdots < \sigma(n).$$

Then

$$\mathrm{I}^{\mathfrak{m}}(0; a_1, \ldots, a_r; 1)\mathrm{I}^{\mathfrak{m}}(0; a_{r+1}, \ldots, a_n; 1) = \sum_{\sigma \in \mathrm{Sh}_{n,r}} \mathrm{I}^{\mathfrak{m}}(0; a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}; 1).$$

v) (Period map) There is a ring homomorphism $\mathrm{per} : (\mathcal{H}, \cdot) \to (\mathbb{C}, \cdot)$, called the period map, sending a motivic iterated integral to the corresponding complex iterated integral when it converges.

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For a tuple of positive integers $(k_1, \ldots, k_d)$, and $\ell \geq 0$, we write $\zeta^{\mathfrak{m}} = \zeta_0^{\mathfrak{m}}$ and

$$\zeta_\ell^{\mathfrak{m}}(k_1, \ldots, k_d) \coloneqq (-1)^d \mathrm{I}^{\mathfrak{m}}(0; \{0\}^\ell, 1, \{0\}^{k_1-1}, \ldots, 1, \{0\}^{k_d-1}; 1), \tag{2.1}$$

where $\{0\}^n$ denotes n repeated zeroes.

Remark 2.2. In the standard definition of motivic multiple zeta values, we have that

$$\mathrm{I}^{\mathfrak{m}}(0; 0; 1) = \mathrm{I}^{\mathfrak{m}}(0; 1; 1) = 0.$$

This, combined with compatibility with the shuffle product, uniquely determines the image of motivic MZVs which do not correspond to convergent iterated integrals. This convention produces what are called shuffle regularised MZVs. It is occasionally convenient to consider other regularisations, such as stuffle regularised MZVs or MZVs regularised so that $\zeta(1)$ is regularised to a nonzero constant [25].

Remark 2.3. The reversal of paths property and the functoriality property give an important relation for motivic MZVs called the *duality* relation:

$$\mathrm{I}^{\mathfrak{m}}(0; a_1, \ldots, a_n; 1) = (-1)^n \mathrm{I}^{\mathfrak{m}}(0; 1-a_n, \ldots, 1-a_1; 1)\,.$$

Let $\mathcal{A} \coloneqq \mathcal{H}/(\zeta^{\mathfrak{m}}(2))$ be the quotient by the ideal generated by $\zeta^{\mathfrak{m}}(2)$, and denote by $\mathrm{I}^{\mathfrak{a}}(a_0; a_1, \ldots, a_n; a_{n+1})$ the image of $\mathrm{I}^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1})$. The formula given below equips $\mathcal{H}$ with the structure of an $\mathcal{A}$-comodule

$$\Delta \colon \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}.$$

Explicitly, $\Delta \mathrm{I}^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1})$ is equal to

$$\sum_{\substack{i_0 < i_1 < \cdots < i_{k+1} \\ i_0 = 0,\, i_{k+1} = n+1}} \left(\prod_{s=0}^{k} \mathrm{I}^{\mathfrak{a}}(a_{i_s}; a_{i_s+1}, \ldots, a_{i_{s+1}-1}; a_{i_{s+1}}) \right) \otimes \mathrm{I}^{\mathfrak{m}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}). \tag{2.2}$$

A linear combination R of motivic multiple zeta values vanishes in $\mathcal{H}$ if and only if

i) $\mathrm{per}(R) = 0$, that is, R holds numerically
ii) $\mathrm{per}(R') = 0$ for all transforms R' under the motivic coaction, that is, the relation is motivic. A transform R' of R under the motivic coaction is obtained by choosing a weight graded basis $\{a_i\}_{i \in I}$ of $\mathcal{A}$ and writing

$$\Delta(R) = \sum_{i \in I} a_i \otimes R_i.$$

Each R_i is referred to as a transform of R.

As the coaction is quite combinatorially complicated, it is often convenient to instead consider the infinitesimal coactions D_r. Define the Lie coalgebra of indecomposables

$$\mathcal{L} \coloneqq \mathcal{A}_{>0}/\mathcal{A}_{>0}\mathcal{A}_{>0},$$

where $\mathcal{A}_{>0}$ denotes the positive weight part of $\mathcal{A}$. Denote by $I^{\mathfrak{l}}(a_0; a_1, \ldots, a_n; a_{n+1})$ the image of $\mathrm{I}^{\mathfrak{a}}(a_0; a_1, \ldots, a_n; a_{n+1})$ in $\mathcal{L}$, and similarly, denote by $\zeta^{\mathfrak{l}}(k_1, \ldots, k_r)$ the image of $\zeta^{\mathfrak{m}}(k_1, \ldots, k_r)$ in $\mathcal{L}$. Let $\mathcal{L}_r$ be the weight r component of $\mathcal{L}$. The infinitesimal coaction is then the composition

$$\mathcal{H} \to \mathcal{A} \otimes \mathcal{H} \to \mathcal{L}_r \otimes \mathcal{H}$$

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of $\Delta - 1 \otimes \mathrm{id}$ with the projection to $\mathcal{L}_r$. Explicitly, $\mathrm{D}_r\, \mathrm{I}^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1})$ is given by

$$\sum_{k=0}^{n-r} I^{\mathfrak{l}}(a_k; a_{k+1}, \ldots, a_{k+r}; a_{k+r+1}) \otimes \mathrm{I}^{\mathfrak{m}}(a_0; a_1, \ldots, a_k, a_{k+r+1}, \ldots, a_n; a_{n+1}). \tag{2.3}$$

These infinitesimal coactions are significantly easier to compute but still encode almost all essential information surrounding motivic MZVs.

Theorem 2.4 (Brown [5, Theorem 3.3]). *Let $N > 2$, and denote by $\mathrm{D}_{<N} = \bigoplus_{3\leq 2r+1<N} \mathrm{D}_{2r+1}$. Then in weight N, the kernel of $\mathrm{D}_{<N}$ is one dimensional:*

$$\ker \mathrm{D}_{<N} \cap \mathcal{H}_N = \mathbb{Q}\zeta^{\mathfrak{m}}(N).$$

Brown proves this result by considering a particular choice of isomorphism of coalgebras

$$(\mathcal{A}, \Delta) \cong (\mathbb{Q}\langle f_3, f_5, f_7, \ldots\rangle, \Delta_{\mathrm{decon}}),$$

which he lifts to an isomorphism of comodules over these coalgebras. We may instead consider the corresponding vector spaces of indecomposables, equipped with the structure of Lie coalgebras by defining the cobracket to be the natural cobracket coming from antisymmetrising the coproduct. We then have an isomorphism

$$(\mathcal{L}, \partial) \cong (\mathbb{Q}\langle f_3, \ldots\rangle_{>0} / \mathbb{Q}\langle f_3, \ldots\rangle_{>0}^{\sqcup\!\sqcup 2}, \partial_{\mathrm{decon}}),$$

which we use to obtain the following standard proposition.

Proposition 2.5. *Denote by*

$$\partial_r : \mathcal{L} \to \mathcal{L}_r \otimes \mathcal{L}$$

the r^{th} infinitesimal cobracket, given in weight N by

$$\partial_r \xi := \pi_r \circ \partial = \mathrm{D}_r\, \xi - \tau\, \mathrm{D}_{N-r}\, \xi,$$

where $\tau(a \otimes b) = b \otimes a$. Let $\partial_{<N} = \bigoplus_{3\leq 2r+1<N} \partial_{2r+1}$. Then, in weight N, the kernel of $\partial_{<N}$ is at most one dimensional:

$$\ker \partial_{<N} \cap \mathcal{L}_N = \mathbb{Q}\zeta^{\mathfrak{l}}(N),$$

where we note that $\zeta^{\mathfrak{l}}(2n) = 0$.

Proof. It is known $\mathcal{L}$ is isomorphic to L the Lie coalgebra of indecomposables of $\mathbb{Q}\langle f_3, f_5, , \ldots\rangle$ with respect to the shuffle product, which is the cofree Lie coalgebra with cogenerators $f_3, f_5, \ldots$. Choosing an isomorphism ϕ, such that $\phi(\zeta^{\mathfrak{l}}(2n+1)) = f_{2n+1}$, it suffices to show that

$$\ker \partial_{<N} \cap L_N = \mathbb{Q}f_N,$$

where we take $f_{2n} := 0$.

As L is graded by f-degree, the vanishing of $\partial_{<N}$ is equivalent to the vanishing of the full cobracket ∂ followed by projection onto odd weight in the first component. This composition is dual to the Lie bracket

$$L^{\vee}_{\mathrm{odd}} \otimes L^{\vee} \to L^{\vee},$$

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where $L^\vee$ is the free Lie algebra on $f_3^\vee, f_5^\vee, \ldots$. As this map is surjective onto the f-degree at least 2 part of $L^\vee$, this implies that $\partial_{<N}$ is injective on the f-degree at least 2 part of L, and hence the kernel is spanned in weight N by f_N. □

Remark 2.6. It is worth noting that this formalism for motivic multiple zeta values extends to more general motivic iterated integrals, in particular, alternating motivic MZVs [19]. We will need this extension for the results of Appendix B, and will introduce the necessary results and concepts as needed.

2.2. *The motivic Lie algebra*

Elements of $\mathcal{L}$ may be viewed as motivic multiple zeta values, modulo products. By considering the weight graded dual, we obtain a Lie algebra $\mathfrak{g}^{\mathfrak{m}}$, called the motivic Lie algebra. From the theory of mixed Tate motives and Tannakian categories, this Lie algebra is equal to a subspace of $\mathbb{Q}\langle e_0, e_1\rangle$, equipped with the Ihara bracket $\{\cdot, \cdot\}$ [5, 26]. Via the pairing

$$\langle I^{\mathfrak{l}}(0; a_1, \ldots, a_m; 1), e_{i_1} \ldots e_{i_n}\rangle = \delta_{m,n}\delta_{a_1,i_1} \ldots \delta_{a_m,i_m},$$

elements of $\mathfrak{g}^{\mathfrak{m}}$ may be viewed as encoding relations among elements of $\mathcal{L}$. For example, in weight 5, $\mathfrak{g}^{\mathfrak{m}}$ is spanned by

$$\sigma_5 = e_1 e_0^4 + \frac{9}{2} e_1 e_0^2 e_1 e_0 + \cdots$$

from which we can conclude that

$$\zeta^{\mathfrak{m}}(3, 2) \equiv \frac{9}{2}\zeta^{\mathfrak{m}}(5) \quad (\text{mod products}).$$

As such, describing relations among motivic MZVs (up to products) is equivalent to describing the elements of $\mathfrak{g}^{\mathfrak{m}}$. In particular, to describe all such relations, it would suffice to describe explicit generators for $\mathfrak{g}^{\mathfrak{m}}$. It is known [14] that the motivic Lie algebra is isomorphic to a free Lie algebra

$$\mathfrak{g}^{\mathfrak{m}} \cong \mathrm{Lie}[\sigma_3, \sigma_5, \ldots],$$

with generators σ_{2k+1} in every odd weight greater than 1. However, this isomorphism is noncanonical, and there does not exist an explicit canonical choice of representatives of σ_{2k+1}. However, we can somewhat remedy this by considering graded relations among motivic MZVs for a certain filtration.

2.3. *The block filtration*

In [4], Brown proposed a new filtration on the space of convergent motivic MZVs, based on the work of the first author [13]. This was expanded to a filtration on all motivic MZVs by the second author in [28, 29], in which the associated graded algebra - and relations therein - is investigated. In this section, we provide a brief summary of this filtration and relations in the associated graded algebra.

Call a word in two letters $\{a, b\}$ alternating if it contains no subword of the form aa or bb. As noted in [13], every word in $\{a, b\}$ then has a unique factorisation into alternating words of maximal length. This allows us to uniquely determine a word w by its first letter and the lengths of the alternating blocks in this factorisation $(x; \ell_1, \ldots, \ell_n)$, $x \in \{a, b\}$. We call this sequence the block decomposition of the word w.

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We define the block degree $\deg_{\mathcal{B}}(w)$ of a word w to be the number of instances of subwords of the form aa or bb in w. This allows us to define the block filtration on the vector space $\mathbb{Q}\langle a, b\rangle$ by defining the n^{th} part

$$\mathcal{B}_n\mathbb{Q}\langle a, b\rangle \coloneqq \langle w \mid \deg_{\mathcal{B}}(w) \leq n\rangle_{\mathbb{Q}}$$

to be the vector subspace spanned by words of degree at most n.

As motivic iterated integrals, and hence motivic MZVs, may be viewed as a quotient of $\mathbb{Q}\langle e_0, e_1\rangle$ via the map

$$e_{i_0}e_{i_1}\ldots e_{i_{n+1}} \mapsto \mathrm{I}^{\mathfrak{m}}(i_0; i_1, \ldots, i_n; i_{n+1}),$$

the space of motivic MZVs inherits the block filtration. We may also define the block degree of an iterated integral, by taking the block degree of the associated word. This turns out to be a very natural filtration to consider, satisfying a number of nice properties, the proofs of which we shall either sketch here, or may be found in [28, 29].

Proposition 2.7. *The block filtration is equal to the coradical filtration induced by the motivic coaction. Furthermore, when restricted to the Hoffman basis, it is the level filtration of Brown [5]. Hence, any MZV of block degree N may be written as a linear combination of Hoffman MZVs with at most N threes.*

Lemma 2.8 [13]. *All MZVs of block degree b and weight N, with b and N of opposite parity, vanish.*

Proof. If $\mathrm{I}^{\mathfrak{m}}(i_0; i_1, \ldots, i_n; i_{N+1})$ has block degree b, then the final letter of $e_{i_0}\ldots e_{i_{N+1}}$ must be equal to the final letter of the alternating word of length $N + 2 - b$, beginning with e_{i_0}. In particular, we must have that $e_{i_{N+2-b}} = e_{i_0}$ if $N + 2 - b$ is odd, that is, N and b are off opposite parity. Hence, $\mathrm{I}^{\mathfrak{m}}(i_0; i_1, \ldots, i_n; i_{n+1}) = 0$, as it has the same start and end points of the integral. □

Analogously to depth graded MZVs [9], we may consider the associated graded algebra

$$\mathrm{gr}^{\mathcal{B}}\,\mathcal{H} \coloneqq \bigoplus_{n\geq 0} \mathcal{B}_n\mathcal{H}/\mathcal{B}_{n-1}\mathcal{H}$$

and consider relations among block graded multiple zeta values. Much like relations among motivic MZVs, modulo products, in the motivic Lie algebra $\mathfrak{g}^{\mathfrak{m}}$, relations among block graded MZVs, modulo products, are encoded in the block Lie algebra

$$\mathfrak{bg} \coloneqq \bigoplus_{n\geq 0} \mathcal{B}^n\mathfrak{g}^{\mathfrak{m}}/\mathcal{B}^{n+1}\mathfrak{g}^{\mathfrak{m}},$$

where the filtration on $\mathfrak{g}^{\mathfrak{m}}$ is induced by the filtration

$$\mathcal{B}^n\mathbb{Q}\langle e_0, e_1\rangle \coloneqq \langle w \mid \deg_{\mathcal{B}}(w) \geq n\rangle_{\mathbb{Q}}$$

via the embedding $\mathfrak{g}^{\mathfrak{m}} \hookrightarrow e_0\mathbb{Q}\langle e_0, e_1\rangle e_1$. We denote by $\mathfrak{bg}_n$ the block degree n part, via the embedding into $\mathcal{B}^n\mathbb{Q}\langle e_0, e_1\rangle/\mathcal{B}^{n+1}\mathbb{Q}\langle e_0, e_1\rangle$.

Proposition 2.9. *As Lie algebras $\mathfrak{g}^{\mathfrak{m}} \cong \mathfrak{bg}$. In particular, they are (noncanonically) isomorphic to* $\mathrm{Lie}[\sigma_3, \sigma_5, \ldots]$.

Proposition 2.10. *Let $\{\sigma_{2k+1}\}_{k\geq 1}$ be a choice of generators for $\mathfrak{g}^{\mathfrak{m}}$. Then the block degree* 1 *piece of the image of $\{\sigma_{2k+1}\}_{k\geq 1}$ in $\mathbb{Q}\langle e_0, e_1\rangle$ is independent of the choice of generators. In particular, we can make a canonical identification between the image of $\mathfrak{bg}$ in $\mathbb{Q}\langle e_0, e_1\rangle$ and the free Lie algebra* $\mathrm{Lie}[\sigma_3, \sigma_5, \ldots]$.

Remark 2.11. It is in these two results that we see a stark contrast to the case of depth graded multiple zeta values [9]. Analogously to the above, one can consider the associated graded Lie algebra for the depth filtration, induced by the e_1-degree of words in $\mathbb{Q}\langle e_0, e_1\rangle$. As for the block graded case, the

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image of the generators $\{\sigma_{2k+1}\}_{k\geq 1}$ in depth 1 is canonical. However, unlike the block graded case, the depth graded Lie algebra $\mathfrak{dg}$ is not free, having quadratic relations and extra generators in depth 4. These quadratic relations are algebraically well understood [9, 36], and give a somewhat mysterious connection to modular forms. Indeed, the quadratic relations are exactly encoded in the period polynomials of cusp forms. This is a relationship that we can discuss in a new light using the approaches of this paper.

As the image of $\mathfrak{bg}$ in $\mathbb{Q}\langle e_0, e_1\rangle$ lies in $e_0\mathbb{Q}\langle e_0, e_1\rangle e_1$, the block decomposition gives an injection of vector spaces

$$\mathfrak{bg} \to \bigoplus_{n\geq 1} \mathbb{Q}[x_1, \ldots, x_n]$$

obtained by sending a word $w = e_0 \ldots$ to $x_1^{\ell_1} \ldots x_n^{\ell_n}$, where $(e_0; \ell_1, \ldots, \ell_n)$ is the block decomposition of w. The image of $\mathfrak{bg}_n$ under this map lies in $(x_1 - x_{n+1})x_1 \ldots x_{n+1}\mathbb{Q}[x_1, \ldots, x_{n+1}]$. We denote by $\mathfrak{rbg}_n$ the image of $\mathfrak{bg}_n$ divided by $(x_1 - x_{n+1})x_1 \ldots x_{n+1}$, and let $\mathfrak{rbg} := \bigoplus_{n\geq 1} \mathfrak{rbg}_n$. Thus, we have reduced the problem of describing relations among block graded MZVs modulo products to describing $\mathfrak{rbg}$ as a subspace of $\bigoplus_{n\geq 2} \mathbb{Q}[x_1, \ldots, x_n]$.

In [29], a number of relations are found. In particular, elements of $\mathfrak{rbg}$ satisfy a functional equation coming from shuffle regularisation, a differential equation, and have a dihedral symmetry.

Proposition 2.12. *If $f(x_1, \ldots, x_n) \in \mathfrak{rbg}$, then*

$$f(x_1, \ldots, x_n) = f(x_n, \ldots, x_1) = (-1)^{n+1} f(x_2, \ldots, x_n, x_1).$$

It turns out these three relations, along with Lemma 2.8, describe most relations among block graded MZVs [29]. In block degree 1, the shuffle regularisation, the differential equation, and this dihedral symmetry, along with Lemma 2.8, describe all relations among block graded motivic multiple zeta values.

Proposition 2.13. *The vector space $\mathfrak{rbg}_1$ is the subspace of $\mathbb{Q}[x_1, x_2]$ given by polynomials $f(x_1, x_2)$, such that*

$$\begin{aligned} f(0, x) &= 2f(x, -x), \\ f(-x_1, -x_2) &= f(x_1, x_2), \\ f(x_1, x_2) &= f(x_2, x_1), \end{aligned} \qquad \frac{\partial^2 f}{\partial x_1^2} = \frac{\partial^2 f}{\partial x_2^2}.$$

Unfortunately, even in block degree 2, this is insufficient, leaving degrees of freedom linear in weight. While a remedy in block degree 2 was given in Proposition 2.8.7 of [28], it turns out that the failure of space cut out by the dihedral symmetry, differential equation, and shuffle regularisation to encode all relations in block degree 2 has an interesting connection to double zeta values, and gives an alternative source of the relations between double zeta values coming from period polynomials. This connection is explored in the next section.

3. Block graded relations among double zeta values

As noted above, relations among block graded motivic multiple zeta values, modulo products, are determined by the coefficients of elements of $\mathfrak{bg}$. However, these relations are also genuine relations among motivic multiple zeta values mod products for motivic MZVs of block degree at most 2. Observe that

$$\mathcal{B}_0\mathcal{L} = \langle \zeta(2n) \mid n \geq 1\rangle_{\mathbb{Q}} = \{0\} \quad (\text{mod products}),$$

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and so, modulo products, $\mathcal{B}_1\mathcal{L}/\mathcal{B}_0\mathcal{L} = \mathcal{B}_1\mathcal{L}$. Similarly, as MZVs of block degree 1 are necessarily of odd weight, and MZVs of block degree 2 are necessarily of even weight, block graded relations among motivic MZVs of block degree 2 are genuine relations, modulo products. As Corollary 1.3 defines an explicit representation of double zeta values in terms of MZVs of block degree 2, relations among double zeta values are determined, modulo products, by the coefficients of elements of $\mathfrak{bg}$. In this section, we will make this precise, and show that all relations among double zeta values are determined by the below relations.

Explicitly, following Remark 9.3 of [29], the weight $2n+2$, block degree 2 piece of $\mathfrak{bg}$ can be identified with a subspace of $V_n \subset \mathbb{Q}[x_1, x_2, x_3]$, where V_n is spanned by polynomials satisfying the following relations.

Theorem 3.1 [29]. *Define $V_n \subset \mathbb{Q}[x_1, x_2, x_3]$ to be the space spanned by polynomials satisfying the block relations:*

$$f(\lambda x_1, \lambda x_2, \lambda x_3) = \lambda^{2n} f(x_1, x_2, x_3) \text{ for all } \lambda \in \mathbb{Q}, \tag{Relation 0}$$

$$f(x_1, x_2, x_3) = f(x_2, x_3, x_1) = -f(x_3, x_2, x_1), \tag{Relation 1}$$

$$\frac{1}{2}\big(f(0, y, z) - f(0, y, -z)\big) = f(-y, y, z) - f(y, -z, z), \tag{Relation 2}$$

$$\frac{\partial^4 f}{\partial x_1^4} + \frac{\partial^4 f}{\partial x_2^4} + \frac{\partial^4 f}{\partial x_3^4} - 2\frac{\partial^4 f}{\partial x_1^2 \partial x_2^2} - 2\frac{\partial^4 f}{\partial x_2^2 \partial x_3^2} - 2\frac{\partial^4 f}{\partial x_3^2 \partial x_1^2} = 0. \tag{Relation 3}$$

Then $\mathfrak{rbg}_{2,2n+2}$, the weight $2n+2$ component of $\mathfrak{rbg}_2$, is a subspace of V_n.

As mentioned previously, this inclusion is strict. Additional relations are necessary in order to completely describe $\mathfrak{rbg}_{2,2n+2}$ as a subspace of $\mathbb{Q}[x_1, x_2, x_3]$. A choice of such relations is given in Proposition 2.8.7 of [28]. For any $f(x_1, x_2, x_3) = \sum_{i+j+k=2n} \alpha_{i,j,k} x_1^i x_2^j x_3^k$, define

$$f_e(x_1, x_2, x_3) := \sum_{i+j+k=n} \alpha_{2i,2j,2k} x_1^{2i} x_2^{2j} x_3^{2k}.$$

One may easily check that if $f(x_1, x_2, x_3) \in V_n$, then both $f_e(x_1, x_2, x_3)$ and $f(x_1, x_2, x_3) - f_e(x_1, x_2, x_3)$ are elements of V_n. As such, the linear map $f(x_1, x_2, x_3) \mapsto f_e(x_1, x_2, x_3)$ defines a projection $V_n \to V_n$.

Proposition 3.2 [28]. *Let V_n be as above, and let $P_e : V_n \to V_n$ denote the projection $f(x_1, x_2, x_3) \mapsto f_e(x_1, x_2, x_3)$. Then*

$$\dim \operatorname{im} P_e \leq \left\lfloor \frac{n}{3} \right\rfloor,$$
$$\dim \ker P_e = \left\lfloor \frac{n-1}{2} \right\rfloor = \dim \mathfrak{g}^{\mathfrak{m}}{}_{2,2n+2},$$

where $\mathfrak{g}^{\mathfrak{m}}{}_{2,2n+2}$ denotes the vector space spanned by $\{\sigma_k, \sigma_\ell\}$ with $k + \ell = 2n+2$.

We delay the proof until the following section. Denote by

$$\Phi_n(x_1, \ldots, x_{n+1}) := \sum_{\ell_1, \ell_2, \ldots, \ell_{n+1}} \mathrm{I}^{\mathfrak{bl}}(\ell_1, \ldots, \ell_{n+1}) \otimes x_1^{\ell_1} \ldots x_{n+1}^{\ell_{n+1}} \in \mathrm{gr}_n^{\mathcal{B}} \mathcal{L} \otimes \mathbb{Q}[x_1, \ldots, x_{n+1}]$$

the generating series of block degree n motivic MZVs modulo products. As a consequence of Lemma 7.4 [29],

$$\Phi_n(x_1, \ldots, x_{n+1}) = x_1 \ldots x_{n+1}(x_1 - x_{n+1})\phi_n(x_1, \ldots, x_{n+1})$$

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for some $\phi_n \in \mathrm{gr}_n^{\mathcal{B}} \mathcal{L} \otimes \mathbb{Q}[x_1, \ldots, x_{n+1}]$. This series is then given by

$$\phi_n(x_1, \ldots, x_{n+1}) = \sum_{k_1, \ldots, k_{n+1} \geq 0} F(k_1, \ldots, k_{n+1}) \otimes x_1^{k_1} \ldots x_{n+1}^{k_{n+1}},$$

where

$$F(k_1, \ldots, k_{n+1}) = -\sum_{i+j=k_1} \mathrm{I}^{\mathfrak{bl}}(i+1, k_2+1, \ldots, k_n+1, k_{n+1}+j+2).$$

Conversely, we have that

$$\mathrm{I}^{\mathfrak{bl}}(\ell_1, \ldots, \ell_{n+1}) = F(\ell_1 - 2, \ell_2 - 1, \ldots, \ell_n - 1, \ell_{n+1} - 1) - F(\ell_1 - 1, \ldots, \ell_n - 1, \ell_{n+1} - 2).$$

Via the pairing

$$\mathrm{gr}_n^{\mathcal{B}} \mathcal{L} \times \mathfrak{bg}_n \to \mathbb{Q},$$

we can view Φ_n as a linear map

$$\mathfrak{bg}_n \to \mathbb{Q}[x_1, \ldots, x_{n+1}],$$

the image of which is precisely the embedding described previously. We may similarly view ϕ_n as a linear isomorphism $\mathfrak{bg}_n \to \mathfrak{rbg}_n$.

Lemma 3.3. *In block degree 2, the coefficients of $\phi_{3,e}(x_1, 0, x_3)$ are equal to motivic double zeta values:*

$$F(a, 0, b) = 4\zeta^{\mathrm{I}}(a+1, b+1)$$

for a, b even nonnegative integers.

Proof. We assume, without loss of generality, that $a \leq b$. From Lemma 4.1 and Corollary 1.3, we have that

$$\zeta^{\mathrm{I}}(2a+1, 2b+3) = \frac{(-1)^{a+b}}{4} \sum_{s=a}^{b} \zeta^{\mathrm{I}}(\{2\}^s, 4, \{2\}^{a+b-s}).$$

The block decomposition of

$$\zeta^{\mathrm{I}}(\{2\}^s, 4, \{2\}^{a+b-s}) = (-1)^{a+b+1} \mathrm{I}^{\mathrm{I}}(0; \{1, 0\}^s, 1, 0, 0, 0, \{1, 0\}^{a+b-s}; 1)$$

is $(\ell_1, \ell_2, \ell_3) = (2s+3, 1, 2a+2b-2s+2)$. Hence

$$\begin{aligned} \zeta^{\mathrm{I}}(2a+1, 2b+3) &= -\frac{1}{4} \sum_{s=a}^{b} \mathrm{I}^{\mathfrak{bl}}(2s+3, 1, 2a+2b-2s+2) \\ &= -\frac{1}{4} \sum_{s=a}^{b} F(2s+1, 0, 2a+2b-2s+1) - F(2s+2, 0, 2a+2b-2s). \end{aligned}$$

The dihedral symmetry of $\mathfrak{rbg}_2$ implies that

$$F(p, q, r) = F(q, r, p) = -F(r, q, p),$$

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and so this sum reduces to

$$\zeta^{\mathrm{I}}(2a+1,2b+3) = \frac{1}{4}F(2b+2,0,2a),$$

from which the claim follows. □

Lemma 3.4. *For every tuple* (a,b,c) *of even, nonnegative integers,*

$$F(a,b,c) = -4\zeta^{\mathrm{I}}{}_{a}(b+1,c+1),$$

where we denote by

$$\zeta^{\mathrm{I}}{}_{a}(b+1,c+1) := I^{\mathrm{I}}(0;\{0\}^{a},1,\{0\}^{b},1,\{0\}^{c};1)$$

the regularized iterated integral modulo products.

Proof. By viewing ϕ_2 as a linear isomorphism $\mathfrak{bg}_2 \to \mathfrak{rbg}_2$, we see that we must have that the vector space

$$\langle F(a,b,c) \mid a,b,c \geq 0,\ a+b+c=2n\rangle_{\mathbb{Q}}$$

is dual to $\mathfrak{rbg}_{2,2n+2}$, and furthermore that

$$\langle\langle F(a,b,c) \mid a,b,c \geq 0 \text{ and even},\ a+b+c=2n\rangle_{\mathbb{Q}}$$

is dual to $P_e(\mathfrak{rbg}_{2,2n+2})$.

Following Brown's conventions [9], $\mathrm{gr}^1_{\mathcal{D}}\,\mathfrak{g}^{\mathfrak{m}}$ may be identified with the space of translation invariant polynomials $s_{2n+1}(x_0,x_1) := (x_0-x_1)^{2n}$. By the work of the second author [29], $\mathfrak{rbg}_1$ is spanned by polynomials

$$p_{2n+1}(x_0,x_1) := \frac{(2^{2n+1}-1)(x_0+x_1)^{2n}-(x_0-x_1)^{2n}}{2^{2n}}.$$

Denoting by $f_e(x_0,x_1)$ the projection of a polynomial $f(x_0,x_1)$ onto $\mathbb{Q}[x_0^2,x_1^2]$, it is easy to see that

$$s_{2n+1,e}(x_0,x_1) = 2p_{2n+1,e}(x_0,x_1)\,.$$

The depth graded Ihara bracket

$$\{\cdot,\cdot\} : \mathrm{gr}^1_{\mathcal{D}}\,\mathfrak{g}^{\mathfrak{m}} \wedge \mathrm{gr}^1_{\mathcal{D}}\,\mathfrak{g}^{\mathfrak{m}} \to \mathrm{gr}^2_{\mathcal{D}}\,\mathfrak{g}^{\mathfrak{m}}$$

is given by

$$\begin{aligned}\{f,g\}(x_0,x_1,x_2) = f(x_0,x_1)(g(x_0,x_2)-g(x_1,x_2)) + f(x_1,x_2)(g(x_0,x_1)-g(x_0,x_2))\\ + f(x_2,x_0)(g(x_1,x_2)-g(x_0,x_1)),\end{aligned}$$

which is identical to the block graded Ihara bracket

$$\{\cdot,\cdot\} : \mathfrak{rbg}_1 \wedge \mathfrak{rbg}_1 \to \mathfrak{rbg}_2.$$

Furthermore, as all the polynomials involved are of even total degree, this commutes with projection onto polynomials even in each variable (where we have formally extended the Ihara bracket to all

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polynomials of even total degree). Thus, we obtain a surjective map

$$\operatorname{gr}^2_{\mathcal{D}} \mathfrak{g}^{\mathfrak{m}} \to P_e(\mathfrak{rdg}_2),$$
$$\{s_{2k+1}, s_{2\ell+1}\} \mapsto 4\{p_{2k+1,e}, p_{2\ell+1,e}\}.$$

Dualising this, we obtain an injective map

$$\langle F(a,b,c) \mid a,b,c \geq 0 \text{ and even}\rangle_{\mathbb{Q}} \to \operatorname{gr}^{\mathcal{D}}_2 \mathcal{L},$$
$$F(a,b,c) \mapsto 4{\zeta^{\mathrm{I}}}_a(b+1,c+1).$$

Since this map is injective, and $\{\zeta^{\mathrm{I}}(b+1,c+1)\}_{b+c=2n}$ spans $\operatorname{gr}^{\mathcal{D}}_2 \mathcal{L}$ in weight $2n+2$, we must have that if (a,b,c) are even nonnegative integers, such that

$${\zeta^{\mathrm{I}}}_a(b+1,c+1) = \sum_{2k+2\ell=a+b+c} \eta_{k,\ell}\zeta^{\mathrm{I}}(2k+1,2\ell+1),$$

then, by Lemma 3.3,

$$F(a,b,c) = \sum_{2k+2\ell=a+b+c} \eta_{k,\ell} F(0,2k,2\ell),$$

and hence

$$F(a,b,c) = \sum_{2k+2\ell=a+b+c} -4\eta_{k,\ell}\zeta^{\mathrm{I}}(2k+1,2\ell+1) = -4{\zeta^{\mathrm{I}}}_a(b+1,c+1).$$

□

Proposition 3.5. *All relations among double zeta values of weight* $2n+2$ *modulo products are determined by* (Relation 1) *and* (Relation 3) *via Corollary 1.3.*

Proof. By corollary 4.2 of [36], we have that the dimension of the space of motivic multiple zeta values of weight $2n+2$ modulo products is equal to the dimension of $\operatorname{gr}^{\mathcal{D}}_2 {\mathfrak{g}^{\mathfrak{m}}}_{2,2n+2}$, which is given by

$$\left\lfloor \frac{n-1}{2} \right\rfloor - \dim S_{2n+2},$$

where S_{2n+2} is the space of cusp forms of weight $2n+2$. It is known that

$$\dim S_{2n+2} = \begin{cases} \frac{n}{6} - 1 & \text{if } n \equiv 0 \pmod 6 \\ \frac{n}{6} & \text{if } n \equiv 1,2,3,4 \pmod 6 \\ \frac{n}{6} + 1 & \text{if } n \equiv 5 \pmod 6). \end{cases}$$

Checking each case, we see

$$\left\lfloor \frac{n-1}{2} \right\rfloor - \dim S_{2n+2} = \left\lfloor \frac{n}{3} \right\rfloor.$$

By Proposition 4.3,

$$\dim P_e V_n \leq \left\lfloor \frac{n}{3} \right\rfloor.$$

As $\operatorname{gr}^{\mathcal{D}}_2 \mathcal{L}$ is spanned by motivic double zetas, Lemma 3.4 implies that

$$\langle F(a,b,c) \mid a,b,c \in 2\mathbb{Z}\rangle_{\mathbb{Q}} \cong \operatorname{gr}^{\mathcal{D}}_2 \mathcal{L}.$$

Hence, the surjection of Lemma 3.4 is an isomorphism

$$P_e \mathfrak{rdg}_{2,2n+2} \cong \mathrm{gr}_2^{\mathcal{D}}\, \mathfrak{g^m}_{2,2n+2}$$

for every $n > 0$. Following Theorem 3.1, we have that $P_e \mathfrak{rdg}_{2,2n+2} \subset P_e V_n$. But

$$\left\lfloor \frac{n}{3} \right\rfloor = \dim \mathrm{gr}_2^{\mathcal{D}}\, \mathfrak{g^m}_{2,2n+2} = \dim P_e \mathfrak{rdg}_{2,2n+2} \leq \dim P_e V_n \leq \left\lfloor \frac{n}{3} \right\rfloor,$$

and hence

$$P_e \mathfrak{rdg}_{2,2n+2} = P_e V_n.$$

As such, all relations in

$$\langle F(a,b,c) \mid a,b,c \in 2\mathbb{Z} \rangle_{\mathbb{Q}} \cong \mathrm{gr}_2^{\mathcal{D}}\, \mathcal{L}$$

are determined by the relations defining V_n. In fact, as (Relation 2) has no even part, all weight graded relations among $F(a,b,c)$ with a,b,c even are determined by (Relation 1) and (Relation 3). □

Notably, Proposition 3.5 tells us that the period polynomial relations among double zetas are a consequence of the dihedral symmetry of the block graded motivic Lie algebra. Using the surjection of Lemma 3.4, we can, in fact, make this quite explicit.

Remark 3.6. Note that this also shows that we can upgrade the results of Proposition 3.2 to

$$\begin{aligned} \dim \operatorname{im} P_e &= \left\lfloor \frac{n}{3} \right\rfloor, \\ \dim \ker P_e &= \left\lfloor \frac{n-1}{2} \right\rfloor = \dim \mathfrak{g^m}_{2,2n+2}, \end{aligned}$$

with equalities in both cases.

3.1. An explicit connection to period polynomials

Recall that the space of even period polynomials W_{2n}^+ of degree $2n$ is defined as the subspace of $\mathbb{Q}[x_1,x_2]$ consisting of polynomials that are homogeneous of degree $2n$, even in each variable, and satisfy

$$\begin{aligned} P(x_1,0) = P(0,x_2) &= 0, \\ P(x_1,x_2) + P(x_2,x_1) &= 0, \\ P(x_1,x_2) + P(x_1,x_1+x_2) + P(x_1+x_2,x_2) &= 0 \end{aligned}$$

See, for example, section 8 of [8] for more detail.

Proposition 3.7. *Denote by e_{2k+1} the projection of the image of the Lie algebra generator σ_{2k+1} in $\mathfrak{rdg}_1$ to $\mathbb{Q}[x_1^2,x_2^2]$. The kernel*

$$K := \ker\left(\{\cdot,\cdot\} : \bigoplus_{k\geq 1} \mathbb{Q}e_{2k+1} \wedge \bigoplus_{\ell \geq 1} \mathbb{Q}e_{2\ell+1} \to \bigoplus_{n\geq 1} P_e V_n \right)$$

is isomorphic to the space of even period polynomials

$$K \cong \bigoplus_{n\geq 1} W_{2n}^+.$$

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Proof. Define a pair of linear maps

$$\begin{aligned}\pi_1 : X^2\mathbb{Q}[X^2] &\to \mathbb{Q}[x_1, x_2],\\ p(X) &\mapsto P_e(p(x_1 - x_2))\end{aligned}$$

and

$$\begin{aligned}\pi_2 : \mathbb{Q}[X^2, Y^2]_{>0} &\to \mathbb{Q}[x_1, x_2, x_3],\\ p(X,Y) &\mapsto P_e(p(x_1 - x_2, x_2 - x_3)),\end{aligned}$$

where we write $\mathbb{Q}[X^2, Y^2]_{>0}$ for the subspace of polynomials of positive degree. We define a basis for the space of antisymmetric polynomials in $\mathbb{Q}[X^2, Y^2]_{>0}$ given by

$$\left\{Q_{2k,2\ell}(X,Y) \coloneqq X^{2k}Y^{2\ell} - X^{2\ell}Y^{2k}\right\}.$$

The first map π_1 defines an isomorphism

$$X^2\mathbb{Q}[X^2] \to \bigoplus_{k\geq 1} \mathbb{Q}e_{2k+1},$$

and it is not difficult to show that we have a commutative diagram

$$\begin{tikzcd} X^2\mathbb{Q}[X^2] \wedge Y^2\mathbb{Q}[Y^2] \arrow[r] \arrow[d] & \bigoplus_{k\geq 1} \mathbb{Q}e_{2k+1} \wedge \bigoplus_{\ell\geq 1} \mathbb{Q}e_{2\ell+1} \arrow[d] \\ \mathbb{Q}[X^2, Y^2]_{>0} \arrow[r] & \mathbb{Q}[x_1, x_2, x_3], \end{tikzcd}$$

where the left vertical arrow is the map

$$X^{2k} \wedge Y^{2\ell} = Q_{2k,2\ell}(X,Y) + Q_{2k,2\ell}(X, X+Y) + Q_{2k,2\ell}(X+Y, Y)$$

and the right vertical arrow is the map induced by the Ihara bracket. By construction, the image of the right vertical arrow is contained in $\bigoplus_{n\geq 1} P_e V_n$, and has kernel K. Note also that

$$\bigoplus_{n\geq 1} W_{2n}^+ \cong \ker\left(X^2\mathbb{Q}[X^2] \wedge Y^2\mathbb{Q}[Y^2] \to \mathbb{Q}[X^2, Y^2]_{>0}\right)$$

via the identification

$$X^{2k} \wedge Y^{2\ell} \mapsto Q_{2k,2\ell}(X,Y),$$

again, by construction. Thus, we have a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & W_{2n}^+ \arrow[r] \arrow[d, "F"] & \bigoplus_{k+l=n} \mathbb{Q}X^{2k} \wedge Y^{2\ell} \arrow[r] \arrow[d, "\cong"] & Q_{2n} \arrow[r] \arrow[d, "G"] & 0 \\ 0 \arrow[r] & K_n \arrow[r] & \bigoplus_{k+l=n} \mathbb{Q}e_{2k+1} \wedge \mathbb{Q}e_{2\ell+1} \arrow[r] & \mathfrak{rbg}_{2,2n+2} \arrow[r] & 0, \end{tikzcd}$$

where we denote by

$$K_n \coloneqq \ker\left(\bigoplus_{k+l=n} \mathbb{Q}e_{2k+1} \wedge \mathbb{Q}e_{2\ell+1} \to P_e V_n\right).$$

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A short diagram chase shows that F is an injection and is G a surjection. Thus, we must have

$$\dim K_n \geq \dim W_{2n}^{+} = \dim S_{2n+2}.$$

If $\dim K_n > \dim W_{2n}^{+}$, then we must have

$$\langle \{e_{2k+1}, e_{2\ell+1}\} \mid k+\ell = n\rangle_{\mathbb{Q}} < \left\lfloor \frac{n-1}{2} \right\rfloor - \dim S_{2n+2} = \dim P_e V_n.$$

But

$$\langle \{e_{2k+1}, e_{2\ell+1}\} \mid k+\ell = n\rangle_{\mathbb{Q}} = P_e \mathfrak{rdg}_{2,2n+2}$$

by definition, and by the previous theorem $P_e \mathfrak{rdg}_{2,2n+2} = P_e V_n$. Hence, $\dim K_n = \dim W_{2n}^{+}$ for every $n > 0$. □

This suggests that a possible approach to study depth graded motivic multiple zeta values and exploring Conjecture 1.1 is to consider the Lie algebra generated by the $\{e_{2k+1}\}_{k\geq 1}$, or equivalently, the projection of $\mathfrak{rdg}$ onto $\bigoplus_{n\geq 2} \mathbb{Q}[x_1^2, x_2^2, \ldots, x_n^2]$. Indeed, the results of this section show that this is isomorphic to the depth graded motivic Lie algebra in depths 1 and 2, though this isomorphism cannot extend to depth 4 as the projection of $\mathfrak{rdg}_4$ onto $\mathbb{Q}[x_1^2, x_2^2, x_3^2, x_4^2, x_5^2]$ is generated by the $\{e_{2k+1}\}_{k\geq 1}$, and hence, we cannot find the 'exceptional' generators in depth 4 referred to in Remark 2.11 required to generate the full depth graded Lie algebra [9, Section 1.4]

4. Proofs of the more technical results

We now explain the proofs of some of the more technical results used in the previous sections. It is worth noting that determining the statement of Lemma 4.1 required computing the full evaluation of $\zeta^{\mathfrak{m}}(\{2\}^a, 4, \{2\}^b)$ described in Appendices A and B. However, as we only use the evaluation of $\zeta^{\mathrm{I}}(\{2\}^a, 4, \{2\}^b)$ in terms of double zetas modulo products, we have elected to give here a simpler direct proof using the motivic formalism.

4.1. Evaluation of $\zeta^{\mathrm{I}}(\{2\}^a, 4, \{2\}^b)$

Lemma 4.1. *The following evaluation holds in the motivic coalgebra*

$$\zeta^{\mathrm{I}}(\{2\}^a, 4, \{2\}^b) = (-1)^{a+b}\Bigg\{-4\zeta^{\mathrm{I}}(2a+2, 2b+2) + 4\zeta^{\mathrm{I}}(2b+1, 2a+3) + \sum_{\substack{i+j=2a+2b \\ i,j\geq 0}} \left(\frac{1}{2^i}\binom{i+1}{2a+1} + \frac{1}{2^j}\binom{j+1}{2b+1}\right)\zeta^{\mathrm{I}}(i+2, j+2)\Bigg\}.$$

Proof. For Z a weight w combination of motivic MZVs, it is sufficient to check, by Proposition 2.5, that

$$\partial_{<w} Z = 0$$

for all $1 < 2r+1 < w$, as this would show that $Z = \alpha\zeta^{\mathfrak{m}}(w) + \text{products}$. If the weight of Z is even, we have that $\zeta^{\mathrm{I}}(w) = 0$; this means checking that $\partial_{<w} Z = 0$ allows us to confirm that $\pi(Z) = 0$ on the nose, where $\pi\colon \mathcal{A} \to \mathcal{L}$ is the natural projection.

Explicitly, it amounts to checking for $1 < 2r+1 < w$ that

$$(\mathrm{id} \otimes \pi)(\mathrm{D}_r\, Z) - \tau(\mathrm{id} \otimes \pi)(\mathrm{D}_{w-r}\, Z) = 0,$$

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where $\tau(a \otimes b) = b \otimes a$. For the case under consideration, we need to check for $3 \leq 2r+1 \leq 2a+2b+1$

$$(\mathrm{id} \otimes \pi)(\mathrm{D}_{2r+1} Z) - \tau(\mathrm{id} \otimes \pi)(\mathrm{D}_{2a+2b+3-2r} Z) \overset{?}{=} 0.$$

It is a straightforward calculation, as explained in Section B.2, to show the following.

$$\begin{aligned}(\mathrm{id} \otimes \pi)\, \mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m}}(\{2\}^a, 4, \{2\}^b) = &-\delta_{r \leq a} \zeta_1^{\mathrm{I}}(\{2\}^r) \otimes \zeta^{\mathrm{I}}(\{2\}^{a-r}, 3, \{2\}^b) \\ &+ \delta_{r \leq b} \zeta_1^{\mathrm{I}}(\{2\}^r) \otimes \zeta^{\mathrm{I}}(\{2\}^a, 3, \{2\}^{b-r}).\end{aligned}$$

Recalling the motivic evaluations of $\zeta_1^{\mathfrak{m}}(\{2\}^r)$ and $\zeta^{\mathfrak{m}}(\{2\}^a, 3, \{2\}^b)$ from [5], we have that

$$\begin{aligned}\zeta^{\mathrm{I}}{}_1(\{2\}^r) &= 2(-1)^r \zeta^{\mathrm{I}}(2r+1), \\ \zeta^{\mathrm{I}}(\{2\}^a, 3, \{2\}^b) &= 2(-1)^{a+b+1}\left(\binom{2a+2b+2}{2a+2} - (1 - 2^{-2a-2b-2})\binom{2a+2b+2}{2b+1}\right)\zeta^{\mathrm{I}}(2a+2b+3).\end{aligned}$$

Therefore

$$\begin{aligned}&(\mathrm{id} \otimes \pi)\, \mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m}}(\{2\}^a, 4, \{2\}^b) \\ &\quad = \Bigg\{ 4\delta_{r \leq a}(-1)^{a+b}\left(\binom{2a+2b+2-2r}{2a-2r+2} - (1 - 2^{2r-2a-2b-2})\binom{2a+2b+2-2r}{2b+1}\right) \\ &\qquad - 4\delta_{r \leq b}(-1)^{a+b}\left(\binom{2a+2b+2-2r}{2a+2} - (1 - 2^{2r-2a-2b-2})\binom{2a+2b+2-2r}{2b-2r+1}\right)\Bigg\} \\ &\qquad \cdot \zeta^{\mathrm{I}}(2r+1) \otimes \zeta^{\mathrm{I}}(2a+2b+3-2r).\end{aligned}$$

Likewise from Section B.1

$$\begin{aligned}(\mathrm{id} \otimes \pi)\, \mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m}}(p, q) = &\left(\delta_{2r+1=p} + (-1)^p \binom{2r}{p-1} - (-1)^q \binom{2r}{q-1}\right) \\ &\cdot \zeta^{\mathrm{I}}(2r+1) \otimes \zeta^{\mathrm{I}}(p+q-2r-1).\end{aligned}$$

So for the purpose of checking

$$(\mathrm{id} \otimes \pi)(\mathrm{D}_{2r+1} Z) - \tau(\mathrm{id} \otimes \pi)(\mathrm{D}_{2a+2b+3-2r} Z) = 0,$$

where Z is the purported evaluation of $\zeta^{\mathrm{I}}(\{2\}^a, 4, \{2\}^b)$ via double zeta values $\zeta^{\mathrm{I}}(n_1, n_2)$, we can project $\zeta^{\mathrm{I}}(2r+1) \otimes \zeta^{\mathrm{I}}(2a+2b+3-2r) \mapsto 1$, and reduce to an identity among binomial coefficients.

After some straightforward simplification of the deltas and binomial coefficients in the expression for

$$\big((\mathrm{id} \otimes \pi)\, \mathrm{D}_{2r+1} - \tau(\mathrm{id} \otimes \pi)\, \mathrm{D}_{2a+2b+3-2r}\big)(\text{LHS Lemma 4.1} - \text{RHS Lemma 4.1}),$$

for the range $3 \leq 2r+1 \leq 2a+2b+1$, and using that i, j have the same parity in the sum, we find the claimed identity to be equivalent to the following purported identity

$$\sum_{\substack{i+j=2a+2b\\ i,j\geq 0}} (-1)^i \left(2^{-i}\binom{i+1}{2a+1} + 2^{-j}\binom{j+1}{2b+1}\right)$$
$$\cdot \left(\binom{2r}{i+1} - \binom{2r}{j+1} - \binom{2a+2b+2-2r}{i+1} + \binom{2a+2b+2-2r}{j+1}\right)$$
$$\stackrel{?}{=} 2^{-(2a+2b+1-2r)}\binom{2a+2b+2-2r}{2b+1} - 2^{-(2a+2b+1-2r)}\binom{2a+2b+2-2r}{2a+1}$$
$$- 2^{-(2r-1)}\binom{2r}{2b+1} + 2^{-(2r-1)}\binom{2r}{2a+1}.$$

This is seen to be an exact identity using the results from Lemma 4.2 below. □

Lemma 4.2. *For* $3 \leq 2r+1 \leq 2k+2\ell-3$*, the following identities hold*

$$\sum_{i=0}^{2k-2} (-2)^{-i}\binom{i+2\ell-1}{2\ell-1}\binom{2r}{i+2\ell-1} = 2^{-(2r+1-2\ell)}\binom{2r}{2\ell-1} \tag{i}$$

$$\sum_{i=0}^{2\ell-2} (-2)^{-i-2k}\binom{i+2k-1}{2k-1}\binom{2r}{2\ell-i-1} = \sum_{i=0}^{2k-2} (-2)^{-i-2\ell}\binom{i+2\ell-1}{2\ell-1}\binom{2r}{2k-i-1}, \tag{ii}$$

that is, the left-hand side is symmetric in $k \leftrightarrow \ell$*.*

Proof. Given the restriction $2r+1 \leq 2k+2\ell-3$, the sum in (i) can be truncated to $i = 2r+1-2\ell$. It is then reduced to the binomial theorem after simplifying the summand.

For (ii), we show that the left-hand side is symmetric in $k \leftrightarrow \ell$, to obtain the equality. We remark, here, that the symmetry is not obvious, as even the number of nonzero therms differs between the two sides. To show the symmetry, note that

$$\sum_{i=0}^{2\ell-2} (-2)^{-i-2k}\binom{i+2k-1}{2k-1}\binom{2r}{2\ell-i-1}$$
$$= -(-2)^{-i-2k}\binom{i+2k-1}{2k-1}\underbrace{\binom{2r}{2\ell-i-1}}_{=1}\Bigg|_{i=2\ell-1} + \sum_{i=0}^{\infty}(-2)^{-i-2k}\binom{i+2k-1}{2k-1}\binom{2r}{2\ell-i-1},$$

since the second binomial vanishes for $i > 2\ell-1$. The first term is equal to the coefficient of $x^{2\ell-1}$ in

$$-(2+x)^{-2k}.$$

Likewise, the second term is the coefficient of $x^{2\ell-1}$ in

$$(1+x)^{2r}(2+x)^{-2k}.$$

Therefore, the left-hand side of (ii) is given by

$$[x^{2\ell-1}]\left(\frac{(1+x)^{2r}-1}{(2+x)^{2k}}\right) = [x^{-1}]\left(\frac{(1+x)^{2r}-1}{(2+x)^{2k}x^{2\ell}}\right).$$

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This is not obviously symmetric in $k \leftrightarrow \ell$; it is, however, equal to the following contour integral around 0 (along a sufficiently small circle $C_\varepsilon(0)$) by the residue theorem

$$= \frac{1}{2\pi \mathrm{i}} \oint_{C_\varepsilon(0)} \frac{(1+z)^{2r}-1}{(2+z)^{2k} z^{2\ell}} \mathrm{d}z.$$

The only poles of the integrand

$$f(z) = \frac{(1+z)^{2r}-1}{(2+z)^{2k} z^{2\ell}}$$

are at $z=-2$, and at $z=0$. Since $2r+1 \leq 2k+2\ell-3$, we see that

$$-\frac{1}{z^2} f\Big(\frac{1}{z}\Big) = \frac{z^{2r}-(1+z)^{2r}}{(1+2z)^{2k}} \cdot z^{2k+2l-2-2r}$$

has no pole at $z=0$, so that our original integrand $f(z)$ has no pole (and hence no residue) at $z=\infty$. We therefore find that the residues at $z=0$ and at $z=-2$ must cancel, giving

$$\frac{1}{2\pi \mathrm{i}} \oint_{C_\varepsilon(0)} \frac{(1+z)^{2r}-1}{(2+z)^{2k} z^{2\ell}} \mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \oint_{C_\varepsilon(-2)} \frac{(1+z)^{2r}-1}{(2+z)^{2k} z^{2\ell}} \mathrm{d}z = 0. \tag{4.1}$$

Now put $z \mapsto -2-w$, with $\mathrm{d}z = -\mathrm{d}w$, in the second integral, and we find

$$\frac{1}{2\pi \mathrm{i}} \oint_{C_\varepsilon(-2)} \frac{(1+z)^{2r}-1}{(2+z)^{2k} z^{2\ell}} \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \oint_{C_\varepsilon(0)} \frac{(-1-w)^{2r}-1}{(-w)^{2k}(-2-w)^{2\ell}} \mathrm{d}w.$$

Putting this back into Equation (4.1) shows that

$$\frac{1}{2\pi \mathrm{i}} \oint_{C_\varepsilon(0)} \frac{(1+z)^{2r}-1}{(2+z)^{2k} z^{2\ell}} \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \oint_{C_\varepsilon(0)} \frac{(1+w)^{2r}-1}{(w)^{2k}(2+w)^{2\ell}} \mathrm{d}w = 0,$$

and so establishes the symmetry in $k \leftrightarrow \ell$ that we claimed. □

4.2. Computing the dimension of $\operatorname{im} P_e$

Proposition 3.2 [28]. *Let V_n be as above, and let $P_e : V_n \to V_n$ denote the projection $f(x_1,x_2,x_3) \mapsto f_e(x_1,x_2,x_3)$. Then*

$$\begin{aligned} \dim \operatorname{im} P_e &\leq \left\lfloor \frac{n}{3} \right\rfloor, \\ \dim \ker P_e &= \left\lfloor \frac{n-1}{2} \right\rfloor = \dim \mathfrak{g}^{\mathfrak{m}}{}_{2,2n+2}, \end{aligned}$$

where $\mathfrak{g}^{\mathfrak{m}}{}_{2,2n+2}$ denotes the vector space spanned by $\{\sigma_k, \sigma_\ell\}$ with $k+\ell = 2n+2$.

Proof. Suppose $q(x_1,x_2,x_3) \in \ker P_e$. Then Equation (Relation 3) implies

$$\begin{aligned} q(x_1,x_2,x_3) = \sum_{i+j=2n} &\alpha_{i,j}(x_1-x_2)^i(x_2-x_3)^j + \beta_{i,j}(x_1+x_2)^i(x_2-x_3)^j \\ &+ \gamma_{i,j}(x_1-x_2)^i(x_2+x_3)^j + \delta_{i,j}(x_1+x_2)^i(x_2+x_3)^j. \end{aligned}$$

Define $q_\star(x_1, x_2, x_3) \coloneqq \frac{1}{4}(q(x_1, x_2, x_3) - q(-x_1, x_2, x_3) - q(x_1, x_2, -x_3) + q(-x_1, x_2, -x_3))$; this is the part of q that is odd in x_1 and x_3, and even in x_2. We can write

$$\begin{aligned} q_\star(x_1, x_2, x_3) = \sum_{\substack{i+j=2n \\ i,j>0}} \rho_{i,j} \big(&(x_1 - x_2)^i (x_2 - x_3)^j + (-1)^{i+1} (x_1 + x_2)^i (x_2 - x_3)^j \\ &- (x_1 - x_2)^i (x_2 + x_3)^j + (-1)^i (x_1 + x_2)^i (x_2 + x_3)^j \big), \end{aligned}$$

where $\rho_{i,j} \coloneqq \alpha_{i,j} + (-1)^{i+1}\beta_{i,j} - \gamma_{i,j} + (-1)^i \delta_{i,j}$. As $q(x_1, x_2, x_3) = -q(x_3, x_2, x_1)$, the same holds for $q_\star(x_1, x_2, x_3)$, and thus $\rho_{i,j} = -\rho_{j,i}$.

Then, as $q_e(x_1, x_2, x_3) = 0$, and $q(x_1, x_2, x_3) = q(x_2, x_3, x_1)$, we must have

$$q(x_1, x_2, x_3) = q_\star(x_1, x_2, x_3) + q_\star(x_2, x_3, x_1) + q_\star(x_3, x_1, x_3).$$

Thus, q is uniquely determined by $q_\star$. We currently have $n - 1$ free variables in $q_\star$, so in order for $\dim \ker P_e$ to be equal to $\lfloor \frac{n-1}{2} \rfloor$, we need (Relation 2) to impose $\lceil \frac{n-1}{2} \rceil$ independent constraints on the $\rho_{i,j}$.

Writing (Relation 2) in terms of $q_\star(x_1, x_2, x_3)$, we find that we must have

$$q_\star(z, 0, y) = 2q_\star(z, y, y) - 2q_\star(y, z, z).$$

Evaluating the coefficient of $y^k z^l$ in this equation, we obtain

$$\rho_{l,k} = \sum_{\substack{0<j\le k \\ i+j=2n}} (-2)^j \binom{i}{l} \rho_{i,j} - \sum_{\substack{0<j\le l \\ i+j=2n}} (-2)^j \binom{i}{k} \rho_{i,j}$$

if k is odd, and $0 = 0$ if k is even, or if $k = l$. As the coefficient of $y^l z^k$ is just the negative of this, this gives us $\lceil \frac{n-1}{2} \rceil$ equations, so it suffices to show that they are independent. As we are solving for rational $\rho_{i,j}$, it is sufficient to show that these equations are independent modulo 2. But modulo 2, we obtain the system of equations

$$\{\rho_{l,k} \equiv 0 \pmod 2\},$$

which are clearly independent. Hence, we have $\lfloor \frac{n-1}{2} \rfloor$ free variables in $q_\star$ and $\dim \ker P_e = \lfloor \frac{n-1}{2} \rfloor$.

Similarly, if $q(x_1, x_2, x_3) \in \operatorname{im} P_e$, then

$$\begin{aligned} q(x_1, x_2, x_3) = \sum_{\substack{i+j=2n \\ i,j\ge 0}} \eta_{i,j} \big(&(x_1 - x_2)^i (x_2 - x_3)^j + (-1)^i (x_1 + x_2)^i (x_2 - x_3)^j \\ &+ (x_1 - x_2)^i (x_2 + x_3)^j + (-1)^i (x_1 + x_2)^i (x_2 + x_3)^j \big). \end{aligned}$$

Indeed, the set

$$\begin{aligned} \mathcal{Q} \coloneqq \big\{ &(x_1 - x_2)^i (x_2 - x_3)^j + (-1)^i (x_1 + x_2)^i (x_2 - x_3)^j \\ &+ (x_1 - x_2)^i (x_2 + x_3)^j + (-1)^i (x_1 + x_2)^i (x_2 + x_3)^j \big\}_{i+j=2n} \end{aligned}$$

forms a basis for the space of totally even solutions of

$$\frac{\partial^4 q}{\partial x_1^4} + \frac{\partial^4 q}{\partial x_2^4} + \frac{\partial^4 q}{\partial x_3^4} - 2\frac{\partial^4 q}{\partial x_1^2 \partial x_2^2} - 2\frac{\partial^4 q}{\partial x_2^2 \partial x_3^2} - 2\frac{\partial^4 q}{\partial x_3^2 \partial x_1^2} = 0.$$

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Then, as $\frac{1}{2}(q(0,y,z) - q(0,y,-z)) = q(-y,y,z) - q(y,-z,z)$ holds trivially for any totally even polynomial satisfying the symmetry conditions, it is sufficient to compute the dimension of the subspace of skew-symmetric polynomials spanned by $\mathcal{Q}$. This is a simple representation theoretic argument: we consider $\mathbf{Span}(\mathcal{Q})$ as a representation of the symmetric group $\mathcal{S}_3$ via the standard polynomial representation, and compute the dimension of the sign representation within this. In particular, representation theory says that

$$\begin{aligned}\dim \operatorname{im} P_e &= \frac{1}{6}\big[\operatorname{Tr}(\mathrm{id}) - 3\operatorname{Tr}((1\,3)) + 2\operatorname{Tr}((1\,2\,3))\big] \\ &= \frac{1}{6}\big[2n+1 - 3\operatorname{Tr}((1\,3)) + 2\operatorname{Tr}((1\,2\,3))\big].\end{aligned}$$

Note that the vector space generated by $\{(x_1-x_2)^i(x_2-x_3)^j\}_{i+j=2n}$ is invariant under the action of $\mathcal{S}_3$, and there is a natural surjection onto $\mathcal{Q}$, so it is sufficient to consider the trace of the action restricted to $\{(x_1-x_2)^i(x_2-x_3)^j\}_{i+j=2n}$ in order to get an upper bound.

Clearly, the trace of $(1\,3)$ is 1, as the only diagonal entry corresponds to $(x_1-x_2)^n(x_2-x_3)^n \mapsto (x_3-x_2)^n(x_2-x_1)^n$. Now, computing the trace of $(1\,2\,3)$, we find that it is given by

$$\sum_{i=0}^{2n}(-1)^i\binom{2n-i}{i}.$$

To compute this, we consider the generating series

$$\begin{aligned}\sum_{n\geq 0}\sum_{i=0}^{n}\binom{n-i}{i}x^iy^n &= \sum_{k\geq 0}\sum_{i\geq 0}\binom{k}{i}(xy)^iy^k \\ &= \sum_{k\geq 0}(1+xy)^ky^k \\ &= \frac{1}{1-y-xy^2}.\end{aligned}$$

Setting $x=-1$, we obtain

$$\begin{aligned}\sum_{n\geq 0}\sum_{i=0}^{n}(-1)^i\binom{n-i}{i}(-y)^n &= \frac{1}{1+y+y^2} \\ &= \frac{1-y}{1-y^3} \\ &= \sum_{m\geq 0} y^{3m} - y^{3m+1}.\end{aligned}$$

Thus,

$$\sum_{i=0}^{2n}(-1)^i\binom{2n-i}{i} = \begin{cases} 1 & \text{if } 2n \equiv 0 \pmod 6 \\ -1 & \text{if } 2n \equiv 4 \pmod 6 \\ 0 & \text{if } 2n \equiv 2 \pmod 6 \end{cases}.$$

Hence

$$\dim \operatorname{im} P_e \leq \frac{1}{6}(2n+1-3+2x),$$

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where x is determined by $2n \pmod 6$. A quick consideration of each case shows we obtain $\lfloor \frac{2n}{6} \rfloor = \lfloor \frac{n}{3} \rfloor$. $\square$

Remark 4.3. Recall that the weight $2n+2$ part of $\mathfrak{rdg}_2$ is a subspace of V_n of dimension $\dim \mathfrak{g}^{\mathfrak{m}}{}_{2,2n+2}$. In order to describe this subspace in terms of relations, we need to find nonzero linear maps $\{R_i : V_n \to W_i\}_{i \in I}$, such that $\mathfrak{rdg}_{2,2n+2} \subset \cap_{i\in I} \ker R_i$. However, the projection $\mathfrak{rdg}_{2,2n+2} \to \ker P_e$ is an isomorphism, and, as a consequence of Proposition 3.5, $P_e(\mathfrak{rdg}_{2,2n+2}) = \operatorname{im} P_e$. As such, no such R can impose any additional relations that restrict nontrivially to either the totally even or not-totally-even parts. Equivalently, any such R must induce a map $\ker P_e \to \operatorname{Gr}(k_R, \operatorname{im} P_e)$ to the space of k_R-dimensional subspaces of $\operatorname{im} P_e$ for some unique integer k_R. From another perspective, if such a description of $\mathfrak{rdg}_{2,2n+2}$ can be found, this would give an alternative proof of Conjecture 1.1 in depth 2.

5. Applications to multiple t values

From [23], we recall the multiple t value $t(n_1, \ldots, n_d)$ is defined by restricting the denominators in the series defining an MZV to be odd. Namely

$$t(k_1, \ldots, k_d) \coloneqq \sum_{0<n_1<\cdots<n_d} \frac{1}{(2n_1-1)^{k_1}\cdots(2n_d-1)^{k_d}}.$$

By inserting the factor $\frac{1}{2}(1-(-1)^{i_j})$ into the numerator, one may extend the sum to all denominators, and obtain the following expression [23, Corollary 4.1] for $t(n_1, \ldots, n_d)$ in terms of alternating MZVs (with various signs) of the same set of indices

$$\begin{aligned} t(k_1, \ldots, k_d) &= \frac{1}{2^d} \sum_{0<n_1<\cdots<n_d} \frac{(1-(-1)^{n_1})\cdots(1-(-1)^{n_d})}{n_1^{k_1}\cdots n_d^{k_d}} \\ &= \frac{1}{2^d} \sum_{\varepsilon_1,\ldots,\varepsilon_d \in \{\pm 1\}} \varepsilon_1 \cdots \varepsilon_d \zeta(\varepsilon_1 \diamond k_1, \cdots \varepsilon_d \diamond k_d). \end{aligned} \tag{5.1}$$

Here, the operator $\diamond$ is defined so that $1 \diamond x = x$ and $-1 \diamond x = \overline{x}$, where, as usual, $\overline{n_j}$ denotes the argument, n_j is accompanied with sign $\varepsilon_i = -1$ in the definition of an alternating MZV (giving character $(-1)^{n_j}$ in the numerator thereof).

From Murakami [32, Theorem 1], we know that every multiple t value with all arguments ≥ 2 – which would *a priori* be a combination of alternating MZVs – satisfies a Galois descent, and is expressible as a $\mathbb{Q}$-linear combination of classical multiple zeta values. Murakami's theorem is actually a statement about motivic multiple t values but gives the same descent for classical MtVs after applying the period map. However, Murakami's result is purely existential and does not give an explicit formula, nor does it put any limits on the depth of the resulting combination. Using the result of Proposition A.3 for alternating double zeta values, we will give explicit formulae in terms of depth 2 classical MZVs for any $t(\mathrm{ev}, \mathrm{ev})$ in Proposition 5.4.

Remark 5.1 (Galois descent of $t(\mathrm{od}, \mathrm{ev})$ and $t(\mathrm{ev}, \mathrm{od})$). Observe that the depth-parity theorem in depth 2 for alternating MZVs [33, Equation 3.5] implies that every multiple $t(a, b)$ value of odd weight (with $a, b \neq 1$) is a polynomial in single zeta values. This already gives an explicit formula for the Galois descent of $t(\mathrm{od}, \mathrm{ev})$ and $t(\mathrm{ev}, \mathrm{od})$. Equivalent formulae were derived in [38, Theorems 4.1, and 4.2] using contour integral techniques (compare [16] for classical MZVs handled in a similar way), namely

$$\begin{aligned} t(2a+1, 2b) &= t(2a+1)t(2b) - \frac{1}{2} t(2a+2b+1) \\ &\quad - \sum_{s=1}^{a+b} \left\{ \binom{2a+2b-2s}{2a} + \binom{2a+2b-2s}{2b-1} \right\} \frac{\zeta(2a+2b+1-2s)}{2^{2a+2b+1-2s}} t(2s), \end{aligned}$$

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$$t(2a,2b+1) = -\frac{1}{2}t(2a+2b+1) + \sum_{s=1}^{a+b}\left\{\binom{2a+2b-2s}{2b} + \binom{2a+2b-2s}{2a-1}\right\}\frac{\zeta(2a+2b+1-2s)}{2^{2a+2b+1-2s}}t(2s)$$

One has that $t(a) = (1-2^{-a})\zeta(a)$, for $a > 1$, which can be applied to rewrite the above purely in terms of Riemann zeta values.

Remark 5.2 (Galois descent of $t(\mathrm{od},\mathrm{od})$). On the other hand, the remaining case involving $t(\mathrm{od},\mathrm{od})$ is less tractable. Using the MZV Data Mine [1], one can check the following relation

$$\begin{aligned} t(3,9) = {} & \frac{9}{128}\zeta(1,1,4,6) + \frac{1305}{4096}\zeta(3,9) - \frac{27}{128}\zeta(2)\zeta(3,7) - \frac{27}{256}\zeta(4)\zeta(3,5) \\ & + \frac{3131}{2048}\zeta(9)\zeta(3) - \frac{321}{1024}\zeta(5)\zeta(7) - \frac{3}{512}\zeta(3)^4 - \frac{45}{64}\zeta(2)\zeta(7)\zeta(3) - \frac{63}{256}\zeta(2)\zeta(5)^2 \\ & + \frac{9}{256}\zeta(4)\zeta(5)\zeta(3) + \frac{81}{512}\zeta(6)\zeta(3)^2 + \frac{353139}{5660672}\zeta(12). \end{aligned}$$

In particular, the (conjecturally) irreducible depth 4 MZV $\zeta(1,1,4,6)$ (or any equivalent choice) is necessary to obtain an expression for the Galois descent of $t(3,9)$ to classical MZVs. This already suggests describing the Galois descent explicitly (with the minimal necessary depth) would be challenging.

We can, conjecturally at least, say that depth 4 MZVs will be sufficient. Indeed, since we may write $t(a,b)$ as a sum of depth 2 alternating MZVs, the alternating analogue of Lemma 1.2 tells us that $t(a,b)$ lies in coradical degree at most 2. Hence, if a Galois descent to classical MZVs exists, it must also lie in coradical degree at most 2. When depth 2 MZVs do not span this space in even weight, the homological version of the Broadhurst-Kreimer Conjecture [9, Conjecture 5] tells us that depth 2 MZVs along with irreducible depth 4 MZVs coming from cusp forms do.

More generally, if a depth d multiple t value has a Galois descent to classical MZVs, the same line of reasoning tells us that we should expect an expression in terms of MZVs of depth at most $2d$.

By combining the usual expression for $t(a,b)$ in terms of alternating MZVs [23, Corollary 4.1], namely

$$t(a,b) = \frac{1}{4}\big(\zeta(a,b) + \zeta(a,\overline{b}) + \zeta(\overline{a},b) + \zeta(\overline{a},\overline{b})\big)$$

with the distribution relation [20, Proposition 2.13]

$$\zeta(a,b) + \zeta(a,\overline{b}) + \zeta(\overline{a},b) + \zeta(\overline{a},\overline{b}) = \frac{1}{2^{a+b-2}}\zeta(a,b),$$

we can write

$$t(a,b) = \frac{1}{2}\zeta(\overline{a},\overline{b}) + \frac{1}{2}\zeta(a,b) - \frac{1}{2^{a+b}}\zeta(a,b) \tag{5.2}$$

More generally, see the alternative expression given by Hoffman, using a sum which inserts only an even number of bars into the argument string [23, Corollary 4.2].

Let us now note the following result from Section A.5, which gives an explicit form for the Galois descent of $\zeta(\overline{2\ell},\overline{2k})$ in terms of classical double MZVs.

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Proposition A.3 (Galois descent of $\zeta(\overline{2\ell},\overline{2k})$). *The alternating double zeta value $\zeta(\overline{2\ell},\overline{2k})$ enjoys a Galois descent to classical depth 2 MZVs as follows*

$$\begin{aligned}\zeta(\overline{2\ell},\overline{2k}) = \sum_{i=2}^{2k+2\ell-2} 2^{-i}\left\{\binom{i-1}{2k-1}\zeta(2k+2\ell-i,i)+\binom{i-1}{2\ell-1}\zeta(i,2k+2\ell-i)\right\} \\ -\zeta(2\ell,2k)+\sum_{r=2}^{2k+2\ell-2}(-2)^{-r}\binom{r-1}{2k-1}\zeta(r)\zeta(2k+2\ell-r) \\ -2^{-2k-2\ell}\left\{2\binom{2k+2\ell-2}{2k-1}+\binom{2k+2\ell-1}{2k-1}\right\}\zeta(2k+2\ell). \qquad \text{(A.10)}\end{aligned}$$

Proof sketch. We recall the notation $\zeta_\ell(k_1,\ldots,k_d)$ is defined by inserting ℓ leading 0's at the start of the integral representation of $\zeta(k_1,\ldots,k_d)$ (c.f. Equations (2.1) or (A.1) for alternating MZVs). Now simultaneously solve the following equations: the dihedral symmetry Equation (A.8)

$$\begin{aligned}\zeta_{2k-1}(1,\overline{2\ell})-\zeta(\overline{2\ell},\overline{2k}) = \binom{2k+2\ell-1}{2k-1}\zeta(\overline{2k+2\ell}) \\ -\sum_{r=1}^{2k+2\ell-2}\left((-1)^r\binom{r-1}{2k-1}+\binom{r-1}{2\ell-1}\right)\zeta(\overline{r})\zeta(2k+2\ell-r),\end{aligned}$$

and the generalised doubling identity [1, Section 4], [42, Section 14.2.5]

$$\begin{aligned}&\zeta(\overline{s},\overline{t})+(-1)^t\zeta_{t-1}(1,\overline{s}) \\ &= \sum_{i=1}^{s}\binom{s+t-i-1}{t-1}2^{1+i-s-t}\zeta(i,s+t-i)+\sum_{i=1}^{t}\binom{s+t-i-1}{s-1}2^{1+i-s-t}\zeta(s+t-i,i) \\ &\quad -\zeta(s,t)+(-1)^t\zeta_{t-1}(s,1)-\sum_{i=1}^{t}\binom{s+t-i-1}{s-1}\zeta(\overline{s+t-i})\zeta(i)-\binom{s+t-1}{s}\zeta(s+t)\end{aligned}$$

(here slightly rewritten, see Section A.5) in the case $t=2k$, $s=2\ell$. □

Now substituting this Galois descent into Equation (5.2), we immediately have the following proposition.

Proposition 5.3. *The multiple t value $t(2\ell,2k)$ is expressed through classical double zeta values as follows*

$$\begin{aligned}t(2\ell,2k) = \sum_{i=2}^{2k+2\ell-2} 2^{-i-1}\left\{\binom{i-1}{2k-1}\zeta(2k+2\ell-i,i)+\binom{i-1}{2\ell-1}\zeta(i,2k+2\ell-i)\right\} \\ -2^{-2k-2\ell}\zeta(2\ell,2k)-\sum_{r=2}^{2k+2\ell-2}(-2)^{-r-1}\binom{r-1}{2k-1}\zeta(r)\zeta(2k+2\ell-r) \\ -2^{-2k-2\ell-1}\left\{2\binom{2k+2\ell-2}{2k-1}+\binom{2k+2\ell-1}{2k-1}\right\}\zeta(2k+2\ell). \qquad \text{(5.3)}\end{aligned}$$

A. Analytic evaluation of $\zeta(\{2\}^a,4,\{2\}^b)$ via double zeta values

The goal of this section is to give an explicit evaluation for $\zeta(\{2\}^a,4,\{2\}^b)$ in terms of double zeta values on the analytic (numerical) level. In Appendix B, we will then lift this to the corresponding identity among motivic MZVs.

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For the numerical evaluation, we need to assemble a number of ingredients. In particular, we need to use the stuffle antipode ([24], [31] or [18, 19]) to convert $\zeta(\{2\}^a, 4, \{2\}^b)$ to a corresponding multiple zeta star value. Then we can apply Zhao's generalised 2-1 Theorem [41] (in the block decomposition form [12] for convenience) to rewrite the zeta star value as an alternating zeta-half value. By application of the parity theorem ([33], or [21]), we reduce this to an explicit combination of depth 2 alternating MZVs. It becomes convenient to write these (combinations of) alternating double zeta values as certain shuffle-regularised alternating double zetas $\zeta_z^{\sqcup\!\sqcup,0}(r, \overline{s})$ with a number of initial zeros; this presentation then manifests a dihedral symmetry modulo products and lower depth [18],[19], which we can describe explicitly. Finally (perhaps surprisingly), by combining this dihedral symmetry with a generalised doubling identity [1, 42], one can explicitly evaluate these alternating double zeta values in terms of classical double zeta values (as opposed to higher depth MZVs which would certainly suffice by the generalised 2-1 Theorem).

Alternating and interpolated MZVs:

Let us recall, again, the notions of alternating MZVs, and of multiple zeta star values and multiple zeta-half values, which will be useful imminently. Given a tuple $(k_1, k_2, \ldots, k_d)$ of positive integers, and a tuple $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d) \in \{\pm 1\}^d$, with $(k_d, \varepsilon_d) \neq (1, 1)$, we define the *alternating MZV* (or Euler sum) with signs $\varepsilon_1, \ldots, \varepsilon_d$ as follows,

$$\zeta\begin{pmatrix}\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d \\ k_1, k_2, \ldots, k_d\end{pmatrix} \coloneqq \sum_{0<n_1<n_2<\cdots<n_d} \frac{\varepsilon_1^{n_1}\varepsilon_2^{n_2}\cdots\varepsilon_d^{n_d}}{n_1^{k_1}n_2^{k_2}\cdots n_d^{k_d}}.$$

One then streamlines the notation by suppressing the ε_i's and writing $\overline{k_i}$ if $\varepsilon_i = -1$, and just k_i if $\varepsilon_i = 1$ otherwise. For example

$$\zeta(k_1, \overline{k_2}, \overline{k_3}) \coloneqq \zeta\begin{pmatrix}1, & -1, & -1 \\ k_1, & k_2, & k_3\end{pmatrix} \coloneqq \sum_{0<n_1<n_2<n_3} \frac{(-1)^{n_2}(-1)^{n_3}}{n_1^{k_1}n_2^{k_2}n_3^{k_3}}.$$

An alternating MZV can be written as an iterated integral in the following way

$$\zeta\begin{pmatrix}\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d \\ k_1, k_2, \ldots, k_d\end{pmatrix} = (-1)^d I^{\mathfrak{m}}(0; \eta_1, \{0\}^{k_1-1}, \eta_2, \{0\}^{k_2-1}, \ldots, \eta_d, \{0\}^{k_d-1}; 1), \tag{A.1}$$

where $\eta_i = \varepsilon_i\varepsilon_{i+1}\cdots\varepsilon_d$.

Next, we have the interpolated multiple zeta values $\zeta^r(k_1, \ldots, k_d)$ introduced by Yamamoto [39],

$$\zeta^r(k_1, \ldots, k_d) \coloneqq \sum_{\circ_i = \text{“+”or“,”}} r^{\#\{i \mid \circ_i = \text{“+”}\}} \zeta(k_1 \circ_1 k_2 \circ_2 \cdots \circ_{r-1} k_d).$$

For example, $\zeta^r(a, b, c) = \zeta(a, b, c) + r\zeta(a+b, c) + r\zeta(a, b+c) + r^2\zeta(a+b+c)$. In the case $r = 0$, only the term with all $\circ_i =$ “,” survives, and so $\zeta^0(k_1, \ldots, k_d) = \zeta(k_1, \ldots, k_d)$. When $r = 1$, then we have $\zeta^1(k_1, \ldots, k_d) = \zeta^\star(k_1, \ldots, k_d)$, where the *multiple zeta star value* (MZSV) is originally defined as

$$\zeta^\star(k_1, \ldots, k_d) \coloneqq \sum_{0<n_1\leq n_2\leq\cdots\leq n_d} \frac{1}{n_1^{k_1}n_2^{k_2}\cdots n_r^{k_d}},$$

and arises by replacing the strict inequalities between n_i, n_{i+1} with a nonstrict inequality. For $r = 1/2$, we then obtain a new variant ‘mid-way’ between ζ and $\zeta^\star$, called the *multiple zeta-half value*.

This formalism can be extended to allow for alternating interpolated MZVs, by replacing ‘+’, above with ‘$\oplus$’, where $a \oplus b$ denotes addition of the absolute values and multiplication of the bars viewed as signs (i.e. if $k \in \mathbb{Z}_{>0}$, we have$|k| = |\overline{k}| = k$ with $\mathrm{sgn}(\overline{k}) = -1$ and $\mathrm{sgn}(k) = 1$). In particular, for

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$\alpha, \beta \in \mathbb{Z}_{>0}$, we have $\alpha \oplus \beta = \alpha + \beta$, $\alpha \oplus \overline{\beta} = \overline{\alpha} \oplus \beta = \overline{\alpha + \beta}$ and $\overline{\alpha} \oplus \overline{\beta} = \alpha + \beta$. Then, for example, we have the following interpolated alternating MZV

$$\begin{aligned}\zeta^r(a, \overline{b}, \overline{c}) &= \zeta(a, \overline{b}, \overline{c}) + r\zeta(a \oplus \overline{b}, \overline{c}) + r\zeta(a, \overline{b} \oplus \overline{c}) + r^2\zeta(\overline{a} \oplus \overline{b} \oplus \overline{c}) \\ &= \zeta(a, \overline{b}, \overline{c}) + r\zeta(\overline{a + b}, \overline{c}) + r\zeta(a, b + c) + r^2\zeta(\overline{a + b + c}).\end{aligned}$$

The case $r = 1/2$ of alternating interpolated MZVs is a convenient way of formulating Zhao's [41] generalised 2-1 Theorem, as we will see below.

A.1. Stuffle antipode

We define $G^\star$, the generating series of $\zeta^\star(\{2\}^a, 4, \{2\}^b)$, G, a related generating series for $\zeta(\{2\}^a, 4, \{2\}^b)$, and $S^\star$ the generating series of $\zeta^\star(\{2\}^n)$ as follows.

$$\begin{aligned}G^\star(x, y) &:= \sum_{a,b=0}^{\infty} \zeta^\star(\{2\}^a, 4, \{2\}^b) x^{2a} y^{2b}, \\ G(x, y) &:= \sum_{a,b=0}^{\infty} (-1)^{a+b} \zeta(\{2\}^a, 4, \{2\}^b) x^{2a} y^{2b}, \\ S^\star(x) &:= \sum_{n=0}^{\infty} \zeta^\star(\{2\}^n) x^{2n}.\end{aligned}$$

Then from [31, Equation 2.4] (in the special case, $a_2 = z_4$, $a_1 = a_3 = z_2$), we have that

$$G^\star(x, y) = G(y, x) * S^\star(x) * S^\star(y). \tag{A.2}$$

This is an identity in the stuffle algebra; in particular, it automatically lifts to a motivic identity since the stuffle product is known to be motivic (see [34] or [37]). Moreover, it is well-known (or readily verifiable by factoring the generating series as a product, see, for example [2, Equation 36], and [40, Equation 44]) that

$$\sum_{n=0}^{\infty} \zeta^\star(\{2\}^n) x^{2n} = \frac{\pi x}{\sin(\pi x)}, \text{ and } \sum_{n=0}^{\infty} \zeta(\{2\}^n) x^{2n} = \frac{\sin(\mathrm{i}\pi x)}{\mathrm{i}\pi x}.$$

By solving Equation (A.2) for $G(y, x)$, and extracting the coefficient of $x^{2a} y^{2b}$, we obtain the following explicit formula for $\zeta(\{2\}^a, 4, \{2\}^b)$ in terms of similar $\zeta^\star$ values,

$$\zeta(\{2\}^a, 4, \{2\}^b) = \sum_{n=0}^{a} \sum_{m=0}^{b} (-1)^{m+n} \zeta^\star(\{2\}^m, 4, \{2\}^n) \zeta(\{2\}^{a-n}) \zeta(\{2\}^{b-m}). \tag{A.3}$$

A similar identity holds for 4 replaced by any value k; these identities give the precise version of the stuffle antipode result

$$\zeta^{\mathrm{I}}(k_1, \ldots, k_d) = (-1)^{d+1} \zeta^{\star,\mathrm{I}}(k_d, \ldots, k_1)$$

considered in [18, Lemma 4.2.2]. Moreover, since the stuffle algebra identity and the evaluations of $\zeta(\{2\}^n)$, $\zeta^\star(\{2\}^n)$ are motivic ([5, Lemma 3.4], [18, Lemma 4.4.3]), Equation (A.3) lifts automatically to a motivic version.

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A.2. Generalised 2-1 Theorem

We now recall the generalised 2-1 Theorem, established by Zhao [41], which evaluates each $\zeta^\star$ value in terms of a certain alternating $\zeta^{1/2}$ value. It is more convenient – and indeed has a closer connection with the goal – to write the generalised 2-1 Theorem in the block decomposition form given in [12, Lemma 3.1].

Let $\underline{\mathbf{s}} = (s_1, \ldots, s_k)$ be a sequence of MZV arguments, and let $\underline{\mathbf{B}} = (\ell_1, \ldots, \ell_n)$ be the corresponding block decomposition (see Section 2.3). Write

$$\widetilde{x} = \begin{cases} x & \text{if } x \text{ odd,} \\ \overline{x} & \text{if } x \text{ even} \end{cases}$$

Recall: $\overline{x}$ denotes that the argument x in an alternating MZV has sign -1. Then

$$\zeta^\star(\underline{\mathbf{s}}) = \varepsilon(\underline{\mathbf{s}}) \cdot 2^n \zeta^{1/2}(\widetilde{\ell_1 - 2}, \widetilde{\ell_2}, \ldots, \widetilde{\ell_n}),$$

where $\varepsilon(\underline{\mathbf{s}}) = 1$ if $s_1 = 1$ and $\varepsilon(\underline{\mathbf{s}}) = -1$ if $s_1 \geq 2$, and if $\ell_1 - 2 = 0$, one should neglect this argument. This follows by combining Zhao's generalised 2-1 Theorem [41], which involves a certain recursively constructed sequence of indices $\underline{\mathbf{s}}^{(i)}$, with the description of the final such index string $\underline{\mathbf{s}}^{(k)}$, given in [12, Lemma 3.1], in terms of the block decomposition. This final string supplies the $\zeta^{1/2}$ arguments in Zhao's formulation of the 2-1 Theorem.

In our case, we want to apply this to $\zeta^\star(\{2\}^a, 4, \{2\}^b)$. The block decomposition of $(\{2\}^a, 4, \{2\}^b)$ is given by $(2a+3, 1, 2b+2)$, and we therefore have

$$\zeta^\star(\{2\}^a, 4, \{2\}^b) = -2^3 \zeta^{1/2}(2a+1, 1, \overline{2b+2}).$$

Then expanding out, by definition of the interpolated $\zeta^{1/2}$, we have

$$\begin{aligned} \zeta^\star(\{2\}^a, 4, \{2\}^b) = & -2\zeta(\overline{2a+2b+4}) - 4\zeta(2a+1, \overline{2b+3}) \\ & - 4\zeta(2a+2, \overline{2b+2}) - 8\zeta(2a+1, 1, \overline{2b+2}). \end{aligned} \tag{A.4}$$

This reduces our task of evaluating $\zeta^\star(\{2\}^a, 4, \{2\}^b)$ to understanding certain explicit depth 3 alternating MZVs.

A.3. The parity theorem in depth 3

The parity theorem for MZVs states roughly that an MZV of weight w and depth d can be reduced to a combination of lower depth MZVs and products, when $w \not\equiv d \pmod 2$. In particular, an MZV of depth 3 and even weight is reducible. This claim actually also holds for alternating MZVs, via the parity theorem for multiple polylogarithms [33], as -1 is its own multiplicative inverse.

An explicit version of the depth 3 parity theorem is given for the multiple polylogarithm functions $\mathrm{Li}_{n_1,n_2,n_3}(z_1, z_2, z_3)$ in [33, Equation 4.3]. By specialising to $z_i = \pm 1$, we recover the claimed reduction of depth 3 alternating MZVs, for any choice of signs z_i (encoded with a 'bar' over the corresponding argument $\overline{n_i}$, if $z_i = -1$), as follows. Namely, if $\alpha+\beta+\gamma$ is even and $\gamma \neq 1$ (although $\gamma = \overline{1}$ is okay), then with $\zeta(0) = \zeta(\overline{0}) = -\frac{1}{2}$ by convention, and stuffle-regularisation if necessary (see [25] for the notion of regularisation, and Remark A.1 below for the behaviour in this case) when $\zeta(1)$ appears, we have

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$$\zeta(\alpha,\beta,\gamma) = \frac{1}{2}\zeta(\alpha)\big(\zeta(\beta,\gamma) - (-1)^{|\beta|+|\gamma|}\zeta(\beta,\gamma)\big) - \zeta(\beta,\alpha)\zeta(\gamma)\delta_{|\gamma|\text{even}}$$
$$- \frac{1}{2}\zeta(\alpha\oplus\beta,\gamma) + \frac{1}{2}\zeta(\beta\oplus\gamma,\alpha)$$
$$+ \sum_{\substack{2s+\nu+\mu=|\alpha| \\ s,\mu,\nu\geq 0}} (-1)^{|\beta|+|\gamma|+\mu+\nu}\zeta(\operatorname{sgn}(\alpha\beta\gamma)\diamond 2s)\binom{-|\beta|}{\mu}\binom{-|\gamma|}{\nu}\zeta(\beta\oplus\mu,\gamma\oplus\nu)$$
$$+ \sum_{\substack{2s+\nu+\mu=|\beta| \\ s,\mu,\nu\geq 0}} (-1)^{\gamma+\mu}\zeta(\operatorname{sgn}(\alpha\beta\gamma)\diamond 2s)\binom{-|\gamma|}{\mu}\binom{-|\alpha|}{\nu}\zeta(\gamma\oplus\mu)\zeta(\alpha\oplus\nu)$$
$$+ \sum_{\substack{2s+\nu+\mu=|\gamma| \\ s,\mu,\nu\geq 0}} \zeta(\operatorname{sgn}(\alpha\beta\gamma)\diamond 2s)\binom{-|\beta|}{\mu}\binom{-|\alpha|}{\nu}\zeta(\beta\oplus\mu,\alpha\oplus\nu)$$

To avoid abuse of notation, we define $1\diamond x := x$, and $-1\diamond x := \overline{x}$, to give the corresponding decoration for signed arguments.

Now specialise to $\alpha = 2a+1, \beta = 1, \gamma = \overline{2b+2}$. We can simplify various binomial coefficients and powers of -1, using $\binom{-k}{\ell} = (-1)^\ell\binom{\ell+k-1}{k-1}$, and expand out the second summation into its two nontrivial terms $(s,\mu,\nu) = (0,1,0),(0,0,1)$. After doing so, and inserting the result into Equation (A.4), we note some simplifications. Firstly, the term $-4\zeta(2a+2,\overline{2b+2})$ in the $\zeta^{1/2}$ cancels with one from the depth 3 reduction; secondly, the term $-4\zeta(2a+1,\overline{2b+3})$ combines with one from the depth 3 reduction to produce

$$\begin{aligned} &-4\zeta(2a+1,\overline{2b+3}) - 4\zeta(\overline{2b+3},2a+1) \\ &\quad = -4\zeta(2a+1)\zeta(\overline{2b+3}) + 4\zeta(\overline{2a+2b+4}). \end{aligned}$$

Overall, this produces the following evaluation for $\zeta(\{2\}^a,4,\{2\}^b)$, as the first main stepping stone, with stuffle-regularisation applied where necessary

$$\begin{aligned} \zeta^\star(\{2\}^a,4,\{2\}^b) &= 2\zeta(\overline{2a+2b+4}) + 8\zeta(1,2a+1)\zeta(\overline{2b+2}) - 8\zeta(2a+1)\zeta(1,\overline{2b+2}) \\ &\quad + 4(2b+1)\zeta(\overline{2b+3})\zeta(2a+1) - 4(2a+1)\zeta(\overline{2b+2})\zeta(2a+2) \\ &\quad + 8\sum_{\substack{2s+\nu+\mu=2a+1 \\ s,\mu,\nu\geq 0}} \zeta(\overline{2s})\binom{\nu+(2b+1)}{\nu}\zeta(1+\mu,\overline{2b+2+\nu}) \\ &\quad - 8\sum_{\substack{2s+\nu+\mu=2b+2 \\ s,\mu,\nu\geq 0}} \zeta(\overline{2s})\binom{\nu+(2a)}{\nu}\zeta(1+\mu,2a+1+\nu). \end{aligned} \tag{A.5}$$

Remark A.1 (Independence of regularisation). Let us note here that the shuffle-regularised and stuffle-regularised versions of this formula agree (and, indeed, also for the depth 3 reduction, with $c \neq 1$), and are independent of the regularisation parameter; we may therefore switch to the shuffle-regularisation at $T = 0$ for later convenience. This is expected since we are reducing a convergent triple zeta value.

More precisely, this is because terms with a single trailing 1 are equal under either regularisation, and in the case $a = 0$, the single term $8\zeta(\overline{2b+1})\zeta(1,1)$ with two trailing 1's, which arises, arising from $(s,\nu,\mu) = (b+1,0,0)$ in the last sum, cancels with the corresponding term on the second line. Otherwise, when $a = 1$, the regularisation parameter T in terms arising from $\nu = 0$ in the last sum (with $\zeta^{*,T}$ explicitly denoting the stuffle-regularised version with $\zeta^{*,T}(1) = T$),

$$\sum_{\substack{2s+\mu=2b+2\\ s\geq 0,\mu>0}} \zeta(\overline{2s})\zeta^{*,T}(1+\mu,1) = \sum_{\substack{2s+\mu=2b+2\\ s\geq 0,\mu>0}} \zeta(\overline{2s})\big(T\zeta(1+\mu) - \zeta(1,1+\mu) - \zeta(2+\mu)\big)$$

can be seen to cancel with that arising from the terms

$$\begin{aligned} &8\zeta^{*,T}(1)\zeta(1,\overline{2b+2}) + 4(2b+1)\zeta(\overline{2b+3})\zeta^{*,T}(1) \\ &\quad = 8T\zeta(1,\overline{2b+2}) + 4(2b+1)\zeta(\overline{2b+3})T. \end{aligned}$$

In fact, this cancellation is equivalent to the following reduction which follows from the depth-parity theorem in depth 2 (see [33, Equation 3.5]):

$$\zeta(1,\overline{2b+2}) = -\sum_{\substack{2s+k=2b+3\\ s\geq 0,k\geq 2}} \zeta(\overline{2s})\zeta(k) + \frac{2b+1}{2}\zeta(\overline{2b+3}). \tag{A.6}$$

A.4. Shuffle-regularisation and dihedral symmetries

Now let us take advantage of the shuffle-regularisation in earnest. Define $\zeta_\ell(k_1,\ldots,k_d)$ by inserting ℓ leading 0's at the start of the iterated integral representation of $\zeta(k_1,\ldots,k_d)$ given in Equation (A.1) (see also Equations (2.1) and (B.1)), and write $\zeta_\ell^{ш,T=0}(k_1,\ldots,k_d)$ for the shuffle regularisation thereof (see Remark 2.2 above), with $\zeta^{ш,T=0}(1) = 0$. Then we have the regularisation formula (see, for example [10, Section 5.1 R2], and the obvious generalisation to alternating MZVs in [18, Equation 2.28])

$$\zeta_\ell^{ш,T=0}(k_1,\ldots,k_d) = (-1)^\ell \sum_{i_1+\cdots+i_d=\ell} \binom{k_1+i_1-1}{i_1}\cdots\binom{k_r+i_r-1}{i_r}\zeta(k_1\oplus i_1,\ldots,k_r\oplus i_r).$$

In particular, using this, we can write

$$\begin{aligned} &\sum_{\substack{\nu+\mu=2a+1-2s\\ \mu,\nu\geq 0}} \binom{\nu+(2b+1)}{\nu}\zeta(1+\mu,\overline{2b+2+\nu}) = -\zeta^{ш,T=0}_{2a+1-2s}(1,\overline{2b+2}) \\ &\sum_{\substack{\nu+\mu=2b+2-2s\\ \mu,\nu\geq 0}} \binom{\nu+(2a)}{\nu}\zeta(1+\mu,2a+1+\nu) = \zeta^{ш,T=0}_{2b+2-2s}(1,2a+1). \end{aligned}$$

Substituting these and Equation (A.6) into Equation (A.5) gives us the following

$$\begin{aligned} \zeta^\star(\{2\}^a,4,\{2\}^b) = {} & 2\zeta(\overline{2a+2b+4}) + 8\zeta(1,2a+1)\zeta(\overline{2b+2}) - 4(2a+1)\zeta(\overline{2b+2})\zeta(2a+2) \\ & + 8\zeta(2a+1)\sum_{k=1}^{b+1}\zeta(2k+1)\zeta(\overline{2b+2-2k}) \\ & - 8\sum_{s=0}^{a}\zeta(\overline{2s})\zeta^{ш,T=0}_{2a+1-2s}(1,\overline{2b+2}) - 8\sum_{s=0}^{b+1}\zeta(\overline{2s})\zeta^{ш,T=0}_{2b+2-2s}(1,2a+1). \end{aligned}$$

Let us finally note that the term $8\zeta(1,2a+1)\zeta(\overline{2b+2})$ cancels with the $s=b+1$ term of the last sum. In particular, we obtain the second stepping stone in our quest to evaluate $\zeta^\star(\{2\}^a,4,\{2\}^b)$. Namely

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$$
\begin{aligned}
\zeta^\star(\{2\}^a, 4, \{2\}^b) = 2\zeta(\overline{2a+2b+4}) &- 4(2a+1)\zeta(\overline{2b+2})\zeta(2a+2) \\
&+ 8\zeta(2a+1)\sum_{k=1}^{b+1}\zeta(2k+1)\zeta(\overline{2b+2-2k}) \qquad \text{(A.7)} \\
&- 8\sum_{s=0}^{a}\zeta(\overline{2s})\zeta^{Ш,T=0}_{2a+1-2s}(1,\overline{2b+2}) - 8\sum_{s=0}^{b}\zeta(\overline{2s})\zeta^{Ш,T=0}_{2b+2-2s}(1,2a+1).
\end{aligned}
$$

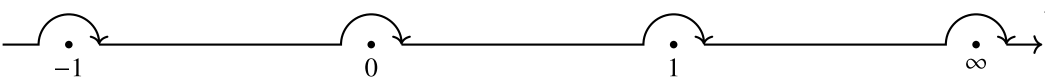
Henceforth, we shall always use the shuffle regularisation with $\zeta^{Ш,T=0}(1) = 0$, and will therefore drop the extraneous $\bullet^{Ш,T=0}$ from our notation. This regularisation is consistent with the regularisation normally used in the motivic framework (c.f. Remark 2.2). Moreover, we will check in Section B.2 that this reduction is indeed motivic.

Now, we recall from [18, Corollary 4.2.6] that depth p alternating zeta star values satisfy a dihedral symmetry of order $p+1$, modulo products and lower depth. More precisely therein, this symmetry is phrased in terms of so-called multiple zeta star-star values, which incorporate the lower depth terms, making the symmetry hold already modulo products. In particular, in our case, we claim that

$$
\begin{aligned}
\zeta_{2k-1}(1,\overline{2\ell}) &\equiv \zeta(\overline{2\ell},\overline{2k}) \quad \text{(mod products)} \\
\zeta_{2k}(1,2\ell+1) &\equiv \zeta(2\ell+1,2k+1) \quad \text{(mod products)}
\end{aligned}
$$

The depth 1 terms in these cases are reducible, as the weight is even. We will not actually use the implicit form of the dihedral symmetry established by Glanois which immediately produces the above; instead, guided by the Glanois's proof, we will establish an exact version in this depth 2 case. However, let us point out some technical issue which apparently occurs when considering the octagon relation, in an attempt to derive a so-called *hybrid relation* (Theorem 4.2.3 in [18]), a key part of the proof of the dihedral symmetry.

Remark A.2 (Regularisation in the octagon relation). The octagon relation for level $N = 2$ multiple zeta values (i.e. alternating MZVs) is obtained by integrating a word in e_0, e_1, e_{-1} around the following closed loop.



However, one must take into account the tangential base points, and how they are transformed under the Möbius transformation which cyclically maps the segments $(0,1) \mapsto (-1,0) \mapsto (\infty,-1) \mapsto (1,\infty)$. More precisely, the Möbius transformation

$$
f(z) = \frac{z-1}{z+1}
$$

transforms the segments as indicated, and therefore the integral $\int_{(-1,0)}$ is related to the integral $\int_{(0,1)}$ via a suitable pullback. However, note that the straight line path

$$
\begin{aligned}
\mathrm{dch}\colon [0,1] &\to [0,1] \\
t &\mapsto t
\end{aligned}
$$

with tangential base points $\vec{0}_1$ and $\vec{1}_{-1}$ is transformed into the path

$$
\begin{aligned}
(f^*\mathrm{dch})(t)\colon [0,1] &\to [-1,0] \\
t &\mapsto \frac{t-1}{t+1}
\end{aligned}
$$

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with tangential base points $\overrightarrow{-1}_2$ and $\overrightarrow{0}_{-\frac{1}{2}}$. In particular, the semicircular integrals evaluate in the following way

$$\int_{\overrightarrow{0}_{-\frac{1}{2}}}^{\overrightarrow{0}_1} \frac{\mathrm{d}t}{t} = \log(2) - \mathrm{i}\pi.$$

So the octagon relation actually takes the form

$$e^{\left(-\frac{\mathbb{L}^{\mathfrak{m}}}{2}+\log^{\mathfrak{m}}(2)\right)e_{-1}}\Phi^{\mathfrak{m}}(e_0,e_{-1},e_1)e^{\left(-\frac{\mathbb{L}^{\mathfrak{m}}}{2}+\log^{\mathfrak{m}}(2)\right)e_0}\Phi^{\mathfrak{m}}(e_0,e_1,e_{-1}) \\ \cdot e^{\left(-\frac{\mathbb{L}^{\mathfrak{m}}}{2}+\log^{\mathfrak{m}}(2)\right)e_1}\Phi^{\mathfrak{m}}(e_\infty,e_1,e_{-1})e^{\left(-\frac{\mathbb{L}^{\mathfrak{m}}}{2}+\log^{\mathfrak{m}}(2)\right)e_\infty}\Phi^{\mathfrak{m}}(e_\infty,e_{-1},e_1) = 1,$$

where e_∞ is defined, such that $e_0 + e_1 + e_\infty + e_{-1} = 0$.

This change should not render Glanois's hybrid identity invalid, as the derivation of the hybrid identity mainly requires the octagon relation modulo products, and these additional terms largely cancel out.

We now turn to the derivation of the exact identities which verify our earlier claim that

$$\zeta_{2k-1}(1,\overline{2\ell}) \equiv \zeta(\overline{2\ell},\overline{2k}) \quad (\text{mod products}) \\ \zeta_{2k}(1,2\ell+1) \equiv \zeta(2\ell+1,2k+1) \quad (\text{mod products}).$$

We treat the first, as the second is exactly analogous; we will, nevertheless, give the full identity in each case. Firstly, apply shuffle regularisation to

$$\zeta_{z-1}(\overline{\alpha},\beta) + \zeta_{z-1}(\beta,\overline{\alpha})$$

to obtain

$$= (-1)^{z-1}\sum_{i+j=z-1}\binom{i+\alpha-1}{i}\binom{j+\beta-1}{j}\left\{\zeta(\overline{i+\alpha},j+\beta)+\zeta(j+\beta,\overline{i+\alpha})\right\}$$

Note, we have combined the two original sums by switching $i \leftrightarrow j$ in the second sum. By the stuffle product (switching to stuffle regularisation is okay, as there is at most a single trailing 1), we have

$$= (-1)^{z-1}\sum_{i+j=z-1}\binom{i+\alpha-1}{i}\binom{j+\beta-1}{j}\left\{\zeta(j+\beta)\zeta(\overline{i+\alpha})-\zeta(\overline{\alpha+\beta+z-1})\right\}.$$

On the other hand, apply the shuffle antipode [21, Equation 29]

$$(-1)^N I(a;x_N,\ldots,x_1;b) + I(a;x_1,\ldots,x_N;b) \\ + \sum_{i=1}^{N-1}(-1)^{N-i}I(a;x_1,\ldots,x_i;b)I(a;x_N,\ldots,x_{i+1};b) = 0$$

which effectively reverses the differential forms in an iterated integral $I(a,x_1,\ldots,x_N;b)$, modulo explicit products terms, to

$$\zeta_{z-1}(\overline{\alpha},\beta) = I(0;\{0\}^{z-1},-1,\{0\}^{\alpha-1},1,\{0\}^{\beta-1};1),$$

and we find

$$\begin{aligned}
&\zeta_{z-1}(\overline{\alpha},\beta)+(-1)^{z-1+\alpha+\beta}\zeta_{\beta-1}(\overline{\alpha},\overline{z})\\
&\quad= I(0;\{0\}^{z-1},-1,\{0\}^{\alpha-1},1,\{0\}^{\beta-1};1)\\
&\qquad+(-1)^{z-1+\alpha+\beta}I(0;\{0\}^{\beta-1},1,\{0\}^{\alpha-1},-1,\{0\}^{z-1};1)\\
&\quad= -\sum_{i=0}^{\alpha-1}(-1)^{z-1+\alpha+i}\binom{z+i-1}{i}\binom{\alpha+\beta-2-i}{\beta-1}\zeta(\overline{z+i})\zeta(\alpha+\beta-1-i).
\end{aligned}$$

Finally, take the difference of these two identities, and set $\alpha = 2\ell, \beta = 1, z = 2k$, we then obtain the dihedral symmetry we claimed

$$\begin{aligned}
\zeta_{2k-1}(1,\overline{2\ell})-\zeta(\overline{2\ell},\overline{2k}) = &-\sum_{i=0}^{2k-1}\binom{i+2\ell-1}{i}\left\{\zeta(\overline{i+2\ell})\zeta(2k-i)-\zeta(\overline{2k+2\ell})\right\}\\
&-\sum_{i=0}^{2\ell-1}(-1)^i\binom{2k+i-1}{i}\zeta(\overline{2k+i})\zeta(2\ell-i).
\end{aligned}$$

A slightly more concise version of this is obtained by extending the sums to negative indices – where the binomial coefficients vanish – in order to combine them into one, and explicitly summing the coefficient of $\zeta(\overline{2k+2\ell})$. This puts the identity in a form closer to that which one could directly check/derive with the motivic derivations, namely

$$\begin{aligned}
\zeta_{2k-1}(1,\overline{2\ell})-\zeta(\overline{2\ell},\overline{2k}) = &\binom{2k+2\ell-1}{2k-1}\zeta(\overline{2k+2\ell})\\
&-\sum_{r=1}^{2k+2\ell-2}\left((-1)^r\binom{r-1}{2k-1}+\binom{r-1}{2\ell-1}\right)\zeta(\overline{r})\zeta(2k+2\ell-r).
\end{aligned}\tag{A.8}$$

In an analogous way, we find the explicit form of the dihedral symmetry in the other case to be

$$\begin{aligned}
\zeta_{2k}(1,2\ell+1)-\zeta(2\ell+1,2k+1) = &-\zeta(2)\delta_{k=\ell=0}-\binom{2k+2\ell+1}{2\ell+1}\zeta(2k+2\ell+2)\\
&+\sum_{r=1}^{2k+2\ell}\left((-1)^r\binom{r-1}{2k}+\binom{r-1}{2\ell}\right)\zeta(r)\zeta(2k+2\ell+2-r)
\end{aligned}\tag{A.9}$$

Here, the term $\delta_{k=\ell=0}$ accounts for the difference in shuffle- and stuffle-regularisation in the case $\zeta(1,1)$.

Both of these identities are easily verified to be motivic, either by direct calculation via D_{2r+1}, or by noting that the ingredients – namely, the shuffle and stuffle products, and the regularisation $\zeta^{ш,0}$ – are themselves motivic in nature.

A.5. Generalised doubling identity

The final ingredient we require for our evaluation is one of the so-called *generalised doubling identities*, as described in [1, Section 4], and [42, Section 14.2.5] (be aware these references use the opposite MZV convention).

In depth 2, the relevant relation is already given explicitly by Zhao, and states (with either shuffle or stuffle regularisation) that

$$\begin{aligned}&\zeta(s,t)+\zeta(\overline{s},\overline{t})\\&\quad=\sum_{i=1}^{s}\binom{s+t-i-1}{t-1}2^{1+i-s-t}\zeta(i,s+t-i)+\sum_{i=1}^{t}\binom{s+t-i-1}{s-1}2^{1+i-s-t}\zeta(s+t-i,i)\\&\qquad-\sum_{i=1}^{t}\binom{s+t-i-1}{s-1}\big\{\zeta(s+t-i,i)+\zeta(\overline{s+t-i},i)\big\}-\binom{s+t-1}{s}2^{1-s-t}\zeta(s+t).\end{aligned}$$

We then flip $\zeta(\overline{a},b)$ to $\zeta(b,\overline{a})$ using the stuffle product, rewrite the double zeta sums that lack powers of 2 using the shuffle regularisation as before, and simplify the resulting coefficient of $\zeta(s+t)$ (The power of 2 does indeed just disappear!) This gives the equivalent identity

$$\begin{aligned}&\zeta(\overline{s},\overline{t})+(-1)^t\zeta_{t-1}(1,\overline{s})\\&\quad=\sum_{i=1}^{s}\binom{s+t-i-1}{t-1}2^{1+i-s-t}\zeta(i,s+t-i)+\sum_{i=1}^{t}\binom{s+t-i-1}{s-1}2^{1+i-s-t}\zeta(s+t-i,i)\\&\qquad-\zeta(s,t)+(-1)^t\zeta_{t-1}(s,1)-\sum_{i=1}^{t}\binom{s+t-i-1}{s-1}\zeta(\overline{s+t-i})\zeta(i)-\binom{s+t-1}{s}\zeta(s+t).\end{aligned}$$

Finally, we note that upon substituting $t=2k, s=2\ell$, we can solve this identity simultaneously with Equation (A.8) to obtain expressions for both $\zeta(\overline{2\ell},\overline{2k})$ and $\zeta_{2k-1}(1,\overline{2\ell})$ individually as combinations of classical depth 2 MZVs and products. In particular, we have established the following proposition (after substituting an expression for $(-1)^t\zeta_{t-1}(s,1)=\zeta_{2k-1}(2\ell,1)\equiv\zeta(2k,2\ell)\pmod{\text{products}}$ using the shuffle antipode, or via a further dihedral symmetry, and simplifying).

Proposition A.3 (Galois descent of $\zeta(\overline{2\ell},\overline{2k})$). *The alternating double zeta value $\zeta(\overline{2\ell},\overline{2k})$ enjoys a Galois descent to classical depth 2 MZVs as follows*

$$\begin{aligned}\zeta(\overline{2\ell},\overline{2k})=&\sum_{i=2}^{2k+2\ell-2}2^{-i}\left\{\binom{i-1}{2k-1}\zeta(2k+2\ell-i,i)+\binom{i-1}{2\ell-1}\zeta(i,2k+2\ell-i)\right\}\\&-\zeta(2\ell,2k)+\sum_{r=2}^{2k+2\ell-2}(-2)^{-r}\binom{r-1}{2k-1}\zeta(r)\zeta(2k+2\ell-r)\\&-2^{-2k-2\ell}\left\{2\binom{2k+2\ell-2}{2k-1}+\binom{2k+2\ell-1}{2k-1}\right\}\zeta(2k+2\ell).\end{aligned}\tag{A.10}$$

Moreover, by a direct calculation, since D_{2r+1} is a tensor product of single-zeta values in this case, we see Proposition A.3 (and the generalised doubling identity itself) lifts to the motivic level. This is checked in detail in Section B.1.

Remark A.4. It is clear from the generalised 2-1 Theorem that $\zeta(\overline{2\ell},\overline{2k})$ descends to a combination of classical MZVs; we can, in fact, easily give an explicit formula

$$\zeta(\overline{2\ell},\overline{2k})=\frac{1}{4}\zeta^\star(1,\{2\}^{\ell-1},3,\{2\}^{k-1})-\frac{1}{2}\zeta(2k+2\ell).$$

However, it is certainly not clear from this expression that depth 2 classical MZVs suffice, and so this would not help us in evaluating $\zeta(\{2\}^a,4,\{2\}^b)$ in any useful manner. However, we do obtain an evaluation for $\zeta^\star(1,\{2\}^{\ell-1},3,\{2\}^{k-1})$ by substituting Proposition A.3 into the above.

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Moreover, since $\zeta(1,1,4,6)$ is – according to the Data Mine [1] – a combination of depth 2 alternating MZVs and products

$$\begin{aligned}\zeta(1,1,4,6) =& \frac{64}{9}\zeta(\overline{3},\overline{9}) + \frac{371}{144}\zeta(3,9) + 3\zeta(2)\zeta(3,7) + \frac{3}{2}\zeta(4)\zeta(3,5) - \frac{3131}{144}\zeta(9)\zeta(3) \\ &+ \frac{107}{24}\zeta(5)\zeta(7) + 10\zeta(2)\zeta(7)\zeta(3) + \frac{7}{2}\zeta(2)\zeta(5)^2 - \frac{1}{2}\zeta(4)\zeta(5)\zeta(3) \\ &- \frac{9}{4}\zeta(6)\zeta(3)^2 + \frac{\zeta(3)^4}{12} - \frac{117713}{132672}\zeta(12),\end{aligned}$$

but apparently irreducible as a classical MZV, one cannot, in general, expect the Galois descent to always respect the depth. However, as pointed out in Remark 5.2, one has – assuming the homological version of the Broadhurst-Kreimer Conjecture [3] (see also Conjecture 1.1 above) – that the depth of an alternating MZV after Galois descent should be at most twice the original; here, the Galois descent of $\zeta(\overline{3},\overline{9})$ involving classical MZVs up to depth 4 corroborates this.

By substituting Proposition A.3 into Equation (A.8), and this result into Equation (A.7), we establish that $\zeta^\star(\{2\}^a,4,\{2\}^b)$ and (via Equation (A.3)) that $\zeta(\{2\}^a,4,\{2\}^b)$ are both expressible in terms of only classical double zeta values.

Theorem A.5 (Nonexplicit form). *Both $\zeta^\star(\{2\}^a,4,\{2\}^b)$ and $\zeta(\{2\}^a,4,\{2\}^b)$ are expressible in terms of classical double zeta values.*

A.6. Generating series

In order to extract an explicit useable identity for $\zeta(\{2\}^a,4,\{2\}^b)$, we will convert everything to a generating series identity as a route to simplifying the resulting combinations.

Let us introduce the following generating series, whose names originate from Zagier's evaluation of $\zeta(\{2\}^a,3,\{2\}^b)$ [40], and some related generating series of even zeta values. The generating series of odd MZVs, and alternating odd MZVs, are given by

$$\begin{aligned}A(z) &= \sum_{r=1}^{\infty}\zeta(2r+1)z^{2r} = \psi(1) - \frac{1}{2}\psi(1-z) - \frac{1}{2}\psi(1+z), \\ B(z) &= \sum_{r=1}^{\infty}(-\zeta(\overline{2r+1}))z^{2r} = \sum_{r=1}^{\infty}(1-2^{-2r})\zeta(2r+1)z^{2r} = A(z) - A(\tfrac{z}{2}),\end{aligned}$$

where $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x)$ is the digamma function, the logarithmic derivative of the Gamma function. We keep with the choice established by Zagier taking negatives in the series for alternating MZVs. The generating series for even versions, using the convention that $\zeta(0) = \zeta(\overline{0}) = -\frac{1}{2}$, are given by

$$E(z) = \sum_{r=0}^{\infty}\zeta(2r)z^{2r-1} = -\frac{\pi}{2}\cot(\pi z), \qquad F(z) = \sum_{r=0}^{\infty}(-\zeta(\overline{2r}))z^{2r-1} = \frac{\pi}{2}\csc(\pi z).$$

$$\widetilde{E}(z) = \sum_{r=1}^{\infty}\zeta(2r)z^{2r-1} = \frac{1}{2x} - \frac{\pi}{2}\cot(\pi z), \qquad \widetilde{F}(z) = \sum_{r=0}^{\infty}(-\zeta(\overline{2r}))z^{2r-1} = -\frac{1}{2x} + \frac{\pi}{2}\csc(\pi z).$$

The $\widetilde{\bullet}$ versions, which are missing the polar term, will be convenient later. Let us introduce the following double zeta generating series

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$$D_{\mathrm{ev}}(x,y) = \sum_{a,b=1}^{\infty} \zeta(2a,2b) x^{2a-1} y^{2b-1}, \quad D_{\mathrm{od}}(x,y) = \sum_{a,b=1}^{\infty} \zeta(2a+1,2b+1) x^{2a} y^{2b},$$

$$D(x,y) = \sum_{\substack{a,b=2 \\ a \equiv b \pmod 2}}^{\infty} \zeta(a,b) x^{a-1} y^{b-1} = D_{\mathrm{od}}(x,y) + D_{\mathrm{ev}}(x,y).$$

As an intermediate step, let us also introduce the following generating series to capture the shuffle-regularised zetas appearing in Equation (A.7), and in the dihedral symmetries in Equations (A.8) and (A.9), as well as the alternating zeta values as part of the Galois descent result

$$K_{\mathrm{alt}}(x,y) = \sum_{a,b=1}^{\infty} \zeta(\overline{2a},\overline{2b}) x^{2a-1} y^{2b-1}, \qquad K_{\mathrm{ev}}(x,y) = \sum_{a,b=1}^{\infty} \zeta_{2b-1}(1,\overline{2a}) x^{2a-1} y^{2b-1},$$

$$K_{\mathrm{od}}(x,y) = \sum_{a,b=0}^{\infty} \zeta_{2b}(1,2a+1) x^{2a} y^{2b}.$$

Note that we sum from $a,b=0$ in K_{od} but will restrict this to start from $a,b=1$ in D_{od}, on account of the well-known reductions of $\zeta(1,2b+1)$, and $\zeta(2a+1,1)$.

Generating series for Equation (A.7).

To obtain the generating series $G^\star(x,y)$, we sum the left-hand side of Equation (A.7) weighted by $x^{2a}y^{2b}$ over all $a,b \geq 0$.

We find then that the generating series of the first term on the right-hand side is

$$\begin{aligned}
\sum_{a,b=0}^{\infty} \zeta(\overline{2a+2b+4}) x^{2a} y^{2b} &= \sum_{r=0}^{\infty} \sum_{s=0}^{r} \zeta(\overline{2r+4}) x^{2r-2s} y^{2s} \\
&= \sum_{r=0}^{\infty} \zeta(\overline{2r+4}) \sum_{s=0}^{r} x^{2r-2s} y^{2s} \\
&= \sum_{r=0}^{\infty} \zeta(\overline{2r+4}) \cdot \frac{x^{2r+2} - y^{2r+2}}{x^2 - y^2} \\
&= -\frac{y\widetilde{F}(x) - x\widetilde{F}(y)}{xy(x^2-y^2)}.
\end{aligned}$$

Likewise, the second leads to

$$\begin{aligned}
\sum_{a,b=0}^{\infty} (2a+1)\zeta(2a+2)\zeta(\overline{2b+2}) x^{2a} y^{2b} &= \sum_{a=0}^{\infty} (2a+1)\zeta(2a+2) x^{2a} \cdot \sum_{b=0}^{\infty} \zeta(\overline{2b+2}) y^{2b} \\
&= \frac{1}{y} \frac{\mathrm{d}\widetilde{E}(x)}{\mathrm{d}x} \cdot \widetilde{F}(y).
\end{aligned}$$

The third and fourth terms are readily summed to give

$$\sum_{a,b=0}^{\infty} \zeta(2a+1) \cdot \sum_{k=1}^{b+1} \zeta(2k+1) \zeta(\overline{2b+2-2k}) x^{2a} y^{2b} = -\frac{1}{y} A(x) A(y) F(y)$$

$$\sum_{a,b=0}^{\infty} \sum_{s=0}^{a} \zeta(\overline{2s}) \zeta_{2a+1-2s}(1,\overline{2b+2}) x^{2a} y^{2b} = -\frac{1}{y} F(x) K_{\mathrm{ev}}(y,x).$$

The last term requires a little care, as the summand $\zeta_0(1, 2a+1)$ does not appear, and so must be discounted via $K_{\mathrm{od}}(x, 0)$, the constant-in-y term of $K_{\mathrm{od}}(x, y)$. That is

$$\sum_{a,b=0}^{\infty}\sum_{s=0}^{b}\zeta(\overline{2s})\zeta_{2b+2-2s}(1,2a+1)x^{2a}y^{2b} = -\frac{1}{y}F(y)(K_{\mathrm{od}}(x,y) - K_{\mathrm{od}}(x,0)).$$

This gives us

$$\begin{aligned} G^{\star}(x,y) =& \frac{8}{y}A(x)A(y)F(y) + \frac{8}{y}F(x)K_{\mathrm{ev}}(y,x) + \frac{8}{y}F(y)(K_{\mathrm{od}}(x,y) - K_{\mathrm{od}}(x,0)) \\ & - 2\cdot\frac{y\widetilde{F}(x) - x\widetilde{F}(y)}{xy(x^2-y^2)} + \frac{4}{y}\frac{\mathrm{d}\widetilde{E}(x)}{\mathrm{d}x}\cdot\widetilde{F}(y). \end{aligned}$$

Generating series for dihedral identities:
The generating series of the form given by summing $\sum_{k,\ell=1}^{\infty}(\bullet)y^{2k-1}x^{2\ell-1}$ (note the order of the variables), over the left-hand side of Equation (A.8) is simply just $K_{\mathrm{ev}}(x,y) - K_{\mathrm{alt}}(x,y)$. Then

$$\begin{aligned} \sum_{k,\ell=1}^{\infty}\binom{2k+2\ell-1}{2k-1}\zeta(\overline{2k+2\ell})y^{2k-1}x^{2\ell-1} &= \sum_{r=2}^{\infty}\zeta(\overline{2r})\sum_{k=1}^{r-1}\binom{2r-1}{2k-1}y^{2k-1}x^{2r-2k-1} \\ &= \sum_{r=2}^{\infty}\zeta(\overline{2r})\Big\{-y^{2r} + \frac{y}{2}(y-x)^{2r-1} + \frac{y}{2}(y+x)^{2r-1}\Big\} \\ &= -\frac{1}{2x}\Big(\widetilde{F}(y-x) + \widetilde{F}(y+x) - 2\widetilde{F}(y)\Big). \end{aligned}$$

Similarly

$$\begin{aligned} &\sum_{k,\ell=1}^{\infty}\sum_{r=1}^{2k+2\ell-2}(-1)^r\binom{r-1}{2k-1}\zeta(\overline{r})\zeta(2k+2\ell-r)\cdot y^{2k-1}x^{2\ell-1} \\ &= \sum_{k,\ell=1}^{\infty}\Bigg(-\sum_{r=0}^{k+\ell-2}\binom{2r}{2k-1}\zeta(\overline{2r+1})\zeta(2k+2l-2r-1) \\ &\quad + \sum_{r=1}^{k+\ell-1}\binom{2r+1}{2k-1}\zeta(\overline{2r})\zeta(2k+2\ell-2r)\Bigg)y^{2k-1}x^{2\ell-1} \\ &= \sum_{r=0}^{\infty}\sum_{s=1}^{\infty}\sum_{k=1}^{r}\Bigg(-\binom{2r}{2k-1}\zeta(\overline{2r+1})\zeta(2s-1) + \binom{2r-1}{2k-1}\zeta(\overline{2r})\zeta(2s)\Bigg)y^{2k-1}x^{2s+2r-2k-1}. \end{aligned}$$

The sum over k can be evaluated explicitly, and (taking care with the $r=0$ terms) one obtains

$$\begin{aligned} &= \sum_{r,s=1}^{\infty}\frac{1}{2}\big((y-x)^{2r} - (y+x)^{2r}\big)\zeta(\overline{2r+1})\zeta(2s-1)x^{2s-2} \\ &\qquad + \sum_{r,s=1}^{\infty}\frac{1}{2}\big((y-x)^{2r-1} + (y+x)^{2r-1}\big)\zeta(\overline{2r})\zeta(2s)x^{2s-1} \\ &= -\frac{1}{2}\big(B(y-x) - B(y+x)\big)A(x) - \frac{1}{2}\big(\widetilde{F}(y-x) + \widetilde{F}(y+x)\big)\widetilde{E}(x). \end{aligned}$$

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Likewise, one finds

$$\sum_{k,\ell=1}^{\infty}\sum_{r=1}^{2k+2\ell-2}\binom{r-1}{2\ell-1}\zeta(\overline{r})\zeta(2k+2\ell-r)\cdot y^{2k-1}x^{2\ell-1}$$
$$=\frac{1}{2}\big(B(y-x)-B(y+x)\big)A(y)+\frac{1}{2}\big(\widetilde{F}(y-x)-\widetilde{F}(y+x)\big)\widetilde{E}(y),$$

which essentially amounts to switching $x \leftrightarrow y$, and switching the sign between the two terms.

Overall, one obtains

$$\begin{aligned}&K_{\mathrm{ev}}(x,y)-K_{\mathrm{alt}}(x,y)\\&\quad=\frac{1}{2}(A(x)-A(y))(B(x-y)-B(x+y))+\frac{1}{2}\widetilde{E}(y)(\widetilde{F}(x-y)+\widetilde{F}(x+y))\\&\qquad-\frac{1}{2}\widetilde{E}(x)(\widetilde{F}(x-y)-\widetilde{F}(x+y))-\frac{\widetilde{F}(x+y)-\widetilde{F}(x-y)-2\widetilde{F}(y)}{2x}.\end{aligned}$$

In exactly the same way, one finds for the second dihedral identity Equation (A.9) – taking care with the terms $\zeta(1,2b+1)$ and $\zeta(2a+1,1)$ missing from $D_{\mathrm{od}}(x,y)$ – that

$$\begin{aligned}&K_{\mathrm{od}}(x,y)-D_{\mathrm{od}}(x,y)\\&\quad=2\zeta(2)+\frac{1}{2}(A(x)-A(y))(A(x)-A(x-y)+A(y)-A(x+y))\\&\qquad+\frac{1}{2}(\widetilde{E}(x)-\widetilde{E}(y))(\widetilde{E}(x)+\widetilde{E}(y)-\widetilde{E}(x-y))+\frac{1}{2}\big(\widetilde{E}(x)+\widetilde{E}(y)\big)\widetilde{E}(x+y)\\&\qquad-\frac{\widetilde{E}(x-y)+\widetilde{E}(x+y)}{2x}-\frac{\widetilde{E}(y)}{y}.\end{aligned}$$

One may observe from this, that

$$K(x,0)=\frac{1}{4}\zeta(2)-\frac{3}{8x^2}-\frac{1}{2}A(x)^2+\Big(\frac{\pi^2}{4}+E(x)^2\Big)\Big(\frac{1}{2}+\frac{\sin(2\pi x)}{2\pi x}\Big).$$

Generating series for Equation (A.10):
Now, we compute the generating series for the identity from Proposition A.3. Again, taking $\sum_{k,\ell=1}^{\infty}(\bullet)y^{2k-1}x^{2\ell-1}$, note the variable order, we find the left-hand side to be just $K_{\mathrm{alt}}(x,y)$. The binomial times double zeta terms can be summed as follows

$$\sum_{k,\ell=1}^{\infty}\sum_{i=2}^{2k+2\ell-2}2^{-i}\binom{i-1}{2k-1}\zeta(2k+2\ell-i,i)\cdot y^{2k-1}x^{2\ell-1}$$
$$=\sum_{r,s=1}^{\infty}\Big\{\zeta(2r,2s)\cdot 2^{-2s}\sum_{k=1}^{s}\binom{2s-1}{2k-1}x^{2r+2s-2k-1}y^{2k-1}$$
$$+\zeta(2r+1,2s+1)\cdot 2^{-2s-1}\sum_{k=1}^{s}\binom{2s}{2k-1}x^{2r+2s+1-2k}y^{2k-1}\Big\}$$

$$
\begin{aligned}
&= \frac{1}{4} \sum_{r,s=1}^{\infty} \Big\{ \zeta(2r,2s) x^{2r-1} \Big\{ \Big(\frac{x+y}{2}\Big)^{2s-1} - \Big(\frac{x-y}{2}\Big)^{2s-1} \Big\} \\
&\quad + \zeta(2r+1,2s+1) x^{2r} \Big\{ \Big(\frac{x+y}{2}\Big)^{2s} - \Big(\frac{x-y}{2}\Big)^{2s} \Big\} \Big\} \\
&= \frac{1}{4} \Big(D\Big(x, \frac{x+y}{2}\Big) - D\Big(x, \frac{x-y}{2}\Big) \Big),
\end{aligned}
$$

The other terms may be handled similarly. Overall, we find the generating series identity

$$
\begin{aligned}
K_{\mathrm{alt}}(x,y) =& \frac{1}{4} \Big(D\Big(x, \frac{x+y}{2}\Big) - D\Big(x, \frac{x-y}{2}\Big) + D\Big(\frac{x+y}{2}, y\Big) - D\Big(-\frac{x-y}{2}, y\Big) \Big) - D_{\mathrm{ev}}(x,y) \\
&+ \frac{1}{4} \Big(\widetilde{E}\Big(\frac{x+y}{2}\Big) - \widetilde{E}\Big(\frac{x-y}{2}\Big) \Big) \widetilde{E}(x) - \frac{1}{4} A(x) \Big(A\Big(\frac{x+y}{2}\Big) - A\Big(\frac{x-y}{2}\Big) \Big) \\
&+ \frac{3x-y}{4x(x-y)} \widetilde{E}\Big(\frac{x-y}{2}\Big) - \frac{3x+y}{4x(x+y)} \widetilde{E}\Big(\frac{x+y}{2}\Big) + \frac{1}{2x} \widetilde{E}\Big(\frac{y}{2}\Big).
\end{aligned}
$$

A.7. *Explicit evaluations for* $\zeta^\star(\{2\}^a, 4, \{2\}^b)$

After substituting the previous generating series into the expression for $G^\star(x,y)$ given via Equation (A.7), performing a nontrivial amount of trigonometric manipulation, we find

$$
\begin{aligned}
G^\star(x,y) = & \frac{2F(x)}{y} \Big(-D\Big(\frac{x-y}{2}, x\Big) - D\Big(y, -\frac{x-y}{2}\Big) + D\Big(y, \frac{x+y}{2}\Big) + D\Big(\frac{x+y}{2}, x\Big) \Big) \\
& - \frac{8F(x)}{y} D_{\mathrm{ev}}(y,x) + \frac{8F(y)}{y} D_{\mathrm{od}}(x,y) \\
& + \frac{2F(x)}{y} \Big\{ -A(y) \cdot \Big(A\Big(\frac{x+y}{2}\Big) - A\Big(\frac{x-y}{2}\Big) \Big) + 2\big(A(x) - A(y)\big) \cdot \big(B(x+y) - B(x-y)\big) \Big\} \\
& + \frac{4F(y)}{y} \Big\{ A(x) \cdot \big(A(x) - 2A(y)\big) + \big(A(x) - A(y)\big) \cdot \big(A(x) + A(y) - A(x-y) - A(x+y)\big) \Big\} \\
& - 3\zeta(2) \cdot \frac{1 - y\widetilde{E}(y)}{y^2} \cdot \sec\Big(\frac{\pi(x-y)}{2}\Big) \sec\Big(\frac{\pi(x+y)}{2}\Big) - \frac{2\big(xF(x) - yF(y)\big)}{y^2(x^2-y^2)} \\
& + \frac{4F(x)}{y} \Big\{ \widetilde{E}(x) \Big(\frac{1}{y} + \widetilde{E}(x+y) - \widetilde{F}(x-y) \Big) - \frac{\widetilde{E}(\frac{1}{2}(x+y))}{x+y} + \frac{\widetilde{E}(\frac{1}{2}(x-y))}{x-y} \Big\} \\
& + \frac{4F(y)}{y} \Big\{ 2\zeta(2) + \widetilde{E}(y)^2 - \frac{2\widetilde{E}(y)}{y} + \frac{x\widetilde{E}(x) - y\widetilde{E}(y)}{x^2-y^2} \\
& - \frac{(x+y)\widetilde{E}(x-y) - (x-y)\widetilde{E}(x+y)}{2xy} \Big\}.
\end{aligned}
$$

If one so desires, the following explicit formula for the individual coefficient $\zeta^\star(\{2\}^a, 4, \{2\}^b)$ can be extracted from the above.

Theorem A.6 (Evaluation of $\zeta^\star(\{2\}^a, 4, \{2\}^b)$ via double zeta values[2]). *Write as shorthand* $\zeta(\overline{n}) = -(1 - 2^{1-n})\zeta(n)$, *and employ the conventions that* $\zeta(0) = \zeta(\overline{0}) = -\frac{1}{2}$ *and* $\zeta(1) = 0$. *However, no further regularisation is necessary. Denote by* E_n *the n-th Euler number, given as the coefficients of*

[2]Computer readable versions as plain text files in `Mathematica` syntax and in `pari/gp` syntax are included in the supplementary materials.

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$\operatorname{sech}(t) = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n$. *Then for any* $a, b \in \mathbb{Z}_{\geq 0}$, *the following evaluation holds, where we assume all summation variables start from* 0,

$$\begin{aligned}
\zeta^\star(\{2\}^a, 4, \{2\}^b) = &\sum_{p+q=a} 8\zeta(\overline{2q})\zeta(2b+2, 2p+2) - \sum_{r+s=b} 8\delta_{a>0}\zeta(\overline{2s})\zeta(2a+1, 2r+3) \\
&- \sum_{2u+i+j=2a+2b} 2\left(\frac{1}{2^i}\binom{i+1}{2b+1} + \frac{1}{2^j}\binom{j+1}{2a+1-2u}\right)\zeta(\overline{2u})\zeta(i+2, j+2) \\
&+ \sum_{\substack{p+q=a \\ r+s=b-1}} \left(\frac{1}{2^{2q+2s}}\zeta(2q+2s+3) - 8\zeta(\overline{2s+2q+3})\right)\binom{2+2q+2s}{1+2s}\zeta(2r+3)\zeta(\overline{2p}) \\
&+ \sum_{u+v+w=a-1} 8\binom{2w+2b+2}{2b+1}\zeta(2u+3)\zeta(\overline{2v})\zeta(\overline{2b+2w+3}) \\
&- \sum_{p+q=a-2} 8\zeta(2p+3)\zeta(2q+3)\zeta(\overline{2b+2}) + \sum_{r+s=b} 8\zeta(2a+1)\zeta(2r+3)\zeta(\overline{2s}) \\
&+ \sum_{u+v+w=b-1} 4\delta_{a=0}\zeta(2u+3)\zeta(2v+3)\zeta(\overline{2w}) \\
&+ \sum_{\substack{p+q=a-1 \\ r+s=b+1}} 8\binom{2q+2s}{2s}\zeta(\overline{2r})\zeta(2p+3)\zeta(2q+2s+1) \\
&- \sum_{u+v+w=b} \binom{2a+2v}{2v} 8\zeta(\overline{2u})\zeta(2w+3)\zeta(2a+2v+1) \\
&- \sum_{\substack{i+j=2a \\ r+s=2b+2}} 3\zeta(2)\frac{(-1)^r E_{i+r}E_{j+s}}{i!j!r!s!}\left(\frac{\mathrm{i}\pi}{2}\right)^{2a+2b+2} \\
&+ \sum_{\substack{i+j=2a \\ r+s+2t=2b}} 3\zeta(2)\frac{(-1)^r E_{i+r}E_{j+s}}{i!j!r!s!}\left(\frac{\mathrm{i}\pi}{2}\right)^{2a+2b-2t}\zeta(2t+2) \\
&+ 2\zeta(\overline{2a+2b+4}) + \sum_{p+q=a+1} \frac{4}{2^{2p+2b}}\binom{2p+2b}{2b+1}\zeta(2p+2b+2)\zeta(\overline{2q}) \\
&+ \sum_{u+v+w=a} 8\binom{2u+2b+1}{2b+1}\zeta(\overline{2w})\zeta(2v+2)\zeta(\overline{2u+2b+2}) \\
&+ \sum_{r+s=b+1} 4\left(\binom{2a+2r+1}{2a+1} - \binom{2a+2r+1}{2r+1}\right)\zeta(2a+2r+2)\zeta(\overline{2s}) \\
&- \sum_{r+s=b+1} 4\zeta(2a+2r+2)\zeta(\overline{2s}) - \sum_{u+v+w=b} 4\delta_{a=0}\zeta(2u+2)\zeta(2v+2)\zeta(\overline{2w}) \\
&+ \sum_{r+s=b+1} 8\delta_{a=0}\zeta(2r+2)\zeta(\overline{2s}) - 8\delta_{a=0}\zeta(2)\zeta(\overline{2b+2}).
\end{aligned}$$

A.8. Explicit evaluations for $\zeta(\{2\}^a, 4, \{2\}^b)$

By substituting the expression for $G^\star(x, y)$ into Equation (A.2), and finding $G(x, y)$ via

$$G(x, y) = \sum_{a,b=0}^{\infty} (-1)^{a+b}\zeta(\{2\}^a, 4, \{2\}^b)x^{2a}y^{2b} = G^\star(y, x)\frac{\sin(\pi x)}{\pi x}\frac{\sin(\pi y)}{\pi y},$$

we obtain the following explicit expression for the generating series,

$$
\begin{aligned}
G(x,y) = {} & \frac{\sin(\pi x)}{\pi x^2 y}\Big(-D\Big(-\frac{x-y}{2},x\Big) - D\Big(x,\frac{x-y}{2}\Big) + D\Big(x,\frac{x+y}{2}\Big) + D\Big(\frac{x+y}{2},y\Big)\Big) \\
& - \frac{4\sin(\pi x)}{\pi x^2 y} D_{\mathrm{ev}}(x,y) + \frac{\sin(\pi y)}{\pi x^2 y} D_{\mathrm{od}}(y,x) \\
& - \frac{\sin(\pi x)}{\pi x^2 y}\Big\{A(x)\cdot\Big(A\Big(\frac{x+y}{2}\Big) - A\Big(\frac{x-y}{2}\Big)\Big) + 2\big(A(x)-A(y)\big)\cdot\big(B(x+y)-B(x-y)\big)\Big\} \\
& - \frac{\sin(\pi y)}{\pi x^2 y}\Big\{A(y)\cdot\big(2A(x)-A(y)\big) + \big(A(x)-A(y)\big)\cdot\big(A(x)+A(y)-A(x-y)-A(x+y)\big)\Big\} \\
& - 3\zeta(2)\cdot\frac{1-x\widetilde{E}(x)}{x^2}\cdot\sec\Big(\frac{\pi(x-y)}{2}\Big)\sec\Big(\frac{\pi(x+y)}{2}\Big)\frac{\sin(\pi x)}{\pi x}\frac{\sin(\pi y)}{\pi y} \\
& - \frac{1}{x^2(x^2-y^2)}\Big(\frac{\sin(\pi x)}{\pi x} - \frac{\sin(\pi y)}{\pi y}\Big) \\
& + \frac{2\sin(\pi x)}{\pi x^2 y}\Big\{\widetilde{E}(y)\Big(\frac{1}{x} + \widetilde{E}(x+y) + \widetilde{F}(x-y)\Big) - \frac{\widetilde{E}(\frac{1}{2}(x+y))}{x+y} + \frac{\widetilde{E}(\frac{1}{2}(x-y))}{x-y}\Big\} \\
& + \frac{2\sin(\pi y)}{\pi x^2 y}\Big\{2\zeta(2) + \widetilde{E}(x)^2 - \frac{2\widetilde{E}(x)}{x} + \frac{x\widetilde{E}(x) - y\widetilde{E}(y)}{x^2-y^2} \\
& + \frac{(x+y)\widetilde{E}(x-y) - (x-y)\widetilde{E}(x+y)}{2xy}\Big\}.
\end{aligned}
$$

If one desires, the following explicit formula for the individual coefficient $\zeta(\{2\}^a, 4, \{2\}^b)$ can be extracted from the above.

Theorem A.7 (Evaluation of $\zeta(\{2\}^a, 4, \{2\}^b)$ via double zeta values[3]). *Write as shorthand $\zeta(\overline{n}) = -(1-2^{1-n})\zeta(n)$, and employ the conventions that $\zeta(0) = \zeta(\overline{0}) = -\frac{1}{2}$ and $\zeta(1) = 0$. However, no further regularisation is necessary. Denote by E_n the n-th Euler number, given as the coefficients of* $\operatorname{sech}(t) = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n$. *Then for any $a, b \in \mathbb{Z}_{\geq 0}$, the following evaluation holds, where we assume all summation variables start from* 0,

$$
\begin{aligned}
& \zeta(\{2\}^a, 4, \{2\}^b) \\
& \quad = (-1)^{a+b}\Bigg\{ -\sum_{p+q=a} 4\zeta(2p+2, 2b+2)\frac{(\mathrm{i}\pi)^{2q}}{(2q+1)!} + \sum_{r+s=b-1} 4\zeta(2r+3, 2a+3)\frac{(\mathrm{i}\pi)^{2s}}{(2s+1)!} \\
& \qquad + \sum_{2u+i+j=2a+2b} \Bigg(\frac{1}{2^i}\binom{i+1}{2a-2u+1} + \frac{1}{2^j}\binom{j+1}{2b+1}\Bigg)\zeta(i+2, j+2)\frac{(\mathrm{i}\pi)^{2u}}{(2u+1)!} \\
& \qquad + \sum_{u+v+w=a-1} 4\binom{2w+2b+2}{2b+1}\Bigg(\zeta(\overline{2b+2w+3}) - \frac{\zeta(2b+2w+3)}{2^{2b+2w+3}}\Bigg)\zeta(2v+3)\frac{(\mathrm{i}\pi)^{2u}}{(2u+1)!} \\
& \qquad - \sum_{\substack{p+q=a \\ r+s=b}} 4\binom{2p+2s}{2s-1}\zeta(2r+3)\zeta(\overline{2s+2p+1})\frac{(\mathrm{i}\pi)^{2q}}{(2q+1)!}
\end{aligned}
$$

[3]Computer readable versions as plain text files in `Mathematica` syntax and in `pari/gp` syntax are included in the supplementary materials.

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$$
\begin{aligned}
&- \sum_{u+v+w=b-1} 4\binom{2a+2v+2}{2v}\zeta(2w+3)\zeta(2a+2v+3)\frac{(\mathrm{i}\pi)^{2u}}{(2u+1)!} \\
&+ \sum_{\substack{p+q=a\\ r+s=b}} 4\binom{2q+2r}{2r}\zeta(2p+3)\zeta(2q+2r+1)\frac{(\mathrm{i}\pi)^{2s}}{(2s+1)!} \\
&- \sum_{p+q=a-1} 2\zeta(2p+3)\zeta(2q+3)\frac{(\mathrm{i}\pi)^{2b}}{(2b+1)!} - \sum_{r+s=b-1} 4\zeta(2a+3)\zeta(2r+3)\frac{(\mathrm{i}\pi)^{2s}}{(2s+1)!} \\
&- \sum_{\substack{i+j+2k=2a+2\\ p+q+2r=2b}} 3\zeta(2)\frac{(-1)^p E_{i+p}E_{j+q}}{i!j!p!q!}\left(\frac{\mathrm{i}\pi}{2}\right)^{2a+2b+2}\frac{2^{2k+2r}}{(2k+1)!(2r+1)!} \\
&+ \sum_{\substack{i+j+2k+2\ell=2a\\ p+q+2r=2b}} 3\zeta(2)\frac{(-1)^p E_{i+p}E_{j+q}}{i!j!p!q!}\left(\frac{\mathrm{i}\pi}{2}\right)^{2a+2b-2\ell}\frac{2^{2k+2r}}{(2k+1)!(2r+1)!}\zeta(2\ell+2) \\
&+ \frac{(\mathrm{i}\pi)^{2a+2b+4}}{(2a+2b+5)!} + 2\zeta(2b+2)\frac{(\mathrm{i}\pi)^{2a+2}}{(2a+3)!} - 4\zeta(2a+4)\frac{(\mathrm{i}\pi)^{2b}}{(2b+1)!} \\
&- \sum_{p+q=a+1} \frac{1}{2^{2p+2b-1}}\binom{2p+2b}{2b+1}\zeta(2p+2b+2)\frac{(\mathrm{i}\pi)^{2q}}{(2q+1)!} \\
&- \sum_{\substack{p+q=a\\ r+s=b}} 4\binom{2p+2r+1}{2p+1}\zeta(\overline{2p+2r+2})\zeta(2s+2)\frac{(\mathrm{i}\pi)^{2q}}{(2q+1)!} \\
&+ \sum_{r+s=b} 2\zeta(2s+2r+4)\left(\binom{2a+2r+3}{2a+3} - \binom{2a+2r+3}{2r+1}\right)\frac{(\mathrm{i}\pi)^{2s}}{(2s+1)!} \\
&+ \sum_{r+s=b} 2\zeta(2a+2r+4)\frac{(\mathrm{i}\pi)^{2s}}{(2s+1)!} + \sum_{p+q=a} 2\zeta(2p+2)\zeta(2q+2)\frac{(\mathrm{i}\pi)^{2b}}{(2b+1)!} \Bigg\}.
\end{aligned}
$$

In particular, we obtain the following corollary on the reduction of $\zeta(\{2\}^a, 4, \{2\}^b)$ modulo products. In essence, it extracts those double zeta terms above, which are not multiplied by a power of π.

Corollary A.8. *Modulo decomposables (i.e. products of MZVs), the following evaluation holds*

$$
\begin{aligned}
\zeta(\{2\}^a, 4, \{2\}^b) = (-1)^{a+b}\Bigg\{&-4\zeta(2a+2, 2b+2) + 4\zeta(2b+1, 2a+3) \\
&+ \sum_{\substack{i+j=2a+2b\\ i,j\geq 0}} \left(\frac{1}{2^i}\binom{i+1}{2a+1} + \frac{1}{2^j}\binom{j+1}{2b+1}\right)\zeta(i+2, j+2)\Bigg\} \quad (\mathrm{mod\ products}).
\end{aligned}
$$

B. Motivic evaluation of $\zeta^{\mathrm{m}}(\{2\}^a, 4, \{2\}^b)$ via motivic double zeta values

In order to verify that the evaluation of $\zeta(\{2\}^a, 4, \{2\}^b)$ in Section A.8 (or at least, the evaluation in Corollary A.8) is motivic, we only need to show that the various ingredients used in Appendix A are motivic. More precisely, we need to show that Proposition A.3 and Equation (A.7) are motivic. All other identities used in the derivation of Corollary A.8 and Theorems A.6 and A.7 were obtained from the shuffle or stuffle product, and so are automatically motivic. The shuffle product is motivic by definition, for the stuffle-product, see [34, 37].

Framework of alternating motivic MZVs:

We note, here, that the motivic MZV framework of Section 2.1 generalises readily to the case of alternating motivic MZVs. For the technical details of this, we refer to [18, 19]; the most important points are the comodule of alternating motivic MZVs $\mathcal{H}^{(2)}$ is obtained by extending Definition 2.1 to allow $a_i \in \{0, \pm 1\}$ (although functoriality in a useful form only applies when all $a_i \in \{0, 1\}$). Then for a tuple $(k_1, \ldots, k_d)$ of positive integers, and $(\varepsilon_1, \ldots, \varepsilon_d) \in \{\pm 1\}^d$ of signs, and $\ell \geq 0$, we define the motivic alternating MZV by

$$\zeta^{\mathfrak{m}}_{\ell}\binom{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d}{k_1, k_2, \ldots, k_d} \coloneqq (-1)^d I^{\mathfrak{m}}(0; \{0\}^{\ell}, \eta_1, \{0\}^{k_1-1}, \eta_2, \{0\}^{k_2-1}, \ldots, \eta_d, \{0\}^{k_d-1}; 1), \tag{B.1}$$

where $\eta_i = \varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_d$, mimicking the integral representation of alternating MZVs in Equation (A.1). One can again streamline the notation by dropping the ε_i's and writing $\overline{k_i}$ if $\varepsilon_i = -1$, and just k_i if $\varepsilon_i = 1$. Then $\mathcal{A}^{(2)} \coloneqq \mathcal{H}^{(2)}/(\zeta^{\mathfrak{m}}(2))$ and $\mathcal{L}^{(2)} = \mathcal{A}^{(2)}_{>0}/\mathcal{A}^{(2)}_{>0}\mathcal{A}^{(2)}_{>0}$ define the obvious extensions of the Hopf algebra and the Lie coalgebra of irreducibles. The coaction $\Delta\colon \mathcal{H}^{(2)} \longrightarrow \mathcal{A}^{(2)} \otimes \mathcal{H}^{(2)}$ is defined by the same formula as in Equation (2.2), and the infinitesimal derivations $\mathrm{D}_r\colon \mathcal{H}^{(2)} \longrightarrow \mathcal{L}^{(2)}_r \otimes \mathcal{H}^{(2)}$, with $\mathcal{L}^{(2)}_r$ the weight r component of $\mathcal{L}^{(2)}$, are given by the same formula as in Equation (2.3).

For alternating motivic MZVs, D_1 plays a nontrivial role, as the weight 1 alternating MZV $\zeta^{\mathfrak{m}}(\overline{1}) = \log^{\mathfrak{m}}(2)$ is nonzero. The analogue of Brown's [5] characterisation of $\ker \mathrm{D}_{<N}$ in the alternating case is given by Glanois as follows.

Theorem B.1 (Glanois, Corollary 2.4.5 [18]). *Let $N \geq 1$, and denote by $\mathrm{D}_{<N} = \bigoplus_{1 \leq 2r+1 < N} \mathrm{D}_{2r+1}$. Then in weight N, the kernel of $D_{<N}$ on alternating motivic MZVs is one dimensional:*

$$\ker \mathrm{D}_{<N} \cap \mathcal{H}^{(2)}_N = \begin{cases} \mathbb{Q}\zeta^{\mathfrak{m}}(\overline{1}) = \mathbb{Q}\log^{\mathfrak{m}}(2) & \text{if } N = 1 \\ \mathbb{Q}\zeta^{\mathfrak{m}}(N) & \text{if } N > 1. \end{cases}$$

Since the identities we wish to lift involve alternating MZV terms in a nontrivial way, we necessarily have to use Glanois's criterion to verify the motivic lift, even if, as it happens, $\mathrm{D}_1 = 0$ in each case.

B.1. *Motivic version of Proposition A.3*

We prove the following proposition which claims that Proposition A.3 lifts to a motivic version.

Proposition B.2 (Motivic Galois descent of $\zeta^{\mathfrak{m}}(\overline{2\ell}, \overline{2k})$). *The alternating motivic double zeta value $\zeta^{\mathfrak{m}}(\overline{2\ell}, \overline{2k})$ enjoys a Galois descent to classical depth 2 motivic MZVs as follows*

$$\begin{aligned} \zeta^{\mathfrak{m}}(\overline{2\ell}, \overline{2k}) = {} & \sum_{i=2}^{2k+2\ell-2} 2^{-i}\left\{\binom{i-1}{2k-1}\zeta^{\mathfrak{m}}(2k+2\ell-i, i) + \binom{i-1}{2\ell-1}\zeta^{\mathfrak{m}}(i, 2k+2\ell-i)\right\} \\ & - \zeta^{\mathfrak{m}}(2\ell, 2k) + \sum_{r=2}^{2k+2\ell-2} (-2)^{-r}\binom{r-1}{2k-1}\zeta^{\mathfrak{m}}(r)\zeta^{\mathfrak{m}}(2k+2\ell-r) \\ & - 2^{-2k-2\ell}\left\{2\binom{2k+2\ell-2}{2k-1} + \binom{2k+2\ell-1}{2k-1}\right\}\zeta^{\mathfrak{m}}(2k+2\ell). \end{aligned} \tag{B.2}$$

Proof. We compute D_{2r+1} of both sides, and verify they agree for $1 \leq r \leq k + \ell - 2$. The case $r = 0$ does not play a role, since D_1 is known to be exactly zero by the Galois descent property established in [18]; alternatively, one can directly compute it and see there is no contribution since the sequences $(0, 1, -1), (0, -1, 1), (-1, 1, 0), (1, -1, 0)$ which give rise to $\log^{\mathfrak{l}}(2)$ are not present in the integral representation of any term. In the case, $r = k + \ell - 1$, $\mathrm{D}_{2k+2\ell-1}$ is quickly checked to vanish, as only $\zeta^{\mathfrak{m}}(1) = 0$ appears in the right-hand tensor factor. ◻

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Computation of $\mathrm{D}_{2r+1}\zeta^{\mathfrak{m}}(\overline{2a},\overline{2b})$ **and** $\mathrm{D}_{2r+1}\zeta^{\mathfrak{m}}(2a,2b)$
We see that only the following subsequences can contribute to the motivic coaction. This is because any subsequence must start or end one of the three nonzero entries; one then checks whether the length $2r+1$ subsequences which start/end at these points actually contribute

$$\zeta^{\mathfrak{m}}(\overline{2a},\overline{2b}) = I^{\mathfrak{m}}(\,0\,;\boxed{1},\{0\}^{2a-1},\boxed{-1},\{0\}^{2b-1},\boxed{1}).$$

We find

$$\begin{aligned}
&\mathrm{D}_{2r+1}\,\zeta^{\mathfrak{m}}(\overline{2a},\overline{2b})\\
&\quad= -\delta_{a\leq r}\zeta^{\mathrm{I}}_{2r+1-2a}(\overline{2a})\otimes\zeta(2a+2b-2r-1)+\delta_{b\leq r}\zeta^{\mathrm{I}}_{2r+1-2b}(\overline{2b})\otimes\zeta^{\mathfrak{m}}(2a+2b-2r-1)\\
&\quad= \left(\binom{2r}{2a-1}-\binom{2r}{2b-1}\right)\zeta^{\mathrm{I}}(\overline{2r+1})\otimes\zeta(2a+2b-2r-1)
\end{aligned}$$

The binomial factors should *a prior* retain the delta factors, but they can be removed as the binomials vanish already for the complementary condition. The corresponding result holds for $\zeta^{\mathfrak{m}}(2a,2b)$ by removing all bars from the above result

$$\mathrm{D}_{2r+1}\,\zeta^{\mathfrak{m}}(2a,2b) = \left(\binom{2r}{2a-1}-\binom{2r}{2b-1}\right)\zeta^{\mathrm{I}}(2r+1)\otimes\zeta(2a+2b-2r-1).$$

Computation of $\mathrm{D}_{2r+1}\,\zeta^{\mathfrak{m}}(2a+1,2b+1)$
We see that only the following subsequences can contribute to the motivic coaction. This is because any subsequence must involve one of the three nonzero entries; one then checks whether the length $2r+1$ subsequences which start/end at these points actually contribute

$$\zeta^{\mathfrak{m}}(2a+1,2b+1) = I^{\mathfrak{m}}(\,0\,;\boxed{1},\{0\}^{2a},\boxed{1},\{0\}^{2b},\boxed{1})\,.$$

We find

$$\begin{aligned}
\mathrm{D}_{2r+1}\,\zeta^{\mathfrak{m}}(2a+1,2b+1) &= \delta_{a=r}\zeta^{\mathrm{I}}(2r+1)\otimes\zeta^{\mathfrak{m}}(2a+2b+1-2r)\\
&\quad+\left(-\delta_{a\leq r}\zeta^{\mathrm{I}}_{2r-2a}(2a+1)+\delta_{b\leq r}\zeta^{\mathrm{I}}_{2r-2b}(2b+1)\right)\otimes\zeta^{\mathfrak{m}}(2a+2b+1-2r)\\
&= \left(\delta_{a=r}-\binom{2r}{2a}+\binom{2r}{2b}\right)\zeta^{\mathrm{I}}(2r+1)\otimes\zeta(2a+2b+1-2r).
\end{aligned}$$

Computation of $\mathrm{D}_{2r+1}\,\zeta^{\mathfrak{m}}(p,q)$**,** $p+q$ **even**
We note that the two cases above can be combined to give the following, for $p+q$ even

$$\mathrm{D}_{2r+1}\,\zeta^{\mathfrak{m}}(p,q) = \left(\delta_{2r+1=p}+(-1)^p\binom{2r}{p-1}-(-1)^q\binom{2r}{q-1}\right)\zeta^{\mathrm{I}}(2r+1)\otimes\zeta(p+q-2r-1).$$

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Verification of Proposition B.2

The claim that D_{2r+1} of both sides agree is equivalent to the following putative identity among binomial coefficients, when $1 \le r \le k+\ell-1$, which arises after projecting $\zeta^{\mathrm{l}}(2r+1) \otimes \zeta^{\mathfrak{m}}(2k+2\ell-2r-1) \mapsto 1$.

$$\begin{aligned}
0 \overset{?}{=} & (1-2^{-2r})\left(\binom{2r}{2\ell-1} - \binom{2r}{2k-1}\right) \\
& + \sum_{i=2}^{2k+2\ell-2} 2^{-i}\binom{i-1}{2k-1}\left(\delta_{2k+2\ell-i=2r+1} + (-1)^i\binom{2r}{2k+2\ell-i-1} - (-1)^i\binom{2r}{i-1}\right) \\
& + \sum_{i=2}^{2k+2\ell-2} 2^{-i}\binom{i-1}{2\ell-1}\left(\delta_{i=2r+1} + (-1)^i\binom{2r}{i-1} - (-1)^i\binom{2r}{2k+2\ell-i-1}\right) \\
& - \left(\binom{2r}{2l-1} - \binom{2r}{2k-1}\right) \\
& + (-2)^{-(2r+1)}\binom{2r}{2k-1} + (-2)^{-(2k+2\ell-2r-1)}\binom{2k+2\ell-2r-2}{2k-1}.
\end{aligned}$$

After some simplification of the right-hand side, and reindexing the sums, we find that the claim is equivalent to the following

$$\begin{aligned}
0 \overset{?}{=} & -2^{-2r-1}\left(\binom{2r}{2\ell-1} - \binom{2r}{2k-1}\right) \\
& + (-2)^{-2k}\sum_{i=0}^{2\ell-2}(-2)^{-i}\binom{i+2k-1}{2k-1}\left(\binom{2r}{2\ell-i-1} - \binom{2r}{2k+i-1}\right) \\
& + (-2)^{-2\ell}\sum_{i=0}^{2k-2}(-2)^{-i}\binom{i+2k-1}{2\ell-1}\left(\binom{2r}{2\ell+i-1} - \binom{2r}{2k-i-1}\right).
\end{aligned}$$

This is verified to be exactly 0 from Lemma 4.2 of Section 4. With that, we have finished the proof of Proposition B.2.

B.2. Motivic version of Equation (A.7)

We prove the following proposition, which claims that Equation (A.7) lifts to a motivic version.

Proposition B.3. *The following identity holds among motivic multiple zeta (star) values*

$$\begin{aligned}
\zeta^{\mathfrak{m},\star}(\{2\}^a, 4, \{2\}^b) = & \, 2\zeta^{\mathfrak{m}}(\overline{2a+2b+4}) - 4(2a+1)\zeta^{\mathfrak{m}}(\overline{2b+2})\zeta^{\mathfrak{m}}(2a+2) \\
& + 8\zeta^{\mathfrak{m}}(2a+1)\sum_{k=1}^{b+1}\zeta^{\mathfrak{m}}(2k+1)\zeta^{\mathfrak{m}}(\overline{2b+2-2k}) \\
& - 8\sum_{s=0}^{a}\zeta^{\mathfrak{m}}(\overline{2s})\zeta^{\mathfrak{m}}_{2a+1-2s}(1, \overline{2b+2}) - 8\sum_{s=0}^{b}\zeta^{\mathfrak{m}}(\overline{2s})\zeta^{\mathfrak{m}}_{2b+2-2s}(1, 2a+1).
\end{aligned} \tag{B.3}$$

Proof. We compute D_{2r+1} of both sides and will show that they agree. The analytic version of this identity, which is given in Equation (A.7) then fixes the remaining coefficient of $\zeta^{\mathfrak{m}}(2a+2b+4)$ (which, here, is expressed as a sum of two terms involving products of even zetas). Notice that the only place

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D_1 could contribute is from $\zeta^{\mathfrak{m}}_{2a+1}(1, \overline{2b+2})$, but we will see momentarily that $\mathrm{D}_1 = 0$, hence, we can take $r > 0$. □

Computation of $\mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m}}_{2a+1}(1, \overline{2b+2})$
We see that only the following subsequences can contribute to the motivic coaction. This is because any subsequence must start or end at one of the three nonzero entries; one then checks whether the length $2r+1$ subsequences which start/end at these points actually contribute

$$\zeta_{2a+1}(1, \overline{2b+2}) = I^{\mathfrak{m}}(\,0\,; \{0\}^{2a+1}, \boxed{-1}, \boxed{-1}, \{0\}^{2b+1}, \boxed{1}).$$

Hence, we have

$$\begin{aligned}
\mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m}}_{2a+1}(1, \overline{2b+2}) &= \delta_{r \le a} \zeta^{\mathrm{I}}_{2r}(1) \otimes \zeta^{\mathfrak{m}}_{2a+1-2r}(\overline{2b+2}) - \delta_{r \le b} \zeta^{\mathrm{I}}_{2r}(1) \otimes \zeta^{\mathfrak{m}}_{2a+1}(\overline{2b+2-2r}) \\
&= \zeta^{\mathrm{I}}(2r+1) \otimes \left(-\delta_{r \le a} \binom{2a+2b+2-2r}{2b+1} + \delta_{r \le b} \binom{2a+2b+2-2r}{2a} \right) \\
&\quad \cdot \zeta^{\mathfrak{m}}(\overline{2a+2b+3-2r}).
\end{aligned}$$

And, in particular, $\mathrm{D}_1 = 0$.

Computation of $\mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m}}_{2b+2}(1, 2a+1)$
Similarly, only the following subsequences can contribute to the motivic coaction

$$\zeta_{2b+2}(1, 2a+1) = I^{\mathfrak{m}}(\,0\,; \{0\}^{2b+2}, \boxed{1}, \boxed{1}, \{0\}^{2a}, \boxed{1})\,.$$

Hence, we have

$$\begin{aligned}
\mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m}}_{2b+2}(1, 2a+1) &= \delta_{r \le b+1} \zeta^{\mathrm{I}}_{2r}(1) \otimes \zeta^{\mathfrak{m}}_{2b+2-2r}(2a+1) - \delta_{r \le a-1} \zeta^{\mathrm{I}}_{2r}(1) \otimes \zeta^{\mathfrak{m}}_{2b+2}(2a+1-2r) \\
&= \zeta^{\mathrm{I}}(2r+1) \otimes \left(-\delta_{r \le b+1} \binom{2a+2b+2-2r}{2a+1} + \delta_{r \le a-1} \binom{2a+2b+2-2r}{2b+1} \right) \\
&\quad \cdot \zeta^{\mathfrak{m}}(2a+2b+3-2r).
\end{aligned}$$

Computation of D_{2r+1} **of right-hand side of Equation (B.3)**
With the above two computations of the motivic coaction on the double zeta values in Equation (B.3), we can readily compute the rest of the coaction using the derivation property of D_{2r+1}, namely, $\mathrm{D}_{2r+1}\, XY = (1 \otimes Y)\, \mathrm{D}_{2r+1}\, X + (1 \otimes X)\, \mathrm{D}_{2r+1}\, Y$, as well as the fact that $\mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m}}(N) = \delta_{N=2r+1} \zeta^{\mathrm{I}}(N) \otimes 1$. Note also the first two terms on the right-hand side of Equation (B.3) are products of even zetas, and so do not contribute. So we find

$$\begin{aligned}
&\mathrm{D}_{2r+1}(\text{RHS Equation (B.3)}) \\
&\quad = 8\delta_{r=a} \zeta^{\mathrm{I}}(2r+1) \otimes \sum_{k=1}^{b+1} \zeta^{\mathfrak{m}}(2k+1) \zeta^{\mathfrak{m}}(\overline{2b+2-2k}) \\
&\quad\quad + 8\delta_{r \le b+1} \zeta^{\mathrm{I}}(2r+1) \otimes \zeta^{\mathfrak{m}}(2a+1) \zeta^{\mathfrak{m}}(\overline{2b+2-2r})
\end{aligned}$$

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$$
\begin{aligned}
&- 8 \sum_{s=0}^{a} \zeta^{\mathrm{I}}(2r+1) \otimes \left(-\delta_{r \leq a-s} \binom{2a+2b+2-2r-2s}{2b+1} + \delta_{r \leq b} \binom{2a+2b+2-2r-2s}{2b+1-2r} \right) \\
&\cdot \zeta^{\mathfrak{m}}(\overline{2a-2s+2b+3-2r})\zeta^{\mathfrak{m}}(\overline{2s}) \\
&- 8 \sum_{s=0}^{b} \zeta^{\mathrm{I}}(2r+1) \otimes \left(\delta_{r \leq b-s+1} \binom{2a+2b+2-2r-2s}{2a} - \delta_{r \leq a-1} \binom{2a+2b+2-2r-2s}{2a-2r} \right) \\
&\cdot \zeta^{\mathfrak{m}}(2a+2b-2s+3-2r)\zeta^{\mathfrak{m}}(\overline{2s}).
\end{aligned}
$$

Computation of D_{2r+1} of left-hand side of Equation (B.3)

We compute the derivation $\mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m},\star}(\{2\}^a, 4, \{2\}^b)$ by first applying the stuffle antipode to obtain an expression involving only $\zeta^{\mathfrak{m}}(\{2\}^j, 4, \{2\}^i)$, which has a simpler coaction.

We see that only the following subsequences can contribute to the motivic coaction of $\mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m}}(\{2\}^a, 4, \{2\}^b)$; all other subsequences will start and end at letters of the same parity

$$
\zeta^{\mathfrak{m}}(\{2\}^a, 4, \{2\}^b) = (-1)^{a+b+1} I^{\mathfrak{m}}(\,0\,; \{\,1\,, 0\}^a, 1, 0\,, \boxed{0}, 0\,, \{\,1\,, 0\}^b, 1\,)\,.
$$

Hence, we have

$$
\begin{aligned}
&\mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m}}(\{2\}^a, 4, \{2\}^b) \\
&\quad = -\delta_{r \leq a} \zeta_1^{\mathrm{I}}(\{2\}^r) \otimes \zeta^{\mathfrak{m}}(\{2\}^{a-r}, 3, \{2\}^b) + \delta_{r \leq b} \zeta_1^{\mathrm{I}}(\{2\}^r) \otimes \zeta^{\mathfrak{m}}(\{2\}^a, 3, \{2\}^{b-r}).
\end{aligned}
$$

Now, with the stuffle antipode formula extracted from Equation (A.2), we compute

$$
\begin{aligned}
&D_{2r+1} \zeta^{\mathfrak{m},\star}(\{2\}^a, 4, \{2\}^b) \\
&\quad = \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j}\, \mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m}}(\{2\}^j, 4, \{2\}^i) \Big(1 \otimes \zeta^{\mathfrak{m},\star}(\{2\}^{a-i}) \zeta^{\mathfrak{m},\star}(\{2\}^{b-j}) \Big) \\
&\quad = \zeta_1^{\mathrm{I}}(\{2\}^r) \otimes \sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \big(-\delta_{r \leq j} \zeta^{\mathfrak{m}}(\{2\}^{j-r}, 3, \{2\}^i) + \delta_{r \leq i} \zeta^{\mathfrak{m}}(\{2\}^j, 3, \{2\}^{i-r}) \big) \\
&\qquad \cdot \zeta^{\mathfrak{m},\star}(\{2\}^{a-i}) \zeta^{\mathfrak{m},\star}(\{2\}^{b-j}).
\end{aligned}
$$

Here, we can apply the motivic evaluation of $\zeta^{\mathfrak{m}}(\{2\}^\alpha, 3, \{2\}^\beta)$ established by Brown [5]. Alternatively, we can apply the stuffle antipode again to rewrite the result instead of involving $\zeta^{\mathfrak{m},\star}(\{2\}^\alpha, 3, \{2\}^\beta)$ and appeal to the motivic evaluation thereof, for a more direct formula (Glanois [18] claims that the motivic evaluation of $\zeta^{\mathfrak{m},\star}(\{2\}^\alpha, 3, \{2\}^\beta)$ requires knowing exactly certain conjectural identities among so-called $\zeta^{\star\star}$ values, however, it seems that the stuffle antipode formula allows one to automatically transfer the $\zeta^{\mathfrak{m}}(\{2\}^\alpha, 3, \{2\}^\beta)$ evaluation to a corresponding $\zeta^{\mathfrak{m},\star}(\{2\}^\beta, 3, \{2\}^\alpha)$ evaluation).

After (separately) shifting i and j by r, which gives the factor $(-1)^r$ below (and taking care with the signs; use the correspondence $j \leftrightarrow b$, $i \to a$), we find

$$
\begin{aligned}
&\mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m},\star}(\{2\}^a, 4, \{2\}^b) \\
&\quad = (-1)^r \zeta_1^{\mathrm{I}}(\{2\}^r) \otimes \big(\delta_{r \leq a} \zeta^{\mathfrak{m},\star}(\{2\}^{a-r}, 3, \{2\}^b) - \delta_{r \leq b} \zeta^{\mathfrak{m},\star}(\{2\}^a, 3, \{2\}^{b-r}) \big).
\end{aligned}
$$

We note that this is essentially the same expression as one obtains with Glanois's setup involving the motivic coaction on $\zeta^\star$ values, after applying the dihedral symmetries to simplify terms in the coalgebra on the left-hand side. One only needs to apply the result that $(-1)^r \zeta_1^{\mathrm{I}}(\{2\}^r) = 2\zeta^{\mathrm{I}}(2r+1) = -\zeta_1^{\mathrm{I},\star}(\{2\}^r)$ to obtain exactly the same formula.

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Now, apply the following motivic evaluations

$$\zeta_1^{\mathrm{I}}(\{2\}^r) = 2(-1)^r \zeta^{\mathrm{I}}(2r+1)$$
$$\zeta^{\mathfrak{m},\star}(\{2\}^a, 3, \{2\}^b) = -2 \sum_{s=1}^{a+b+1} \left[\binom{2s}{2a} - \delta_{s=a} - (1-2^{-2s}) \binom{2s}{2b+1} \right] \zeta^{\star,\mathfrak{m}}(\{2\}^{a+b+1-s}) \zeta^{\mathfrak{m}}(2s+1)$$

along with $\zeta^{\star,\mathfrak{m}}(\{2\}^n) = -2\zeta^{\mathfrak{m}}(\overline{2n})$. We find

$$\begin{aligned} \mathrm{D}_{2r+1}\, \zeta^{\mathfrak{m},\star}(\{2\}^a, 4, \{2\}^b) = 8\zeta^{\mathrm{I}}(2r+1) \otimes & \sum_{s=1}^{a+b+1-r} \left[\binom{2s}{2a-2r} - \delta_{s=a-r} - (1-2^{-2s}) \binom{2s}{2b+1} \right] \\ & \cdot \zeta^{\mathfrak{m}}(\overline{2a+2b+2-2s-2r}) \zeta^{\mathfrak{m}}(2s+1) \\ & - 8\zeta^{\mathrm{I}}(2r+1) \otimes \sum_{s=1}^{a+b+1-r} \left[\binom{2s}{2a} - \delta_{s=a} - (1-2^{-2s}) \binom{2s}{2b-2r+1} \right] \\ & \cdot \zeta^{\mathfrak{m}}(\overline{2a+2b+2-2s-2r}) \zeta^{\mathfrak{m}}(2s+1). \end{aligned}$$

Comparison of left- and right-hand side of Equation (B.3)

Firstly, make the change of variables $s \mapsto a+b+1-s-r$ in the sums for D_{2r+1}(RHS Equation (B.3)); after considering the cases in each resulting delta term – and dropping terms $\zeta^{\mathfrak{m}}(1) = 0$ by regularisation– we find

$$\begin{aligned} \mathrm{D}_{2r+1}(\text{RHS Equation (B.3)}) = 8\delta_{r=a}\zeta^{\mathrm{I}}(2r+1) \otimes & \sum_{k=1}^{b+1} \zeta^{\mathfrak{m}}(2k+1)\zeta^{\mathfrak{m}}(\overline{2b+2-2k}) \\ & + 8\delta_{r\leq b+1}\zeta^{\mathrm{I}}(2r+1) \otimes \zeta^{\mathfrak{m}}(2a+1)\zeta^{\mathfrak{m}}(\overline{2b+2-2r}) \\ & - 8 \sum_{s=\max(1,b-r+1)}^{a+b+1-r} \zeta^{\mathrm{I}}(2r+1) \otimes \left(-\binom{2s}{2b+1} + \binom{2s}{2b+1-2r} \right) \\ & \cdot \zeta^{\mathfrak{m}}(\overline{2s+1})\zeta^{\mathfrak{m}}(\overline{2a+2b+2-2r-2s})) \\ & - 8 \sum_{s=\max(1,a-r+1)}^{a+b+1-r} \zeta^{\mathrm{I}}(2r+1) \otimes \left(\binom{2s}{2a} + \delta_{r=a} - \binom{2s}{2a-2r} \right) \\ & \cdot \zeta^{\mathfrak{m}}(2s+1)\zeta^{\mathfrak{m}}(\overline{2a+2b+2-2r-2s})). \end{aligned}$$

Note, here, that the two terms involving $\delta_{r=a}$ cancel. Then the sums over s may be extended to start at $s = 1$. The first sum needs no correction term, as the numerators of each binomial are strictly greater than the denominators in this case, however, the term $\binom{2s}{2a-2r}$ in the second sum needs to be corrected when $s = a - r$ for $a - r \geq 1$. We obtain

$$\begin{aligned} \mathrm{D}_{2r+1}(\text{RHS Equation (B.3)}) = & \; 8\delta_{r\leq b+1}\zeta^{\mathrm{I}}(2r+1) \otimes \zeta^{\mathfrak{m}}(2a+1)\zeta^{\mathfrak{m}}(\overline{2b+2-2r}) \\ & - 8\delta_{r\leq a-1}\zeta^{\mathrm{I}}(2r+1) \otimes \zeta^{\mathfrak{m}}(2a+1-2r)\zeta^{\mathfrak{m}}(\overline{2b+2}) \\ & - 8 \sum_{s=1}^{a+b+1-r} \zeta^{\mathrm{I}}(2r+1) \otimes \left(-\binom{2s}{2b+1} + \binom{2s}{2b+1-2r} \right) \\ & \cdot \zeta^{\mathfrak{m}}(\overline{2s+1})\zeta^{\mathfrak{m}}(\overline{2a+2b+2-2r-2s})) \end{aligned}$$

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$$- 8 \sum_{s=1}^{a+b+1-r} \zeta^{\mathrm{I}}(2r+1) \otimes \left(\binom{2s}{2a} - \binom{2s}{2a-2r} \right) \cdot \zeta^{\mathfrak{m}}(2s+1) \zeta^{\mathfrak{m}}(\overline{2a+2b+2-2r-2s}).$$

Finally, write $\zeta^{\mathfrak{m}}(\overline{2s+1}) = -(1-2^{2s})\zeta^{\mathfrak{m}}(2s+1)$. It is now straightforward to check that D_{2r+1}(LHS Equation (B.3)) $=$ D_{2r+1}(RHS Equation (B.3)); the two terms outside the sum for D_{2r+1}(RHS Equation (B.3)) above correspond to the deltas terms in the expression for D_{2r+1}(LHS Equation (B.3)).

This completes the proof of Proposition B.3, and shows the reduction of $\zeta^\star(\{2\}^a, 4, \{2\}^b)$ to depth 3 alternating MZVs is motivic.

B.3. Motivic evaluation of $\zeta^{\mathrm{I}}(\{2\}^a, 4, \{2\}^b)$

Now that we have verified all of the ingredients for the evaluations of $\zeta(\{2\}^a, 4, \{2\}^b)$ and $\zeta^\star(\{2\}^a, 4, \{2\}^b)$ are motivic, we may conclude that the identities in Theorems A.6 and A.7 hold for $\zeta^{(\star)}$ replaced by their motivic counterparts, and $\mathrm{i}\pi$ replaced by $\frac{1}{2}\mathbb{L}^{\mathfrak{m}} = (\mathrm{i}\pi)^{\mathfrak{m}}$.

More importantly, the evaluation of $\zeta(\{2\}^a, 4, \{2\}^b)$ modulo products from Corollary A.8 is also motivic, and we obtain the result of Lemma 4.1 as an immediate corollary.

Corollary B.4. *The following evaluation holds in the motivic coalgebra*

$$\zeta^{\mathrm{I}}(\{2\}^a, 4, \{2\}^b) = (-1)^{a+b} \Bigg\{ -4\zeta^{\mathrm{I}}(2a+2, 2b+2) + 4\zeta^{\mathrm{I}}(2b+1, 2a+3) + \sum_{\substack{i+j=2a+2b \\ i,j \geq 0}} \left(\frac{1}{2^i}\binom{i+1}{2a+1} + \frac{1}{2^j}\binom{j+1}{2b+1} \right) \zeta^{\mathrm{I}}(i+2, j+2) \Bigg\}$$

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