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Glued lattices are better quantizers than K_{12}

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***Abstract*—40 years ago, Conway and Sloane proposed using the highly symmetrical Coxeter–Todd lattice K_{12} for quantization, and estimated its second moment. Since then, all published lists identify K_{12} as the best 12-dimensional lattice quantizer. Surprisingly, K_{12} is not optimal: we construct two new 12-dimensional lattices with lower normalized second moments. The new lattices are obtained by gluing together products of two 6-dimensional lattices.**

***Index Terms*—Coxeter–Todd lattice, glue vectors, gluing theory, lattice theory, mean square error, moment of inertia, normalized second moment, product lattice, quantization constant, quantization error, vector quantization, Voronoi region.**

I. Introduction

One of the classical problems in lattice theory is to find the best *lattice quantizer,* i.e., the lattice with minimum normalized second moment in a given dimension [1, Ch. 2]. This problem has applications in data compression [2], [3, Chs. 1, 3–5], geometric shaping of modulation formats [4], [3, Ch. 9], coding for noisy channels [1, p. 70], experimental design [5], and data analysis [6].

The lattice quantizer problem was pioneered by Fejes Tóth, who showed that the hexagonal lattice is optimal in two dimensions [7]. The corresponding optimum in three dimensions is the body-centered cubic lattice, as proved by Barnes and Sloane in 1983 [8]. In higher dimensions n, the optimal lattices are not known. Tables of the best known lattices were presented for $n \leq 5$ in [9], $n \leq 10$ in [10], [11], $n = 1, \ldots, 8, 12, 16, 24$ in [12], [1, pp. 12, 61], [3, p. 135], $9 \leq n \leq 12$ in [13], $n = 1, \ldots, 10, 12, 16, 24$ in [14], $n \leq 15$ in [6], $n \leq 24$ in [15], and $n \leq 48$ in [16].

The above-mentioned tables devote more attention to some dimensions than others. One such dimension is $n = 12$, and the reason for the interest in this dimension is the existence of the highly symmetrical *Coxeter–Todd lattice* K_{12}, which was discovered by Coxeter and Todd in 1953 [17]. Its second moment was computed numerically in 1984 [12] and exactly in 2009 [13]. It is listed as the best 12-dimensional lattice quantizer in all tables we have seen. The lattice was conjectured optimal for both quantization and packing in [1, p. 13]. Its construction, symmetry group, and other properties are described in [18], [1, pp. 127–129].

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In this paper, we prove that K_{12}, contrary to the popular belief, is *not* the optimal lattice quantizer in 12 dimensions. We do this by designing two better lattices and computing their second moments exactly. These new lattices are constructed using *gluing theory,* which was introduced by Conway *et al.* for self-dual block codes in [19] and for integral lattices in [20]. Using this theory, self-dual integral lattices were constructed as the union of a finite number of translates of a given base lattice, which is typically a *product lattice* [21], [1, Sec. 3 in Ch. 4]. Gluing or similar techniques have not, as far as we know, been applied in the quest for good lattice quantizers, which are not necessarily integer lattices. That is the scope of the present paper.

II. Lattice Fundamentals

A *lattice* L is an infinite, countable set of real vectors that forms a finitely generated group under addition. For $n \leq m$, it can be defined by an $n \times m$ real *generator matrix* $\boldsymbol{B}$ with linearly independent rows, such that

$$L = \{\boldsymbol{uB} \colon \boldsymbol{u} \in \mathbb{Z}^n\}, \tag{1}$$

where $\boldsymbol{u}$ are n-dimensional row vectors of integers. The rows of $\boldsymbol{B}$ are called *basis vectors.* The *dimension* of L is n, and the lattice is embedded in the Euclidean space $\mathbb{R}^m$. There exist infinitely many generator matrices for the same lattice.

The $n \times n$ symmetric, positive definite *Gram matrix* $\boldsymbol{A} \triangleq \boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}$ gives the inner products of all basis vectors with each other. There exist infinitely many Gram matrices for the same lattice, but their determinants are equal, and this value is called the *lattice determinant* [1, p. 4].

The lattice generated by $\boldsymbol{B}^* \triangleq \boldsymbol{A}^{-1}\boldsymbol{B}$, which is the transpose of the pseudoinverse of $\boldsymbol{B}$, is the *dual lattice* L^*. The Gram matrix of this dual lattice is $\boldsymbol{A}^{-1}$. If $m = n$, then $\boldsymbol{A}^{-1} = \left(\boldsymbol{B}^{\mathrm{T}}\right)^{-1}\boldsymbol{B}^{-1}$ and $\boldsymbol{B}^* = (\boldsymbol{B}^{\mathrm{T}})^{-1}$. If $L = L^*$, then the lattice is said to be *self-dual*.

The *Voronoi region* of the lattice generated by $\boldsymbol{B}$ is

$$\Omega \triangleq \left\{\boldsymbol{zB} \colon \boldsymbol{z} \in \mathbb{R}^n, \min_{\boldsymbol{u}\in\mathbb{Z}^n} \|\boldsymbol{zB} - \boldsymbol{uB}\|^2 = \|\boldsymbol{zB}\|^2\right\}. \tag{2}$$

This is the set of all vectors (in the space spanned by the lattice vectors) whose closest lattice vector is the all-zero vector $\boldsymbol{0}$. The Voronoi region of any lattice is a convex polytope, which is symmetric under reflection through $\boldsymbol{0}$ and has volume $V = \sqrt{\det \boldsymbol{A}}$. The facets of the Voronoi region lie in $(n-1)$-dimensional planes that are equidistant from $\boldsymbol{0}$ and another lattice vector; these lattice vectors are called *relevant vectors.* The vertices of the Voronoi region are called *holes* of the lattice, and the vertices farthest away from $\boldsymbol{0}$ are called *deep holes.*

For many purposes, it is desirable that the Voronoi region is as spherical as possible according to some criteria. For lattice quantizers, we seek to minimize the second moment (moment of inertia) of the Voronoi region. Using the customary normalization, the relevant figure of merit is the *normalized second moment* (NSM) or *quantizer constant*

$$G = \frac{1}{nV^{1+2/n}} \int_{\Omega} \|\boldsymbol{x}\|^2 \, \mathrm{d}\boldsymbol{x}. \tag{3}$$

The normalization by $V^{1+2/n}$ makes G invariant under rescaling, and normalization by n ensures that a product lattice (defined below) of identical lattices L has the same NSM as L.

The (Cartesian) *product* of two lattices (or any other vector sets) is

$$L_1 \times L_2 \triangleq \{[\boldsymbol{x}_1 \; \boldsymbol{x}_2] \colon \boldsymbol{x}_1 \in L_1, \boldsymbol{x}_2 \in L_2\}. \tag{4}$$

The Voronoi region of a product lattice is the product of the Voronoi regions of each of its lower-dimensional component lattices. Hence, the set of holes of a product lattice is the Cartesian product of the set of holes of each component lattice. The relevant vectors of a product lattice are $[\boldsymbol{x}_1 \, \mathbf{0}]$ and $[\mathbf{0} \, \boldsymbol{x}_2]$, where $\boldsymbol{x}_i$ are the relevant vectors of L_i. Thus, the number of facets of the product lattice is the sum of the number of facets of the component lattices. Product lattices are the currently best known lattice quantizers in many dimensions [9], [15], [16], but they are not optimal, because their NSM can always be decreased by small perturbations of the generator matrix [16, Th. 7]. This motivates a search for systematic ways to improve product lattices. The approach we follow here is provided by gluing theory.

III. Gluing Theory

Gluing theory was developed in the context of the well-studied *integer lattices,* i.e., lattices whose Gram matrix contains only integers. An interesting feature is that an integer lattice L is always a sublattice of its dual L^*. This means that L^* can be written as the union of L and a finite number of its translates

$$L^* = \bigcup_{\boldsymbol{g} \in L^*/L} (L + \boldsymbol{g}). \tag{5}$$

Here L^*/L denotes a (finite) set of coset representatives of L in L^* [1, p. 48].

These coset representatives are called *glue vectors* of L. The number of glue vectors is $\det \boldsymbol{A}$, and by convention we use $\mathbf{0}$ as the coset representative for L itself, so $\mathbf{0}$ is always a glue vector. Glue vectors of the root lattices A_n, D_n, and E_n are listed in [20, Tab. I], [1, Ch. 4]. We say that a vector $\boldsymbol{x} \in L^*$ is of *type* $\boldsymbol{g}$ if $\boldsymbol{x} \in L + \boldsymbol{g}$.

If some glue vectors are omitted from the union in (5), then a more general construction [22]

$$\tilde{L} \triangleq \bigcup_{\boldsymbol{g} \in \Gamma} (L + \boldsymbol{g}) \tag{6}$$

is obtained, where $\{\mathbf{0}\} \subseteq \Gamma \subseteq L^*/L$. We call this construction "gluing." The term was introduced in [19]–[21] for the special case when L is a product code or product lattice; as in [22], we use it more generally for any lattice constructed by (6). In one extreme, (6) yields L and in the other extreme L^*. Intermediate choices of Γ can yield interesting families of lattices or nonlattice packings, such as the Coxeter lattices A_n^r [23], [13, Sec. 5.1] and D_n^+ [1, p. 119]. The so-called *Construction A* can be seen as a special case of (6) with $L = \sqrt{2}\mathbb{Z}^n$ and Γ being a rescaled binary block code [1, pp. 137–141, 182–185].

$\tilde{L}$ is a lattice if and only if Γ is a group under addition modulo L. If so, Γ is called the *glue group* of $\tilde{L}$, and a generator matrix for $\tilde{L}$ can be obtained as follows. Starting with a generator matrix for L, we append the elements of Γ as $|\Gamma|$ additional rows. Then we carry out linear row transformations with integer coefficients to make $|\Gamma|$ rows all-zero, and remove those all-zero rows.

The Voronoi region of $\tilde{L}$ is contained in the Voronoi region of L. Specifically, if some elements of Γ are located in holes of L, then the corresponding vertices of the Voronoi region are "cut away." This intuitively explains why gluing can potentially make Voronoi regions more spherical and create better lattice quantizers.

We exploit this construction technique, selecting L as the product $L = L_1 \times \cdots \times L_k$ of known integer lattices L_i. Then L is also an integer lattice and its glue vectors are Cartesian products of the glue vectors of $L_1, \ldots, L_k$. When written as row vectors, this concatenates them, so the glue vectors of a product lattice L are called *glue words.*

When L is a product lattice, the set of glue words in (5) is the Cartesian product of the sets of glue vectors of the component lattices, $L^*/L = (L_1^*/L_1) \times \cdots \times (L_k^*/L_k)$. There are $(\det \boldsymbol{A}_1) \cdots (\det \boldsymbol{A}_k)$ such glue words, where $\boldsymbol{A}_i$ is a Gram matrix of L_i, and hence many options when constructing new lattices via (6).

Conway and Sloane studied a large number of glued lattices $\tilde{L}$ generated from various product lattices L by (6). Their goal was to construct integer lattices with determinant one, which are self-dual. They successfully enumerated the lattice components and glue words of all such lattices in dimensions up to 24 [20], [21]. In the next section, we use the same technique to find better lattice quantizers.

More generally, the construction (6) is also valid when Γ is not a subset of L^*/L. For example, the 9-dimensional lattice quantizer with the smallest known NSM is $L \cup (L + \boldsymbol{g})$ (corresponding to $\Gamma = \{\mathbf{0}, \boldsymbol{g}\}$), where $L = D_8 \times 2a\mathbb{Z}$, $\boldsymbol{g}$ is a deep hole of L, and a is an algebraic scalar constant [6], [11]. In this case, L is not an integer lattice and L^*/L does not exist. While interesting, such generalizations of (6) are not considered further in this paper.

IV. New Lattices

A. Glued $E_6 \times E_6$

The integer lattice E_6 is often defined as a sublattice of the *Gosset lattice* E_8. This approach gives rise to the rectangular generator matrix in [1, p. 126], where $m = 8$. We find it more

TABLE I
LATTICES $\tilde{L}$ GENERATED BY GLUING $L = E_6 \times E_6$ ACCORDING TO (6).

Glue words Γ	Estimated NSM of $\tilde{L}$	Exact NSM	Comment
$\{\boldsymbol{g}_{00}\}$	0.074336 ± 0.000010	$G_{E_6} \approx 0.074347$	$\tilde{L} = E_6 \times E_6$
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{01}, \boldsymbol{g}_{02}\}$	0.075557 ± 0.000010	$7711/102\,060 \approx 0.075554$	$\tilde{L} = E_6 \times E_6^*$
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{10}, \boldsymbol{g}_{20}\}$	0.075556 ± 0.000010	Same as the previous	$\tilde{L} = E_6^* \times E_6$, equivalent to the previous
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{11}, \boldsymbol{g}_{22}\}$	0.070060 ± 0.000007	Given by (10)	$\tilde{L}$ is a better quantizer than K_{12}!
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{12}, \boldsymbol{g}_{21}\}$	0.070063 ± 0.000007	Same as the previous	Equivalent to the previous
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{01}, \boldsymbol{g}_{02}, \boldsymbol{g}_{10}, \boldsymbol{g}_{11}, \boldsymbol{g}_{12}, \boldsymbol{g}_{20}, \boldsymbol{g}_{21}, \boldsymbol{g}_{22}\}$	0.074237 ± 0.000009	$G_{E_6^*} \approx 0.074244$	$\tilde{L} = E_6^* \times E_6^*$

convenient to work with the square generator matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \tag{7}$$

which is obtained by applying linear row operations to [11, Eq. (38)] and negating the sign of one column.

Since $\det \boldsymbol{A} = (\det \boldsymbol{B})^2 = 3$, the lattice has three glue vectors E_6^*/E_6. These are given in the traditional 8-dimensional representation in [1, p. 126]. The corresponding glue vectors in our 6-dimensional representation are

$$\begin{aligned} \boldsymbol{g}_0 &= [\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\ ], \\ \boldsymbol{g}_1 &= [-\tfrac{1}{2} \quad -\tfrac{1}{2} \quad -\tfrac{1}{2} \quad -\tfrac{1}{2} \quad -\tfrac{1}{2} \quad \tfrac{\sqrt{3}}{6}], \\ \boldsymbol{g}_2 &= [\ \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad -\tfrac{\sqrt{3}}{6}]. \end{aligned} \tag{8}$$

The (Abelian) glue group is the cyclic group on 3 elements Z_3. It has $\boldsymbol{g}_0$ as the identity, and multiplication table $\boldsymbol{g}_1^2 = \boldsymbol{g}_2$, $\boldsymbol{g}_2^2 = \boldsymbol{g}_1$, and $\boldsymbol{g}_1\boldsymbol{g}_2 = \boldsymbol{g}_2\boldsymbol{g}_1 = \boldsymbol{g}_0$, where the group operation is addition modulo E_6.

The Voronoi region of E_6 has 72 facets and 54 vertices. All vertices have the same norm and hence constitute deep holes of the lattice. These holes all lie in E_6^*: 27 are of type $\boldsymbol{g}_1$ and 27 are of type $\boldsymbol{g}_2$.

From the multiplication table, it follows immediately that no proper subset $\Gamma \subset E_6^*/E_6 = \{\boldsymbol{g}_0, \boldsymbol{g}_1, \boldsymbol{g}_2\}$ except $\{\boldsymbol{g}_0\}$ is a group. Thus, for $L = E_6$, the construction (6) yields no lattices other than E_6 and E_6^*. The NSMs of E_6 and E_6^* are $G_{E_6} \triangleq 5/(56 \cdot 3^{1/6}) \approx 0.074347$ [10] and $G_{E_6^*} \triangleq 12\,619/(68\,040 \cdot 3^{5/6}) \approx 0.074244$ [24], respectively.

The product lattice $E_6 \times E_6$ has nine glue words, formed by all Cartesian products $\boldsymbol{g}_{ij} \triangleq [\boldsymbol{g}_i \ \boldsymbol{g}_j]$ for $i, j = 0, 1, 2$. The Voronoi region of $E_6 \times E_6$ has $2 \cdot 72 = 144$ facets and $54^2 = 2916$ vertices. All these vertices are lattice vectors in $E_6^* \times E_6^*$ and are of four different types, namely, the glue words $\boldsymbol{g}_{ij}$ for $i, j = 1, 2$.

We consider all subsets of the nine glue words $(E_6^*/E_6) \times (E_6^*/E_6)$ such that $\tilde{L}$ is a lattice, i.e., subsets Γ that are groups under addition modulo $E_6 \times E_6$. From the group multiplication table it is seen that there are six such subsets, which are listed in Table I. Each of them generates a lattice $\tilde{L}$ via (6).

We estimate the NSMs G of the obtained lattices by Monte Carlo integration using 10^7 independent samples in the Voronoi region of each lattice. To find the lattice vector closest to an arbitrary vector in $\mathbb{R}^{12}$, which is an essential step in generating samples in the Voronoi regions, we use [25, Algorithm 5]. The estimated NSMs are presented in the form $\hat{G} \pm 2\hat{\sigma}$, where $\hat{G}$ is an unbiased estimate of G computed as in [12, Eq. (2)] and $\hat{\sigma}$ is an estimate of the standard deviation of $\hat{G}$ computed as in [26, Eq. (15)]. Because some of the constructed lattices are *equivalent* to each other by rotation and/or reflection, there are only four lattices in the table whose geometric properties such as the NSM differ.

Four of the six groups Γ are direct products of groups of 6-dimensional vectors, namely, $\{\boldsymbol{g}_0\}$ and/or $\{\boldsymbol{g}_0, \boldsymbol{g}_1, \boldsymbol{g}_2\}$, so the corresponding lattices are product lattices. The NSMs of $E_6 \times E_6$ and $E_6^* \times E_6^*$ are G_{E_6} and $G_{E_6^*}$, which are defined above. The NSM of $E_6 \times E_6^*$ and $E_6^* \times E_6$ can be calculated from [16, Prop. 3] as $(3^{1/6} G_{E_6} + 3^{-1/6} G_{E_6^*})/2 = 7711/102\,060 \approx 0.075554$. These exact NSMs are also shown in Table I.

As discussed in Section II, product lattices cannot be optimal quantizers, so it comes as no surprise that the only nonproduct lattice in Table I is also the best quantizer. This lattice is obtained by applying (6) to $L = E_6 \times E_6$ with $\Gamma = \{\boldsymbol{g}_{00}, \boldsymbol{g}_{11}, \boldsymbol{g}_{22}\}$ or equivalently $\{\boldsymbol{g}_{00}, \boldsymbol{g}_{12}, \boldsymbol{g}_{21}\}$. What *is* surprising is that its estimated NSM $\hat{G} \approx 0.070060$ is *below* the NSM of the Coxeter–Todd lattice K_{12}, which was suggested for quantization by Conway and Sloane in 1984 [12] and has remained unsurpassed since then. The exact NSM of K_{12} was computed in [13] and is $797\,361\,941/(6\,567\,561\,000\sqrt{3}) \approx 0.070096$.

To confirm the record, we investigated the new lattice analytically. A generator matrix of an equivalent lattice is

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2\sqrt{3}} \end{bmatrix}, \tag{9}$$

which was obtained as described after (6) followed by negation of some columns for cosmetic reasons. Using the same method as in [26], which builds upon [27], the face hierarchy of the Voronoi region was fully determined. It has in total 11 432 765 485 faces in dimensions 0 through 12, of which 1 is in dimension 12 (the Voronoi region itself), 1602 are in dimension 11 (facets), and 65 665 350 are in dimension 0 (vertices). The faces lie in 702 equivalence classes under the action of the lattice's symmetry group, which has order 10 749 542 400. Using the methods in [26], [27], the exact NSM of the new lattice is determined to be

$$G = \frac{200\,359\,601\,790\,277}{2\,859\,883\,842\,816\,000} \approx 0.070\,058\,650, \tag{10}$$

confirming the numerical estimate in Table I. Its covariance matrix is proportional to the identity, which is a necessary but not sufficient condition for global and local optimality [16], [28]. The complete face catalog is available online [29, Ancillary files].

In comparison with K_{12}, which has a symmetry group of order 78 382 080 [1, p. 129], the lattice (9) is much more symmetric with a symmetry group whose order is about 137 times as large. Despite this, with 11 971 901 593 faces in 809 classes (determined with the methods in [26]), K_{12} has about 5 % more faces than (9).

The Voronoi region inherits some properties from (unglued) $E_6 \times E_6$, having the same packing radius $1/\sqrt{2}$ and kissing number 144. It has however three times the packing density, which is 0.02086. Its covering radius is $2/\sqrt{3}$, which is a factor $\sqrt{2}$ less than the covering radius of $E_6 \times E_6$, and its thickness is 7.502. The lattice is equivalent to its dual, but it is not strictly self-dual, because the dual is a rotated version of the lattice itself.

B. Glued $D_6 \times D_6$

A generator matrix for the integer lattice D_6 is

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

Since $\det \boldsymbol{A} = (\det \boldsymbol{B})^2 = 4$, the lattice has four glue vectors D_6^*/D_6. These can be taken as [1, p. 117]

$$\begin{aligned} \boldsymbol{g}_0 &= [\,0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,], \\ \boldsymbol{g}_1 &= [\,\tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2}\,], \\ \boldsymbol{g}_2 &= [\,1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,], \\ \boldsymbol{g}_3 &= [\,-\tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2}\,]. \end{aligned} \tag{12}$$

This glue group is the point group $C_{2\mathrm{v}}$, which has order four. The group identity is $\boldsymbol{g}_0$, and the multiplication table reads $\boldsymbol{g}_1^2 = \boldsymbol{g}_2^2 = \boldsymbol{g}_3^2 = \boldsymbol{g}_0$, $\boldsymbol{g}_1\boldsymbol{g}_2 = \boldsymbol{g}_3$, $\boldsymbol{g}_1\boldsymbol{g}_3 = \boldsymbol{g}_2$, and $\boldsymbol{g}_2\boldsymbol{g}_3 = \boldsymbol{g}_1$, where the (Abelian) group operation is addition modulo D_6.

The Voronoi region of D_6 has 60 facets and 76 vertices, which are elements of D_6^*. The vertices consist of 64 deep holes, which are of type $\boldsymbol{g}_1$ or $\boldsymbol{g}_3$, and 12 shallow holes of type $\boldsymbol{g}_2$. Setting $L = D_6$ in (6), the glued lattice $\tilde{L}$ is one of D_6, D_6^+, $\mathbb{Z}^6$, or D_6^*, depending on the choice of Γ. In addition, (6) yields several nonlattice packings if Γ is not a group under addition modulo D_6.

TABLE II
LATTICES $\tilde{L}$ GENERATED BY GLUING $L = D_6 \times D_6$ ACCORDING TO (6). PRODUCT LATTICES AND MULTIPLE OCCURRENCES OF EQUIVALENT LATTICES ARE EXCLUDED.

Glue words Γ	Estimated NSM of $\tilde{L}$
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{11}\}$	0.071771 ± 0.000008
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{12}\}$	0.074092 ± 0.000010
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{22}\}$	0.077095 ± 0.000012
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{01}, \boldsymbol{g}_{12}, \boldsymbol{g}_{13}\}$	0.072099 ± 0.000008
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{01}, \boldsymbol{g}_{22}, \boldsymbol{g}_{23}\}$	0.075170 ± 0.000010
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{02}, \boldsymbol{g}_{11}, \boldsymbol{g}_{13}\}$	0.073558 ± 0.000009
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{02}, \boldsymbol{g}_{21}, \boldsymbol{g}_{23}\}$	0.075909 ± 0.000010
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{11}, \boldsymbol{g}_{22}, \boldsymbol{g}_{33}\}$	0.070705 ± 0.000008
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{11}, \boldsymbol{g}_{23}, \boldsymbol{g}_{32}\}$	0.070034 ± 0.000007
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{01}, \boldsymbol{g}_{10}, \boldsymbol{g}_{11}, \boldsymbol{g}_{22}, \boldsymbol{g}_{23}, \boldsymbol{g}_{32}, \boldsymbol{g}_{33}\}$	0.071753 ± 0.000008
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{01}, \boldsymbol{g}_{12}, \boldsymbol{g}_{13}, \boldsymbol{g}_{20}, \boldsymbol{g}_{21}, \boldsymbol{g}_{32}, \boldsymbol{g}_{33}\}$	0.072887 ± 0.000008
$\{\boldsymbol{g}_{00}, \boldsymbol{g}_{02}, \boldsymbol{g}_{11}, \boldsymbol{g}_{13}, \boldsymbol{g}_{20}, \boldsymbol{g}_{22}, \boldsymbol{g}_{31}, \boldsymbol{g}_{33}\}$	0.074801 ± 0.000009

The product lattice $L = D_6 \times D_6$ has 16 glue words, which are all Cartesian products $\boldsymbol{g}_{ij} \triangleq [\boldsymbol{g}_i \; \boldsymbol{g}_j]$ for $i, j = 0, 1, 2, 3$. Its Voronoi region has $2 \cdot 60 = 120$ facets and $76^2 = 5776$ vertices. All vertices are lattice vectors in $D_6^* \times D_6^*$ and are of nine different types, namely, the glue words $\boldsymbol{g}_{ij}$ for $i, j = 1, 2, 3$.

We consider all subsets Γ of the glue words $(D_6^*/D_6) \times (D_6^*/D_6)$ such that $\tilde{L}$ in (6) is a lattice, i.e., subsets that are groups under addition modulo L. There are 67 such subsets. However, several of these subsets are equivalent to each other, in the sense that a rotation and/or reflection operation in the symmetry group of L transforms all elements of one subset into the element of another subset. The relevant symmetry operations are (i) interchanging $\boldsymbol{g}_{ij}$ with $\boldsymbol{g}_{ji}$ throughout Γ, i.e. swapping the first set of 6 coordinates with the last 6; (ii) replacing all occurrences of $\boldsymbol{g}_{1j}$ with $\boldsymbol{g}_{3j}$ and vice versa; and (iii) replacing all occurrences of $\boldsymbol{g}_{i1}$ with $\boldsymbol{g}_{i3}$ and vice versa. If only inequivalent subsets of glue words are considered, 22 subsets remain. We furthermore exclude the 10 subsets that are direct products of 6-dimensional glue groups, which generate product lattices. The remaining 12 glue groups Γ are listed in Table II. The NSM of each corresponding lattice $\tilde{L}$ was estimated as in Section IV-A.

Numerical studies indicated that one lattice stands out among the dozen new lattices. When $\Gamma = \{\boldsymbol{g}_{00}, \boldsymbol{g}_{11}, \boldsymbol{g}_{23}, \boldsymbol{g}_{32}\}$, we estimate an NSM of $\hat{G} \approx 0.070034$, which is the smallest value reported to date for 12-dimensional lattices. It is slightly smaller than for the best $E_6 \times E_6$-based lattice in Section IV-A, and therefore also better than K_{12}.

With this as motivation, we again investigated the new lattice analytically. A generator matrix of an equivalent lattice is

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{13}$$

Using the same method as before, it was found that the Voronoi region has 1912 facets, 21 273 456 vertices, and 10 395 549 553 faces overall. The faces fall into 2 542 equivalence classes under its symmetry group, which has order 1 061 683 200. The face catalog is available online [29, Ancillary files]. The NSM of the new lattice is

$$G = \frac{6\,492\,178\,537\,549}{92\,704\,053\,657\,600} \approx 0.070\,031\,226 \tag{14}$$

and the covariance matrix is proportional to the identity.

The packing radius and kissing number of the lattice (13) are $1/\sqrt{2}$ and 120, respectively, which are the same as for $D_6 \times D_6$. The packing density is 0.02086, four times larger than for $D_6 \times D_6$ and the same as for the lattice (9). The covering radius is $\sqrt{3/2}$, which is a factor of $\sqrt{2}$ smaller than for $D_6 \times D_6$, and the thickness is 15.21. Like (9), the lattice generated by (13) is equivalent to its own dual.

V. Conclusions

Contrary to the common belief, there exist lattices with lower second moments than K_{12}. One such lattice is a union of three translated copies of $E_6 \times E_6$ and another is a union of four translated copies of $D_6 \times D_6$. The latter sets a new record for 12-dimensional lattice quantizers.

As a byproduct, an improved 13-dimensional lattice quantizer is obtained. The previously best published lattice for $n = 13$ is a product of K_{12} and a scaled integer lattice $a\mathbb{Z}$, with an NSM of 0.071035 [15, Tab. I]. Replacing K_{12} with the new best 12-dimensional lattice in a similar product construction yields a slightly improved NSM of 0.070974 (which is however inferior to a yet unpublished laminated lattice [27]).

Applying gluing theory to the design of lattice quantizers clearly holds great promise, and can probably lead to better quantizers in other dimensions as well. Even in 12 dimensions, the fundamental question remains open: Can even better lattice quantizers be found, by gluing theory or other methods?

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