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Baxter Q-operator for the hyperbolic Calogero–Moser system*

Martin Hallnäs

Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, SE-412 96 Gothenburg, Sweden

E-mail: hallnas@chalmers.se

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Abstract
We introduce a Q-operator $\mathcal{Q}_z$ for the hyperbolic Calogero–Moser system as a one-parameter family of explicit integral operators. We establish the standard properties of a Q-operator, i.e. invariance of Hamiltonians, commutativity for different parameter values and that its eigenvalues satisfy an explicitly given first order ordinary difference equation in the parameter z.

Keywords: Calogero–Moser systems, Baxter Q-operators, Heckman–Opdam hypergeometric functions

1. Introduction

Baxter [Bax72, Bax82] first introduced the notion of a Q-operator as a technical tool in his study of the eight vertex model, allowing him to deduce Bethe-like equations for its eigenvalues even though a Bethe ansatz for the eigenvectors was lacking. A similar approach was later developed for the periodic Toda chain, first by Gutzwiller [Gut81], who could handle the $N = 2, 3$ and 4-particle cases. After important simplifications by Sklyanin [Skl85], Pasquier and Gaudin [PG92] proposed a Q-operator for the periodic Toda chain, given as an explicit integral operator, and used it to generalise Gutzwiller's Bethe equations for the energy spectrum to all particle numbers N. By now, Q-operators have been obtained for a number of models.

To be more specific, let us consider a quantum integrable N-particle system given by N independent and pairwise commuting partial differential (or difference) operators H_r, $r = 1,\dots,N$. A corresponding Q-operator $\mathcal{Q}_z$ should depend on a parameter z and its characteristic properties usually include (see Kuznetsov and Sklyanin [KS98])

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(1) invariance of Hamiltonians,

$$[\mathcal{Q}_z, H_r] = 0;$$

(2) commutativity,

$$[\mathcal{Q}_z, \mathcal{Q}_w] = 0;$$

(3) and the fact that its eigenvalues $\phi(z)$ on a joint eigenfunction of H_r and $\mathcal{Q}_z$ satisfy an *ordinary* differential- or difference equation

$$W(z, -i\hbar \mathrm{d}/\mathrm{d}z; \{E_r\})\,\phi(z) = 0,$$

involving the corresponding eigenvalues E_r of H_r.

In this paper, we focus on the A_{N-1} hyperbolic Calogero–Moser system, which describes an arbitrary number of particles N on the line that interact pairwise through the hyperbolic potential

$$U(x_1,\ldots,x_N) = \sum_{1\leqslant i<j\leqslant N} u(x_i - x_j)\,,\quad u(x) \equiv 2g(g-\hbar)\,\mu^2/4\sinh^2(\mu x/2)\,,$$

where $\hbar \equiv h/2\pi > 0$ is the reduced Planck constant, $g > 0$ a coupling constant with dimension [action] and $\mu > 0$ a parameter with dimension $[\text{position}]^{-1}$. A complete set of independent pairwise commuting quantum integrals $H_1,\ldots,H_N$ are given by the following explicit formula:

$$H_r \equiv \frac{1}{(N-r)!}\sum_{0\leqslant s\leqslant [r/2]}\frac{(-1)^s}{4^s s!\,(r-2s)!} \cdot \sum_{\sigma\in S_N}\sigma\left(u(x_1-x_2)\cdots u(x_{2s-1}-x_{2s})\,\hat{p}_{2s+1}\cdots\hat{p}_r\right), \tag{1}$$

with the momentum operators $\hat{p}_i \equiv -\mathrm{i}\hbar\partial_{x_i}$, $i = 1,\ldots,N$; see e.g. [OP83, OS95, Rui99, HR15]. In particular, the Schrödinger operator $H \equiv -\hbar^2\Delta + U$ can be obtained as the linear combination $H_1^2 - 2H_2$.

Introducing the hyperbolic weight function

$$\mathcal{W}_N(g;x) \equiv \prod_{1\leqslant i<j\leqslant N}\left[4\sinh^2(\mu(x_i-x_j)/2)\right]^{g/\hbar}, \tag{2}$$

we recall that the quantum integrals $H_1,\ldots,H_N$ have asymptotically free joint eigenfunctions of the form

$$\Psi_N((p_1,\ldots,p_N), g; (x_1,\ldots,x_N)) = \mathcal{W}_N(g;(x_1,\ldots,x_N))^{1/2} \cdot F_N((p_1/\hbar\mu,\ldots,p_N/\hbar\mu), g/\hbar; (\mu x_1,\ldots,\mu x_N)), \tag{3}$$

with $F_N(u,\lambda;t)$ a function of the $2N+1$ dimensionless quantities

$$(u_1,\ldots,u_N) \equiv (p_1/\hbar\mu,\ldots,p_N/\hbar\mu)\,,\quad \lambda \equiv g/\hbar \tag{4}$$

and

$$(t_1,\ldots,t_N) \equiv (\mu x_1,\ldots,\mu x_N) \tag{5}$$

that is both analytic and S_N-invariant in t. (Here and below we take the positive square root of $\mathcal{W}_N$.) The corresponding eigenvalue of H_r is given by the rth symmetric function

$$S_r(p) \equiv \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant N} p_{i_1} \cdots p_{i_r}$$

of the momenta $p_1, \ldots, p_N$.

In the case of $N = 2$ variables, we have $F_2(u, \lambda; t) = \exp((u_1 + u_2)(t_1 + t_2)/2) F(u_1 - u_2, \lambda; t_1 - t_2)$, where F is essentially equal to the conical (or Mehler) function specialisation of the Gauss hypergeometric function ${}_2F_1$; see e.g. chapter 14 in [Dig10]. In the general-N case, the analog of F was identified for the parameter values $\lambda = d/2$ with $1, 2, 4$ by Olshanetsky and Perelomov [OP83] as the spherical function on $SL_n(\mathbb{F})/SU_n(\mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. The appropriate generalisation of F to arbitrary $\lambda > 0$ (and $N \geqslant 3$) is the Heckman–Opdam hypergeometric function associated with the root system A_{N-1}, first constructed and studied by Heckman and Opdam [HO87] in the context of an arbitrary root system. A corresponding generalised Fourier transform was introduced and developed by Opdam [Opd95] as well as Cherednik [Che97].

As a first result, we show that the joint eigenfunctions Ψ_N (3) satisfy a one-parameter family of integral equations, with integration kernels

$$Q_z(g; x, y) \equiv \exp\left(i \frac{z}{\hbar} \sum_{i=1}^{N} (x_i - y_i)\right) \frac{\mathcal{W}_N(g; x)^{1/2} \mathcal{W}_N(g; y)^{1/2}}{\prod_{i,j=1}^{N} [2\cosh(\mu(x_i - y_j)/2)]^{g/\hbar}} \tag{6}$$

and eigenvalues that are explicit as well. More precisely, choosing the Weyl chamber

$$G_N \equiv \left\{x \in \mathbb{R}^N \mid x_1 < x_2 < \cdots < x_N\right\},$$

we prove the following theorem.

Theorem 1.1. *Letting $z \in \mathbb{R}$ and $(p, g, x) \in \mathbb{R}^N \times [\hbar, \infty) \times \mathbb{R}^N$, we have*

$$\int_{G_N} Q_z(g; x, y) \Psi_N(p, g; y) \mathrm{d}y = \phi(p, g; z) \Psi_N(p, g; x), \tag{7}$$

with eigenvalue

$$\phi(p, g; z) = \prod_{i=1}^{N} \frac{\Gamma(g/2\hbar + \mathrm{i}(p_i - z)/\hbar\mu) \Gamma(g/2\hbar - \mathrm{i}(p_i - z)/\hbar\mu)}{\mu \Gamma(g/\hbar)}. \tag{8}$$

From the account in sections 4.1.1–2 of [HR12], it is readily infered that Q_z satisfies the so-called kernel identities

$$(H_r(x) - H_r(-y)) Q_z(x, y) = 0, \quad r = 1, \ldots, N. \tag{9}$$

For the Schrödinger operator $H \equiv -\hbar^2 \Delta + U$, such an identity was first established by Langmann [Lan00] in the more general elliptic case. The generalisation to all (symmetric elliptic) quantum integrals was later obtained by Ruijsenaars [Rui06]. Using the identities (9) as well as the manifest formal self-adjointness of H_r, we show that the left-hand side of (7) is a joint eigenfunction of the quantum integrals H_r with the same eigenvalue as Ψ_N. Combining this result with a suitable uniqueness result for regular joint eigenfunctions, due to Heckman–Opdam in the (radial) gauge corresponding to F_N, we thus arrive at an integral equation of the form (7). The explicit expression (8) for the eigenvalue is then deduced by computing the dominant asymptotic behaviour of the integral in (7) deep inside the Weyl chamber G_N. We suspect that all $g \geqslant 0$ could be allowed in theorem 1.1, but, just as in [HR15], we need the stronger assumption $g \geqslant \hbar$ in order to control all aspects of its proof.

For $z \in \mathbb{R}$, we define $\mathcal{Q}_z$ as a one-parameter family of integral operators on $L^2(G_N)$ by

$$(\mathcal{Q}_z f)(x) \equiv \int_{G_N} Q_z(g;x,y) f(y)\,\mathrm{d}y. \tag{10}$$

By combining theorem 1.1 with the pertinent generalised Fourier (or eigenfunction) transform, we readily infer that it has all three of the above Properties (1)–(3) of a Q-operator for the hyperbolic Calogero–Moser system. This constitutes our main result and its precise formulation now follows.

Theorem 1.2. *Assuming $g \in [\hbar, \infty)$ and $z \in \mathbb{R}$, the operator $\mathcal{Q}_z$, as defined by (10), is bounded and self-adjoint and satisfies the commutation relations*

$$[\mathcal{Q}_z, H_r] = 0, \quad r = 1, \ldots, N, \tag{11}$$

and

$$[\mathcal{Q}_z, \mathcal{Q}_w] = 0, \quad z, w \in \mathbb{R}. \tag{12}$$

Furthermore, introducing the generating function

$$E(\gamma;p) \equiv \prod_{i=1}^{N} (\gamma + p_i) = \sum_{r=0}^{N} \gamma^{N-r} S_r(p),$$

we have the difference equation

$$E(i\mu(\hbar - g/2) - z)\,\phi(z - i\hbar\mu) = (-1)^N E(i\mu g/2 - z)\,\phi(z). \tag{13}$$

For the trigonometric Calogero–Moser(–Sutherland) system, a Q-operator was obtained by Kuznetsov *et al* [KMS03]. They also realised their Q-operator as a one-parameter family of integral operators, with a structure similar to (10) but acting on a space of symmetric polynomials; and, in particular, proved that it is diagonalised by the Jack polynomials.

In the $N = 2$ case, the results in theorems 1.1 and 1.2 (except the difference equation (13) in the latter), as well as the main ideas behind their proofs, can all be extracted from the paper [HR18]. See also the recent preprint by Belousov *et al* [BDKK23c] for a detailed account of Q-operators for the 2-particle hyperbolic Calogero–Moser system.

While the 2-variable case can be handled using results on classical conical functions, our arguments in this paper rely on the relatively recent theory of Heckman–Opdam hypergeometric functions. On the other hand, in [HR18] we were able to obtain analogous results also for the $N = 2$ hyperbolic relativistic Calogero–Moser (or Ruijsenaars–Schneider) system. We believe that the approach presented in the present paper can be used to generalize these results on the relativistic case to all particle numbers $N > 3$, but, at the time of writing, a suitable uniqueness result for the pertinent joint eigenfunctions is missing.

In a remarkable paper, Belousov *et al* [BDKK23a] recently obtained a Q-operator for the arbitrary-N hyperbolic relativistic Calogero–Moser system; and, in the follow-up paper [BDKK23b], they established an integral equation analogous to (7) for the joint eigenfunctions of the corresponding quantum integrals constructed via an explicit recursive scheme by the author and Ruijsenaars in [HR14].

The results of Belousov *et al* are obtained using methods that are distinctly different from the ones used in this paper. Specifically, their proof of commutativity of Q-operators (i.e. Property (2)) proceeds in a more direct manner using a hypergeometric identity generalising (9); and the integral equation in question is deduced from the explicit recursive construction of the joint eigenfunctions. We note that in the above works, including the papers on the 2-variable case, parameter values analogous to $g>0$ rather than $g \geqslant \hbar$ are typically allowed.

We believe it would be interesting to further develop and compare the different methods indicated above. In particular, while Belousov *et al* already have obtained striking results at the relativistic level, the approach used here does not rely on the recursive construction of joint eigenfunctions, which is available only in the A_{N-1}-case, and could thus potentially be used to obtain interesting results on Calogero–Moser systems associated with other root systems.

We also recall that, when appropriately normalised, the joint eigenfunctions $\Psi_N(p;x)$ (3) satisfy a system of difference equations in the spectral variable p that can be interpreted as joint eigenvalue equations for quantum integrals of a rational Ruijsenaars–Schneider system. Such bispectrality results are available for arbitrary root systems and were obtained, in increasing levels of generality, by Cherednik [Che97], Chalykh [Cha00] and van Diejen and Emsiz [vDE15]. Hence it is natural to expect the existence of a dual Q-operator $\hat{\mathcal{Q}}$, acting in the spectral variable. In the $N=2$ case, such a Q-operator $\hat{\mathcal{Q}}$ can be extracted from [HR18] (see equation (177)) and is considered in some detail in [BDKK23c]. Moreover, on a formal level, a suitable degeneration of the Q-operator for the hyperbolic relativistic Calogero–Moser system, as obtained in [BDKK23a], yields a natural candidate for $\hat{\mathcal{Q}}$ in the general-N case. We note that it is not entirely clear how best to adapt the methods used here to this dual case, since a direct analogue of the Heckman-Opdam uniqueness result for joint eigenfunctions is lacking. We think it would be worthwhile to consider this problem in further detail.

The plan of the paper is as follows. In section 2, we prepare the ground for our proofs of theorems 1.1 and 1.2. Specifically, we recall the connection between the quantum integrals $H_1,\dots,H_N$ and the A_{N-1}-instance of the Heckman–Opdam hypergeometric system of PDEs as well as the asymptotic behaviour of the corresponding hypergeometric function. Section 3 contains the proof of theorem 1.1 and section 4 is devoted to the proof of theorem 1.2.

2. Joint eigenfunctions

In this section, we briefly review properties of the joint eigenfunctions Ψ_N (3) that we rely on in our proofs of theorems 1.1 and 1.2. It will be convenient to work in the (radial) F_N-gauge and with the dimensionless quantities (4) and (5).

Therefore, we consider the PDOs

$$D_r(\lambda;t) \equiv (\hbar\mu)^{-r}\left(\mathcal{W}_N^{-1/2} H_r \mathcal{W}_N^{1/2}\right)(\hbar\lambda;\mu^{-1}t), \quad r=1,\dots,N, \tag{14}$$

which generate a commutative algebra of algebraically independent PDOs, containing, in particular, the second order PDO

$$\begin{aligned} L_2 &\equiv \Delta + 2\lambda \sum_{1\leqslant i<j\leqslant N} \coth\frac{t_i-t_j}{2}\left(\frac{\partial}{\partial t_i}-\frac{\partial}{\partial t_j}\right) \\ &= 2D_2 - D_1^2 - (\rho,\rho), \end{aligned} \tag{15}$$

where $(\cdot,\cdot)$ denotes the standard bilinear form on $\mathbb{C}^N$ and $\rho=\rho(\lambda)$ the Weyl vector for 'multiplicity' λ, given by

$$\rho = \frac{\lambda}{2}(N-1, N-3, \dots, -N+3, -N+1).$$

The system of PDEs

$$D_r(t)F_N = S_r(u)F_N, \quad r = 1,\dots,N, \tag{16}$$

essentially amounts to the Heckman–Opdam hypergeometric system associated with a root system of type A_{N-1}, multiplicity parameter $\lambda \geqslant 0$ and spectral parameter $u \in \mathbb{C}^N$.

We note that the orthogonal projection of $v \in \mathbb{R}^N$ onto the hyperplane $v_1 + \cdots + v_N = 0$ in $\mathbb{R}^N$ equals

$$\pi(v) \equiv v - \frac{1}{N}(v,\underline{1})\underline{1}, \quad \underline{1} \equiv (1,\dots,1).$$

Up to a constant multiple, the hypergeometric system (16) has a unique solution of the form

$$F_N(u,\lambda;t) = \exp\left(\frac{i}{N}(u,\underline{1})(t,\underline{1})\right)F(i\pi(u),\lambda;\pi(t)) \tag{17}$$

that is symmetric and analytic in t on a suitable neighbourhood of the origin. (Here and below we extend π by linearity to all of $\mathbb{C}^N$.) If we impose the normalisation condition $F(u,\lambda;0) = 1$, the function F is precisely Heckman and Opdam's hypergeometric function of type A_{N-1} and, following the terminology of Brennecken and Rösler [BR23], we refer to the corresponding function F_N as the *extended* A_{N-1} Heckman–Opdam hypergeometric function.

We note that, although the original uniqueness result of Heckman and Opdam [HO87] (see also [HO21]) on symmetric analytic solutions of hypergeometric systems concerns F, the above extension to F_N follows as a straightforward corollary; see e.g. section 7 in [HR15] and section 2 in [BR23].

Taking $u \in \mathbb{R}^N$, we proceed to record the dominant asymptotics of the extended hypergeometric function $F_N(u,x)$ for

$$m_N(x) \equiv \max_{i=1,\dots,N-1}(x_i - x_{i+1}) \to -\infty$$

as well as a bound on the remainder, which exhibits its exponential decay; both of which are easily inferred from the asymptotic expansion of $F(\mathrm{i}u,x)$ in Weyl chambers, as established by Heckman and Opdam [HO87] (see also [Opd95]). More precisely, introducing the dominant asymptotics function

$$F_N^{\mathrm{as}}(u,\lambda;x) := \sum_{\sigma\in S_N} c(-\sigma iu,\lambda)\exp((\sigma iu + \rho, x)), \tag{18}$$

with the generalised Harish-Chandra c-function given by

$$c(v,\lambda) = \frac{\tilde{c}(v,\lambda)}{\tilde{c}(\rho,\lambda)}, \quad \tilde{c}(v,\lambda) = \prod_{1\leqslant i<j\leqslant N}\frac{\Gamma(v_i - v_j)}{\Gamma(v_i - v_j + \lambda)},$$

we have the following result.

Proposition 2.1. *Let $\delta > 0$ and assume that $\lambda \geqslant 0$ and that $u \in \mathbb{R}^N$ is regular, in the sense that $u_i - u_j \neq 0$ for all $1 \leqslant i < j \leqslant N$. Then there exists a constant $C_\delta > 0$ such that*

$$|(F_N - F_N^{\mathrm{as}})(u,\lambda;x)| < C_\delta \exp((\rho,x) + m_N(x)) \tag{19}$$

for all $x \in G_N$ with $m_N(x) < -\delta$.

Sketch of proof. Specialising theorem 6.3 in [Opd95] to a root system of type A_{N-1}, we infer the following information on the asymptotic behaviour of F_N in the Weyl chamber G_N:

$$F_N(u,\lambda;x) = \sum_{\sigma\in S_N} c(-i\sigma u,k)\,\phi(i\sigma u+\rho,\lambda;x),$$

where the generalized Harish-Chandra series ϕ is of the form

$$\phi(\xi+\rho,\lambda;x) = \exp((\xi+\rho,x))\sum_{\chi\in Q_A^+}\Delta_\chi(\xi,\lambda)\exp((\chi,x)),$$

with Q_A^+ being the $\mathbb{Z}_+$-span of the simple roots e_i-e_{i+1}, $i=1,\dots,N-1$, and the coefficients Δ_χ satisfying, in particular, the following properties: $\Delta_0\equiv 1$ and, for each $x_0\in -G_N$, there exists $K_{x_0}>0$ such that

$$|\Delta_\chi(\xi,\lambda)|\leqslant K_{x_0}\exp((\chi,x_0)),\quad \chi\in Q_A^+,\ \ \xi\in i\mathbb{R}^N.$$

Choosing $x_0\in -G_N$ such that $0<m_N(x_0)<\delta/2$, we have

$$m_N(x_0+x)\leqslant m_N(x_0)+m_N(x)<-\delta/2$$

whenever $m_N(x)<-\delta$. From these two bounds, it is now a straightforward exercise to deduce that

$$\sum_{\chi\in Q_A^+\setminus\{\underline{0}\}}|\Delta_\chi(\xi,k)|\exp(\langle\chi,x\rangle)<C_\delta\exp(m_N(x))$$

for $m_N(x)<-\delta$, and where $C_\delta>0$ can be chosen to depend continuously on δ, from which the proposition clearly follows. □

3. Integral equation

We turn now to the proof of theorem 1.1. In order to easily make use of results discussed in section 2, we substitute (3) in (7), remove the overall factor $W_N(g;x)^{1/2}$, switch to the dimensionless quantities (4)–(5) and

$$\xi\equiv z/\hbar\mu. \tag{20}$$

We are thus led to consider the integrand

$$I_\xi(u,\lambda;t,s)\equiv\exp\left(i\xi\sum_{i=1}^N(t_i-s_i)\right)K_N(\lambda;t,s)\,F_N(u,\lambda;s)\,W_N(\lambda;s), \tag{21}$$

with kernel function

$$K_N(\lambda;t,s)\equiv\prod_{i,j=1}^N\left[2\cosh\frac{t_i-s_j}{2}\right]^{-\lambda} \tag{22}$$

and weight function

$$W_N(\lambda;s)\equiv\mathcal{W}_N\left(\hbar\lambda;\mu^{-1}s\right)=\prod_{1\leqslant i<j\leqslant N}\left[4\sinh^2\frac{s_i-s_j}{2}\right]^\lambda. \tag{23}$$

Given $(u,\lambda)\in\mathbb{R}^N\times(0,\infty)$, the extended hypergeometric function F_N is known to satisfy the bound

$$|F_N(u,\lambda;s)|\leqslant C\exp((\rho,s))\prod_{1\leqslant i<j\leqslant N}(1+s_j-s_i),\quad s\in G_N, \tag{24}$$

for some constant $C>0$; see corollary 7 in [Saw08], corollary 3.1 & theorem 3.1 in [Sch08] or theorem 6.3 in [HR15]. Rewriting the weight function according to

$$W_N(s)=e^{-2(\rho,s)}\prod_{1\leqslant i<j\leqslant N}\left(1-e^{s_i-s_j}\right)^{2\lambda} \tag{25}$$

and using, in addition, the elementary estimate

$$\left|\cosh\frac{w}{2}\right|^{-\lambda}\leqslant C(\mathrm{Im}\,w)\exp\left(-\frac{\lambda}{2}|\mathrm{Re}\,w|\right),\quad |\mathrm{Im}\,w|<\pi, \tag{26}$$

where C is a continuous function on $(-\pi,\pi)$, as well as the fact that $|\rho_i|\leqslant\lambda(N-1)/2$, we thus infer the bound

$$|I_\xi(u,\lambda;t,s)|\leqslant C(\mathrm{Re}\,t,\mathrm{Im}\,t)\exp\left(-\frac{\lambda}{2}||s||_1\right)\prod_{1\leqslant i<j\leqslant N}(1+s_j-s_i),\quad s\in G_N, \tag{27}$$

with C a continuous function on $\mathbb{R}^N\times(-\pi,\pi)^N$, and where $||s||_1\equiv|s_1|+\cdots+|s_N|$. Since the singularities of the kernel function $K_N(t,s)$ are located at

$$t_i=s_j\pm i\pi(2n+1),\quad i,j=1,\ldots,N,\ \ n\in\mathbb{N},$$

it follows that the function

$$\mathscr{F}_\xi(u,\lambda;t)\equiv\int_{G_N}I_\xi(u,\lambda;t,s)\,\mathrm{d}s,\quad (u,\lambda,t)\in\mathbb{R}^N\times(0,\infty)\times\mathbb{R}^N, \tag{28}$$

is well defined and extends to a holomorphic function of t for $|\mathrm{Im}\,t_i|<\pi$, $i=1,\ldots,N$. (Indeed, using Cauchy's integral formula and the uniform bound (27), it is readily seen that we may differentiate in t under the integral sign.)

From the kernel identities (9), we now infer that $\mathscr{F}_\xi(u,\lambda;t)$ is a solution to the system of (extended) hypergeometric PDEs (16). It is at this point that we need to assume that $\lambda\geqslant 1$, or equivalently that $g\geqslant\hbar$.

Lemma 3.1. *Let $\lambda\geqslant 1$ and $u\in\mathbb{R}^N$. For $t\in\mathbb{C}^N$ with $|\mathrm{Im}\,t_i|<\pi$, $i=1,\ldots,N$, we have the joint eigenfunction property*

$$D_r(t)\mathscr{F}_\xi(u;t)=S_r(u)\mathscr{F}_\xi(u;t),\quad r=1,\ldots,N. \tag{29}$$

Proof. If we substitute $\rho(\mathrm{Re}\,\lambda)$ for ρ in (24), theorem 6.3 in [HR15] implies that the resulting bound holds true as long as $\mathrm{Re}\,\lambda>1$. It follows that $\mathscr{F}_\xi(u,\lambda;t)$ is analytic in λ for $\mathrm{Re}\,\lambda>1$, so that it suffices to prove the lemma for $\lambda>2$, say.

Using (3), (6), (14), (21) and (28), we rewrite the left-hand side of (29) as

$$\hbar^{-r}W_N(\lambda,t)^{-1/2}\int_{G_N}H_r(g,x)Q_z(g;x,y)\Psi_N(p,g;y)\,\mathrm{d}y$$

cf (4) and (5). Invoking the kernel identity (9), we obtain

$$\hbar^{-r}W_N(\lambda,t)^{-1/2}\int_{G_N}\Psi_N(p,g;y)H_r(g,-y)Q_z(g;x,y)\,\mathrm{d}y.$$

We note that $\lambda > 2$, or equivalently $g > 2\hbar$, ensures that $W(y)^{1/2}H_r(-y)W(y)^{1/2}$ is contained in $C(\mathbb{R}^N)$. Indeed, $\mathcal{W}_N(g,y)^{1/2}$ is clearly in $C^2(\mathbb{R}^N)$, each of its factors is differentiated at most twice and each pole of $u(y_i - y_j)$, $1 \leqslant i \neq j \leqslant N$, is matched by a corresponding zero (of order > 2) of $\mathcal{W}_N(g,y)^{1/2}$. Since $H_r(g,-y)$ is manifestly self-adjoint and the remaining factors in both $\Psi_N(p,g;y)$ and $Q_z(g;x,y)$ are smooth, we can thus use integration by parts to get

$$\hbar^{-r}W_N(\lambda,t)^{-1/2}\int_{G_N} Q_z(g;x,y)\,H_r(g,y)\,\Psi_N(p,g;y)\,\mathrm{d}y,$$

where $-y$ has been replaced by y due to the absence of complex conjugation. Finally, appealing to the eigenvalue property $H_r(y)\Psi_N(p;y) = S_r(p)\Psi_N$, substituting (3) and (6) and reverting back to the dimensionless quantities (4) and (5), we arrive at the right-hand side of (29). $\square$

Since $\mathscr{F}_\xi(u,\lambda;t)$ is manifestly symmetric in t and the hypergeometric function $F_N(u,\lambda;t)$ spans the space of symmetric solutions to (16) that are analytic at the origin, we can thus conclude that

$$\mathscr{F}_\xi(u,\lambda;t) = \mu_\xi(u,\lambda)\,F_N(u,\lambda;t) \tag{30}$$

for some function μ_ξ, which remains to be determined. To this end, we proceed to compute the dominant asymptotics of $\mathscr{F}_\xi(u,\lambda;t)$ for $m_N(t) \to -\infty$, which, when compared with (18), will yield an explicit expression for $\mu_\xi(u,\lambda)$.

Using the expression (25) for W_N and employing the bound (19), a simple telescoping argument yields the estimate

$$W_N(s)\,F_N(u;s) = e^{-(\rho,s)}\left[e^{-(\rho,s)}F_N^{\mathrm{as}}(u,s) + O\left(e^{m_N(s)}\right)\right],\quad m_N(s) \to -\infty. \tag{31}$$

If we introduce the function

$$\begin{aligned}\widetilde{K}_N(t,s) &\equiv \prod_{1\leqslant i\neq j\leqslant N}\left[2\cosh\frac{t_i - s_j}{2}\right]^{-\lambda} \\ &= \frac{e^{(\rho,t+s)}}{\prod_{1\leqslant i<j\leqslant N}\left(1+e^{s_i-t_j}\right)^{\lambda}\left(1+e^{t_i-s_j}\right)^{\lambda}}\end{aligned}$$

and take $\alpha \in (0,1)$, we have

$$\widetilde{K}_N(t,s) = e^{(\rho,t+s)}[1 + R_N(t,s)], \tag{32}$$

with the remainder R_N satisfying the bound

$$|R_N(t,s)| < Ce^{(1-\alpha)m_N(t)}, \tag{33}$$

for all $t,s \in G_N$ such that $\max_{i=1,\ldots,N}|s_i - t_i| < -\alpha m_N(t)$, where $C > 0$ is independent of α. Since the remaining factors in $K_N(t,s)$ decay exponentially as $|s_i - t_i| \to \infty$, this state of affairs suggests that the dominant $m_N(t) \to -\infty$ asymptotics of $\mathscr{F}_\xi(u;t)$ is obtained by performing the substitutions $W_N(s)F_N(u;s) \to e^{-2(\rho,s)}F_N^{\mathrm{as}}(u;s)$ and $K_N(t,s) \to e^{(\rho,t+s)}\prod_{i=1}^N[2\cosh\frac{t_i-s_i}{2}]^{-\lambda}$ in I_ξ. In the following lemma, we make this suggestion precise.

Lemma 3.2. *For $\lambda > 0$, we have*

$$\begin{aligned}\mathscr{F}_\xi(u,\lambda;t) = e^{(\rho,t)+\mathrm{i}\xi\sum_{i=1}^N t_i} \\ \cdot\left(\int_{\mathbb{R}^N}\frac{e^{-(\rho,s)-\mathrm{i}\xi\sum_{i=1}^N s_i}F_N^{\mathrm{as}}(u,\lambda;s)}{\prod_{i=1}^N\left[2\cosh\frac{t_i-s_i}{2}\right]^{\lambda}}\mathrm{d}s + O\left(e^{rm_N(t)/2}\right)\right),\end{aligned} \tag{34}$$

as $m_N(t) \to -\infty$, with decay rate

$$r = \min(1, \lambda/4).$$

Proof. For $t \in G_N$, we consider the domain

$$D_N(t) \equiv \left\{ s \in \mathbb{R}^N \mid |s_i - t_i| < -m_N(t)/4, \, i = 1, \ldots, N \right\} \subset G_N.$$

Note that $m_N(s) < m_N(t)/2 < 0$ whenever $t \in G_N$ and $s \in D_N(t)$. From (31) and (32)–(33), we can thus infer

$$\int_{D_N(t)} I_\xi(u;t,s)\,\mathrm{d}s = e^{(\rho,t)+\mathrm{i}\xi\sum_{i=1}^N t_i} \left(\int_{D_N(t)} \frac{e^{-(\rho,s)-\mathrm{i}\xi\sum_{i=1}^N s_i} F_N^{\mathrm{as}}(u;s)}{\prod_{i=1}^N \left[2\cosh\frac{t_i-s_i}{2}\right]^\lambda} \mathrm{d}s + O\left(e^{m_N(t)/2}\right) \right)$$

as $m_N(t) \to -\infty$.

As demonstrated after the lemma, it is desirable to arrive at the integral in the right-hand side over all of $\mathbb{R}^N$, since it can be evaluated explicitly. To this end, we observe that the function $e^{-(\rho,s)} F_N^{\mathrm{as}}(u;s)$ is bounded for $s \in \mathbb{R}^N$. Moreover, if $s \in \mathbb{R}^N \setminus D_N(t)$, then $|s_i - t_i| \geqslant -m_N(t)/4$ for at least one $i = 1, \ldots, N$ and, by the elementary estimate (26), we clearly have

$$\int_{t_i \mp m_N(t)/4}^{\pm\infty} \frac{\mathrm{d}s_i}{\left[2\cosh\frac{t_i-s_i}{2}\right]^\lambda} = O\left(e^{\lambda m_N(t)/8}\right)$$

as $m_N(t) \to -\infty$. From these observations the lemma readily follows. □

At this point we can invoke the Fourier transform formula

$$\int_{\mathbb{R}} \frac{e^{ivw}}{\left[2\cosh\frac{w}{2}\right]^\lambda} \mathrm{d}w = \frac{\Gamma(\lambda/2 + iv)\Gamma(\lambda/2 - iv)}{\Gamma(\lambda)}, \tag{35}$$

which is easily inferred from a standard integral representation for the Beta function, see equation (5.12.7) in [Dig10]. Indeed, when combined with (18), it yields

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{e^{-(\rho,s)-\mathrm{i}\xi\sum_{i=1}^N s_i} F_N^{\mathrm{as}}(u,\lambda;s)}{\prod_{i=1}^N \left[2\cosh\frac{t_i-s_i}{2}\right]^\lambda} \mathrm{d}s \\
&= \sum_{\sigma \in S_N} c(-i\sigma u, \lambda) \prod_{i=1}^N \int_{\mathbb{R}} \frac{e^{\mathrm{i}[(\sigma u)_i - \xi]s_i}}{\left[2\cosh\frac{t_i-s_i}{2}\right]^\lambda} \mathrm{d}s_i \\
&= e^{-(\rho,t)-\mathrm{i}\xi\sum_{i=1}^N t_i} F_N^{\mathrm{as}}(u,\lambda;t) \prod_{i=1}^N \frac{\Gamma(\lambda/2 + \mathrm{i}(u_i - \xi))\Gamma(\lambda/2 - \mathrm{i}(u_i - \xi))}{\Gamma(\lambda)}.
\end{aligned}$$

Substituting this expression in (34), it becomes clear from proposition 2.1 that the function μ_ξ in (30) is given by

$$\mu_\xi(u,\lambda) = \prod_{i=1}^N \frac{\Gamma(\lambda/2 + i(u_i - \xi))\Gamma(\lambda/2 - i(u_i - \xi))}{\Gamma(\lambda)}.$$

Multiplying (30) by $W_N(\lambda;t)^{1/2}$, substituting the above expression for μ_ξ and rewriting the resulting equation in terms of the momenta p and z, coupling constant g and positions x and y (cf (4), (5) and (20)), we arrive at the integral equation for the joint eigenfunctions Ψ_N given by (7) and (8). This concludes the proof of theorem 1.1.

4. *Q*-operator

Introducing the renormalised joint eigenfunctions (cf (3))

$$\begin{aligned}&\widehat{\Psi}_N\left((p_1,\dots,p_N),g;(x_1,\dots,x_N)\right)\\&\quad\equiv\widehat{\mathcal{W}}_N(g;p)^{1/2}\cdot F_N\left((p_1/\hbar\mu,\dots,p_N/\hbar\mu),g/\hbar;(\mu x_1,\dots,\mu x_N)\right)\cdot\mathcal{W}_N(g;x)^{1/2},\end{aligned}$$

where

$$\widehat{\mathcal{W}}_N(g;p)\equiv 1\big/\widehat{C}(g;p)\,\widehat{C}(g;-p),\ \widehat{C}(g;p)\equiv\prod_{1\leqslant i<j\leqslant N}\frac{\Gamma\left(\mathrm{i}\,(p_i-p_j)/\hbar\mu\right)}{\Gamma\left(g/\hbar+i\,(p_i-p_j)/\hbar\mu\right)},$$

we continue with the proof of theorem 1.2. Specifically, generalising the treatment of the $N=2$ case in [HR18], we shall make the Hilbert space properties of the Q-operator $\mathcal{Q}_z$ plain by using the fact that the generalised Fourier transform

$$\mathcal{F}_N(g):C_0^\infty(G_N)\subset L^2(G_N)\to L^2(G_N),\quad g\geqslant 0,$$

defined by

$$(\mathcal{F}_N(g)f)(x)\equiv\frac{1}{h^{N/2}}\int_{G_N}\widehat{\Psi}_N(p,g;x)f(p)\,\mathrm{d}p,$$

extends to a unitary operator on $L^2(G_N)$. The validity of this claim is readily inferred from the product structure (17) of the extended hypergeometric function F_N and the analogous result for the A_{N-1}-instance of the hypergeometric Fourier transform, first introduced and developed by Opdam [Opd95] as well as Cherednik [Che97]; see also e.g. [HO21].

Multiplying (7) by $\widehat{\mathcal{W}}_N(g;p)^{1/2}f(p)$, $f\in C_0^\infty(G_N)$, and integrating over $p\in G_N$, we may change the order of integration in the left-hand side to obtain

$$\mathcal{Q}_z\left(\mathcal{F}_N(f)\right)=\mathcal{F}_N\left(\phi(z)f\right).$$

From the well-known Gamma function properties (see chapter 5 in [Dig10])

$$\overline{\Gamma(z)}=\Gamma(\bar{z}),\quad|\Gamma(s+it)|\leqslant|\Gamma(s)|,$$

it is clear that that the eigenvalue $\phi(p;z)$ is a real-valued bounded function for $p\in\mathbb{R}^N$. Since $C_0^\infty(G_N)$ is dense in $L^2(G_N)$, boundedness and self-adjointness of $\mathcal{Q}_z$ as well as (12) clearly follow.

For $r=1,\dots,N$, the multiplication operator $f\mapsto S_rf$ is self-adjoint when equipped with the domain

$$\mathrm{Dom}(S_r)\equiv\left\{f\in L^2(G_N)\mid S_rf\in L^2(G_N)\right\}.$$

Each formal PDO H_r can thus be promoted to a self-adjoint operator $\mathcal{F}_NS_r\mathcal{F}_N^*$, whose domain $\mathrm{Dom}(H_r)=\mathcal{F}_N(\mathrm{Dom}(S_r))$. Since $\phi(\cdot;z)$ is a bounded function, it is clear that $\mathcal{Q}_z\mathrm{Dom}(H_r)\subset\mathrm{Dom}(H_r)$ and that $Q_zH_rf=H_rQ_zf$ for each $f\in\mathrm{Dom}(H_r)$, which amounts to (11).

Finally, the difference equation (13) is straightforward to establish by a direct computation using the Gamma function recurrence $\Gamma(z+1)=z\Gamma(z)$.

Data availability statement

No new data were created or analysed in this study.

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ORCID iD

Martin Hallnäs https://orcid.org/0000-0002-0496-1699

References

[Bax72] Baxter R J 1972 Partition function of the eight-vertex lattice model *Ann. Phys., NY* **70** 193–228

[Bax82] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (Academic)

[BDKK23a] Belousov N, Derkachov S, Kharchev S and Khoroshkin S 2023 Baxter operators in Ruijsenaars hyperbolic system I: commutativity of Q-operators *Ann. Henri Poincaré* (https://doi.org/10.1007/s00023-023-01364-4)

[BDKK23b] Belousov N, Derkachov S, Kharchev S and Khoroshkin S 2023 Baxter operators in Ruijsenaars hyperbolic system II: bispectral wave functions *Ann. Henri Poincaré* (https://doi.org/10.1007/s00023-023-01385-z)

[BDKK23c] Belousov N, Derkachov S, Kharchev S and Khoroshkin S 2023 Baxter Q-operators in Ruijsenaars–Sutherland hyperbolic systems: one- and two-particle cases *Zap. Nauchn. Sem.* **520** 50–123

[BR23] Brennecken D and Rösler M 2023 The Dunkl–Laplace transform and Macdonald's hypergeometric series *Trans. Am. Math. Soc.* **376** 2419–47

[Cha00] Chalykh O A 2000 Bispectrality for the quantum Ruijsenaars model and its integrable deformation *J. Math. Phys.* **41** 5139–67

[Che97] Cherednik I 1997 Inverse Harish-Chandra transform and difference operators *Int. Math. Res. Not.* **1997** 733–50

[vDE15] van Diejen J F and Emsiz E 2015 Difference equation for the Heckman–Opdam hypergeometric function and its confluent Whittaker limit *Adv. Math.* **285** 1225–40

[Dig10] Digital Library of Mathematical Functions 2010 *NIST Digital Library of Mathematical Functions* (National Institute of Standards and Technology) (Posted Online 7 May 2010) (available at: http://dlmf.nist.gov)

[Gut81] Gutzwiller M C 1981 The quantum mechanical Toda lattice. II *Ann. Phys., NY* **133** 304–31

[HR12] Hallnäs M and Ruijsenaars S N M 2012 Kernel functions and Bäcklund transformations for relativistic Calogero-Moser and Toda systems *J. Math. Phys.* **53** 123512

[HR14] Hallnäs M and Ruijsenaars S N M 2014 Joint eigenfunctions for the relativistic Calogero–Moser Hamiltonians of hyperbolic type: I first steps *Int. Math. Res. Not.* **2014** 4400–56

[HR15] Hallnäs M and Ruijsenaars S 2015 A recursive construction of joint eigenfunctions for the hyperbolic nonrelativistic Calogero–Moser Hamiltonians *Int. Math. Res. Not.* **2015** 10278–313

[HR18] Hallnäs M and Ruijsenaars S 2018 Product formulas for the relativistic and nonrelativistic conical functions *Representation Theory, Special Functions and Painlevé Equations–RIMS 2015* (*Advanced Studies in Pure Mathematics* vol 76) (Mathematical Society of Japan) pp 195–245

[HO87] Heckman G J and Opdam E M 1987 Root systems and hypergeometric functions I *Compos. Math.* **64** 329–52

[HO21] Heckman G J and Opdam E M 2021 Jacobi polynomials and hypergeometric functions associated with root systems *Encyclopedia of Special Functions: The Askey-Bateman Project* vol 2 (Cambridge University Press) ch 8

[KMS03] Kuznetsov V, Mangazeev V V and Sklyanin E K 2003 Q-operator and factorised separation chain for Jack polynomials *Indag. Math.* **14** 451–82

[KS98] Kuznetsov V B and Sklyanin E K 1998 On Bäcklund transformations for many-body systems *J. Phys. A: Math. Gen.* **31** 2241

[Lan00] Langmann E 2000 Anyons and the elliptic Calogero–Sutherland model *Lett. Math. Phys.* **54** 279–89

[OP83] Olshanetsky M A and Perelomov A M 1983 Quantum integrable systems related to Lie algebras *Phys. Rep.* **94** 313–404

[Opd95] Opdam E M 1995 Harmonic analysis for certain representations of graded Hecke algebras *Acta Math.* **175** 75–121

[OS95] Oshima T and Sekiguchi H 1995 Commuting families of differential operators invariant under the action of a Weyl group *J. Math. Sci. Univ. Tokyo* **2** 1–75

[PG92] Pasquier V and Gaudin M 1992 The periodic Toda chain and a matrix generalization of the Bessel function recursion relations *J. Phys. A: Math. Gen.* **25** 5243–52

[Rui99] Ruijsenaars S N M 1999 Systems of Calogero–Moser type *Particles and Fields* (*CRM Series in Mathematical Physics*) (Springer) pp 251–352

[Rui06] Ruijsenaars S N M 2006 Zero-eigenvalue eigenfunctions for differences of elliptic relativistic Calogero-Moser Hamiltonians *Theor. Math. Phys.* **146** 25–33

[Saw08] Sawyer P 2008 A global estimate for the Legendre function for the root systems of type A with arbitrary multiplicities (LU ZONE UL) (available at: https://zone.biblio.laurentian.ca/handle/10219/267) (Accessed 16 December 2022)

[Sch08] Schapira B 2008 Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel *Geom. Funct. Anal.* **18** 222–50

[Skl85] Sklyanin E K 1985 The quantum Toda chain *Non-Linear Equations in Classical and Quantum Field Theory* (*Lecture Notes in Physics* vol 226) ed N Sanchez (Springer)