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Beurling-Fourier algebras and complexification

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ABSTRACT

In this paper, we develop a new approach that allows to identify the Gelfand spectrum of weighted Fourier algebras as a subset of an abstract complexification of the corresponding group for a wide class of groups and weights. This generalizes recent related results of Ghandehari-Lee-Ludwig-Spronk-Turowska [11] about the spectrum of Beurling-Fourier algebras on some Lie groups. In the case of discrete groups we show that the spectrum of Beurling-Fourier algebra is homeomorphic to the group itself.

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1. Introduction

Let G be a locally compact group. The Fourier algebra $A(G)$, introduced by Eymard in [5], is a subalgebra of $C_0(G)$ consisting of the coefficients of the left regular representation λ of G, i.e.

$$A(G) = \{u \in C_0(G) \mid u(s) = \langle \lambda(s)\xi, \eta\rangle, \xi, \eta \in L^2(G)\}.$$

The natural norm on $A(G)$ that makes it a Banach algebra is given by

$$\|u\|_{A(G)} = \inf\{\|\xi\|_2\|\eta\|_2 \mid u(s) = \langle \lambda(s)\xi, \eta\rangle\},$$

where the infimum is taken over all possible representations $u(s) = \langle \lambda(s)\xi, \eta\rangle$. Moreover, $A(G)$ is the unique predual of the group von Neumann algebra $VN(G)$.

The Gelfand spectrum of the algebra is known to be topologically isomorphic to G, giving a non-trivial link between topological groups and Banach algebras. We note that the spectral theory has been an important tool in understanding commutative Banach algebras.

Several authors, including the second author, have been investigating in [18,20,23,12, 19,11] a weighted version of the Fourier algebra, by imposing a weight that changes the norm structure. Weighted Fourier algebras for compact quantum groups were studied in [8]. Recall that if G is abelian with the dual group $\hat{G}$, the Fourier algebra $A(G)$ is isometrically isomorphic via the Fourier transform to $L^1(\hat{G})$. If $w : \hat{G} \to [1, +\infty)$ is a Borel measurable and sub-multiplicative function, i.e.

$$w(st) \le w(s)w(t), \quad s, t \in \hat{G},$$

(such w is called a weight function) then $L^1(\hat{G}, w) := \{f \in L^1(G) \mid fw \in L^1(\hat{G})\}$ is a subalgebra of $L^1(G)$ and is a Banach algebra with respect to the norm $\|f\|_w := \|fw\|_1$, $f \in L^1(\hat{G}, w)$. Its image under inverse Fourier transform gives a weighted version $A(G, w)$ of $A(G)$. We note that for a weight function w on $\hat{G}$ the (unbounded) operator

$$\tilde{w} = \int_{\hat{G}}^{\oplus} w(s)ds$$

defines a closed positive operator affiliated with $L^\infty(\hat{G}) \simeq VN(G)$ which satisfies $\Gamma(\tilde{w}) \le \tilde{w} \otimes \tilde{w}$, where Γ is the comultiplication on $VN(G)$. This model has been taken in [18] and [11] to generalize the notion of weight to general locally compact groups. Accordingly, a weight, called a weight on the dual of G, is a certain unbounded positive operator $\tilde{w}$ affiliated with $VN(G)$, which, if in addition $\tilde{w}$ is bounded below, i.e. $\omega := \tilde{w}^{-1} \in VN(G)$, satisfies

$$\omega \otimes \omega = \Gamma(\omega)\Omega$$

for a contractive 2-cocycle $\Omega \in VN(G\times G)$ (see [11]). One can find numerous examples of non-trivial weights for general compact groups in [20] and certain connected Lie groups in [11].

In this paper, we will work with such weight inverse ω omitting the condition of its positivity. To each ω we will associate a subspace $A(G,\omega)$ of $A(G)$ which becomes a commutative Banach algebra with respect to a new weighted norm and the pointwise multiplication and so it is natural to study its Gelfand spectrum, spec $A(G,\omega)$. When G is compact and ω is a positive central weight, this question was studied in [20]; specific connected Lie groups, namely $SU(N)$, the Heisenberg group, the reduced Heisenberg group, the Euclidean motion group $E(2)$ and its simply connected cover, were treated in the long paper [11]. It has been proved that spec $A(G,\omega)$ is closely related to an (abstract) complexification of G. To establish this fact the strategy in [11] was to find a simpler dense subalgebra $\mathscr{A}$ so that one could easily identify its spectrum, spec $\mathscr{A}$, and get spec $A(G,\omega) \subset$ spec $\mathscr{A}$. If G is compact a natural choice is $\mathscr{A} = \text{Trig}\, G$, the algebra of matrix coefficients of finite-dimensional representations of G; spec $\mathscr{A}$ is then an abstract complexification of G, introduced by McKennon in [21], which coincides in the case of compact connected Lie groups with the universal complexification of G. For non-compact groups, it seems there is no such natural choice of the subalgebra. In [11] the construction of $\mathscr{A}$ is rather technical and each G treated in the paper required an individual approach, which heavily involved in particular the theory of group representations and technique of analytic extensions; the technicalities were an obstacle to develop a general theory applicable to any connected Lie group.

In this paper, we propose a different approach to the problem of identifying the spectrum of $A(G,\omega)$ that allows us to realise spec $A(G,\omega)$ as a subset of an abstract complexification of G for a wide class of groups and weights.

The key idea is the observation that, identifying the dual of $A(G,\omega)$ with $VN(G)$, any multiplicative linear functional corresponds to $\sigma \in VN(G)$ satisfying the same equation as the weight inverse ω, i.e. $\sigma\otimes\sigma = \Gamma(\sigma)\Omega$ for the contractive 2-cocycle $\Omega \in VN(G\times G)$ associated with ω. A simple formal calculation, which we could make to be rigorous under certain conditions, gives the equality $S(\sigma)\sigma = S(\omega)\omega$, where S is the antipode on $VN(G)$. That allows us to define a closed operator T_σ, affiliated with $VN(G)$ and satisfying $\Gamma(T_\sigma) = T_\sigma \otimes T_\sigma$ (Theorem 4.5). It is known that the set of all non-zero $T \in VN(G)$ with $\Gamma(T) = T\otimes T$ coincides with $\lambda(G) = \{\lambda(s) \mid s \in G\}$, providing the embedding of G into the spectrum of $A(G,\omega)$ through the evaluation $u \mapsto u(s) = (\lambda(s), u)$, $s \in G$; the set $G^+_{\mathbb{C},\lambda}$ of all positive solutions $T \in \overline{VN(G)}$ of $\Gamma(T) = T\otimes T$ is the image of the Lie algebra Λ of derivations

$$\Lambda = \{\alpha \in \overline{VN(G)} \mid \alpha^* = -\alpha,\ \Gamma(\alpha) = \alpha\otimes 1 + 1\otimes\alpha\}$$

under the exponential map $\alpha \mapsto \exp(i\alpha)$; $G_{\mathbb{C},\lambda} := \lambda(G) \cdot G^+_{\mathbb{C},\lambda}$ is then the space of all solutions to $\Gamma(T) = T \otimes T$ in $\overline{VN(G)}$ (Proposition 3.3). Here $\overline{VN(G)}$ is the set of unbounded operators affiliated with $VN(G)$. In many cases including connected compact and some nilpotent Lie groups G, $G_{\mathbb{C},\lambda} = \lambda_{\mathbb{C}}(G^u_{\mathbb{C}})$, where $G^u_{\mathbb{C}}$ is the universal complexification of G and $\lambda_{\mathbb{C}}$ is the extension of the left regular representation to $G^u_{\mathbb{C}}$.

The paper is organised as follows. In Section 2, we introduce the notion of a weight inverse ω on the dual of G and use this to define the Beurling-Fourier algebra $A(G, \omega)$ as a subalgebra of $A(G)$ with a modified norm and identify its dual with $VN(G)$. As $VN(G)$ has the unique predual, we show that $A(G, \omega)$ is isometrically isomorphic to $A(G)$ with a modified product $\cdot_\Omega$, depending on the 2-cocycle Ω associated with ω and not the particular weight ω. In Proposition 2.6 we give a necessary and sufficient condition for the inclusion $A(G, \omega_1) \subset A(G, \omega_2)$.

In Section 3, we review some basic concepts on unbounded operators and operators affiliated with a von Neumann algebra, and define the λ-complexification $G_{\mathbb{C},\lambda}$ of G as the set of non-zero (unbounded) closed operators T which are affiliated with $VN(G)$ and satisfy the equation $\Gamma(T) = T \otimes T$.

In Section 4 we investigate the relation that the λ-complexification has to the Gelfand spectrum of $A(G, \omega)$. We prove the embedding of spec $A(G, \omega)$ into the complexification $G_{\mathbb{C},\lambda}$ for a wide class of groups and weights; this comes down to verifying that $S(\sigma)\sigma = S(\omega)\omega$ holds for the points σ in spec $A(G, \omega)$, considered as a subset of $VN(G)$. We also give a heuristic reason why we conjecture that this holds in general. These arguments give immediately the equality for any virtually abelian group and any weight considered on it. The other cases of G and ω, for which the embedding of spec $A(G, \omega)$ into $G_{\mathbb{C},\lambda}$ holds, include, for example, compact, discrete and more general [SIN]-groups with arbitrary weights and general locally compact groups with weights extended from weights on the dual of abelian or compact subgroups. Even though we could not establish the inclusion result in full generality our approach allows us to generalise most of the previous results and avoid the main technicalities in [11] to find a dense subalgebra which plays the role of Trig G for the compact case. Moreover, as the main available source of weights on the dual of non-commutative groups are the weights induced from abelian or compact subgroups, Theorem 4.20 and Theorem 4.22 cover most of the known Beurling-Fourier algebras. For discrete group G we show that the spectrum of the corresponding Beurling-Fourier algebra is homeomorphic to G.

Finally, in Section 5, we discuss some of the questions that arose during our investigation, as well as some examples that show the necessity of certain conditions.

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2. Beurling-Fourier algebras

For a locally compact group G, we let $\lambda : G \to B(L^2(G))$ be the left regular representations on $L^2(G)$. Let $VN(G) \subseteq B(L^2(G))$ be the group von Neumann algebra, $C_r^*(G) \subseteq VN(G)$ the reduced group C^*-algebra and $W \in VN(G)\bar{\otimes}B(L^2(G))$ the fundamental multiplicative unitary, implementing the co-multiplication $\Gamma : VN(G) \to VN(G)\bar{\otimes}VN(G)$ as

$$\Gamma(x) = W^*(I \otimes x)W. \tag{1}$$

Recall that Γ is the unique normal $*$-homomorphism satisfying $\Gamma(\lambda(s)) = \lambda(s) \otimes \lambda(s)$, $s \in G$, and $W \in B(L^2(G \times G))$ is given by the action

$$(W\xi)(s,t) = \xi(ts,t), \quad \text{for } \xi \in L^2(G \times G).$$

The coproduct Γ is co-commutative and satisfies the co-associative law:

$$(\iota \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \iota) \circ \Gamma, \tag{2}$$

where ι is the identity map.

Definition *(Weight inverse).* A $\omega \in VN(G)$ will be called a *weight inverse on the dual of* G (we usually abbreviate this to a *weight inverse*) if

$$\omega\omega^* \otimes \omega\omega^* \leq \Gamma(\omega\omega^*) \tag{3}$$

and

$$\ker \omega = \ker \omega^* = \{0\}. \tag{4}$$

We note that if $\omega \in VN(G)$ is a weight inverse, then $\ker \Gamma(\omega) = \ker \Gamma(\omega^*) = \{0\}$ which follows easily from (1) and (4).

Lemma 2.1. *Let $\omega \in VN(G)$ be a weight inverse. Then there exists an injective $\Omega \in VN(G)\bar{\otimes}VN(G) \simeq VN(G \times G)$ of norm $||\Omega|| \leq 1$, such that*

$$\omega \otimes \omega = \Gamma(\omega)\Omega \tag{5}$$

and Ω satisfies the 2-cocycle relation

$$(\iota \otimes \Gamma)(\Omega)(I \otimes \Omega) = (\Gamma \otimes \iota)(\Omega)(\Omega \otimes I), \tag{6}$$

Proof. $\ker \Gamma(\omega^*) = \{0\}$ and the inequality (3) give a well-defined linear map

$$\Omega^* : \Gamma(\omega^*)x \mapsto (\omega^* \otimes \omega^*)x, \quad \text{for } x \in L^2(G) \otimes L^2(G),$$

that satisfies $||\Omega^*(\Gamma(\omega^*)x)||^2 = ||(\omega^* \otimes \omega^*)x||^2 \leq ||\Gamma(\omega^*)x||^2$. As the range $\mathrm{Ran}(\Gamma(\omega^*))$ is dense in $L^2(G \times G)$, we can extend Ω^* to a bounded linear operator on the whole Hilbert space. Let Ω be its adjoint. Clearly $||\Omega|| \leq 1$ and (5) holds.

It is easy to see from (4) and (5) that Ω must commute with any element in the commutant $(VN(G)\bar{\otimes}VN(G))'$ and thus $\Omega \in VN(G)\bar{\otimes}VN(G)$. By (5), it follows that $\ker \Omega \subset \ker(\omega \otimes \omega) = \{0\}$ and therefore Ω is injective. Using (5), we get

$$(\iota \otimes \Gamma)(\Gamma(\omega))(\iota \otimes \Gamma)(\Omega)(I \otimes \Omega) = \omega \otimes \omega \otimes \omega = (\Gamma \otimes \iota)(\Gamma(\omega))(\Gamma \otimes \iota)(\Omega)(\Omega \otimes I).$$

Finally the co-associativity of Γ and (4) imply (6). □

Remark.

(i) If $\omega \in VN(G)$ satisfies (5) then it satisfies (3):

$$\omega\omega^* \otimes \omega\omega^* = \Gamma(\omega)\Omega\Omega^*\Gamma(\omega)^* \leq \Gamma(\omega\omega^*).$$

Therefore a weight inverse could be also defined as $\omega \in VN(G)$ satisfying (4) and (5) instead.

(ii) It follows from (3) that

$$||\omega||^4 = ||\omega\omega^* \otimes \omega\omega^*|| \leq ||\Gamma(\omega\omega^*)|| = ||\omega||^2,$$

so that $||\omega||^2 \leq 1$ and hence a weight inverse is always a contraction.

(iii) In [11] a (bounded below) weight on the dual of G was defined as an (unbounded) *positive* operator w which is affiliated with $VN(G)$ and admits an inverse $w^{-1} \in VN(G)$ such that $\Gamma(w)(w^{-1} \otimes w^{-1})$ is defined and contractive on a dense subspace, i.e. w^{-1} is a positive weight inverse, in our terminology.

(iv) A weight inverse was considered in [23] as an element in the multiplier algebra $M(C_r^*(G))$ of $C_r^*(G)$ satisfying some additional density conditions. If G is compact, $M(C_r^*(G)) = VN(G)$ and our definition coincides with the one in [23].

(v) The notion of unitary dual 2-cocycle on a compact group was introduced by Landstad [17] and Wassermann [27] in the study of ergodic actions. In the context of quantum groups it was defined by Drinfeld [3]. Their 2-cocycle condition is similar and defined as follows:

$$(I \otimes \Omega)(\iota \otimes \Gamma)(\Omega) = (\Omega \otimes I)(\Gamma \otimes \iota)(\Omega). \tag{7}$$

The cocycle of the form $(u \otimes u)\Gamma(u)^{-1}$ is called a coboundary. The inverse of our 2-cocycle satisfies (7).

Example 2.2. Let w be a bounded below weight function on $G = \mathbb{R}$ or $\mathbb{Z}$ given by $w(x) = e^{i\gamma x}(1 + |x|)^\alpha$ or $w(x) = e^{i\gamma x+\beta|x|}$, $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$. It is easy to check that

$$|w(x + y)| \leq |w(x)||w(y)|, \quad \text{for all } x, y \in \mathbb{R},$$

and moreover that $w(x)^{-1}$ is bounded. As $VN(\mathbb{R}) \simeq L^\infty(\mathbb{R})$ and $VN(\mathbb{T}) \simeq \ell^\infty(\mathbb{Z})$ via the Fourier transform, the image ω of w^{-1} is a weight inverse on the dual of G.

The above weight inverses can be extended to $VN(\mathbb{R}^k \times \mathbb{T}^{n-k})$ by tensoring: $\omega = \omega_1 \otimes \ldots \otimes \omega_n$.

If a group G contains a closed subgroup H isomorphic to $\mathbb{R}^k \times \mathbb{T}^{n-k}$ then any weight inverse ω on the dual of $\mathbb{R}^k \times \mathbb{T}^{n-k}$ can be lifted to $VN(G)$ by considering $\omega_G = \iota_H(\omega)$, where $\iota_H : VN(H) \to VN(G)$ is the injective homomorphism $\lambda_H(s) \mapsto \lambda_G(s)$, here λ_G and λ_H are the left regular representations of G and H respectively; the existence of ι_H is due to Herz's restriction theorem, see for example [14].

For other examples of weights and weight inverses, we refer the reader to [11].

Let $A(G)$ be the unique pre-dual of $VN(G)$. Recall that it can be identified with the space of functions on G:

$$A(G) = \{g * \check{h} \mid g, h \in L^2(G)\} \subset C_0(G),$$

where $\check{h}(s) = h(s^{-1})$, $s \in G$, and

$$g * \check{h} = \int g(t)h(s^{-1}t)dt = \langle \lambda(s)h, \bar{g}\rangle;$$

$A(G)$ becomes a commutative Banach algebra, usually called the Fourier algebra of G, with respect to the pointwise multiplication and the norm given by

$$\|f\|_{A(G)} = \inf \|g\|_2\|h\|_2,$$

where the infimum is taken over all possible decomposition $f = g * \check{h}$, see for example [5,16]. The duality between $VN(G)$ and $A(G)$ is given by

$$(T, u) = \langle T\xi, \eta\rangle$$

for $T \in VN(G)$ and $u(s) = \langle \lambda(s)\xi, \eta\rangle = (\bar{\eta} * \check{\xi})(s) \in A(G)$; here and through the rest of the paper we use $\langle \cdot, \cdot\rangle$ to denote the inner product on a Hilbert space and we keep notation $(\cdot, \cdot)$ for duality pairing between $\mathcal{M}$ and $\mathcal{M}_*$ when $\mathcal{M}$ is a von Neumann algebra.

For $T \in VN(G)$ and $f \in A(G)$, we let $Tf \in A(G)$ be given by

$$(R, Tf) := (RT, f), \quad \text{for } R \in VN(G).$$

The assignment $T, f \mapsto Tf$ turns $A(G)$ into a left $VN(G)$-module.

If ω is a weight inverse, we define

$$A(G,\omega) := \omega A(G) = \{\omega f \mid f \in A(G)\} \subset A(G)$$

and call it the *Beurling-Fourier algebra* of G associated to ω.

Proposition 2.3. *$A(G,\omega)$ is a Banach algebra with respect to the pointwise multiplication and the norm*

$$||\omega f||_\omega := ||f||_{A(G)}.$$

Moreover, $A(G,\omega)$ is a predual of $VN(G)$ with the pairing given by

$$\begin{gathered}(\cdot,\cdot)_\omega : VN(G) \times A(G,\omega) \to \mathbb{C}, \\ (T,\omega f)_\omega = (T,f).\end{gathered} \tag{8}$$

Proof. To see that $||\cdot||_\omega$ is a norm, we should only see that it is well defined. In fact, if $f = \bar{\eta} * \check{\xi}$ then

$$(\lambda(s),\omega f) = \langle \lambda(s)\omega\xi, \eta\rangle = \langle \omega\xi, \lambda(s^{-1})\eta\rangle, \quad \text{for } s \in G.$$

Let $\mathcal{U} = \overline{[\lambda(s)\eta \mid s \in G]}$, the closed linear span of $\lambda(s)\eta$, $s \in G$, and let P be the projection onto $\mathcal{U}$. As $\mathcal{U}$ is invariant with respect to $VN(G)$, we have $P\omega = \omega P$. Assuming now that $\omega f = 0$, we obtain $\langle \omega\xi, \lambda(s^{-1})\eta\rangle = 0$ for any $s \in G$, and hence $\omega P\xi = P\omega\xi = 0$. By (4), $P\xi = 0$ and hence $f(s) = \langle \xi, \lambda(s^{-1})\eta\rangle = 0$ for any $s \in G$. From (5) it follows that $A(G,\omega)$ is a commutative Banach algebra; in fact, we have

$$(\omega u)(\omega v) = \omega(\Gamma_*(\Omega(u \otimes v))), \quad \text{for } u, v \in A(G),$$

and

$$\|(\omega u)(\omega v)\|_\omega = \|\Gamma_*(\Omega(u \otimes v)))\|_{A(G)} \le \|u\|_{A(G)}\|v\|_{A(G)} = \|\omega u\|_\omega \|\omega v\|_\omega, \tag{9}$$

where $\Gamma_* : A(G)\widehat{\otimes}A(G) \to A(G)$ is the predual of the co-multiplication Γ defined on the operator space projective product of $A(G)\widehat{\otimes}A(G)$ (see [4]). The associativity of the product is clear. The completeness of $A(G,\omega)$, as well as that it is a predual of $VN(G)$, is obvious from it being linearly isometrically isomorphic to $A(G)$. □

We note that the previous proposition was proved in [11] for positive weight inverses. Similar arguments can be applied to prove the general case. For the reader's convenience, we have chosen to give its full proof.

The next statement shows that we can restrict ourselves to positive weight inverses.

Proposition 2.4. *If ω is a weight inverse and $\omega^* = U|\omega^*|$ is the polar decomposition of ω^*, then $|\omega^*|$ is a weight inverse and the identity map $\omega u \mapsto \omega u = |\omega^*|(U^*u)$, $u \in A(G)$, defines an isometric isomorphism $A(G,\omega) \to A(G,|\omega^*|)$.*

Proof. Note that $U \in VN(G)$ is unitary by (4). From (5) it is immediate that $|\omega^*|$ is again a weight inverse. Clearly $A(G,\omega) = A(G,|\omega^*|)$ as subsets of $A(G)$, and the identity is an algebra homomorphism. Moreover

$$\|\omega u\|_{|\omega^*|} = \||\omega^*|(U^*u)\|_{|\omega^*|} = \|U^*u\|_{A(G)} = \|u\|_{A(G)} = \|\omega u\|_\omega. \quad \square$$

We will use the following lemma:

Lemma 2.5. *If $\mathcal{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra and $a_1, a_2 \in \mathcal{M}$ satisfy*

$$a_1\mathcal{M}_* \subseteq a_2\mathcal{M}_*,$$

then there is $c \in \mathcal{M}$ such that $a_1 = a_2c$. Moreover, we can assume that $\ker c = \ker a_1$*, and* $\ker a_2 \subseteq \ker c^*$*, and under these assumptions, c is uniquely determined.*

Proof. Let $a_i = S_i|a_i|$, for $i = 1, 2$, be the polar decompositions, and let $P_i = S_i^*S_i$, so that $a_iP_i = a_i$. For $i = 1, 2$, the maps $P_i\mathcal{M}_* \to a_i\mathcal{M}_*$, defined as $f \mapsto a_if$, are bijective linear maps. As $a_1\mathcal{M}_* \subseteq a_2\mathcal{M}_*$, there is for every $f \in P_1\mathcal{M}_*$ a unique $h(f) \in P_2\mathcal{M}_*$ such that $a_1f = a_2h(f)$. Let $R(f) = h(f)$. Clearly R is a linear injective map and moreover for $b \in \mathcal{M}$, we have $R(fb) = R(f)b$. Note that $P_i\mathcal{M}_*$, $i = 1, 2$, are closed subspaces of $\mathcal{M}_*$, thus Banach spaces. We claim that R is closed: let f_n be a sequence such that $f_n \to f$ and $R(f_n) \to h$ as $n \to \infty$. Then

$$a_1f = \lim_{n\to\infty} a_1f_n = \lim_{n\to\infty} a_2R(f_n) = a_2h,$$

so that $R(f) = h$. As R is defined on the whole $P_1\mathcal{M}_*$, it is thus bounded. Extend R to all of $\mathcal{M}_* \cong (I - P_1)\mathcal{M}_* \oplus P_1\mathcal{M}_*$ by the formula $R(f) = R(P_1f)$. Clearly, for this extension we still have $a_2R(f) = a_1f$, as well as $R(fb) = R(f)b$ for all $b \in \mathcal{M}$. Let $R' : \mathcal{M} \to \mathcal{M}$ be the dual of R. Then as

$$(mR'(b), f) = (R'(b), fm) = (b, R(fm)) = (b, R(f)m) = (mb, R(f)) = (R'(mb), f)$$

for all $b, m \in \mathcal{M}$ and $f \in \mathcal{M}_*$, it follows $R'(mb) = mR'(b)$. Thus with $c = R'(I) \in \mathcal{M}$, we have $R'(b) = bc$. We get

$$(b, R(f)) = (R'(b), f) = (bc, f) = (b, cf), \quad \text{for all } b \in \mathcal{M} \text{ and } f \in \mathcal{M}_*,$$

so that $R(f) = cf$. It gives $a_1f = a_2R(f) = a_2cf$, and thus $a_1 = a_2c$. Clearly, $\ker c = \ker a_1$, $\ker c^* \supseteq \ker I - P_2 = (\ker a_2)^\perp$ and that c is the unique element such that $a_1 = a_2c$ with these properties. $\square$

Proposition 2.6. *Let ω_1, ω_2 be two weight inverses on the dual of G. The inclusion $A(G, \omega_1) \subseteq A(G, \omega_2)$ implies that there is $a \in VN(G)$ such that $\omega_1 = \omega_2 a$. Furthermore, we have $A(G, \omega_1) = A(G, \omega_2)$ if and only if $\omega_1 = \omega_2 a$ for an invertible element $a \in VN(G)$.*

Proof. It follows from Lemma 2.5 that if $A(G, \omega_1) \subseteq A(G, \omega_2)$, then there is an $a \in VN(G)$ such that $\omega_1 = \omega_2 a$. Moreover, if actually $A(G, \omega_1) = A(G, \omega_2)$, then we get $a, b \in VN(G)$ such that $\omega_1 = \omega_2 a$ and $\omega_2 = \omega_1 b$. It then follows that $\omega_1(I - ba) = 0$ and $\omega_2(I - ab) = 0$ and as $\ker \omega_i = \{0\}$ for $i = 1, 2$, we get $ba = ab = I$, so that a is invertible. □

Another equivalent model of the Beurling-Fourier algebra, which was given in [11] for positive weights, is defined as follows. For a weight inverse ω and the corresponding 2-cocycle Ω define a new multiplication on $A(G)$ by

$$u \cdot_\Omega v = \Gamma_*(\Omega(u \otimes v)), \quad \text{for } u, v \in A(G). \tag{10}$$

It follows from (9) that $(A(G), \cdot_\Omega)$ is a commutative contractive Banach algebra which is isomorphic to $A(G, \omega)$, showing that $A(G, \omega)$ can be determined by the 2-cocycle Ω rather than the weight inverse ω. Assume $A(G, \omega_1) = A(G, \omega_2)$ and let $a \in VN(G)$ be the invertible operator such that $\omega_1 = \omega_2 a$ which exists due to Proposition 2.6. If Ω_1 and Ω_2 are the corresponding 2-cocycles, then

$$\Gamma(a)\Omega_1 = \Omega_2(a \otimes a) \tag{11}$$

and $(A(G), \cdot_{\Omega_1}) \simeq (A(G), \cdot_{\Omega_2})$. The converse also holds: if Ω_1, $\Omega_2 \in VN(G \times G)$ are 2-cocycles that satisfy (11) and correspond to weight inverses ω_1 and ω_2 respectively, then $u \mapsto au$, $u \in A(G)$, gives the isometric isomorphism $(A(G), \cdot_{\Omega_1}) \simeq (A(G), \cdot_{\Omega_2})$. To see this let u, $v \in A(G)$ and $x \in VN(G)$. Then

$$\begin{aligned}(x, a(u \cdot_{\Omega_1} v)) &= (xa, u \cdot_{\Omega_1} v) = (\Gamma(xa), \Omega_1(u \otimes v)) = (\Gamma(x)\Gamma(a)\Omega_1, u \otimes v) \\ &= (\Gamma(x)\Omega_2(a \otimes a), u \otimes v) = (\Gamma(x), \Omega_2(au \otimes av)) = (x, au \cdot_{\Omega_2} av).\end{aligned}$$

If a is not assumed to be invertible, the map $u \mapsto au$ gives a homomorphism from $(A(G), \cdot_{\Omega_1})$ to $(A(G), \cdot_{\Omega_2})$. We note that any 2-cocycle associated with a weight inverse is symmetric, that is invariant under the 'flip' automorphism $a \otimes b \mapsto b \otimes a$ of $VN(G)\bar{\otimes}VN(G)$.

We finish this section by defining a representation of $(A(G), \cdot_\Omega)$.

Recall the fundamental unitary $W \in VN(G)\bar{\otimes}B(L^2(G))$ and let $f \in A(G)$, $f(\cdot) = \langle \lambda(\cdot)\xi, \eta \rangle$. Then for x, $y \in L^2(G)$ we have

$$\langle (f \otimes \iota)(W)x, y \rangle = \langle W(\xi \otimes x), \eta \otimes y \rangle = \int\limits_{G \times G} \xi(ts)x(t)\overline{\eta(s)y(t)}dtds =$$

$$= \int_G (\int_G \xi(ts)\overline{\eta(s)}ds)x(t)\overline{y(t)}dt = \int_G \langle\lambda(t)^{-1}\xi, \eta\rangle x(t)\overline{y(t)}dt = \langle M_{\check{f}}x, y\rangle,$$

where $\check{f}(t) = f(t^{-1})$ and $M_{\check{f}}$ is the multiplication operator by $\check{f}$.

For $X \in B(L^2(G) \otimes L^2(G))$ write $X_{12} = X \otimes I$ and $X_{23} = I \otimes X$ for operators on $L^2(G) \otimes L^2(G) \otimes L^2(G)$ and define X_{13} similarly. Then W satisfies the pentagonal relation

$$W_{23}W_{12} = W_{12}W_{13}W_{23}. \tag{12}$$

For $f \in A(G)$ define $\lambda_\Omega(f) = (f \otimes \iota)(W\Omega)$.

Lemma 2.7. *The map $f \mapsto \lambda_\Omega(f)$ is a representation of $(A(G), \cdot_\Omega)$ on $B(L^2(G))$, i.e.*

$$\lambda_\Omega(f \cdot_\Omega g) = \lambda_\Omega(f)\lambda_\Omega(g) \text{ for all } f, g \in A(G).$$

Moreover,

$$\omega\lambda_\Omega(f) = M_{\check{\omega f}}\omega, f \in A(G). \tag{13}$$

Proof. Let f, $g \in A(G)$ and ξ, $\eta \in L^2(G)$. Write $\psi_{\xi,\eta}$ for the vector functional given by $\psi_{\xi,\eta}(T) = \langle T\xi, \eta\rangle$, $T \in B(L^2(G))$. Then

$$\begin{aligned}
\langle\lambda_\Omega(f \cdot_\Omega g)\xi, \eta\rangle &= (((f \cdot_\Omega g) \otimes \iota)(W\Omega), \psi_{\xi,\eta}) = ((\Gamma \otimes \iota)(W\Omega), \Omega(f \otimes g) \otimes \psi_{\xi,\eta}) \\
&= ((\Gamma \otimes \iota)(W)(\Gamma \otimes \iota)(\Omega)(\Omega \otimes I), f \otimes g \otimes \psi_{\xi,\eta}) \\
&\overset{(6)}{=} ((\Gamma \otimes \iota)(W)(\iota \otimes \Gamma)(\Omega)(I \otimes \Omega), f \otimes g \otimes \psi_{\xi,\eta}) \\
&= (W_{12}^*W_{23}W_{12}(\iota \otimes \Gamma)(\Omega)(I \otimes \Omega), f \otimes g \otimes \psi_{\xi,\eta}) \\
&\overset{(12)}{=} (W_{13}W_{23}(\iota \otimes \Gamma)(\Omega)(I \otimes \Omega), f \otimes g \otimes \psi_{\xi,\eta}) \\
&= (W_{13}W_{23}W_{23}^*\Omega_{13}W_{23}\Omega_{23}, f \otimes g \otimes \psi_{\xi,\eta}) \\
&= (W_{13}\Omega_{13}W_{23}\Omega_{23}, f \otimes g \otimes \psi_{\xi,\eta}) \\
&= \langle(f \otimes \iota)(W\Omega)(g \otimes \iota)(W\Omega), \xi, \eta\rangle = \langle\lambda_\Omega(f)\lambda_\Omega(g)\xi, \eta\rangle,
\end{aligned}$$

where the first equality in the last line can be seen on elementary tensors and using then linearity and density arguments. In fact, if $X = a \otimes b$, $Y = c \otimes d$, $f(s) = \langle\lambda(s)\xi_1, \eta_1\rangle$ and $g(s) = \langle\lambda(s)\xi_2, \eta_2\rangle$, then

$$\begin{aligned}
&(X_{13}Y_{23}, f \otimes g \otimes \psi_{\xi,\eta}) = \langle(a \otimes I \otimes b)(I \otimes c \otimes d)\xi_1 \otimes \xi_2 \otimes \xi, \eta_1 \otimes \eta_2 \otimes \eta\rangle \\
&= \langle a\xi_1, \eta_1\rangle\langle c\xi_2, \eta_2\rangle\langle bd\xi, \eta\rangle = \langle(f \otimes \iota)(X)(g \otimes \iota)(Y)\xi, \eta\rangle.
\end{aligned}$$

The formula (13) follows from the following calculations

$$\omega\lambda_\Omega(f) = (f\otimes\iota)((I\otimes\omega)W\Omega) = (f\otimes\iota)(W\Gamma(\omega)\Omega)$$
$$= (f\otimes\iota)(W\omega\otimes\omega) = (\omega f\otimes\iota)(W)\omega = M_{\check{\omega f}}\omega. \quad \square$$

3. Complexification of G

3.1. Preliminaries on unbounded operators

We start with some basic material on unbounded operators which will be used in the paper. Our main reference is [24].

Let $\mathscr{H}$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. Recall that a linear operator T defined on a subspace $\mathscr{D}(T)\subset\mathscr{H}$, called a domain of T, is said to be closed if the graph of T, $\{(\xi,T\xi)\mid\xi\in\mathscr{H}\}$, is closed in $\mathscr{H}\oplus\mathscr{H}$. Given linear operators T and S, we write $T\subset S$ if $\mathscr{D}(T)\subset\mathscr{D}(S)$ and $S|_{\mathscr{D}(T)}=T$; we say that S is an extension of T. We have $T=S$ if $T\subset S$ and $S\subset T$. A linear operator T is called closable if it has a closed extension. Clearly, T is closable if and only if the conditions $(\xi_n)_n\in\mathscr{D}(T)$, $\eta\in\mathscr{H}$, $\|\xi_n\|\to 0$ and $\|T\xi_n-\eta\|\to 0$ imply $\eta=0$. The minimal closed extension of a closable T exists and will be denoted by $\bar{T}$. We say that a subspace $\mathscr{U}\subseteq\mathscr{D}(T)$ is a *core* for T if for any $\xi\in\mathscr{D}(T)$, there is a sequence $(\xi_n)_n\subset\mathscr{U}$, such that $\xi_n\to\xi$ and $T\xi_n\to T\xi$. Equivalently, the subspace $\{(\xi,\ T\xi)\mid\xi\in\mathscr{U}\}\subseteq\mathscr{H}\oplus\mathscr{H}$ is dense in the graph of T.

If T is an operator with a dense domain it has a well-defined adjoint operator $T^*:\mathscr{D}(T^*)\to\mathscr{H}$, which is always a closed operator. An operator T is called selfadjoint if $T=T^*$; a selfadjoint operator is positive if it has a nonnegative spectrum. T is essentially selfadjoint if $\bar{T}$ is selfadjoint.

Any selfadjoint T has a spectral measure E_T on the σ- algebra $\mathscr{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$, and

$$T=\int_{\operatorname{spec}T} t\,dE_T(t);$$

if f is a Borel measurable function, we write $f(T)$ for the operator

$$f(T)=\int_{\operatorname{spec}T} f(t)\,dE_T(t),\quad \mathscr{D}(f(A))=\{\xi\in H\mid \int_{\operatorname{spec}T}|f(t)|^2\,d(E(t)\xi,\xi)<\infty\}.$$

If T is a closed operator with dense domain, then T^*T is positive and T has the polar decomposition $T=U|T|$, where $|T|=(T^*T)^{1/2}$ and U is a partial isometry; $|T|$, T and U have the identical initial projections.

We say that a closed operator T defined on a dense domain $\mathscr{D}(T)\subseteq\mathscr{H}$ is *affiliated* with a von Neumann algebra $\mathscr{M}$ of $B(\mathscr{H})$ if $UT\subset TU$ for any unitary operator $U\in\mathscr{M}'$, where $\mathscr{M}'$ as usually stands for the commutant of $\mathscr{M}$. Note that if $T=U|T|$ is the polar decomposition of T then T is affiliated with $\mathscr{M}$ if and only if $U\in\mathscr{M}$ and $|T|$ is affiliated with $\mathscr{M}$, the latter is equivalent that the spectral projections $E_{|T|}(\Delta)$, $\Delta\in\mathscr{B}(\mathbb{R})$, of $|T|$

belong to $VN(G)$. We denote by $\overline{VN(G)}$ the set of all affiliated with $VN(G)$ elements. We write $\overline{VN(G)}^+$ for the set of positive operators in $\overline{VN(G)}$. If $T = T^*$ is affiliated with $VN(G)$, i.e. $E_T(\Delta) \in VN(G)$ for any $\Delta \in \mathscr{B}(\mathbb{R})$, then $f(T) \in \overline{VN(G)}$ for any Borel function f on $\mathbb{R}$.

Let A be a linear operator on $\mathscr{H}$. A vector φ in $\mathscr{H}$ is called analytic for A if $\varphi \in \mathscr{D}(A^n)$ for all $n \in \mathbb{N}$ and if there exists a constant M (depending on φ) such that

$$\|A^n\varphi\| \le M^n n! \quad \text{for all } n \in \mathbb{N}.$$

We write $\mathscr{D}_\omega(A)$ for the set of all analytic vectors of A. If A is selfadjoint with $E_A(\cdot)$ being the spectral measure of A, then $E_A(\Delta)\varphi$ is analytic for A for any $\varphi \in \mathscr{H}$ and any bounded $\Delta \in \mathfrak{B}(\mathbb{R})$, as

$$\|A^n E_A(\Delta)\varphi\| \le M^n\|\varphi\|,$$

if $\Delta \subset [-M, M]$.

It is known (see e.g. [24, Proposition 10.3.4]) that if T is a symmetric operator, i.e. $T \subset T^*$, with a dense set of analytic vectors, then T is essentially selfadjoint.

If $\mathscr{U}$ and $\mathscr{V}$ are subspace of $\mathscr{H}$ we write $\mathscr{U} \odot \mathscr{V}$ for the algebraic tensor product of the subspaces; $\mathscr{H}_1 \otimes \mathscr{H}_2$ is the usual Hillbertian tensor product of two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$.

If T_1, T_2 are closed densely defined operator with the domains $\mathscr{D}(T_1)$ and $\mathscr{D}(T_2) \subset \mathscr{H}$ respectively, then the operator $T_1 \otimes T_2$ with domain $\mathscr{D}(T_1) \odot \mathscr{D}(T_2)$ is closable. In fact, if $\xi_n \to 0$, where $\xi_n \in \mathscr{D}(T_1) \odot \mathscr{D}(T_2)$, and $(T_1 \otimes T_2)\xi_n \to \eta$ then for any $f \in \mathscr{D}(T_1^*) \odot \mathscr{D}(T_2^*)$, we have $\langle (T_1 \otimes T_2)\xi_n, f\rangle = \langle \xi_n, (T_1^* \otimes T_2^*)f\rangle \to 0$, giving $\langle \eta, f\rangle = 0$. As each T_i is closed, $\mathscr{D}(T_i^*)$ is dense in $\mathscr{H}$ and hence $\mathscr{D}(T_1^*) \odot \mathscr{D}(T_2^*)$ is dense in $\mathscr{H} \otimes \mathscr{H}$, showing that $\eta = 0$ and that $T_1 \otimes T_2$ is closable. Unless otherwise stated we will write $T_1 \otimes T_2$ for the corresponding closure.

We say that two selfadjoint operators T_1, T_2 strongly commute, if

$$E_{T_1}(\Delta_1)E_{T_2}(\Delta_2) = E_{T_2}(\Delta_2)E_{T_1}(\Delta_1), \quad \text{for all } \Delta_1, \Delta_2 \in \mathscr{B}(\mathbb{R}),$$

where $E_{T_i}(\cdot)$ is the spectral measure of T_i. We define a product spectral measure $E_{T_1} \times E_{T_2} : \mathscr{B}(\mathbb{R}^2) \to \mathscr{B}(\mathscr{H})$ by letting $E_{T_1} \times E_{T_2}(\Delta_1 \times \Delta_2) = E_{T_1}(\Delta_1)E_{T_2}(\Delta_2)$ for Borel measurable rectangle $\Delta_1 \times \Delta_2$. If $f : \mathbb{R}^2 \to \mathbb{C}$ is Borel measurable we set

$$f(T_1, T_2) = \int_{\mathbb{R}^2} f(x_1, x_2)\, dE_{T_1} \times E_{T_2}(x_1, x_2).$$

It is a selfadjoint operator if f is real-valued.

Let T_i be selfadjoint operators, $i = 1, 2$. Then $T_1 \otimes 1$ and $1 \otimes T_2$ are selfadjoint operators that commute strongly. Then $T_1 \otimes T_2$ is selfadjoint and

$$T_1 \otimes T_2 = f(T_1 \otimes 1, 1 \otimes T_2),$$

where $f(x_1, x_2) = x_1 x_2$. Observe that $T_1 \otimes T_2$ is essentially selfadjoint on $\mathscr{D}(T_1) \odot \mathscr{D}(T_2)$, as $\mathscr{D}_\omega(T_1) \odot \mathscr{D}_\omega(T_2)$ is a dense subset of $\mathscr{H}_1 \otimes \mathscr{H}_2$ and consists of analytic vectors for $T_1 \otimes T_2$.

For closed densely defined operators S_1, S_2 with polar decomposition $S_i = U_i|S_i|$, $i = 1, 2$, we have $S_1 \otimes S_2 = (U_1 \otimes U_2)(|S_1| \otimes |S_2|)$ is the polar decomposition of the closed operator $S_1 \otimes S_2$.

3.2. λ-complexification of a locally compact group

Let G be a locally compact group and let W be the fundamental multiplicative unitary on $L^2(G \times G)$ implementing the coproduct Γ on $VN(G)$. We can extend Γ to $\overline{VN(G)}$ by defining

$$\Gamma(T) = W^*(1 \otimes T)W, \quad \text{for } T \in \overline{VN(G)}.$$

Clearly, the unbounded operator $\Gamma(T)$ is closed. If $T^* = T$ and $E_T(\cdot)$ is the spectral measure of T, then both operators $1 \otimes T$ and $\Gamma(T)$ are selfadjoint with $1 \otimes E_T(\cdot)$ and $\Gamma \circ E_T(\cdot)$ being the corresponding spectral measures. In particular,

$$\Gamma(T) = \int_{\mathbb{R}} t \, d(\Gamma \circ E_T(t)).$$

If $T = U|T|$ is the polar decomposition of T, $\Gamma(T) = \Gamma(U)\Gamma(|T|)$ is the polar decomposition of $\Gamma(T)$.

Definition. By the λ-complexification $G_{\mathbb{C},\lambda}$ of G we shall mean the set of all non-zero (unbounded) operators $T \in \overline{VN(G)}$ such that

$$\Gamma(T) = T \otimes T. \tag{14}$$

We note that $G_{\mathbb{C},\lambda} \cap VN(G) = \{T \in VN(G) \,|\, \Gamma(T) = T \otimes T, T \neq 0\} = \lambda(G)$, see e.g. [25, Chapter 11, Theorem 16] giving an embedding of G into $G_{\mathbb{C},\lambda}$.

Let

$$\Lambda = \{\alpha \in \overline{VN(G)} \mid \alpha^* = -\alpha, \ \alpha \otimes 1 + 1 \otimes \alpha = \Gamma(\alpha)\}. \tag{15}$$

As for $\alpha \in \Lambda$, the operators $i\alpha \otimes 1$ and $1 \otimes i\alpha$ are selfadjoint and strongly commute, the sum $i\alpha \otimes 1 + 1 \otimes i\alpha$, defined via the functional calculus, gives a selfadjoint operator; in (15) we require it to be equal to the selfadjoint operator $\Gamma(i\alpha)$.

If $\alpha \in \Lambda$, we will define $\exp z\alpha$, $z \in \mathbb{C}$, through functional calculus, i.e.

$$\exp z\alpha = \exp(-iz(i\alpha)) = \int_{\mathbb{R}} \exp(-izt)\, dE_{i\alpha}(t).$$

Proposition 3.1. *For $\alpha \in \Lambda$ and $z \in \mathbb{C}$, $\exp z\alpha \in G_{\mathbb{C},\lambda}$.*

Proof. It follows from the functional calculus and definition of Γ that

$$\begin{aligned}\Gamma(\exp z\alpha) &= W^*(1 \otimes \exp z\alpha)W = W^* \int_{\mathbb{R}} \exp(-izt)\, dE_{1\otimes i\alpha}(t) W \\ &= \int_{\mathbb{R}} \exp(-izt)\, dE_{W^*(1\otimes i\alpha)W}(t) = \exp z\Gamma(\alpha) \\ &= \exp z(\alpha \otimes 1 + 1 \otimes \alpha) \\ &= \exp(z\alpha \otimes 1) \exp(1 \otimes z\alpha) = (\exp z\alpha \otimes 1)(1 \otimes \exp z\alpha) \\ &= \exp z\alpha \otimes \exp z\alpha. \quad \square\end{aligned}$$

Proposition 3.2. *The map $\alpha \in \Lambda \mapsto \exp i\alpha$ is a bijection onto $G_{\mathbb{C},\lambda} \cap \overline{VN(G)}^+$.*

Proof. That $\exp i\alpha$ is positive and affiliated with $VN(G)$ follows from the functional calculus.

Let $T \in G_{\mathbb{C},\lambda} \cap \overline{VN(G)}^+$. Let P be the projection onto the closure of the range of T. Then $\Gamma(P) = P \otimes P$ and hence, as noted above, $P \in \lambda(G)$. Therefore, as $T \neq 0$, $P = I$ and hence the range of T is dense and T^{it}, $t \in \mathbb{R}$, is a well-defined unitary operator. Using arguments similar to those in the proof of the previous proposition we obtain

$$\Gamma(T^{it}) = W^*(1 \otimes T^{it})W = \Gamma(T)^{it} = (T \otimes T)^{it} = T^{it} \otimes T^{it}, \quad \text{for } t \in \mathbb{R}.$$

Let $A = \int_{\mathbb{R}^+} \ln t\, dE_T(t)$. Then $T^{it} = \exp itA$ and

$$\exp it\Gamma(A) = \Gamma(\exp itA) = \exp itA \otimes \exp itA = \exp it(A \otimes 1 + 1 \otimes A), \quad \text{for } t \in \mathbb{R}.$$

By Stone's theorem about infinitesimal generator of a strongly continuous unitary group, we obtain $\Gamma(A) = A \otimes 1 + 1 \otimes A$. Set $\alpha = -iA$. Then $T = \exp i\alpha$. $\square$

Proposition 3.3.

$$G_{\mathbb{C},\lambda} = \{\lambda(s) \exp i\alpha \mid \alpha \in \Lambda,\ s \in G\}.$$

Proof. As it was noticed before, if $T = U|T|$ is the polar decomposition of $T \in G_{\mathbb{C},\lambda}$ then $\Gamma(T) = \Gamma(U)\Gamma(|T|)$ and $T \otimes T = (U \otimes U)|T| \otimes |T|$ are the polar decompositions of $\Gamma(T)$ and $T \otimes T$ respectively. Hence, by uniqueness of the polar decomposition, the equality

$\Gamma(T) = T \otimes T$ implies $\Gamma(U) = U \otimes U$ and $\Gamma(|T|) = |T| \otimes |T|$. As $\lambda(G) = \{\lambda(s) \mid s \in G\}$ is precisely the family of non-zero bounded operators in $G_{\mathbb{C},\lambda}$, it gives $U = \lambda(s)$ for some $s \in G$. The statement now follows from Proposition 3.2. $\square$

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra with the exponential map $\exp_G : \mathfrak{g} \to G$. Let $\pi : G \to B(\mathcal{H}_\pi)$ be a unitary representation of G. A vector $\varphi \in \mathcal{H}_\pi$ is called a C^∞-vectors for π if the map $s \to \pi(s)\varphi$ from the C^∞-manifold G to $\mathcal{H}_\pi$ is a C^∞-mapping. We write $\mathcal{D}^\infty(\pi)$ for the set of C^∞-vectors for π. For $X \in \mathfrak{g}$ we define the operator $d\pi(X)$ with domain $\mathcal{D}^\infty(\pi)$ by

$$d\pi(X)\varphi = \tfrac{d}{dt}\pi(\exp_G(tX))\varphi|_{t=0}, \quad \text{for } \varphi \in \mathcal{D}^\infty(\pi).$$

It is known that $id\pi(X)$ is essentially self-adjoint. We denote its self-adjoint closure by $i\partial\pi(X)$ which is the infinitesimal generator of the strongly continuous one-parameter unitary group $t \mapsto \pi(\exp_G(tX))$, i.e.

$$\pi(\exp_G(tX)) = \exp(t\partial\pi(X)).$$

Proposition 3.4. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then $\Lambda = \{\partial\lambda(X) \mid X \in \mathfrak{g}\}$ and $G_{\mathbb{C},\lambda} = \{\lambda(s)\exp(i\partial\lambda(X)) \mid s \in G,\ X \in \mathfrak{g}\}$.*

Proof. If $\alpha \in \Lambda$ then $\{\exp(t\alpha) | \ t \in \mathbb{R}\}$ is a strongly continuous one parameter group in $\lambda(G) \subset VN(G)$. Moreover,

$$\langle \exp(t\alpha), \bar{\eta} * \check{\xi} \rangle = \langle \exp(t\alpha)\xi, \eta \rangle, \quad \text{for } \xi, \eta \in L^2(G),$$

and $\{\exp(t\alpha) | \ t \in \mathbb{R}\}$ is continuous in the weak* topology on $VN(G)$ with the weak*-limit $w^* - \lim_{t\to 0} \exp(t\alpha) = 1$. Since $\lambda : G \to \lambda(G) \subset VN(G)$ is a homeomorphism when $VN(G)$ carries weak* topology it follows that $\lambda^{-1}(\exp t\alpha)$ is a continuous one-parameter subgroup of G. Therefore there exists $X \in \mathfrak{g}$ such that $\lambda^{-1}(\exp(t\alpha)) = \exp(tX)$ and $\lambda(\exp(tX)) = \exp(t\alpha)$, $t \in \mathbb{R}$, giving $\partial\lambda(X) = \alpha$ and

$$\Lambda \subset \{\partial\lambda(X) \mid X \in \mathfrak{g}\}.$$

To see the reverse inclusion, we note that $\partial\lambda(X) \in \overline{VN(G)}$, $\Gamma(\exp(t\partial\lambda(X))) = \exp(t\Gamma(\partial\lambda(X)))$ and

$$\exp(t\Gamma(\partial\lambda(X))) = \exp(t\partial\lambda(X)) \otimes \exp(t\partial\lambda(X)), \quad \text{for } t \in \mathbb{R}.$$

Since $\lim_{t\to 0} t^{-1}[\exp(tV)\varphi - \varphi] = V\varphi$ for any closed skew adjoint operator V and $\varphi \in \mathcal{D}(V)$, we can easily obtain that $\partial\lambda(X) \in \Lambda$. $\square$

Remark. Our definition is motivated by the work of McKennon [21] and Cartwright and McMullen [1], where they developed an abstract Lie theory for general, not necessarily Lie, compact groups. If we choose representatives $\pi_j : G \to B(H_j)$ of the isomorphism classes of irreducible (finite-dimensional) unitary representations of G and identify $VN(G)$ with the ℓ^∞-sum of $B(H_j)$, then $\overline{VN(G)} = \prod_j B(H_j) \simeq \mathrm{Trig}(G)^\dagger$, where $\mathrm{Trig}(G)^\dagger$ is the linear dual of the span of coefficients of irreducible unitary representations of G. In this case we have that $G_{\mathbb{C},\lambda}$ coincides with the complexification $G_{\mathbb{C}}$ from [1,21]. We have that $G_{\mathbb{C}}$ is a group and as $G \simeq \{X \in VN(G) \mid \Gamma(X) = X \otimes X, X \neq 0\}$ ([25]), one has a Cartan decomposition $G_{\mathbb{C}} \simeq G \cdot \exp i\Lambda$. If G is a compact connected Lie group then $G_{\mathbb{C}} = G_{\mathbb{C},\lambda}$ coincides with the well-known construction of the universal complexification of G due to Chevalley and the Lie algebra of $G_{\mathbb{C}}$ is the complexification $\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g} \simeq \Lambda$ of G, where the usual Lie bracket $[X,Y] = XY - YX$ is considered in Λ. For instance $\mathbb{T}_{\mathbb{C}} \simeq \mathbb{C}^*$ and $(SU(n))_{\mathbb{C}} \simeq SL(n,\mathbb{C})$.

The concept of complexification was later generalised from compact to general locally compact groups in [22] by McKennon, where the group W^*-algebra $W^*(G)$ was used instead of $VN(G)$. Our construction is an adaptation of McKennon's idea to the group von Neumann algebra setting. We have chosen this approach as it fits better our purpose to describe the spectrum of Beurling-Fourier algebras. As for the compact group case McKennon's complexification $G_{\mathbb{C}}$ admits a factorisation $G_{\mathbb{C}} = G_\gamma \cdot G_{\mathbb{C}}^+$, where G_γ is the image of G under the canonical monomorphism γ from G to the group of unitary elements of $W^*(G)$ (compare this to the factorisation in Proposition 3.4). However unlike the compact case, the unboundedness of elements in $G_{\mathbb{C}}^+$ and also $G_{\mathbb{C},\lambda} \cap \overline{VN(G)}^+$ causes a problem in considering $G_{\mathbb{C}}$ and $G_{\mathbb{C},\lambda}$ as groups, see [22, section 4]. A relation to the universal complexification of G, when G is a Lie group, is also unclear in general. However, in many interesting examples considered in [11] we have $G_{\mathbb{C},\lambda} = \lambda_{\mathbb{C}}(G_{\mathbb{C}}^u)$, where $G_{\mathbb{C}}^u$ is the universal complexification of G and $\lambda_{\mathbb{C}}$ is the extension of the left regular representation to $G_{\mathbb{C}}^u$; the equality means that for any $\varphi \in G_{\mathbb{C},\lambda}$ there exists $g \in G_{\mathbb{C}}^u$ such that $\varphi = \overline{\lambda_{\mathbb{C}}(g)}$, see the discussion in [11, section 2.3]; in those cases one also has the Cartan decomposition

$$G_{\mathbb{C}}^u \simeq G \cdot \exp_{\mathbb{C}}(i\mathfrak{g}),$$

where $\exp_{\mathbb{C}}$ is the extension of the exponential map to the complexification $\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ of G. It seems an interesting question to investigate the group structure of $G_{\mathbb{C},\lambda}$ but it diverges from the main purpose of this paper.

4. The spectrum of Beurling-Fourier algebra and complexification

In this section we establish sufficient conditions in terms of groups and weight inverses for the inclusion of the Gelfand spectrum of $A(G,\omega)$ into the λ-complexification $G_{\mathbb{C},\lambda}$ of G, generalising some earlier results from [20] and [11].

4.1. Point-spectrum correspondence

Let $\phi : A(G,\omega) \to \mathbb{C}$ be a character of $A(G,\omega)$. By the duality (8), there is a unique $\sigma \in VN(G)$ such that for any $\omega u \in A(G,\omega)$ we have $\phi(\omega u) = (\sigma, \omega u)_\omega = (\sigma, u)$. The multiplicativity of ϕ gives

$$\sigma \otimes \sigma = \Gamma(\sigma)\Omega, \tag{16}$$

and moreover, every $\sigma \in VN(G)$ satisfying (16) gives rise to a unique point in the spectrum spec $A(G,\omega)$. In fact, for u, $v \in A(G)$,

$$\begin{aligned}\phi((\omega u)(\omega v)) &= (\sigma, (\omega u)(\omega v))_\omega = (\sigma, \omega\Gamma_*(\Omega(u \otimes v)))_\omega \\ &= (\sigma, \Gamma_*(\Omega(u \otimes v))) = (\Gamma(\sigma)\Omega, u \otimes v);\end{aligned}$$

on the other hand

$$\phi(\omega u)\phi(\omega v) = (\sigma, \omega u)_\omega (\sigma, \omega v)_\omega = (\sigma, u)(\sigma, v) = (\sigma \otimes \sigma, u \otimes v),$$

giving (16).

We can thus identify spec $A(G,\omega)$ as the set of all non-zero elements $\sigma \in VN(G)$ satisfying (16), i.e.

$$\text{spec } A(G,\omega) = \{\sigma \in VN(G) \mid \Gamma(\sigma)\Omega = \sigma \otimes \sigma, \sigma \neq 0\}.$$

Note that spec $A(G,\omega)$ depends on the 2-cocycle Ω rather than the weight inverse ω. Moreover, by (16), for any $\sigma \in A(G,\omega)$

$$\sigma\sigma^* \otimes \sigma\sigma^* = \Gamma(\sigma)\Omega\Omega^*\Gamma(\sigma)^* \leq \Gamma(\sigma\sigma^*),$$

thus satisfying condition (3) in the definition of a weight inverse. It is a question whether σ also satisfies (4). We will see in this section that in many cases (though we conjecture all) the elements in spec $A(G,\omega)$ are again weight inverses.

We let S be the antipode of $VN(G)$; this is an anti-isomorphism of $VN(G)$ given by $S(\lambda(s)) = \lambda(s^{-1})$, $s \in G$. If W is the multiplicative unitary and $w \in B(L^2(G))_*$, then

$$S((\iota \otimes w)(W^*)) = (\iota \otimes w)(W). \tag{17}$$

We refer to [6] for background on the theory of Hopf-von-Neumann algebras but warn that our notations may differ from those in [6].

Throughout the rest of this section, we use $\mathcal{H}$ for $L^2(G)$ and write $\psi_{\xi,\eta}$ to denote the normal functional on $B(\mathcal{H})$ given by $\psi_{\xi,\eta}(x) = \langle x\xi, \eta\rangle$, $x \in B(\mathcal{H})$.

Lemma 4.1. *Let $\sigma \in \operatorname{spec} A(G,\omega)$. Then*

$$\psi_{\xi,\tilde{\xi}}(S(\iota \otimes \psi_{\sigma^*\eta,\tilde{\eta}}(\Omega^* W^*))) = \langle (1 \otimes \sigma^*) W (S(\sigma^*) \otimes 1)\xi \otimes \eta, \tilde{\xi} \otimes \tilde{\eta} \rangle,$$

for any ξ, η, $\tilde{\xi}$, $\tilde{\eta} \in \mathcal{H}$.

Proof. From (17) and $\Omega^*\Gamma(\sigma^*) = \sigma^* \otimes \sigma^*$, we have

$$\begin{aligned}
&\psi_{\xi,\tilde{\xi}}(S(\iota \otimes \psi_{\sigma^*\eta,\tilde{\eta}}(\Omega^* W^*))) = \psi_{\xi,\tilde{\xi}}(S(\iota \otimes \psi_{\eta,\tilde{\eta}}(\Omega^* W^* (1 \otimes \sigma^*))))\\
&= \psi_{\xi,\tilde{\xi}}(S(\iota \otimes \psi_{\eta,\tilde{\eta}}(\Omega^* \Gamma(\sigma^*) W^*))) = \psi_{\xi,\tilde{\xi}}(S(\iota \otimes \psi_{\eta,\tilde{\eta}}(\sigma^* \otimes \sigma^* W^*)))\\
&= \psi_{\xi,\tilde{\xi}}(S(\sigma^*(\iota \otimes \psi_{\eta,\tilde{\eta}}((1 \otimes \sigma^*) W^*)))) = \psi_{\xi,\tilde{\xi}}(S(\sigma^*(\iota \otimes \psi_{\eta,\sigma\tilde{\eta}}(W^*))))\\
&= \psi_{\xi,\tilde{\xi}}(S(\iota \otimes \psi_{\eta,\sigma\tilde{\eta}}(W^*)) S(\sigma^*)) = \psi_{\xi,\tilde{\xi}}((\iota \otimes \psi_{\eta,\sigma\tilde{\eta}}(W)) S(\sigma^*))\\
&= \psi_{\xi,\tilde{\xi}} \otimes \psi_{\eta,\sigma\tilde{\eta}}(W(S(\sigma^*) \otimes 1)) = \psi_{\xi,\tilde{\xi}} \otimes \psi_{\eta,\tilde{\eta}}((1 \otimes \sigma^*) W (S(\sigma^*) \otimes 1))\\
&= \langle (1 \otimes \sigma^*) W (S(\sigma^*) \otimes 1)\xi \otimes \eta, \tilde{\xi} \otimes \tilde{\eta} \rangle. \quad \square
\end{aligned}$$

Proposition 4.2. *Let G be a locally compact group and let $\sigma \in \operatorname{spec} A(G,\omega)$. Assume that $\sigma^*(\mathcal{H}) \cap \omega^*(\mathcal{H}) \neq \{0\}$. Then*

$$S(\omega)\omega = S(\sigma)\sigma.$$

Remark. It has been known for compact groups ([20]) and some Lie groups with certain weights ([11]) that the operators $\sigma\omega^{-1}$, $\sigma \in \operatorname{spec} A(G,\omega)$, are "points" of the complexification $G_{\mathbb{C}}$. From this, the claim of the proposition becomes intuitively quite clear. Formally, if there is an element $T \in G_{\mathbb{C}}$ such that $\sigma = T\omega$ then, as $S(T) = T^{-1}$ (the antipode "inverts" the elements of G and $G_{\mathbb{C}}$), we would have $S(\sigma)\sigma = S(T\omega)T\omega = S(\omega)T^{-1}T\omega = S(\omega)\omega$.

Proof. Let η and ζ in $\mathcal{H}$ be such that $\sigma^*\zeta = \omega^*\eta \neq 0$. By Lemma 4.1, we have

$$(1 \otimes \omega^*) W (S(\omega^*) \otimes 1)\xi \otimes \zeta = (1 \otimes \sigma^*) W (S(\sigma^*) \otimes 1)\xi \otimes \eta$$

for any $\xi \in \mathcal{H}$.

Multiplying both hand sides of the equality from the left by $\Omega^* W^*$ and using the equality $\Omega^*\Gamma(\sigma)^* = \sigma^* \otimes \sigma^*$ which holds for all $\sigma \in \operatorname{spec} A(G,\omega)$ and in particular for ω, we conclude that

$$\omega^* S(\omega^*)\xi \otimes \omega^*\eta = \sigma^* S(\sigma^*)\xi \otimes \sigma^*\zeta, \quad \text{for all } \xi \in \mathcal{H},$$

and hence $\omega^* S(\omega)^* = \sigma^* S(\sigma)^*$. $\square$

Remark. The following formal calculations support the idea that the above proposition might be true for any $\sigma \in \operatorname{spec} A(G,\omega)$.

Consider $M = (S \otimes \iota)(W\Omega)W\Omega$. From (5) it follows that $(I \otimes \omega)W\Omega = W(\omega \otimes \omega)$, and hence $(I \otimes \omega)(S \otimes \iota)(W\Omega) = (S(\omega) \otimes I)W^*(I \otimes \omega)$. We then calculate

$$\begin{aligned}(I \otimes \omega)M &= (S(\omega) \otimes I)W^*(I \otimes \omega)W\Omega = (S(\omega) \otimes I)W^*W(\omega \otimes \omega)\\ &= (S(\omega)\omega) \otimes \omega = (I \otimes \omega)(S(\omega)\omega \otimes I).\end{aligned}$$

As $\ker \omega = \{0\}$ we get $M = (S(\omega)\omega) \otimes I$. Let now $\sigma \in \operatorname{spec} A(G, \omega)$ be arbitrary. Similarly,

$$(I \otimes \sigma)M = (S(\sigma)\sigma) \otimes \sigma$$

and hence $(S(\sigma)\sigma) \otimes \sigma = (S(\omega)\omega) \otimes \sigma$. Therefore, $S(\sigma)\sigma = S(\omega)\omega$.

The calculations are only formal as S is not a completely bounded map in general and hence $S \otimes \iota$ is not defined on the whole $VN(G)\bar{\otimes}B(\mathscr{H})$. By [7, Proposition 1.5], S is completely bounded if and only if G is virtually abelian, i.e. has an abelian subgroup of finite index. Consequently, for such G, $S(\sigma)\sigma = S(\omega)\omega$ for any $\sigma \in \operatorname{spec} A(G, \omega)$.

Corollary 4.3. *Let $\sigma \in \operatorname{spec} A(G, \omega)$. If $\sigma^*(\mathscr{H}) \cap \omega^*(\mathscr{H}) \neq \{0\}$ then $\ker \sigma = \{0\}$.*

Proof. This follows from Proposition 4.2, as $\ker \sigma \subseteq \ker S(\sigma)\sigma = \ker S(\omega)\omega = \{0\}$. □

A natural question is when $\sigma \in \operatorname{spec} A(G, \omega)$ is again a weight inverse. Clearly,

$$\sigma\sigma^* \otimes \sigma\sigma^* = \Gamma(\sigma)\Omega\Omega^*\Gamma(\sigma^*) \leq \Gamma(\sigma\sigma^*)$$

and hence the first condition (3) of being a weight inverse is fulfilled. The same arguments as in Corollary 4.3 show that if $S(\omega)\omega = S(\sigma)\sigma$ then $\ker \sigma = \{0\}$. An issue is to obtain $\ker \sigma^* = \{0\}$. We will adopt the extra condition $\ker\ \Omega^* = \{0\}$ as a way to guarantee it.

Lemma 4.4. *If $\ker\ \Omega^* = \{0\}$, then $\ker \sigma^* = \{0\}$ for every $\sigma \in \operatorname{spec} A(G, \omega)$.*

Proof. Let $\sigma \in \operatorname{spec} A(G, \omega)$. As $\ker\ \Omega^* = 0$, we have

$$\ker\ \sigma^* \otimes \sigma^* = \ker\ \Omega^*\Gamma(\sigma^*) = \ker\ \Gamma(\sigma^*).$$

Thus, if we let P denote the projection onto the closure of the range of σ, then $P \in VN(G)$ and P satisfies

$$P \otimes P = \Gamma(P). \tag{18}$$

As P is a projection, [25, Chapter 11, Theorem 16] gives either $P = \lambda(e) = I$ or $P = 0$. Having $\sigma \in \operatorname{spec} A(G, \omega)$ and hence non-zero, we obtain $P = I$ and therefore $\ker \sigma^* = \{0\}$. □

Here comes the main result of this section that establishes a connection between (a part of) the spectrum spec $A(G,\omega)$ and $G_{\mathbb{C},\lambda}$.

Theorem 4.5. *Let $\omega \in VN(G)$ be a weight inverse on the dual of G and let $\sigma \in$ spec $A(G,\omega)$ be such that $\sigma^*(\mathcal{H}) \cap \omega^*(\mathcal{H}) \neq \{0\}$. Assume* ker $\Omega^* = \{0\}$. *Then*

(*i*) *σ is a weight inverse,*
(*ii*) *$S(\sigma)\sigma = S(\omega)\omega$,*
(*iii*) *the linear operator*

$$T_\sigma : \omega\xi \mapsto \sigma\xi, \quad \textit{for } \xi \in \mathcal{H},$$

is closable with the closure in $G_{\mathbb{C},\lambda}$.

Proof. (*i*) follows from Corollary 4.3 and Lemma 4.4, together with the earlier observation that $\sigma\sigma^* \otimes \sigma\sigma^* \leq \Gamma(\sigma\sigma^*)$.

(*ii*) follows from Proposition 4.2.

(*iii*) The operator T_σ is well-defined as $\ker\omega = \{0\}$. Let $\xi_n \in \mathcal{H}$ be such that $\omega\xi_n \to 0$ and $\sigma\xi_n \to y$. Then for any $\xi \in \mathcal{H}$,

$$\langle y, S(\sigma^*)\xi\rangle = \lim_{n\to\infty} \langle S(\sigma)\sigma\xi_n, \xi\rangle = \lim_{n\to\infty} \langle S(\omega)\omega\xi_n, \xi\rangle = 0.$$

Therefore $y \perp \overline{S(\sigma^*)(\mathcal{H})}$. By Lemma 4.4, $\ker\sigma^* = \{0\}$. This yields $\ker S(\sigma) = \{0\}$ (since if P is the range projection for $A \in VN(G)$, then $S(P)$ is the range projection for $S(A^*)$). Therefore $S(\sigma^*)\mathcal{H}$ is dense in $\mathcal{H}$ and hence $y = 0$. Consequently, T_σ is closable.

Write T_σ also for the closure. Then T_σ is affiliated with $VN(G)$, and even more, it is affiliated with the von Neumann algebra $\mathcal{N}(\omega,\sigma)$ generated by ω and σ. In fact, let $V \in \mathcal{N}(\omega,\sigma)'$ be a unitary. Then for any $\xi \in \mathcal{H}$ of the form $\xi = \omega\eta$, we have

$$VT_\sigma\xi = V\sigma\eta = \sigma V\eta = T_\sigma\omega V\eta = T_\sigma V\xi$$

and hence $V^*T_\sigma V = T_\sigma$, showing the statement.

The only claim left to prove is that $\Gamma(T_\sigma) = T_\sigma \otimes T_\sigma$. Observe first that

$$(\omega\otimes\omega)(\mathcal{H}\otimes\mathcal{H}) = \Gamma(\omega)\Omega(\mathcal{H}\otimes\mathcal{H}) \subseteq$$
$$\subseteq \Gamma(\omega)(\mathcal{H}\otimes\mathcal{H}) \subseteq \mathcal{D}(\Gamma(T_\sigma)),$$

and

$$\Gamma(T_\sigma)\Gamma(\omega)\Omega(\mathcal{H}\odot\mathcal{H}) = \Gamma(T_\sigma\omega)\Omega(\mathcal{H}\odot\mathcal{H}) = \Gamma(\sigma)\Omega(\mathcal{H}\odot\mathcal{H}) =$$
$$= (\sigma\otimes\sigma)(\mathcal{H}\odot\mathcal{H}) = (T_\sigma\otimes T_\sigma)(\omega\otimes\omega)(\mathcal{H}\odot\mathcal{H}).$$

We have

$$T_\sigma \otimes T_\sigma|_{\omega(\mathscr{H})\odot\omega(\mathscr{H})} = \Gamma(T_\sigma)|_{\omega(\mathscr{H})\odot\omega(\mathscr{H})}.$$

By convention, $T_\sigma \otimes T_\sigma$ is the closure of the operator $T_\sigma \odot T_\sigma$ defined on $\mathscr{D}(T_\sigma)\odot\mathscr{D}(T_\sigma)$ or, equivalently, on $\omega(\mathscr{H})\odot\omega(\mathscr{H})$, as $\omega(\mathscr{H})$ is a core of T_σ. Hence

$$\Gamma(T_\sigma) \supset T_\sigma \otimes T_\sigma.$$

To see the equality, we must prove that $\overline{\Gamma(T_\sigma)|_{\omega(\mathscr{H})\odot\omega(\mathscr{H})}} = \Gamma(T_\sigma)$. To do this we note first that $\Gamma(\omega)(\mathscr{H}\otimes\mathscr{H})$ is a core for $\Gamma(T_\sigma)$ and hence the linear subspace

$$\{(x,\ \Gamma(T_\sigma)x) \mid x \in \Gamma(\omega)(\mathscr{H}\otimes\mathscr{H})\} \tag{19}$$

is dense in the graph of $\Gamma(T_\sigma)$. Therefore, it is enough to see that the closure of

$$\{(x, \Gamma(T_\sigma)x) \mid x \in \Gamma(\omega)\Omega(\mathscr{H}\odot\mathscr{H})\} = \{(x,\ \Gamma(T_\sigma)x) \mid x \in \omega\mathscr{H}\odot\omega\mathscr{H})\}$$

contains (19).

As $\Omega(\mathscr{H}\odot\mathscr{H})$ is dense in $\mathscr{H}\otimes\mathscr{H}$, we have that for any $\Gamma(\omega)\xi$, $\xi \in \mathscr{H}\otimes\mathscr{H}$, there exists $(\xi_n)_n \subset \mathscr{H}\odot\mathscr{H}$ such that $\Omega\xi_n \to \xi$ and hence $\Gamma(\omega)\Omega\xi_n \to \Gamma(\omega)\xi$. Moreover,

$$\Gamma(T_\sigma)\Gamma(\omega)\Omega\xi_n = \Gamma(\sigma)\Omega\xi_n \to \Gamma(\sigma)\xi = \Gamma(T_\sigma)\Gamma(\omega)\xi,$$

showing the claim. $\square$

We remark that

$$\sigma^*(\mathscr{H}) \cap \omega^*(\mathscr{H}) \neq \{0\} \tag{20}$$

for $\sigma \in \operatorname{spec} A(G,\omega)$ means that the domain $\mathscr{D}(T_\sigma^*)$ of the operator $T_\sigma^* = (\sigma\omega^{-1})^* = (\omega^*)^{-1}\sigma^*$ is not zero. The theorem says that in this case $\mathscr{D}(T_\sigma^*)$ is large enough to be dense in $\mathscr{H}$, as the latter is equivalent to the closability of T_σ.

In what follows we shall use the notation T_σ for the closed operator $\bar{T}_\sigma$ when there is no risk of confusion.

We derive now a number of consequences from the previous theorem. We assume that $\ker \Omega^* = \{0\}$.

Corollary 4.6. *For $\sigma \in \operatorname{spec} A(G,\omega)$ as in Theorem 4.5, there is a natural isometric isomorphism*

$$A(G,\sigma) \cong A(G,\omega),$$
$$\sigma f \mapsto \omega f.$$

Proof. This is immediate from the definitions of the norm and product on the corresponding spaces:

$$||\omega f||_\omega = ||f|| = ||\sigma f||_\sigma, \quad \text{for } f \in A(G),$$
$$(\omega g)(\omega h) = \omega\Gamma_*(\Omega(g \otimes h)), \quad (\sigma g)(\sigma h) = \sigma\Gamma_*(\Omega(g \otimes h)), \quad \text{for } g, h \in A(G). \quad \square$$

We remark that the above corollary is also clear from the discussion after the proof of Proposition 2.6.

Corollary 4.7. *For $\sigma \in \operatorname{spec} A(G,\omega)$ we have $\sigma^*(\mathcal{H}) \cap \omega^*(\mathcal{H}) \neq \{0\}$ if and only if $\sigma = T\omega$ for $T \in G_{\mathbb{C},\lambda}$ such that $\omega(\mathcal{H}) \subset \mathcal{D}(T)$. Consequently, if $\sigma^*(\mathcal{H}) \cap \omega^*(\mathcal{H}) \neq \{0\}$ for any $\sigma \in \operatorname{spec} A(G,\omega)$ then*

$$\operatorname{spec} A(G,\omega) \subset \{T\omega \mid T \in G_{\mathbb{C},\lambda}, \omega(\mathcal{H}) \subset \mathcal{D}(T)\}.$$

Proof. The "only if" part follows from Theorem 4.5. If $\sigma = T\omega$ for $T \in G_{\mathbb{C},\lambda}$ then $\sigma^* \supset \omega^* T^*$ giving the "if" part. $\square$

Remark 4.8. In [11] the dual $A(G,\omega)^*$ is identified with the weighted space $VN(G,\omega)$ given by

$$VN(G,\omega) := \{A\omega^{-1} \mid A \in VN(G)\}$$

with the norm $\|A\omega^{-1}\|_{VN(G,\omega)} = \|A\|$ via

$$(A\omega^{-1}, \omega u) := (A, u).$$

Then the spectrum of $A(G,\omega)$ is considered as a subset of $VN(G,\omega)$ instead of $VN(G)$. Clearly we have the isometry $\Phi : VN(G) \to VN(G,\omega)$, $A \mapsto A\omega^{-1}$. With this identification we have that if $\sigma^*(\mathcal{H}) \cap \omega^*(\mathcal{H}) \neq \{0\}$ for any $\sigma \in \operatorname{spec} A(G,\omega)$, then

$$\operatorname{spec} A(G,\omega) \simeq \{T \in G_{\mathbb{C},\lambda} | \, T\omega \in VN(G)\} \subset G_{\mathbb{C},\lambda}. \tag{21}$$

Next, we prove a 'partial converse' of Theorem 4.5, which shows that every element in $G_{\mathbb{C},\lambda}$ can be seen as coming from a weight inverse.

Proposition 4.9. *If $T \in G_{\mathbb{C},\lambda}$ then there exists a weight inverse $\omega \in VN(G)$ and $\sigma \in \operatorname{spec} A(G,\omega)$ such that $T = T_\sigma$.*

Proof. Let $T \in G_{\mathbb{C},\lambda}$ and $U|T|$ be its polar decomposition. Then $U = \lambda(s)$ for some $s \in G$ and $\Gamma(|T|) = |T| \otimes |T|$. Hence $\Gamma(|T|^{it}) = |T|^{it} \otimes |T|^{it}$ and $|T|^{it}$ determines a strongly continuous representation $\psi : \mathbb{R} \to \lambda(G) \subseteq B(\mathcal{H})$ by setting $\psi(t) = |T|^{it}$. By the standard theory, the map

$$\hat{f}(x) = \int_{\mathbb{R}} f(t)e^{ixt}\, dt \;\mapsto\; \int_{\mathbb{R}} f(t)\psi(t)\, dt \in VN(G), \quad \text{for } f \in L^1(\mathbb{R}),$$

extends to a $*$-homomorphism $\varphi : C^*(\mathbb{R}) \cong C_0(\mathbb{R}) \to VN(G)$; we have $\varphi(f) = f(\ln(|T|))$. The image of $C_0(\mathbb{R})$ is clearly non-degenerate, and hence we can extend φ in a unique way to a homomorphism $\overline{\varphi} : C_b(\mathbb{R}) \to VN(G)$. If we let $\overline{\varphi \otimes \varphi}$ denote the extension of the map $\varphi \otimes \varphi : C_0(\mathbb{R}) \otimes C_0(\mathbb{R}) \to VN(G)\bar{\otimes}VN(G)$ to $C_b(\mathbb{R} \times \mathbb{R})$, then it is easy to see from the uniqueness of the extensions that the diagram

$$\begin{array}{ccc} C_b(\mathbb{R} \times \mathbb{R}) & \xrightarrow{\overline{\varphi\otimes\varphi}} & VN(G)\bar{\otimes}VN(G) \\ \Gamma_{\mathbb{R}} \uparrow & & \uparrow \Gamma \\ C_b(\mathbb{R}) & \xrightarrow{\overline{\varphi}} & VN(G) \end{array} \tag{22}$$

is commutative; here we write $\Gamma_{\mathbb{R}}$ for the restriction of the coproduct to $C_b(\mathbb{R})$.

Now if we let $\omega = \varphi(e^{-2|x|})$, then $\omega^2 \otimes \omega^2 \leq \Gamma(\omega^2)$, and the non-degeneracy of φ gives $\ker\, \omega = \{0\}$. Thus ω is a weight inverse in $VN(G)$. Moreover, the 2-cocycle associated to ω is given by

$$\Omega = \overline{\varphi \otimes \varphi}(e^{2|x+y|-2|x|-2|y|}). \tag{23}$$

If we let $\sigma = \lambda(s)\varphi(e^{x-2|x|})$, then it is easy to see from (23) that $\Gamma(\sigma)\Omega = \sigma \otimes \sigma$ and hence $\sigma \in \operatorname{spec} A(G, \omega)$. Moreover, the closure of the unbounded operator

$$\omega\xi = e^{-2|\ln|T||}\xi \mapsto \sigma\xi = \lambda(s)e^{\ln|T|-2|\ln|T||}\xi, \quad \text{for } \xi \in \mathscr{H},$$

is given by T. $\square$

Next we derive some further properties of $\operatorname{spec} A(G, \omega) \cap G_{\mathbb{C},\lambda}\omega$.

Lemma 4.10. *Let $\sigma \in \operatorname{spec} A(G, \omega) \cap G_{\mathbb{C},\lambda}\omega$. Then there exists a unique $s \in G$ such that $\beta = \lambda(s)^*\sigma \in \operatorname{spec} A(G, \omega) \cap G_{\mathbb{C},\lambda}\omega$ and*

$$|T_\sigma| = T_\beta.$$

Proof. Taking the polar decomposition $T_\sigma = U|T_\sigma|$, we conclude, as in the proof of Proposition 3.3, that $U = \lambda(s)$ for a unique $s \in G$. Clearly we have, $\beta = \lambda(s)^*\sigma \in \operatorname{spec} A(G, \omega)$, and $\beta^*(\mathscr{H}) \cap \omega^*(\mathscr{H}) = \sigma^*(\mathscr{H}) \cap \omega^*(\mathscr{H}) \neq \{0\}$. It follows from Theorem 4.5 that the closure of

$$\{(\omega\xi,\ \beta\xi) \mid \xi \in \mathscr{H}\} \subseteq \mathscr{H} \oplus \mathscr{H}$$

is the graph of the positive operator $|T_\sigma|$; on the other hand the closure is the graph of the closed operator T_β. $\square$

Proposition 4.11. *Let $\sigma \in \text{spec } A(G,\omega) \cap G_{\mathbb{C},\lambda}\omega$. If $\sigma = \lambda(s)\beta$ is the decomposition from Lemma 4.10. Then $\psi(t) = \lambda(s)T_\beta^t\omega$ is a continuous function*

$$\psi : [0,1] \to \text{spec } A(G,\omega) \cap G_{\mathbb{C},\lambda}\omega, \quad \psi(0) = \lambda(s)\omega, \quad \psi(1) = \sigma.$$

Proof. By the functional calculus, we have $\omega(\mathcal{H}) \subseteq \mathcal{D}(1 + T_\beta) \subseteq \mathcal{D}(T_\beta^t)$ for every $t \in [0,1]$. Moreover, as $x^{2t} \leq 1 + x^2$ for $x \in \mathbb{R}_+$, we have

$$0 \leq \omega^* T_\beta^{2t}\omega \leq \omega^*\omega + \omega^* T_\beta^2 \omega = \omega^*\omega + \beta^*\beta.$$

It follows that $T_\beta^t\omega$ is bounded for every $t \in [0,1]$, and hence $T_\beta^t\omega$ and $\lambda(s)T_\beta^t\omega$ belong to $VN(G)$, and the function $t \mapsto T_\beta^t\omega$ is strongly continuous; to see the latter we observe that if $P_n = E([0,n])$, where $E(\cdot)$ is the spectral measure of T_β, then $t \mapsto T_\beta^t P_n\omega\xi$ is continuous for every $\xi \in \mathcal{H}$. Moreover,

$$\begin{aligned}
&\|T_\beta^t(P_n\omega\xi - \omega\xi)\|^2 = \langle T_\beta^t(P_n\omega\xi - \omega\xi), T_\beta^t(P_n\omega\xi - \omega\xi)\rangle \\
&= \langle T_\beta^{2t}(P_n\omega\xi - \omega\xi), (P_n\omega\xi - \omega\xi\rangle \leq \langle T_\beta^2(P_n\omega\xi - \omega\xi), P_n\omega\xi - \omega\xi\rangle \\
&= \|T_\beta(P_n\omega\xi - \omega\xi)\|^2 = \|P_n T_\beta\omega\xi - T_\beta\omega\xi\|^2 \to 0
\end{aligned}$$

as $P_n \to I$ strongly. Basic approximation arguments give now that $T_\beta^t\omega\xi$ must depend continuously on $t \in [0,1]$ for each $\xi \in \mathcal{H}$. From this we conclude that $t \mapsto \psi(t)$ is continuous as the map from $[0,1]$ to $VN(G) \simeq A(G,\omega)^*$ with the weak* topology.

It follows from the functional calculus that $\Gamma(T_\beta^t) = T_\beta^t \otimes T_\beta^t$ for all $t \in [0,1]$, and thus

$$\Gamma(T_\beta^t\omega)\Omega = \Gamma(T_\beta^t)(\omega \otimes \omega) = (T_\beta^t\omega) \otimes (T_\beta^t\omega)$$

so that $T_\beta^t\omega$ and hence $\lambda(s)T_\beta^t\omega$ are in spec $A(G,\omega)$.

As the kernel of T_β is trivial, there is $n \in \mathbb{N}$ such that the orthogonal projection $P = E([\frac{1}{n}, n])$ is non-zero. The restriction of T_β^t to the invariant subspace $P\mathcal{H}$ is then invertible for every $t \in [0,1]$ and as $P\lambda(s)^*\psi(t) = PT_\beta^t P\omega$, $t \in [0,1]$, we have

$$\psi(t)^*(\lambda(s)P\mathcal{H}) = \omega^*(PT_\beta^t P\mathcal{H}) = \omega^*(P\mathcal{H}),$$

giving $\psi(t)^*(\mathcal{H}) \cap \omega^*(\mathcal{H}) \supset \omega^*(P\mathcal{H}) \neq \{0\}$. By Corollary 4.7, we obtain $\psi(t) \in$ spec $A(G,\omega) \cap G_{\mathbb{C},\lambda}\omega$ for all $t \in [0,1]$. □

The last results concern a deformation retraction of weight inverses.

Lemma 4.12. *Assume that a weight inverse ω is positive. For every $s \in [0,1]$, the operator ω^s is again a weight inverse.*

Proof. By the Löwner-Heinz inequality: if $0 \leq A \leq B$, then also $0 \leq A^s \leq B^s$ for $s \in [0,1]$. Applying this to the inequality (3), we get

$$\omega^{2s} \otimes \omega^{2s} \le \Gamma(\omega^{2s}), \quad \text{for all } s \in [0,1].$$

The conditions on the kernel(s) are easy to see. □

Proposition 4.13. *Let ω be a positive weight inverse. If Ω_s is the 2-cocycle associated to ω^s, $s \in [0,1]$, then for $0 \le s \le t \le 1$ the following hold*

(*i*)

$$\Gamma(\omega^s)\Omega_t = \Omega_{t-s}(\omega^s \otimes \omega^s); \tag{24}$$

(*ii*) *if* $\ker\ \Omega_t^* = \{0\}$, *then* $\ker \Omega_s^* = \{0\}$*;*

(*iii*) *the map* $\operatorname{spec} A(G,\omega^s) \to \operatorname{spec} A(G,\omega^t)$, *given as* $\sigma \mapsto \sigma\omega^{t-s}$, *is injective and maps* $\operatorname{spec} A(G,\omega^s) \cap G_{\mathbb{C},\lambda}\omega^s$ *to* $\operatorname{spec} A(G,\omega^t) \cap G_{\mathbb{C},\lambda}\omega^t$.

Proof. (*i*) The 2-cocycle Ω_{t-s} is the unique operator which satisfies $\Gamma(\omega^{t-s})\Omega_{t-s} = \omega^{t-s} \otimes \omega^{t-s}$. Hence, since $\ker \omega^s = \{0\}$, it follows that $\Omega_{t-s}(\omega^s \otimes \omega^s)$ is the unique operator that satisfies $\Gamma(\omega^{t-s})\Omega_{t-s}(\omega^s \otimes \omega^s) = \omega^t \otimes \omega^t$. As $\Gamma(\omega^{t-s})\Gamma(\omega^s)\Omega_t = \omega^t \otimes \omega^t$, we obtain (24).

(*ii*) This follows now directly from (24).

(*iii*) If $\sigma \in \operatorname{spec} A(G,\omega^s)$, then by (24)

$$\Gamma(\sigma\omega^{t-s})\Omega_t = \Gamma(\sigma)\Omega_s(\omega^{t-s} \otimes \omega^{t-s}) = (\sigma\omega^{t-s}) \otimes (\sigma\omega^{t-s}),$$

i.e. $\sigma\omega^{t-s} \in \operatorname{spec} A(G,\omega^t)$. If $\sigma^*(\mathscr{H}) \cap \omega^s(\mathscr{H}) \ne \{0\}$ then

$$(\omega^{t-s}\sigma)^*(\mathscr{H}) \cap (\omega^t(\mathscr{H}) = \omega^{t-s}(\sigma^*(\mathscr{H}) \cap \omega^s(\mathscr{H})) \ne \{0\},$$

as the kernel of ω^{t-s} is trivial. The injectivity of $\sigma \mapsto \sigma\omega^{t-s}$ follows from the fact that the range of ω^{t-s} is dense in $\mathscr{H}$. □

4.2. Conditions guaranteeing complexification

In this section we will investigate conditions on the group G and the weight inverse ω for which the inclusion (21) of the spectrum of $A(G,\omega)$ into the complexification $G_{\mathbb{C},\lambda}$ holds true.

First, we present some sufficient conditions for $\ker\ \Omega^* = \{0\}$.

Recall that if H is a closed subgroup of G and λ_H and λ_G are the left regular representations of H and G respectively, then there is a canonical injective $*$-homomorphism $\iota_H : VN(H) \to VN(G)$ given by $\lambda_H(s) \mapsto \lambda_G(s)$, for $s \in H$ ([14]).

We say that a weight inverse ω on the dual of G is *central* if ω is in the centre of $VN(G)$.

Proposition 4.14. *Let ω be a weight inverse on the dual of G. Then* $\ker\ \Omega^* = \{0\}$ *holds provided that any of the following is satisfied:*

1. *G is compact;*
2. *$\omega = \iota_H(\omega_H)$, where ω_H is a central weight inverse on the dual of a closed subgroup H of G.*

Proof. (1) It is known that if G is compact then $VN(G)\bar{\otimes}VN(G) \simeq VN(G \times G)$ can be identified with the ℓ^∞ sum of matrix algebras $M_{n_j}(\mathbb{C})$. Therefore $\ker\, X = \{0\} \Leftrightarrow \ker\, X^* = \{0\}$ for any $X \in VN(G \times G)$. This gives $\ker\, \Omega^* = \{0\}$, as Ω is injective.

(2) Since $\ker\, \Omega = \{0\}$ it is enough to see that

$$\Omega^*\Omega = \Omega\Omega^*,$$

as in this case $\ker\, \Omega = \ker\, \Omega^*$. Assume first that $H = G$. Then being central, ω is a normal operator and therefore so is $\Gamma(\omega)$. Moreover, as $\omega \otimes \omega \in Z(VN(G \times G))$ (the centre of $VN(G \times G)$), we have

$$\Gamma(\omega)\Omega\Gamma(\omega) = (\omega \otimes \omega)\Gamma(\omega) = \Gamma(\omega)(\omega \otimes \omega) = \Gamma(\omega)^2\Omega.$$

Hence, as $\ker\, \Gamma(\omega) = \{0\}$, we have $\Gamma(\omega)\Omega = \Omega\Gamma(\omega)$. By the Fuglede-Putnam theorem it follows that also $\Gamma(\omega)^*\Omega = \Omega\Gamma(\omega)^*$. A calculation now yields

$$\begin{aligned}
&\Gamma(\omega)\Omega\Omega^*\Gamma(\omega)^* = \omega\omega^* \otimes \omega\omega^* = \omega^*\omega \otimes \omega^*\omega \\
&= \Omega^*\Gamma(\omega)^*\Gamma(\omega)\Omega = \Omega^*\Gamma(\omega)\Gamma(\omega)^*\Omega = \Gamma(\omega)\Omega^*\Omega\Gamma(\omega)^*,
\end{aligned}$$

and we get the claim by using again $\ker\, \Gamma(\omega) = \{0\}$.

The proof for general H is similar, if we take into account that $\Gamma \circ \iota_H = (\iota_H \otimes \iota_H) \circ \Gamma_H$, where Γ_H is the comultiplication on $VN(H)$. $\square$

The next simple lemma gives a sufficient condition for $\sigma^*(\mathscr{H}) \cap \omega^*(\mathscr{H}) \neq \{0\}$ to hold for any $\sigma \in \operatorname{spec} A(G, \omega)$, where $\mathscr{H} = L^2(G)$. We assume that $\ker\, \Omega^* = \{0\}$.

Lemma 4.15. *If there is a subspace $\mathscr{K} \subset \mathscr{H}$ such that $VN(G)(\mathscr{K}) \subset \mathscr{K}$ and $\omega|_{\mathscr{K}}$ is invertible, then $\sigma^*(\mathscr{H}) \cap \omega^*(\mathscr{H}) \neq \{0\}$ for any $\sigma \in \operatorname{spec} A(G, \omega)$.*

Proof. As $\mathscr{K}$ is invariant and $\omega|_{\mathscr{K}}$ is invertible, $\omega^*(\mathscr{K}) = \mathscr{K}$. We have

$$\sigma^*(\mathscr{H}) \cap \omega^*(\mathscr{H}) \supset \sigma^*(\mathscr{K}) \cap \omega^*(\mathscr{K}) = \sigma^*(\mathscr{K}),$$

where the latter is non-zero by Lemma 4.4. $\square$

Using Lemma 4.15 we can now list groups and weights for which the spectrum of the associated Beurling-Fourier algebra is in the complexification $G_{\mathbb{C},\lambda}$, meaning that we identify $A(G, \omega)^*$ with $VN(G, \omega)$ as in Remark 4.8; with a slight abuse of notation we write $\operatorname{spec} A(G, \omega) \subset G_{\mathbb{C},\lambda}$.

(1) *G is compact and ω is arbitrary.* If G is compact then it is known that the left regular representation λ on G is a direct sum of irreducible (finite-dimensional) representations and hence there exists a finite-dimensional invariant subspace $\mathcal{K} \subseteq \mathcal{H}$. As $\ker \omega = \{0\}$, ω is invertible on $\mathcal{K}$. By Proposition 4.14, $\ker\, \Omega^* = \{0\}$. By Corollary 4.7, spec $A(G, \omega) \subset G_{\mathbb{C},\lambda}$. In [20] and [11] the result was derived from the "abstract Lie" theory developed in [1,21] showing that the multiplicative linear functionals on $\mathrm{Trig}(G)$, the algebra of coefficient functions with respect to irreducible representations, can be identified with the complexification $G_{\mathbb{C},\lambda}$. As $\mathrm{Trig}(G) \subset A(G, \omega)$, the statement is clear.

(2) *G is an extension of a compact group by abelian group and ω is a weight inverse such that* $\ker\, \Omega^* = \{0\}$. If K is a non-trivial compact normal subgroup, let $P_K \in B(L^2(G))$ be the projection onto the (non-trivial) subspace of functions which are constant on the cosets xK, $x \in G$. As P_K commutes with $\lambda_G(g)$, $g \in G$, the subspace $P_K L^2(G)$ is invariant with respect to λ_G, and as G/K is abelian and $P_K f$ are constant on the cosets, $\lambda_G(g_1 g_2) P_K f = \lambda_G(g_2 g_1) P_K f$, i.e. the von Neumann algebra generated by $\lambda_G(g) P_K$, $g \in G$, is commutative. As $\omega_K := \omega|_{P_K L^2(G)}$ belongs to the von Neumann algebra, there exists a subspace $\mathcal{K}$ (e.g. $\mathcal{K} = E_{|\omega_K|}([\varepsilon, \infty)) P_K L^2(G)$ for some $\varepsilon > 0$) such that $VN(G)\mathcal{K} \subset \mathcal{K}$ and $\omega|_{\mathcal{K}}$ is invertible.

(3) *G is a separable Moore group and ω is arbitrary.* If G is a Moore group, i.e. any irreducible representation of G is finite dimensional, then G is a type I group with the unitary dual $\hat{G}$ being a standard Borel space. Moreover, there is a standard Borel measure μ and a μ-measurable cross section $\xi \to \pi^\xi$ from $\hat{G}$ to concrete irreducible unitary representation acting on $\mathcal{H}_\xi$ such that λ is quasi-equivalent to $\int_{\hat{G}}^{\oplus} \pi^\xi d\mu(\xi)$ so that $VN(G) \simeq L^\infty(\hat{G}, d\mu(\xi); B(\mathcal{H}_\xi))$. With this identification we have $\omega = \int_{\hat{G}}^{\oplus} \omega_\xi d\mu(\xi)$. Let for $\varepsilon > 0$

$$\Delta_\varepsilon = \cap_n \{\xi \in \widehat{G} \mid \langle |\omega_\xi| x_n(\xi), x_n(\xi)\rangle \geq \varepsilon \|x_n(\xi)\|^2\},$$

where $(x_n)_n$ is a sequence such that $(x_n(\xi))_n$ is total in $\mathcal{H}_\xi$ for any ξ. As $\ker \omega = \ker \omega^* = \{0\}$, there exists a null set $M \subset \hat{G}$ such that $\ker \omega_\xi = \ker \omega_\xi^* = \{0\}$ for any $\xi \in \hat{G} \backslash M$. Then, as $\mathcal{H}_\xi$ is finite-dimensional, for each $\xi \in \hat{G} \backslash M$, we have $|\omega_\xi| \geq c_\xi I_\xi$ for some $c_\xi > 0$. Hence $\mu(\Delta_\varepsilon) > 0$ for some $\varepsilon > 0$ and $P_\varepsilon = \int_{\hat{G}} \chi_{\Delta_\varepsilon} I_\xi \, d\mu(\xi)$ is a non-zero projection onto invariant subspace $\mathcal{K}$ such that $|\omega||_{\mathcal{K}} \geq \varepsilon$; $\omega|_{\mathcal{K}}$ is invertible. As $\ker \Omega = \{0\}$ and $G \times G$ is Moore, we can argue as above to conclude that $\ker\, \Omega^* = \{0\}$. Therefore, by Corollary 4.7, we have the inclusion of the spectrum of $A(G, \omega)$ into $G_{\mathbb{C},\lambda}$ as in the previous paragraph.

(4) *G is a separable type I unimodular group and $\omega = \int_{\hat{G}}^{\oplus} \omega_\xi \, d\mu(\xi)$ with ω_ξ invertible on a set $\mathcal{N} \subset \hat{G}$ of positive measure.* We define an invariant subspace $\mathcal{K}$ such that $\omega|_{\mathcal{K}}$ is invertible as above and get the statement of Corollary 4.7 in this case as well if $\ker\, \Omega^* = \{0\}$. Central weights fall in this class. Any weight on G such that the set $\mathcal{N} = \{\xi \in \hat{G} \mid \dim \mathcal{H}_\xi < \infty\}$ has positive μ-measure also satisfies that condition.

Recall that a locally compact group G is called an *[IN]-group* if it has a compact conjugation-invariant neighbourhood of the identity. It is called a *[SIN]-group* if it has a base of conjugate-invariant neighbourhoods of e. We note that any [SIN]-group is [IN]. Typical [SIN]-groups are discrete, compact and abelian groups.

The following result is a part of [26, Proposition 4.2].

Proposition 4.16. *G is an [IN]-group if and only if $VN(G)$ admits a normal tracial state.*

Corollary 4.17. *If G is an [IN]-group and $\omega \in VN(G)$ is a weight inverse such that* $\ker \Omega^* = \{0\}$*, then* $\operatorname{spec} A(G,\omega) \subset G_{\mathbb{C},\lambda}$.

Proof. As $\ker \Omega^* = \{0\}$ if follows from Lemma 4.4 that for all $\sigma \in \operatorname{spec} A(G,\omega)$ we have $\sigma(\mathscr{H})$ is dense in $\mathscr{H}$. Consider the following two inequalities

$$\sigma^*\sigma \leq \sigma^*\sigma + \omega^*\omega, \quad \omega^*\omega \leq \sigma^*\sigma + \omega^*\omega. \tag{25}$$

Letting $R = (\sigma^*\sigma + \omega^*\omega)^{\frac{1}{2}}$, we can deduce from (25), similar to the proof of Lemma 2.1, that there exist $U, V \in VN(G)$ such that

$$UR = \sigma, \quad VR = \omega. \tag{26}$$

Moreover, we have

$$R(U^*U + V^*V)R = \sigma^*\sigma + \omega^*\omega = R^2,$$

so that the density of the range of R (implied by the density of the range of ω) gives

$$U^*U + V^*V = I. \tag{27}$$

In particular, we obtain that U^*U and V^*V commute.

Assume towards a contradiction that $\sigma^*(\mathscr{H}) \cap \omega^*(\mathscr{H}) = \{0\}$. Then by (26) and the injectivity of R we can deduce that also $U^*(\mathscr{H}) \cap V^*(\mathscr{H}) = \{0\}$. Thus

$$(U^*U)(V^*V) = (V^*V)(U^*U) = 0,$$

so that (27) implies that U, V are partial isometries. As $\ker \Omega^* = \{0\}$, it follows from Lemma 4.4 that $\ker \sigma^* = \ker \omega^* = \{0\}$, and by (26), $\ker U^* = \ker V^* = \{0\}$. Thus U^* and V^* are isometries in $VN(G)$ such that (27) holds, i.e. (U, V) is a representation of the Cuntz algebra O_2 in $VN(G)$. This contradicts the claim that $VN(G)$ admits a tracial state ϕ:

$$1 = \phi(I) = \phi(U^*U + V^*V) = \phi(U^*U) + \phi(V^*V) = \phi(UU^*) + \phi(VV^*) = 2. \quad \square$$

Proposition 4.18. *If G is a [SIN]-group then* spec $A(G,\omega) \subset G_{\mathbb{C},\lambda}$ *for any weight inverse* ω.

Proof. By [2, 13.10.5], G is a [SIN]-group if and only if $VN(G)$ is finite. Therefore, as ker $\Omega = \{0\}$, we have ker $\Omega^* = \{0\}$, giving, by Lemma 4.4, $\ker \sigma^* = \{0\}$ and hence by finiteness of $VN(G)$, $\ker \sigma = \{0\}$ for any $\sigma \in \operatorname{spec} A(G,\omega)$. As both $(\sigma^*)^{-1}$ and $(\omega^*)^{-1}$ are densely defined and affiliated with $VN(G)$, and the set of affiliated elements is an algebra, we obtain that $\sigma^*(L^2(G)) \cap \omega^*(L^2(G)) \neq \{0\}$ as the domain of $(\sigma^*)^{-1} + (\omega^*)^{-1} \in \overline{VN(G)}$. Therefore, $\operatorname{spec} A(G,\omega) \subset G_{\mathbb{C},\lambda}$, by Corollary 4.7. $\square$

We remark that $VN(G)$ is finite for all Moore groups G and hence any such G is [SIN].

Corollary 4.19. *If G is discrete, then* spec $A(G,\omega) = G$ *for any weight inverse* ω.

Proof. G clearly does not contain any non-trivial image of a homomorphism $\mathbb{R} \to G$, and we can deduce that the complexification is trivial, i.e. $G_{\mathbb{C},\lambda} = G$. Moreover, as G is a [SIN]-group, by Proposition 4.18 the spectrum of $A(G,\omega)$ is the smallest possible, that is G. $\square$

The statement can be extended to totally disconnected [SIN]-groups which are prodiscrete, i.e. admits arbitrarily small compact open normal subgroups, as it was pointed out to us by the referee.

An important class of weights that has been studied in the literature are weights extended from closed abelian or compact subgroups, see [11, Proposition 3.25]. The next statements show that for all such weights we have the inclusion of the spectrum into the complexification. We first recall the construction of a so called central weight on the dual of a compact group following [11], see also [20].

If H is compact, we have the quasi-equivalence $\lambda \simeq \oplus_{\pi\in\widehat{H}}\pi$ which gives $VN(H) \simeq \oplus_{\pi\in\widehat{H}} M_{d_\pi}$, where d_π is the dimension of the representation space H_π. We have also the Plancherel theorem giving the isomorphism

$$L^2(H) \simeq \oplus^{\ell^2}_{\pi\in\widehat{H}} \sqrt{d_\pi} S^2_{d_\pi}$$

with $\langle \xi,\eta\rangle = \sum_{\pi\in\widehat{H}} d_\pi \operatorname{tr}(\hat{\xi}(\pi)\hat{\eta}(\pi)^*)$, for ξ, $\eta \in L^2(H)$, where S^2_n refers to Hilbert-Schmidt class on ℓ^2_n and $\hat{\xi}(\pi) = \int_H \xi(s)\pi(s^{-1})ds$. Recall also that for $A = (A(\pi))_{\pi\in\widehat{H}} \in VN(H)$ we have

$$\Gamma(A) = \oplus_{\pi,\pi'} \left[U^*_{\pi,\pi'}(\oplus_{\sigma\subseteq\pi\otimes\pi'} A(\sigma))U_{\pi,\pi'}\right]$$

where for $\sigma,\pi,\pi' \in \widehat{H}$, the notation $\sigma \subseteq \pi\otimes\pi'$ means that σ is a subrepresentation of $\pi\otimes\pi'$, and $U_{\pi,\pi'}$ is the unitary appearing in the irreducible decomposition of $\pi\otimes\pi'$.

If ω_H is a central positive weight inverse then $\omega_H \simeq \oplus_{\pi\in\widehat{H}}\omega(\pi)I_\pi$ for a function $\omega : \widehat{H} \to (0,+\infty)$ which satisfies $\omega(\pi)\omega(\rho) \le \omega(\sigma)$ for any σ, π, $\rho \in \widehat{H}$ such that $\sigma \subseteq \pi\otimes\rho$, see [11, 3.3.2]. We refer the reader to [20] and [11, section 5] for numerous examples of central weight inverses.

Recall the conjugate representation $\bar{\pi}$ of $\pi \in \widehat{H}$ which is defined as follows: we denote the linear dual space of H_π by $H_{\bar{\pi}}$ and for $A \in B(H_\pi)$, let A^t in $B(H_{\bar{\pi}})$ be its linear adjoint; for $s \in H$ we define $\bar{\pi}(s) = \pi(s^{-1})^t$; it is a unitary irreducible representation on $H_{\bar{\pi}}$ and $\bar{\bar{\pi}} = \pi$, as equivalence classes.

For the antipode S and the central weight ω_H we have $S(\omega_H) \simeq \oplus_{\pi\in\widehat{H}}\omega(\bar{\pi})I_\pi$. Indeed, we observe first that

$$\langle S(\omega_H)\xi,\eta\rangle = \langle\omega_H\bar{\eta},\bar{\xi}\rangle = \sum_{\pi\in\widehat{H}}\omega(\pi)\mathrm{tr}(\hat{\bar{\eta}}(\pi)\hat{\bar{\xi}}(\pi)^*).$$

Because of the unitary equivalence $\hat{\bar{\xi}}(\pi)^* = \int_H \xi(s)\pi(s)ds \sim \int_H \xi(s)\bar{\pi}(s^{-1})^t ds$ and $\hat{\bar{\eta}}(\pi) = \int_H \bar{\eta}(s)\pi(s^{-1})ds \sim \int_H \bar{\eta}(s)(\bar{\pi}(s^{-1})^*)^t ds$ with the same unitary operator, we obtain

$$\langle S(\omega_H)\xi,\eta\rangle = \sum_{\pi\in\widehat{H}}\omega(\pi)\mathrm{tr}(\hat{\xi}(\bar{\pi})\hat{\eta}(\bar{\pi})^*) = \sum_{\pi\in\widehat{H}}\omega(\bar{\pi})\mathrm{tr}(\hat{\xi}(\pi)\hat{\eta}(\pi)^*),$$

that shows the statement.

If $\sigma \subseteq \pi\otimes\rho$ then $\rho \subseteq \bar{\pi}\otimes\sigma$ which follows from [15, 2.30, 2.34(b,c)]; this gives $\omega(\sigma)\omega(\bar{\pi}) \le \omega(\rho)$ which together with the expression for the antipode gives the inequality

$$(S(\omega_H)\otimes I)\Gamma(\omega_H) \le I\otimes\omega_H, \tag{28}$$

the arguments for this are similar to that given in [11, 3.3.2].

If H is abelian, the weight inverse condition for positive weight inverse ω_H can be also equivalently written as (28) since in this case

$$\omega_H(s^{-1})\omega_H(st) \le \omega_H(t) \quad \text{for almost all } s,t\in\widehat{H}.$$

Theorem 4.20. *Let $H \subseteq G$ be a closed abelian or compact subgroup of G and $\iota_H : VN(H) \to VN(G)$ be the canonical injective homomorphism. Let $\omega_H \in VN(G)$ be a central weight inverse. If $\omega := \iota_H(\omega_H)$, then every $\sigma \in \mathrm{spec}\, A(G,\omega)$ is a weight inverse and*

$$S(\sigma)\sigma = S(\omega)\omega. \tag{29}$$

Moreover, $\mathrm{spec}\, A(G,\omega) \subset G_{\mathbb{C},\lambda}$.

Proof. Let Ω be the 2-cocycle associated to ω. By Proposition 4.14 $\ker\,\Omega^* = \{0\}$ and hence $\ker\sigma^* = \{0\}$ for every $\sigma \in \mathrm{spec}\, A(G,\omega)$ by Lemma 4.4. To show that σ is a weight inverse, it is enough to see the equality (29), which will imply $\ker\sigma = \{0\}$. To prove (29), let us without any loss of generality assume that ω_H is positive. Then ω_H satisfies (28). It is clearly preserved by ι_H giving

$$(S(\omega)\otimes I)\Gamma(\omega)\leq I\otimes\omega.$$

As in the proof of Lemma 2.1 we can conclude that there is a unique element $\Phi\in VN(G)\bar{\otimes}VN(G)$ such that

$$(I\otimes\omega)\Phi=(S(\omega)\otimes I)\Gamma(\omega).$$

If W is the fundamental multiplicative unitary, the latter equality gives

$$(I\otimes\omega)\Phi W^*=(S(\omega)\otimes I)W^*(I\otimes\omega). \tag{30}$$

Let $\xi,\eta,\tilde{\xi},\tilde{\eta}\in\mathscr{H}$. We retain the notation $\psi_{x,y}$ for the normal functional $\psi_{x,y}(T)=\langle Tx,y\rangle$, $T\in B(\mathscr{H})$. By Lemma 4.1,

$$\psi_{\xi,\tilde{\xi}}(S(\iota\otimes\psi_{\sigma^*\eta,\tilde{\eta}}(\Omega^*W^*)))=\langle(1\otimes\sigma^*)W(S(\sigma^*)\otimes 1)\xi\otimes\eta,\tilde{\xi}\otimes\tilde{\eta}\rangle, \tag{31}$$

for any $\sigma\in\operatorname{spec}A(G,\omega)$. In particular, it holds for ω which combined with (30) gives

$$\psi_{\xi,\tilde{\xi}}(S(\iota\otimes\psi_{\omega^*\eta,\tilde{\eta}}(\Omega^*W^*)))=\langle W\Phi^*\xi\otimes\omega^*\eta,\tilde{\xi}\otimes\tilde{\eta}\rangle. \tag{32}$$

Fix $\sigma\in\operatorname{spec}A(G,\omega)$. As the range of ω^* is dense in $\mathscr{H}$, there exists $\{\eta_n\}_n\subset\mathscr{H}$ such that $\omega^*\eta_n\to\sigma^*\eta$. From (31) and (32) we obtain

$$\begin{aligned}&\langle(1\otimes\sigma^*)W(S(\sigma^*)\otimes 1)\xi\otimes\eta,\tilde{\xi}\otimes\tilde{\eta}\rangle=\psi_{\xi,\tilde{\xi}}(S(\iota\otimes\psi_{\sigma^*\eta,\tilde{\eta}}(\Omega^*W^*)))\\&=\lim_{n\to\infty}\psi_{\xi,\tilde{\xi}}(S(\iota\otimes\psi_{\omega^*\eta_n,\tilde{\eta}}(\Omega^*W^*)))=\lim_{n\to\infty}\langle W\Phi^*\xi\otimes\omega^*\eta_n,\tilde{\xi}\otimes\tilde{\eta}\rangle\\&=\langle W\Phi^*\xi\otimes\sigma^*\eta,\tilde{\xi}\otimes\tilde{\eta}\rangle,\end{aligned}$$

giving

$$(I\otimes\sigma)\Phi W^*=(S(\sigma)\otimes I)W^*(I\otimes\sigma).$$

Reasoning as in the remark after the proof of Proposition 4.2, we have that the operator

$$M:=(\Phi W^*)W\Omega$$

satisfies

$$\begin{aligned}(I\otimes\sigma)M&=(S(\sigma)\otimes I)W^*(I\otimes\sigma)W\Omega\\&=(S(\sigma)\otimes I)\Gamma(\sigma)\Omega=S(\sigma)\sigma\otimes\sigma,\end{aligned} \tag{33}$$

for every $\sigma \in \operatorname{spec} A(G,\omega)$. As $\omega \in \operatorname{spec} A(G,\omega)$ and $\ker \omega = \{0\}$, it tells that $M = S(\omega)\omega \otimes I$ and $(I \otimes \sigma)M = (S(\omega)\omega) \otimes \sigma$ which together with (33) give the equality $S(\omega)\omega = S(\sigma)\sigma$. It now implies $\sigma^*(\mathscr{H}) \cap \omega^*(\mathscr{H}) \neq \{0\}$ and hence by Proposition 4.14 and Corollary 4.7, we get the claimed inclusion for the spectrum. □

Theorem 4.21. *Let $H \subseteq G$ be a closed subgroup, $\omega_H \in VN(H)$ be a weight inverse and $\omega := \iota_H(\omega_H)$. Assume that $\operatorname{spec} A(G,\omega) \subset G_{\mathbb{C},\lambda}$. Then every $\sigma \in \operatorname{spec} A(G,\omega)$ is of the form*

$$\sigma = \lambda_G(s)\iota_H(\tilde{\sigma}),$$

for some $s \in G$ and $\tilde{\sigma} \in \operatorname{spec} A(H,\omega_H)$.

Proof. The condition $\operatorname{spec} A(G,\omega) \subset G_{\mathbb{C},\lambda}$ implies that any $\sigma \in \operatorname{spec} A(G,\omega)$ admits a factorisation $\sigma = T\omega$ for some $T \in G_{\mathbb{C},\lambda}$; hence $\sigma^* \supset \omega^* T^*$ showing that $\sigma^* L^2(G) \cap \omega^* L^2(G) \neq \{0\}$. By Proposition 4.2, we get $S(\sigma)\sigma = S(\omega)\omega = \iota_H(S(\omega_H)\omega_H) \in \iota_H(VN(H))$. Moreover, as $T^* = (\sigma\omega^{-1})^* = (\omega^{-1})^*\sigma^*$ and $\ker T^* = \{0\}$, $\ker \sigma^* = \{0\}$ showing that σ is a weight inverse. It follows that

$$\Gamma(S(\sigma)\sigma)\Omega = \Gamma(S(\sigma))(\sigma \otimes \sigma) \in (\iota_H \bar{\otimes} \iota_H)(VN(H) \bar{\otimes} VN(H)), \tag{34}$$

and is independent of particular $\sigma \in \operatorname{spec} A(G,\omega)$. Applying a slice map $\iota \otimes f$, $f \in A(G)$, to (34) and using the fact that the elements of the form $f\sigma$ form a dense subspace in $A(G)$ (as the range of σ is dense) we obtain

$$(\iota \otimes f)(\Gamma(S(\sigma)))\sigma \in \iota_H(VN(H)), \quad \text{for all } f \in A(G). \tag{35}$$

Consider the subspace

$$\mathscr{A} = \overline{\{(\iota \otimes f)(\Gamma(S(\sigma))) \mid f \in A(G)\}}^{w^*} \subseteq VN(G)$$

(the weak* closure). By (35), we have $\mathscr{A}\sigma \subseteq \iota_H(VN(H))$. Let $I_{\mathscr{A}} \subseteq A(G)$ be the preannihilator of $\mathscr{A}$, i.e.

$$I_{\mathscr{A}} = \mathscr{A}_{\perp} := \{f \in A(G) \mid \langle A, f\rangle = 0 \ \forall A \in \mathscr{A}\}.$$

We claim that $I_{\mathscr{A}}$ is equal to the subspace

$$\{f \in A(G) \mid (\iota \otimes f)(\Gamma(S(\sigma))) = 0\},$$

and indeed, this follows from the action of $A(G)$ on $VN(G)$ being commutative. Moreover, the same argument shows that $I_{\mathscr{A}} \subseteq A(G)$ is a non-trivial closed ideal, as $\sigma \neq 0$. By duality, we have

$$\mathscr{A} = (\mathscr{A}_\perp)^\perp = \{x \in VN(G) \mid f(x) = 0, \forall f \in I_\mathscr{A}\}.$$

As $I_\mathscr{A} \neq A(G)$, there is at least one $s \in G$ such that $\lambda_G(s)^*$ annihilates $I_\mathscr{A}$, and hence $\lambda_G(s)^* \in \mathscr{A}$. It follows that

$$\lambda_G(s)^*\sigma \in \iota_H(VN(H)),$$

and moreover that the pre-image $\tilde{\sigma} = \iota_H^{-1}(\lambda_G(s)^*\sigma) \in \operatorname{spec} A(H, \omega_H)$. This gives the statement of the theorem. □

Combining methods in the proofs of Theorem 4.20 and Theorem 4.21 we obtain a generalisation of Theorem 4.20 to weights induced from non-central weights of compact subgroups of G.

Theorem 4.22. *Let $H \subseteq G$ be a compact subgroup, and $\omega_H \in VN(H)$ be a weight inverse on the dual of H. Then with $\omega = \iota_H(\omega_H)$, we have*

$$\operatorname{spec} A(G, \omega) \subseteq G_{\mathbb{C},\lambda}.$$

Moreover, every $\sigma \in \operatorname{spec} A(G, \omega)$ is of the form $\lambda_G(s)\iota_H(\tilde{\sigma})$ for some $s \in G$ and $\tilde{\sigma} \in \operatorname{spec} A(H, \omega_H)$.

Proof. Let $F \subset \widehat{H}$ be finite and set $\tilde{P}_F$ to be the central projection in $VN(H)$ given by $\tilde{P}_F = \oplus_{\pi \in \hat{H}} \chi_F(\pi) I_\pi$, where χ_F is the indicator function of F. Set

$$C_F = \{\pi \in \widehat{H} \,|\, \pi \subseteq \pi_1 \otimes \pi_2, \pi_1, \pi_2 \in F\}.$$

Then using arguments as in [11, 3.3.2], we obtain $\tilde{P}_F \otimes \tilde{P}_F \leq \Gamma(\tilde{P}_{C_F})$ and hence $(\tilde{P}_F \otimes \tilde{P}_F)\Gamma(\tilde{P}_{C_F}) = \tilde{P}_F \otimes \tilde{P}_F$, which gives

$$(\tilde{P}_F \otimes \tilde{P}_F)W^*(I \otimes \tilde{P}_{C_F}) = (\tilde{P}_F \otimes \tilde{P}_F)W^*.$$

As $(\tilde{P}_F \otimes \tilde{P}_F)W^* \in VN(H) \otimes (\oplus_{\pi \in F} M_{d_\pi})$ we can apply $S \otimes \iota$ to the last equality to obtain

$$(I \otimes \tilde{P}_F)W(S(\tilde{P}_F) \otimes \tilde{P}_{C_F}) = (I \otimes \tilde{P}_F)W(S(\tilde{P}_F) \otimes I)$$

and

$$\Gamma(\tilde{P}_F)(S(\tilde{P}_F) \otimes \tilde{P}_{C_F}) = \Gamma(\tilde{P}_F)(S(\tilde{P}_F) \otimes I). \tag{36}$$

In $\overline{VN(H \times H)}$, we consider the element

$$\tilde{\Phi}^* = \Gamma(\omega_H^*)(S(\omega_H^*) \otimes I)(I \otimes (\omega_H^*)^{-1}).$$

Note that for $\xi \in L^2(H)$, $\eta \in \mathscr{D}((\omega_H^*)^{-1})$,

$$\begin{aligned}\Gamma(\tilde{P}_F)\tilde{\Phi}^*(S(\tilde{P}_F)\xi \otimes \eta) &= \Gamma(\tilde{P}_F)\Gamma(\omega_H^*)(S(\omega_H^*) \otimes I)(S(\tilde{P}_F)\xi \otimes (\omega_H^*)^{-1}\eta) \\ &= \Gamma(\omega_H^*)\Gamma(\tilde{P}_F)(S(\tilde{P}_F) \otimes I)(S(\tilde{\omega}^*)\xi \otimes (\omega_H^*)^{-1}\eta) \\ &= \Gamma(\omega_H^*)\Gamma(\tilde{P}_F)(S(\tilde{P}_F) \otimes \tilde{P}_{C_F})(S(\omega_H^*)\xi \otimes (\omega_H^*)^{-1}\eta) \\ &= \Gamma(\omega_H^*\tilde{P}_F)(S(\omega_H^*\tilde{P}_F)\xi \otimes (\omega_H^*)^{-1}\tilde{P}_{C_F}\eta)\end{aligned}$$

from which we conclude that $\Gamma(\tilde{P}_F)\tilde{\Phi}^*(S(\tilde{P}_F) \otimes I)$ extends to a bounded operator in $VN(H \times H)$.

Now let $P_F = \iota_H(\tilde{P}_F)$, $P_{C_F} = \iota_H(\tilde{P}_{C_F})$ and $\Phi^* = (\iota_H \otimes \iota_H)(U)(\iota_H \otimes \iota_H)(|\tilde{\Phi}^*|) \in \overline{VN(G \times G)}$, where $\tilde{\Phi}^* = U|\tilde{\Phi}^*|$ is the polar decomposition of $\tilde{\Phi}^*$, and $(\iota_H \otimes \iota_H)(|\tilde{\Phi}^*|)$ is the extension of $\iota_H \otimes \iota_H$ to the positive operator $|\tilde{\Phi}^*| \in \overline{VN(H \times H)}$, see [11, Section 2].

Applying [11, Proposition 2.1] we can conclude that $|\tilde{\Phi}^*|(I \otimes \tilde{\omega}^*) \in VN(H \times H)$ and $\iota_H \otimes \iota_H(|\tilde{\Phi}^*|)(I \otimes \omega^*) \in VN(G \times G)$, which show that $\Phi^*(I \otimes \omega^*)$ is bounded and

$$\Phi^*(I \otimes \omega^*) = \Gamma(\omega^*)(S(\omega^*) \otimes I).$$

Let Ω be the 2-cocycle associated with ω. As in the proof of Theorem 4.20 we have for $\sigma \in \operatorname{spec} A(G, \omega)$

$$\psi_{\xi,\tilde{\xi}}(S(\iota \otimes \psi_{\sigma^*\eta,\tilde{\eta}}(\Omega^* W^*))) = \langle (1 \otimes \sigma^*)W(S(\sigma^*) \otimes 1)\xi \otimes \eta, \tilde{\xi} \otimes \tilde{\eta}\rangle, \tag{37}$$

and

$$\psi_{\xi,\tilde{\xi}}(S(\iota \otimes \psi_{\omega^*\eta,\tilde{\eta}}(\Omega^* W^*))) = \langle W\Phi^*(\xi \otimes \omega^*\eta), \tilde{\xi} \otimes \tilde{\eta}\rangle, \tag{38}$$

where $\xi, \eta, \tilde{\xi}, \tilde{\eta} \in \mathscr{H}$.

Take $\xi \in S(P_F)L^2(G)$ and $\tilde{\eta} \in P_F L^2(G)$. Then the right-hand side of (38) becomes

$$\langle W(\Gamma(P_F)\Phi^*(S(P_F) \otimes I))(\xi \otimes \omega^*\eta), \tilde{\xi} \otimes \tilde{\eta}\rangle.$$

Fix $\sigma \in \operatorname{spec} A(G, \omega)$. As the range of ω^* is dense in $\mathscr{H}$, there exists $\{\eta_n\}_n \subset \mathscr{H}$ such that $\omega^*\eta_n \to \sigma^*\eta$. From (37) and (38) together with $\Gamma(P_F)\Phi^*(S(P_F) \otimes I) \in VN(G \times G)$, we get

$$\begin{aligned}&\langle (1 \otimes \sigma^*)W(S(\sigma^*) \otimes 1)\xi \otimes \eta, \tilde{\xi} \otimes \tilde{\eta}\rangle = \psi_{\xi,\tilde{\xi}}(S(\iota \otimes \psi_{\sigma^*\eta,\tilde{\eta}}(\Omega^* W^*))) \\ &= \lim_{n\to\infty} \psi_{\xi,\tilde{\xi}}(S(\iota \otimes \psi_{\omega^*\eta_n,\tilde{\eta}}(\Omega^* W^*))) \\ &= \lim_{n\to\infty} \langle W\Gamma(P_F)\Phi^*(S(P_F) \otimes I)\xi \otimes \omega^*\eta_n, \tilde{\xi} \otimes \tilde{\eta}\rangle \\ &= \langle W\Gamma(P_F)\Phi^*(S(P_F) \otimes I)\xi \otimes \sigma^*\eta, \tilde{\xi} \otimes \tilde{\eta}\rangle\end{aligned}$$

giving

$$(\Gamma(P_F)\Phi^*(S(P_F)\otimes I))(I\otimes\sigma^*)=\Gamma(P_F)\Gamma(\sigma^*)(S(\sigma)^*S(P_F)\otimes I). \tag{39}$$

Let M be in the commutant of $\iota_H(VN(H))\bar{\otimes}VN(G)$. Clearly, M commutes with the left-hand side of (39) and as $\tilde{P}_F\to I$ weak*, we obtain that it commutes with $\Gamma(\sigma^*)(S(\sigma)^*\otimes I)$. Therefore,

$$\Gamma(\sigma^*)(S(\sigma)^*\otimes I)\in\iota_H(VN(H))\bar{\otimes}VN(G). \tag{40}$$

If we let $f\in VN(G)_*$ be arbitrary, then it follows from (40)

$$S(\sigma)(\iota\otimes f)(\Gamma(\sigma))\in\iota_H(VN(H)).$$

We now proceed in a similar way as in the proof of Theorem 4.21 and let

$$\mathscr{A}=\overline{\{(\iota\otimes f)(\Gamma(\sigma))\mid f\in A(G)\}}^{w^*}.$$

We can argue as before that the ideal $I_{\mathscr{A}}:=\mathscr{A}_\perp\neq A(G)$ and hence there is $s\in G$ such that $f(s)=0$ for all $f\in I_{\mathscr{A}}$. As $I_{\mathscr{A}}^\perp=\mathscr{A}$, $\lambda_G(s)\in\mathscr{A}$ and therefore $S(\sigma)\lambda_G(s)\in\iota_H(VN(H))$ and $\lambda_G(s^{-1})\sigma\in\iota_H(VN(H))$. It follows that there is an $\tilde{\sigma}\in\operatorname{spec}A(H,\omega_H)$ such that $\lambda_G(s^{-1})\sigma=\iota_H(\tilde{\sigma})$, and hence

$$\sigma=\lambda_G(s)\iota_H(\tilde{\sigma}). \tag{41}$$

As $\operatorname{spec}A(H,\omega_H)\subseteq H_{\mathbb{C},\lambda}$, we conclude that $\operatorname{spec}A(G,\omega)\subseteq G_{\mathbb{C},\lambda}$. □

Let G be a connected simply connected Lie group and $\mathfrak{g}$ its associated Lie algebra. We also fix the symbol H and $\mathfrak{h}$ for a connected closed Lie subgroup of G and its Lie algebra respectively. We write λ_G and λ_H for the left regular representations of G and H respectively. The next statement generalizes [11, Theorem 5.9, Theorem 6.19, Theorem 7.11, Theorem 8.20 and Theorem 9.11], where it was proved for compact connected Lie groups with a weight induced from a closed Lie subgroup, the Heisenberg group, the reduced Heisenberg group, the Euclidean motion group on $\mathbb{R}^2$, and the simply connected cover of it with weights induced from abelian connected Lie subgroups. We note that the proofs of the latter theorems from [11] required lengthy and specific arguments for each particular group. We also answer [11, Question 11.4] as our technique does not require the existence and density of entire vectors for the left regular representation which was essential to prove the mentioned results in [11].

Theorem 4.23. *Let G be a connected simply connected Lie group and let H be either abelian or compact connected closed subgroup of G. Suppose ω_H is a positive weight* *inverse on the dual of H and $\omega = \iota_H(\omega_H)$ is the extended weight inverse on the dual of G. Then*

$$\text{spec } A(G,\omega) \simeq \{\lambda_G(s)\exp i\partial\lambda_G(X) \mid s \in G, X \in \mathfrak{h}, \exp i\partial\lambda_H(X) \in \text{spec } A(H,\omega_H)\}.$$

Proof. By Theorem 4.20, Theorem 4.22 and the remark after Lemma 4.15 we have

$$\text{spec } A(G,\omega) \subset G_{\mathbb{C},\lambda},$$

and hence by Proposition 3.4 for any $\sigma \in \text{spec } A(G,\omega)$ there is a unique $s \in G$ and $X \in \mathfrak{g}$ such that $\text{Ran}(\omega) \subset \mathfrak{D}(\exp i\partial\lambda_G(X))$, $\exp i\partial\lambda_G(X)\omega$ is bounded and

$$(\sigma, u) = (\lambda_G(s)\exp i\partial\lambda_G(X)\omega, u)_\omega$$

for all $u \in A(G,\omega)$. By Theorem 4.21 and Theorem 4.22, we have that

$$\lambda_G(s)\exp i\partial\lambda_G(X)\omega = \lambda_G(t)\iota_H(\tilde{\sigma})$$

for some $\tilde{\sigma} \in \text{spec } A(H,\omega_H)$ and $t \in G$. By assumption of the theorem, there exist $\tilde{s} \in H$ and $\tilde{X} \in \mathfrak{h}$ such that and $\tilde{\sigma} = \lambda_H(\tilde{s})\exp i\partial\lambda_H(\tilde{X})$. As $\iota_H(\exp i\partial\lambda_H(\tilde{X})) = \exp i\partial\lambda_G(\tilde{X})$, we obtain by applying [11, Proposition 2.1] that $\iota_H(\tilde{\sigma}) = \lambda_G(\tilde{s})\exp i\partial\lambda_G(\tilde{X})$ from which we get the inclusion "$\subset$".

Conversely, if $\exp i\partial\lambda_H(X) \in \text{spec } A(H,\omega_H)$, then $\exp i\partial\lambda_G(X) \in \text{spec } A(G,\iota_H(\omega_H))$, which follows from [11, Proposition 2.1]. □

Example 4.24. Consider the "$ax+b$"-group that can be represented as the group G of matrices:

$$G = \left\{ g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}.$$

It is known to be the semidirect product of the subgroups

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\} \text{ and } B = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}.$$

The Lie algebra of G is generated by H and E given by

$$H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \text{ and } E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

that satisfy $[H,E] = E$.

We have the one parameter subgroups $A = \{\exp tH \mid t \in \mathbb{R}\}$ and $B = \{\exp sE \mid s \in \mathbb{R}\}$. The unitary dual of G can be described as follows:

$$\widehat{G} = \{\sigma_\pm\} \cup \{\chi_r : r \in \mathbb{R}\},$$

where χ_r is a one-dimensional representation for $r \in \mathbb{R}$ and $\sigma_\pm$ are two infinite-dimensional representations defined on $L^2(\mathbb{R})$ given as

$$\begin{aligned} \sigma_\pm(g)f(x) &= \exp\left(\pm ise^x\right) f(x+t) \\ \chi_r(g) &= e^{itr} \end{aligned}$$

for $g = \exp sE \exp tH$. Moreover, we have that

$$\begin{cases} \partial\sigma_\pm(H)f = f' \\ \partial\sigma_\pm(E)f(x) = \pm ie^x f(x) \end{cases}$$

We have the following quasi-equivalence of the left regular representation λ:

$$\lambda \simeq \sigma_+ \oplus \sigma_-,$$

and $VN(G) \simeq B(L^2(\mathbb{R})) \oplus B(L^2(\mathbb{R}))$ (see [10, chapter 4.3] and [9, chapter 3.8]).

For a bounded below weight function $w : \widehat{B} \simeq \mathbb{R} \to (0, \infty)$, write $M_{w^{-1}}$ for the multiplication operator on $L^2(\mathbb{R})$ by the function $w^{-1} \in L^\infty(\mathbb{R})$ and consider $\widetilde{M_{w^{-1}}} = \mathcal{F}^{-1} M_{w^{-1}} \mathcal{F} \in VN(\mathbb{R}) \simeq VN(B)$, where $\mathcal{F}$ is the Fourier transform. Let $\omega = \iota_B(\widetilde{M_{w^{-1}}})$ be the extended weight inverse. Then, by [11, Proposition 3.26], $\omega \sim (\omega(\sigma_+), \omega(\sigma_-))$, where $\omega(\sigma_\pm)\xi(x) = w^{-1}(\mp e^x)\xi(x)$. By Theorem 4.23,

$$\text{spec } A(G, \omega) \simeq \{\lambda_G(g) \exp i\partial\lambda_G(X) \mid g \in G, X = sE, e^{sx}/w(x) \in L^\infty(\mathbb{R})\}.$$

In particular, if $w(x) = \beta^{|x|}$, then

$$\text{spec } A(G, \omega) \simeq \{\exp\left(t\partial\lambda_G(H)\right) \exp\left(s\partial\lambda_G(E)\right) \mid t \in \mathbb{R}, |\text{Im } s| \le \ln\beta\}.$$

Similarly, we can start with a bounded below weight $\tilde{w} : \widehat{A} \simeq \mathbb{R} \to (0, \infty)$ and consider $\tilde{\omega} = \iota_A(\widetilde{M_{w^{-1}}})$. We have $\tilde{\omega}(\sigma_\pm) = \mathcal{F}^{-1}\tilde{w}^{-1}(-x)\mathcal{F}$ and if $\tilde{w} = \tilde{\beta}^{|x|}$, then

$$\begin{aligned} \text{spec } A(G, \tilde{\omega}) &\simeq \{\lambda_G(g) \exp i\partial\lambda_G(sH) \mid g \in G, s \in \mathbb{R}, |s| \le \ln\tilde{\beta}\} \\ &= \{\exp\left(t\partial\lambda_G(E)\right) \exp\left(s\partial\lambda_G(H)\right) \mid t \in \mathbb{R}, |\text{Im } s| \le \ln\tilde{\beta}\}. \end{aligned}$$

We note that by [13] the left regular representation does not admit a dense subset of entire vectors, the fact that was an obstacle in [11] for the study of the spectrum of $A(G, \omega)$. The density of the set $\mathcal{H}_w(\lambda)$ of entire vectors was also important for the identification of $A(G, \omega)$ as a subset of the complexification $G_\mathbb{C}$ of G: letting $\lambda_\mathbb{C}(\exp X) = \exp\partial\lambda(X)$, $X \in \mathfrak{g}_\mathbb{C}$, one obtains a representation of $G_\mathbb{C}$ on $\mathcal{H}_w(\lambda)$, see [13, Corollary 2.2]; in general and in particular for the "$ax + b$"-group, it seems there is no natural way for

λ to be continued to a global representation of the complexified group and hence to see $G_{\mathbb{C},\lambda}$ as a group.

We refer the reader to [11] for other specific examples of weights and precise descriptions of the spectrum of the associated Beurling-Fourier algebras (Examples 6.21, 7.13 and 8.22).

5. Some remarks and open questions

In this section, we list open questions and make some remarks. The most pressing question that we left unanswered is of course whether we can extend the point spectrum correspondence to general locally compact groups. Namely,

Question 5.1. *Does* $\operatorname{spec} A(G,\omega) \subset G_{\mathbb{C},\lambda}$ *hold for any locally compact group* G *and any weight inverse* ω*?*

Let Ω be the 2-cocycle, corresponding to ω. Whether we can answer the above question positively seems to rely on whether the following statements are true:

(i) $\ker\ \Omega^* = \{0\}$;
(ii) $S(\sigma)\sigma = S(\omega)\omega$ holds for all $\sigma \in \operatorname{spec}\ A(G,\omega)$;
(iii) $\sigma^*(\mathscr{H}) \cap \omega^*(\mathscr{H}) \neq \{0\}$ for any $\sigma \in \operatorname{spec}\ A(G,\omega)$;
(iv) any $\sigma \in \operatorname{spec}\ A(G,\omega)$ is a weight inverse.

We have the implication $(ii) \Rightarrow (iii)$, as

$$\sigma^* S(\sigma)^*\xi = \omega^* S(\omega)^*\xi \in \sigma^*(\mathscr{H}) \cap \omega^*(\mathscr{H}), \quad \text{for all } \xi \in \mathscr{H}, \tag{42}$$

and as $\ker \omega^* = \ker S(\omega)^* = \{0\}$, thus also $\ker \omega^* S(\omega)^* = \{0\}$, we obtain that the subspace $\sigma^*(\mathscr{H}) \cap \omega^*(\mathscr{H})$ is non-trivial. By Corollary 4.7, (i) and (ii) give the embedding $\operatorname{spec} A(G,\omega) \subset G_{\mathbb{C},\lambda}$; Theorem 4.5 shows that (i) and (iii) imply (ii) and (iv).

As it was noticed in Section 2 the definition of the product in $A(G,\omega)$ depends on the 2-cocycle Ω rather than the weight inverse ω, and $A(G,\omega) \simeq A(G,\Omega)$, where $A(G,\Omega)$ is $A(G)$ (as a Banach space) with the modified product

$$f \cdot_\Omega g = \Gamma_*(\Omega(f \otimes g)), \quad \text{for } f, g \in A(G).$$

We note that the 2-cocycle Ω associated with a weight inverse is always symmetric, i.e. invariant with respect to the flip automorphism on $VN(G)\bar{\otimes}VN(G)$,

Question 5.2. *Can one develop a similar theory for* $A(G,\Omega)$ *with general (symmetric)* 2-*cocycle* Ω*? What are the conditions on* Ω *that guarantee the existence of a weight inverse* ω *such that* $\Gamma(\omega)\Omega = \omega \otimes \omega$*?*

More specific questions are:

Question 5.3. *For which symmetric 2-cocycles Ω is the spectrum of $A(G,\Omega)$ non-empty?*

It seems that it depends on whether or not Ω^*, or perhaps Ω, has a non-trivial kernel. Below we give examples of Ω for which ker $\Omega^* \neq \{0\}$ and spec $A(G,\Omega) = \emptyset$.

Example. Let $G = \mathbb{R}$, so that $VN(\mathbb{R}) \cong L^\infty(\mathbb{R})$. Let

$$\Upsilon(x) = \begin{cases} (1+x)^x, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

and let $\Omega : \mathbb{R}^2 \to \mathbb{C}$ be the measurable function given by

$$\Omega(x,y) = \begin{cases} \frac{\Upsilon(x)\Upsilon(y)}{\Upsilon(x+y)}, & \text{for } x, y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\Upsilon(x)\Upsilon(y) \leq \Upsilon(x+y)$ for $x, y \geq 0$ and hence $\Omega(x,y) \leq 1$. Thus $\Omega(x,y) \in L^\infty(\mathbb{R})\bar{\otimes}L^\infty(\mathbb{R})$. Moreover, it is not hard to see that Ω is a symmetric 2-cocycle. Hence we have a well-defined algebra $A(\mathbb{R},\Omega)$. Using that $A(\mathbb{R}) \cong L^1(\mathbb{R})$ via the Fourier transform, the Ω-modified product between $f, g \in L^1(\mathbb{R})$ is

$$f *_\Omega g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\Omega(x-y,y)\, dy = \int_0^\infty f(x-y)g(y)\frac{\Upsilon(x-y)\Upsilon(y)}{\Upsilon(x)}\, dy. \quad (43)$$

Notice that if $x < 0$, then $\Omega(x-y,y) = 0$ for all $y \in \mathbb{R}$ and hence $f *_\Omega g = 0$ a.e. on $(-\infty, 0)$; in particular, $B := L^1(\mathbb{R}^+)$ is a subalgebra of $(L^1(\mathbb{R}), *_\Omega)$.

Next we will see that spec B is empty. Let $B' = L^1(\mathbb{R}^+, \frac{1}{\Upsilon})$ with the convolution product $(f*g)(x) = \int_0^\infty f(x-y)g(y)dy$. Then

$$f(x) \in B \mapsto f(x)\Upsilon(x) \in B'$$

is an isometric isomorphism. Let ϕ be a linear multiplicative functional on B'. Then there is $m \in L^\infty(\mathbb{R}^+)$ such that $m(x)\Upsilon(x) \in L^\infty(\mathbb{R}^+)$ and

$$\phi(f) = \int_0^\infty m(x)f(x)\, dx, \quad \text{for } f \in L^1(\mathbb{R}^+).$$

As ϕ is multiplicative, $m(x) = e^{ax}$ for some $a \in \mathbb{C}$. As $\lim_{x\to\infty} |e^{ax}\Upsilon(x)| = \lim_{x\to\infty} |e^{ax}(1+x)^x| \to \infty$ for any $a \in \mathbb{C}$, the spectrum of B' and hence of B is empty. To see that this carries over to the actual algebra $A(\mathbb{R},\Omega)$, we use that $f *_\Omega g \in B$, for all $f, g \in L^1(\mathbb{R})$, and hence if we would have a multiplicative linear functional ϕ such that $\phi(f) = 1$ for some $f \in L^1(\mathbb{R})$, then $\phi(f *_\Omega f) = 1$ showing that the restriction of ϕ to B is a non-zero multiplicative functional on B and hence $\phi \in \operatorname{spec} B$, giving a contradiction.

We modify the previous example slightly to obtain a continuous 2-cocycle. Consider the function

$$\nu(x) = \begin{cases} e^{-\frac{1}{x}}, & \text{for } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\nu(x+y) \geq \nu(x)\nu(y)$ for all $x, y \in \mathbb{R}$. Now let

$$L(x) = \begin{cases} \nu(x)\Upsilon(x), & \text{for } x \geq 0, \\ 0, & \text{for } x < 0, \end{cases}$$

and

$$\Theta(x, y) = \begin{cases} \frac{L(x)L(y)}{L(x+y)}, & \text{for } x, y \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

then $\Theta(x, y) \leq 1$ for all $x, y \in \mathbb{R}$. Furthermore, we have $\Theta(x, y) \in C_b(\mathbb{R}^2)$. It is not that hard to see that also $A(\mathbb{R}, \Theta)$ has empty spectrum (the argument is more or less the same as above). If $G_{\mathbb{C},\lambda} \neq G$ then the homomorphism $\bar{\varphi} : C_b(\mathbb{R}) \to VN(G)$ from the proof of Proposition 4.9 intertwines the coproducts and the image $(\bar{\varphi} \otimes \bar{\varphi})(\Theta)$ is then also a 2-cocycle. It seems reasonable to expect that the resulting algebra would also have properties similar to the one above (i.e. not very nice spectrum-vice).

Question 5.4. *What happens if we remove the condition* $\ker \omega = \ker \omega^* = \{0\}$ *from the definition of weight inverse?*

We call such ω a partial weight inverse. A classification of partial weight inverses for discrete G will be given in a separate paper.

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