



## **$L^p$ -ESTIMATES OF EXTENSIONS OF HOLOMORPHIC FUNCTIONS DEFINED ON A NON-REDUCED SUBVARIETY**

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defined on a non-reduced subvariety***

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# **$L^p$ -ESTIMATES OF EXTENSIONS OF HOLOMORPHIC FUNCTIONS DEFINED ON A NON-REDUCED SUBVARIETY**

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ABSTRACT. — Let  $D$  be a strictly pseudoconvex domain in  $\mathbb{C}^N$  and  $X$  a pure-dimensional non-reduced subvariety that behaves well at  $\partial D$ . We provide  $L^p$ -estimates of extensions of holomorphic functions defined on  $X$ .

RÉSUMÉ. — Soit  $D$  un domaine strictement pseudo-convexe de  $\mathbb{C}^N$  et  $X$  une sous-variété non réduite qui se comporte bien en  $\partial D$ . Nous donnons des estimations  $L^p$  des extensions de fonctions holomorphes sur  $X$ .

## **1. Introduction**

Let  $D$  be a pseudoconvex domain in  $\mathbb{C}^N$  and let  $X$  be a smooth submanifold of dimension  $n$ . For any holomorphic function  $\phi$  on  $X$  there is a holomorphic extension  $\Phi$  to  $D$ . The celebrated Ohsawa–Takegoshi theorem, [21], provides very precise weighted  $L^2$ -estimates of such extensions. This theorem, and various variants, have played a decisive role in complex analysis and algebraic geometry during the last decades, see, e.g., [20]. There are also quite recent extension results, see, e.g., [15] and [17], obtained by  $L^2$ -methods, in certain cases when  $X$  is not reduced.

In case  $D$  is strictly pseudoconvex there are  $L^p$ - and  $H^p$ -estimates of extensions from smooth submanifolds, based on integral representation, see [2, 16, 18]. Notably is that if  $D$  is strictly pseudoconvex, and  $X$  behaves reasonably at  $\partial D$ , then any bounded holomorphic function on  $X$  admits a bounded extension. In [1] there are estimates of extensions from

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non-smooth hypersurfaces. These results are based on integral formulas for representing the extensions or for solving  $\bar{\partial}$ -equations in  $D$ .

Let  $i: X \rightarrow D$  be a non-reduced subspace of pure dimension  $n$  of a pseudoconvex domain  $D$ . That is, we have a coherent ideal sheaf  $\mathcal{J} \rightarrow D$  of pure dimension  $n$  so that the sheaf  $\mathcal{O}_X$  of holomorphic functions on  $X$ , the structure sheaf, is isomorphic to  $\mathcal{O}_D/\mathcal{J}$ . We thus have a natural mapping  $i^*: \mathcal{O}_D \rightarrow \mathcal{O}_X$ , and we say that  $\Phi$  is an extension of a function  $\phi$  on  $X$ , or that  $\Phi$  interpolates  $\phi$ , if  $i^*\Phi = \phi$ . In [7] we introduced a pointwise coordinate invariant norm  $|\phi|_X$  of holomorphic functions  $\phi$  on  $X$ . In this paper we will only consider  $X$  such that the underlying reduced space  $Z$ , i.e., the zero set of  $\mathcal{J}$ , is smooth. In this case the norm  $|\phi|_X$  is well-defined on compact subsets up to multiplicative constants. Recall that a holomorphic differential operator  $L$  in  $D$  is Noetherian with respect to  $\mathcal{J}$  if  $L\Psi$  vanishes on  $Z$  as soon as  $\Psi$  is in  $\mathcal{J}$ . Such an  $L$  induces a mapping  $\mathcal{L}: \mathcal{O}_X \rightarrow \mathcal{O}_Z$  that we also call a Noetherian operator. In [7] we introduced a locally finitely generated coordinate invariant  $\mathcal{O}_D$ -sheaf  $\mathcal{N}_X$  of Noetherian operators such that  $\phi = 0$  if and only if  $\mathcal{L}\phi = 0$  for all  $\mathcal{L}$  in  $\mathcal{N}_X$ . We defined the pointwise norm locally as

$$(1.1) \quad |\phi(z)|_X = \sum_j |\mathcal{L}_j\phi(z)|,$$

where  $\mathcal{L}_\ell$  is finite set of generators of  $\mathcal{N}_X$ . For a precise description of  $\mathcal{N}_X$ , see Section 2. Notice that  $|\phi(z)|_X = 0$  in an open set if and only if  $\phi$  vanishes identically there. Roughly speaking  $|\cdot|_X$  is the smallest invariant norm with this property, see Remark 3.3.

By means of  $|\cdot|_X$  we can define  $L^p$ -norms of  $\phi$  in  $\mathcal{O}_X$ . It is then natural to look for  $L^p$ -estimates of extensions of holomorphic functions on  $X$ . In this paper we present a couple of such results when  $D$  is strictly pseudoconvex. We do not look for the most general possible statements but our aim is to point out some new ideas. In order not to conceal them by technicalities we assume that  $X$  behaves well at the boundary of  $D$ . Here is our main result.

**THEOREM 1.1.** — *Let  $D \subset \Omega \subset \mathbb{C}^N$  be a strictly pseudoconvex domain with smooth boundary, and let  $i: X \rightarrow \Omega \subset \mathbb{C}^N$  be a non-reduced subspace of pure dimension  $n$  such that  $Z = X_{\text{red}}$  is smooth and intersects  $\partial D$  transversally. Assume that  $\mathcal{O}_X$  is Cohen–Macaulay at each point on  $Z \cap \partial D$ . Let  $\kappa = N - n$ . Assume that  $1 \leq p < \infty$  and that  $r > -1$ . Let  $\delta(z) = \text{dist}(z, \partial D)$  be the distance to the boundary. Each holomorphic function  $\phi$*

in  $\mathcal{O}(X \cap D)$  admits a holomorphic extension  $\Phi \in \mathcal{O}(D)$  such that

$$(1.2) \quad \int_D \delta^r |\Phi|^p dV_D \leq C_{r,p}^p \int_{Z \cap D} \delta^{\kappa+r} |\phi|_X^p dV_Z,$$

provided that the right hand side is finite.

Here  $dV_D$  and  $dV_Z$  denote some volume forms on  $D$  and  $Z$ , respectively. Since  $X$  is defined in  $\Omega$ , the  $L^p$ -norms are well-defined up to multiplicative constants.

The transversality condition means that if  $\rho$  is a defining function for  $D$  and  $(\zeta, \eta)$  are local coordinates such that  $Z = \{\eta = 0\}$ , then  $\partial\rho \wedge d\eta_1 \wedge \dots \wedge d\eta_\kappa$  is non-vanishing on  $\partial D \cap Z$ . In particular,  $D \cap Z$  is a strictly pseudoconvex domain in  $Z$  with smooth boundary.

Assume that  $D \subset \mathbb{C}_{\zeta, \tau}^{n+\kappa}$  is the unit ball,  $Z = \{\tau = 0\}$  and  $X = Z$  is reduced. If  $\phi(\zeta)$  is holomorphic on  $Z \cap D$  and  $\Phi(\zeta, \eta) = \phi(\zeta)$  is the trivial extension to the entire ball, and  $\delta(\zeta, \tau) = 1 - |\zeta|^2 - |\tau|^2$ , then

$$\int_D \delta^r |\Phi|^p dV_D = c_{r,\kappa} \int_{Z \cap D} \delta^{r+\kappa} |\phi|^p dV_Z,$$

where  $c_{r,\kappa} = \pi^\kappa / (r+1) \cdots (r+\kappa)$ . It follows that the estimate (1.2) is sharp up to the constant  $C_{r,p}$  when  $X$  is reduced. In the non-reduced case it is not, as we will see in our second result.

Assume that  $Z$  is a smooth hypersurface in  $\Omega$  defined by the function  $f$  in  $\Omega$ , i.e.,  $Z = Z(f)$  and  $df \neq 0$  on  $Z$ , let  $\mathcal{J} = \langle f^{M+1} \rangle$  and let  $\mathcal{O}_X = \mathcal{O}_\Omega / \mathcal{J}$ . It turns out that then  $\mathcal{N}_X$  is generated by all differential operators of order at most  $M$ , so that

$$|\phi(z)|_X = \sum_{k=0}^M \sum_{|\beta|=k} |\partial^\beta \phi|_z, \quad z \in Z.$$

**THEOREM 1.2.** — *Let  $D \subset\subset \Omega$  be as in Theorem 1.1. Assume that  $Z$  is a smooth hypersurface in  $\Omega$  that intersect  $\partial D$  transversally. Assume that that  $Z$  is defined by the function  $f$  and let  $\mathcal{O}_X = \mathcal{O}_D / \langle f^{M+1} \rangle$ . Moreover, assume that  $1 \leq p < \infty$ ,  $r > -1$ , and let  $\delta$  be the distance to the boundary. Each function  $\phi$  on  $\mathcal{O}(D \cap X)$  has an extension  $\Phi \in \mathcal{O}(D)$  such that*

$$\int_D \delta^r |\Phi|^p dV_D \leq C_{r,p}^p \sum_{k=0}^M \int_{Z \cap D} \delta^{r+1+k/2} \sum_{|\beta|=k} |\partial^\beta \phi|^p dV_Z,$$

provided that the right hand side is finite.

Thus the requirement is less restrictive for higher derivatives of  $\phi$ .

The extension  $\Phi$  in Theorems 1.1 and 1.2 is obtained by an integral formula, that in turn is constructed by means of the residue currents in [10]

and the division-interpolation formulas in [4]. A main novelty is the technique to carry out the estimates in terms of the norm in [7].

When  $D$  is a ball the extension formula is explicitly given in terms of the residue current associated with  $X$ . In the general case the analogously constructed formula does not provide a holomorphic extension, so it has to be slightly modified by a technique inspired by a classical idea of Kerzman–Stein and Ligocka, see, e.g., [22]. To this end we have to construct a linear solution operator for the  $\bar{\partial}$ -equation for  $\bar{\partial}$ -closed smooth  $(0, 1)$ -forms in  $\mathcal{E}\mathcal{J}$  for a quite arbitrary ideal sheaf  $\mathcal{J}$ , Theorem 9.1.

In Section 2 we recall the definition of the norm  $|\cdot|_X$ , and in Section 3 we give some examples of computations of the norm  $|\cdot|_X$  and applications of Theorem 1.1. In Sections 4 to 6 we recall the residue currents associated with  $X$ , and we make the construction of interpolation-division formulas in strictly pseudoconvex domains. The remaining sections are devoted to the proofs of Theorem 1.1, Theorem 1.2, and Theorem 9.1.

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## 2. The pointwise norm on $X$

Let  $\Omega \subset \mathbb{C}^N$  be an open pseudoconvex domain, let  $Z$  be a submanifold of dimension  $n < N$ , and let  $\kappa = N - n$ . The  $\mathcal{O}_\Omega$ -sheaf of Coleff–Herrera currents,  $\mathcal{CH}_\Omega^Z$ , introduced by Björk, see [14], is the set of  $\bar{\partial}$ -closed  $(N, \kappa)$ -currents  $\mu$  in  $\Omega$  with support on  $Z$  such that  $\bar{h}\mu = 0$  for all holomorphic functions  $h$  that vanishes on  $Z$ . It is well-known that  $\mathcal{CH}_\Omega^Z$  is coherent. Notice that if  $\mathcal{J} \subset \mathcal{O}_\Omega$  is an ideal sheaf with zero set  $Z$ , then  $\text{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$  is the subsheaf of  $\mu$  in  $\mathcal{CH}_\Omega^Z$  that are annihilated by  $\mathcal{J}$ .

*Remark 2.1.* — If  $Z$  is not smooth, then  $\mathcal{CH}_\Omega^Z$  is defined in the same way, but one must impose an additional regularity condition at  $Z_{\text{sing}}$ , see, [14] or, e.g., [8, Section 2.1].

Consider the embedding  $i: X \rightarrow \Omega \subset \mathbb{C}^N$ . Locally, in say  $U \subset \Omega$ , we have coordinates  $(\zeta, \tau) = (\zeta_1, \dots, \zeta_n, \tau_1, \dots, \tau_\kappa)$  so that  $Z \cap U = \{\tau = 0\}$ . Then the mapping  $\pi: U \rightarrow Z \cap U, (\zeta, \tau) \mapsto \zeta$  is a submersion, and locally

any submersion appears in this way. If  $\mu$  is a section of  $\text{Hom}(\mathcal{O}_\Omega/\mathcal{I}, \mathcal{CH}_\Omega^Z)$  in  $U$  and  $\phi$  is a holomorphic function, then  $\pi_*(\phi\mu)$  is a holomorphic  $(0, n)$ -form on  $Z \cap U$  that only depends on the image  $i^*\phi$  of  $\phi$  in  $\mathcal{O}(X \cap U)$ . If  $d\zeta = d\zeta_1 \wedge \dots \wedge d\zeta_n$  thus

$$(2.1) \quad \pi_*(\phi\mu) = \mathcal{L}\phi d\zeta$$

defines a holomorphic differential operator, a Noetherian operator,  $\mathcal{L}: \mathcal{O}(X \cap U) \rightarrow \mathcal{O}(Z \cap U)$ , cf. Section 1. Following [7] we define  $\mathcal{N}_X$  as the set all such local operators  $\mathcal{L}$  obtained from some  $\mu$  in  $\text{Hom}(\mathcal{O}_\Omega/\mathcal{I}, \mathcal{CH}_\Omega^Z)$  and local submersion. It follows from (2.1) that if  $\xi$  is in  $\mathcal{O}_Z$ , then  $\xi\mathcal{L}\phi = \mathcal{L}(\pi^*\xi\phi)$ . Thus  $\mathcal{N}_X$  is a left  $\mathcal{O}_Z$ -module. It is in fact coherent, in particular it is locally finitely generated. Recall that our norm  $|\cdot|_X$  is defined by (1.1).

### 3. Examples

We now give some examples how to compute the Noetherian operators  $\mathcal{L}$ , the norm  $|\cdot|_X$ , and where Theorem 1.1 is applicable. We keep the notation from the previous section.

*Example 3.1.* — Assume that we have an embedding and local coordinates  $(\zeta, \tau)$  as above in  $U \subset \Omega$ . Let  $M = (M_1, \dots, M_\kappa)$  be a tuple of non-negative integers and consider the ideal sheaf

$$\mathcal{I} = \left\langle \tau_1^{M_1+1}, \dots, \tau_\kappa^{M_\kappa+1} \right\rangle.$$

Let  $\widehat{X}$  be the analytic space with structure sheaf  $\mathcal{O}_{\widehat{X}} = \mathcal{O}_\Omega/\mathcal{I}$ . For a multiindex  $m = (m_1, \dots, m_\kappa)$ ,  $m \leq M$  means that  $m_j \leq M_j$  for  $j = 1, \dots, \kappa$ . Any  $\psi$  in  $\mathcal{O}_{\widehat{X}}$  has a unique representative in  $\Omega$  of the form

$$(3.1) \quad \psi = \sum_{m \leq M} \widehat{\psi}_m(\zeta) \otimes \tau^m.$$

The tensor product of currents

$$(3.2) \quad \widehat{\mu} = \bar{\partial} \frac{d\tau_1}{\tau_1^{M_1+1}} \wedge \dots \wedge \bar{\partial} \frac{d\tau_\kappa}{\tau_\kappa^{M_\kappa+1}} \wedge d\zeta =: \bar{\partial} \frac{d\tau}{\tau^{M+1}} \wedge d\zeta,$$

where  $d\tau_j/\tau_j^{M_j+1}$  are principal value currents, is a Coleff–Herrera current in  $U$ . If  $\varphi = \varphi_0(\zeta, \tau) d\bar{\zeta}$  is a test form, then

$$(3.3) \quad \widehat{\mu} \cdot \varphi = \bar{\partial} \frac{d\tau}{\tau^{M+1}} \wedge d\zeta \cdot \varphi = \frac{(2\pi i)^\kappa}{M!} \int_\zeta \frac{\partial^{|M|} \varphi_0}{\partial \tau^M}(\zeta, 0) d\zeta \wedge d\bar{\zeta},$$

where  $M! = M_1! \cdots M_\kappa!$ . In particular,  $\tau_j^{M_j+1} \widehat{\mu} = 0$  for each  $j$  and thus  $\widehat{\mu}$  is in the  $\mathcal{O}_\Omega$ -module (and  $\mathcal{O}_{\widehat{X}}$ -module)  $\mathcal{H}om(\mathcal{O}/\mathcal{I}, \mathcal{CH}_\Omega^Z)$ . It is in fact a generator for this module, see, e.g., [5, Theorem 4.1]. If  $\pi$  is the simple projection  $(\zeta, \tau) \mapsto \zeta$  and  $\psi$  is holomorphic, then it follows from (3.3) that

$$\pi_*(\psi \widehat{\mu}) = \frac{(2\pi i)^\kappa}{M!} \frac{\partial^{|M|} \psi}{\partial \tau^M}(\zeta, 0) d\zeta$$

and hence, cf. (2.1),

$$\mathcal{L}\psi = \frac{(2\pi i)^\kappa}{M!} \frac{\partial^{|M|}(\psi \gamma)}{\partial \tau^M}(\zeta, 0).$$

For a general projection  $\pi$ , its associated Noetherian operator  $\mathcal{L}$  will involve derivatives with respect to  $\zeta$  as well, cf. [7, Section 2].

One can check that the set  $\mu_\alpha := \tau^\alpha \widehat{\mu}$ ,  $\alpha \leq M$ , generates the  $\mathcal{O}_Z$ -module  $\mathcal{N}_{\widehat{X}}$  in  $U$ . If  $\Psi(\zeta, \tau)$  is any representative in  $U$  for  $\psi$  in  $\mathcal{O}_{X'}$ , then it follows from [7, Proposition 4.6], with  $a_k = 1$ , cf. [7, (4.22)], that

$$(3.4) \quad |\psi|_{\widehat{X}} \sim \sum_{m \leq M, |\alpha| \leq |M-m|} \left| \frac{\partial}{\partial \tau^m \partial \zeta^\alpha} \Psi(\zeta, 0) \right|.$$

Possibly after shrinking  $\Omega$  somewhat, there is a finite number  $\mu_1, \dots, \mu_\rho$  of sections that generate the  $\mathcal{O}_\Omega$ -module  $\mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$  in  $\Omega$ . Locally, in  $U \subset \Omega$ , we can choose  $M$  in Example 3.1 so that  $\mathcal{I} \subset \mathcal{J}$  in  $U$ . Since  $\widehat{\mu}$  generates  $\mathcal{H}om(\mathcal{O}_\Omega/\mathcal{I}, \mathcal{CH}_\Omega^Z) \supset \mathcal{H}om(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ , there are holomorphic functions  $\gamma_1, \dots, \gamma_\rho$  such that

$$(3.5) \quad \mu_j = \gamma_j \widehat{\mu}, \quad j = 1, \dots, \rho$$

in  $U$ . It follows from [7, (4.22)] that

$$(3.6) \quad |\phi|_X \sim \sum_{j=1}^\rho |\gamma_j \phi|_{\widehat{X}}.$$

*Example 3.2.* — Let  $f_1, \dots, f_\kappa$  be holomorphic functions in a neighborhood  $\Omega$  of the (closure of the) unit ball  $D$  in  $\mathbb{C}^N$  such that  $df_1 \wedge \dots \wedge df_\kappa \neq 0$  on their common zero set  $Z$ . Moreover, assume that  $Z$  intersect  $\partial D$  transversally. Let  $\mathcal{J} = \langle f_1^{M_1+1}, \dots, f_\kappa^{M_\kappa+1} \rangle$  and let  $X$  be the non-reduced space associated with  $\mathcal{O}_\Omega/\mathcal{J}$ . Then Theorem 1.1 applies to  $X$  and  $D$ .

At a given point  $p \in Z \cap \partial D$  we can assume, possibly after reordering, that the coordinates in  $\mathbb{C}^N$  are  $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_\kappa)$  and that

$$(3.7) \quad df_1 \wedge \dots \wedge df_\kappa \wedge dz_1 \wedge \dots \wedge dz_n \neq 0$$

at  $p$ . In a neighborhood  $U$  of  $p$ , therefore  $\zeta = z$ ,  $\tau = f$  are local coordinates. The assumption (3.7) means that the matrix  $B = \partial f / \partial w$  is invertible, and

we have

$$(3.8) \quad \frac{\partial}{\partial \tau} = B^{-1} \frac{\partial}{\partial w}, \quad \frac{\partial}{\partial \zeta} = \frac{\partial}{\partial z} - \frac{\partial f}{\partial z} B^{-1} \frac{\partial}{\partial w}.$$

If  $\phi$  is a function on  $X$ , then  $|\phi|_X$  is given by (3.4), which can be expressed in terms of the original coordinates  $(z, w)$  by (3.8).

*Remark 3.3 (The pointwise norm when  $X$  is Cohen–Macaulay).* — Assume that we have a local coordinates  $(\zeta, \tau)$  in  $U \subset \Omega$  as above. Assume furthermore that  $\mathcal{O}_X$  is Cohen–Macaulay. Then one can find monomials  $1, \tau^{\alpha_1}, \dots, \tau^{\alpha_{\nu-1}}$  such that each  $\phi$  in  $\mathcal{O}_X$  has a unique representative

$$(3.9) \quad \widehat{\phi} = \widehat{\phi}_0(\zeta) \otimes 1 + \dots + \widehat{\phi}_{\nu-1}(\zeta) \otimes \tau^{\alpha_{\nu-1}},$$

where  $\widehat{\phi}_j$  are in  $\mathcal{O}_Z$ , see, e.g., [8, Corollary 3.3], cf. (3.1). In this way  $\mathcal{O}_X$  becomes a free  $\mathcal{O}_Z$ -module. By [7, Theorem 4.1 (iii)]  $|\cdot|_X$  is the smallest norm such that

$$(3.10) \quad \sum |\widehat{\phi}_j(\zeta)| \leq C |\phi(\zeta)|_X,$$

for any choice of local coordinates and monomial basis.

We now consider a space  $X$  with a non-Cohen–Macaulay point, see [7, Section 5].

*Example 3.4.* — Consider the 2-plane  $Z = \{w_1 = w_2 = 0\}$  in  $\Omega = \mathbb{C}_{z_1, z_2, w_1, w_2}^4$ , let

$$\mathcal{J} = \langle w_1^2, w_2^2, w_1 w_2, w_1 z_2 - w_2 z_1 \rangle.$$

Then the associated non-reduced space  $X$  has pure dimension 2 and is Cohen–Macaulay except at the point 0, see, [8, Example 6.9]. It is also shown there that  $\text{Hom}_{\mathcal{O}}(\mathcal{O}/\mathcal{J}, \mathcal{CH}_{\Omega}^Z)$  is generated by

$$\begin{aligned} \mu_1 &= \bar{\partial} \frac{dw_1}{w_1} \wedge \bar{\partial} \frac{dw_2}{w_2} \wedge dz_1 \wedge dz_2, \\ \mu_2 &= (z_1 w_2 + z_2 w_1) \bar{\partial} \frac{dw_1}{w_1^2} \wedge \bar{\partial} \frac{dw_2}{w_2^2} \wedge dz_1 \wedge dz_2. \end{aligned}$$

It turns out, see [7, Section 5], that the left  $\mathcal{O}_Z$ -module  $\mathcal{N}_X$  is generated by

$$(3.11) \quad 1, z_2 \frac{\partial}{\partial z_1}, z_1 \frac{\partial}{\partial z_2}, z_2 \frac{\partial}{\partial z_2}, z_1 \frac{\partial}{\partial z_2}, z_1 \frac{\partial}{\partial w_1} + z_2 \frac{\partial}{\partial w_2}.$$

Thus we get, cf. (1.1),

$$(3.12) \quad |\phi|_X^2 = |\phi|^2 + |z|^2 \left| \frac{\partial \phi}{\partial z_1} \right|^2 + |z|^2 \left| \frac{\partial \phi}{\partial z_2} \right|^2 + \left| z_1 \frac{\partial \phi}{\partial w_1} + z_2 \frac{\partial \phi}{\partial w_2} \right|^2.$$

*Example 3.5.* — Let  $X$  be the space in Example 3.4 and let  $D$  be the unit ball in  $\mathbb{C}^4$ . Each holomorphic function in  $\phi$  in  $\mathcal{O}(X \cap D)$  can be written

$$(3.13) \quad \phi = \phi_0(z) \otimes 1 + \phi_1(z) \otimes w_1 + \phi_2(z) \otimes w_2,$$

where  $\phi_j$  are holomorphic functions in the unit ball  $\mathbb{B} \subset \mathbb{C}_z^2$ . One can check that

$$(3.14) \quad h(z) = z_1\phi_1(z) + z_2\phi_2(z)$$

is independent of the representation (3.13). In this way we get a one-to-one correspondence between functions in  $\mathcal{O}(X \cap D)$  and pairs  $\phi_0, h$  of holomorphic functions in  $\mathbb{B}$  such that  $h(0) = 0$ . Notice that the rightmost term in (3.12) is precisely  $|h|^2$ . If now  $\phi$  is in  $L^p(\mathbb{B}, \delta^r)$ , then  $h$  is in  $L^p(\mathbb{B}, \delta^r)$ , and  $h(0) = 0$ . It is well-known that one can find  $\phi_1, \phi_2$  in  $L^p(\mathbb{B}, \delta^r)$  such that (3.14) holds. With such choices, (3.13) defines an extension  $\Phi$  of  $\phi$  to the entire ball  $D$  that satisfies the estimate (1.2) in Theorem 1.1, cf. the discussion after that theorem.

If we deform the embedding of  $X$  in  $D$  slightly, or replace the unit ball by a more general strictly pseudoconvex set  $D$ , so that the non-Cohen–Macaulay point is still in the interior of  $D$ , then Theorem 1.1 provides a non-trivial extension. If this point lies on  $\partial D$ , then Theorem 1.1 is not applicable.

### 4. Residue currents associated with a free resolution

If  $\mathcal{J}$  is a coherent ideal sheaf in  $\Omega$ , then we can find a free resolution

$$(4.1) \quad 0 \rightarrow \mathcal{O}(E_\nu) \xrightarrow{f_\nu} \mathcal{O}(E_{\nu-1}) \cdots \xrightarrow{f_1} \mathcal{O} \rightarrow \mathcal{O}/\mathcal{J} \rightarrow 0$$

of  $\mathcal{O}/\mathcal{J}$  in a slightly smaller pseudoconvex domain that we for simplicity denote by  $\Omega$  as well. If the (trivial) vector bundles  $E_k$  are equipped with Hermitian metrics we say that (4.1) is a hermitian resolution. For each Hermitian resolution there are, [10], associated residue currents

$$R = \sum_{k=\kappa}^{\nu} R_k, \quad U = \sum_{\ell, k} U_k^\ell,$$

where  $R_k$  are currents of bidegree  $(0, k)$  with support on  $Z := Z(\mathcal{J})$  that take values in  $\text{Hom}(E_0, E_k) \simeq E_k$ , and  $U_k^\ell$  are  $(0, k - \ell)$ -currents that are smooth outside  $Z$  and take values in  $\text{Hom}(E_\ell, E_k)$ .

*Remark 4.1.* — The currents  $R$  and  $U$  are defined even if (4.1) is just a pointwise generically exact complex. In general then  $R$  has components  $R_k^\ell$  with values in  $\text{Hom}(E_\ell, E_k)$  even for  $\ell \geq 1$ .

If  $\mathcal{J}$  is Cohen–Macaulay, then one can choose (4.1) so that  $\nu = \kappa$ . In that case the components of  $R = R_\kappa$  are in  $\text{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ . If we only assume that  $\mathcal{O}_X$  has pure dimension, then we may have components  $R_k$  for  $k \leq N - 1$ , see, e.g., [8, 9]. They can be written, [8, Lemma 6.2],

$$(4.2) \quad R_k \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_N = a_k \mu,$$

where  $\mu$  is in  $\text{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$  with values in a trivial bundle  $F$  and  $a_k$  are currents in  $\Omega$  that take values in  $\text{Hom}(F, E_k)$ . Moreover,  $a_k$  are smooth outside the Zariski closed set  $W \subset \Omega$  of non-Cohen–Macaulay points on  $Z$ , which has positive codimension on  $Z$ . The currents  $a_k$  are *almost semi-meromorphic* in the terminology from [9, 12]. For us the important point is that

$$(4.3) \quad a_k \mu = \lim_{\epsilon \rightarrow 0} \chi(|f|^2/\epsilon) a_k \mu,$$

if  $f$  is a holomorphic tuple with zero set  $W$  and  $\chi$  is a smooth function on  $[0, 1)$  that is 1 for  $t < 1/2$  and 0 for  $t > 1$ . Notice that for each  $\epsilon > 0$ ,  $\chi(|f|^2/\epsilon) a_k \mu$  is the product of a current and a smooth form.

We have the following duality results.

**PROPOSITION 4.2.** — *If  $\Phi$  is holomorphic, then it is in the ideal  $\mathcal{J}$  if and only if  $R\Phi = 0$ . If  $\mathcal{O}_X$  has pure dimension, then  $\Phi$  is in  $\mathcal{J}$  if and only if  $\mu\Phi = 0$  for all  $\mu$  in  $\text{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$ .*

The first statement is a basic result in [10], and the second one is proved in [6].

### 5. Integral representation of holomorphic functions

Following [3] we recall a formalism to generate representation formulas for holomorphic functions. Let  $z$  be a fixed point in  $\Omega$ , let  $\delta_{\zeta-z}$  be contraction with the vector field

$$2\pi i \sum_{j=1}^N (\zeta_j - z_j) \frac{\partial}{\partial \zeta_j}$$

in  $\Omega$  and let  $\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar{\partial}$ , where  $\bar{\partial}$  only acts on  $\zeta$ . We say that a current  $g = g_{0,0} + \cdots + g_{n,n}$ , where  $g_{k,k}$  has bidegree  $(k, k)$ , is a weight with respect to  $z$ , if  $\nabla_{\zeta-z} g = 0$ ,  $g$  is smooth in a neighborhood of  $z$ , and  $g_{0,0}$  is

1 when  $\zeta = z$ . Notice that if  $g$  and  $g'$  are weights, one of which is smooth, then  $g' \wedge g$  is again a weight. The basic observation is that if  $g$  is a weight (with respect to  $z$ ) with compact support in  $\Omega$ , then

$$(5.1) \quad \phi(z) = \int_{\zeta \in \Omega} g\phi$$

if  $\phi$  is holomorphic in  $\Omega$ , [3, Proposition 3.1].

If  $\Omega$  is pseudoconvex and  $D \subset\subset \Omega$ , then, see [4, Example 1], we can find a weight  $g$ , with respect to  $z \in \bar{D}$ , with compact support in  $\Omega$ , such that  $g$  depends holomorphically on  $z \in \bar{D}$ . If  $D$  and  $\Omega$  are balls with center at  $0 \in \Omega$ , then we can take

$$g = \chi - \bar{\partial}\chi \wedge \frac{\sigma}{\nabla_{\zeta-z}\sigma} = \chi - \bar{\partial}\chi \wedge \sum_{\ell=1}^N \frac{1}{(2\pi i)^\ell} \frac{\zeta \cdot d\bar{\zeta} \wedge (d\zeta \cdot d\bar{\zeta})^{\ell-1}}{(|\zeta|^2 - \bar{\zeta} \cdot z)^\ell},$$

where  $\chi$  that is 1 in a neighborhood of  $\bar{D}$ , with compact support in  $\Omega$ , and

$$\sigma = \frac{1}{2\pi i} \frac{\zeta \cdot d\bar{\zeta}}{|\zeta|^2 - \bar{\zeta} \cdot z}.$$

### 5.1. Division-interpolation formulas

Let  $(E, f)$  be a Hermitian resolution of  $\mathcal{O}/\mathcal{J}$  in  $\Omega$  as in Section 4. In order to construct division-interpolation formulas with respect to  $(E, f)$ , in [4] was introduced the notion of an associated family  $H = (H_k^\ell)$  of Hefer morphisms. The  $H_k^\ell$  are holomorphic  $(k - \ell)$ -forms with values in  $\text{Hom}(E_{\zeta,k}, E_{z,\ell})$  that are connected in the following way: To begin with,  $H_k^\ell = 0$  if  $k - \ell < 0$ , and  $H_\ell^\ell$  is equal to  $I_{E_\ell}$  when  $\zeta = z$ . In general,

$$(5.2) \quad \delta_{\zeta-z} H_{k+1}^\ell = H_k^\ell f_{k+1}(\zeta) - f_{\ell+1}(z) H_{k+1}^{\ell+1}.$$

If  $R$  and  $U$  are the associated currents in Section 4, then

$$HR = \sum H_k^0 R_k, \quad H^1 U = \sum_k H_k^1 U_k^1,$$

are scalar-valued currents, cf. Remark 4.1. It turns out, see [4, Eq. (5.4)], that

$$g' = f_1(z) H^1 U + HR$$

is a weight with respect to  $z$  for each  $z \in \Omega \setminus Z$ . If  $g$  is a smooth weight with respect to  $z \in \bar{D} \subset \Omega$ , depending holomorphically on  $z$ , with compact

support in  $\Omega$  and  $\Psi$  is holomorphic in  $\Omega$ , then  $g' \wedge g$  is a weight with compact support with respect to  $z \in D \setminus Z$ . By (5.1) we therefore have

$$(5.3) \quad \Psi(z) = \int_{\zeta \in \Omega} g' \wedge g \Psi = f_1(z) \int_{\zeta \in \Omega} H^1 U \wedge g \Psi + \int_{\zeta \in \Omega} H R \wedge g \Psi$$

for  $z \in D \setminus Z$ . Since the right hand side has a holomorphic extension across  $Z$ , actually (5.3) holds for all  $z$  in  $D$  by continuity.

Now assume that  $\phi$  is a section of  $\mathcal{O}/\mathcal{J}$  in  $\Omega$ . Since  $\Omega$  is pseudoconvex there is some holomorphic extension  $\Psi$  of  $\phi$  to  $\Omega$ . Since  $\mathcal{J}$  annihilates  $R$ , see Section 4, the current  $R\phi := R\Psi$  is independent of the extension and thus intrinsic. Since  $f_1(z)$  is in  $\mathcal{J}$ , we conclude from (5.3) that

$$(5.4) \quad \Phi(z) = \int_{\zeta \in \Omega} H R \wedge g \phi$$

is a holomorphic function in  $D$  that extends  $\phi$ . In order to obtain interesting estimates however, we must replace  $g$  by a weight with support on  $\bar{D}$ .

For future reference notice that if  $g$  only depends smoothly on  $z \in D$ , then (5.4) is a smooth function in  $D$  such that  $\Phi - \phi$  is in  $\mathcal{E}\mathcal{J}$ , where  $\mathcal{E}$  is the sheaf of smooth functions.

### 6. Integral formulas in strictly pseudoconvex domains

The material in this section is basically well-known but we need it for the construction of our formula. Assume that  $D \subset\subset \Omega \subset \mathbb{C}^N$  is strictly pseudoconvex with smooth boundary. We can assume that  $D = \{\rho < 0\}$ , where  $\rho$  is strictly plurisubharmonic in  $\Omega$ . If  $D$  is the ball we can take  $\rho = |\zeta|^2 - 1$ . If  $D$  is strictly convex, then  $\delta_{\zeta-z}\partial\rho$  is holomorphic in  $z \in D$ , and if  $\rho$  is strictly convex, then

$$2 \operatorname{Re} \delta_{\zeta-z}\partial\rho \geq \rho(\zeta) - \rho(z) + c|\zeta - z|^2$$

for some constant  $c > 0$ . If

$$(6.1) \quad v(\zeta, z) := \delta_{\zeta-z}\partial\rho - \rho(\zeta) = -\rho(\zeta) - \sum_j \frac{\partial\rho}{\partial\zeta_j}(\zeta)(z_j - \zeta_j),$$

because of the strict convexity, therefore

$$(6.2) \quad 2\operatorname{Re} v(\zeta, z) \geq -\rho(z) - \rho(\zeta) + c|\zeta - z|^2,$$

and moreover,

$$(6.3) \quad d(\operatorname{Im} v)|_{\zeta=z} = d^c\rho(z)/4\pi.$$

Altogether it follows that if  $z$  (or  $\zeta$ ) is a fixed point  $p$  on  $\partial D$ , then the level sets of  $|v(\zeta, z)|$  are non-isotropic so-called Koranyi balls around  $p$ . More precisely, if  $x_1 = -\rho(\zeta)$ ,  $x_2 = \text{Im } v(\zeta, z)$ , and  $x_3, \dots, x_{2N}$  are chosen so that  $x_1, \dots, x_{2N}$  is a local (non-holomorphic) coordinate system at  $p$  with  $x(p) = 0$ , and  $y$  are the corresponding coordinates for  $z$ , then

$$(6.4) \quad |v(\zeta, z)| \sim x_1 + y_1 + |x_2 - y_2| + \sum_{j=3}^{2N} (x_j - y_j)^2 + \mathcal{O}(|x - y|^3).$$

One can make a similar construction of  $v$  if  $D$  is strictly pseudoconvex. If  $D$  is the ball and  $\rho = |\zeta|^2 - 1$ , then

$$v(\zeta, z) = 1 - \bar{\zeta} \cdot z$$

which is anti-holomorphic in  $\zeta$ . In general, unfortunately,  $\partial_\zeta v$  will only vanish to first order on the diagonal. We need such a function  $v$  that is (essentially) anti-holomorphic in  $\zeta$  so we must elaborate the construction.

### 6.1. Definition of $v$ in the general case

First assume that  $\rho(z)$  is strictly plurisubharmonic and real-analytic. Then close to the diagonal we choose  $v(\zeta, z)$  so that  $v(\bar{\zeta}, z)$  is the (unique) holomorphic extension of  $-\rho(z)$  from the totally real subspace  $\{\zeta = \bar{z}\}$  of  $\{(\bar{\zeta}, z); (\zeta, z) \in \Omega_\zeta \times \Omega_z\}$ . Then  $\overline{v(z, \zeta)} = v(\zeta, z)$  and  $v$  is anti-holomorphic in  $\zeta$ . We can represent  $v$  by the power series

$$(6.5) \quad v(\zeta, z) = - \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^\alpha \rho}{\partial \zeta^\alpha}(\zeta) (z - \zeta)^\alpha.$$

We claim that

$$(6.6) \quad 2 \text{Re } v = -\rho(\zeta) - \rho(z) + L\rho(\zeta) + \mathcal{O}(|\zeta - z|^3),$$

where  $L\rho(\zeta)$  is the Levi form in the Taylor expansion of  $\rho$  at  $\zeta$ . In fact, from (6.5) we have, using the notation  $\rho_j = \partial\rho/\partial\zeta_j(\zeta)$  etc and  $\eta_j = z_j - \zeta_j$ ,

$$\begin{aligned} 2 \text{Re } v &= -2\rho(\zeta) - 2 \text{Re} \sum_j \rho_j \eta_j - \text{Re} \sum_{jk} \rho_{jk} \eta_j \eta_k + \mathcal{O}(|\eta|^3) \\ &= -\rho(\zeta) + L\rho(\zeta) \\ &\quad - \left( \rho(\zeta) + 2 \text{Re} \sum_j \rho_j(\zeta) \eta_j + \text{Re} \sum_{jk} \rho_{jk} \eta_j \eta_k + L\rho(\zeta) \right) + \mathcal{O}(|\eta|^3) \\ &= -\rho(z) + L\rho(\zeta) - \rho(\zeta) + \mathcal{O}(|\eta|^3). \end{aligned}$$

Since  $\rho$  is strictly plurisubharmonic it follows from (6.6) that (6.2) holds, and since also (6.3) holds, the level sets of  $|v|$  are the Koryani balls discussed above and (6.4) holds. From (6.5) it is easy to find a  $(1, 0)$ -form  $q$ , depending holomorphically on  $z$ , such that

$$(6.7) \quad v = \delta_{\zeta-z}q - \rho(\zeta).$$

We now turn to the case when  $\rho$  is just smooth and strictly plurisubharmonic. Let  $\chi$  be a smooth function on  $[0, \infty)$  that is 1 when  $t < 1/2$  and 0 when  $t > 1$ . We claim that the series

$$(6.8) \quad v(\zeta, z) = - \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{\alpha} \rho}{\partial \zeta^{\alpha}}(\zeta)(z - \zeta)^{\alpha} \chi(c_{|\alpha|}|z - \zeta|)$$

converges uniformly, with all its derivatives, and therefore defines a smooth function in a neighborhood of the diagonal in  $\Omega \times \Omega$ , if  $c_k$  tends to infinity fast enough. In fact, notice that

$$|(z - \zeta)^{\gamma} \chi^{(\tau)}(c_k|z - \zeta|^k) \leq c_k^{-|\gamma|} \sup |\chi^{(\tau)}|.$$

If  $m_k = \sup_{|\alpha| \leq k} |\partial^{\alpha} \rho / \partial \zeta^{\alpha}| / \alpha!$  it is enough to choose  $c_k$  so that for any fixed  $\ell$ ,  $c_k \gg m_{k+\ell}$  for all large enough  $k$  (where  $\gg$  also depends on  $\ell$  derivatives of  $\chi$ ). Thus the choice of  $c_k$  depends on the ultra-differentiable class of  $\rho$ . We also claim that  $v$  is almost anti-holomorphic in  $\zeta$  in the sense that

$$(6.9) \quad \partial_{\zeta} v = \mathcal{O}(|\zeta - z|^{\infty}).$$

To see this, given a positive integer  $\nu$ , let us write (6.8) as  $A + B$ , where  $A$  is the sum over  $|\alpha| \leq \nu$ . Now  $\partial_{\zeta} A$  becomes a telescoping sum plus the terms where  $\partial_{\zeta}$  falls on  $\chi$ . The sum gives rise to terms that are  $\mathcal{O}(|\zeta - z|^{\nu})$ , whereas the remaining terms vanish close to the diagonal. Clearly  $\partial_{\zeta} B = \mathcal{O}(|\zeta - z|^{\nu})$ . Thus (6.9) holds.

*Remark 6.1.* — Notice that  $v(\bar{\zeta}, z)$  is a smooth extension of  $-\rho(\bar{\zeta})$  from the totally real subspace  $\{\zeta = \bar{z}\}$  of  $\{(\bar{\zeta}, z); (\zeta, z) \in \Omega_{\zeta} \times \Omega_z\}$  such that  $\bar{\partial} v(\bar{\zeta}, z) = \mathcal{O}(|z - \bar{\zeta}|^{\infty})$ . That is,  $v(\bar{\zeta}, z)$  is a so-called almost holomorphic extension. Such extensions are well-known in the literature and can be constructed in many ways.

Again one can find  $q$  that is holomorphic in  $z$  such that (6.7) holds. Moreover,  $\overline{v(z, \zeta)} - v(\zeta, z) = \mathcal{O}(|\zeta - z|^{\infty})$  but this property is not used in this paper.

We extend  $v$  to  $\Omega \times \Omega$  by patching with  $|\zeta - z|^2$ , that is, if  $\eta = \zeta - z$  we let

$$\tilde{v} = \chi(|\eta|^2)v + (1 - \chi(|\eta|^2))|\eta|^2, \quad \tilde{q} = \chi(|\eta|^2)q + (1 - \chi(|\eta|^2))\partial|\eta|^2.$$

so that  $\tilde{v} = \delta_{\zeta-z}\tilde{q} - \rho(\zeta)$ . In what follows, for simplicity, we write  $v$  and  $q$  even for the extensions.

*Remark 6.2.* — Assume that we have an embedding  $\psi: \Omega \rightarrow \Omega'$  into a higher dimensional ball  $\Omega' \subset \mathbb{C}^{n'}$ ,  $D' \subset\subset \Omega'$  is the unit ball, and  $D = \psi^{-1}D'$ . Assume in addition that  $\psi(\Omega)$  intersects  $\partial D'$  transversally. Then  $\rho = |\phi|^2 - 1$  is a strictly pseudoconvex defining function for  $D$  and  $v(\zeta, z) = 1 - \bar{\psi}(\zeta) \cdot \psi(z)$  is globally defined in  $\Omega \times \Omega$ , holomorphic in  $z$ , anti-holomorphic in  $\zeta$ , and equal to  $-\rho$  in the diagonal. Moreover, one can find a  $(1, 0)$ -form  $q$  in  $D$ , depending holomorphically on  $z$ , such that (6.7) holds.

*Example 6.3.* — There are non-trivial domains that admit a  $v$  as in Remark 6.2. One can check that  $D = \{z \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 + 4|1 - z_1z_2|^2 < 3\}$  is strictly pseudoconvex. It is the inverse image of the unit ball in  $\mathbb{C}^3$  under the mapping  $\psi(z_1, z_2) = (1/\sqrt{3})(z_1, z_2, 2(1 - z_1z_2))$  and hence there is a global  $v(\zeta, z)$  in  $D$  as in the remark. Notice that  $\alpha = dz_1/z_1$  is a closed 1-form in  $D$  that is not exact since its integral over the cycle  $\theta \mapsto (e^{i\theta}, e^{-i\theta})$  is  $2\pi i \neq 0$ . Thus  $H^1(D, \mathbb{C}) \neq 0$ .

### 6.2. The weight $g^\alpha$

Let  $\alpha$  be any complex number. We claim that for each fixed  $z \in D$ ,

$$(6.10) \quad g^\alpha = \left(1 + \nabla_{\zeta-z} \frac{q}{-\rho}\right)^{-\alpha+1} = \left(\frac{v}{-\rho} + \bar{\partial} \frac{q}{\rho}\right)^{-\alpha+1}$$

is a weight with respect to  $z$ . In fact, the scalar term within the second brackets has positive real part in view of (6.2) and hence  $g^\alpha$  is well-defined by elementary functional calculus, see [3], and  $\nabla_{\zeta-z}g = 0$  since  $\nabla_{\zeta-z}^2 = 0$ . It is also clear that  $g_{0,0}^\alpha = 1$  when  $\zeta = z$ . Thus the claim holds.

A simple computation gives that

$$(6.11) \quad g^\alpha = \sum_{k=0}^N c_{k,\alpha} \frac{(-\rho)^\alpha \beta_k}{v^{\alpha+k+1}},$$

where  $c_{\alpha,k}$  are constants and  $\beta_k$  are  $(k, k)$ -forms that are smooth in  $\Omega$ . If  $\text{Re } \alpha$  is positive, then  $g^\alpha$  vanishes on the boundary of  $D$  for each fixed  $z \in D$ .

In case  $D$  is the ball, or if, e.g., as in Remark 6.2, this weight depends holomorphically on  $z \in D$ . In general it only depends smoothly on  $z$ ; however, (6.9) holds, which is crucial in the proofs of Theorems 1.1 and 1.2.

*Remark 6.4.* — For a given  $X$  in Theorem 1.1 or Theorem 1.2 it is in fact enough for our proofs to choose  $v$  such that  $\partial_\zeta v = \mathcal{O}(|\zeta - z|^\nu)$  for a large enough  $\nu$ . Such a  $v$  is obtained by restricting the sum (6.8) to  $|\alpha| \leq \nu + 1$ ; then of course the factors  $\chi(c_{|\alpha|}|z - \zeta|^2)$  are not needed.

### 7. Proof of Theorem 1.1 when $D$ is the ball

Let us first assume that our  $v(\zeta, z)$  is defined in  $\Omega \times \Omega$ , holomorphic in  $z$  and anti-holomorphic in  $\zeta$ , as in the case with the ball. Recall, cf. (6.11), that for fixed  $z \in D$ , the weight  $g^\alpha$  vanishes to order  $\alpha$  at  $\partial D$ . If we define it as 0 outside  $D$ , it is therefore of class  $C^{\alpha-1}$ .

LEMMA 7.1. — *If  $\alpha$  is large enough and  $\phi$  is holomorphic in a neighborhood of  $X \cap \bar{D}$ , then*

$$(7.1) \quad \Phi(z) = \int_{\zeta \in D} HR \wedge \frac{(-\rho)^\alpha \beta_n}{v^{n+\alpha+1}} \phi$$

is a holomorphic extension of  $\phi$  to  $D$ .

*Proof.* — Assume that  $\alpha$  is larger than the order of the currents  $U$  and  $R$ . Notice that the function that is  $(-\rho)^\alpha$  in  $D$  and 0 outside  $D$  is in  $C^{\alpha-1}$ . For each fixed  $z \in D$ , therefore  $g^\alpha$ , defined as 0 outside  $D$ , is a weight in  $\Omega$  of class  $C^{\alpha-1}$ . Thus (5.3) holds with  $g = g^\alpha$ . As in Section 5.1 we conclude that (5.4), that is, (7.1), is a holomorphic extension of  $\phi$  to  $D$ .  $\square$

We shall now make an a priori estimate of  $\Phi$  in terms of  $\phi$ . Let us assume that  $\phi$  is defined on  $D' \cap X$ , where  $D' \supset \supset D$ . Let us also assume that  $X$  is defined and Cohen–Macaulay in  $\Omega$ . Then we can assume that our Hermitian resolution  $(E, f)$  has length  $\kappa = N - n$ , and hence  $HR = H_\kappa^0 R_\kappa$  has bidegree  $(\kappa, \kappa)$ .

If either  $|\zeta - z| \geq \epsilon$  or  $\zeta$  is far from  $\partial D$ , then  $|v|$  is strictly positive in view of (6.4). By a suitable partition of unity we therefore have to estimate the  $L^p$ -norm of a finite number of terms

$$(7.2) \quad \int_{\zeta \in D} HR \wedge \frac{(-\rho)^\alpha}{v^{n+\alpha+1}} \beta(\zeta, z) \phi,$$

where  $\beta$  is smooth with compact support in a small neighborhood  $U$  of a point  $p \in \partial D \cap Z$ , plus some terms with no singularity at all.

Let us consider a term (7.2). Let us change notation and replace  $\zeta$  by coordinates  $(\zeta, \tau)$  in  $U$  such that  $Z \cap U = \{\tau = 0\}$ . Let  $\mu$  be one of the

components of  $R = R_\kappa$ . It is a Coleff–Herrera current, cf. Section 4, so it can be written

$$\mu = \gamma(\zeta, \tau) \bar{\partial} \frac{d\tau}{\tau^{M+1}} \wedge d\zeta$$

as in (3.5), cf. (3.2). Let us incorporate  $H$  in  $\beta$ . Integrating with respect to  $\eta$ , that is, taking the push-forward  $\pi_*$ , where  $\pi$  is the projection  $(\zeta, \tau) \mapsto \zeta$ , we get, see (3.3),

$$\int_{\zeta \in Z \cap D} \partial_\tau^M |_{\tau=0} \left( \frac{(-\rho)^\alpha}{v^{n+\alpha+1}} \beta(\zeta, \tau; z) \gamma \phi \right).$$

Here  $\partial_\tau^M$  stands for  $\partial^{|M|} / \partial \tau^M$ . In what follows it is understood that we evaluate at  $\tau = 0$  after applying this operator and we thus omit  $|_{\tau=0}$ . Using that  $v$  is anti-holomorphic in  $(\zeta, \tau)$  we have

$$\sum_{m \leq M} \int_{\zeta \in Z \cap D} \frac{1}{v^{n+\alpha+1}} \binom{M}{m} \partial_\tau^{M-m} ((-\rho)^\alpha \beta) \partial_\tau^m (\gamma \phi).$$

Thus we get a sum of terms of the form

$$\int_{\zeta \in Z \cap D} \frac{1}{v^{n+\alpha+1}} (-\rho)^{\alpha-\ell} \beta \partial_\tau^m (\gamma \phi),$$

where  $\ell \leq |M - m|$  and  $\beta$  is smooth. Since  $\rho$  is a defining function we may assume that  $\partial \rho$  is nonzero in  $U$ . If

$$T = \frac{1}{|\partial \rho|^2} \sum_j \frac{\partial \rho}{\partial \bar{\zeta}_j} \frac{\partial}{\partial \zeta_j},$$

then  $T\rho = 1$  and hence

$$(-\rho)^{\alpha-\ell} \beta = \beta' T (-\rho)^{\alpha-\ell+1}.$$

where  $\beta' = \beta / (\ell - \alpha - 1)$ . If  $T'$  is the formal adjoint of  $T$ , again using that  $v$  is anti-holomorphic in  $\zeta$ ,  $(-\rho)^{\alpha-\ell+1} = 0$  on  $\partial D$ , and  $\phi$  is defined on  $X \cap \bar{D}$ , an integration by parts gives

$$\int_{\zeta \in Z \cap D} \frac{1}{v^{n+\alpha+1}} (-\rho)^{\alpha-\ell+1} T' (\beta' \partial_\tau^m (\gamma \phi)).$$

Repeating this procedure  $\ell$  times we see that our extension is  $\Phi$  a finite sum of terms

$$(7.3) \quad A(z) = \int_{\zeta \in Z \cap D} \frac{1}{v^{n+\alpha+1}} (-\rho)^\alpha \beta \partial_\zeta^a \partial_\tau^m (\gamma \phi),$$

where  $a$  is a multiindex such that  $|a| \leq |M - m|$  and  $\beta$  is smooth. It follows from (3.4) and (3.6) that

$$(7.4) \quad |\partial_\zeta^a \partial_\tau^m (\gamma \phi)| \lesssim |\phi|_X.$$

Assume that  $r > -1$ . Provided that  $\alpha$  is large enough, from (7.3), (7.4), and (7.7) in Lemma 7.4 below we have

$$\int_{z \in D} \delta^r |A| \lesssim \int_{\zeta \in Z \cap D} (-\rho)^{N-n+r} |\phi|_X.$$

Summing up all terms  $A$  we get the desired a priori estimate (1.2) in case  $p = 1$ . Now assume that  $1 < p < \infty$ . Let us choose  $\epsilon > 0$  so that  $\alpha - (q/p)\epsilon > -1$ . By Hölder’s equality and (7.8) below,

$$\begin{aligned} |A|^p &\lesssim \left( \int_{\zeta \in Z \cap D} \frac{\delta^{\alpha - \frac{q}{p}\epsilon}}{|v|^{n+\alpha+1}} \right)^{p/q} \int_{\zeta \in Z \cap D} \frac{\delta^{\alpha+\epsilon}}{|v|^{n+\alpha+1}} |\phi|_X^p \\ &\lesssim \delta(z)^{-\epsilon} \int_{\zeta \in Z \cap D} \frac{\delta^{\alpha+\epsilon}}{|v|^{n+\alpha+1}} |\phi|_X^p. \end{aligned}$$

If in addition  $r - \epsilon > -1$ , an application of (7.7) gives (1.2).

If  $\phi$  is just defined in  $X \cap D$  we apply the same construction and argument to the slightly smaller strictly pseudoconvex domains  $D_\epsilon = \{\rho < -\epsilon\}$ . It is not hard to see that the same computation works in  $D_\epsilon$ , with estimates that are uniform in  $\epsilon$ . We thus get  $\Phi_\epsilon$  in  $D_\epsilon$  that interpolate  $\phi$  in  $D_\epsilon \cap X$  such that

$$(7.5) \quad \int_{D_\epsilon} \delta_\epsilon^r |\Phi_\epsilon|^p \, dV_D \leq C_{r,p}^p \int_{Z \cap D_\epsilon} \delta_\epsilon^{r+\epsilon} |\phi|_X^p \, dV_Z,$$

where  $C_{r,p}$  is uniform in  $\epsilon$  and  $\delta_\epsilon \sim -\rho_\epsilon := -(\rho + \epsilon)$  is the distance to  $\partial D_\epsilon$ . Clearly the right hand sides of (7.5) is dominated by the right hand side of (1.2). If this is finite, hence there is a subsequence  $\Phi_{\epsilon_j}$  converging to a function  $\Phi$  in  $D$  uniformly on compact sets in  $D$ . In particular, the convergence is in  $\mathcal{E}(D)$ , and since

$$(\Phi_{\epsilon_j} - \phi)\mu = 0$$

for all  $\mu \in \text{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}_\Omega^Z)$  on compact subsets of  $D$ , this must hold for  $\Phi$  as well, cf. Section 4. Thus  $\phi$  is the image of  $\Phi$  in  $\mathcal{O}_X$ , that is,  $\Phi$  is an extension of  $\phi$ . Clearly  $\Phi$  satisfies (1.2) and thus Theorem 1.1 is proved in case when  $D$  is a ball and  $\mathcal{O}_X$  is Cohen–Macaulay.

*Remark 7.2.* — One can check that the limit

$$\Phi(z) = \int_{\zeta \in D} HR \wedge \frac{(-\rho)^\alpha \beta_n}{v^{n+\alpha+1}} \phi = \lim_{\epsilon \rightarrow 0} \int_{\zeta \in D_\epsilon} HR \wedge \frac{(-\rho_\epsilon)^\alpha \beta_n}{v_\epsilon^{n+\alpha+1}} \phi$$

exists for each  $z \in D$ , and thus it is not necessary to take a subsequence in the argument above. However, we do not need this refinement and omit the details.

We will now point out how to estimate (7.1) if  $\mathcal{O}_X$  has non-Cohen–Macaulay points in  $D$ . Then, cf. Section 4,

$$HR = H_{\kappa}^0 R_{\kappa} + \cdots + H_{N-1}^0 R_{N-1}.$$

Recall the representations (4.2). Since  $\mathcal{O}_X$  is Cohen–Macaulay at points on  $\partial D \cap Z$ ,  $a_k$  are smooth there and hence we can proceed in the same way as before at such points.

Let  $U \subset\subset Z \cap D$  be a small neighborhood of a point on  $Z \cap D$  and let us choose coordinates  $(\zeta, \tau)$  in  $U$  as before. Then we have, cf. (4.2) and (3.5), that

$$H_k^0 R_k = a_k \mu = a_k \gamma \bar{\partial} \frac{d\tau}{\tau^{M+1}}.$$

Since we are far from the boundary  $1/v$  is bounded and thus we get terms like

$$\int_{(\zeta, \tau) \in D} R \wedge \beta \phi = \int_{(\zeta, \tau) \in D} a_k \beta \bar{\partial} \frac{d\tau}{\tau^{M+1}} \wedge \gamma \phi,$$

where  $\beta$  is smooth and has compact support in  $U$ ; also  $H$  is incorporated in  $\beta$  here.

Integrating with respect to  $\tau$ , that is, applying  $\pi_*$ , we get by Lemma 7.3 a sum of terms like

$$(7.6) \quad \int_{\zeta \in Z \cap D} b_m(\cdot, z) \partial_{\tau}^m (\gamma \phi)$$

for  $m \leq M$ , where  $b_m(\zeta, z)$  are currents with compact support in  $U$  that depend holomorphically on  $z$  in  $D$ . By usual Cauchy estimates, (7.6) is controlled by the  $L^p$ -norm of  $\partial_{\tau}^m (\gamma \phi)$  over  $U$ . In view of (3.4) and (3.5) we get the same a priori estimate as before. Thus Theorem 1.1 is fully proved in the case when  $D$  is the ball, except for the following two lemmas.

LEMMA 7.3. — *With the notation in the proof, let  $a = \beta a_k$ , and  $\psi = \gamma \phi$ . Then*

$$\pi_* \left( \bar{\partial} \frac{d\tau}{\tau^{M+1}} a \phi \right) = \sum_{m \leq M} b_m \partial_{\tau}^m \psi|_{\tau=0},$$

where  $b_m$  are currents on  $U$  with compact support in  $U$ . If, in addition,  $\beta$  depends holomorphically on a parameter  $z$ , then also  $b_m$  will do.

*Proof.* — Recall from Section 4, cf. (4.3), that

$$a \bar{\partial} (d\tau / \tau^{M+1}) = \lim_{\epsilon \rightarrow 0} \chi(|f|^2 / \epsilon) d \bar{\partial} (d\tau / \tau^{M+1}),$$

where  $f$  is a holomorphic tuple with zero set  $W$ . It follows that

$$\tau^{M'} a \bar{\partial} (d\tau / \tau^{M+1}) = 0$$

if  $\tau^{M'}$  is in the ideal  $\langle \tau^{M+1} \rangle$ , that is, if  $M'_j \geq M_j + 1$  for some  $j$ . Since  $\psi$  is holomorphic we have

$$\psi(\zeta, \tau) = \sum_{m \leq M} \psi_m(\zeta) \tau^m + \dots$$

where  $\dots$  are terms in  $\langle \tau^{M+1} \rangle$ . It follows that

$$a\psi\bar{\partial}(d\tau/\tau^{M+1}) = \sum_{m \leq M} \psi_m(\zeta) a\bar{\partial}(d\tau/\tau^{M-m+1}),$$

and hence

$$\pi_*(a\psi\bar{\partial}(d\tau/\tau^{M+1})) = \sum_{m \leq M} \psi_m(\zeta) \pi_*(a\bar{\partial}(d\tau/\tau^{M-m+1})).$$

Now the lemma follows, since the last factor depends holomorphically on  $z$ . □

LEMMA 7.4. — *With the notation above we have, for  $s > -1$  and  $b > 0$ , we have the estimates*

$$(7.7) \quad \int_{z \in D} \frac{\delta(z)^s dV(z)}{|v|^{N+1+s+b}} \lesssim \frac{1}{\delta(z)^b}$$

and

$$(7.8) \quad \int_{\zeta \in Z \cap D} \frac{\delta(\zeta)^s dV(\zeta)}{|v|^{n+1+s+b}} \lesssim \frac{1}{\delta(z)^b}.$$

This lemma is well-known and follows in a standard way from the local representation (6.4) of  $|v|$ . For instance, (7.7) is reduced to the elementary estimate

$$\int_{|x| < 1, x_1 > 0} \frac{x_1^s dx_1 \cdots dx_{2N}}{(x_1 + y_1 + |x_2 - y_2| + \sum_{j=3}^{2N} |x_j - y_j|^2)^{N+1+s+b}} \lesssim \frac{1}{y_1^b}.$$

Notice that the “worst case” in (7.8) is when  $z$  lies on  $Z$ . Therefore, it follows from (7.7) applied to  $Z \cap D$ . See, e.g., [22, V.3.3] for a detailed discussion of this kind of estimates.

Remark 7.5. — There is a somewhat different way to construct holomorphic extensions from  $X$ , which is, e.g., used in [2]. Let  $(E, f)$  be a Hermitian resolution of  $\mathcal{O}_D \mathcal{J}$  as before and let  $\nabla_f = f - \bar{\partial}$ , cf. [4, 10]. The associated currents  $U$  and  $R$  are related by the formula  $\nabla_f U^0 = I - R$ , that is,  $f_{k+1} U_{k+1}^0 - \bar{\partial} U_k^0 = I - R_k$ ,  $k = 0, 1, \dots$ . If  $\phi \in \mathcal{O}(X \cap D)$ , then  $R\phi$  is well-defined. By solving a sequence of  $\bar{\partial}$ -equations in  $D$  one can find a current  $V = V_1 + V_2 + \dots + V_N$  in  $D$  such that  $f_{k+1} V_{k+1} - \bar{\partial} V_k = -R_k \phi$ ,  $k \geq 1$ . We claim that  $\Phi = f_1 V_1$  is a holomorphic extension of  $\phi$ . Since one can solve  $\bar{\partial}$  with estimates one get estimates of  $\Phi$ . In case  $\kappa = 1$  there is

just one step in this procedure so that if  $K$  is a solution operator for  $\bar{\partial}$  in  $D$ , then  $\Phi = f_1 K(R_1 \phi)$  is a holomorphic extension of  $\phi$ . However, except for the case when  $X$  is reduced, we cannot see how to obtain Theorems 1.1 or 1.2 with this approach.

Let us sketch a proof of the claim: Let  $\varphi$  be any holomorphic extension of  $\phi$  to  $D$ . Then

$$\nabla_f U^0 \varphi = (I - R)\varphi = \varphi - R\phi.$$

Furthermore,  $\nabla_f V = \Phi - R\phi$ . Hence  $\nabla_f(V - U^0 \varphi) = \varphi - \Phi$ . By solving another sequence of  $\bar{\partial}$ -equations one can find a holomorphic  $w$  such that  $\varphi - \Phi = f_1 w$ . This precisely means that  $\varphi - \Phi$  is in  $\mathcal{J}$ .

### 8. Proof of Theorem 1.1 in the general case

As described in Section 6, in the general case we get a similar function  $v(\zeta, z) = \delta_{\zeta-z} q - \rho(\zeta)$  but instead of being holomorphic in  $z$  and anti-holomorphic in  $\zeta$  we have  $\bar{\partial}_z v = 0$  close to the diagonal  $\Delta$  and the property (6.9), respectively. Notice, cf. (6.8), for future reference, that we can choose  $q$  so that  $\bar{\partial}_z q = 0$  close to  $\Delta$ .

In this section we let the  $\bar{\partial}$  in  $\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar{\partial}$  act on both  $z$  and  $\zeta$ . Thus also anti-holomorphic differentials with respect to  $z$  will occur in  $g^\alpha$ , cf. (6.10), and in

$$(8.1) \quad g := (f_1(z)H^1U + HR) \wedge g^\alpha.$$

However, we only have holomorphic differentials with respect to  $\zeta$ . Then still  $\nabla_{\zeta-z} g^\alpha = 0$  and  $\nabla_{\zeta-z} g = 0$ .

Let  $\Omega$  be a neighborhood of  $\bar{D}$  and assume that  $\phi$  is defined in  $\Omega \cap X$ . Moreover, let  $\Psi$  be a holomorphic extension to  $\Omega$ . Then

$$\Psi(z) = \int_{\zeta} g_{N,N}^{0,0} \Psi, \quad z \in D,$$

where upper and lower indices denote bidegree in  $z$  and  $\zeta$ , respectively. Hence (the  $(0, 0)$ -component  $i$   $z$  of)

$$(8.2) \quad \varphi(z) := \int_{\zeta \in D} HR \wedge g^\alpha(\zeta, z)\phi(\zeta)$$

is a smooth function in  $D$  that interpolates  $\phi$  in the sense that  $\varphi - \phi$  is in  $\mathcal{E}^{0,0} \mathcal{J}$ , cf. Section 5.1.

We shall now modify the kernel in (8.2) so that it produces a holomorphic extension. To this end we invoke a result that should be of independent interest. We formulate and prove a somewhat more general version in Section 9.

PROPOSITION 8.1. — Assume that  $\widehat{D} \subset \subset \widetilde{D}$  are pseudoconvex neighborhoods of  $\overline{D}$ . There is a linear operator  $T: \mathcal{E}^{0,1}(\widetilde{D}) \cap \ker \bar{\partial} \rightarrow \mathcal{E}^{0,0}(\widetilde{D})$  such that  $\bar{\partial}T\xi = \xi$  in  $\widehat{D}$  and furthermore  $T\xi \in \mathcal{E}^{0,0}\mathcal{J}(\widehat{D})$  if  $\xi \in \mathcal{E}^{0,1}\mathcal{J}(\widetilde{D})$ .

Here  $\xi \in \mathcal{E}^{0,1}\mathcal{J}(\widetilde{D})$  means that  $\xi$  is a smooth  $(0, 1)$ -form in  $\widetilde{D}$  such that locally  $\xi$  has a representation  $\xi = \xi_1\eta_1 + \dots + \xi_\nu\eta_\nu$ , where  $\xi_j$  are smooth  $(0, 1)$ -forms and  $\eta_j$  are functions in  $\mathcal{J}$ .

Recall from Section 6 that  $\bar{\partial}_z q = 0$  and  $\bar{\partial}_z v = 0$  in a set  $W = \{|\zeta - z| < \epsilon\}$ . It follows from (6.2) that there is a pseudoconvex neighborhood  $\widetilde{D}$  of  $\overline{D}$  such that  $\bar{\partial}_z g^\alpha$  is smooth in  $D_\zeta \times \widetilde{D}_z$ . It follows that also  $\bar{\partial}_z g$  is smooth in  $\widetilde{D}$  for  $\zeta \in D$ . Since  $\nabla_{\zeta-z} g = 0$ , the component  $g_{N,N}$  of  $g$  of total bidegree  $(N, N)$  is  $\bar{\partial}$ -closed, and hence

$$(8.3) \quad \bar{\partial}_z g_{N,N}^{0,0} + \bar{\partial}_\zeta g_{N,N-1}^{0,1} = 0$$

in  $D \times \widetilde{D}$ . Since  $\bar{\partial}_z q = 0$  in  $W$ , no anti-holomorphic differentials with respect to  $z$  can occur in  $g^\alpha$ , cf. (6.10), there, and hence  $g_{N,N-1}^{0,1} = 0$  in  $W \cap D \times \widetilde{D}$ .

Notice that  $\bar{\partial}_z(HR \wedge g^\alpha) = HR \wedge \bar{\partial}_z g^\alpha$ . We now define

$$(8.4) \quad \mathcal{A}(\zeta, z) = T(H(\zeta, t)R(\zeta) \wedge \bar{\partial}_t g^\alpha(\zeta, t))(z), \quad \zeta \in D, \quad z \in \widehat{D}.$$

Then clearly

$$HR \wedge g^\alpha(\zeta, z) - \mathcal{A}(\zeta, z)$$

is holomorphic in  $z \in D$ . Thus

$$(8.5) \quad \Phi(z) := \int_{\zeta \in D} (HR \wedge g^\alpha(\zeta, z) - \mathcal{A}(\zeta, z))\phi$$

is holomorphic in  $D$ . We claim that  $\Phi$  is indeed an extension of  $\phi$ .

*Proof of the claim.* — As noticed above  $g_{N,N-1}^{0,1}$  vanishes in  $W$ . Hence it is smooth in  $D$  and vanishes to high order at the boundary. Since  $\Psi$  is holomorphic thus

$$\int_{\zeta \in D} \bar{\partial}_\zeta g_{N,N-1}^{0,1} \Psi = 0$$

by Stokes' theorem. In view of (8.3), cf. (8.1), we therefore have

$$(8.6) \quad \int_{\zeta \in D} HR \wedge \bar{\partial}_t g^\alpha \phi = - \int_{\zeta \in D} f_1(t)H^1U \wedge \bar{\partial}_t g^\alpha \Psi.$$

Applying  $T$  we get

$$(8.7) \quad \begin{aligned} \int_{\zeta \in D} \mathcal{A}(\zeta, z)\phi(\zeta) &= T \left( \int_{\zeta \in D} HR \wedge \bar{\partial}_t g^\alpha \phi \right) \\ &= -T \left( \int_{\zeta \in D} f_1(t)H^1U \wedge \bar{\partial}_t g^\alpha \Psi \right). \end{aligned}$$

In fact, the change of order of  $T$  and integration with respect to  $\zeta \in D$  is legitimate since the currents  $U$  and  $R$ , as well as  $(-\rho(\zeta))^r$  go outside and what is left are forms depending on  $t$  that are smooth in  $\tilde{D}$ . Since

$$\int_{\zeta \in D} f_1(t) H^1 U \wedge \bar{\partial}_t g^\alpha \Psi$$

is in  $\mathcal{E}^{0,1} \mathcal{J}(\tilde{D})$  and  $\bar{\partial}_t$ -closed, it follows from Proposition 8.1 that

$$T \left( \int_{\zeta \in D} f_1(t) H^1 U \wedge \bar{\partial}_t g^\alpha \Psi \right)$$

is in  $\mathcal{E}^{0,0} \mathcal{J}(\hat{D})$  with respect to  $z$ . We conclude that (8.7) is in  $\mathcal{E}^{0,0} \mathcal{J}(\hat{D})$ . Thus  $\Phi - \phi$  is in  $\mathcal{E}^{0,0} \mathcal{J}(D)$ , and since  $\Phi$  is holomorphic, therefore  $\Phi - \phi$  is in  $\mathcal{J}$ , see Lemma 8.2. Thus the claim is proved.  $\square$

Now the proof of Theorem 1.1, that is, estimating the extension  $\Phi$ , is concluded in essentially the same way as for the case with the ball in Section 7. Since  $\mathcal{A}$  has no singularities at the diagonal the second term in the definition (8.5) of  $\Phi$  offers no problems at all. The first term is handled as in the proof for the ball. In fact, close to a point  $\partial D \cap Z$  the same arguments as before work. Each time a holomorphic derivative falls on  $v$  we get  $\mathcal{O}(|\zeta - z|^\infty)$  which cancels the singularity in view of (6.2). In a neighborhood of a (possibly non-Cohen–Macaulay) point in  $D \cap Z$  one proceeds precisely as in the the proof of Theorem 1.1 for the ball.

LEMMA 8.2. — *If  $\Phi$  is holomorphic and in  $\mathcal{E}^{0,0} \mathcal{J}$ , then it is in  $\mathcal{J}$ .*

More explicitly, if  $\eta_1, \dots, \eta_\nu$  generate  $\mathcal{J}$ ,  $\Phi = a_1 \eta_1 + \dots + a_\nu \eta_\nu$  for some smooth functions  $a_j$  and  $\Phi$  is holomorphic, then one can choose holomorphic such  $a_j$ .

This lemma should be well-known and it is an immediate consequence of the first part of Proposition 4.2.

### 9. The $\bar{\partial}$ -equation for forms in $\mathcal{E} \mathcal{J}$

In this section  $\mathcal{J}$  is a quite arbitrary ideal sheaf in a pseudoconvex domain  $\Omega \subset \mathbb{C}^N$ .

THEOREM 9.1. — *Let  $\mathcal{J}$  be an ideal sheaf in a pseudoconvex domain  $\Omega \subset \mathbb{C}^N$ , assume that its zero set  $Z$  has codimension  $\kappa > 0$ , and let  $\Omega' \subset\subset \Omega$ . There is a linear operator  $T: \mathcal{E}^{0,1}(\Omega) \cap \text{Ker } \bar{\partial} \rightarrow \mathcal{E}^{0,0}(\Omega')$ , such that  $\bar{\partial} T \xi = \xi$  in  $\Omega'$  and furthermore  $T \xi \in \mathcal{E}^{0,0} \mathcal{J}(\Omega')$  if  $\xi \in \mathcal{E}^{0,1} \mathcal{J}(\Omega)$ .*

*Proof.* — In a possibly slightly smaller pseudoconvex domain, that we denote by  $\Omega$  as well, we can choose a Hermitian free resolution (4.1) of  $\mathcal{O}_\Omega/\mathcal{J}$ . Let  $U$  and  $R$  be the associated currents and let  $H$  be a Hefer morphism associated with (4.1). Moreover, let  $g$  be a smooth weight with respect to  $z \in \Omega'$  with compact support in  $\Omega$ , cf. Section 5. We also assume that  $g$  depends holomorphically on  $z$ . Furthermore, let  $B$  be the component of the full Bochner–Martinelli form, see [3, Section 2], that only has holomorphic differentials with respect to  $\zeta$ . It follows from [4, Section 7.4], see also [8, 9], that if  $v$  is a smooth  $(0, 0)$ -form in  $\Omega$ , then

$$(9.1) \quad v(z) = \int_{\zeta \in \Omega} (f_1(z)H^1U + HR) \wedge g \wedge B \wedge \bar{\partial}v + \int_{\zeta \in \Omega} (f_1(z)H^1U + HR) \wedge gv$$

for  $z \in \Omega'$ . In fact, one can choose regularizations  $U^\epsilon$  and  $R^\epsilon$  of  $U$  and  $R$ , respectively, so that

$$g^\epsilon = f_1(z)H^1U^\epsilon + HR^\epsilon$$

are smooth weights, and then

$$(9.2) \quad v = \int_{\zeta \in \Omega} g^\epsilon \wedge g \wedge B \wedge \bar{\partial}v + \int_{\zeta \in \Omega} g^\epsilon \wedge gv$$

holds for  $\epsilon > 0$ , see Remark 9.3 and, e.g., [9]. Now

$$g^\epsilon \rightarrow g' := f_1(z)H^1U + HR$$

as currents when  $\epsilon \rightarrow 0$ . Notice that  $g' \wedge B$  is a tensor product of currents and hence well-defined in  $\Omega \times \Omega$ , and that  $g^\epsilon \wedge B \rightarrow g' \wedge B$ . Thus (9.1) follows from (9.2).

Let  $\psi$  be a  $\bar{\partial}$ -closed smooth  $(0, 1)$ -form in  $\Omega$  and let  $v$  be a (smooth) solution to  $\bar{\partial}v = \psi$  in  $\Omega$ . Since the second term in (9.1) is holomorphic, it follows that

$$(9.3) \quad T\psi := \int_{\zeta \in \Omega} (f_1(z)H^1U + HR) \wedge g \wedge B \wedge \psi$$

is a solution to  $\bar{\partial}u = \psi$  in  $\Omega'$ . Since two solutions only differ by a holomorphic function it is clear that  $T\psi$  is smooth. This is also seen directly, noticing that

$$(9.4) \quad T\psi = v - \int_{\zeta \in \Omega} (f_1(z)H^1U + HR) \wedge gv.$$

Now assume that, in addition,  $\psi \in \mathcal{E}^{0,1}\mathcal{J}$ . Then  $R\psi = 0$  and thus  $HR \wedge g \wedge B \wedge \psi$  vanishes since it is a tensor product of  $R\psi$  and  $B$  times

smooth forms. Therefore, cf. (9.3),

$$u := T\psi(z) = f_1(z) \int_{\zeta \in \Omega} H^1U \wedge g \wedge B \wedge \psi =: f_1(z)b(z).$$

However, we do not know that  $b$  is smooth; in fact it is (probably) not in general, and hence we cannot conclude directly that  $u \in \mathcal{E}^{0,0}\mathcal{J}$ . Notice for instance that  $1 = f(1/f)$  although  $1$  is not in  $\langle f \rangle$ . To prove that  $u$  is indeed in  $\mathcal{E}^{0,0}\mathcal{J}$  we first use the following lemma.

LEMMA 9.2. — *If  $\psi \in \mathcal{E}^{0,1}\mathcal{J}$  and  $\bar{\partial}\psi = 0$ , then  $Ru = 0$ .*

Since  $u$  is smooth,  $Ru$  is well-defined.

*Proof.* — Let  $R_z$  denote  $R$  depending on  $z$ . First notice that  $R_z \wedge U$  is a well-defined current in  $\Omega_\zeta \times \Omega_z$  since it is a tensor product. Moreover,  $B$  is an almost semi-meromorphic form and therefore, cf. (4.3),

$$R_z \wedge H^1U \wedge B := \lim_{\epsilon \rightarrow 0} R_z \wedge H^1U \wedge B^\epsilon$$

is a well-defined current, where  $B^\epsilon = \chi(|\zeta - z|^2/\epsilon)B$ . See also [8, 9, 12].

Since  $u$  is smooth and  $R_z^\epsilon \rightarrow R_z$  we have that  $R_z^\epsilon u \rightarrow R_z u$ . Moreover,

$$R_z^\epsilon u = \int_{\zeta \in \Omega} R_z^\epsilon \wedge f_1(z)H^1U \wedge B \wedge g\psi.$$

We claim that

$$(9.5) \quad W_k = \lim_{\epsilon \rightarrow 0} R_{z,k}^\epsilon \wedge H^1U \wedge B - R_{z,k} \wedge H^1U \wedge B = 0, \quad k = 0, 1, \dots$$

The proof of this claim relies on the fact that all currents involved are *pseudomeromorphic* and that such currents fulfill the dimension principle: If  $\mu$  is pseudomeromorphic, has bidegree  $(*, \ell)$ , and support on a subvariety of codimension strictly larger than  $\ell$ , then  $\mu$  must vanish. See [9, 11].

*Proof of the claim.* — Since  $R_{z,k} \wedge U$  is a tensor product,  $R_{z,k}^\epsilon \wedge U \rightarrow R_{z,k} \wedge U$ . Since  $B$  is smooth outside the diagonal  $\Delta$ , therefore  $W_k = 0$  there. That is,  $W_k$  has support on  $\Delta$ .

Recall that  $H^1U$  is a sum of currents of bidegree  $(*, *)$  in  $\zeta$  so that  $H^1U \wedge B$  is a sum of currents of bidegree at most  $(N, N - 1)$ . Thus  $W_k$  has bidegree at most  $(N, N - 1 + k)$ . Since  $R_k$  has support on  $Z$  we have that  $W_k$  has support on  $\Delta \cap \Omega \times Z$  which we can think of as  $Z \subset \Delta \subset \Omega \times \Omega$ , and hence it has codimension  $N + \kappa$  in  $\Omega \times \Omega$ . By the dimension principle we conclude that  $W_k = 0$  if  $k \leq \kappa$ .

Next we use the fact that outside a Zariski closed set  $Z_1 \subset Z$  with codimension at least 1 in  $Z$  there is a smooth form  $\alpha_1$  such that  $R_{\kappa+1} = \alpha_1 R_\kappa$ , see, [10]. Outside  $Z_1$  thus  $W_{\kappa+1} = \alpha_1 W_\kappa = 0$ . Thus  $W_{\kappa+1}$  has anti-holomorphic degree at most  $N - 1 + \kappa + 1$  and support on  $Z_1 \subset \Delta \subset \Omega \times \Omega$ .

Again by the dimension principle it must vanish. In general, there are Zariski closed sets  $Z_\ell \subset Z$  of codimension at least  $\ell$  in  $Z$ , and smooth forms  $\alpha_\ell$  outside  $Z_\ell \subset Z$  such that  $R_{\kappa+\ell+1} = \alpha_{\ell+1}R_{\kappa+\ell}$  there. The claim now follows by finite induction.  $\square$

From the claim we conclude that

$$\begin{aligned} R_z T\psi(z) &= \int_{\zeta \in \Omega} R_z f_1(z) H^1 U \wedge g \wedge B \wedge \psi \\ &= \lim_{\epsilon \rightarrow 0} \int_{\zeta \in \Omega} R_z f_1(z) H^1 U \wedge g \wedge B^\epsilon \wedge \psi = 0, \end{aligned}$$

where the last equality holds since  $R_z f_1(z) = 0$  and hence the tensor product (times smooth forms)  $R_z f_1(z) H^1 U \wedge B^\epsilon \wedge \psi$  vanishes as well. Thus the lemma is proved.  $\square$

We can now conclude the proof of Theorem 9.1. Since  $\bar{\partial}u = \psi$ , that is,

$$\partial u / \partial \bar{z}_j = \psi_j, \quad j = 1 \dots, N,$$

where each  $\psi_j$  is in  $\mathcal{E}^{0,0} \mathcal{J}$ , we conclude that

$$(\partial^\alpha u / \partial \bar{z}^\alpha u) R = 0$$

for all  $\alpha \geq 0$ . It now follows from [10, Theorem 5.1] that  $u$  is in  $\mathcal{E}^{0,0} \mathcal{J}$ .  $\square$

*Remark 9.3.* — If  $f$  is a holomorphic tuple that vanishes on  $Z$  and  $\chi(t)$  is as before then one can take  $U^\epsilon = \chi(|f|^2/\epsilon)U$  and then define  $R^\epsilon$  so that  $\nabla_f U^{\epsilon,0} = I - R^\epsilon$ . Notice that  $R_k^\epsilon$  may be non-vanishing for all  $k \geq 0$ .

### 10. Proof of Theorem 1.2

If  $\mathcal{J} = \langle f^{M+1} \rangle$ , then we have the simple resolution

$$0 \rightarrow \mathcal{O}(E_1) \xrightarrow{f^{M+1}} \mathcal{O}(E_0) \rightarrow \mathcal{O}/\mathcal{J} \rightarrow 0,$$

where  $E_1$  and  $E_0$  are trivial line bundles. Moreover,

$$U = \frac{1}{f^{M+1}}, \quad R = R_1 = \bar{\partial} \frac{1}{f^{M+1}},$$

and if  $h$  is a holomorphic  $(1, 0)$ -form in  $\Omega$  for each  $z \in \Omega$  such that  $\delta_{\zeta-z} h = f - f(z)$ , then

$$H = \sum_{k=0}^M f(\zeta)^{M-k} f(z)^k h$$

is a Hefer form for  $f^{M+1}$ , that is,

$$\delta_{\zeta-z} H = f(\zeta)^{M+1} - f(z)^{M+1}.$$

Thus

$$HR = H\bar{\partial}\frac{1}{f^{M+1}} = \sum_{k=0}^M f^k(z)h \wedge \bar{\partial}\frac{1}{f^{k+1}}.$$

Let us first assume that we are in the ball so that  $v(\zeta, z)$  is holomorphic in  $z$  and anti-holomorphic in  $\zeta$ . Then we get our extension

$$\Phi(z) = \int_{\zeta \in D \cap X} \sum_{k=0}^M f^k(z) \bar{\partial} \frac{1}{f^{k+1}} \wedge h \wedge g^\alpha \phi$$

for a suitably large  $\alpha$ . Arguing precisely as in Section 7, cf. (7.3), we see that

$$\Phi(z) = \int_{\zeta \in D \cap Z} \sum_{k=0}^M f^k(z) \frac{(-\rho)^\alpha}{v^{\alpha+n+1}} \beta_k \sum_{|\beta|=k} \partial^\beta \phi,$$

where  $\beta_k$  are smooth forms. If  $\zeta \in Z$ , then  $f(z) = f(z) - f(\zeta) = \mathcal{O}(|\zeta - z|)$  and hence  $|f(z)| \leq \sqrt{|v|}$ . Using the same estimates as in Section 7 now Theorem 1.2 follows in the case with the ball. Combining with the arguments in Section 8 the general case follows.

*Remark 10.1.* — It is reasonable to believe that it is possible to get a similar sharpening of Theorem 1.1, for instance, if  $Z$  has higher codimension and  $\mathcal{J}$  is a jet ideal  $\mathcal{J}_Z^{M+1}$ .

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