A PULLBACK OPERATION ON A CLASS OF CURRENTS

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Håkan Samuelsson Kalm

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ABSTRACT. — For any holomorphic mapping $f\colon X \to Y$ between a complex manifold X and a complex Hermitian manifold Y we extend the pullback f^* from smooth forms to a class of currents. We provide a basic calculus for this pullback and show under quite mild assumptions that it is cohomologically sound. The class of currents we consider contains in particular the Lelong current of any analytic cycle. Our pullback depends in general on the Hermitian structure of Y but coincides with the usual pullback of currents in case f is a submersion. The construction is based on the Gysin mapping in algebraic geometry.

RÉSUMÉ. — Pour toute application holomorphe $f\colon X \to Y$ entre une variété complexe X et une variété hermitienne complexe Y nous étendons le tiré en arrière f^* des formes lisses à une classe de courants. Nous fournissons un calcul de base pour ce tiré en arrière et montrons sous des hypothéses assez faibles qu'il est cohomologiquement correct. La classe de courants que nous considérons contient en particulier le courant de Lelong de tout cycle analytique. Notre tiré en arrière dépend en général de la structure hermitienne de Y mais coïncide avec le tiré en arrière habituel des courants a cas où f est une submersion. La construction est basée sur le morphisme de Gysin dans le géométrie algébrique.

1. Introduction

Calculus of currents is central in complex geometry. One example is the celebrated Monge–Ampère product. Closely related to current products is pullback of currents. The purpose of this paper is to introduce a pullback operation on a class of currents and establish basic properties.

We say that a current μ on a complex manifold Y is a $\mathcal{PS}$-*current* (a pseudosmooth current), $\mu \in \mathcal{PS}(Y)$, if it locally is a finite sum of direct images $g_*\alpha$ where α is a smooth compactly supported form and g is a holomorphic mapping. For instance, Lelong currents of analytic cycles are

$\mathcal{PS}$-currents since if $i\colon V \to Y$ is a subvariety, then $[V] = i_*1$, where $[V]$ is the Lelong current of integration over V_{reg}, and by a partition of unity on V it follows that $[V]$ is a $\mathcal{PS}$-current. The class of $\mathcal{PS}$-currents is closed under exterior products with smooth forms and under d, $\bar{\partial}$, and ∂.

THEOREM 1.1. — *Let X be a complex manifold and let Y be a complex Hermitian manifold. For any holomorphic mapping $f\colon X \to Y$ there is a linear mapping $f^*\colon \mathcal{PS}(Y) \to \mathcal{PS}(X)$ with the following properties: If φ is a smooth form on Y, then $f^*\varphi$ is the usual pullback. If additionally $\mu \in \mathcal{PS}(Y)$, then*

$$(1.1) \qquad\qquad f^*(\varphi \wedge \mu) = f^*\varphi \wedge f^*\mu,$$

$$(1.2) \qquad f^*d\mu = df^*\mu, \quad f^*\bar{\partial}\mu = \bar{\partial}f^*\mu, \quad f^*\partial\mu = \partial f^*\mu.$$

Let $U \subset X$ be an open set. Then $(f^*\mu)|_U = f|_U^*\mu$. Assume that $f(U)$ is a complex manifold and that the mapping $\widetilde{f}\colon U \to f(U)$ induced by f is a submersion. Let $\iota\colon f(U) \to Y$ be the inclusion so that $f|_U = \iota \circ \widetilde{f}$. Then $f|_U^*\mu = \widetilde{f}^*\iota^*\mu$, where $\widetilde{f}^*$ is the usual pullback of currents under a submersion.

Notice in view of (1.1) that $\operatorname{supp} f^*\mu \subset f^{-1}(\operatorname{supp}\mu)$. Notice also that if f is a submersion, then by the last part of Theorem 1.1, our f^* coincides with the usual pullback of currents under a submersion.

For our second main result we will assume that Y has an additional property. We say that a complex manifold Y is *good* if there is a holomorphic section Φ of a holomorphic vector bundle $F \to Y \times Y$ such that Φ defines the diagonal in $Y \times Y$, i.e., Φ vanishes to first order precisely on the diagonal. Many complex manifolds are good. For instance, all projective manifold are good, and any submanifold of a good manifold is good.

THEOREM 1.2. — *Let Y be a good compact complex Hermitian manifold, X a compact complex manifold, and $f\colon X \to Y$ a holomorphic mapping. If $\mu \in \mathcal{PS}(Y)$ is closed, then for any Hermitian metric on Y,*

$$(1.3) \qquad\qquad [f^*\mu]_{\mathrm{dR}} = f^*[\mu]_{\mathrm{dR}},$$

where $[\cdot]_{\mathrm{dR}}$ means de Rham cohomology class and f^* in the right-hand side is the usual pullback of cohomology classes. In particular, $[f^*\mu]_{\mathrm{dR}}$ is independent of the Hermitian structure on Y.

The assumption that Y is good is for proof-technical reasons and we believe that it is not essential for the conclusion of Theorem 1.2 to hold.

Theorem 1.2 implies that our pullback on closed $\mathcal{PS}$-currents is functorial on cohomology level. By this we mean that if f in Theorem 1.2 can be

factorized as $f = f_2 \circ f_1$, where $f_1 \colon X \to X'$ and $f_2 \colon X' \to Y$ are holomorphic mappings and X' is a good compact complex Hermitian manifold, then

$$[f^* \mu]_{\mathrm{dR}} = [f_1^* f_2^* \mu]_{\mathrm{dR}}.$$

This follows indeed by Theorem 1.2 since $f^*[\mu]_{\mathrm{dR}} = f_1^* f_2^* [\mu]_{\mathrm{dR}}$. However, in general $f_1^* f_2^* \mu$ depends on a choice of Hermitian structure on X' and therefore one cannot expect $f^* \mu = f_1^* f_2^* \mu$ to hold in general. Example 7.1 below shows that in general $f^* \mu \neq f_1^* f_2^* \mu$.

A standard approach to pullback of currents is the following. Let X, Y, and f be as in Theorem 1.1, let $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ be the standard projections, and let $i \colon X \to X \times Y$ be the graph embedding defined by $i(x) = (x, f(x))$. If μ is a current in Y, then $\pi_2^* \mu$ is well-defined since π_2 is a projection; in fact, $\pi_2^* \mu = 1 \otimes \mu$. The main step is to give a reasonable meaning to the current product

$$(1.4) \qquad\qquad [i(X)] \wedge \pi_2^* \mu,$$

where $[i(X)]$ is the current of integration over $i(X)$. Then the pullback of μ under f is defined as

$$(1.5) \qquad\qquad (\pi_1)_* ([i(X)] \wedge \pi_2^* \mu).$$

If μ is a smooth form, then (1.4) is canonically defined and (1.5) is the usual pullback.

We follow this approach and the novelty is our definition of (1.4). The definition is modeled on the Gysin mapping introduced in [2]. If $\pi_2^* \mu$ is a (generalized) cycle and $i(X)$ can be defined by a global holomorphic section of some vector bundle over $X \times Y$, then (1.4) indeed is the image of $\pi_2^* \mu$ under the Gysin mapping in [2] associated to the embedding i.

In case Y is good (1.4) can be defined as follows. Let Φ be a global holomorphic section of $F \to Y \times Y$ defining the diagonal. Then $\Psi := (f \times \mathrm{id}_Y)^* \Phi$ is a holomorphic section of $E := (f \times \mathrm{id}_Y)^* \Phi$ over $X \times Y$ defining $i(X)$. Let $N_X \to i(X)$ be the normal bundle of $i(X)$ in $X \times Y$. It is well-known that there is a canonical isomorphism $N_X \simeq \pi_2^* TY|_{i(X)}$. This isomorphism induces a Hermitian metric on N_X since TY is equipped with a Hermitian metric; by Y being Hermitian, or having a Hermitian structure, we mean that TY is equipped with a Hermitian metric. Moreover, Ψ induces an embedding $N_X \hookrightarrow E|_{i(X)}$ (see, e.g., [2, Lemma 7.3] or Section 2.1 below) and we equip E with a Hermitian metric so that this embedding is Hermitian, i.e., is an embedding of Hermitian vector bundles. It then follows from [2,

Proposition 1.5], see Example 2.8 and (2.11) below, that we have

$$(1.6) \qquad\qquad [i(X)] = \widehat{c}(N_X) \wedge M^\Psi,$$

where $\widehat{c}(N_X)$ is the full Chern form of N_X and M^Ψ is a certain current introduced by Andersson in [1]. The current M^Ψ is the sum of currents M_k^Ψ that can be defined as[1]

$$M_k^\Psi = \lim_{\epsilon \to 0} \bar{\partial}\chi(|\Psi|^2/\epsilon) \wedge \frac{\partial \log |\Psi|^2}{2\pi i} \wedge (\mathrm{dd}^c \log |\Psi|^2)^{k-1}$$

if χ is a smooth regularization of the characteristic function of $[1, \infty) \subset \mathbb{R}$. These currents are in fact the restriction to $i(X)$ of Monge–Ampère products $(\mathrm{dd}^c \log |\Psi|^2)^k$, where $(\mathrm{dd}^c \log |\Psi|^2)^k$ are the Monge–Ampère products of Bedford–Taylor and Demailly if $k \leqslant \mathrm{codim}\, i(X) = \dim Y$, and were introduced in [1] if $k > \mathrm{codim}\, i(X)$.

A basic observation now is that if $\mu \in \mathcal{PS}(Y)$, then one can give a natural meaning to the products $M_k^\Psi \wedge \pi_2^*\mu$. Indeed, if χ is as above, then it turns out that the limits

$$(1.7) \quad M_k^\Psi \wedge \pi_2^*\mu := \lim_{\epsilon \to 0} \bar{\partial}\chi(|\Psi|^2/\epsilon) \wedge \frac{\partial \log |\Psi|^2}{2\pi i} \wedge (\mathrm{dd}^c \log |\Psi|^2)^{k-1} \wedge \pi_2^*\mu,$$
$$k = 1, 2, \ldots$$

exist, are independent of the choice of χ, and are in $\mathcal{PS}(X \times Y)$. Clearly, $M_k^\Psi \wedge \pi_2^*\mu$ have support in $i(X)$. The limit

$$(1.8) \qquad\qquad M_0^\Psi \wedge \pi_2^*\mu := \lim_{\epsilon \to 0}(1 - \chi(|\Psi|^2/\epsilon))\pi_2^*\mu$$

also turns out to exist and to have the properties just mentioned for the limit (1.7). We then define (1.4) to be the component of $\widehat{c}(N_X) \wedge M^\Psi \wedge \pi_2^*\mu$ of the "expected bidegree" $(\dim Y, \dim Y) + \mathrm{bidegree}(\mu)$. In general, $\widehat{c}(N_X) \wedge M^\Psi \wedge \pi_2^*\mu$ has components of various different bidegrees, but if μ is a smooth form, then in view of (1.6), $\widehat{c}(N_X) \wedge M^\Psi \wedge \pi_2^*\mu$ is the canonical product $[i(X)] \wedge \pi_2^*\mu$, which has bidegree $(\dim Y, \dim Y) + \mathrm{bidegree}(\mu)$. We can now define $f^*\mu$ by (1.5). Since we are taking the component of the expected bidegree in the definition of (1.4), it follows that $f^*\mu$ has the same bidegree as μ. After a simplification (using (5.6) below) we get the formula

$$(1.9) \qquad\qquad f^*\mu = \sum_{k=0}^{n} f^*\widehat{c}_{n-k}(TY) \wedge (\pi_1)_*(M_k^\Psi \wedge \pi_2^*\mu).$$

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[1] In this paper $d^c = (\partial - \bar{\partial})/4\pi i$ so that if z is the complex coordinate in $\mathbb{C}$ then $\mathrm{dd}^c \log |z|^2 = [0]$.

As an illustration, let $f\colon \operatorname{Bl}_0 \mathbb{C}^2 \to \mathbb{C}^2$ be the blowup of the origin in $\mathbb{C}^2$ and let μ be the Dirac measure at $0 \in \mathbb{C}^2$. With the standard Hermitian structure on $\mathbb{C}^2$ we have $f^*\mu = \omega \wedge [D]$, where $D \simeq \mathbb{P}^1$ is the exceptional divisor and ω is the standard Fubini–Study metric form on that $\mathbb{P}^1$, see Example 5.3 below.

In this example so-called dimensional excess occurs; $\pi_2^*\mu$ and the graph of f do not intersect properly. In such situations it is reasonable that $f^*\mu$ has to depend on some additional structure, e.g., a Hermitian metric as in our case. This can be compared to the intersection theory in [15] where the intersection of two cycles, which do not intersect properly, is determined only up to rational equivalence.

As already mentioned, Lelong currents of analytic cycles are $\mathcal{PS}$-currents but $\mathcal{PS}$-currents also appear naturally in the study of Monge–Ampère products associated to holomorphic sections of Hermitian vector bundles. For instance, in that setting the so-called non-pluripolar part is a $\mathcal{PS}$-current. Moreover, the pullback of a smooth form under a meromorphic mapping is usually not smooth but only a $\mathcal{PS}$-current. Therefore our class of currents may have relevance in the study of the dynamics of such mappings. We believe that the complete generality of the mapping $f\colon X \to Y$ and the properties of our f^* described in Theorems 1.1 and 1.2 can complement and provide a new viewpoint on existing works on pullback of currents. Let us mention a few such works.

The intersection theory in [15] contains in particular a quite general theory of pullback of cycles such that if Z is a cycle in Y then $f^*(Z)$ is a rational equivalence class in X. This is extended to pullback of Green currents in the context of arithmetic intersection theory by Gillet–Soulé in [16]. In complex geometry mainly pullback of positive closed currents has been considered. A positive closed $(1,1)$-current locally has a dd^c-potential, which is a plurisubharmonic function. By using (quasi-)plurisubharmonic dd^c-potentials one can define the pullback of such currents under surjective mappings, see Méo [21]. Dinh and Sibony have considered pullback of positive closed (p,p)-currents in several papers. In [11] is defined an essentially canonical pullback, with good continuity and cohomological properties, of such currents under holomorphic mappings with constant fiber dimension, cf. Example 5.7 below. In [12] and [13] a general theory of super-potentials is developed on compact Kähler manifolds generalizing the case of $(1,1)$-currents. Building on some works by Dinh and Sibony, Truong defines a pullback under dominant meromorphic mappings [23]. Rather recently, in [14], on Kähler manifolds was introduced the notions

of density currents and tangent currents to a positive closed current along a submanifold. We believe that the tangent currents to $\pi_2^* \mu$, where as before $\pi_2 \colon X \times Y \to Y$ is the natural projection, along the graph of f is closely connected to our approach; cf. [19] where Kaufmann and Vu compare density currents to Monge–Ampère-type products. We also mention the recent paper [9] by Barlet; on any reduced pure-dimensional analytic space is introduced a sheaf of $\bar\partial$-closed $(p, 0)$-currents, closed under wedge products and the de Rham differential, so that the pullback under a holomorphic mapping, whose image is not contained in the singular locus, is well-defined and functorial.

The plan of the paper is as follows. Preliminaries and some basic facts about $\mathcal{PS}$-currents are collected in Section 2. Even though we are primarily interested in currents on complex manifolds, it is convenient and natural in our context to recall some basic facts about differential forms and currents on (reduced) analytic spaces. In Section 3 we consider the operator $\mu \mapsto M^\Psi \wedge \mu$ on $\mathcal{PS}$-currents. This operator is used in Section 4 to adapt the Gysin mapping in [2] to our setting. Using this Gysin mapping the pullback operation is introduced in Section 5 and Theorem 1.1 is proved. In Section 5.2 we give an alternative definition of our pullback that can be generalized to give a definition of a pullback under meromorphic mappings and meromorphic correspondences. Theorem 1.2 is proved in Section 6 and functoriality is discussed in Section 7.

Acknowledgment

2. Preliminaries and $\mathcal{PS}$-currents

Let Z be a (reduced) analytic space of pure dimension. Locally Z can be embedded as an analytic subset of an open set Ω of some $\mathbb{C}^N$. A smooth differential form φ in Z_{reg} is smooth in Z, $\varphi \in \mathcal{E}(Z)$, if there is a local embedding $\iota \colon Z \to \Omega$ and a smooth form $\widetilde\varphi$ in Ω such that $\varphi = \iota^* \widetilde\varphi$ on Z_{reg}. It is well-known that this notion of smooth forms in Z is independent of the choice of local embedding. It follows that d, $\bar\partial$, and ∂ are well-defined on $\mathcal{E}(Z)$.

The space of currents, $\mathcal{C}(Z)$, in Z is the dual of the space of test forms, i.e., compactly supported smooth forms in Z. More concretely, if $\iota\colon Z \to \Omega$ is a local embedding, then the currents in Z can be identified with the currents in Ω vanishing on test forms ξ with $i^*\xi = 0$ in Z_{reg}. By duality, d, $\bar\partial$, and ∂ are well-defined on $\mathcal{C}(Z)$. As usual we also have $\mathcal{E}(Z) \subset \mathcal{C}(Z)$ and that $\mathcal{C}(Z)$ is a module over $\mathcal{E}(Z)$.

Let V be an analytic space and $g\colon V \to Z$ a holomorphic mapping. By [10, Corollary 3.2.21] there is a natural pullback mapping $g^*\colon \mathcal{E}(Z) \to \mathcal{E}(V)$. If g is proper it follows that there is a pushforward mapping $g_*\colon \mathcal{C}(V) \to \mathcal{C}(Z)$ defined by $\langle g_*\mu, \xi\rangle = \langle \mu, g^*\xi\rangle$. If $\mu \in \mathcal{C}(V)$ and $\varphi \in \mathcal{E}(Z)$, then we have the following projection formula

$$(2.1) \qquad \varphi \wedge g_*\mu = g_*(g^*\varphi \wedge \mu).$$

If $\mu \in \mathcal{C}(V)$ has compact support, then $g_*\mu$ is defined (in the same way) also if g is not proper, and (2.1) holds.

If $\pi\colon V \to Z$ is a submersion, so that π_* of test forms in V are test forms in Z, and $\mu \in \mathcal{C}(Z)$, then $\pi^*\mu$ is defined by $\langle \pi^*\mu, \xi\rangle := \langle \mu, \pi_*\xi\rangle$. In particular, $\pi_*\mu$ is defined if $\pi\colon W \times Z \to Z$ is the standard projection. In this case we have

$$(2.2) \qquad \pi^*\mu = 1 \otimes \mu.$$

DEFINITION 2.1. — *A current μ in Z is in $\mathcal{PS}(Z)$ if in a neighborhood of each point, μ is a finite sum of currents $g_*\alpha$, where $g\colon V \to Z$ is a holomorphic mapping from a connected complex manifold V and α is a smooth form with compact support in V.*

It is clear from the definition that $\mathcal{PS}$-currents have order 0, and by (2.1) it is clear that $\mathcal{PS}(Z)$ is an $\mathcal{E}(Z)$-module. Since $\bar\alpha$ is smooth if α is smooth, and $\overline{g_*\alpha} = g_*\bar\alpha$ it follows that

$$(2.3) \qquad \overline{\mathcal{PS}(Z)} = \mathcal{PS}(Z).$$

Moreover, it is clear that if $\widetilde{Z}$ is a pure dimensional analytic space and $h\colon \widetilde{Z} \to Z$ is a proper holomorphic mapping, then

$$(2.4) \qquad h_*\colon \mathcal{PS}(\widetilde{Z}) \to \mathcal{PS}(Z).$$

One can let V in Definition 2.1 be an analytic space without changing $\mathcal{PS}(Z)$. In fact, if V is an analytic space and α is a smooth form in V, then one can first assume that V is irreducible by restricting α to each irreducible component. By Hironaka's theorem then there is a modification

$\pi \colon \widetilde{V} \to V$ such that $\widetilde{V}$ is a connected manifold. Since π is a modification,

$$(2.5) \qquad\qquad\qquad \alpha = \pi_* \pi^* \alpha$$

as currents, and so $g_* \alpha = g_* \pi_* \pi^* \alpha$. Hence, one can replace V by $\widetilde{V}$ and α by $g^* \alpha$. By this observation we get

Example 2.2. — Let $V \subset Z$ be an analytic subset and $i \colon V \to Z$ the inclusion. By Lelong's theorem there is a closed current $[V]$ in Z defined as integration over V_{reg}. The function 1 on V is smooth and $i_* 1 = [V]$. Thus, by a smooth partition of unity on V, we see that $[V] \in \mathcal{PS}(Z)$.

It follows that if $\nu = \sum_j a_j V_j$ is an analytic cycle in Z, then the Lelong current $[\nu] = \sum_j a_j [V_j]$ is in $\mathcal{PS}(Z)$. Notice also that there is a $\mu \in \mathcal{PS}(|\nu|)$ such that $[\nu] = j_* \mu$, where $j \colon |\nu| \to Z$ is the inclusion of the support of ν.

Generalized cycles, as introduced in [2], are in $\mathcal{PS}(Z)$ as well.

We are primarily interested in $\mathcal{PS}$-currents on manifolds. We will see that on a manifold, $\mathcal{PS}$-currents are *pseudomeromorphic*, so we recall the definition and some properties that we will use. Pseudomeromorphic currents were introduced in [7] and further developed in [4].

In $\mathbb{C}$ we have the principal value currents $1/z^\ell$ and the associated residue currents $\bar{\partial}(1/z^\ell)$. An *elementary pseudomeromorphic current* is a current in an open set $U \subset \mathbb{C}^N$ of the form

$$\alpha \wedge \frac{1}{z_1^{\ell_1}} \cdots \frac{1}{z_r^{\ell_r}} \bar{\partial}\frac{1}{z_{r+1}^{\ell_{r+1}}} \wedge \cdots \wedge \bar{\partial}\frac{1}{z_s^{\ell_s}},$$

where α is a smooth form with compact support in U, $(z_1, \ldots, z_N)$ are coordinates in U, and the current products are tensor products. A germ of a current τ at a point in Z is *pseudomeromorphic* if it is a finite sum of currents of the form

$$\pi_*^1 \pi_*^2 \cdots \pi_*^k \nu,$$

where each $\pi^j \colon V_j \to V_{j-1}$ is either a modification, a simple projection $V_j \times W \to V_j$, or the inclusion of an open subset, and ν is an elementary pseudomeromorphic current in $V_k \subset \mathbb{C}^N$. The set of such germs turns out to be a sheaf, the sheaf $\mathcal{PM}_Z$ of pseudomeromorphic currents in Z. We refer to [8] for properties of pseudomeromorphic currents.

The sheaf $\mathcal{PM}_Z$ is closed under $\bar{\partial}, \partial$, and under multiplication by smooth forms. Moreover, we have the

Dimension principle. — If $\tau \in \mathcal{PM}(Z)$ has bidegree $(*, q)$ and support in an analytic subset with codimension $> q$, then $\tau = 0$.

Example 2.3. — Let σ be a holomorphic section of a line bundle L and let α be a smooth form with values in L. Then the semi-meromorphic form α/σ has a canonical extension across $\{\sigma = 0\}$ as a pseudomeromorphic current; it can be obtained for instance as a principal value.

If $\tau \in \mathcal{PM}(Z)$ and $V \subset Z$ is a subvariety, then the restriction of τ to the open set $Z \setminus V$ has an extension to a pseudomeromorphic current $\mathbf{1}_{Z \setminus V}\tau$ in Z such that

$$(2.6) \qquad \mathbf{1}_{Z \setminus V}\tau = \lim_{\epsilon \to 0} \chi(|h|^2 v/\epsilon)\tau$$

if χ is a smooth approximation of the characteristic function of $[1, \infty) \subset \mathbb{R}$, h is a holomorphic tuple with $\{h = 0\} = V$, and v is a smooth positive function. It follows that

$$(2.7) \qquad \mathbf{1}_V \tau := \tau - \mathbf{1}_{Z \setminus V}\tau$$

is pseudomeromorphic with support in V. If α is a smooth form, then $\mathbf{1}_V(\alpha \wedge \tau) = \alpha \wedge \mathbf{1}_V \tau$. If $g \colon \widetilde{Z} \to Z$ proper, $\nu \in \mathcal{PM}(\widetilde{Z})$, and $g_* \nu \in \mathcal{PM}(Z)$, then

$$(2.8) \qquad \mathbf{1}_V g_* \nu = g_*(\mathbf{1}_{g^{-1}V}\nu).$$

If ν has compact support, then the same holds for any holomorphic mapping g.

A current a on Z is *almost semi-meromorphic*, $a \in ASM(Z)$, if $a = \pi_*(\alpha/\sigma)$, where $\pi \colon \widetilde{Z} \to Z$ is a modification, σ is a holomorphic section of a line bundle $L \to \widetilde{Z}$, and α is a smooth form with values in L; see, e.g., [8, Section 4]. We have that $ASM(Z) \subset \mathcal{PM}(Z)$. If $a \in ASM(Z)$, then the smallest Zariski closed set outside which a is smooth is called the *Zariski singular support* of a.

LEMMA 2.4 ([8, Theorem 4.8]). — *If $a \in ASM(Z)$ has Zariski singular support V and $\tau \in \mathcal{PM}(Z)$, then there is a unique $T \in \mathcal{PM}(Z)$ such that $T = a \wedge \tau$ in $Z \setminus V$ and $\mathbf{1}_V T = 0$.*

We will denote the current T by $a \wedge \tau$. In view of (2.6) and (2.7), since $\mathbf{1}_V(a \wedge \tau) = 0$ we have

$$(2.9) \qquad a \wedge \tau = \lim_{\epsilon \to 0} \chi(|h|^2 v/\epsilon)a \wedge \tau$$

if χ, h, and v are as above.

LEMMA 2.5 ([8, Theorem 2.25]). — *If Z is a complex manifold, then a germ of a current μ at a point in Z is pseudomeromorphic if and only if it is a finite sum of currents of the form $h_*\tau$, where $h \colon U \to Z$ is a holomorphic mapping and τ is elementary.*

In view of this lemma and Definition 2.1 it is clear that if Z is a manifold, then $\mathcal{PS}(Z) \subset \mathcal{PM}(Z)$. On a manifold thus $\mathcal{PS}$-currents have the properties of pseudomeromorphic currents. In particular, restrictions as in (2.6) and (2.7) are defined on $\mathcal{PS}$-currents. Let us check that such restrictions preserve $\mathcal{PS}(Z)$. It suffices to see that if $V \subset Z$ is a subvariety and $\mu \in \mathcal{PS}(Z)$, then $\mathbf{1}_V \mu \in \mathcal{PS}(Z)$. To see this we can assume that $\mu = g_* \alpha$, where g and α are as in Definition 2.1. By (2.8) thus $\mathbf{1}_V \mu = g_*(\mathbf{1}_{g^{-1}V} \alpha)$. Since α is smooth we have $\mathbf{1}_{g^{-1}V} \alpha = 0$ if $g^{-1}V$ is a proper subvariety, and $\mathbf{1}_{g^{-1}V} \alpha = \alpha$ otherwise. Hence, if $\mu = g_* \alpha$, then either $\mathbf{1}_V \mu$ is 0 or μ, both of which are in $\mathcal{PS}(Z)$.

2.1. Segre forms, Chern forms, and normal bundles

We first give a brief presentation of Segre forms and Chern forms based on the presentation in [2, Section 2].

Recall that if $L \to Z$ is a Hermitian line bundle, then there is the associated first Chern form $\widehat{c}_1(L)$, which is a smooth closed $(1,1)$-form. The first Chern class, $c_1(L)$, of L is the de Rham cohomology class of $\widehat{c}_1(L)$ and is independent of the Hermitian metric on L. Let $E \to Z$ be a Hermitian vector bundle. Over the projectivization $\pi \colon \mathbb{P}(E) \to Z$ of E (the projective bundle of lines through the zero section of E) we have the tautological line bundle $L = \mathcal{O}_E(-1) \subset \pi^* E$ and we equip it with the induced Hermitian metric. The *total Segre form*, $\widehat{s}(E) = 1 + \widehat{s}_1(E) + \widehat{s}_2(E) + \cdots$, is defined by

$$\widehat{s}(E) = \pi_*(1/(1 + \widehat{c}_1(L))) = \pi_*(1/(1 - \widehat{c}_1(L^*))).$$

If rank $E = n$ it follows that

$$(2.10) \qquad \widehat{s}_\ell(E) = \pi_* \widehat{c}_1(L^*)^{\ell + n - 1}.$$

If $n = 1$, then $\mathbb{P}(E) = Z$ so that $\pi = \mathrm{id}_X$, and hence $(1 + \widehat{c}_1(L)) \wedge \widehat{s}(E) = 1$. In the general case we define the *total Chern form*, $\widehat{c}(E) = 1 + \widehat{c}_1(E) + \widehat{c}_2(E) + \cdots$, of E by

$$(2.11) \qquad \widehat{c}(E) \wedge \widehat{s}(E) = 1.$$

It is proved in [22] that this definition coincides with the differential geometric definition. It is well-known that $\widehat{c}_\ell(E) = 0$ if $\ell > n$, and that if $h \colon \widetilde{Z} \to Z$ is a holomorphic mapping, then $h^* \widehat{c}(E) = \widehat{c}(h^* E)$ and $h^* \widehat{s}(E) = \widehat{s}(h^* E)$. The Chern classes $c_j(E)$ are the de Rham cohomology classes of $\widehat{c}_j(E)$ and they are independent of the Hermitian metric on E.

We will primarily consider Chern and Segre forms of normal bundles of submanifolds, and some times also of locally complete intersections, so we

recall a few things that we need about normal bundles. Let $\mathcal{J} \subset \mathcal{O}_Z$ be a locally complete intersection ideal sheaf of codimension n and zero set X. We will use the following ad hoc definition of the normal bundle $N_{\mathcal{J}} \to X$, cf. [2, Section 7]. A section ϕ of $N_{\mathcal{J}}$ is a choice of holomorphic n-tuple $\phi(s)$ locally in X for each local minimal set $s = (s_1, \ldots, s_n)$ of generators for $\mathcal{J}$ such that $A\phi(s) = \phi(As)$ on X for any locally defined holomorphic matrix A that is invertible in a neighborhood of X. Thus, each local minimal set of generators for $\mathcal{J}$ gives a local trivialization of $N_{\mathcal{J}}$.

Assume now that $\mathcal{J}$ is generated by a holomorphic section Ψ of a vector bundle $E \to Z$. Then we have an induced embedding

$$(2.12) \qquad N_{\mathcal{J}} \hookrightarrow E|_X$$

defined as follows, see, e.g., [2, Section 7]. Let s be a local minimal set of generators for $\mathcal{J}$. Since Ψ generates $\mathcal{J}$ there is a local holomorphic section B of $\mathrm{Hom}(\mathbb{C}^n, E)$ such that $\Psi = Bs$ and $B|_X$ is unique and pointwise injective; here $\mathbb{C}^n$ is the trivial vector bundle of rank n over Z. The embedding (2.12) then is given by

$$(2.13) \qquad \phi \mapsto B|_X \phi(s).$$

If $\mathcal{J}$ is principal, then $\mathcal{J}$ defines a divisor, D, and $\Psi = \Psi^0 \Psi'$, where Ψ^0 is a holomorphic section of the line bundle L corresponding to D such that $\mathrm{div}\, \Psi^0 = D$ and Ψ' is a holomorphic section of $L^* \otimes E$. In this case $N_{\mathcal{J}} = L|_{|D|}$ and the embedding (2.12) extends to an embedding

$$(2.14) \qquad L \hookrightarrow E, \quad \sigma \mapsto \sigma \Psi'.$$

For the induced metric $|\cdot|_L$ on L thus $|\Psi^0|_L = |\Psi|$ and so, by the Poincaré–Lelong formula,

$$(2.15) \qquad \mathrm{dd}^c \log |\Psi|^2 = [D] - \widehat{c}_1(L).$$

If Z is a complex manifold and $X \subset Z$ is a submanifold we let $\mathcal{J}_X \subset \mathcal{O}_Z$ be the sheaf of holomorphic functions vanishing on X and we write N_X instead of $N_{\mathcal{J}_X}$.

If $h: \widetilde{Z} \to Z$ is a holomorphic mapping and $\mathcal{J} \subset \mathcal{O}_Z$, then we let $h^* \mathcal{J} \subset \mathcal{O}_{\widetilde{Z}}$ be the sheaf generated by h^* of the generators of $\mathcal{J}$. The following functorial property should be well-known but for the reader's convenience we supply a proof.

LEMMA 2.6. — Let Z and $\widetilde{Z}$ be complex manifolds and $X \subset Z$ a complex submanifold. Assume that $h: \widetilde{Z} \to Z$ is a holomorphic mapping such that $\widetilde{\mathcal{J}} := h^* \mathcal{J}_X$ is a locally complete intersection with zero set $\widetilde{X} \subset \widetilde{Z}$.

(a) *There is a natural embedding $N_{\widetilde{\mathcal{J}}} \hookrightarrow h|_{\widetilde{X}}^* N_X$; it is an equality if* codim $\widetilde{X} = $ codim X.

(b) *Assume that $\mathcal{J}_X$ is generated by a holomorphic section Ψ of a vector bundle $E \to Z$, so that we have an induced embedding $N_X \hookrightarrow E|_X$. Then the composition of the embeddings*

$$(2.16) \qquad N_{\widetilde{\mathcal{J}}} \hookrightarrow h|_{\widetilde{X}}^* N_X \hookrightarrow h|_{\widetilde{X}}^* E|_X,$$

where the first one is the embedding in (a) and the second one is the pullback of $N_X \hookrightarrow E|_X$, is the embedding $N_{\widetilde{\mathcal{J}}} \hookrightarrow (h^*E)|_{\widetilde{X}}$ induced by $h^*\Psi$.

(c) *Assume that $\mathcal{J}_X$ is generated by a holomorphic section Ψ of E as in (b). If E, N_X, and $N_{\widetilde{\mathcal{J}}}$ are equipped with Hermitian metrics such that the embeddings $N_X \hookrightarrow E|_X$ and $N_{\widetilde{\mathcal{J}}} \hookrightarrow (h^*E)|_{\widetilde{X}}$ are Hermitian, then the embeddings in (2.16) are Hermitian.*

Proof. — Let $\widetilde{s} = (\widetilde{s}_1, \ldots, \widetilde{s}_\kappa)$ be a local minimal set of generators of $\widetilde{\mathcal{J}}$ and $s = (s_1, \ldots, s_n)$ a local minimal set of generators of $\mathcal{J}_X$.

(a). — Since by assumption h^*s generates $\widetilde{\mathcal{J}}$ it follows that $\kappa \leqslant n$ and that there is a local holomorphic matrix A such that $h^*s = A\widetilde{s}$ and rank $A|_{\widetilde{X}} = \kappa$. We define the embedding $N_{\widetilde{\mathcal{J}}} \hookrightarrow h|_{\widetilde{X}}^* N_X$ by the local embeddings

$$(2.17) \qquad \phi \mapsto A|_{\widetilde{X}} \phi(\widetilde{s}),$$

where ϕ is a local section of $N_{\widetilde{\mathcal{J}}}$. The right-hand side is independent of the local trivialization of $N_{\widetilde{\mathcal{J}}}$ given by $\widetilde{s}$ and transforms as a section of $h|_{\widetilde{X}}^* N_X$. Thus, the local embeddings (2.17) indeed give a global embedding $N_{\widetilde{\mathcal{J}}} \hookrightarrow h|_{\widetilde{X}}^* N_X$.

Since $n = $ codim X and $\kappa = $ codim $\widetilde{X}$ it follows that if codim $X = $ codim $\widetilde{X}$, then (2.17) is an isomorphism.

(b). — We have that s gives a local trivialization of N_X and hence of $h|_{\widetilde{X}}^* N_X$. Moreover, $\Psi = Bs$ for a local holomorphic section B of $\mathrm{Hom}(\mathbb{C}^n, E)$. In view of (2.13), the second embedding in (2.16) is defined by

$$(2.18) \qquad \xi(s) \mapsto h^* B|_{\widetilde{X}} \xi(s),$$

where $\xi(s)$ is a local section of $h|_{\widetilde{X}}^* N_X$ in the trivialization given by s.

We also have that $h^*\Psi = C\widetilde{s}$ for a local holomorphic section C of $\mathrm{Hom}(\mathbb{C}^\kappa, h^*E)$ such that $C|_{\widetilde{X}}$ is unique. The embedding $N_{\widetilde{\mathcal{J}}} \hookrightarrow (h^*E)|_{\widetilde{X}}$ is given by

$$(2.19) \qquad \phi \mapsto C|_{\widetilde{X}} \phi(\widetilde{s}).$$

Since $\Psi = Bs$ and $h^*s = A\widetilde{s}$ we have $h^*\Psi = h^*Bh^*s = h^*BA\widetilde{s}$. By uniqueness thus

$$(2.20) \qquad C|_{\widetilde{X}} = h^*B|_{\widetilde{X}}A|_{\widetilde{X}}$$

and it follows that (2.19) is the composition of (2.17) and (2.18).

(c). — If the embedding $N_X \hookrightarrow E|_X$ is Hermitian, then it is clear that the second embedding in (2.16), which is given by (2.18), is Hermitian. If the embedding $N_{\widetilde{\mathcal{J}}} \hookrightarrow (h^*E)|_{\widetilde{X}}$ is Hermitian it thus follows from (2.19), (2.20), and (2.18) that

$$|\phi|^2_{N_{\widetilde{\mathcal{J}}}} = \left|C|_{\widetilde{X}}\phi(\widetilde{s})\right|^2_{h^*E} = \left|h^*B|_{\widetilde{X}}A|_{\widetilde{X}}\phi(\widetilde{s})\right|^2_{h^*E} = \left|A|_{\widetilde{X}}\phi(\widetilde{s})\right|^2_{h^*N_X}.$$

By (2.17) thus the first embedding in (2.16) is Hermitian. $\qquad\square$

2.2. The M-operator on pseudomeromorphic currents

Let Ψ be a holomorphic section of a Hermitian vector bundle $E \to Z$ with zero set X. We recall the following lemma from [6, Section 2].

LEMMA 2.7. — *There are unique almost semi-meromorphic currents* m_k^Ψ, $k = 1, 2, \ldots$, *in Z that coincide with* $(2\pi i)^{-1}\partial \log |\Psi|^2 \wedge (\mathrm{dd}^c \log |\Psi|^2)^{k-1}$ *in $Z \setminus X$ and such that* $\mathbf{1}_X m_k^\Psi = 0$.

In view of Lemma 2.4, for $\tau \in \mathcal{PM}(Z)$ we have $m_k^\Psi \wedge \tau \in \mathcal{PM}(Z)$ and we let

$$(2.21) \qquad M_0^\Psi \wedge \tau := \mathbf{1}_X \tau, \quad M_k^\Psi \wedge \tau := \mathbf{1}_X \bar{\partial}(m_k^\Psi \wedge \tau), \ k = 1, 2, \ldots.$$

Since restriction and $\bar{\partial}$ preserve $\mathcal{PM}$ we have $M_k^\Psi \wedge \tau \in \mathcal{PM}(Z)$. Let $\chi_\epsilon = \chi(|\Psi|^2/\epsilon)$, where χ is as in (2.6). In view of (2.6), (2.7), and (2.9),

$$\begin{aligned}
\mathbf{1}_X \bar{\partial}(m_k^\Psi \wedge \tau) &= \lim_{\epsilon \to 0}(1 - \chi_\epsilon)\bar{\partial}(m_k^\Psi \wedge \tau) \\
&= \lim_{\epsilon \to 0} \bar{\partial}((1 - \chi_\epsilon)m_k^\Psi \wedge \tau) + \bar{\partial}\chi_\epsilon \wedge (m_k^\Psi \wedge \tau) \\
&= \bar{\partial}(\mathbf{1}_X m_k^\Psi \wedge \tau) + \lim_{\epsilon \to 0} \bar{\partial}\chi_\epsilon \wedge (m_k^\Psi \wedge \tau).
\end{aligned}$$

By Lemmas 2.4 and 2.7 we have $\mathbf{1}_X m_k^\Psi \wedge \tau = 0$ and so we get, cf. (1.7) and (1.8),

$$(2.22) \ \ M_0^\Psi \wedge \tau = \lim_{\epsilon \to 0}(1 - \chi(|\Psi|^2/\epsilon))\tau, \quad M_k^\Psi \wedge \tau = \lim_{\epsilon \to 0} \bar{\partial}\chi(|\Psi|^2/\epsilon) \wedge m_k^\Psi \wedge \tau.$$

Let $M^\Psi \wedge \tau = M_0^\Psi \wedge \tau + M_1^\Psi \wedge \tau + \cdots$. If φ is a smooth form in Z it follows by (2.22) that

$$(2.23) \qquad M^\Psi \wedge (\varphi \wedge \tau) = \varphi \wedge M^\Psi \wedge \tau.$$

If $g\colon \widetilde{Z} \to Z$ is a holomorphic mapping and $\mu \in \mathcal{PM}(\widetilde{Z})$ are such that $g_*\mu$ is defined and in $\mathcal{PM}(Z)$, then by (2.22) and (2.1),

$$(2.24) \qquad M^\Psi \wedge g_*\mu = g_*(M^{g^*\Psi} \wedge \mu).$$

We will write M^Ψ instead of $M^\Psi \wedge 1$. Notice that if Ψ is generically non-vanishing, then $M_0^\Psi = 0$.

Example 2.8. — Assume that Ψ defines a locally complete intersection ideal sheaf $\mathcal{J}$. Then by (2.12) there is an embedding $N_\mathcal{J} \hookrightarrow E|_X$ and we equip $N_\mathcal{J}$ with the induced metric. By [2, Proposition 1.5],

$$M^\Psi = \widehat{s}(N_\mathcal{J}) \wedge [\mathcal{Z}_\mathcal{J}],$$

where $\mathcal{Z}_\mathcal{J}$ is the fundamental cycle of the locally complete intersection $\mathcal{J}$; see, e.g., [15, Chapter 1.5]. We note two special cases:

(1) If $\mathcal{J}$ is the ideal sheaf of a submanifold X, then $\mathcal{Z}_\mathcal{J} = X$ and so $M^\Psi = \widehat{s}(N_\mathcal{J}) \wedge [X]$.
(2) If $\mathcal{J}$ is principal so that it defines a divisor D, then $\mathcal{Z}_\mathcal{J} = D$. In view of (2.10) thus $M_k^\Psi = \widehat{c}_1(L^*)^{k-1} \wedge [D]$ if L is the line bundle associated with D equipped with the metric induced by (2.14).

In the following, if Ψ generates the ideal sheaf of a submanifold X, then we will usually simply say that Ψ defines X. Similarly, if Ψ generates the principal ideal sheaf of a divisor D we say that Ψ defines D.

3. The M-operator on $\mathcal{PS}$-currents

Let Z be a complex manifold. Then $\mathcal{PS}(Z) \subset \mathcal{PM}(Z)$ by Lemma 2.5. Hence, if $\mu \in \mathcal{PS}(Z)$ and Ψ is a holomorphic section of a Hermitian vector bundle, then $M^\Psi \wedge \mu$ is defined and has the properties in Section 2.2. In this section we will see that $M^\Psi \wedge \mu$ has some additional properties when $\mu \in \mathcal{PS}(Z)$. Lemmas 3.1 and 3.2 below are straightforward adaptions of parts of [2, Theorem 5.2] to $\mathcal{PS}$-currents.

LEMMA 3.1. — *Let Ψ be a holomorphic section of a Hermitian vector bundle $E \to Z$. Let $X = \{\Psi = 0\}$, let $i\colon X \to Z$ be the inclusion, and let $\operatorname{codim} X = n$. If $\mu \in \mathcal{PS}(Z)$, then there are unique $\widehat{s}_{k-n}(\Psi, \mu) \in \mathcal{PS}(X)$ such that*

$$M_k^\Psi \wedge \mu = i_*\widehat{s}_{k-n}(\Psi, \mu), \qquad k = 0, 1, 2, \dots.$$

Proof. — The uniqueness is clear since i_* is injective. It therefore suffices to show the lemma locally in Z. We can thus assume that $\mu = g_*\alpha$, where α and $g\colon V \to Z$ are as in Definition 2.1. By Hironaka's theorem, after a modification we can assume that the ideal sheaf generated by $g^*\Psi$ is principal, cf. (2.5). Let D be the divisor defined by $g^*\Psi$ and let L be the associated line bundle equipped with the metric induced by the embedding $L \hookrightarrow g^*E$, cf. (2.14). Then by (2.24), (2.23), and Example 2.8,

$$(3.1) \qquad M_k^\Psi \wedge \mu = M_k^\Psi \wedge g_*\alpha = g_*(M_k^{g^*\Psi} \wedge \alpha) = g_*(\alpha \wedge M_k^{g^*\Psi})$$
$$= g_*\big(\alpha \wedge \widehat{c}_1(L^*)^{k-1} \wedge [D]\big).$$

Let $W = |D|$, let $j\colon W \to V$ be the inclusion, and consider the fiber diagram

$$(3.2) \qquad\qquad \begin{array}{ccc} W & \xrightarrow{\ j\ } & V \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{\ i\ } & Z. \end{array}$$

By Example 2.2 there is a $\mu_D \in \mathcal{PS}(W)$ such that $[D] = j_*\mu_D$. Hence, by (3.1), (2.1), and commutativity of (3.2),

$$(3.3) \qquad M_k^\Psi \wedge \mu = g_*j_*\big(j^*\alpha \wedge j^*\widehat{c}_1(L^*)^{k-1} \wedge \mu_D\big)$$
$$= i_*h_*\big(j^*\alpha \wedge j^*\widehat{c}_1(L^*)^{k-1} \wedge \mu_D\big).$$

Since the class of $\mathcal{PS}$-currents is closed under multiplication by smooth forms and under direct images of proper holomorphic mappings it follows that

$$\widehat{s}_{k-n}(\Psi, \mu) := h_*\big(j^*\alpha \wedge j^*\widehat{c}_1(L^*)^{k-1} \wedge \mu_D\big)$$

is in $\mathcal{PS}(X)$. The lemma thus follows by (3.3). $\qquad\qquad\square$

It follows from Lemma 3.1 and (2.4) that $M^\Psi \wedge \mu \in \mathcal{PS}(Z)$ if $\mu \in \mathcal{PS}(Z)$.

LEMMA 3.2. — *Let Ψ be a holomorphic section of a Hermitian vector bundle $E \to Z$. If $\mu \in \mathcal{PS}(Z)$, then $\mathrm{d}(M^\Psi \wedge \mu) = M^\Psi \wedge \mathrm{d}\mu$. The same holds with d replaced by $\bar\partial$ or ∂.*

Proof. — This is a local statement so we can assume that $\mu = g_*\alpha$, where α and $g\colon V \to Z$ are as in Definition 2.1. By Hironaka's theorem, in view of (2.5), we can also assume that $g^*\Psi$ defines a principal ideal sheaf. By Example 2.8 thus $M^{g^*\Psi}$ is d-, $\bar\partial$-, and ∂-closed. In view of (2.24) and (2.23),

using that d and g_* commute and that $M^{g^*\Psi}$ is closed, we get

$$\mathrm{d}(M^\Psi \wedge \mu) = \mathrm{d}(M^\Psi \wedge g_*\alpha) = g_*\mathrm{d}(M^{g^*\Psi} \wedge \alpha) = g_*(M^{g^*\Psi} \wedge \mathrm{d}\alpha)$$
$$= M^\Psi \wedge g_*\mathrm{d}\alpha = M^\Psi \wedge \mathrm{d}\mu.$$

The same calculation can be done with d replaced by $\bar\partial$ or ∂. $\square$

In general, if $X \subset Z$ is a complex submanifold one cannot expect there to be a global holomorphic section Ψ of a vector bundle $E \to Z$ defining X. However, for any complex submanifold $X \subset Z$ the next result allows us to define global $\mathcal{PS}$-currents $\widehat{s}_j(N_X, \mu)$ in X for $\mu \in \mathcal{PS}(Z)$ generalizing $\widehat{s}_j(\Psi, \mu)$ in Lemma 3.1 as soon as the normal bundle N_X of X is equipped with a Hermitian metric. This generalizes [2, Definition 5.5] to the setting of $\mathcal{PS}$-currents and when there is no global holomorphic section defining X.

PROPOSITION 3.3. — *Let Z be a complex manifold, $X \subset Z$ a complex submanifold of codimension n, and $i\colon X \to Z$ the inclusion. Assume that the normal bundle $N_X \to X$ of X is equipped with a Hermitian metric. If $\mu \in \mathcal{PS}(Z)$, then there are unique $\widehat{s}_{k-n}(N_X, \mu) \in \mathcal{PS}(X)$ with the following property: If there is a holomorphic section Ψ of a Hermitian vector bundle E in an open $U \subset Z$ such that Ψ defines X in U, and the induced embedding $N_X \hookrightarrow E|_X$ is Hermitian, then in U*

$$(3.4) \qquad i_*\widehat{s}_{k-n}(N_X, \mu) = M_k^\Psi \wedge \mu, \quad k = 0, 1, 2 \dots.$$

Proof. — Locally in Z there are Ψ and E such that Ψ defines X and $N_X \hookrightarrow E|_X$ is Hermitian. For instance, take local coordinates z such that $X = \{z_1 = \cdots = z_n = 0\}$, let E be the trivial rank n-bundle, let $\Psi = (z_1, \dots, z_n)$, and choose a Hermitian metric on E appropriately. Since i_* is injective on currents it follows that $\widehat{s}_{k-n}(N_X, \mu)$ must be unique.

To show the existence of $\widehat{s}_{k-n}(N_X, \mu)$ it thus suffices to show it locally. Locally in Z we can assume that $\mu = g_*\alpha$, where α and $g\colon V \to Z$ are as in Definition 2.1. We can also assume, by Hironaka's theorem and (2.5), that $g^*\mathcal{J}_X$ is principal. Let D be the divisor defined by $g^*\mathcal{J}_X$ and let L be the associated line bundle. Recall that $L|_{|D|}$ is the normal bundle of $g^*\mathcal{J}_X$. By Lemma 2.6(a) we have an embedding

$$(3.5) \qquad L|_{|D|} \hookrightarrow g^*N_X$$

and we equip L with a Hermitian metric so that (3.5) is Hermitian. Consider the fiber diagram (3.2) with $W = |D|$. By Example 2.2 there is a $\mu_D \in \mathcal{PS}(|D|)$ such that $j_*\mu_D = [D]$, and we let

$$(3.6) \qquad \widehat{s}_{k-n}(N_X, \mu) := h_*(\mu_D \wedge j^*\widehat{c}_1(L^*)^{k-1} \wedge j^*\alpha).$$

Since $\mu_D \in \mathcal{PS}(|D|)$, $j^*\widehat{c}_1(L^*)^{k-1} \wedge j^*\alpha$ is smooth, and $h\colon |D| \to X$ is a proper holomorphic mapping it follows that $\widehat{s}_{k-n}(N_X, \mu) \in \mathcal{PS}(X)$.

It remains to check that $\widehat{s}_{k-n}(N_X, \mu)$, as defined in (3.6), have the claimed property. Assume therefore that Ψ is a holomorphic section of a Hermitian vector bundle E that defines X in U and such that the induced embedding $N_X \hookrightarrow E|_X$ is Hermitian. In view of (2.24), (2.23), and Example 2.8,

$$(3.7) \qquad M_k^\Psi \wedge \mu = g_*(M_k^{g^*\Psi} \wedge \alpha) = g_*([D] \wedge \widehat{c}_1(L^*)^{k-1} \wedge \alpha),$$

where $\widehat{c}_1(L^*)$ here is with respect to the metric on L induced by the embedding $L \hookrightarrow g^*E$ given by $g^*\Psi$; cf. (2.14). By Lemma 2.6, on W this metric is the same as the one induced by (3.5). Since $[D] = j_*\mu_D$, by (3.7), commutativity of (3.2), and (3.6), thus

$$M_k^\Psi \wedge \mu = g_* j_*(\mu_D \wedge j^*\widehat{c}_1(L^*)^{k-1} \wedge j^*\alpha) = i_* g_*(\mu_D \wedge j^*\widehat{c}_1(L^*)^{k-1} \wedge j^*\alpha)$$
$$= i_* \widehat{s}_{k-n}(N_X, \mu).$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 3.4. — In the situation of Proposition 3.3, if μ is a smooth form, then

$$(3.8) \qquad\qquad \widehat{s}_{k-n}(N_X, \mu) = \widehat{s}_{k-n}(N_X) \wedge i^*\mu.$$

In particular, if X is a hypersurface with associated line bundle L, then

$$\widehat{s}_{k-1}(N_X, \mu) = \widehat{c}_1(L|_X^*)^{k-1} \wedge i^*\mu$$

since then $N_X = L|_X$. The equality (3.8) follows from (3.4), (2.23), and Example 2.8 since locally there always are Ψ and E as in Proposition 3.3.

PROPOSITION 3.5. — *Let Z be a complex manifold and Ψ a holomorphic section of a Hermitian vector bundle $E \to Z$. Assume that Ψ defines a divisor D and let L be the associated line bundle equipped with the Hermitian metric induced by (2.14). If $\mu \in \mathcal{PS}(Z)$ is closed and of positive degree, then for $k \geqslant 1$, there are currents ν_k in Z such that*

$$M_k^\Psi \wedge \mu = -\widehat{c}_1(L^*)^k \wedge \mathbf{1}_{Z\setminus|D|}\mu + \mathrm{d}\nu_k.$$

Proof. — Let $\chi_\epsilon = \chi(|\Psi|^2/\epsilon)$. By (2.22), a straightforward calculation, and using that $d\mu = 0$, we get

$$
\begin{aligned}
M_k^\Psi \wedge \mu &= \lim_{\epsilon \to 0} \bar{\partial}\chi_\epsilon \wedge \frac{\partial \log |\Psi|^2}{2\pi i} \wedge (dd^c \log |\Psi|^2)^{k-1} \wedge \mu \\
&= \lim_{\epsilon \to 0} d\chi_\epsilon \wedge d^c \log |\Psi|^2 \wedge (dd^c \log |\Psi|^2)^{k-1} \wedge \mu \\
&= \lim_{\epsilon \to 0} -\chi_\epsilon (dd^c \log |\Psi|^2)^k \wedge \mu \\
&\qquad\qquad + d\big(\chi_\epsilon d^c \log |\Psi|^2 \wedge (dd^c \log |\Psi|^2)^{k-1} \wedge \mu\big).
\end{aligned}
$$

Using (2.15) and (2.6) thus

$$
\begin{aligned}
M_k^\Psi \wedge \mu &= \lim_{\epsilon \to 0} -\chi_\epsilon(-\hat{c}_1(L))^k \wedge \mu + d\big(\chi_\epsilon d^c \log |\Psi|^2 \wedge (-\hat{c}_1(L))^{k-1} \wedge \mu\big) \\
&= -\hat{c}_1(L^*)^k \wedge \mathbf{1}_{Z \setminus |D|}\mu + \lim_{\epsilon \to 0} d\big(\chi_\epsilon d^c \log |\Psi|^2 \wedge \hat{c}_1(L^*)^{k-1} \wedge \mu\big).
\end{aligned}
$$

We claim that there are current ν_k in Z such that

$$
\chi_\epsilon d^c \log |\Psi|^2 \wedge \hat{c}_1(L^*)^{k-1} \wedge \mu \to \nu_k, \quad \epsilon \to 0.
$$

Taking the claim for granted the proposition immediately follows. To show the claim, since $\hat{c}_1(L^*)$ is smooth it suffices to see that $\lim_{\epsilon \to 0} \chi_\epsilon d^c \log |\Psi|^2 \wedge \mu$ exists. By Lemma 2.5, since Z is a manifold, μ is pseudomeromorphic. It therefore follows by Lemma 2.7, Lemma 2.4, and (2.9) that

$$
\lim_{\epsilon \to 0} \chi_\epsilon \partial \log |\Psi|^2 \wedge \mu
$$

exists and is a pseudomeromorphic current. By (2.3) and Lemma 2.5 also $\bar{\mu}$ is pseudomeromorphic, and so

$$
\lim_{\epsilon \to 0} \chi_\epsilon \bar{\partial} \log |\Psi|^2 \wedge \mu = \lim_{\epsilon \to 0} \overline{\chi_\epsilon \partial \log |\Psi|^2 \wedge \bar{\mu}}
$$

exists and is the conjugate of a pseudomeromorphic current. Since $d^c = (\partial - \bar{\partial})/4\pi i$ it follows that $\lim_{\epsilon \to 0} \chi_\epsilon d^c \log |\Psi|^2 \wedge \mu$ exists, which shows the claim.   □

4. Gysin mappings

Let $i\colon X \to Z$ be an embedding of a complex m-dimensional manifold X into an $m + n$-dimensional complex manifold Z. Suppose that the normal bundle $N_X \to i(X)$ of the submanifold $i(X)$ is equipped with a Hermitian metric. Notice that rank $N_X = \operatorname{codim} i(X) = n$.

Associated with the embedding $i\colon X \to Z$ there is a Gysin mapping $\mathcal{A}_k(Z) \to \mathcal{A}_{k-n}(X)$, where $\mathcal{A}_k$ is the Chow group of k-cycles modulo rational equivalence; see, e.g., [15, Section 6]. In [2] are introduced generalized

cycles, $\mathcal{GZ}$, which are $\mathcal{PS}$-currents, and a certain quotient group $\mathcal{B}$ of $\mathcal{GZ}$, which can be thought of as an analogue of the Chow group. If $i(X)$ can be defined by a global holomorphic section of some vector bundle, then there are also Gysin mappings $\mathcal{GZ}_k(Z) \to \mathcal{GZ}_{k-n}(X)$ and $\mathcal{B}_k(Z) \to \mathcal{B}_{k-n}(X)$ analogous to the one in [15].

Let $\mu \in \mathcal{PS}(Z)$. Recall from Proposition 3.3 that there are $\widehat{s}_{k-n}(N_X, \mu) \in \mathcal{PS}(X)$ since N_X is equipped with a Hermitian metric. We define Gysin mappings $i^!\colon \mathcal{PS}(Z) \to \mathcal{PS}(X)$ and $i^{!!}\colon \mathcal{PS}(Z) \to \mathcal{PS}(X)$ by

$$
(4.1) \qquad i^!\mu = \sum_{k=0}^{n} i^*\widehat{c}_{n-k}(N_X) \wedge \widehat{s}_{k-n}(N_X, \mu),
$$

$$
(4.2) \qquad i^{!!}\mu = i^*\widehat{c}(N_X) \wedge \widehat{s}(N_X, \mu).
$$

The mapping (4.1) preserves bidegree and is the basis of our pullback. It is straightforward to check that (4.1) is the component of the "full" Gysin mapping (4.2) of the same bidegree as μ; this makes (4.2) useful in some calculations.

Example 4.1. — Suppose that there is a global holomorphic section Ψ of a Hermitian vector bundle $E \to Z$ defining $i(X)$ and that the induced embedding $N_X \hookrightarrow E|_{i(X)}$ (cf. (2.12)) is Hermitian. Then in view of Proposition 3.3 and (2.1),

$$
(4.3) \qquad i_*i^!\mu = \sum_{k=0}^{n} \widehat{c}_{n-k}(N_X) \wedge M_k^{\Psi} \wedge \mu,
$$

$$
(4.4) \qquad i_*i^{!!}\mu = \widehat{c}(N_X) \wedge M^{\Psi} \wedge \mu.
$$

It follows that the restriction of $i^!$ to $\mathcal{GZ}(Z)$ is the Gysin mapping in [2, Equation (1.9)].

Recall from the proof of Proposition 3.3 above that locally in Z there always are Ψ and E with Ψ defining $i(X)$.

Example 4.2. — Let $\mu \in \mathcal{PS}(Z)$ and assume that $\operatorname{supp}\mu \subset i(X)$. Then

$$
(4.5) \qquad \mu = i_*\widehat{s}_{-n}(N_X, \mu) \quad \text{and} \quad i^!\mu = i^*\widehat{c}_n(N_X) \wedge \widehat{s}_{-n}(N_X, \mu).
$$

These are local statements so we can assume that there are Ψ and E as in Example 4.1. Since $\operatorname{supp}\mu \subset i(X)$ it follows by (2.21) and (2.22) that

$$
M_0^{\Psi} \wedge \mu = \mathbf{1}_X\mu = \mu, \qquad M_k^{\Psi} \wedge \mu = 0, k \geqslant 1.
$$

Thus (4.5) follows in view of (4.1) and Proposition 3.3.

PROPOSITION 4.3. — *The Gysin mappings $i^!$ and $i^{!!}$ are linear mappings* $\mathcal{PS}(Z) \to \mathcal{PS}(X)$ *and commute with* d, $\bar{\partial}$, *and* ∂. *If φ is a smooth form in* Z, *then $i^!\varphi = i^{!!}\varphi = i^*\varphi$ and if $\mu \in \mathcal{PS}(Z)$, then*

$$(4.6) \qquad i^!(\varphi \wedge \mu) = i^*\varphi \wedge i^!\mu, \quad i^{!!}(\varphi \wedge \mu) = i^*\varphi \wedge i^{!!}\mu.$$

Proof. — By Proposition 3.3, $\mu \mapsto \widehat{s}_{k-n}(N_X, \mu)$ are mappings $\mathcal{PS}(Z) \to \mathcal{PS}(X)$. By (4.1) and (4.2), since $\mathcal{PS}(X)$ is closed under multiplication by smooth forms, $i^!$ and $i^{!!}$ are mappings $\mathcal{PS}(Z) \to \mathcal{PS}(X)$ as well.

Locally in Z there are Ψ and E as in Example 4.1; we will use this to show the rest of the statements of the proposition. Since $\mu \mapsto M_k^\Psi \wedge \mu$ are linear it follows from Proposition 3.3 that $\mu \mapsto \widehat{s}_{k-n}(N_X, \mu)$ are linear. Thus $i^!$ and $i^{!!}$ are linear. Moreover, using (4.4), that Chern forms are closed, and Lemma 3.2 we have

$$i_* \mathrm{d}i^{!!}\mu = \mathrm{d}i_* i^{!!}\mu = \mathrm{d}(\widehat{c}(N_X) \wedge M^\Psi \wedge \mu) = \widehat{c}(N_X) \wedge M^\Psi \wedge \mathrm{d}\mu = i_* i^{!!}\mathrm{d}\mu.$$

Since i_* is injective it follows that $\mathrm{d}i^{!!}\mu = i^{!!}\mathrm{d}\mu$. Then $\mathrm{d}i^!\mu = i^!\mathrm{d}\mu$ follows by taking the component of the right bidegree. The same calculations can be done with d replaced by $\bar{\partial}$ and ∂.

Let φ be a smooth form in Z. Then by (2.23), Example 2.8, and (2.1)

$$M^\Psi \wedge \varphi = \varphi \wedge M^\Psi \wedge 1 = \varphi \wedge \widehat{s}(N_X) \wedge [X] = \varphi \wedge \widehat{s}(N_X) \wedge i_* 1 = \widehat{s}(N_X) \wedge i_* i^*\varphi.$$

By (4.4) and (2.11) thus,

$$i_* i^{!!}\varphi = \widehat{c}(N_X) \wedge M^\Psi \wedge \varphi = \widehat{c}(N_X) \wedge \widehat{s}(N_X) \wedge i_* i^*\varphi = i_* i^*\varphi.$$

Since i_* is injective it follows that $i^{!!}\varphi = i^*\varphi$ and so, by taking the component of the right bidegree, $i^!\varphi = i^*\varphi$. If $\mu \in \mathcal{PS}(Z)$, by (2.23) and (2.1) we have

$$i_* i^{!!}(\varphi \wedge \mu) = \widehat{c}(N_X) \wedge M^\Psi \wedge (\varphi \wedge \mu) = \varphi \wedge \widehat{c}(N_X) \wedge M^\Psi \wedge \mu = \varphi \wedge i_* i^{!!}\mu$$
$$= i_*(i^*\varphi \wedge i^{!!}\mu).$$

It follows that $i^{!!}(\varphi \wedge \mu) = i^*\varphi \wedge i^{!!}\mu$ and so $i^!(\varphi \wedge \mu) = i^*\varphi \wedge i^!\mu$ as before. This completes the proof. $\qquad\square$

Since by this proposition $i^!\varphi = i^*\varphi$ if φ is a smooth form in Z, it follows that $i_* i^!\varphi = [i(X)] \wedge \varphi$. For a general $\mu \in \mathcal{PS}(Z)$ we define

$$(4.7) \qquad\qquad [i(X)] \wedge \mu := i_* i^!\mu.$$

5. The pullback operation

Throughout this section we use the following notation and setup. Let X be a complex m-dimensional manifold, Y a complex n-dimensional Hermitian manifold, and $f\colon X \to Y$ a holomorphic mapping. We let $Z = X \times Y$ and $\pi_1\colon Z \to X$ and $\pi_2\colon Z \to Y$ are the natural projections. Let also $i\colon X \to Z$, $i(x) = (x, f(x))$, be the graph embedding. The normal bundle $N_X \to i(X)$ of $i(X)$ in Z is naturally isomorphic with TY via

$$(5.1) \qquad N_X = \pi_2^* TY|_{i(X)}.$$

Since Y is Hermitian, TY has a Hermitian metric and we equip N_X with the metric induced by (5.1).

5.1. Definition and basic properties

Let $\mu \in \mathcal{PS}(Y)$. Since π_2 is a projection, $\pi_2^* \mu = 1 \otimes \mu$ is a well-defined current in Z; cf. (2.2). Let us check that $\pi_2^* \mu \in \mathcal{PS}(Z)$. This is a local statement so we can assume that $\mu = g_* \alpha$, where $g\colon V \to Y$ and α are as in Definition 2.1. Let $\{\rho_j\}$ be a locally finite partition of unity on X with ρ_j smooth and compactly supported. Then

$$(5.2) \qquad \pi_2^* \mu = 1 \otimes \mu = \sum_j \rho_j \otimes \mu = \sum_j (\mathrm{id}_X \times g)_* (\rho_j \otimes \alpha),$$

which shows that $\pi_2^* \mu \in \mathcal{PS}(Z)$. We thus have the product $[i(X)] \wedge \pi_2^* \mu$. Since $\pi_1 \circ i = \mathrm{id}_X$ it follows from (4.7) that

$$(\pi_1)_* ([i(X)] \wedge \pi_2^* \mu) = (\pi_1)_* i_* i^! \pi_2^* \mu = i^! \pi_2^* \mu.$$

We use the last expression as the definition of f^*, cf. (1.5).

DEFINITION 5.1. — For $\mu \in \mathcal{PS}(Y)$ we let $f^* \mu = i^! \pi_2^* \mu$, where $i^!$ is the Gysin mapping (4.1).

We also introduce a "full" pullback mapping, $f^\diamond$, using the full Gysin mapping (4.2) by

$$(5.3) \qquad f^\diamond \mu = i^{!!} \pi_2^* \mu.$$

Since $i^! \pi_2^* \mu$ is the component of $i^{!!} \pi_2^* \mu$ of the same bidegree as $\pi_2^* \mu$ it follows that $f^* \mu$ is the component of $f^\diamond \mu$ of the same bidegree as μ. This makes $f^\diamond$ convenient to use in some calculations.

PROPOSITION 5.2. — *The operation f^* is a linear mapping $\mathcal{PS}(Y) \to \mathcal{PS}(X)$ and commutes with d, $\bar{\partial}$, and ∂. If φ is a smooth form in Y and $\mu \in \mathcal{PS}(Y)$, then $f^*\varphi$ is the usual pullback and*

$$(5.4) \qquad f^*(\varphi \wedge \mu) = f^*\varphi \wedge f^*\mu.$$

Proof. — In view of (2.2), π_2^* is a linear mapping $\mathcal{PS}(Y) \to \mathcal{PS}(Z)$ and so it follows from Proposition 4.3 that f^* is a linear mapping $\mathcal{PS}(Y) \to \mathcal{PS}(X)$. Moreover, $\pi_2^* d\mu = d\pi_2^*\mu$ and similarly for $\bar{\partial}$ and ∂. By Proposition 4.3 thus f^* commutes with d, $\bar{\partial}$, and ∂.

Let φ be a smooth form in Y. Then $\pi_2^*\varphi$ is smooth in Z and so, by Proposition 4.3,

$$i^! \pi_2^* \varphi = i^* \pi_2^* \varphi = (\pi_2 \circ i)^* \varphi.$$

Since $\pi_2 \circ i = f$ thus $i^! \pi_2^* \varphi = f^* \varphi$ is the usual pullback. In view of (2.2) we have $\pi_2^*(\varphi \wedge \mu) = \pi_2^* \varphi \wedge \pi_2^* \mu$. Hence, (5.4) follows from (4.6). □

We will now see that if Y is good, then (1.9) holds. Recall that Y is good means that there is a holomorphic section Φ of a holomorphic vector bundle $F \to Y \times Y$ defining the diagonal $\Delta \subset Y \times Y$. Then $\Psi := (f \times \mathrm{id}_Y)^* \Phi$ is a holomorphic section of $E := (f \times \mathrm{id}_Y)^* F$ defining $i(X)$ in Z. We choose a Hermitian metric on E such that the induced embedding $N_X \hookrightarrow E|_{i(X)}$, is a Hermitian. Hence, when Y is good we are in the setting of Example 4.1. Using that $\mathrm{id}_X = \pi_1 \circ i$ it thus follows from (4.3) that

$$(5.5) \qquad f^*\mu = i^! \pi_2^* \mu = (\mathrm{id}_X)_* i^! \pi_2^* \mu = (\pi_1)_* i_* i^! \pi_2^* \mu$$

$$= (\pi_1)_* \sum_{k=0}^{n} \widehat{c}_{n-k}(N_X) \wedge M_k^\Psi \wedge \pi_2^* \mu.$$

Now, $\pi_2|_{i(X)} = f \circ \pi_1|_{i(X)}$ so we can replace π_2^* in (5.1) by $\pi_1^* f^*$. Since the Hermitian metric on N_X is induced by (5.1) it follows by functoriality of Chern forms that

$$(5.6) \qquad \widehat{c}(N_X) = \pi_1^* f^* \widehat{c}(TY)|_{i(X)}.$$

Replacing $\widehat{c}_{n-k}(N_X)$ in (5.5) by $\pi_1^* f^* \widehat{c}_{n-k}(TY)$ and using (2.1) thus (1.9) follows. In the same way one can check that

$$(5.7) \qquad f^\diamond \mu = f^* \widehat{c}(TY) \wedge (\pi_1)_* \big(M^\Psi \wedge \pi_2^* \mu \big).$$

Example 5.3. — Let $Y = \mathbb{C}^2$, with TY equipped with the standard Hermitian metric, and let $f \colon X \to Y$ be the blowup of 0. Let us show that if μ is the Dirac mass at 0 (considered as a $(2,2)$-current), then

$$(5.8) \qquad f^*\mu = \omega \wedge [D], \qquad f^\diamond \mu = \omega \wedge [D] + [D],$$

where $D \simeq \mathbb{P}^1$ is the exceptional divisor of the blowup and ω is the standard Fubini–Study metric form on that $\mathbb{P}^1$.

Notice first that $\pi_2^* \mu = g_* 1$, where $g \colon X \to X \times Y$ is the mapping $g(x) = (x, 0)$. We consider the tuple $\Psi = f(x) - y$ as a section of the trivial rank-2 bundle $E \to X \times Y$ and equip E with the standard Hermitian metric. Clearly, Ψ defines the graph of f in $X \times Y$. One can check that the metric induced on N_X by the embedding $N_X \hookrightarrow E|_{i(X)}$ is the same as the one induced by $N_X = \pi_2^* TY|_{i(X)}$. We can thus use (5.7) to calculate $f^\diamond \mu$. Notice that $\widehat{c}(TY) = 1$ since TY has the standard metric. By (5.7) thus

$$(5.9) \qquad\qquad f^\diamond \mu = (\pi_1)_* \big(M^\Psi \wedge \pi_2^* \mu\big).$$

Since $\pi_2^* \mu = g_* 1$, $g^* \Psi = f(x)$, and f defines D it follows from (2.1) and Example 2.8 that

$$(5.10) \qquad (\pi_1)_*(M^\Psi \wedge \pi_2^* \mu) = (\pi_1)_* g_*(M^{g^* \Psi} \wedge 1) = (\pi_1)_* g_* M^{f(x)}$$
$$= (\pi_1)_* g_* \big([D] + [D] \wedge \widehat{c}_1(L^*)\big),$$

where L is the line bundle corresponding to D equipped with the metric induced by $L \hookrightarrow g^* E$. The embedding $L|_D \hookrightarrow g^* E|_D$ is the standard embedding of $\mathcal{O}(-1)$ into $\mathbb{P}^1 \times \mathbb{C}^2$ and so $\widehat{c}_1(L^*)|_D = \omega$. Since $\pi_1 \circ g = \mathrm{id}_X$, the second equality in (5.8) thus follows from (5.9) and (5.10). The first equality in (5.8) then follows by taking the right bidegree.

Remark 5.4. — Let $f \colon X \to Y$ be the blowup of $0 \in \mathbb{C}^2 = Y$ and μ the Dirac mass at 0 as in Example 5.3. Then no reasonable pullback of currents under f can be continuous. To see this, let a_j be a sequence in Y such that $a_j \to 0$. If μ^{a_j} is the Dirac measure at a_j, then $\mu^{a_j} \to \mu$ as currents. Since f is a biholomorphism outside $f^{-1}(0)$ we have that $f^* \mu^{a_j}$ is the Dirac mass at $f^{-1}(a_j)$ if f^* is a reasonable pullback. If a_j and b_j are sequences going to 0 along different lines in Y, then $f^{-1}(a_j)$ and $f^{-1}(b_j)$ have different limits in X. Hence, $\mu^{a_j} \to \mu$ and $\mu^{b_j} \to \mu$, but $f^* \mu^{a_j}$ and $f^* \mu^{b_j}$ have different limits.

Example 5.5. — Let $f \colon X \to Y$ be the inclusion of an open set $X \subset Y$. Then, for any $\mu \in \mathcal{PS}(Y)$ we have $f^* \mu = f^\diamond \mu = \mu|_X$. To see this, we will show that

$$(5.11) \qquad\qquad i_* f^\diamond \mu = i_*(\mu|_X);$$

recall that $i \colon X \to Z$ is the graph embedding, which in this case is $i(x) = (x, x)$. Since i_* is injective we get $f^\diamond \mu = \mu|_X$ from (5.11). Then $f^* \mu = \mu|_X$ follows by taking the right bidegree.

It suffices to verify (5.11) in a neighborhood of a point $i(x) = (x, x) \in Z$, $x \in X$. To this end we can assume that $X = Y$. We can also assume that there is a holomorphic section Ψ of a Hermitian vector bundle $E \to Z$ defining $i(X)$ such that the induced embedding $N_X \hookrightarrow E|_{i(X)}$ is Hermitian; cf. Example 4.1. By (5.3) and (4.4) thus

$$(5.12) \qquad i_* f^\diamond \mu = \widehat{c}(N_X) \wedge M^\Psi \wedge \pi_2^* \mu.$$

Our considerations are local so we can assume that $\mu = g_* \alpha$, where $g \colon V \to Y$ and α are as in Definition 2.1. Then

$$(5.13) \qquad \pi_2^* \mu = (\mathrm{id}_X \times g)_* (1 \otimes \alpha);$$

cf. (5.2). Recalling that $X = Y$, let $j \colon V \to X \times V$ be the mapping $j(v) = (g(v), v)$. Since Ψ defines $i(X)$, which is the diagonal in $X \times Y$, we have that $(\mathrm{id}_X \times g)^* \Psi$ defines $j(V)$. Let $N_V \to j(V)$ be the normal bundle of $j(V)$ equipped with the Hermitian metric induced by the embedding $N_V \hookrightarrow (\mathrm{id}_X \times g)^* E|_{j(V)}$. Since $\mathrm{codim}\, j(V) = \dim X = \dim Y = \mathrm{codim}\, i(X)$, it follows from Lemma 2.6 that $N_V = (\mathrm{id}_X \times g)^* N_X$ as Hermitian bundles. By functoriality of Chern forms thus $\widehat{c}(N_V) = (\mathrm{id}_X \times g)^* \widehat{c}(N_X)$. Hence, in view of (5.12), (5.13), and (2.24),

$$i_* f^\diamond \mu = (\mathrm{id}_X \times g)_* \big((\mathrm{id}_X \times g)^* \widehat{c}(N_X) \wedge M^{(\mathrm{id}_X \times g)^* \Psi} \wedge (1 \otimes \alpha) \big)$$
$$= (\mathrm{id}_X \times g)_* \big(\widehat{c}(N_V) \wedge M^{(\mathrm{id}_X \times g)^* \Psi} \wedge (1 \otimes \alpha) \big).$$

By (4.4) and Proposition 4.3 thus

$$i_* f^\diamond \mu = (\mathrm{id}_X \times g)_* j_* j^{!!} (1 \otimes \alpha) = (\mathrm{id}_X \times g)_* j_* j^* (1 \otimes \alpha)$$

since $1 \otimes \alpha$ is smooth. It is straightforward to check that $(\mathrm{id}_X \times g) \circ j = i \circ g$ and that $j^*(1 \otimes \alpha) = \alpha$. Hence, $i_* f^\diamond \mu = i_* g_* \alpha = i_* \mu$, which shows (5.11).

PROPOSITION 5.6. — *Let X be a complex manifold, Y a complex Hermitian manifold, and $f \colon X \to Y$ a holomorphic mapping.*

   (a) *If $U \subset X$ is an open subset and $\mu \in \mathcal{PS}(Y)$, then $(f^* \mu)|_U = f|_U^* \mu$.*

   (b) *Assume that $f(X)$ is a complex submanifold of Y and that the induced mapping $\widetilde{f} \colon X \to f(X)$ is a submersion. Let $\iota \colon f(X) \to Y$ be the inclusion. If $\mu \in \mathcal{PS}(Y)$, then $f^* \mu = \widetilde{f}^* \iota^* \mu$, where $\widetilde{f}^*$ is the standard pullback of currents under the submersion $\widetilde{f}$.*

   Proof.

   (a). — Recall that N_X is the normal bundle of $i(X)$ in $Z = X \times Y$ equipped with the metric induced by (5.1). If N_U is the normal bundle of $i(U)$ in $U \times Y$, then $N_U = (N_X)|_{i(U)}$, and we equip N_U with the induced

metric. In view of Proposition 3.3, we have $\widehat{s}_j(N_X, \pi_2^*\mu)|_U = \widehat{s}_j(N_U, \pi_2^*\mu)$. It thus follows from Definition 5.1 and (4.1) that $(f^*\mu)|_U = f|_U^*\mu$.

(b). — We can localize in both X and Y; the statement is local in X by part (a), and in view of Proposition 5.2, by a partition of unity in Y we can assume that μ has support in any given open subset of Y. Since $\widetilde{f}$ is a submersion, after localization in X, we can assume that $X = X' \times X''$, where X' is an open set in some $\mathbb{C}^{m'}$ and X'' is an open subset of $f(X)$, such that $\widetilde{f} \colon X' \times X'' \to X''$ is the projection on the second factor. Thus, the inclusion $\iota \colon f(X) \to Y$ now is $X'' \hookrightarrow Y$.

We will first give an expression for $\iota^\diamond\mu$. Since we can localize in Y we can assume that Y is good and let Ψ be a holomorphic section of a vector bundle $E \to Y \times Y$ such that Ψ defines the diagonal $\Delta \subset Y \times Y$. We equip E with a Hermitian metric so that $TY|_\Delta = N_\Delta \hookrightarrow E|_\Delta$ is Hermitian. Let p_1 and p_2 be the natural projections $X'' \times Y \to X''$ and $X'' \times Y \to Y$, respectively, and let $j \colon X'' \to X'' \times Y$ be the embedding $j(x'') = (x'', \iota(x''))$. We equip the normal bundle $N_{X''} \to j(X'')$ of $j(X'')$ in $X'' \times Y$ with the metric induced by $N_{X''} = p_2^*TY|_{j(X'')}$. The section $(\iota \times \mathrm{id}_Y)^*\Psi$ defines $j(X'')$ and in view of Lemma 2.6, the induced embedding $N_{X''} \hookrightarrow (\iota \times \mathrm{id}_Y)^*E$ is Hermitian. By (5.7) we now have

$$(5.14) \qquad \iota^\diamond\mu = \iota^*\widehat{c}(TY) \wedge (p_1)_*(M^{(\iota\times\mathrm{id}_Y)^*\Psi} \wedge p_2^*\mu).$$

We have a similar expression for $f^\diamond\mu$: The section $(f \times \mathrm{id}_Y)^*\Psi$ defines $i(X)$ in $X \times Y$, and again in view of Lemma 2.6, the induced embedding $N_X \hookrightarrow (f \times \mathrm{id}_Y)^*E$ is Hermitian. By (5.7) thus

$$(5.15) \qquad f^\diamond\mu = f^*\widehat{c}(TY) \wedge (\pi_1)_*(M^{(f\times\mathrm{id}_Y)^*\Psi} \wedge \pi_2^*\mu).$$

Now, since $(f \times \mathrm{id}_Y)^*\Psi = 1 \otimes (\iota \times \mathrm{id}_Y)^*\Psi$ and $\pi_2^*\mu = 1 \otimes p_2^*\mu$ we have

$$(5.16) \qquad M^{(f\times\mathrm{id}_Y)^*\Psi} \wedge \pi_2^*\mu = 1 \otimes \left(M^{(\iota\times\mathrm{id}_Y)^*\Psi} \wedge p_2^*\mu\right).$$

Moreover, since $\widetilde{f} \colon X' \times X'' \to X''$ is the standard projection, $f^*\widehat{c}(TY) = \widetilde{f}^*\iota^*\widehat{c}(TY) = 1 \otimes \iota^*\widehat{c}(TY)$; cf. (2.2). It thus follows by (5.15), (5.16), and (5.14) that

$$f^\diamond\mu = \left(1 \otimes \iota^*\widehat{c}(TY)\right) \wedge (\pi_1)_*\left(1 \otimes \left(M^{(\iota\times\mathrm{id}_Y)^*\Psi} \wedge p_2^*\mu\right)\right)$$
$$= 1 \otimes \left(\iota^*\widehat{c}(TY) \wedge (p_1)_*(M^{(\iota\times\mathrm{id}_Y)^*\Psi} \wedge p_2^*\mu)\right)$$
$$= 1 \otimes \iota^\diamond\mu.$$

Taking the component of the same bidegree as μ we get $f^*\mu = 1 \otimes \iota^*\mu$. The right-hand side here is the standard pullback of the current $\iota^*\mu$ under the projection $\widetilde{f} \colon X' \times X'' \to X''$. Part(b) of the proposition thus follows. $\square$

Proof of Theorem 1.1. — Theorem 1.1 follows from Propositions 5.2 and 5.6. □

Example 5.7. — Let $f\colon X \to Y$ be surjective with constant fiber dimension $m-n$; recall that $m = \dim X$ and $n = \dim Y$. We claim that f^* then is independent of the Hermitian structure on Y and that $f^*\mu = \widehat{s}_0(N_X, \pi_2^*\mu)$ for any $\mu \in \mathcal{PS}(Y)$; recall here the currents $\widehat{s}_j(N_X, \pi_2^*\mu)$ from Proposition 3.3.

To see this, we can localize in both X and Y; cf. the proof of Proposition 5.6. We can thus assume that there are Ψ and E as in Example 4.1. We can also assume that $\mu = \widetilde{g}_*\alpha$, where $\widetilde{g}\colon \widetilde{V} \to Y$ is a holomorphic mapping, $\widetilde{V}$ a connected complex manifold, and α a smooth compactly supported form in $\widetilde{V}$. Consider the fiber diagram (3.2) with V, g, and W defined to be $X \times \widetilde{V}$, $\mathrm{id}_X \times \widetilde{g}$, and $\{g^*\Psi = 0\}$, respectively. Since f is surjective with constant fiber dimension $m-n$ it follows that $\mathrm{codim}_V W = n$. Thus, since $W = \{g^*\Psi = 0\}$, it follows from [2, Theorems 1.1, 1.2] that $M_k^{g^*\Psi} = 0$ if $k < n$ and that $M_n^{g^*\Psi}$ is independent of the metric. Since $\pi_2^*\mu = g_*(1 \otimes \alpha)$, in view of (2.23) and (2.24) we get

$$M_k^{\Psi} \wedge \pi_2^*\mu = g_*\big(M_k^{g^*\Psi} \wedge (1 \otimes \alpha)\big) = g_*\big((1 \otimes \alpha) \wedge M_k^{g^*\Psi}\big).$$

Hence, $M_k^{\Psi} \wedge \pi_2^*\mu$ vanishes if $k < n$ and is independent of the metric if $k = n$. It thus follows by (1.9) and Proposition 3.3 that f^* is independent of the Hermitian structure on Y and that $f^*\mu = \widehat{s}_0(N_X, \pi_2^*\mu)$.

5.2. An alternative approach

We recall the setup; X is a complex manifold, Y a complex Hermitian manifold, $f\colon X \to Y$ a holomorphic mapping, $Z = X \times Y$, and $\pi_1\colon Z \to X$ and $\pi_2\colon Z \to Y$ the natural projections. In this subsection we will also assume that X is Hermitian and let Z have the induced Hermitian structure.

Let $\iota\colon Z \to Z \times Z$ be the diagonal embedding and $N_Z \to \iota(Z)$ the normal bundle of $\iota(Z)$ in $Z \times Z$. Then N_Z is naturally isomorphic with TZ and thus N_Z gets an induced Hermitian metric. The Gysin mappings $\iota^!$ and $\iota^{!!}$ are thus defined.

Recall from the introduction that the main step when defining $f^*\mu$ is to give a reasonable meaning to the product $[i(X)] \wedge \pi_2^*\mu$, where $i(X)$ as before is the graph of f in Z. We have done this by applying the (local) operator(s) M^{Ψ} to $\pi_2^*\mu$, where Ψ is a (local) holomorphic section of some Hermitian bundle defining $i(X)$ (locally) in Z, and then multiplying by

Chern forms. This definition of the product $[i(X)] \wedge \pi_2^* \mu$ is not symmetric in $[i(X)]$ and $\pi_2^* \mu$. The objective of this section is to provide an alternative definition that puts $[i(X)]$ and $\pi_2^* \mu$ on an equal footing. The alternative definition coincides with the product introduced in [3, Definition 5.3] if $\pi_2^* \mu$ is a (generalized) cycle and $\iota(Z)$ can be defined by a global holomorphic section of some vector bundle over $Z \times Z$.

Before stating the main result of this subsection we notice that if $\Gamma \subset Z$ is a submanifold, or merely an analytic subset, and $\mu \in \mathcal{PS}(Y)$, then $[\Gamma] \otimes \pi_2^* \mu \in \mathcal{PS}(Z \times Z)$. In fact, if $\mu = g_* \alpha$, where $g \colon V \to Y$ and α are as in Definition 2.1, then

$$(5.17) \qquad [\Gamma] \otimes \pi_2^* \mu = \tau_*(1 \otimes 1 \otimes \alpha)$$

where $\tau \colon \Gamma \times X \times V \to Z \times Z$, $\tau(\gamma, x, v) = (\gamma, x, g(v))$.

THEOREM 5.8. — *Let X and Y be complex Hermitian manifolds and let $Z = X \times Y$ with the induced Hermitian structure. Let $\iota \colon Z \to Z \times Z$ be the diagonal embedding and equip the normal bundle N_Z of $\iota(Z)$ with the Hermitian metric induced by $N_Z \simeq TZ$. Assume that $\Gamma \subset Z$ is a submanifold such that $\dim \Gamma = \dim X$ and $\pi_1|_\Gamma$ is proper, where $\pi_1 \colon Z \to X$ is the natural projection. Then, with $\pi_2 \colon Z \to Y$ the natural projection,*

$$(5.18) \qquad (\pi_1)_* \iota^!([\Gamma] \otimes \pi_2^* \mu) = (\pi_1|_\Gamma)_*(\pi_2|_\Gamma)^* \mu.$$

Notice that $(\pi_2|_\Gamma)^*$ depends on the Hermitian structure on Y but not on the one on X. The left-hand side of (5.18), which a priori depends on the Hermitian structures on both X and Y, is thus independent of the Hermitian structure on X.

Let us see what Theorem 5.8 means if $\Gamma = i(X)$ is the graph of the mapping $f \colon X \to Y$. After identification of $i(X)$ and X we have $\pi_2|_\Gamma = f$ and $\pi_1|_\Gamma = \mathrm{id}_X$, and so (5.18) becomes

$$(5.19) \qquad (\pi_1)_* \iota^!([i(X)] \otimes \pi_2^* \mu) = f^* \mu.$$

The left-hand side thus is an alternative definition of $f^* \mu$. The product $[i(X)] \wedge \pi_2^* \mu$ in this setting is $\iota^!([i(X)] \otimes \pi_2^* \mu)$, which clearly puts $[i(X)]$ and $\pi_2^* \mu$ on an equal footing.

To prove Theorem 5.8 we will use the following technical lemma.

LEMMA 5.9. — *Let Σ and Ω be complex manifolds and $q \colon \Sigma \to \Omega$ a holomorphic mapping. Let $j \colon \Sigma \to \Sigma \times \Omega$ be the graph embedding, N_Σ the normal bundle of $j(\Sigma)$, and $p \colon \Sigma \times \Omega \to \Sigma$ the natural projection. Let Φ_1 be a holomorphic section of a Hermitian vector bundle $F_1 \to \Sigma$ and assume that there is a holomorphic section Φ_2 of a Hermitian vector*

bundle $F_2 \to \Sigma \times \Omega$ defining $j(\Sigma)$ in $\Sigma \times \Omega$. If N_Σ is equipped with the Hermitian metric induced by $N_\Sigma \hookrightarrow F_2|_{j(\Sigma)}$, then for any $\mu \in \mathcal{PS}(\Sigma)$,

$$j_*(M^{\Phi_1} \wedge \mu) = \widehat{c}(N_\Sigma) \wedge M^{p^*\Phi_1+\Phi_2} \wedge p^*\mu.$$

Proof. — Assume first that μ is a smooth form and that Φ_1 defines a divisor D. Then $p^*\Phi_1$ defines the divisor $D \times \Omega$. The section Φ_2 defines $j(\Sigma)$, which is a locally complete intersection of codimension $\dim \Omega$. It is clear that $D \times \Omega$ and $j(\Sigma)$ intersect properly and it follows that $p^*\Phi_1 + \Phi_2$ defines a locally complete intersection of codimension $\dim \Omega + 1$. The conditions in [2, Proposition 7.6] are then satisfied and the conclusion is that

$$(5.20) \qquad\qquad M^{p^*\Phi_1+\Phi_2} = M^{p^*\Phi_1} \wedge M^{\Phi_2}.$$

Since Φ_2 defines $j(\Sigma)$, in view of Example 2.8 we have $M^{\Phi_2} = \widehat{s}(N_\Sigma) \wedge j_*1$. Hence, by (5.20),

$$(5.21) \qquad\qquad M^{p^*\Phi_1+\Phi_2} = M^{p^*\Phi_1} \wedge \widehat{s}(N_\Sigma) \wedge j_*1.$$

Multiplying by $\widehat{c}(N_\Sigma) \wedge p^*\mu$ and using (2.23), (2.11), and (2.24) we get

$$\widehat{c}(N_\Sigma) \wedge M^{p^*\Phi_1+\Phi_2} \wedge p^*\mu = \widehat{c}(N_\Sigma) \wedge p^*\mu \wedge \widehat{s}(N_\Sigma) \wedge M^{p^*\Phi_1} \wedge j_*1$$
$$= p^*\mu \wedge M^{p^*\Phi_1} \wedge j_*1 = j_*(j^*p^*\mu \wedge M^{j^*p^*\Phi_1}).$$

Since $j^*p^* = (p \circ j)^* = \mathrm{id}_\Sigma^*$ the lemma thus follows in the case when μ is smooth and Φ_1 defines a divisor.

We now consider the general case. The statement is local in $\Sigma \times \Omega$ so we can assume that $\mu = g_*\alpha$, where $g \colon V \to \Sigma$ and α are as in Definition 2.1. Possibly after a modification of V, by Hironaka's theorem we can also assume that $g^*\Phi_1$ defines a divisor.

Let $\widetilde{j} \colon V \hookrightarrow V \times \Omega$ be the embedding $\widetilde{j}(v) = (v, q \circ g(v))$ and let $\widetilde{p} \colon V \times \Omega \to V$ be the natural projection. Notice that $(g \times \mathrm{id}_\Omega)^*\Phi_2$ defines $\widetilde{j}(V)$ in $V \times \Omega$. By the first part of the proof thus

$$(5.22) \qquad \widetilde{j}_*(M^{g^*\Phi_1} \wedge \alpha) = \widehat{c}(N_V) \wedge M^{\widetilde{p}^*g^*\Phi_1+(g\times\mathrm{id}_\Omega)^*\Phi_2} \wedge (\alpha \otimes 1),$$

where N_V is the normal bundle of $\widetilde{j}(V)$ equipped with the metric induced by $N_V \hookrightarrow (g \times \mathrm{id}_\Omega)^*F_2|_{\widetilde{j}(V)}$. The lemma will follow by applying $(g \times \mathrm{id}_\Omega)_*$ to (5.22).

Since $(g \times \mathrm{id}_\Omega) \circ \widetilde{j} = j \circ g$, in view of (2.24) we have

$$(5.23) \qquad (g \times \mathrm{id}_\Omega)_*\widetilde{j}_*(M^{g^*\Phi_1} \wedge \alpha) = j_*g_*(M^{g^*\Phi_1} \wedge \alpha) = j_*(M^{\Phi_1} \wedge \mu).$$

To calculate $(g \times \mathrm{id}_\Omega)_*$ of the right-hand side of (5.22), notice first that by Lemma 2.6 we have an isometry $N_V = (g \times \mathrm{id}_\Omega)^*N_\Sigma$. We can thus replace $\widehat{c}(N_V)$ in the right-hand side of (5.22) by $(g \times \mathrm{id}_\Omega)^*\widehat{c}(N_\Sigma)$. Since

$g \circ \widetilde{p} = p \circ (g \times \mathrm{id}_\Omega)$ we can also replace $\widetilde{p}^* g^* \Phi_1$ by $(g \times \mathrm{id}_\Omega)^* p^* \Phi_1$. After doing this, using (2.1) and (2.24) we get

$$(5.24) \quad (g \times \mathrm{id}_\Omega)_* \Big(\widehat{c}(N_V) \wedge M^{\widetilde{p}^* g^* \Phi_1 + (g \times \mathrm{id}_\Omega)^* \Phi_2} \wedge (\alpha \otimes 1) \Big)$$
$$= \widehat{c}(N_\Sigma) \wedge M^{p^* \Phi_1 + \Phi_2} \wedge (g \times \mathrm{id}_\Omega)_*(\alpha \otimes 1).$$

Since $(g \times \mathrm{id}_\Omega)_*(\alpha \otimes 1) = \mu \otimes 1 = p^* \mu$ the lemma follows by (5.22), (5.23), and (5.24). $\qquad\square$

Proof of Theorem 5.8. — We will show that

$$(5.25) \qquad (\pi_1)_* \iota^{!!}([\Gamma] \otimes \pi_2^* \mu) = (\pi_1|_\Gamma)_* (\pi_2|_\Gamma)^\diamond \mu.$$

The theorem then follows by taking the component of the same bidegree as μ. Let

$$T := (\pi_2|_\Gamma)^\diamond \mu.$$

Then $(\pi_1|_\Gamma)_* T$ is the right-hand side of (5.25). On the other hand, we can factorize $\pi_1|_\Gamma$ as follows

$$\Gamma \xrightarrow{j_1} \Gamma \times Y \xrightarrow{j_2} \Gamma \times X \times Y \xrightarrow{j_3} Z \times X \times Y \xrightarrow{\pi} X,$$

$$j_1(\gamma) = (\gamma, \pi_2(\gamma)), \quad j_2(\gamma, y) = (\gamma, \pi_1(\gamma), y),$$
$$j_3(\gamma, x, y) = (\pi_1(\gamma), \pi_2(\gamma); x, y), \quad \pi(z; x, y) = x.$$

We thus have that $(\pi_1|_\Gamma)_* T = \pi_*(j_3)_*(j_2)_*(j_1)_* T$ and we claim that

$$(5.26) \qquad \pi_*(j_3)_*(j_2)_*(j_1)_* T = (\pi_1)_* \iota^{!!}([\Gamma] \otimes \pi_2^* \mu),$$

which then proves (5.25).

It remains to show (5.26). The claimed identity (5.26) can be checked locally in X and by a partition of unity we can assume that μ has compact support in any given open subset of Y. We can thus assume that both Y and X are good and let Ψ_1 and Ψ_2 be holomorphic sections of vector bundles $E_1 \to Y \times Y$ and $E_2 \to X \times X$ defining the diagonals $\Delta_Y \subset Y \times Y$ and $\Delta_X \subset X \times X$, respectively. We equip E_1 and E_2 with Hermitian metrics such that $TY \simeq N_{\Delta_Y} \hookrightarrow E_1|_{\Delta_Y}$ and $TX \simeq N_{\Delta_X} \hookrightarrow E_2|_{\Delta_X}$ are Hermitian embeddings.

Let $\Phi_1 = (\pi_2|_\Gamma \times \mathrm{id}_Y)^* \Psi_1$ and $F_1 = (\pi_2|_\Gamma \times \mathrm{id}_Y)^* E_1$. Then Φ_1 defines $j_1(\Gamma)$ in $\Gamma \times Y$ and we let N_Γ be the normal bundle equipped with the metric induced by $N_\Gamma \hookrightarrow F_1|_{j_1(\Gamma)}$. In view of Lemma 2.6, this is the same as the metric induced by $N_\Gamma \simeq TY$. Hence, $\widehat{c}(N_\Gamma) = (1 \otimes \widehat{c}(TY))|_{j_1(\Gamma)}$. Now, in view of (5.3) and (2.2) we have $T = (\pi_2|_\Gamma)^\diamond \mu = j_1^{!!}(1 \otimes \mu)$. By (4.4) thus

$$(5.27) \qquad (j_1)_* T = (1 \otimes \widehat{c}(TY)) \wedge M^{\Phi_1} \wedge (1 \otimes \mu).$$

Let F_2 and Φ_2 be the pullback of E_2 and Ψ_2, respectively, under the mapping $\Gamma \times X \times Y \to X \times X$, $(\gamma, x, y) \mapsto (\pi_1(\gamma), x)$. Then Φ_2 defines $j_2(\Gamma \times Y)$, which is the graph of the mapping $\Gamma \times Y \to X$, $(\gamma, y) \mapsto \pi_1(\gamma)$. Let $p \colon \Gamma \times X \times Y \to \Gamma \times Y$ be the natural projection. Since $1 \otimes \widehat{c}(TY) = j_2^*(1 \otimes 1 \otimes \widehat{c}(TY))$ it follows by (5.27), (2.1), and Lemma 5.9 that

$$(j_2)_*(j_1)_*T = (1 \otimes 1 \otimes \widehat{c}(TY)) \wedge \widehat{c}(N_{\Gamma \times Y}) \wedge M^{p^*\Phi_1 + \Phi_2} \wedge p^*(1 \otimes \mu),$$

where the normal bundle $N_{\Gamma \times Y}$ of $j_2(\Gamma \times Y)$ is equipped with the metric induced by $N_{\Gamma \times Y} \hookrightarrow F_2|_{j_2(\Gamma \times Y)}$. In view of Lemma 2.6, $\widehat{c}(N_{\Gamma \times Y}) = (1 \otimes \widehat{c}(TX) \otimes 1)|_{j_2(\Gamma \times Y)}$. Thus, since $p^*(1 \otimes \mu) = 1 \otimes 1 \otimes \mu$,

$$(5.28) \quad (j_2)_*(j_1)_*T = (1 \otimes \widehat{c}(TX) \otimes \widehat{c}(TY)) \wedge M^{p^*\Phi_1 + \Phi_2} \wedge (1 \otimes 1 \otimes \mu).$$

From the definitions of Φ_1, Φ_2, p, and j_3 it is straightforward to check that

$$(5.29) \qquad\qquad p^*\Phi_1 + \Phi_2 = j_3^*(\Pi^*\Psi_1 + \widetilde{\Pi}^*\Psi_2),$$

where $\Pi \colon Z \times Z \to Y \times Y$ and $\widetilde{\Pi} \colon Z \times Z \to X \times X$ are the natural projections. Moreover, since $TZ = TX \oplus TY$ we have $\widehat{c}(TZ) = \widehat{c}(TX) \otimes \widehat{c}(TY)$. It follows that

$$(5.30) \qquad\qquad 1 \otimes \widehat{c}(TX) \otimes \widehat{c}(TY) = j_3^*(1 \otimes \widehat{c}(TZ)).$$

Replacing $p^*\Phi_1 + \Phi_2$ and $1 \otimes \widehat{c}(TX) \otimes \widehat{c}(TY)$ in (5.28) by the right-hand sides of (5.29) and (5.30), respectively, it follows by (2.1) and (2.24) that

$$(j_3)_*(j_2)_*(j_1)_*T = (1 \otimes \widehat{c}(TZ)) \wedge M^{\Pi^*\Psi_1 + \widetilde{\Pi}^*\Psi_2} \wedge (j_3)_*(1 \otimes 1 \otimes \mu).$$

Since $(1 \otimes \widehat{c}(TZ))|_{\iota(Z)} = \widehat{c}(N_Z)$ and $(j_3)_*(1 \otimes 1 \otimes \mu) = [\Gamma] \otimes \pi_2^*\mu$ we have

$$(5.31) \qquad (j_3)_*(j_2)_*(j_1)_*T = \widehat{c}(N_Z) \wedge M^{\Pi^*\Psi_1 + \widetilde{\Pi}^*\Psi_2} \wedge ([\Gamma] \otimes \pi_2^*\mu).$$

Now, $\Pi^*\Psi_1 + \widetilde{\Pi}^*\Psi_2$ defines $\iota(Z)$ in $Z \times Z$ and the corresponding embedding $N_Z \hookrightarrow (\Pi^*E_1 \oplus \widetilde{\Pi}^*E_2)|_{\iota(Z)}$ is Hermitian. In view of (5.31) and (4.4) thus

$$(j_3)_*(j_2)_*(j_1)_*T = \iota_*\iota^{!!}([\Gamma] \otimes \pi_2^*\mu).$$

By applying π_*, using that $\pi_*\iota_* = (\pi_2)_*$ since $\pi \circ \iota = \pi_2$, thus (5.26) follows. This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 5.10. — As mentioned in the introduction, the alternative definition (5.19) of f^* can be generalized to define a pullback under meromorphic mappings and meromorphic correspondences:

If $h \colon X \dashrightarrow Y$ is a meromorphic mapping, then the closure of the graph of h is an analytic subset of Z, and we let Γ be this analytic subset. If

$\mu \in \mathcal{PS}(Y)$, then $[\Gamma] \otimes \pi_2^* \mu \in \mathcal{PS}(Z \times Z)$ and we can define $h^* \colon \mathcal{PS}(Y) \to \mathcal{PS}(X)$ by letting

$$(5.32) \qquad h^* \mu := (\pi_1)_* \iota^! ([\Gamma] \otimes \pi_2^* \mu).$$

If Γ instead is a meromorphic correspondence, i.e., X and Y have the same dimension m and Γ is an effective m-cycle such that π_1 and π_2 restricted to each irreducible component of $|\Gamma|$ are surjective and proper, then the right-hand side of (5.32) makes sense and is a reasonable definition of a pullback of μ under the correspondence Γ.

6. Compatibility with cohomology

The objective of this section is to prove Theorem 1.2. We need a few preliminary results. For a complex manifold Z we let $H^*(Z)$ be the de Rham cohomology groups. Recall that $H^*(Z)$ can be defined using either closed smooth forms or closed currents. If μ is a closed smooth form or a closed current we let $[\mu]_{\mathrm{dR}}$ be the corresponding cohomology class. If $\mu = [W]$ is the current of integration along $W \subset Z$, then we write $[W]_{\mathrm{dR}}$ instead of $[[W]]_{\mathrm{dR}}$.

If $g \colon Z \to \widetilde{Z}$ is a holomorphic mapping, then the pullback of smooth forms induces a mapping $g^* \colon H^*(\widetilde{Z}) \to H^*(Z)$. If $h \colon W \to Z$ is a proper holomorphic mapping, then the pushforward of currents induces a mapping $h_* \colon H^*(W) \to H^{*+2d}(Z)$, where $d = \dim_{\mathbb{C}} Z - \dim_{\mathbb{C}} W$. The wedge product on smooth forms induces the cup product in $H^*(Z)$; we will write $\wedge$ for the cup product of cohomology classes. Notice that if ξ is a smooth closed form and μ is a closed current, then

$$[\xi \wedge \mu]_{\mathrm{dR}} = [\xi]_{\mathrm{dR}} \wedge [\mu]_{\mathrm{dR}}.$$

We will also use that the projection formula (2.1), with φ and μ denoting cohomology classes, holds.

LEMMA 6.1. — *Let Z be a complex manifold and let $W \subset Z$ be an analytic subset. If $\mu \in \mathcal{PS}(Z)$, then*

$$(6.1) \qquad \mathrm{d} \mathbf{1}_W \mu = \mathbf{1}_W \mathrm{d}\mu, \quad \mathrm{d} \mathbf{1}_{Z \setminus W} \mu = \mathbf{1}_{Z \setminus W} \mathrm{d}\mu.$$

If $\mathbf{1}_W \mu = 0$ and $\mathbf{1}_{Z \setminus W} \mathrm{d}\mu = 0$, then $\mathrm{d}\mu = 0$.

Proof. — Since Z is a manifold, μ is pseudomeromorphic and so by [8, Proposition 3.8] we have $\partial \mathbf{1}_W \mu = \mathbf{1}_W \partial \mu$. Since $\mathcal{PS}(Z)$ is closed under

conjugation, cf. (2.3), and the operator $\mathbf{1}_W$ is real we get $\bar{\partial}\mathbf{1}_W\mu = \mathbf{1}_W\bar{\partial}\mu$. The first equality in (6.1) thus follows. We now get

$$\mathrm{d}\mathbf{1}_{Z\setminus W}\mu = \mathrm{d}(\mu - \mathbf{1}_W\mu) = \mathrm{d}\mu - \mathbf{1}_W\mathrm{d}\mu = \mathbf{1}_{Z\setminus W}\mathrm{d}\mu,$$

which proves the second equality in (6.1). Moreover, if $\mathbf{1}_W\mu = 0 = \mathbf{1}_{Z\setminus W}\mathrm{d}\mu$, then

$$\mathrm{d}\mu = \mathbf{1}_W\mathrm{d}\mu + \mathbf{1}_{Z\setminus W}\mathrm{d}\mu = \mathrm{d}\mathbf{1}_W\mu = 0.$$

The last statement in the lemma is thus also proved. □

PROPOSITION 6.2. — *Let X and Z be complex manifolds and $i\colon X \hookrightarrow Z$ an embedding. Let $\pi\colon \mathrm{Bl}_X \to Z$ be the blowup along $i(X)$, let $D \subset \mathrm{Bl}_X$ be the exceptional divisor, and $L \to \mathrm{Bl}_X$ the associated line bundle.*

(a) *If $\mu \in \mathcal{PS}(Z)$ and $\mathbf{1}_{i(X)}\mu = 0$, then there is a unique $\nu \in \mathcal{PS}(\mathrm{Bl}_X)$ such that $\pi_*\nu = \mu$ and $\mathbf{1}_D\nu = 0$. If $\mathrm{d}\mu = 0$, then $\mathrm{d}\nu = 0$.*

(b) *Assume that the normal bundle $N_X \to i(X)$ is equipped with a Hermitian metric and equip L with a Hermitian metric such that $L|_D \hookrightarrow \pi|_D^* N_X$ (cf. Lemma 2.6) is a Hermitian embedding. If $\nu \in \mathcal{PS}(\mathrm{Bl}_X)$, $\mathbf{1}_D\nu = 0$, and $n = \mathrm{codim}\, i(X)$, then*

(6.2) $$i^!\pi_*\nu = \sum_{k=1}^{n} i^*\widehat{c}_{n-k}(N_X) \wedge (\pi|_D)_*\widehat{s}_{k-1}(L|_D, \nu).$$

Recall that $L|_D$ is the normal bundle of D in Bl_X; see Proposition 3.3 for the currents $\widehat{s}_{k-1}(L|_D, \nu)$.

Proof.

(a). — The uniqueness of such a ν is clear since π is a biholomorphism outside D and $\mathbf{1}_D\nu = 0$. To show the existence it thus suffices to see that if $U \subset Z$ is open, then there is such a ν in $\pi^{-1}(U)$.

In U we can assume that $\mu = g_*\alpha$, where $g\colon V \to U$ and α are as in Definition 2.1. Since $\mathbf{1}_{i(X)}\mu = 0$ we can assume that $g^{-1}i(X) \neq V$; otherwise $\mu = 0$ in view of (2.8). After a modification of V, by Hironaka's theorem we can then assume that $g^{-1}i(X)$ is a hypersurface. Thus, by the defining property of Bl_X there is a holomorphic mapping $\widetilde{g}\colon V \to \pi^{-1}(U)$ such that $g = \pi \circ \widetilde{g}$. Then $\nu := \widetilde{g}_*\alpha \in \mathcal{PS}(\pi^{-1}(U))$ and

$$\mu|_U = g_*\alpha = \pi_*\widetilde{g}_*\alpha = \pi_*\nu.$$

Moreover, by (2.8),

$$\mathbf{1}_D\nu = \widetilde{g}_*(\mathbf{1}_{\widetilde{g}^{-1}(D)}\alpha) = 0$$

since $\widetilde{g}^{-1}(D) = g^{-1}i(X) \neq V$. Thus ν has the required properties in $\pi^{-1}(U)$.

If $d\mu = 0$, then clearly $\mathbf{1}_{\mathrm{Bl}_X \setminus D} d\nu = 0$, and so $d\nu = 0$ by Lemma 6.1. Part (a) is thus proved.

(b). — We have the commutative diagram

$$(6.3) \qquad \begin{array}{ccc} D & \xrightarrow{\ j\ } & \mathrm{Bl}_Z \\ {\scriptstyle \pi|_D} \downarrow & & \downarrow {\scriptstyle \pi} \\ X & \xrightarrow{\ i\ } & Z. \end{array}$$

Locally in Z we can assume that we have a holomorphic section Ψ of a Hermitian vector bundle E such that Ψ defines $i(X)$ and the embedding $N_X \hookrightarrow E|_{i(X)}$ is Hermitian; cf. Example 4.1. Then $\pi^*\Psi$ defines D and since $L|_D$ is the normal bundle of D it follows by Lemma 2.6 that the metric induced on $L|_D$ by $L|_D \hookrightarrow \pi^*E|_D$ is the same as the one induced by $L|_D \hookrightarrow \pi|_D^* N_X$. By Proposition 3.3, (2.24) and commutativity of (6.3) thus

$$(6.4) \qquad i_* \widehat{s}_{k-n}(N_X, \pi_*\nu) = M_k^\Psi \wedge \pi_*\nu = \pi_*(M_k^{\pi^*\Psi} \wedge \nu)$$
$$= \pi_* j_* \widehat{s}_{k-1}(L|_D, \nu) = i_*(\pi|_D)_* \widehat{s}_{k-1}(L|_D, \nu)$$

locally in Z. Since $\widehat{s}_{k-n}(N_X, \pi_*\nu)$ and $\widehat{s}_{k-1}(L|_D, \nu)$ are globally defined in X and D, respectively, (6.4) holds in Z.

Since $\mathbf{1}_D \nu = 0$ it follows by (2.8) and (2.21) that $M_0^\Psi \wedge \pi_*\nu = 0$. By Proposition 3.3 and injectivity of i_* thus $\widehat{s}_{-n}(N_X, \pi_*\nu) = 0$. Hence, (6.2) follows by (4.1), (6.4), and injectivity of i_*. $\qquad\square$

In the remaining part of this section we let $f\colon X \to Y$ be a holomorphic mapping between compact complex manifolds X and Y of dimensions m and n, respectively. We let $Z = X \times Y$ and $\pi_1\colon Z \to X$ and $\pi_2\colon Z \to Y$ be the natural projection. We also let $i\colon X \to Z$ be the embedding defined by $i(x) = (x, f(x))$.

LEMMA 6.3. — *Assume that τ and ν are closed currents in X and Z, respectively. Then $[\tau]_{\mathrm{dR}} = i^*[\nu]_{\mathrm{dR}}$ if and only if $i_*[\tau]_{\mathrm{dR}} = [i(X)]_{\mathrm{dR}} \wedge [\nu]_{\mathrm{dR}}$.*

Proof. — Let φ be a smooth representative of $[\nu]_{\mathrm{dR}}$. Then, by definition, $i^*[\nu]_{\mathrm{dR}} = [i^*\varphi]_{\mathrm{dR}}$. Therefore, if $[\tau]_{\mathrm{dR}} = i^*[\nu]_{\mathrm{dR}}$ we have

$$i_*[\tau]_{\mathrm{dR}} = i_*[i^*\varphi]_{\mathrm{dR}} = [i_* i^*\varphi]_{\mathrm{dR}} = [[i(X)] \wedge \varphi]_{\mathrm{dR}} = [i(X)]_{\mathrm{dR}} \wedge [\varphi]_{\mathrm{dR}}$$
$$= [i(X)]_{\mathrm{dR}} \wedge [\nu]_{\mathrm{dR}}.$$

Conversely, assume that $i_*[\tau]_{\mathrm{dR}} = [i(X)]_{\mathrm{dR}} \wedge [\nu]_{\mathrm{dR}}$. Since, by the preceding calculation, $[i(X)]_{\mathrm{dR}} \wedge [\nu]_{\mathrm{dR}} = i_*[i^*\varphi]_{\mathrm{dR}}$ it follows that

$$i_*[\tau - i^*\varphi]_{\mathrm{dR}} = 0.$$

This precisely means that for any smooth closed form ξ in Z we have $(\tau - i^*\varphi) \cdot i^*\xi = 0$. Let ξ' be a smooth closed form in X. Since $\pi_1 \circ i = \mathrm{id}_X$ and $\pi_1^*\xi'$ is closed in Z we get

$$(\tau - i^*\varphi) \cdot \xi' = (\tau - i^*\varphi) \cdot i^*\pi_1^*\xi' = 0.$$

Hence, $[\tau - i^*\varphi]_{\mathrm{dR}} = 0$, and thus $[\tau]_{\mathrm{dR}} = i^*[\varphi]_{\mathrm{dR}} = i^*[\nu]_{\mathrm{dR}}$. $\square$

LEMMA 6.4. — *Suppose that Y is good and Hermitian. Equip the normal bundle N_X of $i(X)$ with the Hermitian metric induced by (5.1). Let $\pi \colon \mathrm{Bl}_X \to Z$ be the blowup along $i(X)$ and let $D \subset \mathrm{Bl}_X$ be the exceptional divisor. If $\nu, \nu' \in \mathcal{PS}(\mathrm{Bl}_X)$ are closed and such that $\mathbf{1}_D\nu = \mathbf{1}_D\nu' = 0$ and $[\nu]_{\mathrm{dR}} = [\nu']_{\mathrm{dR}}$, then*

$$i_*[i^! \pi_*\nu]_{\mathrm{dR}} = i_*[i^! \pi_*\nu']_{\mathrm{dR}}.$$

Proof. — Let L be the line bundle associated with D and equip $L|_D$ with the Hermitian metric induced by the embedding $L|_D \hookrightarrow \pi|_D^* N_X$; cf. Lemma 2.6. We claim that

(6.5) $i_*(\pi|_D)_*[\widehat{s}_{k-1}(L|_D, \nu)]_{\mathrm{dR}} = i_*(\pi|_D)_*[\widehat{s}_{k-1}(L|_D, \nu')]_{\mathrm{dR}}, \quad k \geqslant 1.$

Taking the claim for granted the lemma follows by applying i_* to (6.2), use (2.1), and take the cohomology class.

It remains to show the claim. Since Y is good there is a holomorphic section Φ of a holomorphic vector bundle F defining the diagonal in $Y \times Y$. Then $\Psi := (f \times \mathrm{id}_Y)^*\Phi$ is a holomorphic section of $E := (f \times \mathrm{id}_Y)^* F$ defining $i(X)$. We equip E with a Hermitian metric so that the embedding $N_X \hookrightarrow E|_{i(X)}$ is Hermitian.

Now, $\pi^*\Psi$ defines D and by (2.14) there is an induced Hermitian metric on L. By Lemma 2.6, on $L|_D$ this metric is the same as the one induced by the embedding $L|_D \hookrightarrow \pi|_D^* N_X$. Let $j \colon D \to \mathrm{Bl}_X$ be the inclusion. In view of Propositions 3.3 and 3.5, since $\mathbf{1}_D\nu = \mathbf{1}_D\nu' = 0$, for $k \geqslant 1$ we have

$$j_*\widehat{s}_{k-1}(L|_D, \nu) = M_k^{\pi^*\Psi} \wedge \nu = -\widehat{c}(L^*)^k \wedge \mathbf{1}_{\mathrm{Bl}_X \setminus D}\nu + \mathrm{d}\nu_k = -\widehat{c}(L^*)^k \wedge \nu + \mathrm{d}\nu_k$$

for some currents ν_k in Bl_X. For some currents ν'_k the same formula with ν and ν_k replaced by ν' and ν'_k, respectively, holds. Hence, since $[\nu]_{\mathrm{dR}} = [\nu']_{\mathrm{dR}}$,

(6.6) $j_*[\widehat{s}_{k-1}(L|_D, \nu)]_{\mathrm{dR}} = -c(L^*)^k \wedge [\nu]_{\mathrm{dR}} = j_*[\widehat{s}_{k-1}(L|_D, \nu')]_{\mathrm{dR}}, \quad k \geqslant 1.$

By applying π_* to (6.6) and using that $\pi_* j_* = i_*(\pi|_D)_*$, cf. (6.3), then (6.5) follows. The claim is thus proved and the lemma follows. $\square$

We will use the following well-known facts about the blowup $\pi\colon \mathrm{Bl}_X \to Z$ along $i(X)$, see, e.g., [17, Chapter 4.6]. See (6.3) for the notation we will use. There is an exact sequence of de Rham cohomology groups

$$(6.7) \qquad 0 \to H^k(Z) \longrightarrow H^k(\mathrm{Bl}_X) \oplus H^k(X) \longrightarrow H^k(D) \to 0;$$

the mapping $H^k(Z) \to H^k(\mathrm{Bl}_X) \oplus H^k(X)$ is given by $a \mapsto (\pi^*a, i^*a)$ and the mapping $H^k(\mathrm{Bl}_X) \oplus H^k(X) \to H^k(D)$ is given by $(a, b) \mapsto j^*a - (\pi|_D)^*b$. Moreover, $H^*(D)$ is an $H^*(X)$-algebra via $(\pi|_D)^*$ and as such it is generated by $j^*c_1(L^*)$, where L is the line bundle corresponding to D. This means that if β is a smooth closed r-form in D, then there are smooth closed ℓ-forms β_ℓ on X such that

$$(6.8) \qquad [\beta]_{\mathrm{dR}} = \sum_{\ell=0}^{r} [(\pi|_D)^*\beta_\ell]_{\mathrm{dR}} \wedge j^*c_1(L^*)^{r-\ell}.$$

In the $H^*(X)$-algebra $H^*(D)$ there is the relation

$$(6.9) \qquad \sum_{k=0}^{n} (\pi|_D)^* i^* c_{n-k}(N_X) \wedge j^*c_1(L^*)^k = 0.$$

THEOREM 6.5. — *Let X be a compact complex manifold, Y a good compact complex Hermitian manifold, and $f\colon X \to Y$ a holomorphic mapping. Let $Z = X \times Y$ and $i\colon X \to Z$ the embedding $i(x) = (x, f(x))$. Equip the normal bundle N_X of $i(X)$ with the Hermitian metric induced by (5.1). If $\mu \in \mathcal{PS}(Z)$ is closed, then*

$$(6.10) \qquad [i^!\mu]_{\mathrm{dR}} = i^*[\mu]_{\mathrm{dR}}, \qquad i_*[i^!\mu]_{\mathrm{dR}} = [i(X)]_{\mathrm{dR}} \wedge [\mu]_{\mathrm{dR}}.$$

Proof. — To begin with, notice that the two identities in (6.10) are equivalent by Lemma 6.3. We have $\mu = \mathbf{1}_{i(X)}\mu + \mathbf{1}_{Z \setminus i(X)}\mu$ and by Lemma 6.1 both $\mathbf{1}_{i(X)}\mu$ and $\mathbf{1}_{Z \setminus i(X)}\mu$ are closed. Thus, since $\mathrm{supp}(\mathbf{1}_{i(X)}\mu) \subset i(X)$ and $\mathbf{1}_{i(X)}\mathbf{1}_{Z \setminus i(X)}\mu = 0$, cf. (2.6) and (2.7), it suffices to show the theorem for a μ such that either $\mathrm{supp}\,\mu \subset i(X)$ or $\mathbf{1}_{i(X)}\mu = 0$.

We first assume that $\mathrm{supp}\,\mu \subset i(X)$. Then, by Example 4.2, $i^!\mu = i^*\widehat{c}_n(N_X) \wedge \widehat{s}_{-n}(N_X, \mu_1)$ and $i_*\widehat{s}_{-n}(N_X, \mu) = \mu$. It is well-known that $i^*c_n(N_X) = i^*[i(X)]_{\mathrm{dR}}$ and so, in view of (2.1),

$$i_*[i^!\mu]_{\mathrm{dR}} = i_*\big(i^*c_n(N_X) \wedge [\widehat{s}_{-n}(N_X, \mu)]_{\mathrm{dR}}\big)$$
$$= i_*\big(i^*[i(X)]_{\mathrm{dR}} \wedge [\widehat{s}_{-n}(N_X, \mu)]_{\mathrm{dR}}\big)$$
$$= [i(X)]_{\mathrm{dR}} \wedge i_*[\widehat{s}_{-n}(N_X, \mu)]_{\mathrm{dR}} = [i(X)]_{\mathrm{dR}} \wedge [\mu]_{\mathrm{dR}}.$$

The theorem thus follows in case $\mathrm{supp}\,\mu \subset i(X)$.

It remains to prove the theorem in case $\mathbf{1}_{i(X)}\mu = 0$. Let $\pi\colon \mathrm{Bl}_X \to Z$ be the blowup along $i(X)$, D the exceptional divisor, and L the associated

line bundle. Equip L with a Hermitian metric such that $L|_D \hookrightarrow \pi|_D^* N_X$ is Hermitian, cf. Lemma 2.6.

Assume first that $\mu = \pi_* \widehat{c}_1(L^*)^r$. It is clear that the first identity in (6.10) holds if $r = 0$ since in that case $\mu = 1$. For $r \geqslant 1$ and $k \geqslant 1$, in view of Example 3.4,

$$\widehat{s}_{k-1}\big(L|_D, \widehat{c}_1(L^*)^r\big) = \widehat{c}_1(L|_D^*)^{k-1} \wedge j^* \widehat{c}_1(L^*)^r = j^* \widehat{c}_1(L^*)^{r+k-1},$$

where $j\colon D \to \mathrm{Bl}_X$ is the inclusion. Hence, if $\mu = \pi_* \widehat{c}_1(L^*)^r$, in view of Proposition 6.2(b), (2.1), and (6.9) we get

$$
\begin{aligned}
[i^! \mu]_{\mathrm{dR}} &= \left[\sum_{k=1}^n i^* \widehat{c}_{n-k}(N_X) \wedge (\pi|_D)_* j^* \widehat{c}_1(L^*)^{r+k-1} \right]_{\mathrm{dR}} \\
&= (\pi|_D)_* \left(\sum_{k=1}^n (\pi|_D)^* i^* c_{n-k}(N_X) \wedge j^* c_1(L^*)^k \wedge j^* c_1(L^*)^{r-1} \right) \\
&= -(\pi|_D)_* \big((\pi|_D)^* i^* c_n(N_X) \wedge j^* c_1(L^*)^{r-1} \big).
\end{aligned}
$$

Since $i^* c_n(N_X) = i^*[i(X)]_{\mathrm{dR}}$, by (2.1), and commutativity of (6.3) thus

$$
\begin{aligned}
(6.11) \qquad i_*[i^! \mu]_{\mathrm{dR}} &= -i_*\big(i^* c_n(N_X) \wedge (\pi|_D)_* j^* c_1(L^*)^{r-1} \big) \\
&= -[i(X)]_{\mathrm{dR}} \wedge i_*(\pi|_D)_* j^* c_1(L^*)^{r-1} \\
&= -[i(X)]_{\mathrm{dR}} \wedge \pi_* j_* j^* c_1(L^*)^{r-1}.
\end{aligned}
$$

However, by (2.15),

$$(6.12) \quad \pi_* j_* j^* c_1(L^*)^{r-1} = \pi_*(c_1(L^*)^{r-1} \wedge [D]_{\mathrm{dR}}) = -\pi_* c_1(L^*)^r = -[\mu]_{\mathrm{dR}}.$$

By (6.11) and (6.12) thus μ satisfies the second identity in (6.10) in case $\mu = \pi_* \widehat{c}_1(L^*)^r$.

Next, in view of Proposition 4.3 we notice that the theorem now follows if $\mu = \alpha_\ell \wedge \pi_* \widehat{c}_1(L^*)^{r-\ell}$, where α_ℓ is a smooth closed form in Z and ℓ is an integer, $0 \leqslant \ell \leqslant r$.

Let us now show the theorem for a general μ such that $\mathbf{1}_{i(X)} \mu = 0$. In view of Proposition 6.2(a) there is a closed $\nu \in \mathcal{PS}(\mathrm{Bl}_X)$ such that $\mathbf{1}_D \nu = 0$ and $\pi_* \nu = \mu$. Let r be the degree of ν. We claim that there are smooth closed ℓ-forms α_ℓ, $\ell = 0, \ldots, r$, in Z such that

$$(6.13) \qquad\qquad [\nu]_{\mathrm{dR}} = \left[\sum_{\ell=0}^r \pi^* \alpha_\ell \wedge \widehat{c}_1(L^*)^{r-\ell} \right]_{\mathrm{dR}}.$$

Taking the claim for granted for the moment we can finish the proof: By Lemma 6.4 and (2.1) we then have

$$i_*[i^!\pi_*\nu]_{\mathrm{dR}} = i_* \left[i^!\pi_* \sum_{\ell=0}^r \pi^*\alpha_\ell \wedge \widehat{c}_1(L^*)^{r-\ell} \right]_{\mathrm{dR}}$$

$$= i_* \left[i^! \sum_{\ell=0}^r \left(\alpha_\ell \wedge \pi_*\widehat{c}_1(L^*)^{r-\ell} \right) \right]_{\mathrm{dR}}.$$

Hence, since (6.10) holds for each $\alpha_\ell \wedge \pi_*\widehat{c}_1(L^*)^{r-\ell}$, in view of (2.1) and (6.13),

$$i_*[i^!\mu]_{\mathrm{dR}} = i_*[i^!\pi_*\nu]_{\mathrm{dR}} = [i(X)]_{\mathrm{dR}} \wedge \left[\sum_{\ell=0}^r \alpha_\ell \wedge \pi_*\widehat{c}_1(L^*)^{r-\ell} \right]_{\mathrm{dR}}$$

$$= [i(X)]_{\mathrm{dR}} \wedge \pi_* \left[\sum_{\ell=0}^r \pi^*\alpha_\ell \wedge \widehat{c}_1(L^*)^{r-\ell} \right]_{\mathrm{dR}}$$

$$= [i(X)]_{\mathrm{dR}} \wedge \pi_*[\nu]_{\mathrm{dR}} = [i(X)]_{\mathrm{dR}} \wedge [\mu]_{\mathrm{dR}},$$

which shows (6.10).

It remains to prove the claim. Let α be a smooth closed r-form on Bl_X such that $[\alpha]_{\mathrm{dR}} = [\nu]_{\mathrm{dR}}$. Let $\beta = j^*\alpha$, and take smooth closed ℓ-forms β_ℓ on X such that (6.8) holds. Letting $\alpha_\ell = \pi_1^*\beta_\ell$, where $\pi_1\colon Z \to X$ is the projection on the first factor, we get $\beta_\ell = i^*\alpha_\ell$ since $\pi_1 \circ i = \mathrm{id}_X$. Since $i \circ \pi|_D = \pi \circ j$ thus $(\pi|_D)^*\beta_\ell = j^*\pi^*\alpha_\ell$; cf. (6.3). It thus follows by (6.8) that

$$j^* \left[\alpha - \sum_{\ell=0}^r \pi^*\alpha_\ell \wedge \widehat{c}_1(L^*)^{r-\ell} \right]_{\mathrm{dR}} = \left[\beta - \sum_{\ell=0}^r (\pi|_D)^*\beta_\ell \wedge j^*\widehat{c}_1(L^*)^{r-\ell} \right]_{\mathrm{dR}} = 0.$$

Hence, since (6.7) is exact we can add a smooth closed r-form in Z to α_r and get

$$\left[\alpha - \sum_{\ell=0}^r \pi^*\alpha_\ell \wedge \widehat{c}_1(L^*)^{r-\ell} \right]_{\mathrm{dR}} = 0.$$

Since $[\alpha]_{\mathrm{dR}} = [\nu]_{\mathrm{dR}}$ this proves the claim and concludes the proof. □

Proof of Theorem 1.2. — If $\mu \in \mathcal{PS}(Y)$ is closed, then $\pi_2^*\mu = 1 \otimes \mu \in \mathcal{PS}(Z)$ is closed; recall that $\pi_2\colon Z \to Y$ is the projection on the second factor. Using $\pi_2^*\mu$ as μ in Theorem 6.5 we get

$$[i^!\pi_2^*\mu]_{\mathrm{dR}} = i^*[\pi_2^*\mu]_{\mathrm{dR}}.$$

The left-hand side is $[f^*\mu]_{\mathrm{dR}}$ by Definition 5.1. Since π_2 is a simple projection and since $f = \pi_2 \circ i$, the right-hand side is $i^*\pi_2^*[\mu]_{\mathrm{dR}} = f^*[\mu]_{\mathrm{dR}}$. □

Remark 6.6. — If there where a Poincaré lemma for $\mathcal{PS}$-currents on a complex manifold Z, then $H^*(Z)$ could be defined using closed $\mathcal{PS}$-currents. Theorem 1.2 would then follow by (1.2).

7. The obstruction to functoriality

We begin with an example showing that our pullback is not functorial in general.

Example 7.1. — Let $g\colon \mathbb{C}^2 \to \mathbb{C}^2$, $g(v_1, v_2) = (v_1 v_2, v_2^2)$, and let α be a smooth function with compact support in $\mathbb{C}^2$. Then $\mu := g_*\alpha \in \mathcal{PS}(\mathbb{C}^2)$ and has bidegree $(0,0)$. Let

$$f\colon \{\mathrm{pt}\} \to \mathbb{C}^2, \quad f(\mathrm{pt}) = 0,$$

be inclusion of a point. The mapping f can be factorized as $f = f_2 \circ f_1$, where

$$f_1\colon \{\mathrm{pt}\} \to \mathbb{C}, \ f_1(\mathrm{pt}) = 0, \quad \text{and} \quad f_2\colon \mathbb{C} \to \mathbb{C}^2, \ f_2(w) = (w, w^2).$$

We claim that if $\mathbb{C}^2$ and $\mathbb{C}$ are equipped with their standard Hermitian structures, then

(7.1) $f^*\mu = f^\diamond\mu = 2\alpha(0,0)$ and $f_1^* f_2^*\mu = f_1^\diamond f_2^\diamond\mu = \alpha(1,0) + \alpha(-1,0)$,

which shows that $f^*\mu \neq f_1^* f_2^*\mu$ and $f^\diamond\mu \neq f_1^\diamond f_2^\diamond\mu$ if $2\alpha(0,0) \neq \alpha(1,0) + \alpha(-1,0)$.

To begin with, notice that $f^*\mu = f^\diamond\mu$ for degree reasons. Let us calculate $f^\diamond\mu$. We consider $z = (z_1, z_2)$ as a section of the trivial Hermitian rank-2 bundle over $\{\mathrm{pt}\} \times \mathbb{C}^2_z$. This section defines the graph of f in $\{\mathrm{pt}\} \times \mathbb{C}^2$. Let $\pi_1\colon \{\mathrm{pt}\} \times \mathbb{C}^2 \to \{\mathrm{pt}\}$ and $\pi_2\colon \{\mathrm{pt}\} \times \mathbb{C}^2 \to \mathbb{C}^2$ be the standard projections. We will make the identification $\{\mathrm{pt}\} \times \mathbb{C}^2 \simeq \mathbb{C}^2$ and then these projections are identified with $\mathbb{C}^2 \to \{\mathrm{pt}\}$ and $\mathrm{id}_{\mathbb{C}^2}$, respectively. In particular, $\pi_2^*\mu = \mu$. Moreover, in view of (2.21) and the Dimension principle, $M_0^z \wedge \mu = 0 = M_1^z \wedge \mu$. Since $\hat{c}(T\mathbb{C}^2) = 1$, it now follows by (5.7) that

(7.2) $f^\diamond\mu = f^*\hat{c}(T\mathbb{C}^2) \wedge (\pi_1)_*(M^z \wedge \pi_2^*\mu) = (\pi_1)_*(M_2^z \wedge \mu).$

To compute the right-hand side, notice that $\mu = g_*\alpha = g_* p_* p^*\alpha$, where $p\colon \mathrm{Bl}_0\, \mathbb{C}^2 \to \mathbb{C}^2$ is the blowup of 0. The section $p^* g^* z$ defines the divisor $2D + H$, where $D \subset \mathrm{Bl}_0\, \mathbb{C}^2$ is the exceptional divisor and H is the strict transform of $\{v_2 = 0\}$. If L is the line bundle associated with $2D + H$, then $L \hookrightarrow \mathrm{Bl}_0\, \mathbb{C}^2 \times \mathbb{C}^2$, cf. (2.14), and we equip L with the induced Hermitian

metric. In view of (2.24) and Example 2.8(2), since $p^*\alpha$ is smooth, it follows that

$$(7.3) \quad M_2^z \wedge \mu = g_*p_*(M_2^{p^*g^*z} \wedge p^*\alpha) = g_*p_*\big(\widehat{c}_1(L^*) \wedge (2[D] + [H]) \wedge p^*\alpha\big).$$

A direct calculation in the standard local coordinates on $\mathrm{Bl}_0\,\mathbb{C}^2$ shows that $\widehat{c}_1(L^*) \wedge [H] = 0$ and $\widehat{c}_1(L^*) \wedge [D] = \omega \wedge [D]$, where ω is the Fubini–Study form on $D \simeq \mathbb{P}^1$. By (7.2) and (7.3) thus

$$f^\diamond\mu = (\pi_1)_*g_*p_*\big(2\omega \wedge [D] \wedge p^*\alpha\big) = (\pi_1)_*g_*(2[0] \cdot \alpha(0))$$
$$= 2\alpha(0)(\pi_1)_*[0] = 2\alpha(0).$$

The first two equalities in (7.1) now follow.

Let us now calculate $f_2^\diamond\mu$. Let p_1 and p_2 be the standard projections from $\mathbb{C}_w \times \mathbb{C}_z^2$ to the first and second factor, respectively. We consider $f_2(w) - z$ as a section of the trivial Hermitian rank-2 bundle over $\mathbb{C}_w \times \mathbb{C}_z^2$. Then $f_2(w) - z$ defines the graph of f_2 in $\mathbb{C}_w \times \mathbb{C}_z^2$. In a similar way as above, since $p_2^*\mu = (\mathrm{id}_\mathbb{C} \times g)_*(1 \otimes \alpha)$,

$$(7.4) \quad f_2^\diamond\mu = (p_1)_*(M_2^{f_2(w)-z} \wedge p_2^*\mu) = (p_1)_*(\mathrm{id}_\mathbb{C} \times g)_*(M_2^{f_2(w)-g(v)} \wedge (1 \otimes \alpha)).$$

We have that $f_2(w) - g(v) = (w - v_1v_2, w^2 - v_2^2)$, which defines a locally complete intersection ideal $\mathcal{J}$ in $\mathbb{C}_w \times \mathbb{C}_v^2$. The fundamental cycle $\mathcal{Z}_\mathcal{J}$ of $\mathcal{J}$ is the proper intersection of the divisors $\mathrm{div}(w - v_1v_2)$ and $\mathrm{div}(w^2 - v_2^2)$. A standard calculation gives

$$\mathrm{div}(w^2 - v_2^2) \cdot \mathrm{div}(w - v_1v_2)$$
$$= 2[w = v_2 = 0] + [w = v_2, v_1 = 1] + [w = -v_2, v_1 = -1].$$

Thus, in view of (7.4), (2.23), and Example 2.8,

$$f_2^\diamond\mu = p_{1*}(\mathrm{id}_\mathbb{C} \times g)_*$$
$$\big((1 \otimes \alpha) \wedge (2[w = v_2 = 0] + [w = v_2, v_1 = 1] + [w = -v_2, v_1 = -1])\big).$$

Since $p_1 \circ (\mathrm{id}_\mathbb{C} \times g)$ is the natural projection $\mathbb{C}_w \times \mathbb{C}_v^2 \to \mathbb{C}_w$ it follows that

$$f_2^\diamond\mu = \alpha(1, w) + \alpha(-1, -w).$$

For degree reasons we have $f_2^\diamond\mu = f_2^*\mu$.

Since $\alpha(1, w) + \alpha(-1, -w)$ is smooth we have $f_1^\diamond(\alpha(1, w) + \alpha(-1, -w)) = f_1^*(\alpha(1, w)) + \alpha(-1, -w)) = \alpha(1, 0) + \alpha(-1, 0)$. Hence, the last two equalities in (7.1) follow.

Next we give a sufficient condition in terms of M-operators for functoriality of the $\diamond$-pullback. On can formulate a similar condition for our

∗-pullback but it becomes a bit more technical so we omit giving the details.

Recall that we say that a complex manifold X is good if there is a holomorphic section Φ of a vector bundle $F \to X \times X$ defining the diagonal Δ in $X \times X$. If in addition X is Hermitian we say that such a Φ is a *Hermitian defining section of the diagonal* if F is equipped with a Hermitian metric such that the induced embedding $TX = N_\Delta \hookrightarrow F|_\Delta$, where N_Δ is the normal bundle of Δ, is Hermitian.

PROPOSITION 7.2. — *Let $f_1\colon X_1 \to X_2$ and $f_2\colon X_2 \to Y$ be holomorphic mappings between complex manifolds. Assume that X_2 and Y are good and Hermitian. Let Φ and Ψ be Hermitian defining sections of the diagonals in $X_2 \times X_2$ and $Y \times Y$, respectively. If $\mu \in \mathcal{PS}(Y)$ and if*

$$M^{\Phi(f_1(x_1),x_2)} \wedge M^{\Psi(f_2(x_2),y)} \wedge (1 \otimes 1 \otimes \mu)$$
$$= M^{\Phi(f_1(x_1),x_2)+\Psi(f_2 \circ f_1(x_1),y)} \wedge (1 \otimes 1 \otimes \mu)$$

in $X_1 \times X_2 \times Y$, then $f_1^\diamond f_2^\diamond \mu = (f_2 \circ f_1)^\diamond \mu$.

Proof. — Let $\beta = f_1^* \widehat{c}(TX_2) \otimes f_2^* \widehat{c}(TY) \otimes 1$, which is a smooth form in $X_1 \times X_2 \times Y$. Let $\Pi\colon X_1 \times X_2 \times Y \to X_1$ be the natural projection. We will show

(7.5) $\Pi_* \big(\beta \wedge M^{\Phi(f_1(x_1),x_2)} \wedge M^{\Psi(f_2(x_2),y)} \wedge (1 \otimes 1 \otimes \mu) \big) = f_1^\diamond f_2^\diamond \mu,$

(7.6) $\Pi_* \big(\beta \wedge M^{\Phi(f_1(x_1),x_2)+\Psi(f_2 \circ f_1(x_1),y)} \wedge (1 \otimes 1 \otimes \mu) \big) = (f_2 \circ f_1)^\diamond \mu.$

From (7.6) and (7.5) the proposition immediatly follows.

To prove (7.5) we factorize Π as

$$X_1 \times X_2 \times Y \xrightarrow{\Pi_2} X_1 \times X_2 \xrightarrow{\Pi_1} X_1.$$

Notice that $\Pi_2 = \mathrm{id}_{X_1} \times \pi_2$, where $\pi_2\colon X_2 \times Y \to X_2$ is the natural projection. For notational convenience, let

$$T = M^{\Phi(f_1(x_1),x_2)} \wedge M^{\Psi(f_2(x_2),y)} \wedge (1 \otimes 1 \otimes \mu).$$

By a small abuse of notation we have $\Phi(f_1(x_1),x_2) = \Pi_2^* \Phi(f_1(x_1),x_2)$ and so

$$\beta \wedge T = \Pi_2^* \Pi_1^* f_1^* \widehat{c}(TX_2) \wedge M^{\Pi_2^* \Phi(f_1(x_1),x_2)}$$
$$\wedge \big(1 \otimes (\pi_2^* f_2^* \widehat{c}(TY) \wedge M^{\Psi(f_2(x_2),y)} \wedge (1 \otimes \mu)) \big).$$

By applying $(\Pi_2)_*$ and using (2.1), (2.24), and that $(\Pi_2)_* = (\mathrm{id}_{X_1} \times \pi_2)_*$, in view of (5.7) we get

$$(\Pi_2)_*(\beta \wedge T) = \Pi_1^* f_1^* \widehat{c}(TX_2) \wedge M^{\Phi(f_1(x_1),x_2)} \wedge \big(1 \otimes f_2^\diamond \mu \big).$$

Since $\Pi_* = (\Pi_1)_*(\Pi_2)_*$, by applying $(\Pi_1)_*$ now (7.5) follows in view of (5.7) and (2.1).

Let us now show (7.6). Let $f = f_2 \circ f_1$. We first check that

$$(7.7) \qquad j^*\beta = j^*\big((f^*\widehat{c}(TY) \otimes 1 \otimes 1) \wedge \widehat{c}(N)\big),$$

where

$$j\colon X_1 \times Y \to X_1 \times X_2 \times Y, \quad j(x_1, y) = (x_1, f_1(x_1), y)$$

and $N \to j(X_1 \times Y)$ is the normal bundle of $j(X_1 \times Y)$ in $X_1 \times X_2 \times Y$ equipped with the metric induced by the natural isomorphism $N \simeq TX_2$. Then $j^*\widehat{c}(N) = j^*(1 \otimes \widehat{c}(TX_2) \otimes 1)$. From the special form of j we have $j^*(a \otimes b \otimes c) = (a \wedge f_1^*b) \otimes c$ if a, b, and c are smooth forms on X_1, X_2, and Y, respectively. Thus,

$$(7.8) \;\; j^*(f_1^*\widehat{c}(TX_2) \otimes 1 \otimes 1) = f_1^*\widehat{c}(TX_2) \otimes 1 = j^*(1 \otimes \widehat{c}(TX_2) \otimes 1) = j^*\widehat{c}(N).$$

Moreover,

$$(7.9) \qquad j^*(1 \otimes f_2^*\widehat{c}(TY) \otimes 1) = f_1^*f_2^*\widehat{c}(TY) \otimes 1 = f^*\widehat{c}(TY) \otimes 1$$
$$= j^*(f^*\widehat{c}(TY) \otimes 1 \otimes 1).$$

Now (7.7) follows from (7.8) and (7.9).

Let

$$\widetilde{T} = M^{\Phi(f_1(x_1), x_2) + \Psi(f(x_1), y)} \wedge (1 \otimes 1 \otimes \mu).$$

By our assumption in the proposition of course $\widetilde{T} = T$, but (7.6) holds without this assumption so we here distinguish between $\widetilde{T}$ and T. In view of Lemma 3.1, $\widetilde{T} = j_*\nu$ for some ($\mathcal{PS}$-)current ν. By (7.7) and (2.1) thus

$$\beta \wedge \widetilde{T} = (f^*\widehat{c}(TY) \otimes 1 \otimes 1) \wedge \widehat{c}(N) \wedge \widetilde{T}.$$

Hence, since $\Phi(f_1(x_1), x_2)$ defines $j(X_1 \times Y)$ in $X_1 \times X_2 \times Y$ it follows by Lemma 5.9 and (2.1) that

$$(7.10) \qquad \beta \wedge \widetilde{T} = j_*\big(j^*(f^*\widehat{c}(TY) \otimes 1 \otimes 1) \wedge M^{\Psi(f(x_1), y)} \wedge (1 \otimes \mu)\big);$$

Lemma 5.9 is indeed applicable since, in view of Lemma 2.6, the metric we have on N is the same as the one induced by the embedding $N \hookrightarrow h^*F|_{j(X_1 \times Y)}$, where $h(x_1, x_2, y) = (f_1(x_1), x_2)$ and $F \to X_2 \times X_2$ is the Hermitian vector bundle of which Φ is a section. Let $P\colon X_1 \times Y \to X_1$ be the natural projection. Then $P_* = \Pi_*j_*$ and moreover $P^*f^*\widehat{c}(TY) = j^*(f^*\widehat{c}(TY) \otimes 1 \otimes 1)$. Hence, by applying Π_* to (7.10), in view of (5.7) we get

$$\Pi_*(\beta \wedge \widetilde{T}) = \Pi_*j_*\big(P^*f^*\widehat{c}(TY) \wedge M^{\Psi(f(x_1), y)} \wedge (1 \otimes \mu)\big)$$
$$= P_*\big(P^*f^*\widehat{c}(TY) \wedge M^{\Psi(f(x_1), y)} \wedge (1 \otimes \mu)\big) = f^\diamond\mu.$$

This proves (7.6) and the proposition follows. □

Whether the assumed M-identity in Proposition 7.2 holds depends on how singular μ is compared to the composition $f_2 \circ f_1$. To illustrate this, assume that $\mu = g_* \alpha$, where $g \colon V \to Y$ and α are as in Definition 2.1. Then one can show that if the fiber product $V' := X_2 \times_Y V$ is a locally complete intersection in $X_2 \times V$ and the fiber product $X_1 \times_{X_2} V'$ is a locally complete intersection in $X_1 \times V'$, then the assumed identity in Proposition 7.2 holds. From this it follows in particular that if μ in Proposition 7.2 is a smooth form, so that we can take $V = Y$, then the M-identity in the proposition holds.

We conclude by a comment about the assumption in Proposition 7.2 that X_2 and Y are good. The conclusion $f_1^\diamond f_2^\diamond \mu = (f_2 \circ f_1)^\diamond \mu$ of Proposition 7.2 is a local statement in X so we can replace X by a small neighborhood of a given $x \in X$. In view of Proposition 5.6(a) one can then replace X_2 and Y in Proposition 7.2 by small neighborhoods of $f_1(x)$ and $f_2(f_1(x))$, respectively. Locally any complex manifold is good, cf. the proof of Proposition 3.3. The assumption that X_2 and Y are good thus becomes superfluous after these localizations.

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Håkan SAMUELSSON KALM
Department of Mathematical Sciences, Chalmers
University of Technology and University of
Gothenburg, SE-412 96 Göteborg (Sweden)
hasam@chalmers.se