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Modelling Wave Propagation: Mathematical Theory and Numerical Analysis, in Memory of V. Dougalis

On the convergence of a linearly implicit finite element method for the nonlinear Schrödinger equation

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Abstract
We consider a model initial- and Dirichlet boundary–value problem for a nonlinear Schrödinger equation in two and three space dimensions. The solution to the problem is approximated by a conservative numerical method consisting of a standard conforming finite element space discretization and a second-order, linearly implicit time stepping, yielding approximations at the nodes and at the midpoints of a nonuniform partition of the time interval. We investigate the convergence of the method by deriving optimal-order error estimates in the L^2 and the H^1 norm, under certain assumptions on the partition of the time interval and avoiding the enforcement of a courant-friedrichs-lewy (CFL) condition between the space mesh size and the time step sizes.

KEYWORDS
convergence, finite element method, linearly implicit time stepping, nonlinear Schrödinger equation, nonuniform mesh, optimal-order error estimates, stability

1991 MATHEMATICS SUBJECT CLASSIFICATION
65M12, 65M60 (primary), 35Q55, 35Q60 (secondary)

1 | INTRODUCTION

1.1 | Formulation of the problem

Let $T > 0$ be a final time and $D \subset \mathbb{R}^d$ be a bounded convex domain with smooth boundary ∂D, where $d \in \{2, 3\}$. Then, we consider the following initial- and Dirichlet boundary–value problem for a nonlinear Schrödinger (NLS) equation: Find a function $u : [0, T] \times \overline{D} \to \mathbb{C}$ such that

$$u_t = \mathrm{i}\,\Delta u + \mathrm{i}\, f\left(|u|^2\right) u \quad \forall\,(t, x) \in (0, T] \times D, \tag{1}$$

$$u(t, \cdot)|_{\partial D} = 0 \qquad \forall\, t \in (0, T], \tag{2}$$

$$u(0, x) = u_0(x) \qquad \forall\, x \in \overline{D}, \tag{3}$$

where $f : [0, +\infty) \to \mathbb{R}$ is a real-valued smooth function and $u_0 : \overline{D} \to \mathbb{C}$ is a complex-valued smooth function with $u_0|_{\partial D} = 0$.

Equation (1) is a well-known mathematical model in several areas of physics such as nonlinear optics, nonlinear water waves, plasma physics, and Bose–Einstein condensates (see, e.g., Refs. [11, 13, 16, 21, 25, 33–35, 38, 45]). The general formulation above includes the cubic and the cubic-quintic (NLS) equation under the choice $f(x) = \alpha\, x + \beta\, x^2$, where α and β are real constants, and the saturated focusing nonlinearity $f(x) = \frac{x}{1+x}$. For a sample of mathematical results related to the problem above, we refer the reader to Refs. [8, 12, 38] and [20], and the references therein.

Hereafter, we assume that the problem above has a unique solution, which is regular enough on $[0, T] \times \overline{D}$ for our purposes, and hence the case of a solution that blows-up in finite time is excluded.

1.2 | Notation and preliminaries

Let $L^2(D)$ be the space of all Lebesgue measurable complex-valued functions, which have the second power of their absolute value integrable on D with respect to Lebesgue's measure dx, provided with the standard norm $\|v\| := \left(\int_D |v(x)|^2\, dx\right)^{1/2}$ for $v \in L^2(D)$, induced by the standard, nonsymmetric, inner product $(v, w) := \int_D v(x)\, \overline{w}(x)\, dx$ for $v,\, w \in L^2(D)$. To simplify the notation, we extend the norm $\|\cdot\|$ and the inner product $(\cdot, \cdot)$ on vectors of $L^2(D)$-functions, by setting $\|F\| := \|\, |F|_{\mathbb{C}^d} \,\|$ for $F \in (L^2(D))^d$, and $(V, W) := \sum_{j=1}^{d} (V_j, W_j)$ for $V, W \in (L^2(D))^d$, where $|\cdot|_{\mathbb{C}^d}$ is the usual Euclidean norm on $\mathbb{C}^d$.

Let $\mathbb{N}_0$ be the set of all nonnegative integers. For $\kappa \in \mathbb{N}_0$, we denote by $H^\kappa(D)$ the Sobolev space of all complex-valued functions which belong, along with their generalized derivatives up to order

κ, to $L^2(D)$ (see, e.g., Ref. [1]). Then, we denote by $\|\cdot\|_\kappa$ its usual norm, that is,

$$\|v\|_\kappa := \left(\sum_{\alpha\in\mathbb{N}_0^d,\,|\alpha|\le\kappa} \|\partial^\alpha v\|^2 \right)^{1/2} \quad \forall\, v \in H^\kappa(D),$$

and by $|\cdot|_\kappa$ the corresponding seminorm, that is,

$$|v|_\kappa := \left(\sum_{\alpha\in\mathbb{N}_0^d,\,|\alpha|=\kappa} \|\partial^\alpha v\|^2 \right)^{1/2} \quad \forall\, v \in H^\kappa(D),$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_d$ for $\alpha = (\alpha_1, \dots, \alpha_d)^T \in \mathbb{N}_0^d$. Thus, we have the identical notation $\|\cdot\|_0 = \|\cdot\|$. Also, by $H_0^1(D)$, we denote the subspace of $H^1(D)$ consisting of all functions that vanish at the boundary ∂D of D in the sense of trace. To simplify the notation, we set $\mathbb{H}^\kappa(D) := H^\kappa(D) \cap H_0^1(D)$.

We denote by $L^\infty(D)$ the space of all Lebesgue measurable functions, which have their essential supremum bounded on D, equipped with the standard norm $|v|_\infty := \operatorname{ess\,sup}_D |v|$ for $v \in L^\infty(D)$. Further, for $\kappa \in \mathbb{N}$, we denote by $W^{\kappa,\infty}(D)$ the Sobolev space of complex-valued functions, which belong, along with their generalized derivatives up to order κ, to $L^\infty(D)$, provided with the norm

$$\|v\|_{\kappa,\infty} := \max_{\alpha\in\mathbb{N}_0^d,\,|\alpha|\le\kappa} |\partial^\alpha v|_\infty \quad \forall\, v \in W^{\kappa,\infty}(D)$$

and hence we have the identical notation $\|\cdot\|_{0,\infty} := |\cdot|_\infty$. Since $d \in \{2,3\}$, we recall (see, e.g., Ref. [1]) that $H^\kappa(D) \subset C^{\kappa-2}(\overline{D})$ for $\kappa \ge 2$, and there exists a positive constant $\mathsf{C}_{\mathrm{SV},\kappa}$ such that

$$|v|_{\kappa-2,\infty} \le \mathsf{C}_{\mathrm{SV},\kappa}\, \|v\|_\kappa \quad \forall\, v \in H^\kappa(D). \tag{4}$$

For $\kappa \in \mathbb{N}_0$ with $\kappa \ge 2$, let $\mathsf{T} : H^{\kappa-2}(D) \to \mathbb{H}^\kappa(D)$ be the solution operator of the elliptic boundary value problem, that is, for given $w \in H^{\kappa-2}(D)$ find $\mathsf{T}w \in \mathbb{H}^\kappa(D)$ such that $\Delta(\mathsf{T}w) = w$ on D (see, e.g., Ref. [18]). According to the well-known elliptic regularity result (see, e.g., Ref. [18]), there exists a positive constant $\mathsf{C}_{\mathrm{ER}}^\kappa$ such that

$$\|\mathsf{T}w\|_\kappa \le \mathsf{C}_{\mathrm{ER}}^\kappa\, \|w\|_{\kappa-2} \quad \forall\, w \in H^{\kappa-2}(D), \tag{5}$$

and, hence, it follows that

$$\|\mathsf{T}(\mathsf{T}(w))\|_{2\kappa} \le \mathsf{C}_{\mathrm{ER}}^{2\kappa}\, \|\mathsf{T}(w)\|_{2\kappa-2} \le \mathsf{C}_{\mathrm{ER}}^{2\kappa}\,\mathsf{C}_{\mathrm{ER}}^{2\kappa-2}\, \|w\|_{2\kappa-4} \quad \forall\, w \in H^{2\kappa-4}(D). \tag{6}$$

Remark 1. Let $v \in \mathbb{H}^\kappa(D)$ and $w = \Delta v \in H^{\kappa-2}(D)$ on D. Obviously, we have $\mathsf{T}(w) = v$ and (5) yields $\|v\|_\kappa \le \mathsf{C}_{\mathrm{ER}}^\kappa\, \|\Delta v\|_{\kappa-2}$. Also, let $v \in \mathbb{H}^{2\kappa}(D)$ with $\Delta v \in \mathbb{H}^{2\kappa-2}(D)$, and $w = \Delta^2 v \in H^{2\kappa-4}(D)$. Then, $\mathsf{T}(w) = \Delta v$ and $\mathsf{T}(\mathsf{T}(w)) = v$ and, thus, (6) yields $\|v\|_{2\kappa} \le \mathsf{C}_{\mathrm{ER}}^{2\kappa}\,\mathsf{C}_{\mathrm{ER}}^{2\kappa-2}\, \|\Delta^2 v\|_{2\kappa-4}$.

1.3 | The finite element spaces framework

Let $\mathsf{S}_h^r \subset \mathbb{H}^1(D) \cap C(\overline{D})$ be a finite element space consisting of functions, which are continuous on $\overline{D}$ and piecewise polynomials of degree at most $r \ge 1$ over a shape regular partition of D in triangles or polyhedrals with maximum diameter h (see, e.g., Refs. [7, 14]). Then, we introduce a discrete Laplace operator $\Delta_h : H^1(D) \to \mathsf{S}_h^r$ by

$$(\Delta_h w, \chi) = (\nabla w, \nabla \chi) \quad \forall \chi \in \mathsf{S}_h^r, \quad \forall w \in H^1(D),$$

the $L^2(D)$-projection operator $\mathsf{P}_h : L^2(D) \to \mathsf{S}_h^r$ by

$$(\mathsf{P}_h w, \chi) = (w, \chi) \quad \forall \chi \in \mathsf{S}_h^r, \quad \forall w \in L^2(D),$$

and the elliptic projection operator $\mathsf{R}_h : H^1(D) \to \mathsf{S}_h^r$ by

$$(\nabla \mathsf{R}_h w, \nabla \chi) = (\nabla w, \nabla \chi) \quad \forall \chi \in \mathsf{S}_h^r, \quad \forall w \in H^1(D).$$

It is well known (see, e.g., Ref. [7]) that for the usual Lagrange interpolant $\mathcal{I}_h : C(\overline{D}) \to \mathsf{S}_h^r$, there exist positive constants $\mathsf{C}_{\mathtt{IP1}}$ and $\mathsf{C}_{\mathtt{IP2}}$, independent of the partition of D, such that

$$|\mathcal{I}_h w - w|_\infty \leq \mathsf{C}_{\mathtt{IP1}}\, h^{s-\frac{d}{2}}\, \|w\|_s \quad \forall w \in \mathbb{H}^s(D), \quad s = 2, \dots, r+1, \tag{7}$$

and

$$\|\mathcal{I}_h w - w\| + h\, \|\mathcal{I}_h w - w\|_1 \leq \mathsf{C}_{\mathtt{IP2}}\, h^s\, \|w\|_s \quad \forall w \in \mathbb{H}^s(D), \quad s = 2, \dots, r+1. \tag{8}$$

Following Ref. [40] and using (8), we conclude that there exists a positive constant $\mathsf{C}_{\mathtt{EP1}}$, independent of the partition of D, such that

$$\|\mathsf{R}_h w - w\| + h\, \|\mathsf{R}_h w - w\|_1 \leq \mathsf{C}_{\mathtt{EP1}}\, h^s\, \|w\|_s \quad \forall w \in \mathbb{H}^s(D), \quad s = 2, \dots, r+1. \tag{9}$$

Also, we assume that the triangulation of D is quasiuniform and thus (see, e.g., Ref. [7]) there exist positive constants $\mathsf{C}_{\mathtt{INV1}}$ and $\mathsf{C}_{\mathtt{INV2}}$, independent of the partition of D, such that

$$|\chi|_\infty \leq \mathsf{C}_{\mathtt{INV1}}\, h^{-\frac{d}{2}}\, \|\chi\| \quad \forall \chi \in \mathsf{S}_h^r \tag{10}$$

and

$$\|\chi\|_1 \leq \mathsf{C}_{\mathtt{INV2}}\, h^{-1}\, \|\chi\| \quad \forall \chi \in \mathsf{S}_h^r. \tag{11}$$

Finally, we combine the estimates above to conclude the following maximum norm error estimate for the elliptic projection (see, e.g., Ref. [23]).

Lemma 1. *Assuming that the triangulation of D is quasiuniform, there exists a positive constant $\mathsf{C}_{\mathtt{EP2}}$, independent of the partition of D, such that*

$$|\mathsf{R}_h v - v|_\infty \leq \mathsf{C}_{\mathtt{EP2}}\, h^{2-\frac{d}{2}}\, \|v\|_2 \quad \forall v \in \mathbb{H}^2(D). \tag{12}$$

Proof. Let $v \in \mathbb{H}^2(D)$. Using (10), (7), (9), and (8), we obtain

$$\begin{aligned}
|\mathsf{R}_h v - v|_\infty \leq &|\mathsf{R}_h v - \mathcal{I}_h v|_\infty + |\mathcal{I}_h v - v|_\infty \\
\leq &\mathsf{C}_{\mathtt{INV1}}\, h^{-\frac{d}{2}}\, \|\mathsf{R}_h v - \mathcal{I}_h v\| + \mathsf{C}_{\mathtt{IP1}}\, h^{2-\frac{d}{2}}\, \|v\|_2 \\
\leq &C \left[h^{-\frac{d}{2}} (\|\mathsf{R}_h v - v\| + \|v - \mathcal{I}_h v\|) + h^{2-\frac{d}{2}}\, \|v\|_2 \right] \\
\leq &C\, h^{2-\frac{d}{2}} \|v\|_2.
\end{aligned}$$

□

1.4 | Fully discrete approximations

Let $N \in \mathbb{N}$ and $\mathcal{P}$ be a partition of the time interval $[0, T]$ in subintervals with nodes $(t_n)_{n=0}^N$, that is, $t_0 = 0$, $t_N = T$, and $t_n < t_{n+1}$ for $n = 0, \ldots, N-1$. Then, we set $\mathsf{k}_n := t_n - t_{n-1}$ for $n = 1, \ldots, N$, and proceed as the following steps (see Ref. [5]):

`Step FD1.` Set

$$\mathsf{U}^0 = \mathsf{R}_h u_0. \tag{13}$$

`Step FD2.` For $n = 1, \ldots, N$, first we define $\mathsf{U}^{n-\frac{1}{2}} \in \mathsf{S}_h^r$ by

$$\mathsf{U}^{n-\frac{1}{2}} - \mathsf{U}^{n-1} + \mathrm{i}\,\frac{\mathsf{k}_n}{4}\,\Delta_h\left(\mathsf{U}^{n-\frac{1}{2}} + \mathsf{U}^{n-1}\right) = \mathrm{i}\,\frac{\mathsf{k}_n}{4}\,\mathsf{P}_h\left[f\left(|\mathsf{U}^{n-1}|^2 \right) (\mathsf{U}^{n-\frac{1}{2}} + \mathsf{U}^{n-1}) \right] \tag{14}$$

and then we find $U^n \in \mathsf{S}_h^r$ such that

$$\mathsf{U}^n - \mathsf{U}^{n-1} + \mathrm{i}\,\frac{\mathsf{k}_n}{2}\,\Delta_h\left(\mathsf{U}^n + \mathsf{U}^{n-1}\right) = \mathrm{i}\,\frac{\mathsf{k}_n}{2}\,\mathsf{P}_h\left[f\left(|\mathsf{U}^{n-\frac{1}{2}}|^2 \right) (\mathsf{U}^n + \mathsf{U}^{n-1}) \right]. \tag{15}$$

Remark 2. The method produces in total $2N$ approximations of the solution at the nodes and at the midpoints of the partition $\mathcal{P}$. The computation of each of those approximations requires the numerical solution of a linear system of algebraic equations, the matrix of which depends on the basis of the finite element space S_h^r involved.

Remark 3. Taking the $L^2(D)$-inner product of (14) by $\mathsf{U}^{n-\frac{1}{2}} + \mathsf{U}^{n-1}$ and of (15) by $\mathsf{U}^n + \mathsf{U}^{n-1}$, and then taking the real parts of the equalities obtained, it follows that $\|\mathsf{U}^n\| = \|\mathsf{U}^{n-1}\|$ and $\|\mathsf{U}^{n-\frac{1}{2}}\| = \|\mathsf{U}^{n-1}\|$ for $n = 1, \ldots, N$. By a simple induction argument, we conclude that $\|\mathsf{U}^n\| = \|\mathsf{U}^{n-\frac{1}{2}}\| = \|\mathsf{U}^0\|$ for $n = 1, \ldots, N$. Thus, the numerical method (13)–(15) conserves the $L^2(D)$ norm.

Remark 4. The existence and uniqueness of $(\mathsf{U}^n)_{n=1}^N$ and $(\mathsf{U}^{n-\frac{1}{2}})_{n=1}^N$ follows, unconditionally, by observing that the operator $\mathsf{T}_\varepsilon : \mathsf{S}_h^r \to \mathsf{S}_h^r$ defined by $\mathsf{T}_\varepsilon \chi = \chi + \mathrm{i}\,\varepsilon\,\Delta_h \chi$ for $\chi \in \mathsf{S}_h^r$ and $\varepsilon > 0$ is invertible.

1.5 Motivation, main results, and bibliography

The application of implicit time-stepping methods for the numerical approximation of the solution to the (NLS) equation gives birth to nonlinear systems of algebraic equations that one has to solve numerically by applying an iterative method (see, e.g., Refs. [2–4, 17, 24, 26–28, 36, 41, 43, 44]). Alternatively, the use of an explicit time-stepping method is not attractive because stability is guaranteed only if a rather restrictive CFL condition is satisfied (see, e.g., Ref. [36]). Another way is the development of unconditionally stable, linearly implicit time-stepping methods, where, at every time level, only the solution of a linear system of algebraic equations is required (see, e.g., Refs. [6, 9, 10, 19, 39, 42, 46]). Nevertheless, it is easily observed that the second order, linearly implicit methods proposed in the literature have been constructed and analyzed over a uniform partition of the time interval, mainly because the linearization of the nonlinear term is achieved by using approximations already computed at the previous time nodes (see, e.g., Refs. [6, 10, 19, 39, 42, 46]). Even though a reformulation of the aforementioned methods is possible in order to achieve a second-order consistency error over a nonuniform time partition (see, e.g., Ref. [29]), it seems that the existing convergence theories are not directly applicable.

Here, we focus on an alternative second-order, linearly implicit time discretization of the (NLS) equation (see Ref. [5]) that is far from the idea of using approximations computed at the previous time nodes and close to the idea of computing extra intermediate approximations within each partition interval $[t_{n-1}, t_n]$ (cf., e.g., Ref. [37]). Indeed, the method performs a linearly implicit *half* time-step from t_{n-1} to $t_{n-1} + \frac{\mathsf{k}_n}{2}$ by constructing an approximation $\mathsf{U}^{n-\frac{1}{2}}$ of the solution u at the midpoint $t_{n-1} + \frac{\mathsf{k}_n}{2}$ (see (14)), which then is used to linearize the usual Crank–Nicolson method from t_{n-1} to t_n (see (15)) and thus there is no contribution of the previous time levels in the linearization process. However, there is an additional computational cost, which is finally acceptable because the extra intermediate approximations $\mathsf{U}^{n-\frac{1}{2}}$ are second-order approximations of the solution u at the midpoints, something that is not standard among the Runge–Kutta methods, where intermediate approximations are also used. Here, investigating the aforementioned method, we focus on how to provide an L^∞ bound for the fully discrete approximations, on how the nonuniform partition of the time interval influences the error estimation in the L^2 and H^1 norm, and on how to avoid the enforcement of CFL conditions, which appear frequently in the bibliography (see, e.g., Refs. [2, 3, 22, 26–28, 36, 41, 46]).

Since f is a locally Lipschitz function on $[0, +\infty)$, we build up a convergence analysis of the numerical method under investigation through the derivation of an L^∞ bound for the numerical approximations, which is independent of the discretization parameters. Moving to this direction, we begin by formulating a modified time discrete (MTD) method, which is based on a proper mollification of the nonlinear term (cf. Ref. [46]) and has the following property: When the (MTD) approximations are *close* to the solution u to the problem in the $W^{2,\infty}$ norm, then the mollifier acts as an identity (see (39)–(41)). Analyzing the convergence of the (MTD) method, we provide optimal, second-order in time error estimates in the H^1 and H^2 norm, without imposing conditions on the variable time stepping (see (43) and (44)). In addition, we obtain an error bound in the H^4 norm (see (45)), assuming that there exists a constant $\widehat{\mathsf{C}}_1$, independent of the partition $\mathcal{P}$, such that

$$\mathsf{K}(\mathcal{P}) := \sum_{\ell=1}^{N} (N+1-\ell)\,\mathsf{k}_\ell^3 \leq \widehat{\mathsf{C}}_1, \tag{16}$$

which can be viewed, conditionally, as a suboptimal, first-order in time error estimate (see Remark 5).

Our next step is the formulation of a modified fully discrete (MFD) method, which is based on the mollification of the nonlinear term by a special cut-off function (cf. Ref. [46]) and is characterized by the following property: When the (MFD) approximations are *close* to the solution u to the problem in the L^∞ norm, then they are identical to the fully discrete approximations defined by (13)–(15) (see (104)–(106)). Developing a convergence analysis for the (MFD) method, we focus on the estimation of the error between *the (MFD) approximations* and *the elliptic projection of the (MTD) approximations* in the L^2 norm. In particular, for that error, we derive an $O(h^2)$ estimate when $r \geq 1$ and (16) holds (see (129)), and a higher order $O(h^{\min\{r+1,4\}})$ estimate when $r \geq 2$ and the partition $\mathcal{P}$ satisfies

$$\max_{1\leq \ell \leq N} \mathsf{k}_\ell \leq \mathsf{C} \min_{1\leq \ell \leq N} \mathsf{k}_\ell \tag{17}$$

(see (130)), which is stronger than (16) (see Remark 5). In light of all available error estimates and of the finite element properties, we apply an argument proposed in Ref. [31] (see also Refs. [23, 42]) to conclude the desired L^∞ boundedness of the fully discrete approximations, without the enforcement of a CFL condition. Using the latter result, we derive an $O(\tau^2 + h^{r+1})$ error estimate in the L^2 norm and an $O(\tau^2 + h)$ error estimate in the H^1 norm when (16) holds, where $\tau := \max_{1\leq \ell \leq N} \mathsf{k}_\ell$. Also, assuming that (17) holds, we arrive at a higher order $O(\tau^2 + h^{\min\{r,3\}})$ error estimate in the H^1 norm, which is optimal for $r \in \{2, 3\}$. The latter limitation in the order of convergence in the H^1 norm is due to the, by construction, limited regularity of the (MTD) approximations.

In the convergence analysis we develop here, we use results from the convergence analysis of the corresponding time discrete approximations as a tool to avoid the enforcement of CFL conditions. This technique has been used in the error estimation: of a linearized semi-implicit finite element method for a nonlinear parabolic problem,[31] of a two-step linearly implicit finite element method approximation for an (NLS) equation over a uniform partition of the time interval,[42] and, in a different setting, of a Backward Euler finite element method for a linear stochastic parabolic problem.[30] Within this framework, in order to arrive at an optimal-order error estimate in the L^2 norm, it is sufficient to derive a first-order error estimate in the H^2 norm for the time discrete approximations. However, since the partition $\mathcal{P}$ is not uniform, such an H^2 error estimate leads to the enforcement of the mesh condition (17) in order to bound, in the H^2 norm, the discrete time derivative of the time discrete approximations that appears in the analysis of the fully discrete approximations. This restriction motivated us to push the error analysis up to the H^4 norm by introducing a properly defined modified version of the time discrete method, and thus arriving at the milder and less restrictive mesh condition (16) in light of which we are able to derive an optimal-order error estimate in the L^2 norm, for all r, and in the H^1 norm, for $r = 1$.

Let us close this section by giving an overview of the paper. In Section 2, we define and estimate the consistency error of the time discretization to which our convergence analysis heavily relies. Section 3 concerns time approximations to the solution, where employing a parameter-dependent mollifier, we construct time discrete approximations and investigate their convergence in higher Sobolev norms. Section 4 is devoted to fully discrete approximations, where following the path in Section 3 (now relying on a parameter dependent smooth cut-off function), we construct (MFD) approximations and prove optimal-order convergence, in the L^2 and H^1 norms, of the whole, combined, approximations introduced by (13)–(15). Finally, we summarize our results in Section 5.

2 | CONSISTENCY OF THE TIME DISCRETIZATION

Let $u^n := u(t_n,\cdot)$ for $n=0,\dots,N$, $t_{n-\frac{1}{2}} := \frac{t_n+t_{n-1}}{2}$ for $n=1,\dots,N$, and $u^{n-\frac{1}{2}} := u(t_{n-\frac{1}{2}},\cdot)$ for $n=1,\dots,N$. Then, for $n=1,\dots,N$, we define the consistency errors $\eta^{n-\frac{1}{2}}$ and η^n by

$$\frac{u^{n-\frac{1}{2}}-u^{n-1}}{\mathsf{k}_n/2} = \mathrm{i}\,\Delta\left(\frac{u^{n-\frac{1}{2}}+u^{n-1}}{2}\right) + \mathrm{i}\,f\left(|u^{n-1}|^2\right)\frac{u^{n-\frac{1}{2}}+u^{n-1}}{2} + \eta^{n-\frac{1}{2}} \tag{18}$$

and

$$\frac{u^n-u^{n-1}}{\mathsf{k}_n} = \mathrm{i}\,\Delta\left(\frac{u^n+u^{n-1}}{2}\right) + \mathrm{i}\,f\left(|u^{n-\frac{1}{2}}|^2\right)\frac{u^n+u^{n-1}}{2} + \eta^n. \tag{19}$$

Below we derive some estimates of $\eta^{n-\frac{1}{2}}$ and η^n in terms of k_n, which we will use later in the convergence analysis of the numerical method.

Proposition 1. *We assume that $f \in C^2([0,+\infty),\mathbb{R})$ and*

$$u \in C^3([0,T],\mathbb{H}^2(D)) \cap C^2([0,T],\mathbb{H}^4(D)). \tag{20}$$

Then, it holds that

$$u(t,\cdot),\ f\left(|u(t,\cdot)|^2\right) \in C^2(\overline{D}) \quad \forall\, t\in[0,T], \tag{21}$$

$$\Delta u(t,\cdot) \in \mathbb{H}^2(D) \quad \forall\, t\in[0,T], \tag{22}$$

$$\eta^{n-\frac{1}{2}},\ \eta^n \in \mathbb{H}^2(D), \quad n=1,\dots,N, \tag{23}$$

and, there exist positive constants C_{CE1}, C_{CE2}, C_{CE3}, C_{CE4}, C_{CE5}, and C_{CE6}, independent of $(\mathsf{k}_n)_{n=1}^N$ and N, such that

$$\|\eta^{n-\frac{1}{2}}\| \le C_{\mathrm{CE1}}\,\mathsf{k}_n\left[\max_{[0,T]}\|\partial_t^2 u\| + \max_{[0,T]}|f\left(|u|^2\right)|_\infty \max_{[0,T]}\|\partial_t u\| + \max_{[0,T]}\|\partial_t u\|_2\right], \tag{24}$$

$$\|\eta^{n}\| \le C_{\mathrm{CE2}}\,\mathsf{k}_n^2\left[\max_{[0,T]}\|\partial_t^3 u\| + \max_{[0,T]}|f\left(|u|^2\right)|_\infty \max_{[0,T]}\|\partial_t^2 u\| + \max_{[0,T]}\|\partial_t^2 u\|_2\right], \tag{25}$$

$$\begin{aligned}\|\nabla\eta^{n-\frac{1}{2}}\| \le C_{\mathrm{CE3}}\,\mathsf{k}_n\Bigg[&\max_{[0,T]}\|\partial_t^2 u\|_1 + \max_{[0,T]}|f\left(|u|^2\right)|_\infty \max_{[0,T]}\|\partial_t u\|_1 + \max_{[0,T]}\|\partial_t u\|_3\\ &+ \max_{[0,T]}|f\left(|u|^2\right)|_{1,\infty}\max_{[0,T]}\|\partial_t u\|\Bigg],\end{aligned} \tag{26}$$

$$\begin{aligned}\|\nabla\eta^{n}\| \le C_{\mathrm{CE4}}\,\mathsf{k}_n^2\Bigg[&\max_{[0,T]}\|\partial_t^3 u\|_1 + \max_{[0,T]}|f\left(|u|^2\right)|_\infty \max_{[0,T]}\|\partial_t^2 u\|_1 + \max_{[0,T]}\|\partial_t^2 u\|_3\\ &+ \max_{[0,T]}|f\left(|u|^2\right)|_{1,\infty}\max_{[0,T]}\|\partial_t^2 u\|\Bigg],\end{aligned} \tag{27}$$

$$\|\Delta\eta^{n-\frac{1}{2}}\| \leq C_{\texttt{CE5}}\, \mathsf{k}_n \Bigg[\max_{[0,T]} \|\partial_t^2 u\|_2 + \max_{[0,T]} |f(|u|^2)|_\infty \max_{[0,T]} \|\partial_t u\|_2 + \max_{[0,T]} \|\partial_t u\|_4 + \max_{[0,T]} |f(|u|^2)|_{2,\infty} \max_{[0,T]} \|\partial_t u\| + \max_{[0,T]} |f(|u|^2)|_{1,\infty} \max_{[0,T]} \|\partial_t u\|_1 \Bigg], \tag{28}$$

and

$$\|\Delta\eta^{n}\| \leq C_{\texttt{CE6}}\, \mathsf{k}_n^2 \Bigg[\max_{[0,T]} \|\partial_t^3 u\|_2 + \max_{[0,T]} |f(|u|^2)|_\infty \max_{[0,T]} \|\partial_t^2 u\|_2 + \max_{[0,T]} \|\partial_t^2 u\|_4 + \max_{[0,T]} |f(|u|^2)|_{2,\infty} \max_{[0,T]} \|\partial_t^2 u\| + \max_{[0,T]} |f(|u|^2)|_{1,\infty} \max_{[0,T]} \|\partial_t^2 u\|_1 \Bigg] \tag{29}$$

for $n = 1, \dots, N$.

Proof. Since $f \in C^2([0,+\infty),\mathbb{R})$ and (20) holds, we have $\partial_t u(t,\cdot) \in \mathbb{H}^4(D) \subset C^2(\overline{D})$, $u(t,\cdot) \in \mathbb{H}^4(D) \subset C^2(\overline{D})$, $\Delta u(t,\cdot) \in H^2(D)$, and $f(|u(t,\cdot)|^2) \in C^2(\overline{D})$ for $t \in [0,T]$, which, along with (1), yield (21) and

$$\Delta u(t,\cdot)|_{\partial D} = 0 \quad \forall\, t \in [0,T], \tag{30}$$

and thus, $\Delta u(t,\cdot) \in \mathbb{H}^2(D)$ for $t \in [0,T]$. Finally, combining (30) with (18) and (19), we obtain $\eta^{n-\frac{1}{2}}, \eta^n \in \mathbb{H}^2(D)$ for $n = 1, \dots, N$.

Let $n \in \{1, \dots, N\}$. Now, we subtract the (NLS) equation (1) at time $t = t_{n-\frac{1}{2}}$ from (19) and at time $t = t_{n-1}$ from (18) to get

$$\eta^{n-\frac{1}{2}} = \mathfrak{L}_1^{n-\frac{1}{2}} - \mathrm{i}\,\mathfrak{L}_2^{n-\frac{1}{2}} - \mathrm{i}\,\mathfrak{L}_3^{n-\frac{1}{2}} \quad \text{and} \quad \eta^n = \mathfrak{L}_1^n - \mathrm{i}\,\mathfrak{L}_2^n - \mathrm{i}\,\mathfrak{L}_3^n, \tag{31}$$

where

$$\mathfrak{L}_1^{n-\frac{1}{2}} := \frac{u^{n-\frac{1}{2}} - u^{n-1}}{\mathsf{k}_n/2} - u_t(t_{n-1},\cdot), \qquad \mathfrak{L}_2^{n-\frac{1}{2}} := \frac{\Delta u^{n-\frac{1}{2}} + \Delta u^{n-1}}{2} - \Delta u^{n-1},$$

$$\mathfrak{L}_3^{n-\frac{1}{2}} := f(|u^{n-1}|^2)\left(\frac{u^{n-\frac{1}{2}} + u^{n-1}}{2} - u^{n-1}\right),$$

$$\mathfrak{L}_1^{n} := \frac{u^{n} - u^{n-1}}{\mathsf{k}_n} - u_t(t_{n-\frac{1}{2}},\cdot), \qquad \mathfrak{L}_2^{n} := \frac{\Delta u^{n} + \Delta u^{n-1}}{2} - \Delta u^{n-\frac{1}{2}},$$

$$\mathfrak{L}_3^{n} := f\left(|u^{n-\frac{1}{2}}|^2\right)\left(\frac{u^{n} + u^{n-1}}{2} - u^{n-\frac{1}{2}}\right).$$

Next, we Taylor expand u^n and u^{n-1} about $t_{n-\frac{1}{2}}$ to arrive at

$$\begin{aligned}
\mathfrak{L}_1^{n-\frac{1}{2}} &= \frac{\mathsf{k}_n}{2} \int_0^1 (1-t)\,\partial_t^2 u\Big(t_{n-1} + \frac{\mathsf{k}_n}{2}\,t,\cdot\Big)\,dt,\\
\mathfrak{L}_2^{n-\frac{1}{2}} &= \frac{\mathsf{k}_n}{4} \int_0^1 \partial_t \Delta u\Big(t_{n-1} + \frac{\mathsf{k}_n}{2}\,t,\cdot\Big)\,dt,\\
\mathfrak{L}_3^{n-\frac{1}{2}} &= \frac{\mathsf{k}_n}{4}\, f\big(|u^{n-1}|^2\big) \int_0^1 \partial_t u\Big(t_{n-1} + \frac{\mathsf{k}_n}{2}\,t,\cdot\Big)\,dt,
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
\mathfrak{L}_1^{n} &= \frac{\mathsf{k}_n^2}{8}\left[\int_0^1 t^2\,\partial_t^3 u\Big(t_n - \frac{\mathsf{k}_n}{2}\,t,\cdot\Big)\,dt + \int_0^1 t^2\,\partial_t^3 u\Big(t_{n-1} + \frac{\mathsf{k}_n}{2}\,t,\cdot\Big)\,dt\right],\\
\mathfrak{L}_2^{n} &= \frac{\mathsf{k}_n^2}{8}\left[\int_0^1 t\,\partial_t^2 \Delta u\Big(t_n - \frac{\mathsf{k}_n}{2}\,t,\cdot\Big)\,dt + \int_0^1 t\,\partial_t^2 \Delta u\Big(t_{n-1} + \frac{\mathsf{k}_n}{2}\,t,\cdot\Big)\,dt\right],\\
\mathfrak{L}_3^{n} &= f\Big(|u^{n-\frac{1}{2}}|^2\Big)\,\frac{\mathsf{k}_n^2}{8}\left[\int_0^1 t\,\partial_t^2 u\Big(t_n - \frac{\mathsf{k}_n}{2}\,t,\cdot\Big)\,dt + \int_0^1 t\,\partial_t^2 u(t_{n-1} + \frac{\mathsf{k}_n}{2}\,t,\cdot)\,dt\right].
\end{aligned} \tag{33}$$

Finally, we use (31), (33), and (32), to obtain the consistency error bounds (24) and (25). The estimates (26)–(29) follow by a similar manipulation after applying the operators ∇ and Δ to both sides of (31). $\square$

3 | TIME DISCRETE APPROXIMATIONS

3.1 | Constructing a mollifier

Let $\xi : (0,+\infty)\times\mathbb{R} \to [0,1]$ be defined by

$$\xi(\lambda, x) := \begin{cases} 1, & \text{if } x \le \lambda,\\ \frac{2\lambda - x}{\lambda}, & \text{if } x \in (\lambda, 2\lambda], \quad \forall\, x \in \mathbb{R}, \quad \forall\, \lambda > 0.\\ 0, & \text{if } x > 2\lambda, \end{cases} \tag{34}$$

Then, for $\lambda > 0$ and $t \in [0,T]$, we define (cf. Ref. [46]) a mollifier $\mathsf{M}_{2,\infty}^{\lambda}(t) : H^4(D) \to H^4(D)$ by

$$\mathsf{M}_{2,\infty}^{\lambda}(t)v := v\;\xi\big(\lambda, \|v - u(t,\cdot)\|_{2,\infty}\big) + u(t,\cdot)\,\big[1 - \xi\big(\lambda, \|v - u(t,\cdot)\|_{2,\infty}\big)\big] \quad \forall\, v \in H^4(D), \tag{35}$$

provided that the solution u to the problem (1)–(3) satisfies $u(t,\cdot) \in H^4(D)$ for $t \in [0,T]$.

In the lemma below, we present some basic properties of the mollifier $\mathsf{M}_{2,\infty}^{\lambda}(t)$.

Lemma 2. *Let $\lambda > 0$, $B_\lambda(v) := \{w \in H^4(D) : \ \|v - w\|_{2,\infty} \le \lambda\}$ for $v \in H^4(D)$, $\widetilde{\nu}$ be a seminorm on $H^4(D)$ and $\mu := \max_{[0,T]} \|u\|_{2,\infty}$. Then, for $t \in [0,T]$, it holds that*

$$\mathsf{M}^\lambda_{2,\infty}(t)v = v \qquad \forall\, v \in B_\lambda(u(t,\cdot)), \quad \forall\, \lambda > 0, \tag{36}$$

$$\big\|\mathsf{M}^\lambda_{2,\infty}(t)v\big\|_{2,\infty} < 3\,\lambda \qquad \forall\, v \in H^4(D), \quad \forall\, \lambda \ge \mu, \tag{37}$$

$$\widetilde{\nu}\Big(\,\mathsf{M}^\lambda_{2,\infty}(t)v - u(t,\cdot)\,\Big) \le \widetilde{\nu}(\,v - u(t,\cdot)\,) \quad \forall\, v \in H^4(D), \quad \forall\, \lambda > 0. \tag{38}$$

Proof. Equality (36) is a simple outcome of the definition (34). The proofs for (37) and (38) follow easily, by proceeding along the lines of the proof of Lemmata 3.1 and 3.2 in [47], and thus are omitted. □

3.2 | The (MTD) approximations

Let $\lambda > 0$, $f \in C^2([0,+\infty),\mathbb{R})$, $u_0 \in \mathbb{H}^4(D)$, and $\Delta u_0 \in \mathbb{H}^2(D)$. Then, we construct (MTD) approximations of u following the steps below:

`Step MTD1.` Set

$$\mathrm{Y}^0_{4,\lambda} := u_0 \in \mathbb{H}^4(D). \tag{39}$$

`Step MDT2.` For $m = 1,\dots,N$, first seek $\mathrm{Y}^{m-\frac12}_{4,\lambda} \in \mathbb{H}^4(D)$ such that

$$\mathrm{Y}^{m-\frac12}_{4,\lambda} - \mathrm{Y}^{m-1}_{4,\lambda} = \mathrm{i}\,\tfrac{\mathsf{k}_m}{4}\,\Delta\Big(\mathrm{Y}^{m-\frac12}_{4,\lambda} + \mathrm{Y}^{m-1}_{4,\lambda}\Big) + \mathrm{i}\,\tfrac{\mathsf{k}_m}{4}\,f\Big(|\mathsf{M}^{m-1}_{2,\infty,\lambda}|^2\Big)\Big(\mathrm{Y}^{m-\frac12}_{4,\lambda} + \mathrm{Y}^{m-1}_{4,\lambda}\Big) \tag{40}$$

with $\mathsf{M}^{m-1}_{2,\infty,\lambda} := \mathsf{M}^\lambda_{2,\infty}(t_{m-1})\mathrm{Y}^{m-1}_{4,\lambda}$, and then seek $\mathrm{Y}^m_{4,\lambda} \in \mathbb{H}^4(D)$ such that

$$\mathrm{Y}^m_{4,\lambda} - \mathrm{Y}^{m-1}_{4,\lambda} = \mathrm{i}\,\tfrac{\mathsf{k}_m}{2}\,\Delta\Big(\mathrm{Y}^m_{4,\lambda} + \mathrm{Y}^{m-1}_{4,\lambda}\Big) + \mathrm{i}\,\tfrac{\mathsf{k}_m}{2}\,f\Big(|\mathsf{M}^{m-\frac12}_{2,\infty,\lambda}|^2\Big)\Big(\mathrm{Y}^m_{4,\lambda} + \mathrm{Y}^{m-1}_{4,\lambda}\Big) \tag{41}$$

with $\mathsf{M}^{m-\frac12}_{2,\infty,\lambda} := \mathsf{M}^\lambda_{2,\infty}(t_{m-\frac12})\mathrm{Y}^{m-\frac12}_{4,\lambda}$.

Below, we discuss the existence and uniqueness of the (MTD) approximations defined above.

Lemma 3. *Let $\lambda > 0$, $f \in C^2([0,+\infty),\mathbb{R})$, $u_0 \in \mathbb{H}^4(D)$, and $\Delta u_0 \in \mathbb{H}^2(D)$. Then, the (MTD) approximations $(\mathrm{Y}^{m-\frac12}_{4,\lambda})_{m=1}^N$ and $(\mathrm{Y}^m_{4,\lambda})_{m=1}^N$ are well defined in $\mathbb{H}^4(D)$ by (40) and (41), and $\Delta\mathrm{Y}^{m-\frac12}_{4,\lambda}, \Delta\mathrm{Y}^m_{4,\lambda} \in \mathbb{H}^2(D)$ for $m = 1,\dots,N$.*

Proof. The proof is based on an induction argument. First, we observe that, by our assumptions, we have $\mathrm{Y}^0_{4,\lambda} \in \mathbb{H}^4(D)$ and $\Delta\mathrm{Y}^0_{4,\lambda} \in \mathbb{H}^2(D)$. Now, let us assume that for a given $\ell \in \{1,\dots,N\}$, the (MTD) approximation $\mathrm{Y}^{\ell-1}_{4,\lambda}$ is well defined in $\mathbb{H}^4(D) \subset C^2(\overline{D})$ and satisfies $\Delta\mathrm{Y}^{\ell-1}_{4,\lambda} \in \mathbb{H}^2(D)$. Then, we define a linear elliptic operator $\Lambda : \mathbb{H}^2(D) \to L^2(D)$ by

$$\Lambda v := v - \mathrm{i}\,\frac{\mathsf{k}_\ell}{4}\,\Delta v - \mathrm{i}\,\frac{\mathsf{k}_\ell}{4}\,f\left(\left|\mathsf{M}^{\ell-1}_{2,\infty,\lambda}\right|^2\right) v \qquad \forall\, v \in \mathbb{H}^2(D)$$

and the corresponding bilinear form $\mathfrak{B} : \mathbb{H}^1(D) \times \mathbb{H}^1(D) \to \mathbb{C}$ by

$$\mathfrak{B}(v,w) := (v,w) + \mathrm{i}\,\frac{\mathsf{k}_\ell}{4}\,(\nabla v, \nabla w) - \mathrm{i}\,\frac{\mathsf{k}_\ell}{4}\left(f\left(|\mathsf{M}^{\ell-1}_{2,\infty,\lambda}|^2\right) v, w\right) \qquad \forall\, v, w \in \mathbb{H}^1(D).$$

According to (40), we are looking for a $\mathrm{Y}^{\ell-\frac{1}{2}}_{4,\lambda} \in \mathbb{H}^4(D)$ such that

$$\Lambda \mathrm{Y}^{\ell-\frac{1}{2}}_{4,\lambda} = \Phi^{\ell-\frac{1}{2}}, \tag{42}$$

where

$$\Phi^{\ell-\frac{1}{2}} := \mathrm{Y}^{\ell-1}_{4,\lambda} + \mathrm{i}\,\frac{\mathsf{k}_\ell}{4}\,\Delta \mathrm{Y}^{\ell-1}_{4,\lambda} + \mathrm{i}\,\frac{\mathsf{k}_\ell}{4}\,f\left(\left|\mathsf{M}^{\ell-1}_{2,\infty,\lambda}\right|^2\right) \mathrm{Y}^{\ell-1}_{4,\lambda} \in \mathbb{H}^2(D).$$

Since $\mathrm{Re}\,[\mathfrak{B}(v,v)] = \|v\|^2$ for $v \in \mathbb{H}^1(D)$, the Fredholm Alternative Theorem (see, e.g., Ref. [18]) yields existence and uniqueness of a weak solution $\mathrm{Y}^{\ell-\frac{1}{2}}_{4,\lambda} \in \mathbb{H}^1(D)$. Since $\Phi^{\ell-\frac{1}{2}} \in H^2(D)$, the standard theory of elliptic regularity yields, in addition, that $\mathrm{Y}^{\ell-\frac{1}{2}}_{4,\lambda} \in \mathbb{H}^4(D)$, and hence it is the solution of (42). Since $\Phi^{\ell-\frac{1}{2}} \in \mathbb{H}^2(D)$, $\mathrm{Y}^{\ell-\frac{1}{2}}_{4,\lambda} \in \mathbb{H}^2(D)$, and $f\left(\left|\mathsf{M}^{\ell-1}_{2,\infty,\lambda}\right|^2\right) \in C^2(\overline{D})$, it follows easily from (42) that $\Delta \mathrm{Y}^{\ell-\frac{1}{2}}_{4,\lambda} \in \mathbb{H}^2(D)$. Proceeding in an analogous manner, we show, also, that there exists unique $\mathrm{Y}^{\ell}_{4,\lambda} \in \mathbb{H}^4(D)$ solving (41) and satisfying $\Delta \mathrm{Y}^{\ell}_{4,\lambda} \in \mathbb{H}^2(D)$. $\square$

3.3 | Convergence of the (MTD) approximations

Here, we investigate convergence properties of the (MTD) approximations in various norms.

Theorem 1. *Let us assume that $f \in C^3([0,+\infty),\mathbb{R})$, $u_0 \in \mathbb{H}^4(D)$, $\Delta u_0 \in \mathbb{H}^2(D)$,*

$$u \in C^3([0,T],\mathbb{H}^2(D)) \cap C^2([0,T],\mathbb{H}^4(D)),$$

$\lambda > 1 + 3\max_{[0,T]} \|u\|_{2,\infty}$, and $\tau := \max_{1\le m\le N} \mathsf{k}_m$. Then, there exist positive constants $\mathsf{C}^{\mathrm{I}}_\lambda$, $\mathsf{C}^{\mathrm{II}}_\lambda$, and $\mathsf{C}^{\mathrm{III}}_\lambda$, independent of $(\mathsf{k}_m)_{m=1}^N$ and N, such that

$$\max_{1\le m\le N} \left\|u^{m-\frac{1}{2}} - \mathrm{Y}^{m-\frac{1}{2}}_{4,\lambda}\right\|_1 + \max_{0\le m\le N} \|u^m - \mathrm{Y}^m_{4,\lambda}\|_1 \le \mathsf{C}^{\mathrm{I}}_\lambda\,\tau^2, \tag{43}$$

$$\max_{1\le m\le N} \left\|\Delta u^{m-\frac{1}{2}} - \Delta \mathrm{Y}^{m-\frac{1}{2}}_{4,\lambda}\right\| + \max_{0\le m\le N} \|\Delta u^m - \Delta \mathrm{Y}^m_{4,\lambda}\| \le \mathsf{C}^{\mathrm{II}}_\lambda\,\tau^2, \tag{44}$$

and

$$\max_{1\le m\le N} \|\Delta^2 u^{m-\frac{1}{2}} - \Delta^2 \mathrm{Y}^{m-\frac{1}{2}}_{4,\lambda}\| + \max_{0\le m\le N} \|\Delta^2 u^m - \Delta^2 \mathrm{Y}^m_{4,\lambda}\| \le \mathsf{C}^{\mathrm{III}}_\lambda\,[\tau + \mathsf{K}(\mathcal{P})], \tag{45}$$

where

$$\mathsf{K}(\mathcal{P}) := \sum_{m=1}^{N}\sum_{\ell=1}^{m} \mathsf{k}_\ell^3 = \sum_{\ell=1}^{N}(N+1-\ell)\,\mathsf{k}_\ell^3. \tag{46}$$

Proof. We simplify notation by setting $\mathsf{E}^{m-\frac{1}{2}} := u^{m-\frac{1}{2}} - \mathrm{Y}_{4,\lambda}^{m-\frac{1}{2}} \in \mathbb{H}^4(D)$ for $m = 1, \dots, N$, and $\mathsf{E}^m := u^m - \mathrm{Y}_{4,\lambda}^m \in \mathbb{H}^4(D)$ for $m = 0, \dots, N$. Also, we set $c_{1,m} := m-1$, $c_{2,m} := m - \frac{1}{2}$, $\ell_{1,m} := m - \frac{1}{2}$, and $\ell_{2,m} := m$ for $m = 1, \dots, N$.

In the sequel, we will use the symbol C to denote a generic constant that is independent of $(\mathsf{k}_m)_{m=1}^N$, N and λ, and may be different at different appearances. Also, we will use the symbol C_λ (with or without additional indexes) to denote a generic constant that depends on λ but is independent of $(\mathsf{k}_m)_{m=1}^N$ and N, not necessarily the same at each occurrence. We note that the constants C and C_λ may depend on the solution u and its derivatives.

`Part 1:` Subtracting (40) from (18) and (41) from (19), we arrive at the following error equations:

$$\mathsf{E}^{m-\frac{1}{2}} - \mathsf{E}^{m-1} = \mathrm{i}\,\frac{\mathsf{k}_m}{4}\,\Delta\left(\mathsf{E}^{m-\frac{1}{2}} + \mathsf{E}^{m-1}\right) + \frac{\mathrm{i}\,\mathsf{k}_m}{4}\left(\mathsf{A}^{1,m} + \mathsf{B}^{1,m}\right) + \frac{\mathsf{k}_m}{2}\,\eta^{m-\frac{1}{2}}, \tag{47}$$

$$\mathsf{E}^{m} - \mathsf{E}^{m-1} = \mathrm{i}\,\frac{\mathsf{k}_m}{2}\,\Delta\left(\mathsf{E}^{m} + \mathsf{E}^{m-1}\right) + \frac{\mathrm{i}\,\mathsf{k}_m}{2}\left(\mathsf{A}^{2,m} + \mathsf{B}^{2,m}\right) + \mathsf{k}_m\,\eta^{m}, \tag{48}$$

where

$$\mathsf{A}^{j,m} := \left[f\left(|u^{c_{j,m}}|^2\right) - f\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right)\right](u^{\ell_{j,m}} + u^{m-1}) \in \mathbb{H}^2(D),$$

$$\mathsf{B}^{j,m} := f(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2)\left(\mathsf{E}^{\ell_{j,m}} + \mathsf{E}^{m-1}\right) \in \mathbb{H}^2(D),$$

for $m = 1, \dots, N$ and $j = 1, 2$.

`Part 2:` Here, we deal with the $L^2(D)$-estimation of the terms $\mathsf{A}^{1,m}$ and $\mathsf{A}^{2,m}$, appearing on the right-hand side of (47) and (48). First, using the mean value theorem and (37), we obtain

$$\begin{aligned}
\|\mathsf{A}^{j,m}\| &\leq \left(|u^{\ell_{j,m}}|_\infty + |u^{m-1}|_\infty \right) \left\| f\left(|u^{c_{j,m}}|^2\right) - f\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right)\right\| \\
&\leq C \left(\|u^{\ell_{j,m}}\|_2 + \|u^{m-1}\|_2 \right) \left\| f\left(|u^{c_{j,m}}|^2\right) - f\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right)\right\| \\
&\leq C \left\| \left[\int_0^1 f'\left(\rho\,|u^{c_{j,m}}|^2 + (1-\rho)\,|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) d\rho\right] \left(|u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \right)\right\| \\
&\leq C \left| \int_0^1 f'\left(\rho\,|u^{c_{j,m}}|^2 + (1-\rho)\,|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) d\rho\right|_\infty \left\|\, |u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \,\right\| \\
&\leq C \max_{\rho\in[0,1]} \left| f'\left(\rho\,|u^{c_{j,m}}|^2 + (1-\rho)\,|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right)\right|_\infty \left\|\, |u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \,\right\| \\
&\leq C \max_{|x|\in[0,3\lambda]} |f'(x^2)| \left\|\, |u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \,\right\| \\
&\leq C_\lambda \left\|\, |u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \,\right\|, \quad m = 1, \dots, N, \quad j = 1, 2.
\end{aligned} \tag{49}$$

Next, we apply (37) and (38) to get

$$\begin{aligned}\big\|\,|u^{c_{j,m}}|^2-|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\,\big\| &\leq \big\|\,|u^{c_{j,m}}+\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|\,|u^{c_{j,m}}-\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|\,\big\|\\ &\leq \left(\,\big|u^{c_{j,m}}\big|_\infty+\big|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\big|_\infty\right)\big\|u^{c_{j,m}}-\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\big\|\\ &\leq C_\lambda\,\big\|u^{c_{j,m}}-\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\big\|\\ &\leq C_\lambda\,\|u^{c_{j,m}}-\mathrm{Y}_{4,\lambda}^{c_{j,m}}\|\\ &\leq C_\lambda\,\|\mathsf{E}^{c_{j,m}}\|,\quad m=1,\dots,N,\quad j=1,2.\end{aligned}\tag{50}$$

Thus, from (49) and (50), it follows that

$$\|\mathsf{A}^{j,m}\|\leq\,C_\lambda\,\|\mathsf{E}^{c_{j,m}}\|,\quad m=1,\dots,N,\quad j=1,2.\tag{51}$$

`Part 3:` Taking the $L^2(D)$-inner product of both sides of (47) and (48), with $\mathsf{E}^{m-\frac{1}{2}}+\mathsf{E}^{m-1}$ and $\mathsf{E}^m+\mathsf{E}^{m-1}$, respectively, and then integrating by parts, taking real parts, applying the Cauchy–Schwarz inequality, and using (51), we obtain

$$\begin{aligned}\|\mathsf{E}^{m-\frac{1}{2}}\|^2-\|\mathsf{E}^{m-1}\|^2&=\tfrac{\mathsf{k}_m}{2}\,\operatorname{Re}(\eta^{m-\frac{1}{2}},\mathsf{E}^{m-\frac{1}{2}}+\mathsf{E}^{m-1})-\tfrac{\mathsf{k}_m}{4}\,\operatorname{Im}(\mathsf{A}^{1,m},\mathsf{E}^{m-\frac{1}{2}}+\mathsf{E}^{m-1})\\ &\leq\mathsf{k}_m\left(\|\eta^{m-\frac{1}{2}}\|+C_\lambda\,\|\mathsf{E}^{m-1}\|\right)\left(\|\mathsf{E}^{m-\frac{1}{2}}\|+\|\mathsf{E}^{m-1}\|\right)\end{aligned}$$

and

$$\begin{aligned}\|\mathsf{E}^m\|^2-\|\mathsf{E}^{m-1}\|^2&=\mathsf{k}_m\,\operatorname{Re}(\eta^m,\mathsf{E}^m+\mathsf{E}^{m-1})-\tfrac{\mathsf{k}_m}{2}\,\operatorname{Im}(\mathsf{A}^{2,m},\mathsf{E}^m+\mathsf{E}^{m-1})\\ &\leq\mathsf{k}_m\left(\|\eta^m\|+C_\lambda\,\|\mathsf{E}^{m-\frac{1}{2}}\|\right)\left(\|\mathsf{E}^m\|+\|\mathsf{E}^{m-1}\|\right)\end{aligned}$$

for $m=1,\dots,N$ (where we have used that $(\mathsf{B}^{j,m},\mathsf{E}^{j,m}+\mathsf{E}^{m-1})$ is real), which, along with (24) and (25), yields

$$\|\mathsf{E}^{m-\frac{1}{2}}\|\leq\,(1+C_\lambda\,\mathsf{k}_m)\,\|\mathsf{E}^{m-1}\|+C\,\mathsf{k}_m^2,\tag{52}$$

$$\|\mathsf{E}^m\|\leq\,\|\mathsf{E}^{m-1}\|+C_\lambda\,\mathsf{k}_m\,\|\mathsf{E}^{m-\frac{1}{2}}\|+C\,\mathsf{k}_m^3\tag{53}$$

for $m=1,\dots,N$. Next, we combine (52) and (53), to conclude

$$\|\mathsf{E}^m\|\leq\,(1+C_\lambda\,\mathsf{k}_m)\,\|\mathsf{E}^{m-1}\|+C_\lambda\,\mathsf{k}_m^3,\quad m=1,\dots,N.\tag{54}$$

In view of $\mathsf{E}^0=0$, we apply a standard discrete Gronwall argument on (54), to arrive at

$$\|\mathsf{E}^m\|\leq\,C_\lambda\left(\sum_{\ell=1}^m\mathsf{k}_\ell^3\right),\quad m=1,\dots,N,\tag{55}$$

which, along with (52), yields

$$\|\mathsf{E}^{m-\frac{1}{2}}\| \leq C_\lambda \left(\mathsf{k}_m^2 + \sum_{\ell=1}^{m-1} \mathsf{k}_\ell^3 \right), \quad m = 1, \ldots, N. \tag{56}$$

`Part 4:` First, we take the $L^2(D)$-inner product of both sides of (47) by $\Delta(\mathsf{E}^{m-\frac{1}{2}} + \mathsf{E}^{m-1})$, and of (48) by $\Delta(\mathsf{E}^m + \mathsf{E}^{m-1})$. Then, in light of (23), we integrate by parts to get

$$\begin{aligned}\left(\nabla\left(\mathsf{E}^{\ell_{j,m}} - \mathsf{E}^{m-1}\right), \nabla\left(\mathsf{E}^{\ell_{j,m}} + \mathsf{E}^{m-1}\right)\right) = &-\mathrm{i}\, 2^{j-3}\, \mathsf{k}_m \,\|\Delta\left(\mathsf{E}^{\ell_{j,m}} + \mathsf{E}^{m-1}\right)\|^2 \\ &+ 2^{j-2}\, \mathsf{k}_m \left(\nabla \eta^{\ell_{j,m}}, \nabla\left(\mathsf{E}^{\ell_{j,m}} + \mathsf{E}^{m-1}\right)\right) \\ &+ \mathrm{i}\; 2^{j-3}\, \mathsf{k}_m \left(\nabla \mathsf{A}^{j,m}, \nabla\left(\mathsf{E}^{\ell_{j,m}} + \mathsf{E}^{m-1}\right)\right) \\ &+ \mathrm{i}\; 2^{j-3}\, \mathsf{k}_m \left(\nabla \mathsf{B}^{j,m}, \nabla\left(\mathsf{E}^{\ell_{j,m}} + \mathsf{E}^{m-1}\right)\right)\end{aligned} \tag{57}$$

for $m = 1, \ldots, N$ and $j = 1, 2$. Taking the real parts of (57), and then using (26), (27), and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}|\mathsf{E}^{m-\frac{1}{2}}|_1^2 - |\mathsf{E}^{m-1}|_1^2 \leq &\, C \left(\mathsf{k}_m^2 + \mathsf{k}_m \,\|\nabla \mathsf{A}^{1,m}\| \right) \left(|\mathsf{E}^{m-\frac{1}{2}}|_1 + |\mathsf{E}^{m-1}|_1 \right) \\ &- \frac{\mathsf{k}_m}{4}\, \mathrm{Im}(\nabla \mathsf{B}^{1,m}, \nabla(\mathsf{E}^{m-\frac{1}{2}} + \mathsf{E}^{m-1}))\end{aligned} \tag{58}$$

and

$$\begin{aligned}|\mathsf{E}^m|_1^2 - |\mathsf{E}^{m-1}|_1^2 \leq &\, C \left(\mathsf{k}_m^3 + \mathsf{k}_m \,\|\nabla \mathsf{A}^{2,m}\| \right) \left(|\mathsf{E}^m|_1 + |\mathsf{E}^{m-1}|_1 \right) \\ &- \frac{\mathsf{k}_m}{2}\, \mathrm{Im}(\nabla \mathsf{B}^{2,m}, \nabla(\mathsf{E}^m + \mathsf{E}^{m-1}))\end{aligned} \tag{59}$$

for $m = 1, \ldots, N$. Using (37), we have

$$\begin{aligned}\left|\partial_{x_\kappa}\left(f(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2)\right)\right|_\infty = &\, 2\left| f'(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2)\; \mathrm{Re}\left(\partial_{x_\kappa} \mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\; \overline{\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}}\right)\right|_\infty \\ \leq &\, 2 \max_{x\in[0,3\lambda]} |f'(x^2)| \left|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\right|_\infty \left|\partial_{x_\kappa} \mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\right|_\infty \\ \leq &\, 2 \max_{x\in[0,3\lambda]} |f'(x^2)| \,\|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\|_{1,\infty}^2 \\ \leq &\, C_\lambda, \quad \kappa = 1, \ldots, d, \quad m = 1, \ldots, N, \quad j = 1, 2,\end{aligned} \tag{60}$$

which is used to obtain

$$
\begin{aligned}
\left|\operatorname{Im}\left(\nabla \mathsf{B}^{j,m}, \nabla\left(\mathsf{E}^{\ell\,j,m} + \mathsf{E}^{m-1}\right)\right)\right| &= \left|\operatorname{Im}\left(\left(\mathsf{E}^{\ell\,j,m} + \mathsf{E}^{m-1}\right)\nabla(f(|\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2)), \nabla\left(\mathsf{E}^{\ell\,j,m} + \mathsf{E}^{m-1}\right)\right)\right| \\
&\leq \left\|\left(\mathsf{E}^{\ell\,j,m} + \mathsf{E}^{m-1}\right)\nabla(f(|\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2)\right\| \left|\mathsf{E}^{\ell\,j,m} + \mathsf{E}^{m-1}\right|_1 \\
&\leq C \max_{1\leq\kappa\leq d} \left|\partial_{x_\kappa}\left(f(|\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2)\right)\right|_\infty \|\mathsf{E}^{\ell\,j,m} + \mathsf{E}^{m-1}\| \, |\mathsf{E}^{\ell\,j,m} + \mathsf{E}^{m-1}|_1 \\
&\leq C_\lambda \left(\|\mathsf{E}^{\ell\,j,m}\| + \|\mathsf{E}^{m-1}\|\right)\left(|\mathsf{E}^{\ell\,j,m}|_1 + |\mathsf{E}^{m-1}|_1\right)
\end{aligned}
\tag{61}
$$

for $m = 1, \ldots, N$ and $j = 1, 2$ (where we have used that $\operatorname{Im}(f(|\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2)\nabla(\mathsf{E}^{\ell\,j,m} + \mathsf{E}^{m-1}), \nabla(\mathsf{E}^{\ell\,j,m} + \mathsf{E}^{m-1})) = 0$). Observing that

$$
\begin{aligned}
\nabla \mathsf{A}^{j,m} =& (u^{\ell\,j,m} + u^{m-1})\left(f'\left(|u^{c_{j,m}}|^2\right) - f'\left(|\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2\right)\right)\nabla|u^{c_{j,m}}|^2 \\
&+ (u^{\ell\,j,m} + u^{m-1})\, f'\left(|\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2\right)\nabla\left(|u^{c_{j,m}}|^2 - |\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2\right) \\
&+ \left[f\left(|u^{c_{j,m}}|^2\right) - f\left(|\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2\right)\right]\nabla(u^{\ell\,j,m} + u^{m-1})
\end{aligned}
$$

and moving along the lines of (49) and (50), we obtain

$$
\begin{aligned}
\|\nabla \mathsf{A}^{j,m}\| \leq& C_\lambda \|\mathsf{E}^{c_{j,m}}\| + C\left\|f'\left(|\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2\right)\nabla\left(|u^{c_{j,m}}|^2 - |\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2\right)\right\| \\
\leq& C_\lambda \|\mathsf{E}^{c_{j,m}}\| + C\left|f'\left(|\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2\right)\right|_\infty \left\|\nabla\left(|u^{c_{j,m}}|^2 - |\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2\right)\right\| \\
\leq& C_\lambda \|\mathsf{E}^{c_{j,m}}\| + C \max_{|x|\in[0,3\lambda]} |f'(x^2)| \left\|\nabla\left(|u^{c_{j,m}}|^2 - |\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2\right)\right\| \\
\leq& C_\lambda \left[\|\mathsf{E}^{c_{j,m}}\| + \left\|\nabla\left(|u^{c_{j,m}}|^2 - |\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2\right)\right\|\right]
\end{aligned}
\tag{62}
$$

for $m = 1, \ldots, N$ and $j = 1, 2$. In light of Lemma 2 and of the following relation

$$
\begin{aligned}
\nabla\left(|u^{c_{j,m}}|^2 - |\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2\right) =& \operatorname{Re}\left[\overline{\left(u^{c_{j,m}} + \mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}\right)}\nabla\left(u^{c_{j,m}} - \mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}\right)\right] \\
&+ \operatorname{Re}\left[\overline{\left(u^{c_{j,m}} - \mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}\right)}\nabla(u^{c_{j,m}} + \mathsf{M}^{c_{j,m}}_{2,\infty,\lambda})\right],
\end{aligned}
\tag{63}
$$

we get

$$
\begin{aligned}
\left\|\nabla\left(|u^{c_{j,m}}|^2 - |\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|^2\right)\right\| \leq& \left(|u^{c_{j,m}}|_\infty + |\mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}|_\infty\right)\|\nabla(u^{c_{j,m}} - \mathsf{M}^{c_{j,m}}_{2,\infty,\lambda})\| \\
&+ C \max_{1\leq\kappa\leq d} |\partial_{x_\kappa}(u^{c_{j,m}} + \mathsf{M}^{c_{j,m}}_{2,\infty,\lambda})|_\infty \|u^{c_{j,m}} - \mathsf{M}^{c_{j,m}}_{2,\infty,\lambda}\| \\
\leq& C_\lambda \left(\|\mathsf{E}^{c_{j,m}}\| + |\mathsf{E}^{c_{j,m}}|_1\right)
\end{aligned}
\tag{64}
$$

for $m = 1, \dots, N$ and $j = 1, 2$. Thus, (62) and (64) yield

$$\|\nabla \mathsf{A}^{j,m}\| \leq C_\lambda \left(\|\mathsf{E}^{c_{j,m}}\| + |\mathsf{E}^{c_{j,m}}|_1 \right), \quad m = 1, \dots, N, \;\; j = 1, 2. \tag{65}$$

Now, we use (58), (59), (61), (65), and (52), to conclude

$$|\mathsf{E}^{m-\frac{1}{2}}|_1 \leq (1 + C_\lambda\, \mathsf{k}_m)\, |\mathsf{E}^{m-1}|_1 + C_\lambda\, (\mathsf{k}_m^2 + \mathsf{k}_m\, \|\mathsf{E}^{m-1}\|) \tag{66}$$

and

$$|\mathsf{E}^m|_1 \leq |\mathsf{E}^{m-1}|_1 + C_\lambda\, \mathsf{k}_m \left(\|\mathsf{E}^m\| + \|\mathsf{E}^{m-1}\| + |\mathsf{E}^{m-\frac{1}{2}}|_1 + \mathsf{k}_m^2 \right) \tag{67}$$

for $m = 1, \dots, N$. Finally, (66) and (67) yield

$$|\mathsf{E}^m|_1 \leq (1 + C_\lambda\, \mathsf{k}_m)\, |\mathsf{E}^{m-1}|_1 + C_\lambda\, \mathsf{k}_m\, (\mathsf{k}_m^2 + \|\mathsf{E}^m\| + \|\mathsf{E}^{m-1}\|), \quad m = 1, \dots, N. \tag{68}$$

In light of $\mathsf{E}^0 = 0$, we apply a standard discrete Gronwall argument on (68) and use (55), to arrive at

$$|\mathsf{E}^m|_1 \leq C_\lambda \left(\sum_{\ell=1}^{m} \mathsf{k}_\ell^3 + \max_{0 \leq \ell \leq m} \|\mathsf{E}^\ell\| \right) \leq C_\lambda \left(\sum_{\ell=1}^{m} \mathsf{k}_\ell^3 \right), \quad m = 1, \dots, N, \tag{69}$$

which, along with (66) and (55), yields

$$|\mathsf{E}^{m-\frac{1}{2}}|_1 \leq C_\lambda \left(\mathsf{k}_m^2 + \sum_{\ell=1}^{m-1} \mathsf{k}_\ell^3 \right), \quad m = 1, \dots, N. \tag{70}$$

Thus, (43) follows as a simple outcome of (55), (56), (69), and (70).

`Part 5:` Here, for simplicity, we set $\mathsf{Z}^{m-\frac{1}{2}} := \Delta \mathsf{E}^{m-\frac{1}{2}} \in \mathbb{H}^2(D)$ for $m = 1, \dots, N$, and $\mathsf{Z}^m := \Delta \mathsf{E}^m \in \mathbb{H}^2(D)$ for $m = 0, \dots, N$ (see Lemma 3 and (22)). Then, from (47) and (48), we obtain

$$\mathsf{Z}^{m-\frac{1}{2}} - \mathsf{Z}^{m-1} = \mathrm{i}\, \tfrac{\mathsf{k}_m}{4}\, \Delta(\mathsf{Z}^{m-\frac{1}{2}} + \mathsf{Z}^{m-1}) + \mathrm{i}\, \tfrac{\mathsf{k}_m}{4} \sum_{\ell=1}^{7} \mathsf{K}_\ell^{1,m} + \tfrac{\mathsf{k}_m}{2}\, \Delta \eta^{m-\frac{1}{2}}, \tag{71}$$

$$\mathsf{Z}^m - \mathsf{Z}^{m-1} = \mathrm{i}\, \tfrac{\mathsf{k}_m}{2}\, \Delta\big(\mathsf{Z}^m + \mathsf{Z}^{m-1}\big) + \mathrm{i}\, \tfrac{\mathsf{k}_m}{2} \sum_{\ell=1}^{7} \mathsf{K}_\ell^{2,m} + \mathsf{k}_m\, \Delta \eta^m \tag{72}$$

for $m = 1, \dots, N$, where

$$\begin{aligned}
\mathsf{K}_1^{j,m} &:= \Delta\Big[f\big(|u^{c_{j,m}}|^2\big) - f\Big(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\Big)\Big] \Big(u^{\ell_{j,m}} + u^{m-1}\Big),\\
\mathsf{K}_2^{j,m} &:= \Big[f\big(|u^{c_{j,m}}|^2\big) - f\Big(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\Big)\Big]\, \Delta(u^{\ell_{j,m}} + u^{m-1}),\\
\mathsf{K}_3^{j,m} &:= 2\, f'\big(|u^{c_{j,m}}|^2\big) \Big[\nabla(|u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2) \cdot \nabla(u^{\ell_{j,m}} + u^{m-1}) \Big],\\
\mathsf{K}_4^{j,m} &:= \Big(f'\big(|u^{c_{j,m}}|^2\big) - f'\Big(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\Big)\Big) \Big[\nabla |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \cdot \nabla(u^{\ell_{j,m}} + u^{m-1}) \Big],
\end{aligned}$$

and

$$\mathsf{K}_5^{j,m} := \Delta(f(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2))\left(\mathsf{E}^{\ell\, j,m}+\mathsf{E}^{m-1}\right),\quad \mathsf{K}_6^{j,m} := 2\,\nabla(f(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2))\cdot\nabla\left(\mathsf{E}^{\ell\, j,m}+\mathsf{E}^{m-1}\right),$$
$$\mathsf{K}_7^{j,m} := f(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2)\,(\mathsf{Z}^{\ell\, j,m}+\mathsf{Z}^{m-1}).$$

`Part 6`: Taking the $L^2(D)$-inner product of both sides of (71) by $\mathsf{Z}^{m-\frac{1}{2}}+\mathsf{Z}^{m-1}$ and of both sides of (72) by $\mathsf{Z}^m+\mathsf{Z}^{m-1}$, integrating by parts, taking real parts, observing that

$$\mathrm{Im}\left[(\mathsf{K}_7^{j,m},\mathsf{Z}^{\ell\, j,m}+\mathsf{Z}^{m-1})\right]=0,\quad m=1,\dots,N,\quad j=1,2,$$

using (28) and (29), and applying the Cauchy–Schwarz inequality, we get

$$\|\mathsf{Z}^{\ell\, j,m}\|-\|\mathsf{Z}^{m-1}\|\le C\left(\mathsf{k}_m^{j+1}+\mathsf{k}_m\sum_{\ell=1}^{6}\|\mathsf{K}_\ell^{j,m}\|\right),\quad m=1,\dots,N,\quad j=1,2. \tag{73}$$

In view of Lemma 2, we have

$$\begin{aligned}
\left|\Delta(f(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2))\right| &\le \sum_{\kappa=1}^{d}\left|f''(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2)\left(\partial_{x_\kappa}|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right)^2+f'(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2)\,\partial_{x_\kappa}^2|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right|\\
&\le 4\max_{|x|\in[0,3\lambda]}|f''(x^2)|\,|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|_\infty^2\sum_{\kappa=1}^{d}|\partial_{x_\kappa}\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|_\infty^2\\
&\quad+2\max_{|x|\in[0,3\lambda]}|f'(x^2)|\sum_{\kappa=1}^{d}\left(|\partial_{x_\kappa}^2\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|_\infty\,|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|_\infty+|\partial_{x_\kappa}\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|_\infty^2\right)\\
&\le C_\lambda\left[\|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\|_{1,\infty}^4+\|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\|_{2,\infty}^2+\|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\|_{1,\infty}^2\right]\\
&\le C_\lambda
\end{aligned} \tag{74}$$

and

$$\begin{aligned}
\left|\nabla|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\cdot\nabla\left(u^{\ell\, j,m}+u^{m-1}\right)\right| &\le \sum_{\kappa=1}^{d}|\partial_{x_\kappa}|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2|_\infty\,|\partial_{x_\kappa}(u^{\ell\, j,m}+u^{m-1})|_\infty\\
&\le C\,|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|_\infty\sum_{\kappa=1}^{d}|\partial_{x_\kappa}\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|_\infty\\
&\le C\,\|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\|_{1,\infty}^2\\
&\le C_\lambda
\end{aligned} \tag{75}$$

for $m=1,\dots,N$ and $j=1,2$. Using (74) and (60), we have

$$\begin{aligned}
\|\mathsf{K}_5^{j,m}\| &\le \left|\Delta\left(f\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right)\right)\right|_\infty\|\mathsf{E}^{\ell\, j,m}+\mathsf{E}^{m-1}\|\\
&\le C_\lambda\left(\|\mathsf{E}^{\ell\, j,m}\|+\|\mathsf{E}^{m-1}\|\right),
\end{aligned} \tag{76}$$

$$
\begin{aligned}
\|\mathsf{K}_7^{j,m}\| &\le \left| f\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \right) \right|_\infty \|\mathsf{Z}^{\ell_{j,m}} + \mathsf{Z}^{m-1}\| \\
&\le \max_{|x|\in[0,3\lambda]} |f(x^2)|\, \|\mathsf{Z}^{\ell_{j,m}} + \mathsf{Z}^{m-1}\| \\
&\le C_\lambda \left(\|\mathsf{Z}^{\ell_{j,m}}\| + \|\mathsf{Z}^{m-1}\| \right),
\end{aligned}
\tag{77}
$$

and

$$
\begin{aligned}
\|\mathsf{K}_6^{j,m}\| &\le C \max_{1\le\kappa\le d} \left|\partial_{x_\kappa}\left(f\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \right)\right)\right|_\infty \|\nabla\left(\mathsf{E}^{\ell_{j,m}} + \mathsf{E}^{m-1} \right)\| \\
&\le C_\lambda \left(|\mathsf{E}^{\ell_{j,m}}|_1 + |\mathsf{E}^{m-1}|_1 \right)
\end{aligned}
\tag{78}
$$

for $m = 1, \dots, N$ and $j = 1, 2$. Using (49), (50), (63), (64), and (75), we obtain

$$
\begin{aligned}
\|\mathsf{K}_2^{j,m}\| &\le \left| \Delta(u^{\ell_{j,m}} + u^{m-1}) \right|_\infty \|f(|u^{c_{j,m}}|^2) - f(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2)\| \\
&\le C_\lambda \left(\|u^{\ell_{j,m}}\|_4 + \|u^{m-1}\|_4 \right) \|\mathsf{E}^{c_{j,m}}\| \\
&\le C_\lambda \|\mathsf{E}^{c_{j,m}}\|,
\end{aligned}
\tag{79}
$$

$$
\begin{aligned}
\|\mathsf{K}_3^{j,m}\| &\le C \max_{|x|\in[0,3\lambda]} |f'(x^2)| \max_{1\le\kappa\le d} |\partial_{x_\kappa}(u^{\ell_{j,m}} + u^{m-1})|_\infty \|\nabla(|u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2)\| \\
&\le C_\lambda \left(\|u^{\ell_{j,m}}\|_3 + \|u^{m-1}\|_3 \right) (\|\mathsf{E}^{c_{j,m}}\| + |\mathsf{E}^{c_{j,m}}|_1) \\
&\le C_\lambda \|\mathsf{E}^{c_{j,m}}\|_1,
\end{aligned}
\tag{80}
$$

and

$$
\begin{aligned}
\|\mathsf{K}_4^{j,m}\| &\le \left| \nabla|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \cdot \nabla(u^{\ell_{j,m}} + u^{m-1}) \right|_\infty \|f'(|u^{c_{j,m}}|^2) - f'(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2)\| \\
&\le C_\lambda \|f'(|u^{c_{j,m}}|^2) - f'(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2)\| \\
&\le C_\lambda \|\mathsf{E}^{c_{j,m}}\|
\end{aligned}
\tag{81}
$$

for $m = 1, \dots, N$ and $j = 1, 2$. To estimate $\mathsf{K}_1^{j,m}$, we observe that

$$
\|\mathsf{K}_1^{j,m}\| \le |u^{\ell_{j,m}} + u^{m-1}|_\infty \sum_{\ell=1}^{4} \|\mathsf{K}_{1,\ell}^{j,m}\| \le C \sum_{\ell=1}^{4} \|\mathsf{K}_{1,\ell}^{j,m}\|, \quad m = 1, \dots, N, \quad j = 1, 2, \tag{82}
$$

where

$$
\begin{aligned}
\mathsf{K}_{1,1}^{j,m} &:= \left[f''\left(|u^{c_{j,m}}|^2\right) - f''\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) \right] \left|\nabla\left(|u^{c_{j,m}}|^2\right)\right|^2, \\
\mathsf{K}_{1,2}^{j,m} &:= f''\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) \left[\nabla\left(|u^{c_{j,m}}|^2 + |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) \cdot \nabla\left(|u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right)\right], \\
\mathsf{K}_{1,3}^{j,m} &:= \Delta\left(|u^{c_{j,m}}|^2\right) \left[f'\left(|u^{c_{j,m}}|^2\right) - f'\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) \right], \\
\mathsf{K}_{1,4}^{j,m} &:= f'\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) \Delta\left(|u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right).
\end{aligned}
$$

Moving along the lines of (49) and (50), and using (37) and (64), we have

$$
\begin{aligned}
\|\mathsf{K}_{1,1}^{j,m}\| &\leq \left(\sum_{\kappa=1}^{d} \left|\partial_{x_\kappa} |u^{c_{j,m}}|^2\right|_\infty^2 \right) \left\| f''\left(|u^{c_{j,m}}|^2\right) - f''\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) \right\| \\
&\leq C\, |u^{c_{j,m}}|_\infty^2\, \|u^{c_{j,m}}\|_{1,\infty}^2 \left\| f''\left(|u^{c_{j,m}}|^2\right) - f''\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) \right\| \\
&\leq C_\lambda\, \|\mathsf{E}^{c_{j,m}}\|,
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
\|\mathsf{K}_{1,2}^{j,m}\| &\leq \left| f''\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right)\right|_\infty \left\|\nabla\left(|u^{c_{j,m}}|^2 + |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) \cdot \nabla\left(|u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right)\right\| \\
&\leq C \max_{|x|\in[0,3\lambda]} |f''(x^2)| \left(\|u^{c_{j,m}}\|_{1,\infty}^2 + \|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\|_{1,\infty}^2 \right) \left\|\nabla\left(|u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right)\right\| \\
&\leq C_\lambda \left\|\nabla\left(|u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right)\right\| \\
&\leq C_\lambda\, \|\mathsf{E}^{c_{j,m}}\|_1,
\end{aligned}
\tag{84}
$$

and

$$
\begin{aligned}
\|\mathsf{K}_{1,3}^{j,m}\| &\leq \left| \sum_{\kappa=1}^{d} \partial_{x_\kappa}^2 |u^{c_{j,m}}|^2 \right|_\infty \left\| f'\left(|u^{c_{j,m}}|^2\right) - f'\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) \right\| \\
&\leq C \left(|u^{c_{j,m}}|_\infty \|u^{c_{j,m}}\|_4 + \|u^{c_{j,m}}\|_3^2 \right) \left\| f'\left(|u^{c_{j,m}}|^2\right) - f'\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) \right\| \\
&\leq C_\lambda\, \|\mathsf{E}^{c_{j,m}}\|
\end{aligned}
\tag{85}
$$

for $m = 1, \ldots, N$ and $j = 1, 2$. Observing that

$$
\begin{aligned}
\Delta\left(|u^{c_{j,m}}|^2 - |\mathrm{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\right) &= 2\mathrm{Re}\left[\Delta u^{c_{j,m}} \overline{\left(u^{c_{j,m}} - \mathrm{M}_{2,\infty,\lambda}^{c_{j,m}}\right)} + \Delta\left(u^{c_{j,m}} - \mathrm{M}_{2,\infty,\lambda}^{c_{j,m}}\right) \overline{\mathrm{M}_{2,\infty,\lambda}^{c_{j,m}}} \right] \\
&\quad + 2\mathrm{Re}\left[\nabla\left(u^{c_{j,m}} + \mathrm{M}_{2,\infty,\lambda}^{c_{j,m}}\right) \cdot \nabla\overline{\left(u^{c_{j,m}} - \mathrm{M}_{2,\infty,\lambda}^{c_{j,m}}\right)} \right]
\end{aligned}
$$

and using Lemma 2, we obtain

$$
\begin{aligned}
\|\mathsf{K}_{1,4}^{j,m}\| \leq & \max_{|x|\in[0,3\lambda]} |f'(x^2)| \, \left\|\Delta\Big(|u^{c_{j,m}}|^2 - |\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2\Big)\right\| \\
\leq & C_\lambda \Big[|\Delta u^{c_{j,m}}|_\infty \, \|u^{m-1} - \mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\| + \big|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\big|_\infty \, \left\|\Delta\Big(u^{c_{j,m}} - \mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\Big)\right\| \\
& + \max_{1\leq\kappa\leq d} \big|\partial_{x_\kappa}(u^{c_{j,m}} + \mathsf{M}_{2,\infty,\lambda}^{c_{j,m}})\big|_\infty \, \left\|\nabla\Big(u^{c_{j,m}} - \mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}\Big)\right\| \Big] \\
\leq & C_\lambda \left(\|\mathsf{E}^{c_{j,m}}\|_1 + \|\mathsf{Z}^{c_{j,m}}\| \right)
\end{aligned} \tag{86}
$$

for $m = 1, \dots, N$ and $j = 1, 2$. Thus, from the inequalities (82), (83), (84), (85), (86), (79), (80), (81), (76), and (78), it follows that

$$
\sum_{\ell=1}^{6} \|\mathsf{K}_\ell^{j,m}\| \leq C_\lambda \left(\|\mathsf{E}^{\ell_{j,m}}\|_1 + \|\mathsf{E}^{m-1}\|_1 + \|\mathsf{E}^{c_{j,m}}\|_1 + \|\mathsf{Z}^{c_{j,m}}\| \right), \quad m = 1, \dots, N, \quad j = 1, 2. \tag{87}
$$

Combining (73), (87), (52), and (66), we arrive at

$$
\|\mathsf{Z}^{m-\frac{1}{2}}\| \leq (1 + C_\lambda \, \mathsf{k}_m) \, \|\mathsf{Z}^{m-1}\| + C_\lambda \, \mathsf{k}_m \left(\|\mathsf{E}^{m-\frac{1}{2}}\|_1 + \|\mathsf{E}^{m-1}\|_1 \right) + C \, \mathsf{k}_m^2, \tag{88}
$$

$$
\|\mathsf{Z}^{m}\| \leq \|\mathsf{Z}^{m-1}\| + C_\lambda \, \mathsf{k}_m \left(\|\mathsf{E}^{m}\|_1 + \|\mathsf{E}^{m-1}\|_1 + \|\mathsf{E}^{m-\frac{1}{2}}\|_1 + \|\mathsf{Z}^{m-\frac{1}{2}}\| \right) + C \, \mathsf{k}_m^3 \tag{89}
$$

for $m = 1, \dots, N$. Now, using (88), (89), (52), and (66), we get

$$
\|\mathsf{Z}^{m-\frac{1}{2}}\| \leq (1 + C_\lambda \, \mathsf{k}_m) \, \|\mathsf{Z}^{m-1}\| + C_\lambda \, \mathsf{k}_m \, \|\mathsf{E}^{m-1}\|_1 + C_\lambda \, \mathsf{k}_m^2, \tag{90}
$$

$$
\|\mathsf{Z}^{m}\| \leq (1 + C_\lambda \, \mathsf{k}_m) \, \|\mathsf{Z}^{m-1}\| + C_\lambda \, \mathsf{k}_m \left(\|\mathsf{E}^{m}\|_1 + \|\mathsf{E}^{m-1}\|_1 \right) + C_\lambda \, \mathsf{k}_m^3, \tag{91}
$$

for $m = 1, \dots, N$. Since $\mathsf{Z}^0 = 0$, after applying a standard discrete Gronwall argument on (91) and then using (55) and (69), we arrive at

$$
\|\mathsf{Z}^{m}\| \leq C_\lambda \left(\sum_{\ell=1}^{m} \mathsf{k}_\ell^3 + \max_{0\leq\ell\leq m} \|\mathsf{E}^{\ell}\|_1 \right) \leq C_\lambda \left(\sum_{\ell=1}^{m} \mathsf{k}_\ell^3 \right), \quad m = 1, \dots, N, \tag{92}
$$

which, along with (90), (55), and (69), yields

$$
\|\mathsf{Z}^{m-\frac{1}{2}}\| \leq C_\lambda \left(\mathsf{k}_m^2 + \sum_{\ell=1}^{m-1} \mathsf{k}_\ell^3 \right), \quad m = 1, \dots, N. \tag{93}
$$

Hence, (44) follows, easily, from (92) and (93).

`Part 7`: Taking the $L^2(D)$-inner product of both sides of (71) by $\Delta(\mathsf{Z}^{m-\frac{1}{2}} - \mathsf{Z}^{m-1})$, and of (72) by $\Delta(\mathsf{Z}^{m} - \mathsf{E}^{m-1})$, and then integrating by parts, we have

$$
\begin{aligned}
-\|\nabla(\mathsf{Z}^{\ell\, j,m} - \mathsf{Z}^{m-1})\|^2 = {} & \mathrm{i}\, 2^{j-3}\, \mathsf{k}_m \left(\Delta(\mathsf{Z}^{\ell\, j,m} + \mathsf{Z}^{m-1}), \Delta(\mathsf{Z}^{\ell\, j,m} - \mathsf{Z}^{m-1})\right) \\
& + \mathrm{i}\, 2^{j-3}\, \mathsf{k}_m \sum_{\ell=1}^{7} \left(\mathsf{K}_{\ell}^{j,m}, \Delta(\mathsf{Z}^{\ell\, j,m} - \mathsf{Z}^{m-1})\right) \\
& + 2^{j-2}\, \mathsf{k}_m \left(\Delta\eta^{\ell\, j,m}, \Delta(\mathsf{Z}^{\ell\, j,m} - \mathsf{Z}^{m-1})\right)
\end{aligned}
$$

for $m = 1, \dots, N$ and $j = 1, 2$, which, after taking their imaginary parts, applying the Cauchy–Schwarz inequality, and using (28) and (29) yields

$$
\begin{aligned}
\|\Delta\mathsf{Z}^{\ell\, j,m}\|^2 - \|\Delta\mathsf{Z}^{m-1}\|^2 = {} & -\sum_{\ell=1}^{7} \mathrm{Re}\left(\mathsf{K}_{\ell}^{j,m}, \Delta(\mathsf{Z}^{\ell\, j,m} - \mathsf{Z}^{m-1})\right) \\
& - 2\ \mathrm{Im}\left(\Delta\eta^{\ell\, j,m}, \Delta(\mathsf{Z}^{\ell\, j,m} - \mathsf{Z}^{m-1})\right) \\
\leq {} & C\left(\mathsf{k}_m^{j} + \sum_{\ell=1}^{7} \|\mathsf{K}_{\ell}^{j,m}\|\right)\left(\|\Delta\mathsf{Z}^{\ell\, j,m}\| + \|\Delta\mathsf{Z}^{m-1}\|\right)
\end{aligned}
\tag{94}
$$

for $m = 1, \dots, N$ and $j = 1, 2$. Using (87), (77), (52), (66), (90), (55), (69), and (92), it follows that

$$
\begin{aligned}
\|\Delta\mathsf{Z}^{m-\frac{1}{2}}\| - \|\Delta\mathsf{Z}^{m-1}\| \leq {} & C_\lambda\left(\mathsf{k}_m + \|\mathsf{E}^{m-\frac{1}{2}}\|_1 + \|\mathsf{Z}^{m-\frac{1}{2}}\| + \|\mathsf{E}^{m-1}\|_1 + \|\mathsf{Z}^{m-1}\|\right) \\
\leq {} & C_\lambda\left(\mathsf{k}_m + \|\mathsf{E}^{m-1}\|_1 + \|\mathsf{Z}^{m-1}\|\right)
\end{aligned}
\tag{95}
$$

and

$$
\begin{aligned}
\|\Delta\mathsf{Z}^{m}\| - \|\Delta\mathsf{Z}^{m-1}\| \leq {} & C_\lambda\Big(\mathsf{k}_m^2 + \|\mathsf{E}^{m-\frac{1}{2}}\|_1 + \|\mathsf{Z}^{m-\frac{1}{2}}\| + \|\mathsf{E}^{m}\|_1 + \|\mathsf{Z}^{m}\| \\
& + \|\mathsf{E}^{m-1}\|_1 + \|\mathsf{Z}^{m-1}\|\Big) \\
\leq {} & C_\lambda\left(\mathsf{k}_m^2 + \|\mathsf{E}^{m}\|_1 + \|\mathsf{Z}^{m}\| + \|\mathsf{E}^{m-1}\|_1 + \|\mathsf{Z}^{m-1}\|\right) \\
\leq {} & C_\lambda\left(\mathsf{k}_m^2 + \sum_{\ell=1}^{m} \mathsf{k}_{\ell}^3\right)
\end{aligned}
\tag{96}
$$

for $m = 1, \dots, N$. In light of $\Delta \mathsf{Z}^0 = 0$, we sum with respect to m both sides of (96), to obtain

$$\begin{aligned} \|\Delta \mathsf{Z}^m\| &\leq C_\lambda \left[\sum_{\ell'=1}^{m} \mathsf{k}_{\ell'}^2 + \sum_{\ell'=1}^{m} \sum_{\ell=1}^{\ell'} \mathsf{k}_\ell^3 \right] \\ &\leq C_\lambda \left[\sum_{\ell'=1}^{N} \mathsf{k}_{\ell'}^2 + \sum_{\ell'=1}^{N} \sum_{\ell=1}^{\ell'} \mathsf{k}_\ell^3 \right] \\ &\leq C_\lambda \left[\tau + \mathsf{K}(\mathcal{P}) \right], \quad m = 1, \dots, N. \end{aligned} \tag{97}$$

Also, we use (95) along with (97), (92), (55), and (69), to get

$$\max_{1 \leq m \leq N} \|\Delta \mathsf{Z}^{m-\frac{1}{2}}\| \leq C_\lambda \left[\tau + \mathsf{K}(\mathcal{P}) \right]. \tag{98}$$

Thus, (45) follows, easily, from (97) and (98). □

Remark 5. The error bound (45) turns into a first-order error estimate, when there exists a constant C, independent of N and the partition $\mathcal{P}$ of the time interval, such that

$$\max_{1 \leq \ell \leq N} \mathsf{k}_\ell \leq \mathsf{C} \min_{1 \leq \ell \leq N} \mathsf{k}_\ell \tag{99}$$

or

$$\max_{1 \leq \ell \leq N} \left[\mathsf{k}_\ell \, (N + 1 - \ell) \right] \leq \mathsf{C}, \tag{100}$$

which are both valid when the partition $\mathcal{P}$ is uniform. Indeed, using (99) and (46), we obtain

$$\mathsf{K}(\mathcal{P}) = \sum_{\ell'=1}^{N} \sum_{\ell=1}^{\ell'} \mathsf{k}_\ell^3 \leq \sum_{\ell'=1}^{N} \mathsf{k}_{\ell'} \sum_{\ell=1}^{\ell'} \frac{\mathsf{k}_\ell^3}{\mathsf{k}_{\ell'}} \leq \mathsf{C} \sum_{\ell'=1}^{N} \mathsf{k}_{\ell'} \sum_{\ell=1}^{\ell'} \mathsf{k}_\ell^2 \leq \mathsf{C}\tau \sum_{\ell'=1}^{N} t_{\ell'} \mathsf{k}_{\ell'} \leq \mathsf{C}\, T^2 \,\tau$$

and (100) along with (46) yields

$$\mathsf{K}(\mathcal{P}) = \sum_{\ell'=1}^{N} \sum_{\ell=1}^{\ell'} \mathsf{k}_\ell^3 = \sum_{\ell'=1}^{N} (N + 1 - \ell') \mathsf{k}_{\ell'}^3 \leq \mathsf{C} \sum_{\ell'=1}^{N} \mathsf{k}_{\ell'}^2 \leq \mathsf{C}\, T \,\tau.$$

We note that if (99) holds, then (100) is satisfied because $\min_{1 \leq \ell \leq N} \mathsf{k}_\ell \leq \frac{1}{N}$ and

$$\max_{1 \leq \ell \leq N} \left[\mathsf{k}_\ell \, (N + 1 - \ell) \right] \leq N \max_{1 \leq \ell \leq N} \mathsf{k}_\ell \leq \mathsf{C} N \min_{1 \leq \ell \leq N} \mathsf{k}_\ell \leq \mathsf{C}.$$

However, (99) and (100) are not equivalent, and we can verify it by an counterexample. Let us choose $\mathsf{k}_\ell = \frac{1}{N+1-\ell} \frac{T}{S_N}$ for $\ell = 1, \dots, N$, where $S_N = \sum_{\ell=1}^{N} \frac{1}{\ell}$. Then, we conclude that (100) holds because

$$\max_{1 \leq \ell \leq N} \left[\mathsf{k}_\ell \, (N + 1 - \ell) \right] = \frac{T}{S_N} \leq \frac{T}{\ln(N+1)}$$

and that (99) does not hold since $\frac{\max_{1 \leq \ell \leq N} \mathsf{k}_\ell}{\min_{1 \leq \ell \leq N} \mathsf{k}_\ell} = N$.

4 | CONVERGENCE OF THE FULLY DISCRETE APPROXIMATIONS

4.1 | A smooth cut-off function

Let $\delta > 0$ and $\gamma_\delta \in C^1(\mathbb{R}, \mathbb{R})$ (see, e.g, Ref. [46]) be an odd auxiliary function defined by

$$\gamma_\delta(s) := \begin{cases} s, & \text{if } s \in [0, \delta], \\ q(s), & \text{if } s \in (\delta, 2\delta], \quad \forall s \geq 0, \\ 2\delta, & \text{if } s > 2\delta, \end{cases} \tag{101}$$

where $q \in \mathbb{P}^3[\delta, 2\delta]$ is a polynomial satisfying: $q(\delta) = \delta$, $q'(\delta) = 1$, $q(2\delta) = 2\delta$, and $q'(2\delta) = 0$. Obviously it holds that $\gamma_\delta(s) = s$ when $|s| \leq \delta$, and we can show (see, e.g., Ref. [32]) that

$$\sup_{\mathbb{R}} |\gamma_\delta| = 2\delta, \quad \sup_{\mathbb{R}} |\gamma'_\delta| \leq \frac{4}{3}. \tag{102}$$

We extend γ_δ on $\mathbb{C}$, by setting $g_\delta(z) := \gamma_\delta(\operatorname{Re}(z)) + \mathrm{i}\,\gamma_\delta(\operatorname{Im}(z))$ for $z \in \mathbb{C}$. Then, in view of (101) and (102), it holds that

$$\begin{aligned} g_\delta(z) &= z \quad \forall z \in \mathbb{C} \quad \text{with} \quad |z| < \delta, \\ |g_\delta(z)| &< 3\delta \quad \forall z \in \mathbb{C}, \\ |g_\delta(z) - g_\delta(w)| &\leq \frac{4}{3}|z - w| \quad \forall z, w \in \mathbb{C}. \end{aligned} \tag{103}$$

4.2 | The (MFD) approximations

To investigate the convergence of the fully discrete approximations defined in Section 1.4, we introduce the (MFD) approximations of u, which are defined, for given $\delta > 0$, in the following way (cf. Ref. [46]):

`Step MFD1.` Set

$$U_\delta^0 = \mathsf{U}^0. \tag{104}$$

`Step MFD2.` For $n = 1, \ldots, N$, first we define $U_\delta^{n-\frac{1}{2}} \in \mathsf{S}_h^r$ such that

$$U_\delta^{n-\frac{1}{2}} - U_\delta^{n-1} + \mathrm{i}\,\frac{\mathsf{k}_n}{4}\,\Delta_h\left(U_\delta^{n-\frac{1}{2}} + U_\delta^{n-1}\right) = \mathrm{i}\,\frac{\mathsf{k}_n}{4}\,\mathsf{P}_h\left[f\left(|g_\delta(U_\delta^{n-1})|^2\right)\left(U_\delta^{n-\frac{1}{2}} + U_\delta^{n-1}\right)\right] \tag{105}$$

and then we find $U_\delta^n \in \mathsf{S}_h^r$ such that

$$U_\delta^n - U_\delta^{n-1} + \mathrm{i}\,\frac{\mathsf{k}_n}{2}\,\Delta_h\big(U_\delta^n + U_\delta^{n-1}\big) = \mathrm{i}\,\frac{\mathsf{k}_n}{2}\,\mathsf{P}_h\left[f\left(|g_\delta(U_\delta^{n-\frac{1}{2}})|^2\right)(U_\delta^n + U_\delta^{n-1})\right]. \tag{106}$$

Remark 6. The existence and uniqueness of the (MFD) approximations follows, unconditionally, according to Remark 4.

4.3 | Convergence of the fully discrete approximations

Theorem 2. *Let* $\lambda_\star := 1 + 3 \max_{[0,T]} \|u\|_{2,\infty}$, $\tau := \max_{1\le m\le N} \mathsf{k}_m$, $\mathsf{C}_{\mathtt{SV},4}$ *be the constant in (4) for* $\kappa = 4$, $\mathsf{C}_{\mathtt{ER}}^2$ *and* $\mathsf{C}_{\mathtt{ER}}^4$ *be the constant in (5) for* $\kappa = 2, 4$, *respectively,* $\mathsf{C}_{\lambda_\star}^{\mathtt{III}}$ *be the constant in (45) for* $\lambda = \lambda_\star$, *and* $(\mathsf{U}^m)_{m=0}^N$ *and* $(\mathsf{U}^{m-\frac{1}{2}})_{m=1}^N$ *be the finite element approximations defined by (13)–(15). Also, let us assume that* $f \in C^3([0,+\infty),\mathbb{R})$, $u_0 \in \mathbb{H}^4(D)$, $\Delta u_0 \in \mathbb{H}^2(D)$, $u \in C^3([0,T],\mathbb{H}^2(D)) \cap C^2([0,T],\mathbb{H}^4(D))$,

$$\mathsf{C}_{\mathtt{SV},4}\,\mathsf{C}_{\mathtt{ER}}^4\,\mathsf{C}_{\mathtt{ER}}^2\,\mathsf{C}_{\lambda_\star}^{\mathtt{III}}\,[\tau + \mathsf{K}(\mathcal{P})] \le \frac{\lambda_\star}{3}. \tag{107}$$

Then, there exists a constant $h_\star > 0$ *such that:*

(i) if $h \in (0, h_\star]$ *and* $u \in C^3([0,T],\mathbb{H}^2(D)) \cap C^2([0,T],\mathbb{H}^4(D)) \cap C^1([0,T],\mathbb{H}^{r+1}(D))$, *then*

$$\max_{1\le m\le N} \|\mathsf{U}^{m-\frac{1}{2}} - u^{m-\frac{1}{2}}\| + \max_{0\le m\le N} \|\mathsf{U}^m - u^m\| \le C\,(\tau^2 + h^{r+1}); \tag{108}$$

(ii) if $h \in (0, h_\star]$, *then*

$$\max_{1\le m\le N} \|\mathsf{U}^{m-\frac{1}{2}} - u^{m-\frac{1}{2}}\|_1 + \max_{0\le m\le N} \|\mathsf{U}^m - u^m\|_1 \le C\,(\tau^2 + h); \tag{109}$$

(iii) if $h \in (0, h_\star]$, $r \ge 2$, *and there exists a constant* $\mathsf{C}_{\mathtt{MS}} > 0$, *independent of* N *and* $(\mathsf{k}_m)_{m=1}^N$, *such that*

$$\max_{1\le \ell\le N} \mathsf{k}_\ell \le \mathsf{C}_{\mathtt{MS}} \min_{1\le \ell\le N} \mathsf{k}_\ell, \tag{110}$$

then

$$\max_{1\le m\le N} \|\mathsf{U}^{m-\frac{1}{2}} - u^{m-\frac{1}{2}}\|_1 + \max_{0\le m\le N} \|\mathsf{U}^m - u^m\|_1 \le C\,\left(\tau^2 + h^{\min\{3,r\}}\right). \tag{111}$$

Proof. Let $\delta_\star := 1 + 3\,\lambda_\star$, $(U_{\delta_\star}^m)_{m=0}^N$ and $(U_{\delta_\star}^{m-\frac{1}{2}})_{m=1}^N$ be the (MFD) approximations specified by (104)–(106) for $\delta = \delta_\star$, $(\mathrm{Y}_{4,\lambda_\star}^m)_{m=0}^N$ and $(\mathrm{Y}_{4,\lambda_\star}^{m-\frac{1}{2}})_{m=1}^N$ be the (MTD) approximations specified by (39)–(41) for $\lambda = \lambda_\star$, $\theta^{m-\frac{1}{2}} := \mathsf{R}_h(\mathrm{Y}_{4,\lambda_\star}^{m-\frac{1}{2}}) - U_{\delta_\star}^{m-\frac{1}{2}} \in \mathsf{S}_h^r$ and $\Lambda^{m-\frac{1}{2}} := \mathsf{R}_h(u^{m-\frac{1}{2}}) - U_{\delta_\star}^{m-\frac{1}{2}} \in \mathsf{S}_h^r$ for $m = 1, \dots, N$, and $\theta^m := \mathsf{R}_h(\mathrm{Y}_{4,\lambda_\star}^m) - U_{\delta_\star}^m \in \mathsf{S}_h^r$ and $\Lambda^m := \mathsf{R}_h(u^m) - U_{\delta_\star}^m \in \mathsf{S}_h^r$ for $m = 0, \dots, N$. Also, we recall the previously introduced index notation $c_{1,m} := m - 1$, $c_{2,m} := m - \frac{1}{2}$, $\ell_{1,m} := m - \frac{1}{2}$, and $\ell_{2,m} := m$ for $m = 1, \dots, N$.

In the sequel, we will use the symbol C to denote a generic constant that is independent of $(\mathsf{k}_m)_{m=1}^N$, N and h, and may change values from one place to the other. We note that the constant C may depend on the solution u and its derivatives.

`Round I`: Using (4), Remark 1, (45), and (107), we have

$$
\begin{aligned}
\max_{\mu\in\{m-\frac{1}{2},m\}} \|\mathrm{Y}^{\mu}_{4,\lambda_\star}-u^{\mu}\|_{2,\infty} &\leq \mathsf{C}_{\mathtt{SV},4} \max_{\mu\in\{m-\frac{1}{2},m\}} \|\mathrm{Y}^{\mu}_{4,\lambda_\star}-u^{\mu}\|_{4}\\
&\leq \mathsf{C}_{\mathtt{SV},4}\,\mathsf{C}^4_{\mathtt{ER}}\,\mathsf{C}^2_{\mathtt{ER}} \max_{\mu\in\{m-\frac{1}{2},m\}} \|\Delta^2\mathrm{Y}^{\mu}_{4,\lambda_\star}-\Delta^2 u^{\mu}\|\\
&\leq \mathsf{C}_{\mathtt{SV},4}\,\mathsf{C}^4_{\mathtt{ER}}\,\mathsf{C}^2_{\mathtt{ER}}\,\mathsf{C}^{\mathrm{III}}_{\lambda_\star}\,[\tau+\mathsf{K}(\mathcal{P})] \leq \frac{\lambda_\star}{3}
\end{aligned}
\tag{112}
$$

and

$$
\begin{aligned}
\max_{\mu\in\{m-\frac{1}{2},m\}} \|\mathrm{Y}^{\mu}_{4,\lambda_\star}\|_{2,\infty} &\leq \max_{\mu\in\{m-\frac{1}{2},m\}} \|u^{\mu}\|_{2,\infty} + \max_{\mu\in\{m-\frac{1}{2},m\}} \|\mathrm{Y}^{\mu}_{4,\lambda_\star}-u^{\mu}\|_{2,\infty}\\
&\leq \frac{\lambda_\star}{3} + \max_{\mu\in\{m-\frac{1}{2},m\}} \|\mathrm{Y}^{\mu}_{4,\lambda_\star}-u^{\mu}\|_{2,\infty} \leq \frac{2\lambda_\star}{3}
\end{aligned}
\tag{113}
$$

for $m=1,\dots,N$. Observing that $\|\mathrm{Y}^0_{4,\lambda_\star}\|_{2,\infty}=\|u_0\|_{2,\infty}<\lambda_\star$, we use (113) to conclude that

$$
\max\left\{\max_{1\leq m\leq N} \|\mathrm{Y}^{m-\frac{1}{2}}_{4,\lambda_\star}\|_{2,\infty},\ \max_{0\leq m\leq N} \|\mathrm{Y}^{m}_{4,\lambda_\star}\|_{2,\infty}\right\} < \lambda_\star < \delta_\star,
\tag{114}
$$

which, along with (103), yields

$$
g_{\delta_\star}\left(\mathrm{Y}^{m-\frac{1}{2}}_{4,\lambda_\star}\right)=\mathrm{Y}^{m-\frac{1}{2}}_{4,\lambda_\star} \quad\text{and}\quad g_{\delta_\star}\left(\mathrm{Y}^{m-1}_{4,\lambda_\star}\right)=\mathrm{Y}^{m-1}_{4,\lambda_\star}
\tag{115}
$$

for $m=1,\dots,N$. Also, from (112) and (36), we conclude that

$$
\mathsf{M}^{m-\frac{1}{2}}_{2,\infty,\lambda_\star}=\mathrm{Y}^{m-\frac{1}{2}}_{4,\lambda_\star} \quad\text{and}\quad \mathsf{M}^{m-1}_{2,\infty,\lambda_\star}=\mathrm{Y}^{m-1}_{4,\lambda_\star}
\tag{116}
$$

for $m=1,\dots,N$.

`Round II`: In light of (116) and (115), we combine (105) and (106) (with $\delta=\delta_\star$), with (40) and (41) (with $\lambda=\lambda_\star$), respectively, to get

$$
\left(\theta^{\ell\,j,m}-\theta^{m-1},\chi\right)+\mathrm{i}\,2^{j-3}\,\mathsf{k}_m\,(\nabla(\theta^{\ell\,j,m}+\theta^{m-1}),\nabla\chi)=\mathsf{k}_m\sum_{\ell=1}^{4}\left(\mathfrak{C}^{j,m}_{\ell},\chi\right)\quad \forall\,\chi\in\mathsf{S}^r_h
\tag{117}
$$

for $m=1,\dots,N$ and $j=1,2$, where

$$
\begin{aligned}
\mathfrak{C}^{j,m}_1 &:= \mathsf{R}_h\left[\frac{\mathrm{Y}^{\ell\,j,m}_{4,\lambda_\star}-\mathrm{Y}^{m-1}_{4,\lambda_\star}}{\mathsf{k}_m}\right]-\left[\frac{\mathrm{Y}^{\ell\,j,m}_{4,\lambda_\star}-\mathrm{Y}^{m-1}_{4,\lambda_\star}}{\mathsf{k}_m}\right],\\
\mathfrak{C}^{j,m}_2 &:= \mathrm{i}\,2^{j-3}\left[f\left(|g_{\delta_\star}(\mathrm{Y}^{c\,j,m}_{4,\lambda_\star})|^2\right)-f\left(|g_{\delta_\star}(U^{c\,j,m}_{\delta_\star})|^2\right)\right]\left(\mathrm{Y}^{\ell\,j,m}_{4,\lambda_\star}+\mathrm{Y}^{m-1}_{4,\lambda_\star}\right),\\
\mathfrak{C}^{j,m}_3 &:= \mathrm{i}\,2^{j-3}\,f\left(|g_{\delta_\star}(U^{c\,j,m}_{\delta_\star})|^2\right)\left[\left(\mathrm{Y}^{\ell\,j,m}_{4,\lambda_\star}+\mathrm{Y}^{m-1}_{4,\lambda_\star}\right)-\mathsf{R}_h\left(\mathrm{Y}^{\ell\,j,m}_{4,\lambda_\star}+\mathrm{Y}^{m-1}_{4,\lambda_\star}\right)\right],\\
\mathfrak{C}^{j,m}_4 &:= \mathrm{i}\,2^{j-3}\,f\left(|g_{\delta_\star}(U^{c\,j,m}_{\delta_\star})|^2\right)\left(\theta^{\ell\,j,m}+\theta^{m-1}\right).
\end{aligned}
$$

Keep the real parts of (117) after setting $\chi = \theta^{\ell\, j,m} + \theta^{m-1}$, to have

$$\|\theta^{\ell\, j,m}\|^2 - \|\theta^{m-1}\|^2 = \mathsf{k}_m \sum_{\ell=1}^{3} \mathrm{Re}\Big[(\mathfrak{E}_{\ell}^{j,m}, \theta^{\ell\, j,m} + \theta^{m-1})\Big], \quad m = 1, \dots, N, \quad j = 1, 2, \tag{118}$$

where we have used that $\mathrm{Re}\left[(\mathfrak{E}_4^{j,m}, \theta^{\ell\, j,m} + \theta^{m-1})\right] = 0.$

Now, we use (118) and the Cauchy–Schwarz inequality, to obtain

$$\|\theta^{m-\frac{1}{2}}\| \le \|\theta^{m-1}\| + \mathsf{k}_m \sum_{\ell=1}^{3} \|\mathfrak{E}_{\ell}^{1,m}\|, \tag{119}$$

$$\|\theta^{m}\| \le \|\theta^{m-1}\| + \mathsf{k}_m \sum_{\ell=1}^{3} \|\mathfrak{E}_{\ell}^{2,m}\| \tag{120}$$

for $m = 1, \dots, N$

`Round III`: Using (9) (with $s = 2$), Remark 1, Lemma 3, (40), (41), (45), and (107), we have

$$\begin{aligned}
\|\mathfrak{E}_1^{j,m}\| \le& C\, h^2 \left\| \frac{\mathrm{Y}_{4,\lambda_\star}^{\ell\, j,m} - \mathrm{Y}_{4,\lambda_\star}^{m-1}}{\mathsf{k}_m} \right\|_2 \le C\, h^2 \left\| \Delta \left[\frac{\mathrm{Y}_{4,\lambda_\star}^{\ell\, j,m} - \mathrm{Y}_{4,\lambda_\star}^{m-1}}{\mathsf{k}_m} \right] \right\| \\
\le& C\, h^2 \left[\left\| \Delta^2 \left(\mathrm{Y}_{4,\lambda_\star}^{\ell\, j,m} + \mathrm{Y}_{4,\lambda_\star}^{m-1} \right) \right\| + \left\| \Delta \left[f\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \right) \left(\mathrm{Y}_{4,\lambda_\star}^{\ell\, j,m} + \mathrm{Y}_{4,\lambda_\star}^{m-1} \right) \right] \right\| \right] \\
\le& C\, h^2 \left(C + \sum_{\ell=1}^{3} \|\Xi_{\ell}^{j,m}\| \right), \quad m = 1, \dots, N, \quad j = 1, 2,
\end{aligned} \tag{121}$$

where

$$\begin{aligned}
\Xi_1^{j,m} :=& \Delta f\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \right) \left(\mathrm{Y}_{4,\lambda_\star}^{\ell\, j,m} + \mathrm{Y}_{4,\lambda_\star}^{m-1} \right), \quad \Xi_2^{j,m} := f\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \right) \Delta \left(\mathrm{Y}_{4,\lambda_\star}^{\ell\, j,m} + \mathrm{Y}_{4,\lambda_\star}^{m-1} \right), \\
\Xi_3^{j,m} :=& \nabla f\left(|\mathsf{M}_{2,\infty,\lambda}^{c_{j,m}}|^2 \right) \nabla \left(\mathrm{Y}_{4,\lambda_\star}^{\ell\, j,m} + \mathrm{Y}_{4,\lambda_\star}^{m-1} \right).
\end{aligned}$$

Combining, (121), (74), (37), (44), (60), and (43), we arrive at

$$\|\mathfrak{E}_1^{j,m}\| \le C\, h^2, \quad m = 1, \dots, N, \quad j = 1, 2. \tag{122}$$

Using (9), Lemma 3, Remark 1, (45) (with $\lambda = \lambda_\star$), (107), and (103), we have

$$\begin{aligned}
\|\mathfrak{E}_3^{j,m}\| \le& C\, h^{\mathrm{s}} \max_{|x| \in [0, 3\delta_\star]} |f(x^2)|\, \|\mathrm{Y}_{4,\lambda_\star}^{\ell\, j,m} + \mathrm{Y}_{4,\lambda_\star}^{m-1}\|_{\mathrm{s}} \\
\le& C\, h^{\mathrm{s}}\, \|\mathrm{Y}_{4,\lambda_\star}^{\ell\, j,m} + \mathrm{Y}_{4,\lambda_\star}^{m-1}\|_4 \\
\le& C\, h^{\mathrm{s}}\, \|\Delta^2(\mathrm{Y}_{4,\lambda_\star}^{\ell\, j,m}) + \Delta^2(\mathrm{Y}_{4,\lambda_\star}^{m-1})\| \\
\le& C\, h^{\mathrm{s}}, \quad \mathrm{s} = 2, \dots, \min\{4, r+1\}, \quad m = 1, \dots, N, \quad j = 1, 2.
\end{aligned} \tag{123}$$

Also, in light of (114), (103), (9), Remark 1, (45), and (107), we get

$$
\begin{aligned}
\|\mathfrak{E}_2^{j,m}\| &\leq \left(|\mathrm{Y}_{4,\lambda_\star}^{\ell\,j,m}|_\infty + |\mathrm{Y}_{4,\lambda_\star}^{m-1}|_\infty\right) \|f\left(|\mathrm{g}_{\delta_\star}(\mathrm{Y}_{4,\lambda_\star}^{c_{j,m}})|^2\right) - f\left(|\mathrm{g}_{\delta_\star}(U_{\delta_\star}^{c_{j,m}})|^2\right)\| \\
&\leq C\,\|f\left(|\mathrm{g}_{\delta_\star}\left(\mathrm{Y}_{4,\lambda_\star}^{c_{j,m}}\right)|^2\right) - f\left(|\mathrm{g}_{\delta_\star}(U_{\delta_\star}^{c_{j,m}})|^2\right)\| \\
&\leq C \max_{\rho\in[0,1]} \left|f'\left(\rho\,|\mathrm{g}_{\delta_\star}\left(\mathrm{Y}_{4,\lambda_\star}^{c_{j,m}}\right)|^2 + (1-\rho)\,|\mathrm{g}_{\delta_\star}(U_{\delta_\star}^{c_{j,m}})|^2\right|_\infty \|\mathrm{g}_{\delta_\star}\left(\mathrm{Y}_{4,\lambda_\star}^{c_{j,m}}\right) - \mathrm{g}_{\delta_\star}\left(U_{\delta_\star}^{c_{j,m}}\right)\| \\
&\leq C \max_{|x|\in[0,3\delta_\star]} |f'(x^2)|\,\|\mathrm{Y}_{4,\lambda_\star}^{c_{j,m}} - U_{\delta_\star}^{c_{j,m}}\| \\
&\leq C\left(\|\mathrm{Y}_{4,\lambda_\star}^{c_{j,m}} - \mathsf{R}_h(\mathrm{Y}_{4,\lambda_\star}^{c_{j,m}})\| + \|\theta^{c_{j,m}}\|\right) \\
&\leq C\left(h^{\mathsf{s}}\,\|\mathrm{Y}_{4,\lambda_\star}^{c_{j,m}}\|_4 + \|\theta^{c_{j,m}}\|\right) \\
&\leq C\left(h^{\mathsf{s}}\,\|\Delta^2\mathrm{Y}_{4,\lambda_\star}^{c_{j,m}}\| + \|\theta^{c_{j,m}}\|\right) \\
&\leq C\,(h^{\mathsf{s}} + \|\theta^{c_{j,m}}\|), \quad \mathsf{s} = 2,\dots,\min\{4, r+1\}, \quad m = 1,\dots,N, \quad j = 1,2.
\end{aligned}
\tag{124}
$$

Assuming in addition that (110) holds, we can obtain a higher order, with respect to h, estimate of $\mathfrak{E}_1^{j,m}$ by using Lemma 3, (9), Remark 1, (45) (with $\lambda = \lambda_\star$), and Remark 5, as follows:

$$
\begin{aligned}
\|\mathfrak{E}_1^{j,m}\| &\leq C\,h^{\mathsf{s}} \left\|\frac{\mathrm{Y}_{4,\lambda_\star}^{\ell\,j,m} - \mathrm{Y}_{4,\lambda_\star}^{m-1}}{\mathsf{k}_m}\right\|_{\mathsf{s}} \leq C\,h^{\mathsf{s}} \left\|\frac{\mathrm{Y}_{4,\lambda_\star}^{\ell\,j,m} - \mathrm{Y}_{4,\lambda_\star}^{m-1}}{\mathsf{k}_m}\right\|_4 \leq C\,h^{\mathsf{s}} \left\|\Delta^2\left[\frac{\mathrm{Y}_{4,\lambda_\star}^{\ell\,j,m} - \mathrm{Y}_{4,\lambda_\star}^{m-1}}{\mathsf{k}_m}\right]\right\| \\
&\leq C\,\frac{h^{\mathsf{s}}}{\mathsf{k}_m}\left(\left\|\Delta^2(\mathrm{Y}_{4,\lambda_\star}^{\ell\,j,m} - u^{\ell\,j,m})\right\| + \left\|\Delta^2(u^{\ell\,j,m} - u^{m-1})\right\| + \left\|\Delta^2(u^{m-1} - \mathrm{Y}_{4,\lambda_\star}^{m-1})\right\|\right) \\
&\leq C_\star\,h^{\mathsf{s}}\,\frac{\mathsf{k}_m + \tau}{\mathsf{k}_m} \\
&\leq C_\star\,h^{\mathsf{s}}, \quad \mathsf{s} = 2,\dots,\min\{4, r+1\}, \quad m = 1,\dots,N, \quad j = 1,2.
\end{aligned}
\tag{125}
$$

`Round IV`: Using (119), (120), (121), (123), (124), and (125), we obtain

$$\|\theta^{m-\frac{1}{2}}\| \leq (1 + C\,\mathsf{k}_m)\,\|\theta^{m-1}\| + C\,\mathsf{k}_m\,h^\nu, \tag{126}$$

$$\|\theta^m\| \leq \|\theta^{m-1}\| + C\,\mathsf{k}_m\left(\|\theta^{m-\frac{1}{2}}\| + h^\nu\right) \tag{127}$$

for $m = 1,\dots,N$, where $\nu = 2$, or, $\nu = \min\{4, r+1\}$ under the assumption (110). Combining (126) and (127), we obtain

$$\|\theta^m\| \leq (1 + C\,\mathsf{k}_m)\,\|\theta^{m-1}\| + C\,\mathsf{k}_m\,h^\nu, \quad m = 1,\dots,N. \tag{128}$$

In light of $\theta^0 = 0$, we apply a standard Gronwall argument on (128) to get

$$\max_{0\leq m\leq N} \|\theta^m\| \leq C\,h^\nu,$$

which, along with (126), yields

$$\max_{1\le m\le N}\|\theta^{m-\frac{1}{2}}\|\le C\,h^{\nu}.$$

Thus, we conclude that there exists a constant $\mathsf{C}_{\mathtt{A}}>0$, independent of N, h, and $(\mathsf{k}_m)_{m=1}^N$, such that

$$\max_{0\le m\le N}\|\theta^{m}\|+\max_{1\le m\le N}\|\theta^{m-\frac{1}{2}}\|\le \mathsf{C}_{\mathtt{A}}\,h^{2}. \tag{129}$$

Also, for $r\ge 2$, assuming that (110) holds, the error estimate (129) is improved as

$$\max_{0\le m\le N}\|\theta^{m}\|+\max_{1\le m\le N}\|\theta^{m-\frac{1}{2}}\|\le \mathsf{C}_{\mathtt{B}}\,h^{\min\{4,r+1\}}, \tag{130}$$

where $\mathsf{C}_{\mathtt{B}}>0$ is a constant independent of N, h and $(\mathsf{k}_m)_{m=1}^N$.

`Round V:` Let $m\in\{1,\dots,N\}$ and $\mu\in\{m-\frac{1}{2},m,0\}$. Also, we recall that there exists positive constant $\mathsf{C}_{2,\infty}$ such that $\|v\|_2\le \mathsf{C}_{2,\infty}\,\|v\|_{2,\infty}$ for $v\in H^4(D)$. Then, we use (10), (12), (114), and (129), to get

$$\begin{aligned}
|U^{\mu}_{\delta_\star}|_\infty &\le \big|U^{\mu}_{\delta_\star}-\mathsf{R}_h(\mathrm{Y}^{\mu}_{4,\lambda_\star})\big|_\infty+\big|\mathsf{R}_h(\mathrm{Y}^{\mu}_{4,\lambda_\star})-\mathrm{Y}^{\mu}_{4,\lambda_\star}\big|_\infty+\big|\mathrm{Y}^{\mu}_{4,\lambda_\star}\big|_\infty\\
&\le \mathsf{C}_{\mathtt{INV1}}\;h^{-\frac{d}{2}}\,\|U^{\mu}_{\delta_\star}-\mathsf{R}_h(\mathrm{Y}^{\mu}_{4,\lambda_\star})\|+\mathsf{C}_{\mathtt{EP2}}\,h^{2-\frac{d}{2}}\,\|\mathrm{Y}^{\mu}_{4,\lambda_\star}\|_2+\lambda_\star\\
&\le \mathsf{C}_{\mathtt{INV1}}\,\mathsf{C}_{\mathtt{A}}\,h^{2-\frac{d}{2}}+\lambda_\star+\mathsf{C}_{\mathtt{EP2}}\,\mathsf{C}_{2,\infty}\,h^{2-\frac{d}{2}}\,\|\mathrm{Y}^{\mu}_{4,\lambda_\star}\|_{2,\infty}\\
&\le \lambda_\star+\big(\mathsf{C}_{\mathtt{INV1}}\,\mathsf{C}_{\mathtt{A}}+\mathsf{C}_{2,\infty}\,\mathsf{C}_{\mathtt{EP2}}\big)\,h^{2-\frac{d}{2}}\;\lambda_\star.
\end{aligned} \tag{131}$$

Now, from (131), we conclude that there exists $h_\star>0$ such that if $h\in(0,h_\star]$, then

$$\max_{1\le m\le N}\big|U^{m-\frac{1}{2}}_{\delta_\star}\big|_\infty+\max_{0\le m\le N}\big|U^{m}_{\delta_\star}\big|_\infty\le 2\,\lambda_\star<\delta_\star, \tag{132}$$

which, along with (103), yields that

$$g_{\delta_\star}(U^{m-\frac{1}{2}}_{\delta_\star})=U^{m-\frac{1}{2}}_{\delta_\star}\quad\text{and}\quad g_{\delta_\star}(U^{m-1}_{\delta_\star})=U^{m-1}_{\delta_\star},\quad m=1,\dots,N. \tag{133}$$

Thus, if $h\in(0,h_\star]$, in light of (133), (13)–(15), and (104)–(106), we conclude that

$$U^{m-\frac{1}{2}}_{\delta_\star}=\mathsf{U}^{m-\frac{1}{2}}\quad\text{and}\quad U^{m}_{\delta_\star}=\mathsf{U}^{m},\quad m=1,\dots,N. \tag{134}$$

`Round VI:` Let us assume that $u\in C^3([0,T],\mathbb{H}^2(D))\cap C^2([0,T],\mathbb{H}^4(D))\cap C^1([0,T],\mathbb{H}^{r+1}(D))$. Using (105) and (106) (with $\delta=\delta_\star$) along with (18) and (19), we have

$$\Big(\Lambda^{\ell\,j,m}-\Lambda^{m-1},\chi\Big)+\frac{\mathrm{i}\,\mathsf{k}_m}{2^{3-j}}\,(\nabla(\Lambda^{\ell\,j,m}+\Lambda^{m-1}),\nabla\chi)=\sum_{\ell=1}^{5}(\mathfrak{X}^{j,m}_{\ell},\chi)\quad\forall\,\chi\in\mathsf{S}^r_h, \tag{135}$$

for $m = 1, \dots, N$ and $j = 1, 2$, where

$$
\begin{aligned}
\mathfrak{X}_1^{j,m} &:= \mathsf{R}_h\left(u^{\ell_{j,m}} - u^{m-1}\right) - \left(u^{\ell_{j,m}} - u^{m-1}\right),\\
\mathfrak{X}_2^{j,m} &:= \mathrm{i}\,\frac{2^{j-1}\,\mathsf{k}_m}{4}\left[f\left(|u^{c_{j,m}}|^2\right) - f\left(|\mathrm{g}_{\delta_\star}(U_{\delta_\star}^{c_{j,m}})|^2\right)\right](u^{\ell_{j,m}} + u^{m-1}),\\
\mathfrak{X}_3^{j,m} &:= \mathrm{i}\,\frac{2^{j-1}\,\mathsf{k}_m}{4}\, f\left(|\mathrm{g}_{\delta_\star}(U_{\delta_\star}^{c_{j,m}})|^2\right)\left[\left(u^{\ell_{j,m}} + u^{m-1}\right) - \mathsf{R}_h\left(u^{\ell_{j,m}} + u^{m-1}\right)\right],\\
\mathfrak{X}_4^{j,m} &:= \mathrm{i}\,\frac{2^{j-1}\,\mathsf{k}_m}{4}\, f\left(|\mathrm{g}_{\delta_\star}(U_{\delta_\star}^{c_{j,m}})|^2\right)\left(\Lambda^{\ell_{j,m}} + \Lambda^{m-1}\right),\\
\mathfrak{X}_5^{j,m} &:= \mathrm{i}\,\frac{2^{j-1}\,\mathsf{k}_m}{2}\,\eta^{\ell_{j,m}}.
\end{aligned}
$$

Setting $\chi = \Lambda^{\ell_{j,m}} + \Lambda^{m-1}$ in (135), and then taking real parts, we get

$$
\|\Lambda^{m-\frac{1}{2}}\|^2 - \|\Lambda^{m-1}\|^2 = \sum_{\ell=1}^{5} \mathrm{Re}\left[(\mathfrak{X}_\ell^{1,m}, \Lambda^{m-\frac{1}{2}} + \Lambda^{m-1})\right], \tag{136}
$$

$$
\|\Lambda^{m}\|^2 - \|\Lambda^{m-1}\|^2 = \sum_{\ell=1}^{5} \mathrm{Re}\left[(\mathfrak{X}_\ell^{2,m}, \Lambda^{m} + \Lambda^{m-1})\right] \tag{137}
$$

for $m = 1, \dots, N$. First, we observe that

$$
\mathrm{Re}\left[\left(\mathfrak{X}_4^{j,m}, \Lambda^{\ell_{j,m}} + \Lambda^{m-1}\right)\right] = 0, \quad m = 1, \dots, N, \quad j = 1, 2. \tag{138}
$$

Then, we use (9) and (103), to have

$$
\begin{aligned}
\|\mathfrak{X}_1^{j,m}\| &\le C\,h^{r+1}\,\|u^{\ell_{j,m}} - u^{m-1}\|_{r+1}\\
&\le C\,h^{r+1}\,\mathsf{k}_m,
\end{aligned} \tag{139}
$$

$$
\begin{aligned}
\|\mathfrak{X}_3^{j,m}\| &\le C\,\mathsf{k}_m\,h^{r+1} \max_{|x|\in[0,3\delta_\star]} |f(x^2)|\,\|u^{\ell_{j,m}} + u^{m-1}\|_{r+1}\\
&\le C_\star\,\mathsf{k}_m\,h^{r+1},
\end{aligned} \tag{140}
$$

and

$$
\begin{aligned}
\|\mathfrak{X}_2^{j,m}\| &\le C\,\mathsf{k}_m\,\left\|f\left(|u^{c_{j,m}}|^2\right) - f\left(|\mathrm{g}_{\delta_\star}(U_{\delta_\star}^{c_{j,m}})|^2\right)\right\|\\
&\le C\,\mathsf{k}_m \max_{\rho\in[0,1]}\left|f'\left(\rho\,|\mathrm{g}_{\delta_\star}(u^{c_{j,m}})\right|^2 + (1-\rho)\,|\mathrm{g}_{\delta_\star}(U_{\delta_\star}^{c_{j,m}})|^2\Big|_\infty \|\mathrm{g}_{\delta_\star}(u^{c_{j,m}}) - \mathrm{g}_{\delta_\star}\left(U_{\delta_\star}^{c_{j,m}}\right)\|\\
&\le C\,\mathsf{k}_m \max_{|x|\in[0,3\delta_\star]} |f'(x^2)|\,\|u^{c_{j,m}} - U_{\delta_\star}^{c_{j,m}}\|\\
&\le C\,\mathsf{k}_m\left(\|u^{c_{j,m}} - \mathsf{R}_h(u^{c_{j,m}})\| + \|\Lambda^{c_{j,m}}\|\right)\\
&\le C\,\mathsf{k}_m\left(h^{r+1} + \|\Lambda^{c_{j,m}}\|\right)
\end{aligned} \tag{141}
$$

for $m = 1, \dots, N$ and $j = 1, 2$. (We note that to obtain (141), we have used that, in light of $\max_{[0,T]} |u|_\infty < \lambda_\star < \delta_\star$ and (103), we have $g_{\delta_\star}(u^{c_{j,m}}) = u^{c_{j,m}}$.)

Combining (136)–(141), (24), and (25), we obtain

$$\|\Lambda^{m-\frac{1}{2}}\| \leq (1 + C\,\mathsf{k}_m)\,\|\Lambda^{m-1}\| + C\,(\mathsf{k}_m^2 + \mathsf{k}_m\,h^{r+1}), \tag{142}$$

$$\|\Lambda^{m}\| \leq \|\Lambda^{m-1}\| + C\,\mathsf{k}_m\left(\|\Lambda^{m-\frac{1}{2}}\| + h^{r+1} + \mathsf{k}_m^2\right) \tag{143}$$

for $m = 1, \dots, N$. Now, from (142) and (143), it follows that

$$\|\Lambda^{m}\| \leq (1 + C\,\mathsf{k}_m)\,\|\Lambda^{m-1}\| + C\,\mathsf{k}_m\,(h^{r+1} + \mathsf{k}_m^2), \quad m = 1, \dots, N. \tag{144}$$

Applying a discrete Gronwall argument on (144) and using that $\Lambda^0 = 0$, we obtain

$$\max_{0\leq m\leq N} \|\Lambda^{m}\| \leq C\,(h^{r+1} + \tau^2), \tag{145}$$

which, along with (142), yields

$$\max_{1\leq m\leq N} \|\Lambda^{m-\frac{1}{2}}\| \leq C\,(h^{r+1} + \tau^2). \tag{146}$$

Thus, (108) follows, in a standard way, from (145), (146), (9), and (134).

`Round VII`: Let $m \in \{1, \dots, N\}$ and $\mu \in \{m, n - \frac{1}{2}\}$. Using (134), (43), (9), (11), Remark 1, (45), and (107), we obtain

$$\begin{aligned}
\|u^\mu - \mathsf{U}^\mu\|_1 &= \|u^\mu - U^\mu_{\delta_\star}\|_1 \\
&\leq \|u^\mu - \mathrm{Y}^\mu_{4,\lambda_\star}\|_1 + \|\mathrm{Y}^\mu_{4,\lambda_\star} - \mathsf{R}_h \mathrm{Y}^\mu_{4,\lambda_\star}\|_1 + \|\mathsf{R}_h \mathrm{Y}^\mu_{4,\lambda_\star} - U^\mu_{\delta_\star}\|_1 \\
&\leq C\left(\tau^2 + h^{\min\{3,r\}}\,\|\mathrm{Y}^\mu_{4,\lambda_\star}\|_{\min\{4,r+1\}}\right) + \|\theta^\mu\|_1 \\
&\leq C\left(\tau^2 + h^{\min\{3,r\}}\,\|\mathrm{Y}^\mu_{4,\lambda_\star}\|_4 + h^{-1}\,\|\theta^\mu\|\right) \\
&\leq C\left(\tau^2 + h^{\min\{3,r\}}\,\|\Delta^2 \mathrm{Y}^\mu_{4,\lambda_\star}\| + h^{-1}\,\|\theta^\mu\|\right) \\
&\leq C\left(\tau^2 + h^{\min\{3,r\}} + h^{-1}\,\|\theta^\mu\|\right).
\end{aligned} \tag{147}$$

Thus, (109) follows, easily, as an outcome of (147) and (129). When $r \geq 2$ and (110) holds, we obtain (111) by applying (147) and (130). □

Remark 7. We would like to mention that the error estimate (108) is, also, concluded in Ref. [5], by developing a different stability argument requiring a restriction of the size of the time steps, under the assumption that an L^∞ bound for the fully discrete approximations is available without addressing its derivation. Moreover, the H^1 error estimate presented in Ref. [5] is suboptimal and follows by imposing a CFL condition.

Remark 8. Assuming further that there exists a constant C such that $\|\mathsf{P}_h v\|_1 \leq \mathsf{C}\,\|v\|_1$ for $v \in \mathbb{H}^1(D)$ (see, e.g. Ref. [15]), and starting from the error equations (135), we can derive, easily, an optimal-order error estimate in the H^1 norm under condition (107) (cf. Refs. [23, 27]).

5 | CONCLUSIONS

We consider the approximation of the solution to the (NLS) equation by an L^2-conservative, second-order in time, linearly implicit finite element method that constructs approximations at the nodes of a nonuniform partition of the time interval along with their midpoints formulated in Ref. [5]. In its error analysis, we heavily employ *(MTD) approximations* and *(MFD) approximations* (cf. Ref. [46]) as a standby for the efficient treatment of the nonlinear term in order to arrive at an L^∞ bound of the fully discrete approximations (see (132) and (134)). In the light of (16), we derive an optimal, $O(\tau^2 + h^{r+1})$, error estimate in the L^2 norm and an optimal, $O(\tau^2 + h)$, error estimate in the H^1 norm for linear finite elements, without imposing CFL conditions (cf. Refs. [31, 42]). Also, assuming that (17) holds, we conclude an optimal, $O(\tau^2 + h^r)$, error estimate in the H^1 norm for higher order finite elements with $r \in \{2, 3\}$, avoiding again the enforcement of CFL conditions. However, the latter result can be improved by connecting the construction of the (MTD) approximations to a properly chosen higher order Sobolev norm. Future research includes the investigation of the convergence of numerical methods for partial differential equations with more complex nonlinearities.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no data sets were generated or analyzed during this study.

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