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(article starts on next page)

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The geometry of the flow of forces and moments across a shell

Chris J K WILLIAMS* and Emil ADIELS

* Architecture and Civil Engineering
Chalmers tekniska högskola, Göteborg, Sweden
christopher.williams@chalmers.se

Abstract

We describe the flow of forces and moments across a shell starting from the Günther-Schaefer stress functions for a Cosserat continuum in 3 dimensions and Schaefer's concept of a Krustenschale or crust shell. We apply the idea of the flow of forces and moments to a shell in the shape of Enneper's minimal surface.

Keywords: Günther-Schaefer stress functions, equilibrium of shells, Enneper's minimal surface.

1. Introduction

In a paper[1] recently published in the journal *Meccanica* we show how the internal forces and moments in a shell can be described by 4 vectors, each of which has 3 components in 3 dimensional space, making 12 quantities in total. In this paper we derive the same equations, but in a completely different way, starting from the Günther-Schaefer stress functions for a Cosserat continuum and Schaefer's concept of a Krustenschale or crust shell. This may or may not make the derivations easier to understand, but it does link our equations with the Günther-Schaefer stress functions, which is interesting in its own right.

The internal forces and moments in a shell under a given load are controlled by elastic stiffness, together with the effects of prestressing, creep, temperature changes, plastic deformation and any prestressing. Some of these factors can be modified, for example varying the thickness varies the stiffness in different parts of the shell. Thus it is important to be able to understand the equilibrium of shells independent of elastic analysis, such as that developed by Koiter[2] or a finite element analysis.

We show how the 4 vectors can help us understand the equilibrium of a shell carrying a point load in which both internal forces and internal moments are necessary for equilibrium.

2. A 3 dimensional Cosserat continuum

2.1. The equilibrium equations

In this section we give a brief introduction to the equilibrium equations of a 3 dimensional Cosserat continuum[3], which are relevant to shell structures where the material is concentrated into a relatively thin curved layer. A Cosserat continuum contains couple-stresses, that is moments acting per unit area as well as the more usual stresses, that is force per unit area. An example of a 2 dimensional Cosserat continuum is a woven wire mesh (figure 1a) - if we assume that the mesh is sufficiently fine to be considered a continuum. If we consider forces only acting in the $x - y$ plane of the mesh then the Cosserat moments are acting about the z axis perpendicular to the mesh and are clearly important in the behaviour of the mesh in its own plane.

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(a) Woven wire mesh. Photo: Robinson Wire Cloth Ltd.

(b) Half an onion and a shell structure made from a scale extracted from the other half

(c) A shell structure in the form of Enneper's minimal surface

(d) Point load test of a model from a packet of Pringles® using a BIC® Biro or ballpoint pen.

Figure 1: Wire mesh, an onion, a shell and a model test

Let us begin by reproducing equations (5.1) from Carlson's paper on the Günther Cosserat stress functions[4] in 3 dimensional space:

$$t_{ij,i} + b_j = 0 \quad \text{and} \tag{1}$$

$$m_{ij,i} + \epsilon_{jkl} t_{kl} + c_j = 0. \tag{2}$$

(1) is the equation of equilibrium of forces and (2) is the equation of equilibrium of moments. These equations are written in the Cartesian tensor notation[5], and for who are unfamiliar with this notation, let us rewrite the 1st of the 3 equations (1) for $j = 1$ in engineering notation as used in Timoshenko & Goodier[6]:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + b_x = 0. \tag{3}$$

b_x, is the applied body loads per unit volume in the x direction. The most common body load is $b_z = -\rho g$.

t_{11}, that is $i = 1$ and $j = 1$, in t_{ij} in (1) is σ_x in (3) and $t_{11,1}$ is $\frac{\partial \sigma_x}{\partial x}$. Similarly t_{21} is τ_{yx} and $t_{21,2}$ is $\frac{\partial \tau_{yx}}{\partial y}$ and so on. The comma indicates partial differentiation. The repeated i in $t_{ij,i}$ is interpreted as $\sum_{i=1}^{3} t_{ij,i}$ by the Einstein summation convention. Thus we can express all the information in (3), together with the equations in the y and z directions much more briefly as (1), and that is the whole point of the Cartesian tensor notation.

Similarly we can write the 3rd of the 3 equations of equilibrium of moments (2) by setting $j = 3$,

$$\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + \frac{\partial m_{zz}}{\partial z} + \tau_{xy} - \tau_{yx} + c_z = 0. \tag{4}$$

c_z are the applied body couples per unit volume about the z axis and m_{xz} and m_{yz} are the components of Cosserat moment[7]. m_{xz} is the moment per unit area on the face of a little cube whose normal points in the x direction acting about the z axis. Note that in plate bending the sign convention is normally that m_{xx} and not m_{xy} is the moment acting about the y axis.

Equation (4) tells us that in general $\tau_{xy} \neq \tau_{yx}$ so that complementary shear stresses are not equal. Mindlin[3] derives equation (4) using readily understood reasoning and simple diagrams in 2 dimensions. He uses μ_x instead of m_{xz} and μ_y instead of m_{yz}.

The ϵ_{jkl} in (2) are the components of the permutation tensor or Levi-Civita tensor: $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{321} = \epsilon_{132} = \epsilon_{213} = -1$ and all other $\epsilon_{jkl} = 0$. We shall explain in due course what we mean by a tensor. By the summation convention $\epsilon_{jkl} t_{kl} = \sum_{k=1}^{3} \sum_{l=1}^{3} \epsilon_{jkl} t_{kl}$ in (2) and this produces the $\tau_{xy} - \tau_{yx}$ in (4) because $j = 3$ in that equation and so we only get contributions from $k = 1$ and $l = 2$ and from $k = 2$ and $l = 1$.

2.2. The Günther-Schaefer Cosserat stress functions

In the *Note Added in Proof* at the end of Carlson's paper on the Günther Cosserat stress functions[4] he refers to equations given in a lecture given by Professor Schaefer in September 1965 in Augustów, Poland, which we repeat here:

$$t_{ij} = \epsilon_{ipq} F_{qj,p} + f_{j,i} \quad \text{and} \tag{5}$$

$$m_{ij} = \epsilon_{ipq} G_{qj,p} + \delta_{ij} F_{pp} - F_{ji} + \epsilon_{ijp} f_p + g_{j,i} \tag{6}$$

and we shall see that these equations automatically satisfy (1) and (2), provided that

$$\nabla^2 f_i = -b_i \quad \text{and} \tag{7}$$

$$\nabla^2 g_i = -c_i. \tag{8}$$

The Kronecker delta, $\delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases}$ and the Laplacian, $\nabla^2 = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \right)$.

If we substitute (5) into (1) we obtain $\left(\epsilon_{ipq} F_{qj,p} + f_{j,i}\right)_{,i} + b_j = \epsilon_{ipq} F_{qj,ip} + f_{j,ii} + b_j = \nabla^2 f_j + b_j = 0$ which is (7). The ϵ_{ipq} are constant in Cartesian coordinates and $F_{qj,ip}$ is symmetric in i and p, while ϵ_{ipq} is antisymmetric, so that $\epsilon_{ipq} F_{qj,ip} = 0$.

If we substitute (5) and (6) into (2) we obtain
$\left(\epsilon_{ipq} G_{qj,p} + \delta_{ij} F_{pp} - F_{ji} + \epsilon_{ijp} f_p + g_{j,i}\right)_{,i} + \epsilon_{jkl} \left(\epsilon_{kpq} F_{ql,p} + f_{l,k}\right) + c_j = g_{j,ii} + c_j = 0$ which is (8). Note the use of the summed k in writing $\epsilon_{jkl} \epsilon_{kpq} F_{ql,p} = F_{jl,l} - F_{ll,j}$.

Let us now count up equations and unknowns. If we know b_i and c_i we can in principal solve (7) and (8) for f_i and g_i. The second order tensors with components t_{ij} and m_{ij} each have 9 components in 3 dimensions as do the second order tensors with components F_{ij} and G_{ij}. Thus we still have 18 unknowns, even though we have automatically satisfied the 3 equations of equilibrium of forces and the 3 equations of equilibrium of moments.

If $b_j = 0$, $c_j = 0$ and $m_{ij} = 0$ for the unloaded non-Cosserat case then we have $f_j = 0$ and $g_j = 0$ so that (6) becomes $\epsilon_{ipq} G_{qj,p} + \delta_{ij} F_{pp} - F_{ji} = 0$. However, then $\epsilon_{rpq} G_{qr,p} + \delta_{rr} F_{pp} - F_{rr} = \epsilon_{rpq} G_{qr,p} + 2F_{pp} = 0$ since $\delta_{rr} = 3$. Therefore $F_{ji} = \epsilon_{ipq} G_{qj,p} - \delta_{ij} \epsilon_{rpq} G_{qr,p}$ or $F_{qj} = \epsilon_{jab} G_{bq,a} - \delta_{jq} \epsilon_{rab} G_{br,a}$ and (5) becomes $t_{ij} = \epsilon_{ipq} \epsilon_{jab} G_{bq,pa} - \epsilon_{ipj} \epsilon_{rab} G_{br,pa}$.

We know that $t_{ij} = t_{ji}$ in the non-Cosserat case and therefore for a non-Cosserat material we need to set $G_{br} = G_{rb}$ so that $t_{ij} = \epsilon_{ipq} \epsilon_{jab} G_{bq,pa}$ and G_{bq} reduce to the Beltrami stress functions.

2.3. Tensors

A tensor is physical object. Scalars are 0^{th} order tensors, vectors are 1^{st} order tensors and quantities such as stress are 2^{nd} order tensors. In 3 dimensions a tensor of order n has 3^n components. The stress tensor can be written $\mathbf{t} = \sigma_x \mathbf{ii} + \tau_{xy} \mathbf{ij} + \tau_{xz} \mathbf{ik} + \tau_{yx} \mathbf{ji} + \sigma_y \mathbf{jj} + \tau_{xz} \mathbf{jk} + \tau_{zx} \mathbf{ki} + \tau_{yz} \mathbf{kj} + \sigma_z \mathbf{kk} = t_{ij} \mathbf{e}_i \mathbf{e}_j$ where $\mathbf{i} = \mathbf{e}_1$, $\mathbf{j} = \mathbf{e}_2$ and $\mathbf{k} = \mathbf{e}_3$ are the Cartesian base vectors.

When we write something like $\mathbf{ij}$ we mean the tensor product of the two vectors $\mathbf{i}$ and $\mathbf{j}$. In general the tensor product $\mathbf{uv}$ of the vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by $\mathbf{uv} \cdot \mathbf{w} = (\mathbf{uv}) \cdot \mathbf{w} = \mathbf{u} (\mathbf{v} \cdot \mathbf{w})$ and $\mathbf{w} \cdot \mathbf{uv} = \mathbf{w} \cdot (\mathbf{uv}) = (\mathbf{w} \cdot \mathbf{u}) \mathbf{v}$ where $\mathbf{w}$ is any vector and the $\cdot$ represents the inner or scalar product. Thus $\mathbf{uv} \cdot \mathbf{w}$ is a vector in the direction of $\mathbf{u}$, but with a magnitude $(\mathbf{v} \cdot \mathbf{w})$ times that of $\mathbf{u}$. The tensor product is sometimes written $\mathbf{u} \otimes \mathbf{v}$ and is also called the dyadic product or the outer product. By adding repeated tensor products of vectors we can produce tensors of any order.

Let us define a small area within a continuum by the vector $\mathbf{A}$. The magnitude of the area is $|\mathbf{A}|$ and $\mathbf{A}$ is normal to the area. The force crossing $\mathbf{A}$ due to stress is $\mathbf{P} = A_x \left(\sigma_x \mathbf{i} + \tau_{xy} \mathbf{j} + \tau_{xz} \mathbf{k}\right) + A_y \left(\tau_{yx} \mathbf{i} + \sigma_y \mathbf{j} + \tau_{xz} \mathbf{k}\right) + A_z \left(\tau_{zx} \mathbf{i} + \tau_{yz} \mathbf{j} + \sigma_z \mathbf{k}\right) = (A_k \mathbf{e}_k) \cdot (t_{ij} \mathbf{e}_i \mathbf{e}_j) = A_i t_{ij} \mathbf{e}_j$ or simply $\mathbf{P} =$

$\mathbf{A} \cdot \mathbf{t}$ which is the best way to write it because the relationship applies in the absence of any imagined coordinate system. In the presence of Cosserat moments $\mathbf{t}$ will not be symmetric, so that in general $\mathbf{A} \cdot \mathbf{t} \neq \mathbf{t} \cdot \mathbf{A}$.

2.4. Curvilinear coordinates

It is often convenient to use curvilinear coordinates, and in structural mechanics we usually convect the coordinates with the material, so that even if the coordinates start off rectilinear, they will become curvilinear as the material deforms. The exception to this rule is when the displacements can be considered 'small'.

Let us suppose that we have three curvilinear coordinates, θ^i where $i = 1, i = 2$ or $i = 3$. We will explain shortly why they have superscripts, and θ^i does *not* mean θ raised to the power i[8, 9]. A point in space is defined by the intersection of the 3 surfaces $\theta^1 =$ constant, $\theta^2 =$ constant and $\theta^3 =$ constant. In the case of spherical polar coordinates $\theta^1 =$ constant is a plane, $\theta^2 =$ constant is a cone and $\theta^3 =$ constant is a sphere. In this special case the surfaces intersect at right angles, but in general this will not be the case.

We need some base vectors to resolve forces and so on and here we have two choices, firstly what are called the covariant base vectors, $\mathbf{g}_i = \dfrac{\partial x}{\partial \theta^i}\mathbf{i} + \dfrac{\partial y}{\partial \theta^i}\mathbf{j} + \dfrac{\partial z}{\partial \theta^i}\mathbf{k}$ and secondly what are called the contravariant base vectors, $\mathbf{g}^i = \dfrac{\partial \theta^i}{\partial x}\mathbf{i} + \dfrac{\partial \theta^i}{\partial y}\mathbf{j} + \dfrac{\partial \theta^i}{\partial z}\mathbf{k}$. This shows why we use both subscripts and superscripts.

$\mathbf{g}_3$ is tangent to the curve of intersection of the surfaces $\theta^1 =$ constant and $\theta^2 =$ constant and $\mathbf{g}^3$ is normal to the surface $\theta^3 =$ constant. Thus unless the surfaces intersect at right angles $\mathbf{g}_i$ and $\mathbf{g}^i$ will be in different directions. In addition $\mathbf{g}_i$ and $\mathbf{g}^i$ will in general not be unit vectors.

One would have thought that the sensible thing to do would be decide to use only the $\mathbf{g}_i$ or only the $\mathbf{g}^i$, but in fact it makes sense to use both because we find all sorts of useful properties which follow from $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases}$. Thus, if $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ is any vector in which $v_x = \mathbf{v} \cdot \mathbf{i}$ etc.,

$$\mathbf{v} = v^i\mathbf{g}_i = v_i\mathbf{g}^i, \quad v^i = \mathbf{v} \cdot \mathbf{g}^i = g^{ij}v_j, \quad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j, \quad v_i = \mathbf{v} \cdot \mathbf{g}_i = g_{ij}v^j \quad \text{and} \quad g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j. \tag{9}$$

g_{ij} are the called the components of the metric tensor because the square of the distance between two adjacent points separated by $d\theta^i$ is $g_{ij}d\theta^i d\theta^j$. Note that the summation convention now *always* involves one subscript and one superscript, which it why Cartesian tensor equations look plain wrong when one is used to curvilinear coordinates.

The scalar product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u} \cdot \mathbf{v} = u_i v^i = u^i v_i = g_{ij}u^i v^i = g^{ij}u_i v_i$ and the vector product of two vectors, $\mathbf{w} = w^k\mathbf{g}_k = w_k\mathbf{g}^k = \mathbf{u} \times \mathbf{v} = u_i v_j \chi^{ijk}\mathbf{g}_k = u^i v^j \chi_{ijk}\mathbf{g}^k = (\mathbf{uv}) : \boldsymbol{\chi} = \mathbf{v} \cdot \boldsymbol{\chi} \cdot \mathbf{u}$ - note the use of the double dot : notation. $\boldsymbol{\chi}$ is the third order permutation tensor or Levi-Civita tensor and we have used the symbol $\boldsymbol{\chi}$, rather than the more common $\boldsymbol{\epsilon}$ because we want to us $\boldsymbol{\epsilon}$ for the second order permutation tensor on a surface. We have $\chi_{ijk} = -\chi_{jik} = -\chi_{ikj} = (\mathbf{g}_i \times \mathbf{g}_j) \cdot \mathbf{g}_k$ and $\chi^{ijk} = -\chi^{jik} = -\chi^{ikj} = \left(\mathbf{g}^i \times \mathbf{g}^j\right) \cdot \mathbf{g}^k$ where $\chi^{123} = \sqrt{g}$, $\chi_{123} = 1/\sqrt{g}$ and $g = \det g_{ij}$. In Cartesian coordinates, $\boldsymbol{\chi} = \mathbf{ijk} + \mathbf{jki} + \mathbf{kij} - \mathbf{kji} - \mathbf{jik} - \mathbf{ikj}$.

We can also differentiate vectors, which means differentiating both their components and the base vectors, $\mathbf{v}_{,i} = \left(v^j\mathbf{g}_j\right)_{,i} = v^j_{,i}\mathbf{g}_j + v^j\mathbf{g}_{j,i} = \nabla_i v^j\mathbf{g}_j = \left(v_j\mathbf{g}^j\right)_{,i} = v_{j,i}\mathbf{g}^j + v_j\mathbf{g}^j_{,i} = \nabla_i v_j\mathbf{g}^j$ where the covariant derivatives, $\nabla_i v^j = v^j_{,i} + v^k\Gamma^j_{ki}$ and $\nabla_i v_j = v_{j,i} - v_k\Gamma^k_{ij}$ in which $\Gamma^k_{ij} = \mathbf{g}^k \cdot \mathbf{g}_{i,j}$ are the Christoffel symbols. The covariant derivatives $\nabla_i v^j$ and $\nabla_i v_j$ are sometimes written $v^j_{:i}$ and $v_{j:i}$ or $v^j|_i$ and $v_j|_i$.

We can extend this process to differentiate higher order tensors, introducing more Christoffel symbols. $\nabla_i v^j$ and $\nabla_i v_j$ are the components of a second order tensor, known as the gradient of $\mathbf{v}$, which is equal to $\nabla \mathbf{v} = \mathbf{g}^i \mathbf{v}_{,i} = \nabla_i v^j \mathbf{g}^i \mathbf{g}_j = \nabla_i v_j \mathbf{g}^i \mathbf{g}^j$. Neither $v^j_{,i}$ and $v_{j,i}$ nor Γ^k_{ij} are the components of tensors because they do not represent anything with physical meaning independent of the coordinate system, which can be seen by examining their behaviour when the coordinate system changes. ∇ is pronounced nabla or del. Wilson's book based on J. Willard Gibbs' lectures[10] and published in 1901 gives the following as equation (21) of Chapter III: $\nabla V = \mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial y} + \mathbf{k}\frac{\partial V}{\partial z}$ for the gradient of the scalar V, which is exactly equivalent to our definition, except we have applied the operator ∇ to a vector.

We can define the gradient of a tensor of any order, $\mathbf{Q}$, independent of a coordinate system by writing $\delta\mathbf{Q} = \delta\theta^i \mathbf{Q}_{,i} = \delta\mathbf{r} \cdot \mathbf{g}^i \mathbf{Q}_{,i} = \delta\mathbf{r} \cdot \nabla\mathbf{Q}$ where $\delta\mathbf{Q}$ is the change in $\mathbf{Q}$ in moving from the point $\mathbf{r}$ in space to the adjacent point $\mathbf{r} + \delta\mathbf{r}$. $\nabla\mathbf{Q}$ is a tensor of order 1 higher than that of $\mathbf{Q}$. We can form the gradient of a scalar (or zeroth order tensor) in exactly the same way.

The divergence, $\nabla \cdot \mathbf{Q} = \mathbf{g}^i \cdot \mathbf{Q}_{,i}$ is a tensor of order one lower than $\mathbf{Q}$, and so in this case $\mathbf{Q}$ cannot be a scalar.

3. The equilibrium equations and the Günther-Schaefer Cosserat stress functions in curvilinear coordinates and in coordinate-free notation

We are now in a position to write equations (5.1) from Carlson's paper on the Günther Cosserat stress functions[4], which we reproduced as (1) and (2), in curvilinear coordinates

$$\nabla_i t^{ij} + B^j = 0 \quad \text{and} \tag{10}$$

$$\nabla_i m^i_{\cdot j} + \chi_{jkl} t^{kl} + C_j = 0. \tag{11}$$

We have a certain freedom to move indices up or down, so that $m^i_{\cdot j} = g^{ik} m_{kj} = g_{jk} m^{ik}$. The dot in $m^i_{\cdot j}$ is there to show that $\mathbf{m} = m^i_{\cdot j} \mathbf{g}_i \mathbf{g}^j$. We have used B^j instead of b^j and C^j instead of c^j because we want to reserve the characters b and c for other uses.

We can write the same equations with no coordinates,

$$\nabla \cdot \mathbf{t} + \mathbf{B} = 0 \quad \text{and} \tag{12}$$

$$\nabla \cdot \mathbf{m} + \boldsymbol{\chi} : \mathbf{t} + \mathbf{C} = 0 \tag{13}$$

and we can see that (10) implies (12) and (11) implies (13) and vice versa.

In the same way we can write the Günther-Schaefer Cosserat stress functions, our equations (5) and (6), as $t^{ij} = \chi^{ipq} \nabla_p F^{\cdot j}_q + \nabla^i f_j$ and $m^i_{\cdot j} = \chi^{ipq} \nabla_p G^{\cdot j}_q + \delta^j_i F^{\cdot p}_p - F^{\cdot i}_j + \chi^{ikp} g_{kj} f_p + \nabla^i h_j$ in which $\nabla^i f_j = \nabla_k g^{ik} f_j$ and we have replaced the g_j in (6) by h_j because the letter g is already in use in (9).

Again we can write these equations without coordinates,

$$\mathbf{t} = \boldsymbol{\chi} : \nabla\mathbf{F} + \nabla\mathbf{f} \quad \text{and} \tag{14}$$

$$\mathbf{m} = \boldsymbol{\chi} : \nabla\mathbf{G} + \mathbf{I}\,\mathrm{tr}\,\mathbf{F} - \mathbf{F}^{\mathrm{T}} + \boldsymbol{\chi} \cdot \mathbf{f} + \nabla\mathbf{h} \tag{15}$$

in which $\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} = \mathbf{g}^i \mathbf{g}_i = g^{ij} \mathbf{g}_i \mathbf{g}_j$ is the unit tensor, $\mathbf{F}^{\mathrm{T}}$ is the transpose of $\mathbf{F}$ and $\mathrm{tr}\,\mathbf{F} = F^{\cdot p}_p$ is the trace of $\mathbf{F}$.

Finally, the equilibrium equations look similar in both Cartesian and curvilinear tensor notation, $\nabla^2 f_i = -B_i$ and $\nabla^2 h_i = -C_i$ and with no coordinates, $\nabla^2 \mathbf{f} = -\mathbf{B}$ and $\nabla^2 \mathbf{h} = -\mathbf{C}$.

4. An onion analogy for shells

Thus far we have only considered the equilibrium of a 3 dimensional continuum containing Cosserat moments. Figure 1b shows half an onion and a shell structure made from an onion scale extracted from the other half and we can think of the scale as part of continuum, provided that $\mathbf{n} \cdot \mathbf{t} = 0$ and $\mathbf{n} \cdot \mathbf{m} = 0$

so that there are no forces and moments from the adjoining scales. $\mathbf{n}$ is the unit normal to the scale. The scale or shell is still loaded by the body loads $\mathbf{B}$ and body couples $\mathbf{C}$. The onion is only introduced as a way in which we can envisage a shell as part of an imaginary 3 dimensional continuum.

We can write the unit normal as $\mathbf{n} = \nabla\eta/\sqrt{\nabla\eta\cdot\nabla\eta}$ where η is a scalar which is constant on any one onion scale. If we assume a unit change in η across a scale then the thickness of a scale is equal to $1/\sqrt{\nabla\eta\cdot\nabla\eta}$, which may vary.

The force per unit width and the moment per unit width in a scale are

$$\boldsymbol{\sigma} = \frac{\mathbf{t}}{\sqrt{\nabla\eta\cdot\nabla\eta}} \quad \text{and} \tag{16}$$

$$\boldsymbol{\mu} = \frac{\mathbf{m}}{\sqrt{\nabla\eta\cdot\nabla\eta}} \tag{17}$$

respectively and these equations form the definitions of $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$. Note that we are effectively assuming that $\mathbf{t}$ and $\mathbf{m}$ are constant though the thickness of the scale or shell. We did not make this assumption in our *Meccanica* paper[1], nor did we assume that the shell was thin, or that it did not contain ribs or voids.

If we imagine that the shell is infinitely thin, then we have to imagine that $\sqrt{\nabla\eta\cdot\nabla\eta}$, $\mathbf{t}$ and $\mathbf{m}$ tend to infinity, corresponding to Schaefer's[11, 12] concept of a Krustenschale (crust shell) and therefore to discontinuities across the shell in the Günther-Schaefer stress functions[13, 14].

The Günther-Schaefer stress function $\mathbf{F}$ is a second order tensor which we will now write as

$$\mathbf{F} = \nabla\eta\boldsymbol{\psi}. \tag{18}$$

This did not appear in Schaefer's paper[11, 12] and it is the most significant contribution of the current paper.

The corresponding terms in (15) are

$$\mathbf{I}\,\mathrm{tr}\,\mathbf{F} - \mathbf{F}^{\mathrm{T}} = \sqrt{\nabla\eta\cdot\nabla\eta}\;\boldsymbol{\psi}\cdot(\mathbf{n}\mathbf{I} - \mathbf{I}\mathbf{n}) \tag{19}$$

and in (14) are

$$\boldsymbol{\chi} : (\nabla\eta\boldsymbol{\psi}) = \boldsymbol{\chi} : (\nabla\nabla\eta\boldsymbol{\psi} + \nabla\boldsymbol{\psi}\nabla\eta) = \sqrt{\nabla\eta\cdot\nabla\eta}\;\boldsymbol{\epsilon}\cdot\nabla\boldsymbol{\psi}. \tag{20}$$

Note that $\boldsymbol{\chi} : \nabla\nabla\eta = 0$, even though we are dealing with a curved surface. Usually for a curved surface the order of covariant differentiation is important, and the reason for that is that the contribution of the curvature of the surface is ignored. However we are actually forming the covariant derivative in flat 3 dimensional Euclidean space and so order of covariant differentiation is immaterial. $\boldsymbol{\chi} : \nabla\nabla\eta = 0$ leads to the Peterson-Mainardi-Codazzi equations, $\boldsymbol{\epsilon} : \nabla\mathbf{b} = 0$ where $\mathbf{b} = -(\mathbf{I} - \mathbf{n}\mathbf{n})\cdot\nabla\mathbf{n}$ is the symmetric normal curvature tensor, or shape operator, whose covariant components $\mathbf{b}$ are known as the coefficients of the 2$^{\text{nd}}$ fundamental form[15, 16]. $\boldsymbol{\epsilon} : \nabla\mathbf{b} = 0$ is in effect only 2 equations since the normal component is satisfied identically.

Corresponding to (18), we can also write

$$\mathbf{G} = \nabla\eta\boldsymbol{\phi}. \tag{21}$$

Let us now introduce the notation $\overline{\nabla}\mathbf{Q} = (\mathbf{I} - \mathbf{n}\mathbf{n})\cdot\nabla\mathbf{Q}$ which is the gradient of $\mathbf{Q}$ in the directions tangential to the surface. Thus $\mathbf{b} = -\overline{\nabla}\mathbf{n}$ and the mean curvature,

$$H = \frac{1}{2}\,\mathrm{tr}\,\mathbf{b} = -\frac{1}{2}(\mathbf{I} - \mathbf{n}\mathbf{n}) : \nabla\mathbf{n} = -\frac{1}{2}\nabla\cdot\mathbf{n} = -\frac{1}{2}\overline{\nabla}\cdot\mathbf{n}. \tag{22}$$

The reason that we can use $\nabla\cdot\mathbf{n}$ or $\overline{\nabla}\cdot\mathbf{n}$ in this special case is that $\nabla\mathbf{n}\cdot\mathbf{n} = 0$ because $\mathbf{n}$ is a unit vector.

Now let us consider the vector $\mathbf{p}$, which we shall see is actually the load per unit area on the shell. In order to define $\mathbf{p}$ we need to introduce another another, $\boldsymbol{\beta}$ to give

$$\begin{aligned} -\mathbf{p} &= \sqrt{\nabla\eta\cdot\nabla\eta}\;\overline{\nabla}\cdot\overline{\nabla}\boldsymbol{\beta} = \sqrt{\nabla\eta\cdot\nabla\eta}\;\overline{\nabla}^2\boldsymbol{\beta} = \sqrt{\nabla\eta\cdot\nabla\eta}\left(\mathbf{g}^i - \left(\mathbf{n}\cdot\mathbf{g}^i\right)\mathbf{n}\right)\cdot\left((\mathbf{I} - \mathbf{n}\mathbf{n})\cdot\nabla\boldsymbol{\beta}\right)_{,i} \\ &= \sqrt{\nabla\eta\cdot\nabla\eta}\left(\nabla\cdot\left((\mathbf{I} - \mathbf{n}\mathbf{n})\cdot\nabla\boldsymbol{\beta}\right) + \mathbf{n}\cdot\nabla\mathbf{n}\cdot(\mathbf{I} - \mathbf{n}\mathbf{n})\cdot\nabla\boldsymbol{\beta}\right). \end{aligned}$$

However,
$$\mathbf{n}\cdot\nabla\mathbf{n}\cdot(\mathbf{I}-\mathbf{nn}) = \frac{1}{\nabla\eta\cdot\nabla\eta}\left(\nabla\eta\cdot\nabla\nabla\eta - \frac{(\nabla\eta\cdot\nabla\nabla\eta\cdot\nabla\eta)\,\nabla\eta}{\nabla\eta\cdot\nabla\eta}\right)\cdot(\mathbf{I}-\mathbf{nn}) \qquad (23)$$
$$= \frac{\nabla\nabla\eta\cdot\nabla\eta\cdot(\mathbf{I}-\mathbf{nn})}{\nabla\eta\cdot\nabla\eta} = \frac{\nabla\left(\sqrt{\nabla\eta\cdot\nabla\eta}\right)\cdot(\mathbf{I}-\mathbf{nn})}{\sqrt{\nabla\eta\cdot\nabla\eta}}$$

and so
$$-\mathbf{p} = \sqrt{\nabla\eta\cdot\nabla\eta}\,\overline{\nabla}^2\boldsymbol{\beta} = \nabla\cdot\left(\sqrt{\nabla\eta\cdot\nabla\eta}\,(\mathbf{I}-\mathbf{nn})\cdot\nabla\boldsymbol{\beta}\right) = \nabla\cdot\left(\sqrt{\nabla\eta\cdot\nabla\eta}\,\overline{\nabla}\boldsymbol{\beta}\right). \qquad (24)$$

Now in order to use the equation of equilibrium of moments, (13) we need

$$\begin{aligned}
&\nabla\cdot\left(\sqrt{\nabla\eta\cdot\nabla\eta}\,\boldsymbol{\beta}\cdot(\mathbf{n}\boldsymbol{\epsilon}+\boldsymbol{\epsilon}\mathbf{n})\right) = \nabla\cdot\left(\sqrt{\nabla\eta\cdot\nabla\eta}\,(\mathbf{I}-\mathbf{nn})\cdot(\boldsymbol{\beta}\cdot\boldsymbol{\chi})\right)\\
&= \nabla\left(\sqrt{\nabla\eta\cdot\nabla\eta}\right)\cdot(\mathbf{I}-\mathbf{nn})\cdot(\boldsymbol{\beta}\cdot\boldsymbol{\chi})\\
&+ \sqrt{\nabla\eta\cdot\nabla\eta}\mathbf{g}^i\cdot\left((\mathbf{I}-\mathbf{nn})\cdot\left(\boldsymbol{\beta}_{,i}\cdot\boldsymbol{\chi}\right) - (\mathbf{n}_{,i}\mathbf{n}+\mathbf{n}\mathbf{n}_{,i})\cdot(\boldsymbol{\beta}\cdot\boldsymbol{\chi})\right)\\
&= \sqrt{\nabla\eta\cdot\nabla\eta}\mathbf{n}\cdot\nabla\mathbf{n}\cdot(\mathbf{I}-\mathbf{nn})\cdot(\boldsymbol{\beta}\cdot\boldsymbol{\chi})\\
&+ \sqrt{\nabla\eta\cdot\nabla\eta}\left(\mathbf{g}^i\cdot(\mathbf{I}-\mathbf{nn})\cdot((\mathbf{g}_i\cdot\nabla\boldsymbol{\beta})\cdot\boldsymbol{\chi}) - (\nabla\cdot\mathbf{nn}+\mathbf{n}\cdot\nabla\mathbf{n})\cdot(\boldsymbol{\beta}\cdot\boldsymbol{\chi})\right)\\
&= \sqrt{\nabla\eta\cdot\nabla\eta}\left(-\left((\mathbf{I}-\mathbf{nn})\cdot\mathbf{g}^i\mathbf{g}_i\cdot\nabla\boldsymbol{\beta}\right):\boldsymbol{\chi} + 2H\mathbf{n}\cdot(\boldsymbol{\chi}\cdot\boldsymbol{\beta})\right)\\
&= -\boldsymbol{\chi}:\left(\sqrt{\nabla\eta\cdot\nabla\eta}\,\overline{\nabla}\boldsymbol{\beta}\right) + 2\sqrt{\nabla\eta\cdot\nabla\eta}H\boldsymbol{\epsilon}\cdot\boldsymbol{\beta}
\end{aligned} \qquad (25)$$

where we have again used (23) as well as (22).

Corresponding to (18) and (21) we can write the Günther-Schaefer stress function $\mathbf{f} = \sqrt{\nabla\eta\cdot\nabla\eta}\,\boldsymbol{\beta}\cdot(\mathbf{n}\boldsymbol{\epsilon}+\boldsymbol{\epsilon}\mathbf{n})$ and introducing the vector $\boldsymbol{\alpha}$ we can write $\mathbf{h} = \sqrt{\nabla\eta\cdot\nabla\eta}\,\boldsymbol{\alpha}\cdot(\mathbf{n}\boldsymbol{\epsilon}+\boldsymbol{\epsilon}\mathbf{n})$.
We are now in a position to use (16) and (14) to write the internal forces in a shell as

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\epsilon}\cdot\overline{\nabla}\boldsymbol{\psi} + \overline{\nabla}\boldsymbol{\beta}} \qquad (26)$$

and to use (17) and (15) to write the internal moments as

$$\boxed{\boldsymbol{\mu} = \boldsymbol{\epsilon}\cdot\overline{\nabla}\boldsymbol{\phi} + \overline{\nabla}\boldsymbol{\alpha} + \boldsymbol{\psi}\cdot(\mathbf{nI}-\mathbf{In}) + \boldsymbol{\beta}\cdot(\mathbf{n}\boldsymbol{\epsilon}+\boldsymbol{\epsilon}\mathbf{n}).} \qquad (27)$$

However, this was not a straightforward substitution and we have to justify (26) and (27), by examining the corresponding stresses (14) $\mathbf{t} = \sqrt{\nabla\eta\cdot\nabla\eta}\left(\boldsymbol{\epsilon}\cdot\overline{\nabla}\boldsymbol{\psi}+\overline{\nabla}\boldsymbol{\beta}\right) = \boldsymbol{\chi}:\nabla\mathbf{F} + \sqrt{\nabla\eta\cdot\nabla\eta}\,\overline{\nabla}\boldsymbol{\beta}$ in which we have used (18) and (20). The Cosserat moments (15) become

$$\begin{aligned}
\mathbf{m} &= \sqrt{\nabla\eta\cdot\nabla\eta}\left(\boldsymbol{\epsilon}\cdot\overline{\nabla}\boldsymbol{\phi} + \overline{\nabla}\boldsymbol{\alpha} + \boldsymbol{\psi}\cdot(\mathbf{nI}-\mathbf{In}) + \boldsymbol{\beta}\cdot(\mathbf{n}\boldsymbol{\epsilon}+\boldsymbol{\epsilon}\mathbf{n})\right)\\
&= \boldsymbol{\chi}:\nabla\mathbf{G} + \mathbf{I}\,\mathrm{tr}\,\mathbf{F} - \mathbf{F}^{\mathrm{T}} + \sqrt{\nabla\eta\cdot\nabla\eta}\left(\overline{\nabla}\boldsymbol{\alpha} + \boldsymbol{\beta}\cdot(\mathbf{n}\boldsymbol{\epsilon}+\boldsymbol{\epsilon}\mathbf{n})\right).
\end{aligned} \qquad (28)$$

in which we have used (21) and (19).

The terms containing $\mathbf{F}$ and $\mathbf{G}$ will automatically disappear from the equations of equilibrium, as they did previously. Thus the equation of equilibrium of forces (12) becomes $\nabla\cdot\left(\sqrt{\nabla\eta\cdot\nabla\eta}\,\overline{\nabla}\boldsymbol{\beta}\right) + \mathbf{B} = 0$ and so using (24) we have

$$\boxed{\overline{\nabla}^2\boldsymbol{\beta} + \mathbf{p} = 0} \qquad (29)$$

where $\mathbf{p} = \sqrt{\nabla\eta\cdot\nabla\eta}\,\mathbf{B}$ and $\overline{\nabla}^2\boldsymbol{\beta} = \overline{\nabla}\cdot\overline{\nabla}\boldsymbol{\beta}$ is the Laplacian of the vector $\boldsymbol{\beta}$.

To find the equation of equilibrium of moments we first use (24) with $\boldsymbol{\alpha}$ instead of $\boldsymbol{\beta}$ and (25) to give

$$\begin{aligned}
\nabla\cdot\left(\mathbf{m} - \left(\boldsymbol{\chi}:\nabla\mathbf{G} + \mathbf{I}\,\mathrm{tr}\,\mathbf{F} - \mathbf{F}^{\mathrm{T}}\right)\right) &= \nabla\cdot\left(\sqrt{\nabla\eta\cdot\nabla\eta}\left(\overline{\nabla}\boldsymbol{\alpha} + \boldsymbol{\beta}\cdot(\mathbf{n}\boldsymbol{\epsilon}+\boldsymbol{\epsilon}\mathbf{n})\right)\right)\\
&= \sqrt{\nabla\eta\cdot\nabla\eta}\,\overline{\nabla}^2\boldsymbol{\alpha} - \sqrt{\nabla\eta\cdot\nabla\eta}\,\boldsymbol{\chi}:\overline{\nabla}\boldsymbol{\beta} + 2\sqrt{\nabla\eta\cdot\nabla\eta}H\boldsymbol{\epsilon}\cdot\boldsymbol{\beta}.
\end{aligned} \qquad (30)$$

Then we need
$$\boldsymbol{\chi}:(\mathbf{t} - \boldsymbol{\chi}:\nabla\mathbf{F}) = \sqrt{\nabla\eta\cdot\nabla\eta}\,\boldsymbol{\chi}:\overline{\nabla}\boldsymbol{\beta} \qquad (31)$$

and combining (30) and (31), the equation of equilibrium of moments (13) becomes

$$\boxed{\overline{\nabla}^2\boldsymbol{\alpha} + 2H\boldsymbol{\epsilon}\cdot\boldsymbol{\beta} + \mathbf{c} = 0.} \qquad (32)$$

where $\mathbf{c} = \sqrt{\nabla\eta \cdot \nabla\eta}\,\mathbf{C}$ is the loading couple per unit surface area of shell.

Equations (29) and (32) are the equations of equilibrium of forces and equilibrium of moments of a shell corresponding to internal forces and internal moments given by (26) and (27). These 4 equations are identical to those in our *Meccanica* paper[1], except here we have used $\overline{\nabla}$ as the gradient operator in the tangent plane to the surface, whereas in the *Meccanica* paper we used just ∇ as the gradient operator in the tangent plane to the surface. The reason that we were able to do that in the *Meccanica* paper is that only used the gradient in the plane of the surface, and did not need the normal component of the gradient. Equations (29), (32), (26) and (27) do not use the component of the gradient in the direction of the normal, as we would expect in shell theory.

Thus we have completed our aim of reconciling the *Meccanica* paper[1] with the Günther-Schaefer Cosserat stress functions[4].

We should point out that the internal moments in (27) contain a component of Cosserat moment about the normal, as we discussed concerning the wire mesh in figure 1a. These moments could be important for certain grid shells, but to follow conventional shell theory we would need to impose the condition that $\boldsymbol{\mu} \cdot \mathbf{n} = 0$. Again in conventional shell theory it is often assumed that 'complementary moments' are equal. For this to be the case we need to impose the condition that $\boldsymbol{\mu} \cdot \boldsymbol{\epsilon}$ is symmetric, that is $\operatorname{tr} \boldsymbol{\mu} = 0$.

5. Elements of force and moment, and the boundary conditions at a free edge

We can use (26) and (27) to find the element of force $d\mathbf{f}$ and moment $d\mathbf{m}$ crossing an imaginary cut $d\mathbf{r}$ in the surface representing the shell, $d\mathbf{f} = d\boldsymbol{\psi} - d\mathbf{r} \cdot \boldsymbol{\epsilon} \cdot \nabla\boldsymbol{\beta}$ and

$$d\mathbf{m} = d\boldsymbol{\phi} - d\mathbf{r} \cdot \boldsymbol{\epsilon} \cdot (\nabla\boldsymbol{\alpha} + \boldsymbol{\psi} \cdot (\mathbf{nI} - \mathbf{In}) + \boldsymbol{\beta} \cdot (\mathbf{n}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{n}))\,. \tag{33}$$

Here $\mathbf{m}$ is a vector, of which $d\mathbf{m}$ is an element, whereas the quantity $\mathbf{m}$ in (28) is a second order tensor corresponding to the components m_{ij} in (2) .

It should be noted that the Cartesian components of $\boldsymbol{\beta}$ and $\boldsymbol{\psi}$ only contribute to the same component of $d\mathbf{f}$, so that β_z only contributes to df_z and so on. The vertical forces 'flow' across the surface in the directions normal to the equipotentials of β_z and parallel to the streamlines of ψ_z. The same applies to the x and y components of force.

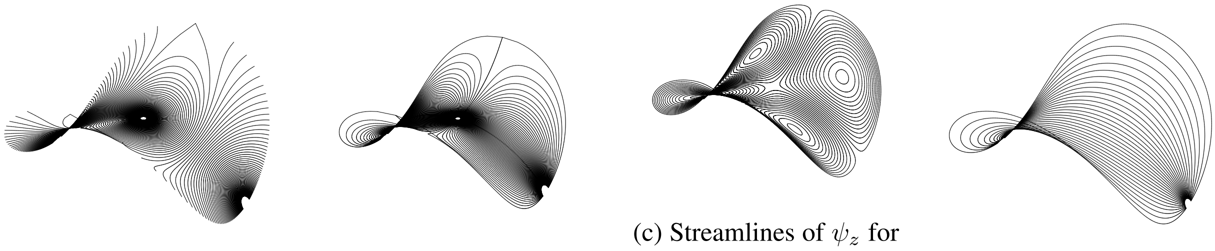
The situation is more complicated for the moment. The Cartesian components of $d\mathbf{m}$ are due to only the same components of $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$, but if we consider the special case when $d\mathbf{m} = -d\mathbf{r} \cdot \boldsymbol{\epsilon} \cdot (\psi_z\mathbf{k} \cdot (\mathbf{nI} - \mathbf{In}) + \beta_z\mathbf{k} \cdot (\mathbf{n}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{n}))$ then $d\mathbf{m} \cdot \mathbf{k} = 0$ so that vertical forces due to β_z and ψ_z produce moments about horizontal axes, as one would expect.

Along a free edge $d\mathbf{f} = 0$ and $d\mathbf{m} = 0$, which is $2 \times 3 = 6$ boundary conditions, although in the conventional theory of plates and shells this is reduced by combining normal shear forces and twisting moments using virtual work[17].

An internal rib of a shell or an edge beam simply represent a concentration of force and moment. These concentrations correspond to step changes in $\boldsymbol{\psi}$, $\boldsymbol{\phi}$, $\overline{\nabla}\boldsymbol{\alpha}$ and $\overline{\nabla}\boldsymbol{\beta}$.

6. An example of a shell in the form of Enneper's minimal surface

Figure 1c shows a shell structure in the form of Enneper's minimal surface. All minimal surfaces can be written using the Enneper-Weierstrass parameterization[15], $x = \Re\left\{\int \left(1 - Y^2\right) Zdw\right\}$, $y = \Re\left\{\int i\left(1 + Y^2\right) Zdw\right\}$ and $z = \Re\left\{\int 2YZdw\right\}$ where $Y = Y(w)$ and $Z = Z(w)$ are analytic functions of the complex variable $w = u + iv$. $\Re\{\}$ means the real part of and $\Im\{\}$ means the imaginary part of. Enneper's minimal surface is obtained by writing $Y = w$ and $Z = 1$ to give $x = u + uv^2 - u^3/3$, $y = -v - u^2v + v^3/3$ and $z = u^2 - v^2$. The curves $u =$ constant and $v =$ constant follow the directions



(a) Equipotentials of β_z for vertical forces
(b) Streamlines of ψ_z for vertical forces
(c) Streamlines of ψ_z for purely internal vertical forces
(d) Streamlines of ψ_x for horizontal thrust

Figure 2: Equipotentials and streamlines

of the principal curvatures and form curvilinear squares on the surface. The boundary of the shell in 1c is the circle in $u - v$ space, $u^2 + v^2 = R^2$ where $R = 0.5$. We have chosen to use a minimal surface but we could equally well have used some other surface, such as the hyperbolic paraboloid.

Figure 1d shows a test of a model of the structure carrying a central point load. The structure undergoes significant bending deformation during the test prior to failure, together with sliding of the supports. Tests on a surface with a higher coefficient of friction show a greater stiffness and failure load.

6.1. Internal forces

The vertical force travels from the load to the support in a direction normal to the equipotentials of β_z shown in 2a which satisfy Laplace's equation $\overline{\nabla}^2\beta_z = 0$ away from the point load and are obtained by setting $\beta_z = \xi$ where $\zeta = \xi + i\gamma$ and $2e^{\zeta} = R/w - w/R$.

We could alternatively use $\psi_z = \gamma$ to transmit the vertical forces to the support, but this time we do not actually need to satisfy Laplace's equation because $\boldsymbol{\psi}$ does not appear in (29). Note that there is then a step change in ψ_z if we follow a path all the way around the point load and get back to where we started, like climbing a spiral staircase. But the gradient of ψ_z is continuous. However this discontinuity in ψ_z does cause some complications when we come to consider moments, so it is better to use β_z.

Either way we satisfy the free boundary condition because no vertical force crosses the edge of the shell.

We can add internal forces corresponding to any system of streamlines of ψ_z which do not cross the boundary without affecting vertical equilibrium or the boundary conditions. However, we would expect symmetry conditions to apply and the streamlines in figure 2c have the correct antisymmetry about both the x and y axes, that is with highs in the north-east and south-west and lows in the north-west and south-east or vice-versa.

Figure 2d shows possible streamlines for ψ_x corresponding to boundary thrusts in the x direction, obtained by plotting $\psi_x = \gamma$ in $e^{\eta} = (R + w) / (R - w)$. Thus, again we satisfy Laplace's equation, even though we do not need to. Again we can add purely internal forces, although ψ_x has to be symmetric about the x axis and antisymmetric about the y axis, and the other way around for ψ_y, which would correspond to internal forces in the y direction.

6.2. Internal moments

The applied loading couple $\mathbf{c}$ is almost invariably taken as zero and the mean curvature $H = 0$ of a minimal surface. Thus $\boldsymbol{\alpha}$ only has to satisfy Laplace's equation in (32), and there is no reason to not simply take $\boldsymbol{\alpha} = 0$. Thus minimal surfaces are simpler than other surfaces, and in this instance an Enneper minimal surface is simpler than hyperbolic paraboloid.

If we set $d\mathbf{m} = 0$ and $\boldsymbol{\alpha} = 0$ in (33) we obtain $d\boldsymbol{\phi} = d\mathbf{r} \cdot \boldsymbol{\epsilon} \cdot (\boldsymbol{\psi} \cdot (\mathbf{nI} - \mathbf{In}) + \boldsymbol{\beta} \cdot (\mathbf{n}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{n}))$ and so if we know $\boldsymbol{\beta}$ and $\boldsymbol{\psi}$ around the boundary from the considerations in §6.1., then we can calculate $\boldsymbol{\phi}$ around the boundary. We can then choose any distribution of $\boldsymbol{\phi}$ we like within the shell without affecting equilibrium. Again we would naturally impose symmetry conditions, and we might impose the requirements $\boldsymbol{\mu} \cdot \mathbf{n} = 0$ and $\operatorname{tr} \boldsymbol{\mu} = 0$.

7. Conclusions

The ideas in this paper are a little complicated, and perhaps can be simplified by further examination. Nevertheless we have gone some way in understanding what happens when we load a Pringle® with a Biro.

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