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The phase transition of the Marcu-Fredenhagen ratio in the abelian lattice Higgs model

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Abstract

The Marcu-Fredenhagen ratio is a quantity used in the physics literature to differentiate between phases in lattice Higgs models. It is defined as the limit of a ratio of expectations of Wilson line observables as the length of these lines go to infinity while the parameters of the model are kept fixed. In this paper, we show that the Marcu-Fredenhagen ratio exists in all predicted phases of the model, and show that it indeed undergoes a phase transition. In the Higgs phase of the model we do a more careful analysis of the ratio to deduce its first order behaviour and also give an upper bound on its rate of convergence. Finally, we also present a short and concise proof of the exponential decay of correlations in the Higgs phase.

Keywords: lattice gauge theory; abelian lattice Higgs model; Marcu-Fredenhagen ratio.
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1 Introduction

Lattice gauge theories are spin models on the directed edges of lattices, which takes spins in some group G, referred to as the structure group or gauge group. Lattice gauge theories were introduced independently by Wilson [26], as lattice approximations of the quantum field theories that appear in the standard model (known as Yang-Mills theories), and by Wegner in [25], as an example of a spin system with a phase transition without a local order parameter. The lattice Higgs model is a lattice gauge theory coupled to an external field. Since their introduction, lattice gauge theories and the lattice Higgs model have attracted great interest in the physics community, and have been successfully used both for simulations and as toy models for the Yang-Mills model [15, 24].

The natural observables in lattice Higgs models are Wilson loop observables, Wilson line observables, and ratios of such observables, such as the Marcu-Fredenhagen ratio ρ (see, e.g., [3, 4, 8, 13, 18, 20, 21, 22, 24]), which is the main focus of this paper. These are all natural observables from a physics perspective (see, e.g. [4, 21]), but are also

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interesting from a mathematical standpoint since they are believed to undergo phase transitions [24]. We draw the conjectured phase diagram (see, e.g., [7, 15, 19]) of the lattice Higgs model with gauge group $\mathbb{Z}_2$, also known as the Ising lattice Higgs model, in Figure 1. For further background, as well as more references, we refer the reader

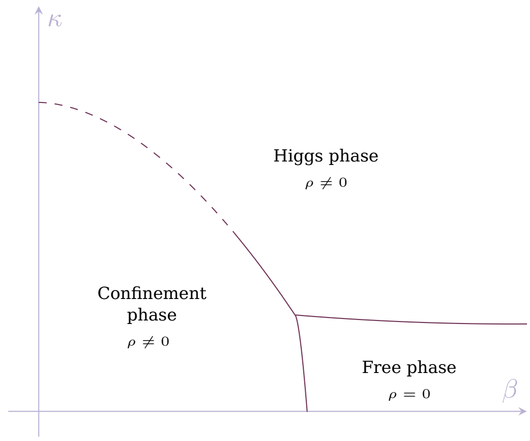


Figure 1: The conjectured phase diagram of the Ising lattice Higgs model. In the Higgs phase and the confinement phase, the Marcu-Fredenhagen ratio is believed to be non-zero, and one expects exponential decay of correlations. In contrast, in the free phase, the Marcu-Fredenhagen ratio is believed to be identically zero, and expects exponential decay of correlations with polynomial correction.

to [15] and [24].

In recent years, there has been a renewed interest in both lattice gauge theories and the lattice Higgs model in the mathematical community. In particular, in [1,5,6,10,11,13], the asymptotic behavior of Wilson loop observables was described, and in [9], similar results were obtained for Wilson line observables. Further, ideas from disagreement percolation were used to understand the rate of the decay of correlations in [2, 14]. Unfortunately, the methods applied in these papers cannot be used to understand the Marcu-Fredenhagen ratio, which requires letting the length of the involved Wilson lines tend to infinity while the parameters of the models are kept fixed. This problem was the main motivation for the current paper. Our main results show that the Marcu-Fredenhagen ratio is non-zero in non-trivial subsets of the Higgs and confinement phases, while identically zero in a non-trivial subset of the free phase. As a consequence, it follows that the model undergo at least one phase transition. One of the main tools of the paper are various special cases of the cluster expansion in [17]. This was inspired by the use of such expansions for pure lattice gauge theories in [13], but the use of these are more complicated when a Higgs field is added to the model and also needs to be different for the different phases of the model. To our knowledge, cluster expansions have not been used to study neither the Marcu-Fredenhagen ratio nor Wilson line observables prior to this paper. In particular, Wilson line observables need special handling in the free phase, where the natural cluster expansion does not converge. Finally, we obtain a very short proof of exponential decay of correlations, which gives an alternative proof of the main results of [14] and [2] in the case $G = \mathbb{Z}_2$, and also extends these from Wilson loops to the more general Wilson lines. For simplicity we state and prove all our results for $G = \mathbb{Z}_2$, but expect the proof ideas to work in more general settings, such as for finite abelian structure groups, with small modifications.

1.1 Preliminary notation

For $m \geqslant 2$, a graph naturally associated to $\mathbb{Z}^m$ has a vertex at each point $x \in \mathbb{Z}^m$ with integer coordinates and oriented edges between nearest neighbors. When e_1 and e_2 are two oriented edges between the same vertices but with opposite orientation, we write $e_2 = -e_1$.

Let $d\mathbf{e}_1 \coloneqq (1,0,0,\ldots,0)$, $d\mathbf{e}_2 \coloneqq (0,1,0,\ldots,0)$, $\ldots$, $d\mathbf{e}_m \coloneqq (0,\ldots,0,1)$ be oriented edges corresponding to the unit vectors in $\mathbb{Z}^m$. We say that an oriented edge e is *positively oriented* if it is equal to a translation of one of these unit vectors, i.e., if there is a $v \in \mathbb{Z}^m$ and a $j \in \{1,2,\ldots,m\}$ such that $e = v + d\mathbf{e}_j$. If $v \in \mathbb{Z}^m$ and $j_1 < j_2$, then $p = (v + d\mathbf{e}_{j_1}) \wedge (v + d\mathbf{e}_{j_2})$ is a positively oriented 2-cell, also known as a *positively oriented plaquette*. We let $C_0(\mathbb{Z}^m)$, $C_1(\mathbb{Z}^m)$, and $C_2(\mathbb{Z}^m)$ denote the sets of oriented vertices, edges, and plaquettes. Next, we let B_N denote the set $[-N,N]^m \cap \mathbb{Z}^m$, and we let $C_0(B_N)$, $C_1(B_N)$, and $C_2(B_N)$ denote the sets of oriented vertices, edges, and plaquettes, respectively, whose endpoints are all in B_N.

Whenever we talk about a lattice gauge theory we do so with respect to some (abelian) group $(G,+)$, referred to as the *structure group*, together with a unitary and faithful representation ρ of $(G,+)$.

Now assume that a structure group $(G,+)$, a unitary representation ρ of $(G,+)$, and an integer $N \geqslant 1$ are given. We let $\Omega^1(B_N,G)$ denote the set of all G-valued 1-forms σ on $C_1(B_N)$, i.e., the set of all G-valued functions $\sigma\colon e \mapsto \sigma(e)$ on $C_1(B_N)$ such that $\sigma(e) = -\sigma(-e)$ for all $e \in C_1(B_N)$. Similarly, we let $\Omega^0(B_N,G)$ denote the set of all G-valued functions $\phi\colon x \mapsto \phi(x)$ on $C_0(B_N)$ which are such that $\phi(x) = -\phi(-x)$ for all $x \in C_1(B_N)$. When $\sigma \in \Omega^1(B_N,G)$ and $p \in C_2(B_N)$, we let ∂p denote the formal sum of the four edges e_1, e_2, e_3, and e_4 in the oriented boundary of p, and define

$$d\sigma(p) \coloneqq \sigma(\partial p) \coloneqq \sum_{e \in \partial p} \sigma(e) \coloneqq \sigma(e_1) + \sigma(e_2) + \sigma(e_3) + \sigma(e_4).$$

Similarly, when $\phi \in \Omega^0(B_N,G)$ and $e \in C_1(B_N)$ is an edge from x_1 to x_2, we let ∂e denote the formal sum $x_2 - x_1$, and define $d\phi(e) \coloneqq \phi(\partial e) \coloneqq \phi(x_2) - \phi(x_1)$.

For $k \in \{0,1,\ldots,m\}$, a k-chain is a formal sum of positively oriented k-cells with integer coefficients. The support of a 1-chain γ, written $\operatorname{supp}\gamma$, is the set of directed edges with non-zero coefficient in γ.

1.2 The abelian lattice Higgs model

In this paper, we will consider the abelian lattice Higgs model in the fixed length limit (also known as the London limit). Given $\beta, \kappa \geqslant 0$, the action $S_{N,\beta,\kappa}$ for the abelian lattice Higgs model on B_N (in the fixed length limit) is, for $\sigma \in \Omega^1(E_N,G)$, and $\phi \in \Omega^0(B_N,G)$, defined by

$$S_{N,\beta,\kappa}(\sigma,\phi) \coloneqq -\beta \sum_{p \in C_2(B_N)} \operatorname{tr}\rho\big(d\sigma(p)\big) - \kappa \sum_{e \in C_1(B_N)} \operatorname{tr}\rho\big(\sigma(e) - \phi(\partial e)\big). \tag{1.1}$$

Elements $\sigma \in \Omega^1(B_N,G)$ will be referred to as *gauge field configurations*, and elements $\phi \in \Omega^0(B_N,G)$ will be referred to as *Higgs field configurations*. The quantity β is known as the *gauge coupling constant*, and κ is known as the *hopping parameter*. For a discussion of this action, see [11].

The Gibbs measure $\mu_{N,\beta,\kappa}$ on $\Omega^1(B_N,G) \times \Omega^0(B_N,G)$ corresponding to the action $S_{N,\beta,\kappa}$ is given by

$$\mu_{N,\beta,\kappa}(\sigma,\phi) \coloneqq Z_{N,\beta,\kappa}^{-1} e^{-S_{N,\beta,\kappa}(\sigma,\phi)}, \qquad \sigma \in \Omega^1(B_N,G),\ \phi \in \Omega^0(B_N,G),$$

where $Z_{N,\beta,\kappa}$ is a normalizing constant. We refer to this lattice gauge theory as the *(fixed length) lattice Higgs model*. We let $\mathbb{E}_{N,\beta,\kappa}$ denote the expectation corresponding to $\mu_{N,\beta,\kappa}$.

Whenever $f\colon \Omega^1(B_m,G)\times\Omega^0(B_m,G)\to\mathbb{R}$ for some $m\geqslant 1$, then, as a consequence of the Ginibre inequalities (see, e.g., [9][Section 2.6]), the infinite volume limit

$$\big\langle f(\sigma,\phi)\big\rangle_{\beta,\kappa} := \lim_{N\to\infty}\mathbb{E}_{N,\beta,\kappa}\big[f(\sigma,\phi)\big]$$

exists and is translation invariant.

We say that a 1-chain with finite support is a path if it has coefficients in $\{-1,0,1\}$. We say that a path is a *loop* if it has empty boundary $\partial\gamma$ (see Section 2). For example, any rectangular loop, as well as any finite disjoint union of such loops, corresponds to such a loop. We say that a path is an *open path* from $x_1\in C_0^+(B_N)$ to $x_2\in C_0^+(B_N)$ if it has boundary $\partial\gamma := x_2 - x_1$.

Given a path γ, a gauge field configuration $\sigma\in\Omega^1(B_N,G)$, and a Higgs field configuration $\phi\in\Omega^0(B_N,G)$, the *Wilson line observable* $W_\gamma(\sigma,\phi)$ is defined by

$$W_\gamma(\sigma,\phi) := \operatorname{tr}\rho\big(\sigma(\gamma)-\phi(\partial\gamma)\big) = \operatorname{tr}\rho\Big(\sum_{e\in\gamma}\sigma(e) - \sum_{v\in\partial\gamma}\phi(v)\Big).$$

If γ is an open path from x_1 to x_2, then $\phi(\partial\gamma)=\phi(x_2)-\phi(x_1)$, and if γ is a closed loop, then $\phi(\partial\gamma)=0$. If γ is a loop, then $W_\gamma(\sigma) := W_\gamma(\sigma,\phi)$ is referred to as a *Wilson loop observable*.

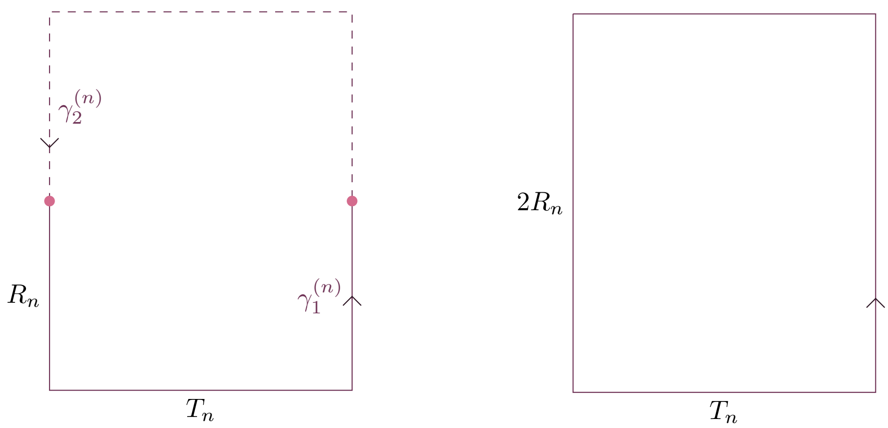
1.3 The Marcu-Fredenhagen ratio

Assume that $(R_n)_{n\geqslant 1}$ and $(T_n)_{n\geqslant 1}$ are increasing sequences of positive integers. For each $n\geqslant 1$, let $\gamma^{(n)}$ be a rectangular loop with side lengths $2R_n$ and T_n (see Figure 2b). Let $\gamma_1^{(n)}$ be as in Figure 2a, and let $\gamma_2^{(n)} = \gamma^{(n)} - \gamma_1^{(n)}$.

Define

$$\rho(\gamma_1^{(n)},\gamma_2^{(n)}) := \frac{\langle W_{\gamma_1^{(n)}}(\sigma,\phi)\rangle_{\beta,\kappa}\langle W_{\gamma_2^{(n)}}(\sigma,\phi)\rangle_{\beta,\kappa}}{\langle W_{\gamma^{(n)}}(\sigma)\rangle_{\beta,\kappa}}. \tag{1.2}$$

The limit $\lim_{n\to\infty}\rho(\gamma_1^{(n)},\gamma_2^{(n)})$ is referred to as the *Marcu-Fredenhagen order parameter*



(a) The open paths $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$. (b) The loop $\gamma^{(n)}$.

Figure 2: The open paths $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ and the rectangular loop $\gamma^{(n)}$ that appear in (1.2).

in the physics literature (see, e.g., [8, 22]). Note that it is not obvious that this limit exists, nor that it is independent of the choice of $(R_n)_{n\geqslant 1}$ and $(T_n)_{n\geqslant 1}$. If this limit (assuming it exists) is zero, the corresponding model is argued to have charged states, and no confinement, whereas if the limit is non-zero, then there should be no charged states and confinement, see, e.g., [4, 16, 19, 22, 24].

Several ratios similar to (1.2) has been considered in the physics literature, see, e.g., [18], and the main ideas in this paper can be adapted to cover also these cases.

1.4 Main results

Our first main result considers the Marcu-Fredenhagen ratio in the Higgs phase.

Theorem 1.1. *Let $G=\mathbb{Z}_2$, $\beta \geqslant 0$, and $\kappa \geqslant \kappa_0^{(Higgs)}$, where $\kappa_0^{(Higgs)} = \kappa_0^{(Higgs)}(m) > 0$ is defined in* (3.4)*. Further, let $(R_n)_{n\geqslant 1}$ and $(T_n)_{n\geqslant 1}$ be increasing sequences of positive integers such that $\limsup_{n\to\infty} T_n/R_n < \infty$, and for each $n \geqslant 1$, let $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ be as in Figure 2. Then the following hold.*

(i) The limit

$$\rho = \rho_{\beta,\kappa} \coloneqq \lim_{n\to\infty} \rho(\gamma_1^{(n)}, \gamma_2^{(n)}) \tag{1.3}$$

exists and is independent of $(R_n)_{n\geqslant 1}$ and $(T_n)_{n\geqslant 1}$.

(ii) The limit ρ is strictly positive, i.e., $\rho > 0$.

(iii) For all $n \geqslant 1$ and $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\begin{aligned} \big|\log \rho_n - \log \rho\big| \leqslant 4C_\varepsilon \sum_{j=1}^{\infty} e^{-4\max(j,\min(R_n,T_n))(\kappa-\kappa_0-\varepsilon)} \\ + 2C_\varepsilon \max(T_n - 2R_n, 0) e^{-4\max(2R_n,\min(R_n,T_n))(\kappa-\kappa_0-\varepsilon)}. \end{aligned} \tag{1.4}$$

Here C_ε is defined in (3.12) *and does not depend on β nor κ.*

In other words, Theorem 1.1 says that the Marcu-Fredenhagen parameter exists and is strictly positive when $\kappa \geqslant \kappa_0^{(\text{Higgs})}$. Also, it gives a upper bound on the convergence rate, thus stating how large an estimate for $-\log \rho_n$ has to be for one to be able to conclude that $\rho > 0$.

We note that the assumption that $\limsup_{n\to\infty} T_n/R_n < \infty$ is needed to guarantee that the right-hand side of (1.4) goes to zero as $n \to \infty$.

Our next result complements our first theorem, Theorem 1.1, by showing that the Marcu-Fredenhagen ratio is non-zero also in parts of the confinement regime, i.e., when β and κ are both sufficiently small.

Theorem 1.2. *Let $G=\mathbb{Z}_2$, $0 < \beta < \beta_0^{(conf)}$, and $\kappa > 0$, where $\beta_0^{(conf)} = \beta_0^{(conf)}(m) > 0$ is defined in* (4.3)*. Further, let $(R_n)_{n\geqslant 1}$ and $(T_n)_{n\geqslant 1}$ be increasing sequences of positive integers such that $\limsup_{n\to\infty} T_n/R_n < \infty$, and for each $n \geqslant 1$, let $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ be as in Figure 2. Then the following hold.*

(i) The limit

$$\rho = \rho_{\beta,\kappa} \coloneqq \lim_{n\to\infty} \rho(\gamma_1^{(n)}, \gamma_2^{(n)})$$

exists and is independent of $(R_n)_{n\geqslant 1}$ and $(T_n)_{n\geqslant 1}$.

(ii) The limit ρ is strictly positive, i.e., $\rho > 0$.

Theorem 1.2 shows that in at least parts of the confinement phase of the lattice Higgs model, the Marcu-Fredenhagen ratio is strictly positive.

Our next result concerns the Marcu-Fredenhagen ratio in the free phase.

Theorem 1.3. *Let $G = \mathbb{Z}_2$, $\beta > 0$ be suffciently large, and $\kappa > 0$ be sufficiently small. Further, let $(R_n)_{n\geqslant 1}$ and $(T_n)_{n\geqslant 1}$ be strictly increasing sequences of positive integers, and for each $n \geqslant 1$, let $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ be as in Figure 2. Then*

(i) The limit

$$\rho = \rho_{\beta,\kappa} := \lim_{n\to\infty} \rho(\gamma_1^{(n)}, \gamma_2^{(n)})$$

exists and is independent of $(R_n)_{n\geqslant 1}$ and $(T_n)_{n\geqslant 1}$.

(ii) The limit ρ is identically zero, i.e., $\rho = 0$.

The most important consequence of Theorems 1.1, 1.2 and 1.3 is that they together prove that the Marcu-Fredenhagen ratio indeed has a phase transition, implying in particular that it can be used as an order parameter.

Our last result gives an upper bound on the decay of correlation in the Higgs phase of the abelian lattice Higgs model. This result extends the results in [2] and [14] to Wilson line observables in the case $G = \mathbb{Z}^2$. However, the main reason we include this result here is that the methods used in this paper yields a very short proof which is very different to the proofs in [2] and [14].

Theorem 1.4. *Let $G = \mathbb{Z}_2$, $\beta \geqslant 0$, and $\kappa \geqslant \kappa_0^{\text{(Higgs)}}$. Further, let γ_1 and γ_2 be two paths. Then, for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that*

$$\bigl|\langle W_{\gamma_1+\gamma_2}\rangle_{\beta,\kappa} - \langle W_{\gamma_1}\rangle_{\beta,\kappa}\langle W_{\gamma_2}\rangle_{\beta,\kappa}\bigr| \leqslant C_\varepsilon \bigl|\operatorname{supp}\gamma_1\bigr| e^{-4(\kappa-\kappa_0-\varepsilon)\operatorname{dist}(\gamma_1,\gamma_2)},$$

Here C_ε is defined in (3.12) *and does not depend on β nor κ, and $\operatorname{dist}(\gamma_1, \gamma_2)$ is the ℓ_0-distance between the supports of γ_1 and γ_2.*

Remark 1.5. The proof of Theorem 1.4 can easily be adapted to show that the covariance decays at most exponentially also in the confinement phase and the free phase. However, we note that in the free phase such a result would not be sharp since the rate of decay in the free phase is believed to be exponential with polynomial corrections.

1.5 Relation to other work

The main result of [9] gives the asymptotic decay rate of Wilson loops and lines γ in the Higgs phase under the two additional assumptions that $6\beta > \kappa > \kappa_0$ and $|\operatorname{supp}\gamma| e^{-24\beta-4\kappa} \ll \infty$. In [12], similar results are given in the confinement phase. However, we are aware of no results about the decay of Wilson lines in the free phase, other than the well known universal lower bound $(\tanh 2\kappa)^{|\gamma|}$. For Wilson loops, several papers contain similar results, see e.g. [1, 5, 10]. In all papers mentioned above, assumptions are made on the parameters so that at least one of them tend to either infinity or zero as $|\gamma|$ grows. As a consequence, these results cannot be applied to deduce anything about the Marcu-Fredenhagen parameter or similar ratios since this limit involves letting $|\gamma|$ tend to infinity while keeping the parameters β and κ fixed.

In this paper, we use high temperature expansions and cluster expansions for the Higgs phase, the confinement phase, and the free phase respectively. The cluster expansions are special cases of the cluster expansion presented in [17]. In [13], we used a similar cluster expansion for a lattice gauge theory, but there only the case $\kappa = 0$ was considered. This expansion is similar to the expansion we use here in the Higgs phase. However, in both the confinement and the Higgs phase we here first use high temperature expansions, while the free phase requires additional work when setting up the cluster expansion.

In the mathematical literature, the decay of correlations in lattice gauge theories has been studied in [2] and [14]. In both of these papers, only observables consisting of combinations of Wilson loops were considered, and the proofs rely on couplings and giving upper bounds on events describing the vortices in the model. The proof method here is very different from this approach, and yields a much shorter proof. In [2, 14], decay of correlations was proven for any finite structure group and any finite abelian structure group respectively. In this paper, for simplicity, we only give a proof for the structure group $\mathbb{Z}_2$, but the same ideas should with some work be possible to translate to any finite abelian structure group.

1.6 Structure of paper

In Section 2, we introduce the notation and definitions we will use throughout the rest of the paper. Section 3, Section 4, and Section 5 contains our results for the three conjectured phases of the model; the Higgs phase, the confinement phase, and the free phase respectively.

Section 3 contains the relevant expansion and the proofs of our main results in the Higgs phase. In Section 3.1, we present the cluster expansion we will use to prove our main result. We also use this cluster expansion to express the Marcu-Fredenhagen ratio and covariance in terms of the cluster expansion. In Section 3.2, we give upper bounds of natural events in terms of the cluster expansion. In Sections 3.3 and 3.4, we give proofs of Theorem 1.1 and Theorem 1.4.

Section 4 contains the relevant expansions and proofs of our main results in the confinement phase. In Section 4.1, we present a high temperature expansion in both parameters which will be useful in this phase. In Section 4.2, we present a cluster expansion of the model obtained from the high temperature expansion. Section 4.3 contains a upper bounds which will be useful in the proof of the main result of this section, and, finally, Section 4.4 contains the proof of Theorem 1.2.

Section 5 contains the relevant expansions and proofs of our main results in the free phase. In Section 5.1, we present a high temperature expansion in κ which will be useful in this phase. In Section 5.2, we present a cluster expansion of a model related to the model obtained from the high temperature expansion. In Section 5.3, we give upper bounds of natural events in terms of the cluster expansion. Finally, Section 5.4 contains the proof of Theorem 1.3.

2 Preliminaries

Even though we later work with $G = \mathbb{Z}_2$, in this section we allow G to be a general finite abelian group since this entails no additional work. We assume that a one-dimensional unitary representation of G has been fixed.

2.1 Discrete exterior calculus

Below we present the notation from discrete exterior calculus that we need in this paper. In order to keep the background section of this paper short, and since these definitions have appeared in several recent papers, we will refer the reader to [11] for further details.

- We will work with the square lattice $\mathbb{Z}^m$, where we assume that the dimension $m \geq 3$ throughout. We write $B_N = [-N, N]^m \cap \mathbb{Z}^m$. Since m will always be fixed, we suppress the dependency on m in this notation.
- For $k = 0, 1, \ldots, m$, write $C_k(B_N)$ and $C_k(B_N)^+$ for the set of unoriented and positively oriented k-cells, respectively (see [10, Sect. 2.1.2]).
- Formal sums of positively oriented k-cells with integer coefficients are called k-chains, and the space of k-chains is denoted by $C_k(B_N, \mathbb{Z})$, (see [10, Sect. 2.1.2])

- Let $k \geqslant 2$ and $c = \frac{\partial}{\partial x^{j_1}}\big|_a \wedge \cdots \wedge \frac{\partial}{\partial x^{j_k}}\big|_a \in C_k(B_N)$. The *boundary* of c is the $(k-1)$-chain $\partial c \in C_{k-1}(B_N, \mathbb{Z})$ defined as the formal sum of the $(k-1)$-cells in the (oriented) boundary of c. The definition is extended to k-chains by linearity. See [10, Sect. 2.1.4].

- If $k \in \{0, 1, \ldots, n-1\}$ and $c \in C_k(B_N)$ is an oriented k-cell, we define the *coboundary* $\hat\partial c \in C_{k+1}(B_N)$ of c as the $(k+1)$-chain $\hat\partial c := \sum_{c' \in C_{k+1}(B_N)} \big(\partial c'[c]\big) c'$. See [10, Sect. 2.1.5].

- We let $\Omega^k(B_N, G)$ denote the set of G-valued (discrete differential) k-forms (see [10, Sect 2.3.1]); the exterior derivative $d : \Omega^k(B_N, G) \to \Omega^{k+1}(B_N, G)$ is defined for $0 \le k \le m-1$ (see [10, Sect. 2.3.2]).

- We write $\operatorname{supp} \omega = \{c \in C_k(B_N) : \omega(c) \neq 0\}$ for the support of a k-form ω. Similarly, we write $(\operatorname{supp} \omega)^+ = \{c \in C_k(B_N)^+ : \omega(c) \neq 0\}$.

- A 1-chain $\gamma \in C_1(B_N, \mathbb{Z})$ with finite support $\operatorname{supp} \gamma$ is called a *path* if for all $e \in C_1(B_N)$, we have that $\gamma[e] \in \{-1, 0, 1\}$. We write $|\gamma| = |\operatorname{supp} \gamma|$, and let $\Lambda_0 \subseteq C^1(B_N, \mathbb{Z})$ be the set of all paths. A path γ is said to be a *closed path* or a *loop* if $\partial \gamma = 0$. A path γ is said to be an *open path* if $|\partial \gamma| = 2$.

- When γ_1 and γ_2 are two paths, we let $\operatorname{dist}(\gamma_1, \gamma_2)$ be ℓ_0-distance between $\operatorname{supp} \gamma_1$ and $\operatorname{supp} \gamma_2$. Equivalently, $\operatorname{dist}(\gamma_1, \gamma_2)$ is the length of the shortest path that connects the supports of γ_1 and γ_2.

2.2 Unitary gauge

In this section, we recall the notion of gauge transforms, and describe how these can be used to rewrite the Wilson line expectation as an expectation with respect to a slightly simpler probability measure. For more details on gauge transforms and unitary gauge, we refer the reader to [9].

For $\eta \in \Omega^0(B_N, G)$, let the map

$$\tau := \tau_\eta := \tau_\eta^{(1)} \times \tau_\eta^{(2)} \colon \Omega^1(B_N, G) \times \Omega^0(B_N, G) \to \Omega^1(B_N, G) \times \Omega^0(B_N, G)$$

be defined by

$$\begin{cases} \sigma(e) \mapsto -\eta(x) + \sigma(e) + \eta(y), & e = (x, y) \in C_1(B_N), \\ \phi(x) \mapsto \phi(x) + \eta(x), & x \in C_0(B_N), \end{cases} \tag{2.1}$$

where $\sigma \in \Omega^1(B_N, G)$ and $\phi \in \Omega^0(B_N, G)$. A map τ of this form is called a *gauge transformation*, and functions $f : \Omega^1(B_N, G) \times \Omega^0(B_N, G) \to \mathbb{C}$ which are invariant under such mappings in the sense that $f = f \circ \tau$ are said to be *gauge invariant*.

For $\beta, \kappa \geqslant 0$ and $\sigma \in \Omega^1(B_N, G)$, we define the probability measure

$$\mu^{(U)}_{N,\beta,\kappa}(\sigma) := (Z^{(U)}_{N,\beta,\kappa})^{-1} \exp\Big(\beta \sum_{p \in C_2(B_N)} \rho\big(d\sigma(p)\big) + \kappa \sum_{e \in C_1(B_N)} \rho\big(\sigma(e)\big)\Big), \tag{2.2}$$

where $Z^{(U)}_{N,\beta,\kappa}$ is a normalizing constant. We let $\mathbb{E}^{(U)}_{N,\beta,\kappa}$ denote the corresponding expectation.

The following lemma, which is considered well-known in the physics literature, will be crucial in the analysis of the lattice Higgs model.

Lemma 2.1 (Corollary 2.17 in [9]). *Let $\beta \geqslant 0$, $\kappa \geqslant 0$, and let γ be a path in $C_1(B_N)$. Then*

$$\mathbb{E}_{N,\beta,\kappa}\big[W_\gamma(\sigma, \phi)\big] = \mathbb{E}^{(U)}_{N,\beta,\kappa}\big[W_\gamma(\sigma, 1)\big] = \mathbb{E}^{(U)}_{N,\beta,\kappa}\Big[\rho\big(\sigma(\gamma)\big)\Big].$$

The main idea of the proof of Lemma 2.1 is to perform a change of variables, where we for each pair (σ, ϕ) apply the gauge transformation $\tau_{-\phi}$, thus mapping ϕ to 0. This maps $\mu_{N,\beta,\kappa}$ to $\mu^{(U)}_{N,\beta,\kappa}$ and $W_\gamma(\sigma, \phi)$ to $W_\gamma(\tau_\phi \sigma, 1)$. Using the Poincaré lemma (see, e.g. [6, Lemma 2.2]), one can show that this map is k-to-1 for some $k \in \mathbb{N}$ that depends on N, and from this the conclusion of the lemma follows. After having applied this gauge transformation, we say we are working in *unitary gauge*.

We point out that Lemma 2.1 holds more generally also for 1-chains in the sense that for any 1-chain $c \in C_1(B_N)$, one has

$$\mathbb{E}_{N,\beta,\kappa}\Big[\rho\big(\sigma(c) - \phi(\partial c)\big)\Big] = \mathbb{E}^{(U)}_{N,\beta,\kappa}\Big[\rho\big(\sigma(c)\big)\Big].$$

However, this more general version of Lemma 2.1 will not be used in this paper.

With the current section in mind, we will work with $\sigma \sim \mu^{(U)}_{N,\beta,\kappa}$ rather than with $(\sigma, \phi) \sim \mu_{N,\beta,\kappa}$ throughout the rest of this paper, together with the observable

$$W_\gamma(\sigma) \coloneqq W_\gamma(\sigma, 1) = \prod_{e \in \gamma} \rho\big(\sigma(e)\big) = \rho(\sigma(\gamma)).$$

2.3 Existence of the infinite volume limit

In this section, we recall a result which shows existence and translation invariance of the infinite volume limit $\langle W_\gamma(\sigma, \phi)\rangle_{\beta,\kappa}$ defined in the introduction. This result is well-known, and is often mentioned in the literature as a direct consequence of the Ginibre inequalities. A full proof of this result in the special case $\kappa = 0$ was included in [10], and the general case can be proven completely analogously, hence we omit the proof here and the refer the reader to [10].

Proposition 2.2. *Let $G = \mathbb{Z}_n$, $M \geqslant 1$, and let $f \colon \Omega^1(B_M, G) \times \Omega^0(B_M, G) \to \mathbb{R}$. For $M' \geqslant M$, we abuse notation and let f denote the natural extension of f to $\Omega^1(B_{M'}, G) \times \Omega^0(B_{M'}, G)$, i.e., the unique function such that $f(\sigma) = f(\sigma|_{C_1(B_M)}, \phi|_{C_0(B_M)})$ for all $(\sigma, \phi) \in \Omega^1(B_{M'}, G) \times \Omega^0(B_{M'}, G)$. Further, let $\beta \in [0, \infty]$ and $\kappa \geqslant 0$. Then the limit $\langle f(\sigma, \phi)\rangle_{\beta,\kappa} = \lim_{N\to\infty} \mathbb{E}_{N,\beta,\kappa}\big[f(\sigma, \phi)\big]$ exists and is translation invariant.*

2.4 The activity

For $a \geqslant 0$ and $g \in G$, we set

$$\phi_a(g) \coloneqq e^{a \operatorname{Re}(\rho(g) - \rho(0))}.$$

Since ρ is unitary, for any $g \in G$ we have $\rho(g) = \overline{\rho(-g)}$, and hence $\operatorname{Re}\rho(g) = \operatorname{Re}\rho(-g)$. In particular, for any $g \in G$

$$\phi_a(g) = e^{a(\operatorname{Re}\rho(g) - \rho(0))} = e^{a(\operatorname{Re}\rho(-g) - \rho(0))} = \phi_a(-g). \tag{2.3}$$

For $\sigma \in \Omega^1(B_N, G)$ and $\beta, \kappa \geqslant 0$ we define the *activity* of σ by

$$\phi_{\beta,\kappa}(\sigma) \coloneqq \prod_{p \in C_2(B_N)} \phi_\beta\big(d\sigma(p)\big) \prod_{e \in C_1(B_N)} \phi_\kappa\big(\sigma(e)\big).$$

Note that for $\sigma \in \Omega^1(B_N, G)$, the probability measure corresponding to the Wilson action lattice gauge theory can be written an

$$\mu^{(U)}_{N,\beta,\kappa}(\sigma) = \frac{\phi_{\beta,\kappa}(\sigma)}{\sum_{\sigma \in \Omega^1(B_N, G)} \phi_{\beta,\kappa}(\sigma)}. \tag{2.4}$$

Moreover, in the case $G = \mathbb{Z}_2$, for $\sigma \in \Omega^1(B_N, \mathbb{Z}_2)$ we have

$$\begin{aligned}\phi_{\beta,\kappa}(\sigma) &= \prod_{p\in C_2(B_N)} \phi_\beta\bigl(d\sigma(p)\bigr) \prod_{e\in C_1(B_N)} \phi_\kappa\bigl(\sigma(e)\bigr)\\ &= \prod_{p\in C_2(B_N)} e^{-2\beta \mathbb{1}\left(d\sigma(p)=1\right)} \prod_{e\in C_1(B_N)} e^{-2\kappa \mathbb{1}(\sigma(e)=1)}\\ &= e^{-2\beta \sum_{p\in C_2(B_N)} \mathbb{1}(d\sigma(p)=1)} e^{-2\kappa \sum_{e\in C_1(B_N)} \mathbb{1}(\sigma(e)=1)}\\ &= e^{-2\beta|\operatorname{supp} d\sigma|} e^{-2\kappa|\operatorname{supp}\sigma|}.\end{aligned}$$

We note that, by definition, if $\sigma, \eta \in \Omega^1(B_N, G)$, $\operatorname{supp}\eta \subseteq \operatorname{supp}\sigma$, $\sigma|_{\operatorname{supp}\eta} = \eta$ and $(d\sigma)|_{\operatorname{supp} d\eta} = d\eta$, then

$$\phi_{\beta,\kappa}(\sigma) = \phi_{\beta,\kappa}(\eta)\phi_{\beta,\kappa}(\sigma - \eta).$$

2.5 Additional notation

Let $D_0 = D_0(m)$ be a universal constant such that for any $e \in C_1(B_N)^+$ and any $j \geqslant 1$, there are at most $D_0 j^{m-1}$ positively oriented plaquettes at distance j from e.

When $(\gamma_1^{(n)})_{n\geqslant 1}$ and $(\gamma_2^{(n)})_{n\geqslant 1}$ are as in Figure 2, we let

$$\Gamma_n := \bigl\{0, \gamma_1^{(n)}, \gamma_2^{(n)}, \gamma_1^{(n)} + \gamma_2^{(n)}\bigr\}. \tag{2.5}$$

Let $\mathcal{G}_0$ be the graph with vertex set $C_1(B_N)^+$ and an edge between two distinct edges $e_1, e_2 \in C_1(B_N)^+$ if $\operatorname{supp}\partial e_1 \cap \operatorname{supp}\partial e_2 \neq \emptyset$ (written $e_1 \sim e_2$). Note that any $e \in C_1(B_N)^+$ has degree at most $M_0 := 4m-1$ in $\mathcal{G}_0$. When $\gamma \in \Lambda_0$, we let $\mathcal{G}_0(\gamma)$ be the subgraph of $\mathcal{G}_0$ induced by $\operatorname{supp}\gamma$. We say that a path $\gamma \in \Lambda_0$ is connected if $\mathcal{G}_1(\gamma)$ is a connected graph, and let

$$\Lambda_1 := \bigl\{\gamma \in \Lambda_0 \colon \gamma \text{ is connected}\bigr\}.$$

3 The Higgs phase (κ large)

3.1 A cluster expansion

In this section we describe a cluster expansion for the lattice Higgs model on a finite box B_N in the Higgs phase. The material here is for the most part well-known and is a natural special case of the cluster expansion as presented in [17]. The expansion we use here is similar to the expansion in [13, 23] but uses polymers in the Higgs field instead of polymers in the gauge field. This is the reason that we need $\kappa \geqslant \kappa_0^{\text{(Higgs)}}$ here instead of $\beta \geqslant \beta_0$ for some $\beta_0 > 0$ as in [13].

Throughout this section and in the rest of the paper, we will assume that $G = \mathbb{Z}_2$.

3.1.1 Polymers

Let $\mathcal{G}_1$ be the graph with vertex set $C_1(B_N)^+$ and an edge between two distinct vertices e_1, e_2 iff $\operatorname{supp}\hat\partial e_1 \cap \operatorname{supp}\hat\partial e_2 \neq \emptyset$, i.e. if e_1 and $\pm e_2$ are both in boundary of some common plaquette.

Since any edge $e \in C_1(B_N)^+$ in B_N is in the boundary of at most $2(m-1)$ plaquettes, and any such plaquettes has exactly three edges in its boundary that are not equal to e, it follows that there are at most

$$M_1 := 3 \cdot 2(m-1) = 6(m-1) \tag{3.1}$$

edges $e' \in C_1(B_N)^+ \smallsetminus \{e\}$ with $\operatorname{supp}\hat\partial e \cap \operatorname{supp}\hat\partial e' \neq \emptyset$. As a consequence, it follows that each vertex in $\mathcal{G}_1$ has degree at most M_1.

The graph $\mathcal{G}_1$ will be useful when we, in the following sections, describe the cluster expansion we will use in the Higgs phase. This graph is analog to the graph introduced in [13, Section 2.3] but has $C_1(B_N)^+$ as its vertex set instead of $C_2(B_N)^+$ which was used in [13].

When $\sigma \in \Omega^1(B_N, G)$, we let $\mathcal{G}_1(\sigma)$ be the subgraph of $\mathcal{G}_1$ induced by $(\operatorname{supp} \sigma)^+$. We let Λ be the set of all $\sigma \in \Omega^1(B_N, G)$ such that $\mathcal{G}_1(\sigma)$ has exactly one connected component. The spin configurations in Λ will be referred to as *polymers*.

3.1.2 Polymer interaction

For $\sigma, \sigma' \in \Lambda$, we write $\sigma \sim \sigma'$ if $\mathcal{G}_1(\sigma) \cup \mathcal{G}_1(\sigma')$ is a connected subgraph of $\mathcal{G}_1$.

In the notation of [17, Chapter 3], the model given by (4.2) corresponds to a model of polymers with polymers described in Section 3.1.1 and interaction function $\iota(\sigma_1, \sigma_2) \coloneqq \zeta(\sigma_1, \sigma_2) + 1$, where

$$\zeta(\sigma_1, \sigma_2) \coloneqq \begin{cases} -1 & \text{if } \sigma_1, \sigma_2 \in \Lambda \text{ and } \sigma_1 \sim \sigma_2 \\ 0 & \text{else.} \end{cases}$$

3.1.3 Clusters of polymers

Consider a multiset

$$\mathcal{S} = \{\underbrace{\eta_1, \ldots, \eta_1}_{n_{\mathcal{S}}(\eta_1) \text{ times}}, \underbrace{\eta_2, \ldots, \eta_2}_{n_{\mathcal{S}}(\eta_2) \text{ times}}, \ldots, \underbrace{\eta_k, \ldots, \eta_k}_{n_{\mathcal{S}}(\eta_k) \text{ times}}\} = \{\eta_1^{n(\eta_1)}, \ldots, \eta_k^{n(\eta_k)}\},$$

where $\eta_1, \ldots, \eta_k \in \Lambda$ are distinct and $n(\eta) = n_{\mathcal{S}}(\eta)$ denotes the number of times η occurs in $\mathcal{S}$. Following [17, Chapter 3], we say that $\mathcal{S}$ is *decomposable* if it is possible to partition $\mathcal{S}$ into disjoint multisets. That is, if there exist non-empty and disjoint multisets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ such that $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ and such that for each pair $(\eta_1, \eta_2) \in \mathcal{S}_1 \times \mathcal{S}_2$, $\eta_1 \nsim \eta_2$. If $\mathcal{S}$ is not decomposable, we say that $\mathcal{S}$ is a *cluster*. We stress that a cluster is unordered and may contain several copies of the same polymer. Given a cluster $\mathcal{S}$, we define

$$\|\mathcal{S}\|_1 = \sum_{\eta \in \Lambda} n_{\mathcal{S}}(\eta) \big|(\operatorname{supp} \eta)^+\big|, \quad \|\mathcal{S}\|_2 = \sum_{\eta \in \Lambda} n_{\mathcal{S}}(\eta) \big|(\operatorname{supp} d\eta)^+\big|,$$
$$n(\mathcal{S}) = \sum_{\eta \in \Lambda} n_{\mathcal{S}}(\eta), \quad \text{and} \quad \operatorname{supp} \mathcal{S} = \bigcup_{\eta \in \mathcal{S}} \operatorname{supp} \eta.$$

For a $1-$ chain $c \in C_1(B_N, \mathbb{Z})$, we define

$$\mathcal{S}(c) = \sum_{\eta \in \Lambda} n_{\mathcal{S}}(\eta) \eta(c).$$

We let Ξ be the set of all clusters.

To simplify notation in the rest of this section, for $e \in C_1(B_N)^+$, $\gamma \in C^1(B_N, \mathbb{Z})$, and $i, j, k \geqslant 1$, we define

$$\Xi_{i,j,k,e} \coloneqq \big\{\mathcal{S} \in \Xi \colon n(\mathcal{S}) = i,\, \|\mathcal{S}\|_1 = j,\, \|\mathcal{S}\|_2 = k,\, e \in \operatorname{supp} \mathcal{S}\big\},$$
$$\Xi_i \coloneqq \big\{\mathcal{S} \in \Xi \colon n(\mathcal{S}) = i\big\},$$
$$\Xi_{i,j} \coloneqq \big\{\mathcal{S} \in \Xi \colon n(\mathcal{S}) = i,\, \|\mathcal{S}\|_1 = j\big\},$$
$$\Xi_{i,j,e} \coloneqq \big\{\mathcal{S} \in \Xi \colon n(\mathcal{S}) = i,\, \|\mathcal{S}\|_1 = j,\, e \in \operatorname{supp} \mathcal{S}\big\},$$

and

$$\Xi_{i,j,\gamma} \coloneqq \big\{\mathcal{S} \in \Xi \colon n(\mathcal{S}) = i,\, \|\mathcal{S}\|_1 = j,\, \mathcal{S}(\gamma) \neq 0\big\}.$$

Further, we let

$$\Xi_{i^+} := \{\mathcal{S} \in \Xi \colon n(\mathcal{S}) \geqslant i\},$$

and define $\Xi_{i^+,j}$, Ξ_{i,j^+}, Ξ_{i^+,j^+}, etc. analogously.

We note that the sets defined above depend on N but we usually suppress this in the notation. When we want to emphasize this dependence, we write $\Xi(B_N)$, $\Xi_i(B_N)$, $\Xi_{i,j}(B_N)$, etc.

The following lemma gives an upper bound on the number of clusters of a given size and with a given edge in its support.

Lemma 3.1. *For any $k \geqslant 1$ and $e \in C_1(B_N)^+$, we have $|\Xi_{1,k,e}| \leqslant M_1^{2k-2}$.*

Proof. Let $k \geqslant 1$ and $e \in C_1(B_N)$. Let $\mathcal{P}$ be the set of all paths in $\mathcal{G}_1$ that starts at e and have length $2k-1$. Since each vertex in $\mathcal{G}_1$ has degree at most M_1, we have $|\mathcal{P}| \leqslant M_1^{2k-2}$.

For $\{\eta\} \in \Xi_{1,k,e}$, let G_η be the subgraph of $\mathcal{G}_1$ induced by the set $(\operatorname{supp}\eta)^+$. Then G_η is connected, and hence G_η has a spanning path $T_\eta \in \mathcal{P}$ of length $2k-1 = 2\big|(\operatorname{supp}\eta)^+\big| - 1$ which starts at e. Since the map $\eta \mapsto T_\eta$ is an injective map from $\Xi_{1,k,e}$ to $\mathcal{P}$ and $|\mathcal{P}| \leqslant M_1^{2k-2}$, the desired conclusion immediately follows. □

3.1.4 The activity of clusters

We extend the notion of activity from Section 2.4 to clusters $\mathcal{S} \in \Xi$ by letting

$$\phi_{\beta,\kappa}(\mathcal{S}) = \prod_{\eta \in \mathcal{S}} \phi_{\beta,\kappa}(\eta)^{n_{\mathcal{S}}(\eta)} = e^{-4\beta\|\mathcal{S}\|_2 - 4\kappa\|\mathcal{S}\|_1}.$$

3.1.5 Ursell functions

In the cluster expansion in [17], which is valid for many different spin models, special functions known as Ursell functions play an important role. We define the Ursell function relevant for our setting below.

For $k \geqslant 1$, we let $\mathcal{G}^k$ be the set of all connected graphs G with vertex set $V(G) = \{1, 2, \ldots, k\}$. Whenever G is a graph, we let $E(G)$ be the (undirected) edge set of G.

Definition 3.2 (The Ursell functions)**.** *For $k \geqslant 1$ and $\eta_1, \eta_2, \ldots, \eta_k \in \Lambda$, we let*

$$U(\eta_1, \ldots, \eta_k) := \frac{1}{k!} \sum_{G \in \mathcal{G}^k} (-1)^{|E(\mathcal{G})|} \prod_{(i,j) \in E(\mathcal{G})} \mathbb{1}(\eta_i \sim \eta_j).$$

Note that this definition is invariant under permutations of the polymers $\eta_1, \eta_2, \ldots, \eta_k$.

For $\mathcal{S} \in \Xi_k$, and any enumeration $\eta_1, \ldots, \eta_k$ (with multiplicities) of the polymers in $\mathcal{S}$, we define

$$U(\mathcal{S}) = k!\, U(\eta_1, \ldots, \eta_k). \tag{3.2}$$

Note that for any $\mathcal{S} \in \Xi_1$, we have $U(\mathcal{S}) = 1$, and for any $\mathcal{S} \in \Xi_2$, we have $U(\mathcal{S}) = -1$.

3.1.6 Cluster expansion of the partition function

The partition function for the abelian lattice Higgs model with parameters $\beta, \kappa \geqslant 0$ is given by

$$Z^{(U)}_{N,\beta,\kappa} = \sum_{\sigma \in \Omega^1(B_N, G)} \phi_{\beta,\kappa}(\sigma).$$

Since N is finite, this is a finite sum. An alternative representation of $Z_{\beta,\kappa,N}$ is given by the *cluster partition function* which is defined by the following (formal) expression:

$$Z^*_{N,\beta,\kappa} = \exp\left(\sum_{\mathcal{S} \in \Xi} \Psi_{\beta,\kappa}(\mathcal{S})\right), \tag{3.3}$$

where for $\mathcal{S} \in \Xi$, we define

$$\Psi_{\beta,\kappa}(\mathcal{S}) := U(\mathcal{S})\phi_{\beta,\kappa}(\mathcal{S})$$

and U is the Ursell function as defined in Section 3.1.5.

It is not obvious that the series in the exponent of (3.3) is convergent but this follows from the next lemma, assuming κ is sufficiently large, and we verify below that in this case $\log Z^{(U)}_{N,\beta,\kappa} = \log Z^*_{N,\beta,\kappa}$. In this lemma, the following constants will be used. Recalling the definition of M_1 from (3.1), we define

$$\alpha = \alpha^{(\mathrm{Higgs})} := \operatorname*{arg\,min}_{\alpha' \in (0,1)} \frac{\log(M_1^2 + 1/\alpha')}{4(1-\alpha')} \quad \text{and} \quad \kappa_0^{(\mathrm{Higgs})} := \frac{\log(M_1^2 + 1/\alpha)}{4(1-\alpha)}. \tag{3.4}$$

Lemma 3.3. *Let $\beta \geqslant 0$ and $\kappa > \kappa_0^{(Higgs)}$. Then, for any $\eta \in \Lambda$, we have*

$$\sum_{\mathcal{S} \in \Xi \colon \eta \in \mathcal{S}} \Psi_{\beta,\kappa}(\mathcal{S}) \leqslant \big(\phi_{0,\kappa}(\eta)\big)^{-\alpha} \phi_{\beta,\kappa}(\eta).$$

Moreover, the series in (3.3) *is absolutely convergent.*

Proof. Let $\alpha > 0$ be such that $\kappa \geqslant \frac{\log(M_1^2+1/\alpha)}{4(1-\alpha)}$. We will prove that for each $\eta \in \Lambda$ we have

$$\sum_{\eta' \in \Lambda} |\phi_{\beta,\kappa}(\eta')| e^{\alpha |\operatorname{supp} \eta'|} \mathbb{1}(\eta \sim \eta') \leqslant \alpha\kappa |\operatorname{supp} \eta| = \alpha \log \phi_{0,\kappa}(\eta).$$

Given this, the conclusion of the lemma follows from [17, Theorem 5.4] choosing the function $a(\eta) := -\alpha \log \phi_{0,\kappa}(\eta)$.

By the choice of α, we have

$$M_1^2 e^{-4\kappa(1-\alpha)} < 1$$

and

$$\frac{e^{-4\kappa(1-\alpha)}}{1 - M_1^2 e^{-4\kappa(1-\alpha)}} \leqslant \alpha.$$

Thus, for any $\eta \in \Lambda$, we have

$$\begin{aligned}\sum_{\eta' \in \Lambda} \phi_{\beta,\kappa}(\eta') e^{a(\eta')} \mathbb{1}(\eta \sim \eta') &= \sum_{\eta' \in \Lambda \colon \eta \sim \eta'} \phi_{\beta,\kappa}(\eta') \phi_{0,\kappa}(\eta')^{-\alpha} \leqslant \sum_{\eta' \in \Lambda \colon \eta \sim \eta'} \phi_{0,\kappa}(\eta')^{1-\alpha} \\ &\leqslant \sum_{e \in (\operatorname{supp} \eta)^+} \sum_{\{\eta'\} \in \Xi_{1,1^+,e}} \phi_{0,\kappa}(\eta')^{1-\alpha} = \sum_{e \in (\operatorname{supp} \eta)^+} \sum_{j=1}^{\infty} |\Xi_{1,j,e}| \left(e^{-4\kappa j}\right)^{1-\alpha}.\end{aligned}$$

Using Lemma 3.1, we bound the right hand side of the previous equation from above by

$$\big|(\operatorname{supp} \eta)^+\big| \sum_{j=1}^{\infty} M_1^{2j-2} e^{-4\kappa j(1-\alpha)} = \big|(\operatorname{supp} \eta)^+\big| \frac{e^{-4\kappa(1-\alpha)}}{1 - M_1^2 e^{-4\kappa(1-\alpha)}}.$$

The desired conclusion now follows from the choice of α. □

Lemma 3.4. *Let $\beta \geqslant 0$ and $\kappa > \kappa_0^{(Higgs)}$. Then*

$$\log Z^{(U)}_{N,\beta,\kappa} = \log Z^*_{N,\beta,\kappa} = \sum_{\mathcal{S} \in \Xi} \Psi_{\beta,\kappa}(\mathcal{S}). \tag{3.5}$$

Proof. The set $\Omega^1(B_N, G)$ is in bijection with the set of subsets of Λ with the property that the polymers in each subset have pairwise disjoint supports and that their differentials have pairwise disjoint supports. Therefore, for any β and κ we can write

$$Z^{(U)}_{N,\beta,\kappa} = \sum_{\Lambda' \subset \Lambda} \phi_\beta(\Lambda') \prod_{\{\nu,\nu'\}\subset\Lambda'} \mathbb{1}(\operatorname{supp}\nu \cap \operatorname{supp}\nu' = \emptyset).$$

On the other hand, if $\kappa > \kappa_0^{(\text{Higgs})}$, we can apply Proposition 5.3 of [17] to see that the right-hand side in the last display equals $\log Z^*_{N,\beta,\kappa}$ as defined in (3.3). From this the desired conclusion follows. □

All results in this section will assume that $\beta \geqslant 0$ and $\kappa \geqslant \kappa_0^{(\text{Higgs})}$, and using this assumption, we from now on write $Z^{(U)}_{N,\beta,\kappa}$ also for the cluster partition function $Z^*_{N,\beta,\kappa}$.

3.1.7 Cluster expansion of Wilson line observables

Consider the weighted cluster partition function

$$Z^{(U)}_{N,\beta,\kappa}[\gamma] := \exp\left(\sum_{\mathcal{S}\in\Xi} \Psi_{\beta,\kappa,\gamma}(\mathcal{S})\right), \tag{3.6}$$

where

$$\Psi_{\beta,\kappa,\gamma}(\mathcal{S}) := U(\mathcal{S})\phi_{\beta,\kappa}(\mathcal{S})\rho\big(\mathcal{S}(\gamma)\big).$$

The series on the right-hand side is absolutely convergent when $\beta \geqslant 0$ and $\kappa \geqslant \kappa_0^{(\text{Higgs})}$ by the proof of Lemma 3.3 since $\left|\rho\big(\mathcal{S}(\gamma)\big)\right| = 1$ for each $\mathcal{S} \in \Xi$. As in the proof of Lemma 3.4, using [17, Proposition 5.3], replacing the weight $\phi_\beta(\mathcal{S})$ by $\phi_\beta(\mathcal{S})\rho\big(\mathcal{S}(\gamma)\big)$, we have

$$\log Z^{(U)}_{\beta,\kappa,N}[\gamma] = \sum_{\sigma\in\Omega^1(B_N,G)} \phi_{\beta,\kappa}(\sigma)\rho\big(\sigma(\gamma)\big).$$

Combining (3.6) and Lemma 3.4 we obtain the following result, which give us an alternative expression for $\log \mathbb{E}^{(U)}_{N,\beta,\kappa}[W_\gamma]$.

Proposition 3.5. *Let $\beta \geqslant 0$ and $\kappa > \kappa_0^{(\text{Higgs})}$. Then for all N such that $\operatorname{supp}\gamma \subset B_N$,*

$$-\log \mathbb{E}^{(U)}_{N,\beta,\kappa}[W_\gamma] = \sum_{\mathcal{S}\in\Xi}\big(\Psi_{\beta,\kappa}(\mathcal{S}) - \Psi_{\beta,\kappa,\gamma}(\mathcal{S})\big) = \sum_{\mathcal{S}\in\Xi}\Psi_{\beta,\kappa}(\mathcal{S})\Big(1-\rho\big(\mathcal{S}(\gamma)\big)\Big). \tag{3.7}$$

Proof. Using (3.6) and Lemma 3.4 we conclude that

$$\log \mathbb{E}^{(U)}_{N,\beta,\kappa}[W_\gamma] = \log \frac{Z^{(U)}_{N,\beta,\kappa}[\gamma]}{Z^{(U)}_{N,\beta,\kappa}},$$

which is what we wanted to prove. □

Remark 3.6. Notice that Proposition 3.5 implies that $\mathbb{E}^{(U)}_{N,\beta,\kappa}[W_\gamma] \in (0,1]$ when $\beta \geqslant 0$ and $\kappa \geqslant \kappa_0^{(\text{Higgs})}$. This fact is not a priori clear since $W_\gamma \in \{-1,1\}$ for every $\sigma \in \Omega^1(B_N,\mathbb{Z})$.

3.1.8 Cluster expansion of the Marcu-Fredenhagen ratio

In this section, we assume that two non-trivial paths γ_1 and γ_2 are given, with disjoint support and with the same endpoints so that $\gamma := \gamma_1 + \gamma_2$ is a loop. The goal of this section is to use the cluster expansions of (3.3) and (3.6) to give an expression of the Marcu-Fredenhagen ratio which uses the cluster expansions.

To simplify notation, we define

$$\rho_N(\gamma_1,\gamma_2) \coloneqq \frac{\mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_1}]\,\mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_2}]}{\mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_1+\gamma_2}]}.$$

We note that by the Ginibre inequality (see [9, Section 2.6]), the limit

$$\rho(\gamma_1,\gamma_2) \coloneqq \lim_{N\to\infty} \rho_N(\gamma_1,\gamma_2)$$

exists and is translation invariant.

Lemma 3.7. *Let $\beta \geqslant 0$ and $\kappa > \kappa_0^{(Higgs)}$. Then*

$$\log \rho_N(\gamma_1,\gamma_2) = -4 \sum_{\mathcal{S}\in\Xi} \Psi_{\beta,\kappa}(\mathcal{S}) \mathbb{1}\Big(\mathcal{S}(\gamma_1) = \mathcal{S}(\gamma_2) = 1\Big). \tag{3.8}$$

Proof. Using Proposition 3.5, we obtain

$$\begin{aligned}
\log \rho_N(\gamma_1,\gamma_2) &= \log \mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_1}] + \log \mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_2}] - \log \mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma}] \\
&= \sum_{\mathcal{S}\in\Xi} \Psi_{\beta,\kappa}(\mathcal{S}) \Big(\rho\big(\mathcal{S}(\gamma_1+\gamma_2)\big) - \rho\big(\mathcal{S}(\gamma_1)\big) - \rho\big(\mathcal{S}(\gamma_2)\big) + 1\Big) \\
&= \sum_{\mathcal{S}\in\Xi} \Psi_{\beta,\kappa}(\mathcal{S}) \Big(\rho\big(\mathcal{S}(\gamma_1)\big)\rho\big(\mathcal{S}(\gamma_2)\big) - \rho\big(\mathcal{S}(\gamma_1)\big) - \rho\big(\mathcal{S}(\gamma_2)\big) + 1\Big) \\
&= \sum_{\mathcal{S}\in\Xi} \Psi_{\beta,\kappa}(\mathcal{S}) \Big(\rho\big(\mathcal{S}(\gamma_1)\big) - 1\Big)\Big(\rho\big(\mathcal{S}(\gamma_2)\big) - 1\Big) \\
&= -4 \sum_{\mathcal{S}\in\Xi} \Psi_{\beta,\kappa}(\mathcal{S}) \mathbb{1}\Big(\mathcal{S}(\gamma_1) = \mathcal{S}(\gamma_2) = 1\Big).
\end{aligned}$$

This concludes the proof. □

3.1.9 Cluster expansion of the covariance of two Wilson lines

In this section, we use Lemma 3.7 to give an upper bound of the covariance of two Wilson lines as a sum over clusters. This result, Lemma 3.8 below, will be the main ingredient in the proof of Theorem 1.4.

Lemma 3.8. *Let $\beta \geqslant 0$ and $\kappa > \kappa_0^{(Higgs)}$. Further, let γ_1 and γ_2 be two paths with disjoint support. Then*

$$\big|\mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_1+\gamma_2}] - \mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_1}]\mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_2}]\big| \leqslant 4 \Big|\sum_{\mathcal{V}\in\Xi} \Psi_\beta(\mathcal{V}) \mathbb{1}\big(\mathcal{S}(\gamma_1) = \mathcal{S}(\gamma_2) = 1\big)\Big|. \tag{3.9}$$

Proof. Since the function $x \mapsto e^x$ is convex for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
&\big|\mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_1+\gamma_2}] - \mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_1}]\mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_2}]\big| \\
&\qquad \leqslant \big|\log \mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_1+\gamma_2}] - \log \mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_1}] - \log \mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_2}]\big| = \big|\log \rho_N(\gamma_1,\gamma_2)\big|.
\end{aligned}$$

Using Lemma 3.7, the desired conclusion immediately follows. □

One interpretation of the right-hand-side of (3.9) is that a cluster $\mathcal{S} \in \Xi$ makes a non-zero contribution to the covariance on the left-hand-side of (3.9) only if its support connects the two paths γ_1 and γ_2.

3.2 Upper bound for clusters

In this section we give upper bounds on sums over the activity of sets of clusters which naturally arise in the proofs of Theorem 1.1 and Theorem 1.4.

Lemma 3.9. *Let $\beta \geqslant 0$ and $\kappa > \kappa_0^{(Higgs)}$, and let $e \in C_1(B_N)^+$. Then*

$$\sum_{\mathcal{S} \in \Xi_{1+,1+,e}} \left|\Psi_{\beta,\kappa}(\mathcal{S})\right| \leqslant \frac{e^{-(2\kappa-\alpha)2}}{1-4M_1^2 e^{-2(2\kappa-\alpha)}},$$

where α is as in (3.4).

Proof. By Lemma 3.3, for any $\eta \in \Lambda$, we have

$$\sum_{\mathcal{S} \in \Xi \colon \eta \in \mathcal{S}} \left|\Psi_{\beta,\kappa}(\mathcal{S})\right| \leqslant \phi_{\beta,\kappa}(\eta)\big(\phi_{0,\kappa}(\eta)\big)^{-\alpha},$$

and hence

$$\sum_{\mathcal{S} \in \Xi_{1+,1+,e}} \left|\Psi_{\beta,\kappa}(\mathcal{S})\right| \leqslant \sum_{\{\eta\} \in \Xi_{1,1+,e}} \sum_{\mathcal{S} \in \Xi \colon \eta \in \mathcal{S}} \left|\Psi_{\beta,\kappa}(\mathcal{S})\right| \leqslant \sum_{\{\eta\} \in \Xi_{1,1+,e}} \phi_{\beta,\kappa}(\eta)\big(\phi_{0,\kappa}(\eta)\big)^{-\alpha}, \tag{3.10}$$

By Lemma 3.1, we have

$$\sum_{\{\eta\} \in \Xi_{1,1+,e}} \phi_{\beta,\kappa}(\eta)\big(\phi_{0,\kappa}(\eta)\big)^{-\alpha} \leqslant \sum_{j=1}^{\infty} |\Xi_{1,j,e}| e^{-2(2\kappa-\alpha)j} = \sum_{j=1}^{\infty} (2M_1)^{2j-2} e^{-(2\kappa-\alpha)2j}. \tag{3.11}$$

Combining (3.10) and (3.11), the desired conclusion immediately follows. □

Lemma 3.10. *Let $\beta \geqslant 0$ and $\kappa > \kappa_0^{(Higgs)}$. Further, let $k \geqslant 1$ and $e \in C_1(B_N)$. Then, for any $\varepsilon > 0$, we have*

$$\sum_{\mathcal{S} \in \Xi_{1+,k+,e}} \left|\Psi_{\beta,\kappa}(\mathcal{S})\right| \leqslant 4^{-1} C_\varepsilon e^{-4k(\kappa-\kappa_0^{(Higgs)}-\varepsilon)},$$

where C_ε is defined by

$$C_\varepsilon \coloneqq 4 \sup_{N \geqslant 1} \sup_{e \in C_1(B_N)} \sum_{\mathcal{S} \in \Xi_{1+,1+,e}} \left|\Psi_{0,\kappa_0^{(Higgs)}+\varepsilon}(\mathcal{S})\right| < \frac{4e^{-(2\kappa-\alpha)2}}{1-4M_1^2 e^{-2(2\kappa-\alpha)}}. \tag{3.12}$$

Proof. Let $\varepsilon > 0$. From Lemma 3.9 we immediately get the upper bound on C_ε in (3.12), and hence C_ε is well defined.

For any $\mathcal{S} \in \Xi$, we have $\phi_{\beta,\kappa}(\mathcal{S}) = e^{-2\beta\|\mathcal{S}\|_2 - 2\kappa\|\mathcal{S}\|_1}$ and $\Psi_{\beta,\kappa}(\mathcal{S}) = U(\mathcal{S})\psi_{\beta,\kappa}(\mathcal{S})$, where $U(\mathcal{S})$ does not depend on β and κ, and hence

$$\Psi_{\beta,\kappa}(\mathcal{S}) = e^{-2(\kappa-\kappa_0^{(\mathrm{Higgs})}-\varepsilon)\|\mathcal{S}\|_1} \Psi_{\beta,\kappa_0^{(\mathrm{Higgs})}+\varepsilon}(\mathcal{S}).$$

Using this observation, we obtain

$$\begin{aligned}\sum_{\mathcal{S} \in \Xi_{1+,k+,e}} \left|\Psi_{\beta,\kappa}(\mathcal{S})\right| &\leqslant e^{-4(\kappa-\kappa_0^{(\mathrm{Higgs})}-\varepsilon)k} \sum_{\mathcal{S} \in \Xi_{1+,k+,e}} \left|\Psi_{\beta,\kappa_0^{(\mathrm{Higgs})}+\varepsilon}(\mathcal{S})\right| \\ &\leqslant e^{-4(\kappa-\kappa_0^{(\mathrm{Higgs})}-\varepsilon)k} \sum_{\mathcal{S} \in \Xi_{1+,1+,e}} \left|\Psi_{\beta,\kappa_0^{(\mathrm{Higgs})}+\varepsilon}(\mathcal{S})\right|.\end{aligned}$$

This concludes the proof. □

3.3 Proof of Theorem 1.4

In this section we give a proof of Theorem 1.4, which gives an exponential upper bound on the decay of correlations.

Proof of Theorem 1.4. If $\mathcal{S} \in \Xi$ is such that $\mathcal{S}(\gamma_1) = \mathcal{S}(\gamma_2) = 1$, then there is some edge $e \in \operatorname{supp} \gamma_1$ such that $\mathcal{S} \in \Xi_{1^+, \operatorname{dist}(\gamma_1,\gamma_2), e}$. Using Lemma 3.8, it follows that

$$\Big|\mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_1+\gamma_2}] - \mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_1}]\mathbb{E}^{(U)}_{N,\beta,\kappa}[W_{\gamma_2}]\Big| \leqslant 4\Big|\sum_{\mathcal{S}\in\Xi} \Psi_{\beta,\kappa}(\mathcal{S})\mathbb{1}\big(\mathcal{S}(\gamma_1) = \mathcal{S}(\gamma_2) = 1\big)\Big|$$
$$\leqslant 4 \sum_{e \in \operatorname{supp} \gamma} \sum_{\mathcal{S}\in\Xi_{1^+,\operatorname{dist}(\gamma_1,\gamma_2),e}} \big|\Psi_{\beta,\kappa}(\mathcal{S})\big|.$$

Using Lemma 3.10, the desired conclusion immediately follows. □

3.4 Proof of Theorem 1.1

Before giving a proof of Theorem 1.1 we state and prove a useful lemma.

Lemma 3.11. *In the setting of Theorem 1.1, for any $n \geqslant 1$ and $k \geqslant 1$, we have*

$$\sum_{\mathcal{S}\in\Xi_{1^+,k^+}} \Big|\Psi_{\beta,\kappa}(\mathcal{S})\mathbb{1}\Big(\mathcal{S}(\gamma_1^{(n)}) = \mathcal{S}(\gamma_2^{(n)}) = 1\Big)\Big|$$
$$\leqslant 2^{-1} C_\varepsilon \sum_{j=1}^{\infty} e^{-4\max(j,k)(\kappa-\kappa_0^{(Higgs)}-\varepsilon)} + \max(T_n - 2R_n, 0) 4^{-1} C_\varepsilon e^{-4\max(2R_n,k)(\kappa-\kappa_0^{(Higgs)}-\varepsilon)}$$

where C_ε is defined in (3.12).

Proof. Let $k \geqslant 1$, $n \geqslant 1$, and let $\gamma_1 = \gamma_1^{(n)}$ and $\gamma_2 = \gamma_2^{(n)}$. Further, let be such that $\mathcal{S} \in \Xi_{1^+,k^+}$ and $\mathcal{S}(\gamma_1) = \mathcal{S}(\gamma_2) = 1$. Since $\mathcal{S}(\gamma_1) = 1$, there is $e \in \operatorname{supp}\gamma_1 \cap \operatorname{supp}\mathcal{S}$. Since $\mathcal{S}(\gamma_2) = 1$, we also have $\operatorname{supp}\gamma_2 \cap \operatorname{supp}\mathcal{S} \neq \emptyset$. Consequently, we must have $\big|(\operatorname{supp}\mathcal{S})^+\big| \geqslant \operatorname{dist}(e,\gamma_2)$, and hence

$$\sum_{\mathcal{S}\in\Xi_{1^+,k^+}} \Big|\Psi_{\beta,\kappa}(\mathcal{S})\mathbb{1}\Big(\mathcal{S}(\gamma_1) = \mathcal{S}(\gamma_2) = 1\Big)\Big| \leqslant \sum_{\mathcal{S}\in\Xi_{1^+,k^+}} \big|\Psi_{\beta,\kappa}(\mathcal{S})\big|\mathbb{1}\Big(\mathcal{S}(\gamma_1) = \mathcal{S}(\gamma_2) = 1\Big)$$
$$\leqslant \sum_{e\in\gamma_1} \sum_{\mathcal{S}\in\Xi_{1^+,k^+,e}} \big|\Psi_{\beta,\kappa}(\mathcal{S})\big|\mathbb{1}\Big(\big|(\operatorname{supp}\mathcal{S})^+\big| > \operatorname{dist}(e,\gamma_2)\Big)$$
$$\leqslant \sum_{j=1}^{\infty} \sum_{\substack{e\in\gamma_1:\\ \operatorname{dist}(e,\gamma_2)=j}} \sum_{\mathcal{S}\in\Xi_{1^+,\max(j,k)^+,e}} \big|\Psi_{\beta,\kappa}(\mathcal{S})\big|.$$

Note that for $j \geqslant 1$, if $j \neq 2R_n$ then there are at most two edges $e \in \gamma_1$ such that $\operatorname{dist}(e,\gamma_2) = j$. Also, there are at most $\max(T_n - 2R_n, 0)$ edges $e \in \gamma_1$ such that $\operatorname{dist}(e,\gamma_2) = 2R_n$. Using Lemma 3.10, we thus obtain the desired conclusion. □

Proof of Theorem 1.1. Let $k \geqslant 1$, and let $n_k \geqslant 1$ be such that $R_n, T_n > k$ for all $n \geqslant n_k$. Further, assume that N is large enough to guarantee that $\operatorname{dist}(\gamma^{(n)}, \partial B_N) > k$. Then, for any $n \geqslant n_k$, we have that

$$a_k \coloneqq \sum_{\mathcal{S}\in\Xi_{1^+,k^-}} \Psi_{\beta,\kappa}(\mathcal{S})\mathbb{1}\Big(\mathcal{S}(\gamma_1^{(n)}) = \mathcal{S}(\gamma_2^{(n)}) = 1\Big)$$

is well defined, and does not depend on n nor the choice of $(R_n)_{n\geqslant 1}$ and $(T_n)_{n\geqslant 1}$. Moreover, by Lemma 3.11, the limit $\lim_{k\to\infty} a_k$ exists. Since we for all $k \geqslant 1$ have

$$a_k < \delta_{1,\varepsilon} < \infty,$$

it follows that $\lim_{k\to\infty} a_k < \infty$.

We now show that (i) and (ii) holds. To this end, let $n \geqslant n_k$. Then, by Lemma 3.7, we have

$$-4^{-1}\log \rho_N(\gamma_1^{(n)},\gamma_2^{(n)}) = a_k + \sum_{\mathcal{S}\in\Xi_{1+,k+}} \Psi_{\beta,\kappa}(\mathcal{S})\mathbb{1}\Big(\mathcal{S}(\gamma_1^{(n)}) = \mathcal{S}(\gamma_2^{(n)}) = 1\Big).$$

Using Lemma 3.11 and the Ginibre inequality, it follows that

$$\lim_{n\to\infty} -4^{-1}\log \rho(\gamma_1^{(n)},\gamma_2^{(n)})$$

exists and is equal to $\lim_{k\to\infty} a_k$, and hence

$$\rho = \lim_{n\to\infty} \rho(\gamma_1^{(n)},\gamma_2^{(n)})$$

exists and is equal to $e^{-4\lim_{k\to\infty} a_k}$. Since $\lim_{k\to\infty} a_k < \infty$, it follows that $\rho > 0$. This completes the proof of (i) and (ii).

We now show that the (iii) holds. To this end, let $n \geqslant 1$, and let $k = \min(R_n, T_n)$ (this guarantees that $n \geqslant n_k$). Then

$$\begin{aligned}\big|\log\rho(\gamma_1^{(n)},\gamma_2^{(n)}) - \log\rho\big| &= \big|\log\rho(\gamma_1^{(n)},\gamma_2^{(n)}) - (-4\lim_{k\to\infty} a_k)\big| \\ &= 4\big|-4^{-1}\log\rho(\gamma_1^{(n)},\gamma_2^{(n)}) - \lim_{k\to\infty} a_k\big| \\ &\leqslant 4\Big(\big|-4^{-1}\log\rho(\gamma_1^{(n)},\gamma_2^{(n)}) - a_k\big| + \big|a_k - \lim_{k\to\infty} a_k\big|\Big)\end{aligned}$$

Using Lemma 3.11, we thus obtain

$$\begin{aligned}&\big|\log\rho(\gamma_1^{(n)},\gamma_2^{(n)}) - \log\rho\big| \\ &\quad\leqslant 4C_\varepsilon \sum_{j=1}^{\infty} e^{-4\max(j,k)(\kappa-\kappa_0^{(\mathrm{Higgs})}-\varepsilon)} + 2C_\varepsilon \max(T_n - 2R_n, 0) e^{-4\max(2R_n,k)(\kappa-\kappa_0^{(\mathrm{Higgs})}-\varepsilon)}.\end{aligned}$$

This shows that (iii) holds, and thus completes the proof. □

4 The confinement phase (β and κ both small)

In this section, we prove our main result for the confinement phase, Theorem 1.2. The proof strategy is similar to that of Theorem 1.1, with the main difference that we here need to use a high temperature expansion before we can find a convergent cluster expansion.

4.1 A high temperature expansion

In this section, we recall the high temperature expansion of Ising lattice gauge theory from [12]. To this end, for $a \geqslant 0$ and $j \in \{0,1\}$, we define

$$\varphi_a(j) := \begin{cases} 1 & \text{if } j = 0 \\ \tanh 2a & \text{if } j = 1. \end{cases}$$

Let γ be a path. For $\omega \in \Omega^2(B_N, \mathbb{Z}_2)$, we let

$$\varphi_{\beta,\kappa}^{\gamma}(\omega) := \prod_{p\in C_2(B_N)^+} \varphi_\beta\big(\omega(p)\big) \prod_{e\in C_1(B_N)^+} \frac{\varphi_\kappa\big(\delta\omega(e) + \gamma[e]\big)}{\varphi_\kappa(\gamma[e])}. \tag{4.1}$$

Further, we let

$$\hat{Z}_{N,\beta,\kappa}[\gamma] := \varphi_\kappa(1)^{|\gamma|} \sum_{\omega\in\Omega^2(B_N,\mathbb{Z}_2)} \varphi^\gamma_{\beta,\kappa}(\omega). \tag{4.2}$$

The high temperature expansion of the lattice Higgs model is given by the following lemma.

Lemma 4.1 (Proposition 4.1 in [12]). *Let $\beta,\kappa \geqslant 0$. Then, for any path γ, we have*

$$\mathbb{E}^{(U)}_{N,\beta,\kappa}[W_\gamma] = \frac{\varphi_\kappa(1)^{|\gamma|}\hat{Z}_{N,\beta,\kappa}[\gamma]}{\hat{Z}_{N,\beta,\kappa}[0]},$$

and thus

$$\rho_N(\gamma_1,\gamma_2) = \frac{\hat{Z}_{N,\beta,\kappa}[\gamma_1]\hat{Z}_{N,\beta,\kappa}[\gamma_2]}{\hat{Z}_{N,\beta,\kappa}[\gamma_1+\gamma_2]\hat{Z}_{N,\beta,\kappa}[0]}.$$

For a path γ, we also define

$$\hat{Z}^\gamma_{N,\beta,\kappa} := \sum_{\substack{\omega\in\Omega^2(B_N,\mathbb{Z}_2)\colon \\ \nexists p\in\operatorname{supp}\omega\colon\, p\sim\gamma}} \varphi^0_{\beta,\kappa}(\sigma).$$

We note that for any path γ, we have

$$\hat{Z}^\gamma_{N,\beta,\kappa} \leqslant \hat{Z}_{N,\beta,\kappa}.$$

4.2 A cluster expansion

In this section we will describe the cluster expansion of [17] for the model obtained from the high temperature expansion in the previous section. The setup for this expansion will be very similar with the corresponding setup in the Higgs phase, and for this reason we will aim to keep the exposition short here.

4.2.1 Polymers

In this section, we assume that a path $\gamma \in \Lambda_0$ is given.

Let $\mathcal{G}_2$ be the graph with vertex set $C_2(B_N)^+$, an edge between two distinct plaquettes $p_1,p_2 \in C_2(B_N)^+$ if $\operatorname{supp}\partial p \cap \operatorname{supp}\partial p' \neq \emptyset$ (written $p_1 \sim p_2$). In other words, $p_1 \sim p_2$ if they have some common edge in their boundaries. Note that any $p \in C_2(B_N)^+$ has degree at most $M_2 := 4\cdot(2(m-1)-1) = 8m-12$ in $\mathcal{G}_2$.

For $\omega \in \Omega^2(B_N,\mathbb{Z}_2)$, we let $\mathcal{G}_2(\omega)$ be the subgraph of $\mathcal{G}_2$ induced by $(\operatorname{supp}\omega)^+$. In other words, $\mathcal{G}_2$ is the graph with vertex set $(\operatorname{supp}\omega)^+$ and an edge between two vertices $p_1,p_2 \in (\operatorname{supp}\omega)^+$ if $p_1\sim p_2$.

In this section, we let

$$\Lambda := \bigl\{\omega \in \Omega^2(B_N,\mathbb{Z}_2)\colon \mathcal{G}_2(\omega) \text{ has exactly one connected component}\bigr\}.$$

The spin configurations in Λ will be referred to as *polymers*.

4.2.2 Polymer interaction

For $\omega,\omega' \in \Lambda$, we write $\omega\sim\omega'$ if $\mathcal{G}_2(\omega)\cup\mathcal{G}_2(\omega')$ is a connected subgraph of $\mathcal{G}_2$.

In the notation of [17, Chapter 3], the model given by (4.2) corresponds to a model of polymers with polymers described in Section 4.2.1 and interaction function $\iota(\omega_1,\omega_2) := \zeta(\omega_1,\omega_2)+1$, where

$$\zeta(\omega_1,\omega_2) := \begin{cases} -1 & \text{if } \omega_1\sim\omega_2 \\ 0 & \text{else,} \end{cases} \qquad \omega_1,\omega_2\in\Lambda.$$

For $\omega \in \Omega^2(B_N, \mathbb{Z}_2)$, we write $\omega \sim \gamma$ if there is $p \in \operatorname{supp} \omega$ such that $\operatorname{supp} \partial p \cap \operatorname{supp} \gamma \neq \emptyset$. Finally, we also write $\gamma \sim \gamma$.

4.2.3 Clusters of polymers

Consider a multiset

$$\mathcal{S} = \{\underbrace{\eta_1, \ldots, \eta_1}_{n_{\mathcal{S}}(\eta_1) \text{ times}}, \underbrace{\eta_2, \ldots, \eta_2}_{n_{\mathcal{S}}(\eta_2) \text{ times}}, \ldots, \underbrace{\eta_k, \ldots, \eta_k}_{n_{\mathcal{S}}(\eta_k) \text{ times}}\} = \{\eta_1^{n(\eta_1)}, \ldots, \eta_k^{n(\eta_k)}\},$$

where $\eta_1, \ldots, \eta_k \in \Lambda$ are distinct and $n(\eta) = n_{\mathcal{S}}(\eta)$ denotes the number of times η occurs in $\mathcal{S}$. Following [17, Chapter 3], we say that $\mathcal{S}$ is *decomposable* if it is possible to partition $\mathcal{S}$ into disjoint multisets. That is, if there exist non-empty and disjoint multisets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ such that $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ and such that for each pair $(\eta_1, \eta_2) \in \mathcal{S}_1 \times \mathcal{S}_2$, $\eta_1 \nsim \eta_2$. If $\mathcal{S}$ is not decomposable, we say that $\mathcal{S}$ is a *cluster*. We stress that a cluster is unordered and may contain several copies of the same polymer.

We let Ξ be the set of all clusters.

For $\mathcal{S} \in \Xi$, we let $\mathcal{S}^0$ be the the set of all $\eta \in \mathcal{S}$ such that $\omega \nsim \gamma$. We let $\mathcal{S}^\gamma \coloneqq \mathcal{S} \smallsetminus \mathcal{S}^0$.

Given a cluster $\mathcal{S} \in \Xi$, we let

$$\|\mathcal{S}\| = \sum_{\eta \in \Lambda} n_{\mathcal{S}}(\eta) \big|(\operatorname{supp} \eta)^+\big|,$$

and

$$\|\mathcal{S}\|_\gamma \coloneqq \sum_{\eta \in \Lambda} n_{\mathcal{S}^\gamma}(\eta) \big|\operatorname{supp} \delta\eta \cap \operatorname{supp} \gamma\big|.$$

We also let

$$\operatorname{supp} \mathcal{S} = \bigcup_{\eta \in \Lambda} (\operatorname{supp} \eta)^+.$$

For $p \in C_2(B_N)^+$ and $j \geqslant 1$, we let

$$\Xi_{1,j,p} \coloneqq \big\{\{\eta\} \colon \eta \in \Lambda \text{ and } p \in \operatorname{supp} \eta \text{ and } \|\{\omega\}\| = j\big\}$$

and

$$\Xi_{1+,j+,p} \coloneqq \big\{\mathcal{S} \in \Xi \colon p \in \operatorname{supp} \mathcal{S} \text{ and } \|S\| \geqslant j\big\}.$$

As in Section 3.1.3, these sets depends on N, and when this is important, we write $\Xi(B_N)$, $\Xi_{1,j,p})(B_N)$, and $\Xi_{1+,j+,p})(B_N)$ respectively.

The following lemma is analogous to Lemma 3.1 and gives an upper bound on the number of clusters of a given size and with a given plaquette in its support.

Lemma 4.2. *For any $k \geqslant 1$ and $p \in C_2(B_N)^+$, we have $|\Xi_{1,k,p}| \leqslant M_2^{2k-2}$.*

Proof. Let $k \geqslant 1$ and $p \in C_2(B_N)^+$. Let $\mathcal{P}$ be the set of all paths in $\mathcal{G}_2$ that starts at p and have length $2k-1$. Since each vertex in $\mathcal{G}_2$ has degree at most M_2, we have $|\mathcal{P}| \leqslant M_2^{2k-2}$.

For any $\{\omega\} \in \Xi_{1,k,p}$, the graph $\mathcal{G}_2(\omega)$ is connected, and hence $\mathcal{G}_2(\omega)$ has a spanning path $T_\omega \in \mathcal{P}$ of length $2k-1 = 2\big|(\operatorname{supp} \omega)^+\big| - 1$ which starts at p. Since the map $\omega \mapsto T_\omega$ is an injective map from $\Xi_{1,k,p}$ to $\mathcal{P}$ and $|\mathcal{P}| \leqslant M_2^{2k-2}$, the desired conclusion immediately follows. $\square$

4.2.4 The activity of clusters

We extend the notion of activity from (4.1) to clusters $\mathcal{S} \in \Xi$ as follows.

$$\varphi^{\gamma}_{\beta,\kappa}(\mathcal{S}) \coloneqq \prod_{\eta \in \mathcal{S}} \varphi^{\gamma}_{\beta,\kappa}(\eta).$$

Note that

$$\varphi^{\gamma}_{\beta,\kappa}(\mathcal{S}) = \varphi^{0}_{\beta,\kappa}(\mathcal{S}) \varphi_\kappa(1)^{-\|\mathcal{S}\|_\gamma}.$$

The following lemma, which we now state and prove, will be useful later.

Lemma 4.3. *Let $\beta, \kappa > 0$, let $\gamma \in \Gamma_n$, and let $\omega \in \Lambda$. Then*

$$\varphi^{\gamma}_{\beta,\kappa}(\omega) \leqslant \varphi_\beta(1)^{(1-o_n(1))|(\operatorname{supp}\omega)^+|}.$$

Proof. For any $\omega \in \Lambda$, we have

$$\varphi^{\gamma}_{\beta,\kappa}(\omega) = \varphi_\beta(1)^{|(\operatorname{supp}\omega)^+|} \varphi_\kappa(1)^{|(\operatorname{supp}\delta\omega)^+ \smallsetminus \operatorname{supp}\gamma| - |(\operatorname{supp}\delta\omega)^+ \cap \operatorname{supp}\gamma|}.$$

Assume first that $\omega \in \Lambda$ is such that $|(\operatorname{supp}\omega)^+| \geqslant R_n T_n$. In this case we have

$$\begin{aligned} \varphi^{\gamma}_{\beta,\kappa}(\omega) &\leqslant \varphi_\beta(1)^{|(\operatorname{supp}\omega)^+|} \varphi_\kappa(1)^{-|\gamma|} \\ &\leqslant \varphi_\beta(1)^{(1-o_n)(1)|(\operatorname{supp}\omega)^+|} \varphi_\beta(1)^{-o_n(1) R_n T_n} \varphi_\kappa(1)^{-(R_n+T_n)}. \end{aligned}$$

Since $R_n, T_n \to \infty$, $R_n T_n$ tends to infinity much quicker than $R_n + T_n$, and hence the desired conclusion follows if $o_n(1)$ goes to zero slowly enough.

Now instead assume that we are given $|(\operatorname{supp}\omega)^+| < R_n T_n$ and $|\operatorname{supp}\delta\omega \cap \operatorname{supp}\gamma| \neq 0$. Then $|\operatorname{supp}\delta\omega|$ is minimized if $\operatorname{supp}\omega$ is a subset of the flat surface q that spans $\gamma_1^{(n)} + \gamma_2^{(n)}$. Assume that this is the case. For each edge $e \in \operatorname{supp}\delta\omega \cap \operatorname{supp}\gamma$, either all plaquettes in $\operatorname{supp} q$ between e and the opposite side of $\gamma_1 + \gamma_2$ are in $\operatorname{supp}\omega$, or there is some such plaquette that is not in $\operatorname{supp}\omega$, and in this case there there is at leat two edges parallell to e that is in $(\operatorname{supp}\delta\omega)^+ \smallsetminus \operatorname{supp}\gamma$. Note that in the first of these cases, there are a total of at least $\min(R_n, T_n)$ plaquettes that are in $\operatorname{supp}\omega$. Moreover, each plaquette appearing in one of these sets are in sets corresponding to at most four edges. Hence

$$|(\operatorname{supp}\delta\omega)^+ \cap \operatorname{supp}\gamma| - |(\operatorname{supp}\delta\omega)^+ \smallsetminus \operatorname{supp}\gamma| \leqslant \frac{4|(\operatorname{supp}\omega)^+|}{\min(R_n, T_n)}.$$

Consequently, in this case, we have

$$\varphi^{\gamma}_{\beta,\kappa}(\omega) \leqslant \varphi_\beta(1)^{|(\operatorname{supp}\omega)^+|} \varphi_\kappa(1)^{-\frac{4|(\operatorname{supp}\omega)^+|}{\min(R_n,T_n)}}.$$

From this the desired conclusion immediately follows. □

4.2.5 Ursell functions

The Ursell functions we will use in the confinement phase are analogous to the Ursell functions used for the Higgs phase (see Definition 3.2), except the set Λ is different and we use the graph $\mathcal{G}_2$ instead of $\mathcal{G}_1$.

4.2.6 Cluster expansion of the partition function

For $\beta \geqslant 0$, $\kappa \geqslant 0$, $\gamma \in \Lambda$, and $\mathcal{S} \in \Xi$, we let

$$\Psi^{\gamma}_{\beta,\kappa}(\mathcal{S}) \coloneqq U(\mathcal{S}) \varphi^{\gamma}_{\beta,\kappa}(\mathcal{S}).$$

Further, we let

$$\beta_0^{(\mathrm{conf})} \coloneqq \sup\bigl\{\beta \geqslant 0 \colon M_2^2\varphi_\beta(1) < 1 \text{ and } \exists \alpha \in (0,1) \text{ s.t. } \frac{M_2^3\varphi_\beta(1)^{1-\alpha}}{1-M_2^2\varphi_\beta(1)^{1-\alpha}} < 2\alpha\bigr\} \tag{4.3}$$

and for $\beta > \beta_0^{(\mathrm{conf})}$, we let

$$\alpha \coloneqq \alpha_\beta \coloneqq \inf\{\alpha \in (0,1) \colon \frac{M_2^3\varphi_\beta(1)^{1-\alpha}}{1-M_2^2\varphi_\beta(1)^{1-\alpha}} < 2\alpha\}. \tag{4.4}$$

Lemma 4.4. *Let $0 < \beta < \beta_0^{(conf)}$ and $\kappa > 0$, and let $\gamma \in \Gamma_n$. Then, for any $\omega \in \Xi$, we have*

$$\sum_{\mathcal{S}\in\Xi \colon \omega\in\mathcal{S}} \bigl|\Psi_{\beta,\kappa}^\gamma(\mathcal{S})\bigr| \leqslant \varphi_{\beta,\kappa}^\gamma(\omega)^{1-\alpha}.$$

Moreover,

$$\log \hat{Z}_{N,\beta,\kappa}[\gamma] = \sum_{\mathcal{S}\in\Xi^\gamma} \Psi_{\beta,\kappa}^\gamma(\mathcal{S}) \tag{4.5}$$

and

$$\log \hat{Z}_{N,\beta,\kappa}^\gamma = \sum_{\mathcal{S}\in\Xi \colon \mathcal{S}^\gamma=\emptyset} \Psi_{\beta,\kappa}^0(\mathcal{S}). \tag{4.6}$$

Furthermore, the series on the right-hand sides of (4.5) *and* (4.6) *are both absolutely convergent.*

Proof. For $\omega \in \Omega^2(B_N, \mathbb{Z}_2)$, let $a(\omega) \coloneqq -\alpha \log \varphi_{\beta,\kappa}^\gamma(\omega)$. We need to show that for any $\omega \in \Lambda$ we have

$$\sum_{\omega'\in\Lambda} \varphi_{\beta,\kappa}^\gamma(\omega') e^{a(\omega')} \mathbf{1}(\omega \sim \omega') \leqslant \alpha |\operatorname{supp}\omega|. \tag{4.7}$$

Given this, the conclusion follows from [17, Theorem 5.4].

We now show that (4.7) holds. To this end, fix any $\omega \in \Lambda$. Then

$$\begin{aligned}\sum_{\omega'\in\Lambda} \varphi_{\beta,\kappa}^\gamma(\omega') e^{a(\omega')} \mathbf{1}(\omega \sim \omega') &= \sum_{\omega'\in\Lambda} \varphi_{\beta,\kappa}^\gamma(\omega')^{1-\alpha} \mathbf{1}(\omega \sim \omega') \\ &\leqslant \sum_{p\in(\operatorname{supp}\omega)^+} \sum_{p'\sim p} \sum_{\substack{\omega'\in\Lambda \colon \\ p'\in\operatorname{supp}\omega}} \varphi_{\beta,\kappa}^\gamma(\omega')^{1-\alpha}.\end{aligned}$$

Since any $p \in C_2(B_N)^+$ has degree at most M_2 in $\mathcal{G}_2$, we can bound the previous expression from above by

$$\begin{aligned}&M_2|(\operatorname{supp}\omega)^+| \sum_{j=1}^\infty \max_{p\in C_2(B_N)^+} \sum_{\{\omega'\}\in\Xi_{1,j,p}} \varphi_{\beta,\kappa}^\gamma(\omega')^{1-\alpha} \\ &\quad\leqslant M_2|(\operatorname{supp}\omega)^+| \sum_{j=1}^\infty \max_{p\in C_2(B_N)^+} |\Xi_{1,j,p}| \max_{\{\omega'\}\in\Xi_{1,j,p}} \varphi_{\beta,\kappa}^\gamma(\omega')^{1-\alpha}.\end{aligned}$$

Using Lemma 4.3 and Lemma 4.2, we obtain

$$\sum_{\{\omega'\}\in\Xi} \varphi_{\beta,\kappa}^\gamma(\omega') e^{a(\omega')} \mathbf{1}(\omega \sim \omega') \leqslant M_2|(\operatorname{supp}\omega)^+| \sum_{j=1}^\infty M_2^{2j-2} \varphi_\beta(1)^{(1-o_n(1))(1-\alpha)j}.$$

The desired conclusion now follows from the choice of α. ☐

4.3 Upper bounds for clusters

In this section we give upper bounds on sums over the activity of sets of clusters which naturally arise in the proofs of Theorem 1.2.

Lemma 4.5. *Let $0 < \beta < \beta_0^{(conf)}$ and $\kappa > 0$, let $\gamma \in \Gamma_n$, and let $p \in C_2(B_N)^+$. Then*

$$\sum_{\mathcal{S} \in \Xi_{1^+,1^+,p}} \left|\Psi^\gamma_{\beta,\kappa}(\mathcal{S})\right| \leqslant \frac{\varphi_\beta(1)^{(1-o_n(1))(1-\alpha)}}{1 - M_2^2 \varphi_\beta(1)^{(1-o_n(1))(1-\alpha)}} =: C^{(2)}_{\varepsilon,n} \tag{4.8}$$

where α is as in (4.4)*.*

Proof. By Lemma 4.4, for any $\omega \in \Xi$, we have

$$\sum_{\mathcal{S} \in \Xi \colon \omega \in \mathcal{S}} \left|\Psi^\gamma_{\beta,\kappa}(\mathcal{S})\right| \leqslant \varphi^\gamma_{\beta,\kappa}(\omega)^{1-\alpha}$$

and hence

$$\sum_{\mathcal{S} \in \Xi_{1^+,1^+,p}} \left|\Psi^\gamma_{\beta,\kappa}(\mathcal{S})\right| \leqslant \sum_{\{\omega\} \in \Xi_{1,1^+,p}} \sum_{\mathcal{S} \in \Xi \colon \omega \in \mathcal{S}} \left|\Psi^\gamma_{\beta,\kappa}(\mathcal{S})\right| \leqslant \sum_{\{\omega\} \in \Xi_{1,1^+,p}} \varphi^\gamma_{\beta,\kappa}(\omega)^{1-\alpha}. \tag{4.9}$$

Next, by Lemma 4.2 and Lemma 4.3, we have

$$\sum_{\{\omega\} \in \Xi_{1,1^+,p}} \varphi^\gamma_{\beta,\kappa}(\omega)^{1-\alpha} \leqslant \sum_{j=1}^\infty |\Xi_{1,j,p}| \max_{\omega \in \Xi_{1,j,p}} \varphi^\gamma_{\beta,\kappa}(\omega)^{1-\alpha} \leqslant \sum_{j=1}^\infty M_2^{2j-2} \varphi_\beta(1)^{(1-o_n(1))(1-\alpha)j}. \tag{4.10}$$

Combining (4.9) and (4.10), the desired conclusion immediately follows. □

Lemma 4.6. *Let $0 < \beta < \beta_0^{(conf)}$ and $\kappa > 0$, let $\gamma \in \Gamma_n$, and let $p \in C_2(B_N)^+$. Further, let $k \geqslant 1$ and $\varepsilon \in (0, \beta_0^{(conf)} - \beta)$. Then*

$$\sum_{\mathcal{S} \in \Xi_{1^+,k^+,p}} \left|\Psi^\gamma_{\beta,\kappa}(\mathcal{S})\right| \leqslant C^{(2)}_{\varepsilon,n} \Big(\frac{\varphi_\beta(1)}{\varphi_{\beta+\varepsilon}(1)}\Big)^k,$$

where $C^{(2)}_{\varepsilon,n}$ is defined in (4.8)*.*

Proof. Note first that, by definition, we have

$$\Psi^\gamma_{\beta,\kappa}(\mathcal{S}) = \Big(\frac{\varphi_\beta(1)}{\varphi_{\beta+\varepsilon}(1)}\Big)^{\|\mathcal{S}\|} \Psi^\gamma_{\beta+\varepsilon,\kappa}(\mathcal{S}).$$

Using this observation, we obtain

$$\begin{aligned}\sum_{\mathcal{S} \in \Xi_{1^+,k^+,p}} \left|\Psi^\gamma_{\beta,\kappa}(\mathcal{S})\right| &\leqslant \Big(\frac{\varphi_\beta(1)}{\varphi_{\beta+\varepsilon}(1)}\Big)^k \sum_{\mathcal{S} \in \Xi_{1^+,k^+,p}} \left|\Psi^\gamma_{\beta+\varepsilon,\kappa}(\mathcal{S})\right| \\ &\leqslant \Big(\frac{\varphi_\beta(1)}{\varphi_{\beta+\varepsilon}(1)}\Big)^k \sum_{\mathcal{S} \in \Xi_{1^+,1^+,p}} \left|\Psi^\gamma_{\beta+\varepsilon,\kappa}(\mathcal{S})\right|.\end{aligned}$$

Applying Lemma 4.5, we obtain the desired conclusion. □

4.4 Proof of Theorem 1.2

In this section we prove Theorem 1.2. Before doing so we provide a useful lemma.

Lemma 4.7. *Let* $0 < \beta < \beta_0^{(conf)}$ *and* $\kappa > 0$. *Further, let* $\gamma \in \Gamma_n$, *let* $k \geqslant 0$, *and let* $\varepsilon \in (0, \beta_0^{(conf)} - \beta)$. *Then*

$$\sum_{\substack{\mathcal{S}\in\Xi\colon \|\mathcal{S}\|\geqslant k,\\ \mathcal{S}^{\gamma_1}\neq\emptyset,\, \mathcal{S}^{\gamma_2}\neq\emptyset}} \bigl|\Psi^{\gamma}_{\beta,\kappa}(\mathcal{S})\bigr| \leqslant 2(m-1)C^{(2)}_{\varepsilon,n}\biggl(2\sum_{j=1}^{\infty}\Bigl(\frac{\varphi_\beta(1)}{\varphi_{\beta+\varepsilon}(1)}\Bigr)^{\max(j,k)} + \max(0,T_n-2R_n)\Bigl(\frac{\varphi_\beta(1)}{\varphi_{\beta+\varepsilon}(1)}\Bigr)^{2R_n}\biggr),$$

where $C^{(2)}_{\varepsilon,n}$ *is defined in* (4.8).

Proof. Note first that

$$\sum_{\substack{\mathcal{S}\in\Xi\colon \|\mathcal{S}\|\geqslant k,\\ \mathcal{S}^{\gamma_1}\neq\emptyset,\, \mathcal{S}^{\gamma_2}\neq\emptyset}} \bigl|\Psi^{\gamma}_{\beta,\kappa}(\mathcal{S})\bigr| \leqslant \sum_{e\in\gamma_1}\sum_{p\in\operatorname{supp}\hat\partial e}\sum_{\substack{\mathcal{S}\in\Xi_{1+,k+,p}\colon\\ \|S\|\geqslant \operatorname{dist}(e,\gamma_2)}} \bigl|\Psi^{\gamma}_{\beta,\kappa}(\mathcal{S})\bigr|. \tag{4.11}$$

Using Lemma 4.6 and noting that $|\operatorname{supp}\hat\partial e| \leqslant 2(m-1)$ for any $e \in C_1(B_N)$, we can upper bound the right hand side of (4.11) by

$$\begin{aligned}
&2(m-1)C^{(2)}_{\varepsilon,n}\sum_{e\in\gamma_1}\Bigl(\frac{\varphi_\beta(1)}{\varphi_{\beta+\varepsilon}(1)}\Bigr)^{\max(k,\operatorname{dist}(e,\gamma_2))}\\
&\quad\leqslant 2(m-1)C^{(2)}_{\varepsilon,n}\biggl(2\sum_{j=1}^{\infty}\Bigl(\frac{\varphi_\beta(1)}{\varphi_{\beta+\varepsilon}(1)}\Bigr)^{\max(j,k)} + \max(0,T_n-2R_n)\Bigl(\frac{\varphi_\beta(1)}{\varphi_{\beta+\varepsilon}(1)}\Bigr)^{2R_n}\biggr).
\end{aligned}$$

This concludes the proof. □

Proof of Theorem 1.2. By combining Lemma 4.1 and Lemma 4.4, we see that

$$\log \rho_N(\gamma_1^{(n)},\gamma_2^{(n)}) = \log \frac{\hat Z_{N,\beta,\kappa}[\gamma_1]\hat Z_{N,\beta,\kappa}[\gamma_2]}{\hat Z_{N,\beta,\kappa}[\gamma_1+\gamma_2]\hat Z_{N,\beta,\kappa}}.$$

To simplify notation, we now let $\gamma_1 \coloneqq \gamma_1^{(n)}$ and $\gamma_2 \coloneqq \gamma_2^{(n)}$.

Let $\varepsilon \in (0, \beta_0^{(\mathrm{conf})} - \beta)$. For each path $\gamma \in \Gamma_n$ we have

$$\begin{aligned}
\log \hat Z_{N,\beta,\kappa}[\gamma] = &\sum_{\mathcal{S}\in\Xi\colon\, \mathcal{S}^{\gamma_1+\gamma_2}=\emptyset} U(\mathcal{S})\varphi^{\gamma}_{\beta,\kappa}(\mathcal{S}) + \sum_{\mathcal{S}\in\Xi\colon\, \mathcal{S}^{\gamma_1}=\emptyset,\, \mathcal{S}^{\gamma_2}\neq\emptyset} U(\mathcal{S})\varphi^{\gamma}_{\beta,\kappa}(\mathcal{S})\\
&+ \sum_{\mathcal{S}\in\Xi\colon\, \mathcal{S}^{\gamma_1}\neq\emptyset,\, \mathcal{S}^{\gamma_2}=\emptyset} U(\mathcal{S})\varphi^{\gamma}_{\beta,\kappa}(\mathcal{S}) + \sum_{\mathcal{S}\in\Xi\colon\, \mathcal{S}^{\gamma_1}\neq\emptyset,\, \mathcal{S}^{\gamma_2}\neq\emptyset} U(\mathcal{S})\varphi^{\gamma}_{\beta,\kappa}(\mathcal{S}).
\end{aligned}$$

Note that if we e.g. have $\mathcal{S}^{\gamma_1} = \emptyset$, then $\varphi^{\gamma_1}_{\beta,\kappa}(\mathcal{S}) = \varphi^{0}_{\beta,\kappa}(\mathcal{S})$ and $\varphi^{\gamma_1+\gamma_2}_{\beta,\kappa}(\mathcal{S}) = \varphi^{\gamma_2}_{\beta,\kappa}(\mathcal{S})$. Using these observations, it follows that

$$\begin{aligned}
&\log \frac{\hat Z_{N,\beta,\kappa}[\gamma_1]\hat Z_{N,\beta,\kappa}[\gamma_2]}{\hat Z_{N,\beta,\kappa}[\gamma_1+\gamma_2]\hat Z_{N,\beta,\kappa}}\\
&\quad= \sum_{\mathcal{S}\in\Xi\colon\, \mathcal{S}^{\gamma_1}\neq\emptyset,\, \mathcal{S}^{\gamma_2}\neq\emptyset} U(\mathcal{S})\bigl(\varphi^{\gamma_1}_{\beta,\kappa}(\mathcal{S}) + \varphi^{\gamma_2}_{\beta,\kappa}(\mathcal{S}) - \varphi^{\gamma_1+\gamma_2}_{\beta,\kappa}(\mathcal{S}) - \varphi^{0}_{\beta,\kappa}(\mathcal{S})\bigr)\\
&\quad= \sum_{\mathcal{S}\in\Xi\colon\, \mathcal{S}^{\gamma_1}\neq\emptyset,\, \mathcal{S}^{\gamma_2}\neq\emptyset} \bigl(\Psi^{\gamma_1}_{\beta,\kappa}(\mathcal{S}) + \Psi^{\gamma_2}_{\beta,\kappa}(\mathcal{S}) - \Psi^{\gamma_1+\gamma_2}_{\beta,\kappa}(\mathcal{S}) - \Psi^{0}_{\beta,\kappa}(\mathcal{S})\bigr).
\end{aligned}$$

Now note that for any $k \geqslant 1$, $n \geqslant 1$ such that $R_n, T_n > k$, and N sufficiently large,

$$\sum_{\substack{\mathcal{S}\in\Xi\colon \|\mathcal{S}\|\leqslant k\\ \mathcal{S}^{\gamma_1}\neq\emptyset,\, \mathcal{S}^{\gamma_2}\neq\emptyset}} \big(\Psi^{\gamma_1}_{\beta,\kappa}(\mathcal{S}) + \Psi^{\gamma_2}_{\beta,\kappa}(\mathcal{S}) - \Psi^{\gamma_1+\gamma_2}_{\beta,\kappa}(\mathcal{S}) - \Psi^{0}_{\beta,\kappa}(\mathcal{S})\big)$$

is independent on n. Using Lemma 4.7, it follows that the limit $\lim_{n\to\infty}\lim_{N\to\infty}\rho_N(\gamma_1^{(n)}, \gamma_2^{(n)})$ exists. This completes the proof of the first part of Theorem 1.2.

Next, note that by Lemma 4.7, we have

$$\sum_{\mathcal{S}\in\Xi\colon \mathcal{S}^{\gamma_1}\neq\emptyset,\, \mathcal{S}^{\gamma_2}\neq\emptyset} \big(\Psi^{\gamma_1}_{\beta,\kappa}(\mathcal{S}) + \Psi^{\gamma_2}_{\beta,\kappa}(\mathcal{S}) - \Psi^{\gamma_1+\gamma_2}_{\beta,\kappa}(\mathcal{S}) - \Psi^{0}_{\beta,\kappa}(\mathcal{S})\big)$$
$$\geqslant -8(m-1)C^{(2)}_{\varepsilon,n}\bigg(2\sum_{j=1}^{\infty}\Big(\frac{\varphi_\beta(1)}{\varphi_{\beta+\varepsilon}(1)}\Big)^j + \max(0, T_n - 2R_n)\Big(\frac{\varphi_\beta(1)}{\varphi_{\beta+\varepsilon}(1)}\Big)^{2R_n}\bigg).$$

Define

$$C^{(2)}_{\varepsilon} \coloneqq \lim_{n\to\infty} C^{(2)}_{\varepsilon,n}. \tag{4.12}$$

Letting first $N\to\infty$ and then $n\to\infty$, we obtain

$$\lim_{n\to\infty}\log\rho(\gamma_1^{(n)}, \gamma_2^{(n)}) > \frac{-16(m-1)C^{(2)}_{\varepsilon}\tanh 2\beta}{\tanh(2\beta+2\varepsilon)-\tanh 2\beta}.$$

This concludes the proof. □

5 The free phase (β large and κ small)

In this section, we provide a proof of Theorem 1.3. The proof strategy is similar to that of Theorem 1.2, but here we use a different high temperature expansion and also have to deal with a few additional complications before we can use a cluster expansion.

5.1 A high temperature expansion

In this section, we use a high temperature expansion of the Ising lattice Higgs model to obtain alternative expressions for $Z_{N,\beta,\kappa}[\gamma]$ that are useful when κ is small and β is large.

When γ is a closed loop, we let q_γ be a corresponding oriented surface. We note that by the discrete Stoke's theorem (see, e.g., [10, Section 2.3.2]), when $\omega \in \Omega^2(B_N, \mathbb{Z}_2)$ is such that $d\omega = 0$, then $\omega(q_\gamma)$ does not depend on the choice of q_γ.

For a path $\gamma \in \Lambda_0$, we define

$$\check{Z}_{N,\beta,\kappa}[\gamma] \coloneqq \sum_{\substack{\omega\in\Omega^2(B_N,\mathbb{Z}_2)\colon\\ d\omega=0}} \sum_{\substack{\gamma'\in\Lambda_0\colon\\ \delta(\gamma+\gamma')=0}} e^{\beta\sum_{p\in C_2(B_N)}\rho(\omega(p))}(\tanh 2\kappa)^{|\gamma'|}\rho(\omega(q_{\gamma+\gamma'})). \tag{5.1}$$

The following lemma gives a connection between $Z^{(U)}_{N,\beta,\kappa}[\gamma]$ and $\check{Z}_{N,\beta,\kappa}[\gamma]$.

Lemma 5.1. *Let $\beta, \kappa \geqslant 0$. Then*

$$Z^{(U)}_{N,\beta,\kappa}[\gamma] = (\cosh 2\kappa)^{|C_1(B_N)^+|}\frac{|\Omega^1(B_N,\mathbb{Z}_2)|}{|\{\omega\in\Omega^2(B_N,\mathbb{Z}_2)\colon d\omega=0\}|}\check{Z}_{N,\beta,\kappa}[\gamma].$$

Proof. For any $\sigma \in \Omega^1(B_N, \mathbb{Z}_2)$, we have

$$e^{\kappa \sum_{e \in C_1(B_N)} \rho(\sigma(e))} = e^{2\kappa \sum_{e \in C_1(B_N)^+} \rho(\sigma(e))} = \prod_{e \in C_1(B_N)^+} (\cosh 2\kappa + \rho(\sigma(e)) \sinh 2\kappa)$$
$$= (\cosh 2\kappa)^{|C_1(B_N)^+|} \prod_{e \in C_1(B_N)^+} \Big(1 + \rho\big(\sigma(e)\big) \tanh \kappa\Big)$$
$$= (\cosh 2\kappa)^{|C_1(B_N)^+|} \sum_{\gamma' \in \Lambda_0} (\tanh 2\kappa)^{|\gamma'|} \rho\big(\sigma(\gamma')\big).$$

Using the definition of $Z^{(U)}_{N,\beta,\kappa}[\gamma]$, it follows that

$$Z^{(U)}_{N,\beta,\kappa}[\gamma] = \sum_{\sigma \in \Omega^1(B_N, \mathbb{Z}_2)} e^{\beta \sum_{p \in C_2(B_N)} \rho(d\sigma(p)) + \kappa \sum_{e \in C_1(B_N)} \rho(\sigma(e))} \rho(\sigma(\gamma))$$
$$= (\cosh 2\kappa)^{|C_1(B_N)^+|} \sum_{\sigma \in \Omega^1(B_N, \mathbb{Z}_2)} e^{\beta \sum_{p \in C_2(B_N)} \rho(d\sigma(p))} \sum_{\gamma' \in \Lambda_0} (\tanh 2\kappa)^{|\gamma'|} \rho(\sigma(\gamma + \gamma')).$$

Now note that if $\gamma' \in \Lambda_0$ is such that $\operatorname{supp} \gamma \cap \operatorname{supp} \gamma' \neq \emptyset$ and $\delta(\gamma + \gamma') \neq 0$, then by gauge invariance, we have

$$\sum_{\sigma \in \Omega^2(B_N, \mathbb{Z}_2)} e^{\beta \sum_{p \in C_2(B_N)} \rho(d\sigma(p))} (\tanh 2\kappa)^{|\gamma'|} \rho(\sigma(\gamma + \gamma')) = 0.$$

Using this observation it follows that

$$Z^{(U)}_{N,\beta,\kappa}[\gamma] = (\cosh 2\kappa)^{|C_1(B_N)^+|} \sum_{\sigma \in \Omega^2(B_N, \mathbb{Z}_2)} e^{\beta \sum_{p \in C_2(B_N)} \rho(d\sigma(p))}$$
$$\times \sum_{\gamma' \in \Lambda_0 \colon \delta(\gamma + \gamma') = 0} (\tanh 2\kappa)^{|\gamma'|} \rho(\sigma(\gamma + \gamma')).$$

Finally, using the Poincaré lemma (see, e.g., [6, Lemma 2.2]), we obtain

$$Z^{(U)}_{N,\beta,\kappa}[\gamma] = (\cosh 2\kappa)^{|C_1(B_N)^+|} \frac{|\Omega^1(B_N, \mathbb{Z}_2)|}{|\{\omega \in \Omega^2(B_N, \mathbb{Z}_2) \colon d\omega = 0\}|}$$
$$\cdot \sum_{\substack{\omega \in \Omega^2(B_N, \mathbb{Z}_2) \colon \\ d\omega = 0}} \sum_{\substack{\gamma' \in \Lambda_0 \colon \\ \delta(\gamma + \gamma') = 0}} e^{\beta \sum_{p \in C_2(B_N)} \rho(\omega(p))} (\tanh 2\kappa)^{|\gamma'|} \rho(\omega(q_{\gamma + \gamma'}))$$

as desired. This concludes the proof. □

When γ is a closed path, we verify in Section 5.2 that $\check{Z}_{N,\beta,\kappa}[\gamma]$ has a cluster expansion. However, when γ is an open path, this argument fails. For this reason, we now give an alternative expression for (5.1). To this end, for a closed loop γ and an open path γ_0, we let

$$\check{Z}_{N,\beta,\kappa}[\gamma, \gamma_0] \coloneqq \sum_{\substack{\omega \in \Omega^2(B_N, \mathbb{Z}_2) \colon \\ d\omega = 0}} \sum_{\substack{\gamma' \in \Omega^1(B_N, \mathbb{Z}_2) \colon \\ \gamma' \nsim \gamma_0,\, \delta\gamma' = 0}} e^{\beta \sum_{p \in C_2(B_N)} (\rho(\omega(p)) - 1)} (\tanh 2\kappa)^{|\gamma'|} \rho(\omega(q_{\gamma + \gamma'})).$$

Further, for any open connected path, we let $\mathcal{L}_\gamma$ be the set of all connected paths γ_0 such that $\gamma + \gamma_0$ is closed (see Figure 3a).

Lemma 5.2. *Let $\beta, \kappa \geqslant 0$. Then the following holds.*

(i) For any closed path γ, we have

$$\check{Z}_{N,\beta,\kappa}[\gamma] = \check{Z}_{N,\beta,\kappa}[\gamma, 0].$$

(ii) For any open connected path γ, we have

$$\check{Z}_{N,\beta,\kappa}[\gamma] = \sum_{\gamma_0 \in \mathcal{L}_\gamma} (\tanh 2\kappa)^{|\gamma_0|} \check{Z}_{N,\beta,\kappa}[\gamma+\gamma_0, \gamma_0].$$

Proof. Assume first that γ is a closed path. Then, by definition, we have

$$\check{Z}_{N,\beta,\kappa}[\gamma] = \sum_{\substack{\omega \in \Omega^2(B_N,\mathbb{Z}_2)\colon \\ d\omega=0}} \sum_{\substack{\gamma' \in \Lambda_0\colon \\ \delta\gamma'=0}} e^{\beta \sum_{p \in C_2(B_N)} \rho(\omega(p))} (\tanh 2\kappa)^{|\gamma'|} \rho(\omega(q_{\gamma+\gamma'})).$$

and hence (i) holds.

Now instead assume that γ is a connected open path with $|\operatorname{supp} \hat\partial \gamma| = 2$. Then, by definition, we have

$$\check{Z}_{N,\beta,\kappa}[\gamma] = \sum_{\gamma_0 \in \mathcal{L}_\gamma} (\tanh 2\kappa)^{|\gamma_0|} \cdot \sum_{\substack{\omega \in \Omega^2(B_N,\mathbb{Z}_2)\colon \\ d\omega=0}} \sum_{\substack{\gamma' \in \Lambda_0\colon \gamma' \nsim \gamma_0 \\ \delta\gamma'=0}} e^{\beta \sum_{p \in C_2(B_N)} \rho(\omega(p))} (\tanh 2\kappa)^{|\gamma'|} \rho(\omega(q_{\gamma+\gamma_0+\gamma'})).$$

This completes the proof of (ii). □

5.2 A cluster expansion

In this section, using the high temperature expansion of Section 5.1, we present a cluster expansion which is useful in the free phase.

5.2.1 Polymers

Let $\mathcal{G}_3$ be the graph with vertex set $C_2(B_N)^+$ and an edge between two distinct plaquettes $p_1, p_2 \in C_2(B_N)^+$ if $\operatorname{supp} \hat\partial p \cap \operatorname{supp} \hat\partial p' \neq \emptyset$ (written $p_1 \sim p_2$). Note that any $p \in C_2(B_N)^+$ has degree at most $M_3 := 10(m-2)$ in $\mathcal{G}_3$. For $\omega, \omega' \in \Omega^2(B_N, \mathbb{Z}_2)$, we write $\omega \sim \omega'$ if the subgraph of $\mathcal{G}_3$ induced by $(\operatorname{supp} \omega)^+ \cup (\operatorname{supp} \omega')^+$ is connected. When $\omega \in \Omega^2(B_N, \mathbb{Z}_2)$, we let $\mathcal{G}_3(\omega_2)$ be the subgraph of $\mathcal{G}_3$ induced by $(\operatorname{supp} \omega)^+$. We let

$$\Lambda_2 := \big\{\omega \in \Omega^2(B_N, \mathbb{Z}_2)\colon \mathcal{G}_3(\omega) \text{ has exactly one connected component}\big\}.$$

Recall the definitions of $\mathcal{G}_0$, M_0, and Λ_1 from Section 2.5.

For two paths $\gamma, \gamma' \in \Lambda_0$, we write $\gamma \sim \gamma'$ if the subgraph of $\mathcal{G}_0$ induced by $\operatorname{supp} \gamma \cup \operatorname{supp} \gamma'$ is connected.

In this section, thee elements in Λ_1 and Λ_2 will be referred to as *polymers*.

5.2.2 Polymer interaction

For $\omega, \omega' \in \Lambda_2$, we write $\omega \sim \omega'$ if $\mathcal{G}_3(\omega) \cup \mathcal{G}_3(\omega')$ is a connected subset of $\mathcal{G}_3$.

For $\gamma, \gamma' \in \Lambda_1$, we recall that $\gamma \sim \gamma'$ if $\mathcal{G}_0(\gamma) \cup \mathcal{G}_0(\gamma')$ is a connected subset of $\mathcal{G}_0$.

For $\gamma \in \Lambda_1$ and $\omega \in \Lambda_2$, we write $\gamma \sim \omega$ and $\omega \sim \gamma$ if $\rho(\omega(q_\gamma)) = -1$.

In the notation of [17, Chapter 3], the model described by $\check{Z}[\gamma, \gamma_0]$ corresponds to a model of polymers with polymers described in Section 5.2.1 and interaction function $\iota(\eta_1, \eta_2) := \zeta(\eta_1, \eta_2) + 1$, where

$$\zeta(\eta_1, \eta_2) := \iota(\eta_1, \eta_2) - 1 = \begin{cases} -2 & \text{if } \eta_1 \in \Lambda_1,\ \eta_2 \in \Lambda_2 \text{ and } \rho(\eta_2(\eta_1)) = -1 \\ -2 & \text{if } \eta_1 \in \Lambda_2,\ \eta_2 \in \Lambda_1 \text{ and } \rho(\eta_1(\eta_2)) = -1 \\ -1 & \text{if } \eta_1, \eta_2 \in \Lambda_1 \text{ and } \eta_1 \sim \eta_2 \\ -1 & \text{if } \eta_1, \eta_2 \in \Lambda_2 \text{ and } \eta_1 \sim \eta_2 \\ 0 & \text{else.} \end{cases}$$

5.2.3 Clusters of polymers

Consider a multiset

$$\mathcal{S} = \{\underbrace{\eta_1,\ldots,\eta_1}_{n_{\mathcal{S}}(\eta_1)\text{ times}}, \underbrace{\eta_2,\ldots,\eta_2}_{n_{\mathcal{S}}(\eta_2)\text{ times}}, \ldots, \underbrace{\eta_k,\ldots,\eta_k}_{n_{\mathcal{S}}(\eta_k)\text{ times}}\} = \{\eta_1^{n(\eta_1)},\ldots,\eta_k^{n(\eta_k)}\},$$

where $\eta_1,\ldots,\eta_k \in \Lambda_1 \cup \Lambda_2$ are distinct and $n(\eta) = n_{\mathcal{S}}(\eta)$ denotes the number of times η occurs in $\mathcal{S}$. Following [17, Chapter 3], we say that $\mathcal{S}$ is *decomposable* if it is possible to partition $\mathcal{S}$ into disjoint multisets. That is, if there exist non-empty and disjoint multisets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ such that $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ and such that for each pair $(\eta_1,\eta_2) \in \mathcal{S}_1 \times \mathcal{S}_2$, $\eta_1 \nsim \eta_2$. If $\mathcal{S}$ is not decomposable, we say that $\mathcal{S}$ is a *cluster of polymers*. We stress that such a cluster is unordered and may contain several copies of the same polymer.

In this section, we let Ξ be the set of all clusters.

When $\mathcal{S} \in \Xi$, we let $\mathcal{S}_1$ denote the multiset $\{\eta^{n(\eta)}\}_{\eta\in\mathcal{S}\colon \eta\in\Lambda_1}$ and analogously let $\mathcal{S}_2$ denote the multiset $\{\eta^{n(\eta)}\}_{\eta\in\mathcal{S}\colon \eta\in\Lambda_2}$. Further, we let

$$\|\mathcal{S}\| \coloneqq \sum_{\eta\in\mathcal{S}} n(\eta)|(\operatorname{supp}\eta)^+| \quad \text{and} \quad n(\mathcal{S}) \coloneqq \sum_{\eta\in\mathcal{S}} n(\mathcal{S}).$$

When $\gamma \in \Lambda_0$ and $v \in C_0(B_N)^+$, we write $v \sim \gamma$ if there is $e \in \gamma$ such that $v \in (\operatorname{supp}\partial e)^+$. Similarly, if $\omega \in \Omega^2(B_N,\mathbb{Z}_2)$ and $p \in C_2(B_N)^+$ we write $\omega \sim p$ if there is $p' \in (\operatorname{supp}\omega)^+$ such that $\operatorname{supp}\hat\partial p \cap \operatorname{supp}\hat\partial p' \neq \emptyset$.

When $\mathcal{S} \in \Xi$ and γ_0 is a path, we write $\mathcal{S}_1 \sim \gamma_0$ if there is $\gamma' \in \mathcal{S}_1$ such that $\gamma \sim \gamma_0$.

For $i \geqslant 0$, we let

$$\Xi_i \coloneqq \big\{\mathcal{S} \in \Xi\colon n(\mathcal{S}) = 1\big\}.$$

As in Sections 3.1.3 and 4.2.3, the sets Ξ and Ξ_i depend on N, but we usually suppress this dependency.

5.2.4 The activity of clusters

For a closed path $\gamma \in \Lambda_0$ and a path $\gamma_0 \in \Lambda_0$, we define the activity of clusters $\mathcal{S} \in \Xi$ by

$$\varphi^{\gamma,\gamma_0}_{\beta,\kappa}(\mathcal{S}) \coloneqq \prod_{\omega\in\mathcal{S}_2} \rho(\omega(\gamma)) e^{\beta\sum_p(\rho(\omega(p))-1)} \prod_{\gamma'\in\mathcal{S}_1} (\tanh 2\kappa)^{|\gamma'|}\mathbf{1}(\gamma' \nsim \gamma_0).$$

5.2.5 Ursell functions

The Ursell function which is relevant in the free phase, which we define below, is slightly different than the Ursell function associated to hard-core interaction which was used in the Higgs phase and the confinement phase.

Recall the definition of $\mathcal{G}^k$ from Section 3.1.5.

Definition 5.3 (The Ursell functions). *For $k \geqslant 1$ and $\eta_1,\eta_2,\ldots,\eta_k \in \Lambda_1\cup\Lambda_2$, we let*

$$U(\eta_1,\ldots,\eta_k) \coloneqq \frac{1}{k!}\sum_{G\in\mathcal{G}^k} (-1)^{|E(\mathcal{G})|} \prod_{\substack{(i,j)\in E(\mathcal{G})\colon\\ \eta_i,\eta_j\in\Lambda_1}} \mathbb{1}(\eta_i \sim \eta_j) \prod_{\substack{(i,j)\in E(\mathcal{G})\colon\\ \eta_i,\eta_j\in\Lambda_2}} \mathbb{1}(\eta_i \sim \eta_j) \cdot \prod_{\substack{(i,j)\in E(\mathcal{G})\colon\\ |\{\eta_i,\eta_j\}\cap\Lambda_1|=1}} 2\cdot\mathbb{1}(\eta_i \sim \eta_j).$$

Note that this definition is invariant under permutations of the polymers $\eta_1,\eta_2,\ldots,\eta_k$.

For $\mathcal{S} \in \Xi_k$, and any enumeration $\eta_1,\ldots,\eta_k$ (with multiplicities) of the polymers in $\mathcal{S}$, we define

$$U(\mathcal{S}) = k!\,U(\eta_1,\ldots,\eta_1). \tag{5.2}$$

Note that for any $\mathcal{S} \in \Xi_1$, we have $U(\mathcal{S}) = 1$, and for any $\mathcal{S} \in \Xi_2$, we have either $U(\mathcal{S}) = -1$ or $U(\mathcal{S}) = -2$.

5.2.6 Cluster expansion of the partition function

Before we state and prove the main result of this section, we will state and prove a few useful lemmas.

Lemma 5.4. *Let $\beta \geqslant 0$ and $\kappa \geqslant 0$. Further, let γ be a closed path, let γ_0 be a path, and let γ' be a non-empty path. Then, for any $a \in (0,1)$, we have*

$$\sum_{\substack{\gamma'' \in \Lambda_1 : \\ \gamma'' \sim \gamma'}} \varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\gamma''\})^a \leqslant \bigl|\{v \in C_0(B_N)^+ : v \sim \gamma'\}\bigr| \sum_{j=2}^{\infty} 2j(2m)^{2j} (\tanh 2\kappa)^{2aj}.$$

Proof. If $\gamma'' \in \Lambda_1$ is such that $\gamma'' \sim \gamma'$, then there must exist some $v \in C_0(B_N)^+$ such that $v \sim \gamma'$ and $v \in \gamma''$. Hence

$$\sum_{\substack{\gamma'' \in \Lambda_1 : \\ \gamma'' \sim \gamma'}} \varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\gamma''\})^a \leqslant \sum_{\substack{\gamma'' \in \Lambda_1 : \\ \gamma'' \sim \gamma'}} \varphi_{\beta,\kappa}^{0,0}(\{\gamma''\})^a = \sum_{\substack{\gamma'' \in \Lambda_1 : \\ \gamma'' \sim \gamma'}} (\tanh 2\kappa)^{a|\gamma''|}.$$

Since any $\gamma'' \in \Lambda_1$ is closed, $|\gamma''|$ is even. Moreover, any non-trivial $\gamma'' \in \Lambda_1$ satisfies $|\gamma''| \geqslant 4$. Combining these observations, we obtain

$$\sum_{\substack{\gamma'' \in \Lambda_1 : \\ \gamma'' \sim \gamma'}} (\tanh 2\kappa)^{a|\gamma''|} \leqslant \sum_{v \sim \gamma'} \sum_{j=2}^{\infty} \bigl|\{\gamma'' \in \Lambda_1 : v \sim \gamma'' \text{ and } |\gamma''| = 2j\}\bigr| (\tanh 2\kappa)^{2aj}.$$

Since any $v' \in C_0(B_N)^+$ has degree at most $2m$, for any $j \geqslant 1$ there can be at most $2j(2m)^j$ paths $\gamma'' \in \Lambda_1$ such that $v \in \gamma''$ and $|\gamma''| = j$, and hence

$$\bigl|\{\gamma'' \in \Lambda_1 : v \sim \gamma'' \text{ and } |\gamma''| = 2j\}\bigr| \leqslant 2j(2m)^{2j}.$$

Combining the above inequalities, the desired conclusion now follows. □

Lemma 5.5. *Let $\beta \geqslant 0$ and $\kappa \geqslant 0$. Further, let γ be a closed path, let γ_0 be a path, let $\omega \in \Lambda_2$, and let $a \in (0,1)$. Then*

$$\sum_{\substack{\omega' \in \Lambda_2 : \\ \omega' \sim \omega}} \bigl|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\omega'\})\bigr|^a \leqslant \bigl|(\operatorname{supp} \omega)^+\bigr| \sum_{j=2(m-1)}^{\infty} M_3^{2j+1} e^{-4\beta a j}.$$

Proof. Since any non-trivial $\omega' \in \Lambda_2$ satisfies $|(\operatorname{supp} \omega')^+| \geqslant 2(m-1)$ (see, e.g., [6]), we can write

$$\sum_{\substack{\omega' \in \Lambda_2 : \\ \omega' \sim \omega}} \bigl|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\omega'\})\bigr|^a = \sum_{\substack{\omega' \in \Lambda_2 : \\ \omega' \sim \omega}} e^{-2\beta a |\operatorname{supp} \omega'|}$$
$$\leqslant \sum_{p \in (\operatorname{supp} \omega)^+} \sum_{j=2(m-1)}^{\infty} \bigl|\{\omega' \in \Lambda_2 : \omega' \sim p \text{ and } |(\operatorname{supp} \omega')^+| = j\}\bigr| e^{-4\beta a j}.$$

If $\omega' \in \Lambda_2$ and $p \in (\operatorname{supp} \omega)^+$ are such that $\omega' \sim p$, then $\{p\} \cup (\operatorname{supp} \omega')^+$ is a induces a connected subgraph G of $\mathcal{G}_3$. Since G is a connected graph, it has a spanning path of length at most $2|(\operatorname{supp} \omega')^+| + 1$ that starts at p. The number of paths in $\mathcal{G}_3$ of length $2|(\operatorname{supp} \omega')^+| + 1$ that starts at p is at most $M_3^{2|(\operatorname{supp} \omega')^+|+1}$. Hence

$$\bigl|\{\omega' \in \Lambda_2 : \omega' \sim p \text{ and } |(\operatorname{supp} \omega')^+| = j\}\bigr| \leqslant M_3^{2j+1}.$$

Combining the above inequalities, we obtain the desired conclusion. □

Lemma 5.6. *Let $\beta \geqslant 0$ and $\kappa \geqslant 0$. Further, let γ be a closed path, let γ_0 be a path, let $\gamma' \in \Lambda_1$, and let $a \in (0,1)$. Then*

$$\sum_{\substack{\omega' \in \Lambda_2 :\\ \omega' \sim \gamma'}} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\omega'\})\right|^a \leqslant |\gamma'| \sum_{j=1}^{\infty} D_0 j^3 \sum_{k=\max(4j,2(d-1))}^{\infty} M_3^{2k-1}\, e^{-4\beta a k}.$$

Proof. Note first that

$$\sum_{\substack{\omega' \in \Lambda_2 :\\ \omega' \sim \gamma'}} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\omega'\})\right|^a \leqslant \sum_{e \in \gamma'} \sum_{j=1}^{\infty} \sum_{\substack{p \in C_2(B_N)^+\\ \operatorname{dist}(p,e)=j}} \sum_{\substack{\omega' \in \Lambda_2 :\, p \in \operatorname{supp} \omega',\\ \operatorname{dist}(\omega',\gamma') \geqslant j}} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\omega'\})\right|^a.$$

If $\omega' \in \Lambda_2$ is such that $\omega' \sim \gamma'$, then every oriented surface q with $\partial q = \gamma'\ \omega$ must intersect q, and hence the support of ω' must loop around γ'. Consequently, if $\omega' \in \Lambda_2$ is such that $\operatorname{dist}(\omega', \gamma') = j$ and $\omega' \sim \gamma'$, then we must have $|(\operatorname{supp} \omega')^+| \geqslant 4j$. Moreover, for any $\omega' \in \Lambda_2$, we have $|(\operatorname{supp} \omega')| \geqslant 2(m-1)$ (see, e.g., [6]). Consequently, for any $j \geqslant 1$ and $p \in C_2(B_N)^+$ such that $\operatorname{dist}(p,e) = j$, we have

$$\sum_{\substack{\omega' \in \Lambda_2 :\, p \in \operatorname{supp} \omega',\\ \operatorname{dist}(\omega',\gamma') \geqslant j}} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\omega'\})\right|^a$$
$$\leqslant \sum_{k=\max(4j,2(d-1))}^{\infty} \left|\{\omega' \in \Lambda_2 \colon p \in \operatorname{supp} \omega' \text{ and } |(\operatorname{supp} \omega')^+| = k\}\right| e^{-2\beta a \cdot 2k}.$$

Now note that any $\omega' \in \Lambda_2$ corresponds to a connected component in $\mathcal{G}_3$ which has a spanning path of length at most $2|(\operatorname{supp} \omega')^+| - 1$ that starts at some given $p \in \operatorname{supp} \omega'$. This implies in particular that for any $p \in C_2(B_N)^+$ and any $k \geqslant 1$, we have

$$\left|\{\omega' \in \Lambda_2 \colon p \in \operatorname{supp} \omega' \text{ and } |(\operatorname{supp} \omega')^+| = k\}\right| \leqslant M_3^{2k-1}.$$

Finally, we note that for any edge $e \in C_1(B_N)^+$ and $j \geqslant 1$, we have

$$\left|\{p \in C_2(B_N)^+ \colon \operatorname{dist}(e,p) = j\}\right| \leqslant D_0 j^3.$$

Combining the above equations, the desired conclusion follows. □

Lemma 5.7. *Let $\beta \geqslant 0$ and let $\kappa \geqslant 0$. Further, let γ be a closed path, let γ_0 be a path, and let $\omega \in \Lambda_2$. Further, let $a \in (0,1)$. Then*

$$\sum_{\substack{\gamma' \in \Lambda_1 :\\ \gamma' \sim \omega}} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\gamma'\})\right|^a \leqslant |(\operatorname{supp} \omega)^+| \sum_{j=1}^{\infty} D_0 j^3 \sum_{k=4j}^{\infty} (2m)^k (\tanh 2\kappa)^{ak}.$$

As the proof of Lemma 5.7 is completely analogous to the proof of Lemma 5.6, we omit it here.

For $\beta, \kappa \geqslant 0$, a closed loop γ, a path γ_0, and $\mathcal{S} \in \Xi$, we let

$$\Psi_{\beta,\kappa}^{\gamma,\gamma_0}(\mathcal{S}) := U(\mathcal{S}) \varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\mathcal{S}).$$

Note that with this notation, we have

$$\Psi_{\beta,\kappa}^{\gamma,\gamma_0}(\mathcal{S}) = \Psi_{\beta,\kappa}^{0,0}(\mathcal{S}) \prod_{\gamma' \in \mathcal{S}} \mathbf{1}(\gamma' \nsim \gamma_0) \prod_{\omega \in \mathcal{S}} \rho(\omega(\gamma)).$$

Proposition 5.8. *For $\alpha \in (0,1)$, there are $\beta_0^{(free)}(\alpha) > 0$ and $\kappa_0^{(free)}(\alpha) > 0$ such that the following holds.*

1. *For all $\alpha \in (0,1)$, $\beta > \beta_0^{(free)}(\alpha)$, $\kappa < \kappa_0^{(free)}(\alpha)$, $\gamma \in \Lambda_1$, $\gamma_0 \in \Lambda_0$, and $\eta \in \Xi$, we have*
$$\sum_{\mathcal{S} \in \Xi \colon \eta \in \mathcal{S}} \left|\Psi_{\beta,\kappa}^{\gamma,\gamma_0}(\mathcal{S})\right| \leqslant \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\eta)\right|^{1-\alpha}.$$

2. *Let $\beta > \beta_0^{(free)}(\alpha)$ and $\kappa < \kappa_0^{(free)}(\alpha)$ for some $\alpha \in (0,1)$. Then, for any $\gamma \in \Lambda$, and $\gamma_0 \in \Lambda_0$ we have*
$$\log \check{Z}[\gamma,\gamma_0] = \sum_{\mathcal{S}\in\Xi} \Psi_{\beta,\kappa}^{\gamma,\gamma_0}(\mathcal{S}). \tag{5.3}$$
Furthermore, series on the right-hand side of (5.3) *is absolutely convergent.*

Proof. We will show that for all $\alpha \in (0,1)$, if β is sufficiently large and κ is sufficiently small, then, if we for $\eta \in \Lambda_1 \cup \Lambda_2$ let $a(\eta) \coloneqq -\alpha \log\left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\eta)\right|$, we have
$$\sum_{\substack{\eta' \in \Lambda_1 \cup \Lambda_2 \colon \\ \eta' \sim \eta}} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\eta'\})\zeta(\eta,\eta')\right| e^{a(\eta')} \leqslant \alpha |(\operatorname{supp} \eta')^+|. \tag{5.4}$$
Given this, the conclusion of the proposition follows from [17, Theorem 5.4]. To this end, let $\gamma' \in \Lambda_1$ and $\omega \in \Lambda_2$. Note that there are four different cases in (5.4), corresponding to (i) $\eta,\eta' \in \Lambda_1$, (ii) $\eta_1 \in \Lambda_1$ and $\eta_2 \in \Lambda_2$, (iii) $\eta_1 \in \Lambda_2$ and $\eta_2 \in \Lambda_1$, and (iv) $\eta_1,\eta_2 \in \Lambda_2$ respectively. We now treat these cases separately.

(i) Let $\gamma' \in \Lambda_1$. Since γ' is closed, we have $\left|\{v \colon v \sim \gamma'\}\right| \leqslant |\gamma'|$. Using, by Lemma 5.4, is follows that for any $\alpha \in (0,1)$ we have
$$\begin{aligned}\sum_{\substack{\gamma'' \in \Lambda_1 \colon \\ \gamma'' \sim \gamma'}} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\gamma''\})\zeta(\gamma',\gamma'')\right| e^{a(\gamma'')} &= \sum_{\substack{\gamma'' \in \Lambda_1 \colon \\ \gamma'' \sim \gamma'}} \varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\gamma''\})^{1-\alpha} \\ &\leqslant |\gamma'| \sum_{j=2}^{\infty} 2j(2m)^{2j} (\tanh 2\kappa)^{2(1-\alpha)j}.\end{aligned}$$
(ii) Let $\omega \in \Lambda_2$. Then, by Lemma 5.5, for any $\alpha \in (0,1)$ we have
$$\begin{aligned}\sum_{\omega' \in \Lambda_2} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\omega'\})\zeta(\omega,\omega')\right| e^{a(\omega')} &= \sum_{\substack{\omega' \in \Lambda_2 \colon \\ \omega' \sim \omega}} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\omega'\})\right|^{1-\alpha} \\ &\leqslant \left|(\operatorname{supp}\omega)^+\right| \sum_{j=2(m-1)}^{\infty} M_3^{2j+1} e^{-4\beta(1-\alpha)j}.\end{aligned}$$
(iii) Let $\gamma' \in \Lambda_1$. Then, by Lemma 5.6, for any $\alpha \in (0,1)$ we have
$$\begin{aligned}\sum_{\omega' \in \Lambda_2} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\omega'\})\zeta(\gamma,\omega')\right| e^{a(\omega')} &= 2 \sum_{\substack{\omega' \in \Lambda_2 \colon \\ \gamma' \sim \omega'}} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\omega'\})\right|^{1-\alpha} \\ &\leqslant 2|\gamma'| \sum_{j=1}^{\infty} D_0 j^3 \sum_{k=\max(4j,2(d-1))}^{\infty} M_3^{2k-1}\, e^{-4\beta a k}\end{aligned}$$
(iv) Let $\omega \in \Lambda_2$. Then, by Lemma 5.7, for any $\alpha \in (0,1)$ we have
$$\begin{aligned}\sum_{\gamma' \in \Lambda_1} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\{\gamma'\})\zeta(\omega,\gamma')\right| e^{a(\gamma')} &= 2 \sum_{\substack{\gamma' \in \Lambda_1 \colon \\ \gamma' \sim \omega}} \varphi_{\beta,\kappa}^{0,0}(\{\gamma'\})^{1-\alpha} \\ &\leqslant 2|(\operatorname{supp}\omega)^+| \sum_{j=1}^{\infty} D_0 j^3 \sum_{k=4j}^{\infty} (2m)^k (\tanh 2\kappa)^{ak}.\end{aligned}$$

Note that for any $\alpha \in (0,1)$, if β is sufficiently large and κ is sufficiently small, then the upper bounds for all cases are finite. In particular, for any $\alpha \in (0,1)$, if β is sufficiently large and κ is sufficiently small, then for any $\eta \in \Lambda_1 \cup \Lambda_2$ we have the upper bound

$$\sum_{\eta' \in \Lambda_1 \cup \Lambda_2} \left|\varphi_{\beta,\kappa}^{\gamma,\gamma_0}(\eta')\zeta(\eta,\eta')\right| e^{a(\eta')} \leqslant \alpha |(\operatorname{supp} \eta)^+|.$$

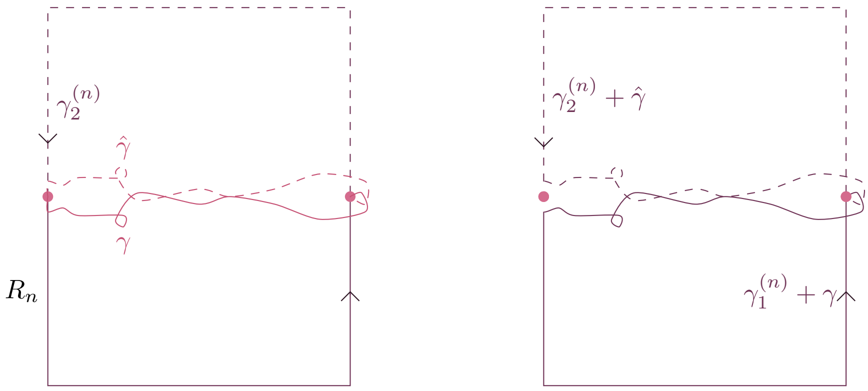
This concludes the proof. □

5.3 Upper bounds for clusters

Since the limiting measure $\langle \cdot \rangle_{\beta,\kappa}$ is translation invariant, without loss of generality we can and will assume that for each $n \geqslant 1$ and all N sufficiently large, we have $\check{Z}_{N,\beta,\kappa}[\gamma_1] = \check{Z}_{N,\beta,\kappa}[\gamma_2]$. Also, without loss of generality, we can and will assume that for each $n \geqslant 1$, $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$ lie in the x_1x_2-plane and has its endpoints at the x_1-axis. When $\gamma \in \mathcal{L}_{\gamma_1^{(n)}}$, we will let $\hat{\gamma}$ denote the path defined for each edge $((x_1, x_2, \ldots, x_m), ((x_1', x_2', \ldots, x_m'))) \in C_1(B_N)^+$ by

$$\hat{\gamma}[((x_1, x_2, \ldots, x_m), (x_1', x_2', \ldots, x_m'))] = -\gamma[((x_1, -x_2, \ldots, x_m), (x_1', -x_2', \ldots, x_m'))]$$

(see Figure 3).



(a) The paths $\gamma_1^{(n)}$, $\gamma_2^{(n)}$, γ, and $\hat{\gamma}$. (b) The paths $\gamma_1^{(n)} + \gamma$ and $\gamma_2^{(n)} + \hat{\gamma}$.

Figure 3: In the two figures above we illustrate the setting of the Proof of Theorem 1.3. In both pictures, we draw the open paths $\gamma_1^{(n)}$ and $\gamma_2^{(n)}$, and also draw a path $\gamma \in \mathcal{L}_{\gamma_1^{(n)}}$.

Before we prove Theorem 1.3 we will state and prove three lemmas.

Lemma 5.9. *Let $\alpha \in (0,1)$, and let $\beta > \beta_0^{(free)}(\alpha)$ and $\kappa < \kappa_0^{(free)}(\alpha)$. Further, let $\gamma \in \Lambda_0$ Then*

$$\sum_{\mathcal{S} \in \Xi \colon \mathcal{S}_1 \sim \gamma} \left|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})\right| \leqslant \big(|\gamma| + 1\big) \sum_{j=2}^{\infty} 2j(2m)^{2j} (\tanh 2\kappa)^{2(1-\alpha)j}.$$

Proof. By Proposition 5.8, we have

$$\sum_{\mathcal{S} \in \Xi \colon \mathcal{S}_1 \sim \gamma} \left|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})\right| \leqslant \sum_{\gamma' \in \Lambda_1 \colon \gamma' \sim \gamma} \sum_{\mathcal{S} \in \Xi \colon \gamma' \in \mathcal{S}} \left|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})\right| \leqslant \sum_{\gamma' \in \Lambda_1 \colon \gamma' \sim \gamma} \left|\varphi_{\beta,\kappa}^{0,0}(\{\gamma'\})\right|^{1-\alpha}.$$

Applying Lemma 5.4 with $a = 1 - \alpha$, we obtain the desired conclusion. □

Lemma 5.10. *Let $\alpha \in (0,1)$, and let $\beta > \beta_0^{\text{(free)}}(\alpha)$ and $\kappa < \kappa_0^{\text{(free)}}(\alpha)$. Further, let γ be a closed path. Then*

$$\sum_{\mathcal{S}\in\Xi\colon \mathcal{S}_2\sim\gamma} \left|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})\right| \leqslant D_0|\gamma| \sum_{j=1}^{\infty} j^3 \sum_{k=\max(4j,2(d-1))}^{\infty} M_3^{2k-1}\, e^{-4\beta(1-\alpha)k}.$$

Proof. By Proposition 5.8, we have

$$\sum_{\mathcal{S}\in\Xi\colon \mathcal{S}_2\sim\gamma} \left|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})\right| \leqslant \sum_{\omega\in\Lambda_2\colon \omega\sim\gamma} \sum_{\mathcal{S}\in\Xi\colon \omega\in\mathcal{S}} \left|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})\right| \leqslant \sum_{\omega\in\Lambda_2\colon \omega\sim\gamma} \left|\varphi_{\beta,\kappa}^{0,0}(\{\omega\})\right|^{1-\alpha}$$

Applying Lemma 5.6 with $1 = 1-\alpha$, we obtain the desired conclusion. □

Lemma 5.11. *Let $\alpha \in (0,1)$, and let $\beta > \beta_0^{\text{(free)}}(\alpha)$ and $\kappa < \kappa_0^{\text{(free)}}(\alpha)$. Further, let $\gamma \in \mathcal{L}_{\gamma_1}$, let $k \geqslant 2(m-1)$, and let $\varepsilon > 0$ be such that $(1-\varepsilon)\beta > \beta_0^{(free)}(\alpha)$ and $(\tanh 2\kappa)^{1-\varepsilon} < \tanh(2\kappa_0^{(free)}(\alpha))$. Then*

$$\sum_{\substack{\mathcal{S}\in\Xi\colon \mathcal{S}_2\sim\gamma_1+\hat\gamma,\\ \mathcal{S}_2\sim\gamma_2+\hat\gamma}} \left|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})\right| \leqslant 2D_0|\gamma| e^{-8\beta\varepsilon(m-1)} \sum_{k=2(m-1)}^{\infty} M_3^{2k-1} e^{-4(1-\varepsilon)\beta k}$$
$$\cdot \sum_{j=1}^{\infty} j^3 \max(e^{-4\beta}, \tanh 2\kappa)^{\varepsilon \max(4j-2(m-1),0)}.$$

Proof. To simplify notation, let $\beta' \coloneqq (1-\varepsilon)\beta$ and $\kappa' \coloneqq \tanh^{-1}((\tanh 2\kappa)^{1-\varepsilon})/2$. Note that, by assumption, we have $\beta' > \beta_0^{(free)}(\alpha)$ and $\kappa' < \kappa_0^{(free)}(\alpha)$. Further, note that with this notation, for any $\mathcal{S} \in \Xi$ we have

$$|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})|^{1-\varepsilon} = |\Psi_{\beta',\kappa'}^{0,0}(\mathcal{S})|.$$

Let $j \geqslant 1$ and $\mathcal{S} \in \Xi$ be such that $\mathcal{S}_2 \sim \gamma_1 + \gamma$, $\mathcal{S}_2 \sim \gamma_2 + \hat\gamma$, and $\operatorname{dist}(\mathcal{S}_2, \{\gamma, \hat\gamma\}) = j$. Then we must have $\|\mathcal{S}\| \geqslant 4j$ and $\|\mathcal{S}_2\| \geqslant 2(m-1)$, and hence

$$\begin{aligned}\left|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})\right| &= \left|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})\right|^{\varepsilon}\left|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})\right|^{1-\varepsilon} = \left|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})\right|^{\varepsilon}\left|\Psi_{\beta',\kappa'}^{0,0}(\mathcal{S})\right| \\ &\leqslant e^{-8\beta\varepsilon(m-1)} \max(e^{-4\beta}, \tanh 2\kappa)^{\varepsilon\max(4j-2(m-1),0)}\left|\Psi_{\beta',\kappa'}^{0,0}(\mathcal{S})\right|.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\sum_{\substack{\mathcal{S}\in\Xi\colon \mathcal{S}_2\sim\gamma_1+\gamma,\\ \mathcal{S}_2\sim\gamma_2+\hat\gamma}} \left|\Psi_{\beta,\kappa}^{\gamma,\gamma_0}(\mathcal{S})\right| &= \sum_{j=1}^{\infty} \sum_{\substack{\mathcal{S}\in\Xi\colon \mathcal{S}_2\sim\gamma_1+\gamma,\gamma_2+\hat\gamma,\\ \operatorname{dist}(\mathcal{S}_2,\{\gamma,\hat\gamma\})=j}} \left|\Psi_{\beta,\kappa}^{0,0}(\mathcal{S})\right| \\ &\leqslant e^{-8\beta\varepsilon(m-1)} \sum_{j=1}^{\infty} \max(e^{-4\beta}, \tanh 2\kappa)^{\varepsilon\max(4j-2(m-1),0)} \sum_{\substack{\mathcal{S}\in\Xi\colon\\ \operatorname{dist}(\mathcal{S}_2,\{\gamma,\hat\gamma\})=j}} \left|\Psi_{\beta',\kappa'}^{0,0}(\mathcal{S})\right|.\end{aligned}$$

By Proposition 5.8, we have

$$\begin{aligned}\sum_{\substack{\mathcal{S}\in\Xi\colon\\ \operatorname{dist}(\mathcal{S}_2,\{\gamma,\hat\gamma\})=j}} \left|\Psi_{\beta',\kappa'}^{0,0}(\mathcal{S})\right| &\leqslant \sum_{\substack{\omega\in\Lambda_2\colon\\ \operatorname{dist}(\mathcal{S}_2,\{\gamma,\hat\gamma\})=j}} \sum_{\mathcal{S}\in\Xi\colon \omega\in\mathcal{S}} \left|\Psi_{\beta',\kappa'}^{0,0}(\mathcal{S})\right| \\ &\leqslant \sum_{\substack{\omega\in\Lambda_2\colon\\ \operatorname{dist}(\mathcal{S}_2,\{\gamma,\hat\gamma\})=j}} \left|\varphi_{\beta',\kappa'}^{0,0}(\omega)\right|.\end{aligned}$$

Recall that there are at most $2D_0 j^3|\gamma|$ positively oriented plaquettes at distance j from $\operatorname{supp}\gamma \cup \operatorname{supp}\hat\gamma$. Also, note that each $\omega \in \Lambda_2$ corresponds to a connected subgraph of $\mathcal{G}_3$ which has a spanning path of length at most $M_3^{2|(\operatorname{supp}\omega)^+|-1}$. Combining these observations, we obtain the upper bound

$$\sum_{\substack{\omega\in\Lambda_2:\\ \operatorname{dist}(\mathcal{S}_2,\{\gamma,\hat\gamma\})=j}} \left|\varphi^{0,0}_{\beta',\kappa'}(\omega)\right| \leqslant 2D_0 j^3|\gamma| \sum_{k=2(m-1)}^{\infty} M_3^{2k-1}e^{-4\beta' k}.$$

Combining the above equations, the desired conclusion follows. □

5.4 Proof of Theorem 1.3

Proof of Theorem 1.3. For some $\alpha \in (0,1)$, let $\beta > \beta_0^{(\text{free})}(\alpha)$ and $\kappa < \kappa_0^{(\text{free})}(\alpha)$

Fix $n \geqslant 0$. To simplify notation, let $\gamma_1 = \gamma_1^{(n)}$ and $\gamma_2 = \gamma_2^{(n)}$.

By Proposition 5.8 that $\check Z_{N,\beta,\kappa}[\gamma_1]$, $\check Z_{N,\beta,\kappa}[\gamma_2]$, $\check Z_{N,\beta,\kappa}[\gamma_1+\gamma_2]$, and $\check Z_{N,\beta,\kappa}[0]$ are all strictly positive. By combining Lemma 5.1 and Lemma 5.2, we can thus write

$$\begin{aligned}\rho_N(\gamma_1,\gamma_2)^{1/2} &= \left(\frac{\check Z_{N,\beta,\kappa}[\gamma_1]\check Z_{N,\beta,\kappa}[\gamma_2]}{\check Z_{N,\beta,\kappa}[\gamma_1+\gamma_2]\check Z_{N,\beta,\kappa}}\right)^{1/2} = \frac{\check Z_{N,\beta,\kappa}[\gamma_1]}{(\check Z_{N,\beta,\kappa}[\gamma_1+\gamma_2]\check Z_{N,\beta,\kappa})^{1/2}}\\ &= \sum_{\gamma\in\mathcal{L}_{\gamma_1}} (\tanh 2\kappa)^{|\gamma|}\frac{\check Z_{N,\beta,\kappa}[\gamma_1+\gamma,\gamma]}{\left(\check Z_{N,\beta,\kappa}[\gamma_1+\gamma_2,0]\check Z_{N,\beta,\kappa}[0,0]\right)^{1/2}}.\end{aligned}$$

Now fix any $\gamma \in \mathcal{L}_{\gamma_1}$ and let

$$\rho_N(\gamma_1,\gamma_2,\gamma) := \frac{\check Z_{N,\beta,\kappa}[\gamma_1+\gamma,\gamma]^2}{\check Z_{N,\beta,\kappa}[\gamma_1+\gamma_2,0]\check Z_{N,\beta,\kappa}[0,0]}.$$

By Proposition 5.8, we have

$$\begin{aligned}\log\rho_N(\gamma_1,\gamma_2,\gamma) &= 2\log \check Z_{N,\beta,\kappa}[\gamma_1+\gamma,\gamma] - \log \check Z_{N,\beta,\kappa}[\gamma_1+\gamma_2,0] - \log \check Z_{N,\beta,\kappa}[0,0]\\ &= 2\sum_{\mathcal{S}\in\Xi}\Psi^{\gamma_1+\gamma,\gamma}_{\beta,\kappa}(\mathcal{S}) - \sum_{\mathcal{S}\in\Xi}\Psi^{\gamma_1+\gamma_2,0}_{\beta,\kappa}(\mathcal{S}) - \sum_{\mathcal{S}\in\Xi}\Psi^{0,0}_{\beta,\kappa}(\mathcal{S}).\end{aligned}$$

Now note that

$$\begin{aligned}&2\sum_{\mathcal{S}\in\Xi}\Psi^{\gamma_1+\gamma,\gamma}_{\beta,\kappa}(\mathcal{S}) - \sum_{\mathcal{S}\in\Xi}\Psi^{\gamma_1+\gamma_2,0}_{\beta,\kappa}(\mathcal{S}) - \sum_{\mathcal{S}\in\Xi}\Psi^{0,0}_{\beta,\kappa}(\mathcal{S})\\ &\quad= \sum_{\mathcal{S}\in\Xi}\Psi^{0,0}_{\beta,\kappa}(\mathcal{S})\Big(2\prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma_1+\gamma})\big)\prod_{\gamma'\in\mathcal{S}}\mathbf{1}(\gamma'\nsim\gamma) - \prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma_1+\gamma_2})\big) - 1\Big)\\ &\quad= -2\sum_{\mathcal{S}\in\Xi}\Psi^{0,0}_{\beta,\kappa}(\mathcal{S})\Big(\prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma+\gamma_1})\big)\Big)\Big(1-\prod_{\gamma'\in\mathcal{S}}\mathbf{1}(\gamma'\nsim\gamma)\Big) \qquad (=: A_0)\\ &\qquad\quad+\sum_{\mathcal{S}\in\Xi}\Psi^{0,0}_{\beta,\kappa}(\mathcal{S})\Big(2\prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma_1+\gamma})\big) - \prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma_1+\gamma_2})\big) - 1\Big).\end{aligned}$$

Further, we have

$$\begin{aligned}&\sum_{\mathcal{S}\in\Xi}\Psi^{0,0}_{\beta,\kappa}(\mathcal{S})\Big(2\prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma_1+\gamma})\big) - \prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma_1+\gamma_2})\big) - 1\Big)\\ &\quad= \sum_{\mathcal{S}\in\Xi}\Psi^{0,0}_{\beta,\kappa}(\mathcal{S})\Big(\prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma_1+\gamma})\big) + \prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma_2+\hat\gamma})\big) - \prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma_1+\gamma_2})\big) - 1\Big)\\ &\quad= -\sum_{\mathcal{S}\in\Xi}\Psi^{0,0}_{\beta,\kappa}(\mathcal{S})\Big(\prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma_1+\gamma})\big)-1\Big)\Big(\prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma_2+\hat\gamma})\big)-1\Big) \qquad (=: A_1)\\ &\qquad\quad-\sum_{\mathcal{S}\in\Xi}\Psi^{0,0}_{\beta,\kappa}(\mathcal{S})\prod_{\omega\in\mathcal{S}}\rho\big(\omega(q_{\gamma_1+\gamma_2})\big)(1-\prod_{\omega\in\mathcal{S}_2}\rho(\omega(q_{\gamma+\hat\gamma})))\Big) \qquad (=: A_2).\end{aligned}$$

Consequently, if we define A_0, A_1, and A_2 as above, then

$$\log \rho_N(\gamma_1, \gamma_2, \gamma) = A_0 + A_1 + A_2.$$

We now give upper bounds for A_0, A_1, and A_2. To this end, note first that by Lemma 5.9, we have

$$|A_0| \leqslant \sum_{\mathcal{S} \in \Xi:\, \mathcal{S}_1 \sim \gamma} \left|\Psi^{0,0}_{\beta,\kappa}(\mathcal{S})\right| \leqslant \big(|\gamma| + 1\big) \sum_{j=2}^{\infty} 2j(2m)^{2j} (\tanh 2\kappa)^{2(1-\alpha)j}.$$

Next, by Lemma 5.10, we have

$$|A_2| \leqslant D_0 |\gamma| \sum_{j=1}^{\infty} j^3 \sum_{k=\max(4j, 2(d-1))}^{\infty} M_3^{2k-1}\, e^{-4\beta(1-\alpha)k}.$$

Finally, by Lemma 5.11, we have

$$\begin{aligned} |A_1| &\leqslant 4 \sum_{\substack{\mathcal{S} \in \Xi:\, \mathcal{S}_2 \sim \gamma_1 + \gamma, \\ \mathcal{S}_2 \sim \gamma_2 + \hat{\gamma}}} \left|\Psi^{0,0}_{\beta,\kappa}(\mathcal{S})\right| \\ &\leqslant 8 D_0 |\gamma| e^{-8\beta\varepsilon(m-1)} \sum_{k=2(m-1)}^{\infty} M_3^{2k-1} e^{-4(1-\varepsilon)\beta k} \\ &\quad \cdot \sum_{j=1}^{\infty} j^3 \max(e^{-4\beta}, \tanh 2\kappa)^{\varepsilon \max(4j - 2(m-1), 0)}. \end{aligned}$$

for some small $\varepsilon > 0$. Combining the three previous equations, it follows that as $\beta \to \infty$ and $\kappa \to 0$, we have

$$\frac{\log \rho_N(\gamma_1, \gamma_2, \gamma)}{|\gamma|} \leqslant o_\beta(1) + o_\kappa(1), \tag{5.5}$$

where the right hand side is independent of γ_1, γ_2, γ, N, and n. In particular, if β is sufficiently large and κ is sufficiently small, then the right-hand-side of (5.5) is strictly smaller than $1/(2m \tanh 2\kappa)$. Finally, we note that since any vertex of B_N has degree $2m$, we have

$$\lim_{n\to\infty} \sum_{\gamma \in \mathcal{L}_{\gamma_1^{(n)}}} (\tanh 2\kappa)^{|\gamma|} \leqslant \lim_{n\to\infty} \sum_{j=T_n}^{\infty} (2m)^j (\tanh 2\kappa)^j = 0.$$

Noting that

$$\rho_N(\gamma_1^{(n)}, \gamma_2^{(n)})^{1/2} = \sum_{\gamma \in \mathcal{L}_{\gamma_1^{(n)}}} (\tanh 2\kappa)^{|\gamma|} \rho_N(\gamma_1, \gamma_2, \gamma),$$

the desired conclusion immediately follows. □

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