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Citation for the original published paper (version of record):
Fedosova, K., Rowlett, J., Zhang, G. (2024). Casimir energy of hyperbolic orbifolds with conical singularities. Journal of Mathematical Physics, 65(10). http://dx.doi.org/10.1063/5.0186488

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RESEARCH ARTICLE | OCTOBER 23 2024

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J. Math. Phys. 65, 101701 (2024)
https://doi.org/10.1063/5.0186488

Journal of
Mathematical Physics

ARTICLE

pubs.aip.org/aip/jmp

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Cite as: J. Math. Phys. 65, 101701 (2024); doi: 10.1063/5.0186488
Submitted: 6 November 2023 • Accepted: 27 September 2024 •
Published Online: 23 October 2024

View Online Export Citation CrossMark

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ABSTRACT

In this article, we obtain the explicit expression of the Casimir energy for compact hyperbolic orbifold surfaces in terms of the geometrical data of the surfaces with the help of zeta-regularization techniques. The orbifolds may have finitely many conical singularities. In computing the contribution to the energy from a conical singularity, we derive an expression of an elliptic orbital integral as an infinite sum of special functions. We prove that this sum converges exponentially fast. Additionally, we show that under a natural assumption known to hold asymptotically on the growth of the lengths of primitive closed geodesics of the $(2, 3, 7)$-triangle group orbifold, its Casimir energy is positive (repulsive).

I. INTRODUCTION

The Casimir energy is named after the Dutch physicist, Hendrik B. G. Casimir who showed in 1948 that two uncharged parallel metal plates alter the vacuum fluctuations in such a way as to attract each other. This is now referred to as the Casimir effect. The energy density between the plates, now known as the Casimir energy, was calculated to be negative. The plates essentially reduce the fluctuations in the gap between them creating negative energy and pressure, which pulls the plates together. For this reason, negative Casimir energy is associated with an attractive force.

The Casimir effect in different spacetimes is an important concept in cosmology,[1–4] quantum field theory,[5–9] supergravity,[10–12] superstring theory,[13,14] hadronic physics,[15] and acoustic scattering.[16] The evaluation of the Casimir effect for massless scalar fields (or spinor fields) has been obtained in, e.g., Refs. 17 and 18. Moreover, in Ref. 19, the authors calculate the Casimir energy for several hyperbolic manifolds that include, among others, the Bolza surface. However, the aforementioned spacetimes do not allow singularities. Hence, they exclude the possibility of exciting and crucial physical objects like Schwarzschild black holes and cosmic strings. Geometrically, these both would create a conical singularity,[20] which is not featured in smooth geometric settings.

Nonetheless, the Casimir energy in spacetimes that may have conical singularities has been studied by several authors.[21–23] However, the geometric context in the aforementioned works is somewhat restrictive. Hence there is motivation to understand the Casimir energy in a broader context. Here we calculate the Casimir energy in two dimensional spacetimes that admit an orbifold structure and may have finite many conical singularities.

A Riemannian orbifold is singular generalization of a Riemannian manifold which is locally modeled on the quotient of a manifold under a finite group of isometries. The orbifolds were first introduced by Satake[24] and, in the coming years, became important not only in mathematics, but also in cosmology and physics. For example, an $SU(3) \times SU(2) \times U(1)$ supersymmetric theory is constructed with an orbifold $\mathbb{S}^1/(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}')$; the orbifold fixed points are crucial for the description of supersymmetric Yukawa interactions.[25,26]

Several articles are dedicated to the calculation of the Casimir energy, e.g., Ref. 27 or 28, in the case of an orbifold. However, in the mentioned articles the authors have closed expressions for the vacuum modes (that is, closed expressions for the eigenvalues of the Laplace operator). However in most cases it is impossible to have precise formulas for the eigenvalues and consequently it is a natural problem to study the Casimir energy in the absence of such formulas.

To describe the orbifold surfaces in this work, let Γ be a discrete subgroup of the group of orientation-preserving isometries, $PSL_2(\mathbb{R})$, acting on the hyperbolic upper half plane, $\mathbb{H}$. Moreover, assume that the orbifold, $X = \Gamma\backslash\mathbb{H}$, obtained by taking the quotient of the hyperbolic upper half plane with Γ, is compact. We denote its volume by $\mathrm{vol}(\Gamma\backslash\mathbb{H})$. Then,

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty \tag{1.1}$$

are the eigenvalues of the associated Laplace operator Δ acting on X. The spectral zeta function, $\zeta_\Gamma(s)$, of X is defined for $\mathrm{Re}(s)$ sufficiently large as

$$\zeta_\Gamma(s) = \sum_{n\in\mathbb{N}} \lambda_n^{-s}.$$

With the help of the Selberg trace formula it is possible to show that the spectral zeta function admits a meromorphic continuation to $s \in \mathbb{C}$. Its value at $s = -1/2$ is referred to as the *Casimir energy*. Our goal is to give an expression for $\zeta_\Gamma(-1/2)$ in terms of geometric data of the orbifold. The geometry of the orbifold is determined by the elements of the group Γ. These are classed in the following two types.

1. A non-identity element, $R \in \Gamma$, is *elliptic* if it is of finite order. We note that any cyclic subgroup, $\mathcal{R}$ of finite order in Γ is generated by a *primitive elliptic element* R_0 of order $m_\mathcal{R} \in \mathbb{N}$. This element R_0 may be chosen in $PSL_2(\mathbb{R})$ to be conjugate to

$$\begin{pmatrix} \cos(\pi/m_\mathcal{R}) & -\sin(\pi/m_\mathcal{R}) \\ \sin(\pi/m_\mathcal{R}) & \cos(\pi/m_\mathcal{R}) \end{pmatrix}.$$

The angle, $\theta_\mathcal{R} = \pi/m_\mathcal{R}$, is the smallest positive angle among all such angles determined by the elements of the group generated by R_0. We denote the set of all primitive elliptic elements of Γ by $\{\mathcal{R}\}_p$.

2. An element $P \in \Gamma$ is *hyperbolic* if it is $\mathrm{PSL}_2(\mathbb{R})$-conjugate to

$$\begin{pmatrix} a(P) & 0 \\ 0 & a(P)^{-1} \end{pmatrix},$$

such that $1 < a(P)$. The norm of P is defined to be $NP := |a(P)|^2$. The element P gives rise to a closed geodesic in $\Gamma\backslash\mathbb{H}$, which has length $\ell_P = \log NP$. We let k be the biggest positive integer such that $P = P_0^k$ for some $P_0 \in \Gamma$. If $k = 1$, we say that $P = P_0$ is a *primitive hyperbolic element*.

We denote the set of Γ-conjugacy classes of all hyperbolic elements, respectively primitive hyperbolic elements, by $\{\mathcal{P}\}$, respectively $\{\mathcal{P}\}_p$. We additionally note that if Γ has no elliptic elements, the set $\{\mathcal{P}\}$ is in 1-to-1 correspondence with the set of oriented closed geodesics of X.

We note that the compactness of X excludes the possibility that Γ contains so-called *parabolic* elements. This is equivalent to saying that there are no elements $\gamma \in \Gamma$ that are $PSL_2(\mathbb{R})$-conjugated to

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

for some $x \in \mathbb{R}\backslash\{0\}$. As many hyperbolic surfaces of interest are not compact, several authors studied values of spectral zeta function for such groups. For example, in Ref. 29, the author investigated certain values of the spectral zeta function in the presence of parabolic elements. However, since the presence of parabolic elements causes the surface to be non-compact, this changes the structure of the Laplace spectrum. In particular it is no longer discrete. Consequently, a modified approach is required to investigate the energy in that case which shall be the subject of future work.

We further recall the Struve function of the second kind and the modified Bessel function of the second kind.

Definition 1.1. We denote by $\boldsymbol{K}_j$ the jth Struve function of the second kind,

$$\boldsymbol{K}_j(z) = \boldsymbol{H}_j(z) - Y_j(z), \quad z \in \mathbb{C}.$$

Here, $\boldsymbol{H}_j$ is the jth Struve function of the first kind as defined in Ref. 30, Sec. 10.4 (see also Ref. 31, Sec. 11.2), and Y_j the Bessel function of the second kind, also known as the Weber Bessel function defined in Ref. 30, Sec. 3.53 (see also Ref. 31, Sec. 10.2). The modified Bessel function of the second kind of order j is denoted K_j and defined in Ref. 30, p. 64 (see also Ref. 31, Secs. 10.27 and 10.31).

With these preparations, we may now state our first main result.

Theorem 1.2. For $s \in \mathbb{C}\backslash\mathbb{N}_{\geq 1}$ the spectral zeta function of the orbifold $\Gamma\backslash\mathbb{H}$ is

$$\begin{aligned}\zeta_\Gamma(s) = {}& \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{8(s-1)} 4^s \pi^{s-1}\Gamma(2-s)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(n+1)(-1)^k}{2^{n+3/2+s}}\binom{n}{k}\frac{\boldsymbol{K}_{3/2-s}(\pi+\pi k)}{(1+k)^{3/2-s}} \\ &+ \sum_{\{\mathcal{R}\}_p}\sum_{\ell=1}^{m_{\mathcal{R}}-1}\frac{2^{s-3/2}\sqrt{\pi}\Gamma(1-s)}{m_{\mathcal{R}}\sin(\pi\ell m_{\mathcal{R}})}\sum_{n=0}^{\infty}2^{-n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\boldsymbol{K}_{-s+1/2}\left(\frac{\pi\ell}{m_{\mathcal{R}}}+\pi k\right)}{\left(\frac{\pi\ell}{m_{\mathcal{R}}}+\pi k\right)^{-s+1/2}} \\ &+ \frac{(4\pi)^{-1/2}}{\Gamma(s)}\sum_{\{\mathcal{P}\}_p}\sum_{n=1}^{\infty}(\ell_\gamma/n)^{1/2}\mathrm{csch}\left(\frac{n\ell_\gamma}{2}\right)(n\ell_\gamma)^s K_{1/2-s}\left(\frac{n\ell_\gamma}{2}\right).\end{aligned}$$

The identity holds for $s \in \mathbb{N}_{\geq 2}$ in the sense that the right side has removable singularities at these points. For $s = 1$, $\zeta_\Gamma(s)$ has a simple pole. Specifying to $s = -1/2$ we obtain the Casimir energy,

$$\begin{aligned}\zeta_\Gamma(-1/2) = {}& \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{\pi}\sum_{n=0}^{\infty}\frac{n+1}{2^{n+6}}\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{\boldsymbol{K}_2[\pi(1+k)]}{(1+k)^2} \\ &+ \sum_{\{\mathcal{R}\}_p}\sum_{\ell=1}^{m_{\mathcal{R}}-1}\frac{1}{8m_{\mathcal{R}}\sin(\pi\ell m_{\mathcal{R}})}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\boldsymbol{K}_1\left[(k+\frac{\ell}{m_{\mathcal{R}}})\pi\right]}{k+\ell m_{\mathcal{R}}} \\ &- \frac{1}{4\pi}\sum_{\{\mathcal{P}\}_p}\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{csch}\left(\frac{n\ell_\gamma}{2}\right)K_1\left(\frac{n\ell_\gamma}{2}\right).\end{aligned}$$

The first two lines of the right hand side converge exponentially fast. Moreover, it is possible to evaluate the Struve functions with the help of systems of computer algebra with an arbitrary precision. Consequently, this form is extremely convenient for calculations.

It is the presence of conical singularities, corresponding to the elliptic elements of the group, that allows for the possibility of positive Casimir energy, corresponding to a repulsive force. Without these elements the Casimir energy is always strictly negative. We show that under a natural assumption on the lengths of closed geodesics, an assumption that is known to hold asymptotically, the Casimir energy may be positive. The groups of which we are aware that may give rise to orbifolds with positive Casimir energy are so-called triangle groups. Fix $p, q, r \in \mathbb{N}$ with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$. Define a (p,q,r)-*triangle group* as in Ref. 32, Definition 10.6.3 and denote it by $\Gamma(p,q,r)$. For such values of (p,q,r), the group $\Gamma(p,q,r)$ is a discrete co-compact subgroup of $PSL(2,\mathbb{R})$. The area of $\Gamma(p,q,r)\backslash\mathbb{H}$ is equal to Ref. 32, p. 280,

$$2\pi\left(1-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)\right). \tag{1.2}$$

There has been a significant amount of research dedicated to triangle groups.[33–38]

One of the most significant triangle groups is the (2,3,7)-triangle group, $\Gamma(2,3,7)$. It is related to a special type of surface, named after Adolf Hurwitz. A Hurwitz surface is a compact Riemann surface of genus g with precisely $84(g-1)$ automorphisms. This number is maximal by virtue of Hurwitz's theorem on automorphisms.[39] This group of automorphisms is called a Hurwitz group. By uniformization, a Hurwitz surface admits a hyperbolic structure wherein the automorphisms act by isometries. Such isometries descend from the (2,3,7)-triangle group acting on the universal cover $\mathbb{H}$. Here, we aim to show that under a natural assumption on the closed geodesics of the (2,3,7)-triangle group orbifold, which is known to hold asymptotically, the Casimir energy is positive.

Conjecture 1. Under the assumption (5.3), the Casimir energy of the (2,3,7)-triangle group orbifold $\Gamma(2,3,7)\backslash\mathbb{H}$ *is larger than 0.01.*

With the standard sign convention, negative Casimir energy physically represents an attracting force, whereas positive Casimir energy physically represents a repelling force.[40] In Lemma 3.1 we show that the first term in the expression for the Casimir energy, $\zeta_\Gamma(-1/2)$ given in Theorem 1.2 is strictly negative. It is also apparent that the last term is strictly negative. The middle term is the contribution of the elliptic elements. This shows that the Casimir energy is always negative for smooth compact hyperbolic surfaces without conical singularities since they have no elliptic elements. Moreover, in the case of the (2,3,7)-orbifold surface it shows that the presence of conical singularities has a profound effect, to the extent that their contribution to the energy is the dominant term. We conjecture that for many, perhaps even most, surfaces obtained as a quotient by a (p,q,r)-triangle group, the Casimir energy is positive, but we postpone that investigation to future work.

TABLE I. The first several lengths of primitive hyperbolic closed geodesics of the (2,3,7)-triangle group orbifold together with their representations and their contribution to the Casimir energy. The right column shows that (5.3) holds for all j from 2 to 51.

$\ell_\gamma \approx$	$s(\gamma)$	γ	$A(\gamma) \approx$	$\log j + \log\log j, j$
0.983 99	1	$R.L$	−0.288 955	Undefined for $j = 1$
1.736 01	1	$R.R.L.L$	−0.064 746	0.326 634, $j = 2$
2.131 11	2	$R.L.R.L.L$	−0.069 526	1.192 66, 1.712 93, $j = 3,4$
2.661 93	2	$R.L.R.R.L.L$	−0.032 848	2.085 32, 2.374 96, $j = 5,6$
2.898 15	2	$R.L.L.R.R.L.L$	−0.024 028	2.611 64, 2.811 54, $j = 7,8$
3.154 82	2	$R.L.R.L.R.L.L$	−0.017 289	2.984 42, 3.136 62, $j = 9,10$
3.542 71	1	$R.L.R.R.L.R.L.L$	−0.005 342 9	3.272 49, $j = 11$
3.627 32	2	$R.L.R.L.R.R.L.L$	−0.009 641 6	3.395 14, 3.506 89, $j = 12, 13$
3.804 70	2	$R.L.R.R.L.R.R.L.L$	−0.007 787 9	3.609 48, 3.704 28, $j = 14, 15$
3.935 95	2	$R.L.R.L.L.R.R.L.L$	−0.006 660 8	3.792 37, 3.874 62, $j = 16, 17$
4.151 97	2	$R.L.R.L.R.L.R.L.L$	−0.005 163 5	3.951 76, 4.024 36, $j = 18, 19$
4.201 81	1	$R.L.L.R.R.L.R.R.L.L$	−0.002 435 5	4.092 92, $j = 20$
4.391 46	2	$R.L.R.R.L.L.R.R.L.L$	−0.003 906 8	4.157 87, 4.219 55, $j = 21, 22$
4.489 26	2	$R.L.R.L.R.R.L.R.L.L$	−0.003 489 4	4.278 28, 4.334 32, $j = 23, 24$
4.604 73	2	$R.L.R.L.R.L.R.R.L.L$	−0.003 055 5	4.387 91, 4.439 24, $j = 25, 26$
4.654 01	2	$R.L.L.R.R.L.L.R.R.L.L$	−0.002 887 7	4.4885, 4.535 84, $j = 27, 28$
4.760 43	2	$R.L.R.L.R.R.L.R.R.L.L$	−0.002 557 1	4.581 41, 4.625 32, $j = 29, 30$
4.841 80	4	$R.L.R.L.L.R.L.R.R.L.L$	−0.004 661 7	4.667 71, 4.708 66, $j = 31, 32$
				4.748 27, 4.786 63, $j = 33, 34$
4.938 76	2	$R.L.R.L.R.L.L.R.R.L.L$	−0.002 087 9	4.8238, 4.859 86, $j = 35, 36$
5.013 22	2	$R.L.R.L.L.R.L.L.R.R.L.L$	−0.001 919 2	4.894 88, 4.928 91, $j = 37, 38$
5.140 68	2	$R.L.R.L.R.L.R.L.R.L.L$	−0.001 662 2	4.962, 4.9942, $j = 39, 40$
5.208 02	2	$R.L.R.L.L.R.R.L.R.R.L.L$	−0.001 540 9	5.025 57, 5.056 13, $j = 41, 42$
5.288 90	2	$R.L.R.L.R.L.L.R.L.R.L.L$	−0.001 407 2	5.085 94, 5.115 02, $j = 43, 44$
5.288 90	2	$R.L.R.R.L.R.L.L.R.R.L.L$	−0.001 407 2	5.143 42, 5.171 15, $j = 45, 46$
5.351 46	2	$R.L.R.L.R.R.L.L.R.R.L.L$	−0.001 312 0	5.198 26, 5.224 77, $j = 47, 48$
5.426 80	1	$R.L.R.L.R.R.L.R.L.R.L.L$	−0.000 602 98	5.2507, $j = 49$
5.459 43	2	$R.L.R.L.R.L.R.R.L.R.L.L$	−0.001 162 8	5.276 08, 5.300 93, $j = 50, 51$

A. Numerics

Some of the calculations in this paper were performed with the help of PARI/GP[41] using a multiple-precision arithmetic with the precision of 500 significant digits. To be more precise, we used it in the Proof of Lemma 3.5, (5.4) and Table I. The code is available upon request.

B. Organization

In Sec. II we recall basic properties of triangle groups and the spectral zeta function, the Selberg trace formula, and standard notation. We continue in Sec. III with the calculation of the orbital integrals arising from the elliptic elements. One interesting observation that follows from Lemma 3.4 is that as the angle of the elliptic element tends to zero, the contribution to the Casimir energy is positive and tends to infinity on the order of θ^{-2} for an angle of measure θ. We then calculate to six significant figures the elliptic contribution to the Casimir energy of the (2,3,7)-triangle group orbifold. In Sec. IV we calculate the identity contribution in general and demonstrate an estimate for the (2,3,7)-triangle group orbifold in particular. In Sec. V we consider the hyperbolic contribution to the Casimir energy in general and then specialize to the case of the (2,3,7)-orbifold surface. Using Vogeler's explicit calculations of the first 50 primitive closed geodesics[42] we calculate to six significant figures their contribution to the Casimir energy. Next, under assumption (5.3) on the remaining geodesic lengths, we estimate the contribution of all but the first 50 primitive closed geodesics. We conclude this section with a proof of Conjecture 1 under this assumption, noting that the assumption holds asymptotically. In Sec. VI we conclude with implications and further directions.

II. PRELIMINARIES

Here we recall additional facts about the geometry of compact hyperbolic surfaces. In Secs. II A and II B, we discuss the Selberg trace formula, sketch the proof of the meromorphic continuation of $\zeta_\Gamma(s)$ and obtain the elliptic orbital integrals.

A. Selberg trace formula

As in (1.1), we let $\{\lambda_n\}_{n\in\mathbb{N}_0}$ be the eigenvalues of the Laplace operator acting on X. It is convenient to introduce a sequence of numbers $r_n \in \mathbb{C}$ such that the following holds:

$$\lambda_n = 1/4 + r_n^2, \text{ for } n \in \mathbb{N}_0. \tag{2.1}$$

To state the Selberg trace formula as in Ref. 43, assume that the function $r \mapsto h(r)$ is analytic on $|\mathrm{Im}\,(r)| \le \frac{1}{2} + \delta$ for some $\delta > 0$. Assume further that h is even, that is $h(-r) = h(r)$, and that h satisfies an estimate $|h(r)| \le M(1 + \mathrm{Re}(r))^{-2-\delta}$ for a constant M. We define the Fourier transform of h to be

$$g(u) = \frac{1}{2\pi}\int_{\mathbb{R}} h(r)e^{-iru}dr. \tag{2.2}$$

Then, with this setup, the Selberg trace formula is the following identity, Ref. 43, pp. 351–352:

$$\begin{aligned}\sum_{n\ge 0} h(r_n) = {} & \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}\int_{\mathbb{R}} rh(r)\tanh(\pi r)dr \\ & + \sum_{\{\mathcal{P}\}} \frac{\log NP_0}{NP^{1/2} - NP^{-1/2}} g(\log(NP)) \\ & + \sum_{\{\mathcal{R}\}_p}\sum_{\ell=1}^{m_{\mathcal{R}}-1} \frac{1}{2m_{\mathcal{R}}\sin(\pi\ell m_{\mathcal{R}})}\int_{\mathbb{R}} \frac{e^{-2r\frac{\pi\ell}{m_{\mathcal{R}}}}}{1+e^{-2\pi r}} h(r)dr.\end{aligned} \tag{2.3}$$

The sums and integrals in the above expression are all absolutely convergent. We note that if one compares the above identity to Ref. 43, pp. 351–352, the representation χ of the fundamental group we have here is the trivial representation, so the traces appearing in Ref. 43 are all equal to one.[44]

B. Spectral zeta function

In Refs. 45, (6.10) and (6.11), 46, and 47, (3), the respective authors study the meromorphic continution of $\zeta_\Gamma(s)$ to $s \in \mathbb{C}$. To avoid the zero in the denominator of the first summand, corresponding to the eigenvalue $\lambda_0 = 0$, they choose $\varepsilon > 0$ and introduce

$$\zeta_{\Gamma,\varepsilon}(s) = \sum_{n\in\mathbb{N}_0} \frac{1}{(\lambda_n+\varepsilon)^s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\sum_{n\in\mathbb{N}_0} e^{-t(\lambda_n+\varepsilon)}dt. \tag{2.4}$$

Next, they consider the function

$$h(r) = e^{-t(r^2+1/4+\varepsilon)}$$

and apply the Selberg trace formula, (2.3), to this function to express

$$\sum_{n\in\mathbb{N}_0} e^{-t(\lambda_n+\varepsilon)}$$

in terms of geometric data of X. The following step is to substitute this sum into (2.4) to obtain $\zeta_{\Gamma,\varepsilon}(s)$. Finally, $\zeta_\Gamma(s)$ is obtained from a limiting procedure by letting ε go to 0. Repeating their proof with the only modification that now we have to take elliptic elements into consideration, we obtain for $\mathrm{Re}(s) < 0$,

$$\begin{aligned}\zeta_\Gamma(s) = {} & \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{8(s-1)}\int_{\mathbb{R}} \left(\frac{1}{4}+r^2\right)^{1-s}\mathrm{sech}^2(\pi r)dr \\ & + \frac{(4\pi)^{-1/2}}{\Gamma(s)}\sum_{\{\mathcal{P}\}_p}\sum_{n=1}^{\infty} (\ell_\gamma/n)^{1/2}\mathrm{csch}\left(\frac{n\ell_\gamma}{2}\right)(n\ell_\gamma)^s K_{1/2-s}\left(\frac{n\ell_\gamma}{2}\right) \\ & + \sum_{\{\mathcal{R}\}_p}\sum_{\ell=1}^{m_{\mathcal{R}}-1} \frac{1}{2m_{\mathcal{R}}\sin(\pi\ell m_{\mathcal{R}})}\int_{\mathbb{R}} \frac{e^{-2r\frac{\pi\ell}{m_{\mathcal{R}}}}}{1+e^{-2\pi r}}\left(\frac{1}{4}+r^2\right)^{-s}dr.\end{aligned} \tag{2.5}$$

One can obtain the same result *formally*, that is, non-rigorously, by taking $h(r) = (1/4 + r^2)^{-s}$; of course, in that case h does not satisfy the growth condition for $\mathrm{Re}(s) < 0$. For such h, the left hand side of the Selberg trace formula, (2.3), formally coincides with the spectral zeta function, as each summand reads $h(r_n) = (1/4 + r_n^2)^{-s} = \lambda_n^{-s}$. Although this may be a useful heuristic, the derivation following Refs. 45, (6.10) and (6.11) and 47, (3) is fully rigorous.

C. Notation

We recall the following notation:

- $f \leq_{a,b,c,...} g$ means $\exists C > 0$ that depends only on the (finitely many) parameters $a, b, c, \ldots$ such that $f \leq Cg$,
- $f \lesssim g$ means $\exists C$ (independent of any parameters) such that $f \leq Cg$,
- a function $f(x)$ is $\mathcal{O}(g(x))$ as $x \to 0$ if there exist $C, \varepsilon > 0$ such that $|f(x)| \leq C|g(x)|$ for all $x \in (0, \varepsilon)$,
- $\Gamma(\cdot)$ is the Gamma function, $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and for $s \in \mathbb{C}\backslash\mathbb{Z}_{\leq 0}$ by meromorphic continuation,
- $\Gamma(a, s)$ is the incomplete Gamma function, $\Gamma(a, s) = \int_a^\infty t^{s-1}e^{-t}dt$,
- the polylogarithm is

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad s \in \mathbb{C}, |z| < 1.$$

It admits a definition via analytic continuation for $|z| \geq 1$, but for our purposes it suffices to consider $|z| < 1$.

III. ELLIPTIC CONTRIBUTION

In this section, we demonstrate an identity that we use to obtain an expression for the contribution of elliptic elements to the spectral zeta function in terms of special functions. This identity is of independent interest as it may be useful for other calculations due to its rapid convergence. Here, we use it to evaluate the contribution of the elliptic elements in $\Gamma(2,3,7)\backslash\mathbb{H}$ to its Casimir energy.

Lemma 3.1. Let $C > 0$, $D \geq 0$, and $s \in \mathbb{C}\backslash\mathbb{N}$. Then,

$$\int_0^\infty \frac{e^{-Cy}}{e^{-Dy}+1}(1+y^2)^{-s}dy = \sqrt{\pi}2^{-s-1/2}\Gamma(1-s)\sum_{n=0}^{\infty} 2^{-n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k \boldsymbol{K}_{-s+1/2}(C+Dk)}{(C+Dk)^{-s+1/2}}. \tag{3.1}$$

Above, $\boldsymbol{K}_{-s+1/2}$ is the Struve function of the second kind. In particular, for $s = -1/2$, the right hand side of (3.1) becomes

$$\pi\sum_{n=0}^{\infty} 2^{-n-2}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k \boldsymbol{K}_1(C+Dk)}{C+Dk}.$$

The series converges exponentially fast; more precisely, the absolute value of the difference between the left hand side of (3.1) and the right hand side, restricted to $n \in \{0, N\}$, is bounded by

$$\frac{1}{2^N}\int_0^1 \frac{(1+\log(t)^2)^{-\mathrm{Re}(s)}t^{C-1}}{(t^D+1)}dt. \tag{3.2}$$

For $s = -1/2$ in particular, we have the following bound:

$$\frac{1}{2^N}\frac{\boldsymbol{K}_1(C)\pi}{2C}. \tag{3.3}$$

Proof. We make the change of variables $t = e^{-y}$ and rewrite the integral:

$$\int_0^\infty \frac{e^{-Cy}}{e^{-Dy}+1}(1+y^2)^{-s}dy = \int_0^1 \frac{t^{C-1}}{t^D+1}(1+\log(t)^2)^{-s}dt. \tag{3.4}$$

For $|x-1| < 2$,

$$\frac{1}{x+1} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n 2^{-n-1} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k 2^{-n-1}x^k\binom{n}{k}. \tag{3.5}$$

If $x = t^D \in [0,1]$, the series above converges uniformly. Moreover, reversing the substitution,

$$\int_0^\infty e^{-Cy}(1+y^2)^{-s}dy = \int_0^1 t^{C-1}(1+\log(t)^2)^{-s}dt.$$

Since $C > 0$, the L^1 norm of the function $x \mapsto x^{C-1}(1+\log^2(x))^{-s}$ is finite on $[0,1]$. This allows us to substitute (3.5) into (3.4) and exchange the summation and the integration to obtain that (3.4) is equal to

$$\sum_{n=0}^{\infty} 2^{-n-1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_0^1 t^{C+Dk-1}(1+\log(t)^2)^{-s}\, dt.$$

As a consequence of Ref. 31, (11.5.2), the sum above is equal to

$$\sqrt{\pi} 2^{-s-1/2} \Gamma(1-s) \sum_{n=0}^{\infty} 2^{-n-1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k \boldsymbol{K}_{-s+1/2}(C+Dk)}{(C+Dk)^{-s+1/2}},$$

that concludes the proof of (3.1).

It remains to prove that the convergence is exponentially fast. Observe that

$$\frac{1}{x+1} - \sum_{n=0}^{N-1} (-1)^n (x-1)^n 2^{-n-1} = \frac{(1-x)^N}{2^N (x+1)}.$$

We obtain that the absolute differences between the right and the left hand sides of (3.1) is bounded from above by

$$\begin{aligned} &\left| \int_0^1 \frac{(1-t^D)^N}{2^N (t^D+1)} t^{C-1} (1+\log(t)^2)^{-s} dt \right| \\ &\le \frac{1}{2^N} \int_0^1 \frac{(1+\log(t)^2)^{-\mathrm{Re}(s)} t^{C-1}}{(t^D+1)} dt. \end{aligned} \tag{3.6}$$

The right hand side decays exponentially fast as $N \to \infty$ therewith proving the exponential convergence in (3.1). We note that for $C > 0, D > 0$, and $s = -1/2$, we can estimate the right hand side of (3.6) from above by

$$\frac{1}{2^N} \int_0^1 (1+\log(t)^2)^{1/2} t^{C-1} dt = \frac{1}{2^N} \frac{\boldsymbol{K}_1(C) \sqrt{\pi} \Gamma(3/2)}{C} = \frac{1}{2^N} \frac{\boldsymbol{K}_1(C)\pi}{2C}.$$

Here we used Ref. 31, (11.5.2). □

Remark 3.2. In the lemma above, we imposed the condition $s \notin \mathbb{N}$. However, (3.1) holds for $s \in \mathbb{N}$ as an equality with removable singularities, since

$$\boldsymbol{K}_{-s+1/2}(x) = 0, \quad s \in \mathbb{N}.$$

The above identity follows from the definition of Struve functions of the second kind. It might not be very convenient to use the right hand side of (3.1) to calculate the value of the integral for such s, because instead of evaluating Struve functions, we would have to resort to their derivatives. Closed expressions for the latter can be found in Ref. 48, however we prefer not to include them in the manuscript to keep it concise.

Lemma 3.3. The contribution of elliptic elements to the Casimir energy is equal to

$$\sum_{\{\mathcal{R}\}_p} \sum_{\ell=1}^{m_{\mathcal{R}}-1} \frac{1}{8 m_{\mathcal{R}} \sin(\pi \ell m_{\mathcal{R}})} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\boldsymbol{K}_1\left[(k+\frac{\ell}{m_{\mathcal{R}}})\pi\right]}{k+\ell m_{\mathcal{R}}}.$$

Proof. We recall from (2.5) that the contribution from elliptic elements to $\zeta_\Gamma(s)$ is equal to

$$\sum_{\{\mathcal{R}\}_p} \sum_{\ell=1}^{m_{\mathcal{R}}-1} \frac{1}{2 m_{\mathcal{R}} \sin(\pi \ell m_{\mathcal{R}})} \int_{\mathbb{R}} \frac{e^{-2r \frac{\pi \ell}{m_{\mathcal{R}}}}}{1+e^{-2\pi r}} \left(\frac{1}{4}+r^2\right)^{-s} dr,$$

where $\{\mathcal{R}\}_p$ and $m_{\mathcal{R}}$ are defined as in Sec. I. Making the substitution $t = 2r$, the integral

$$\int_{\mathbb{R}} \frac{e^{-2r \frac{\pi \ell}{m_{\mathcal{R}}}}}{1+e^{-2\pi r}} \left(\frac{1}{4}+r^2\right)^{-s} dr = 4^s \int_0^{\infty} \frac{e^{-t\pi \ell / m_{\mathcal{R}}}}{1+e^{-\pi t}} (1+t^2)^{-s} dt. \tag{3.7}$$

Since $\ell < m_{\mathcal{R}}$, we may apply Lemma 3.1 to conclude that this integral is equal to

$$4^s\sqrt{\pi}2^{-s-1/2}\Gamma(1-s)\sum_{n\geq 0}2^{-n-1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\boldsymbol{K}_{-s+1/2}(\pi\ell/m_{\mathcal{R}}+\pi k)}{(\pi\ell/m_{\mathcal{R}}+\pi k)^{-s+1/2}}.$$

Setting $s=-1/2$ we obtain

$$\frac{1}{4}\sum_{n=0}^{\infty}2^{-n-1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\boldsymbol{K}_1(\pi\ell/m_{\mathcal{R}}+\pi k)}{\ell/m_{\mathcal{R}}+k}$$

that concludes the proof. □

A. Elliptic elements in triangle groups

Let $\Gamma(p,q,r)$ be the arithmetic (p,q,r)-triangle group; see Ref. 49 for the classification of all arithmetic triangle surfaces. In Ref. 50, the lengths, ℓ_1,ℓ_2,ℓ_3, of the first three geodesics for $q\geq p\geq r\geq 3$ are given as

$$\ell_1 = 2\,\mathrm{arcosh}\left(2\cos\frac{\pi}{r}\cos\frac{\pi}{p}+\cos\frac{\pi}{q}\right),$$
$$\ell_2 = 2\,\mathrm{arcosh}\left(2\cos\frac{\pi}{q}\cos\frac{\pi}{r}+\cos\frac{\pi}{p}\right),$$
$$\ell_3 = 2\,\mathrm{arcosh}\left(2\cos\frac{\pi}{p}\cos\frac{\pi}{q}+\cos\frac{\pi}{r}\right).$$

We note that the group $\Gamma(p,q,r)$ has [up to conjugacy in $\Gamma(p,q,r)$] three cyclic subgroups of finite orders with $m_{\mathcal{R}}\in\{p,q,r\}$ (that statement is also true when $r=2$). For more details, we refer to Refs. 51, p. 163 and 52, pp. 98–99.

In the following Lemma we show that for large values of m_R and, respectively, small values of $\theta=\frac{\pi}{m_R}$, the contribution of elliptic elements to the Casimir energy becomes large.

Lemma 3.4. For $\theta\searrow 0$,

$$\int_0^\infty \frac{e^{-\theta y}}{e^{-\pi y}+1}(1+y^2)^{1/2}dy \gtrsim \theta^{-2}. \tag{3.8}$$

Proof. Using (3.4), we estimate the left hand side of (3.8) from below by

$$\frac{1}{2}\int_0^1 t^{\theta-1}(1+\log(t)^2)^{1/2}dt = \frac{\pi\boldsymbol{K}_1(\theta)}{4\theta},$$

with the equality a consequence of Ref. 31, (11.5.2). Around $\theta=0$, by Ref. 31, 11.2.1,

$$\boldsymbol{H}_1(\theta)=\mathcal{O}(\theta^2),\quad \theta\to 0.$$

By Ref. 31, 10.7.4,

$$Y_1(\theta)=\frac{-1}{2\pi\theta}+\mathcal{O}(1),\quad \theta\to 0.$$

Since

$$\boldsymbol{K}_1(\theta)=\boldsymbol{H}_1(\theta)-Y_1(\theta),$$

we therefore have

$$\frac{\pi\boldsymbol{K}_1(\theta)}{4\theta}=\frac{1}{8\theta^2}+\mathcal{O}(1),\quad \theta\to 0.$$

□

Lemma 3.5. The elliptic contribution to the Casimir energy of $\Gamma(2,3,7)\backslash\mathbb{H}$ rounded to six decimal places is equal to 0.875 676.

Proof. We note that the only elliptic elements in this group are those of order 2, 3, and 7, Ref. 42, Proposition 2.1. Thus, we are interested in the sum

$$\sum_{m_{\mathcal{R}}=2,3,7}\sum_{k=1}^{m_{\mathcal{R}}-1}\frac{1}{2m_{\mathcal{R}}\sin\left(k\frac{\pi}{m_{\mathcal{R}}}\right)}\int_{\mathbb{R}}\frac{e^{-2k\frac{\pi}{m_{\mathcal{R}}}y}}{e^{-2\pi y}+1}\left(\frac{1}{4}+y^2\right)^{1/2}dy. \tag{3.9}$$

We can use Lemma 3.1 to evaluate integrals in (3.9). In order to choose N that would provide a sufficiently accurate approximation, we recall (3.7). Its evaluation is equivalent to the evaluation of the integral in Lemma 3.1 for $D = \pi$ and various values of

$$C \in \left\{ \frac{\pi \ell}{m_{\mathcal{R}}}, \ell \in \{1, \ldots, m_{\mathcal{R}} - 1\}, m_{\mathcal{R}} \in \{2, 3, 7\} \right\}.$$

We also note that the right hand side of (3.6) is a decreasing function of C, thus it will suffice to find the error for $C = \pi/7$. Further, we note that for $N = 100$, by Lemma 3.1, we may estimate the error by

$$\frac{1}{2^N} \frac{7}{4} \frac{\boldsymbol{K}_1(\pi/7)}{m_{\mathcal{R}} \sin(k\pi/m_{\mathcal{R}})}, \qquad \frac{1}{2^N} \frac{7}{4} \boldsymbol{K}_1(\pi/7)\Big|_{N=100} < 10^{-29}.$$

Given that we only want to evaluate the elliptic contribution up to six significant digits, this certainly suffices, but we need to take into account the accumulation of errors that will appear once we find the total sum of on the order of 10^4 summands. Consequently, that will slightly decrease the precision to the order of 10^{-25}. Moreover,

$$\max_{m_{\mathcal{R}}=2,3,7} \max_{k=1,\ldots,m_{\mathcal{R}}} \left| \frac{1}{2m_{\mathcal{R}} \sin\left(k \frac{\pi}{m_{\mathcal{R}}}\right)} \right| < 1.$$

Thus, using Lemma 3.1 with $N = 100$ would be sufficient to evaluate (3.9) up to six significant figures, and we obtain the value 0.875 676. □

IV. IDENTITY CONTRIBUTION

It is possible to rewrite an identity contribution to $\zeta_\Gamma(s)$ as an infinite sum of special functions in the same spirit as we did for the elliptic contribution in Lemma 3.1.

Lemma 4.1. For $s \in \mathbb{C} \backslash \mathbb{N}$, the identity contribution to the spectral zeta function,

$$\frac{\mathrm{vol}(\Gamma \backslash \mathbb{H})}{8(s-1)} \int_{\mathbb{R}} \left(\frac{1}{4} + r^2\right)^{1-s} \mathrm{sech}^2(\pi r) dr$$

is equal to

$$\frac{\mathrm{vol}(\Gamma \backslash \mathbb{H})}{8(s-1)} 4^s \pi^{s-1} \Gamma(2-s) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(n+1)(-1)^k}{2^{n+3/2+s}} \binom{n}{k} \frac{\boldsymbol{K}_{3/2-s}(\pi + \pi k)}{(1+k)^{3/2-s}}.$$

This sum converges exponentially fast. In particular, for $s = -1/2$ this is equal to

$$-\frac{\mathrm{vol}(\Gamma \backslash \mathbb{H})}{\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(n+1)(-1)^k}{2^{n+6}} \binom{n}{k} \frac{\boldsymbol{K}_2(\pi + \pi k)}{(1+k)^2}.$$

Remark 4.2. Similar to Remark 3.2, Lemma 4.1 holds as an identity with removable singularities for $s \in \mathbb{N}_{\geq 2}$, since $\frac{\Gamma(2-s)}{s-1}$ has simple poles at $s \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{K}_{2/3-s}$ vanishes for $s \in \mathbb{N}_{\geq 2}$. As it is well-known in the literature, the identity contribution to the spectral zeta function has a simple pole at $s = 1$.

Proof of Lemma 4.1. Observe that for any constant $D > 0$,

$$\mathrm{sech}^2(Dr) = \frac{4e^{-2Dr}}{(1+e^{-2Dr})^2}.$$

Then,

$$\int_{\mathbb{R}} \left(\frac{1}{4} + r^2\right)^{1-s} \mathrm{sech}^2(Dr) dr = \int_{\mathbb{R}} \left(\frac{1}{4} + r^2\right)^{1-s} \frac{4e^{-2Dr}}{(1+e^{-2Dr})^2} dr$$
$$= 4^{s-1} \int_{\mathbb{R}} (1 + (2r)^2)^{1-s} \frac{4e^{-2Dr}}{(1+e^{-2Dr})^2} dr = 4^s \int_0^\infty (1+y^2)^{1-s} \frac{e^{-Dy}}{(1+e^{-Dy})^2} dy.$$

Above, we used the substitution $y = 2r$ and the fact that the integrand is even. For $|x - 1| < 2$, [compare with (3.5)]

$$\frac{1}{(1+x)^2} = -\frac{d}{dx} \frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n (n+1) 2^{-n-2}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k(n+1)2^{-n-2}x^k\binom{n}{k}.$$

We use this together with the absolute convergence of the integral (since $D > 0$) to obtain (with $x = e^{-Dy}$)

$$4^s\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\frac{n+1}{2^{n+2}}\binom{n}{k}\int_0^{\infty}(1+y^2)^{1-s}e^{-(D+kD)y}dy.$$

By Ref. 31, 11.5.2, this is equal to

$$4^s\sqrt{\pi}\Gamma(2-s)2^{1/2-s}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\frac{n+1}{2^{n+2}}\binom{n}{k}(D+Dk)^{s-3/2}\boldsymbol{K}_{3/2-s}(D+Dk).$$

Setting $D = \pi$ this becomes

$$4^s\pi^{s-1}\Gamma(2-s)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(n+1)(-1)^k}{2^{n+3/2+s}}\binom{n}{k}\frac{\boldsymbol{K}_{3/2-s}(\pi+\pi k)}{(1+k)^{3/2-s}}.$$

Recalling the factor of

$$\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{8(s-1)}$$

completes the first statement of the Lemma. Estimates analogous to the Proof of Lemma 3.1 show the exponential rate of convergence. Specializing to $s = -1/2$ we obtain that the identity contribution to the Casimir energy is

$$-\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(n+1)(-1)^k}{2^{n+6}}\binom{n}{k}\frac{\boldsymbol{K}_2(\pi+\pi k)}{(1+k)^2}.$$

□

For our purposes, we do not need the full precision of the expression in the preceding Lemma. As we will see in Corollary 4.4, specialized to the (2,3,7)-triangle group orbifold, the estimate we obtain in Lemma 4.3 below is sufficient to show that under a natural assumption on the lengths of the closed geodesics of the surface, the Casimir energy is positive.

Lemma 4.3. The identity contribution to the Casimir energy is contained in the interval

$$(-0.011\,017\,1\mathrm{vol}(\Gamma\backslash\mathbb{H}), -0.010\,702\,2\mathrm{vol}(\Gamma\backslash\mathbb{H})).$$

Proof. By (2.5), the contribution from the identity element to $\zeta(-1/2)$ is

$$\begin{aligned} &\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{8(s-1)}\int_{-\infty}^{\infty}\left(\frac{1}{4}+r^2\right)^{1-s}\mathrm{sech}^2(\pi r)dr\Big|_{s=-1/2} \\ &= -\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{12}\int_{-\infty}^{\infty}\left(\frac{1}{4}+r^2\right)^{\frac{3}{2}}\mathrm{sech}^2(\pi r)dr \\ &= -\frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{6}\int_{0}^{\infty}\left(\frac{1}{4}+r^2\right)^{\frac{3}{2}}\mathrm{sech}^2(\pi r)dr. \end{aligned} \quad (4.1)$$

We consider the integral over $[0,1]$ first. We note that

$$\frac{2r^4}{4}+\frac{3r^2}{4}+\frac{1}{8} \le \left(r^2+\frac{1}{4}\right)^{3/2} \le \frac{3r^4}{4}+\frac{3r^2}{4}+\frac{1}{8}, \quad 0 \le r \le 1, \quad (4.2)$$

All integrals $\int_0^1 r^k\,\mathrm{sech}^2(\pi r)dr$ for integer values of $k \ge 0$ can be evaluated, for example,

$$\int_0^1 \mathrm{sech}^2(\pi r)dr = \frac{\tanh(\pi)}{\pi}, \qquad \int_0^1 r^2\,\mathrm{sech}^2(\pi r)dr = \frac{\tanh(\pi)}{\pi} - \frac{2}{\pi}\int_0^1 r\tanh(\pi r)dr,$$

and

$$\int_0^1 r\tanh(\pi r)dr = -\frac{37}{24} + \frac{\log(1+e^{2\pi})}{\pi} - \frac{\mathrm{Li}_2(-e^{-2\pi})}{2\pi^2}.$$

Thus,

$$-\frac{1}{6}\int_0^1\left(\frac{3r^2}{4}+\frac{1}{8}\right)\operatorname{sech}^2(\pi r)dr = \frac{24\pi\log(1+e^{2\pi})-12\mathrm{Li}_2(-e^{-2\pi})-\pi^2(37+14\tanh(\pi))}{96\pi^3}.$$

We further calculate that

$$\int_0^1 r^4\operatorname{sech}^2(\pi r)dr = \frac{1}{\pi^5}\times\Big(-4\pi^3\log(1+e^{2\pi})+6\pi^2\mathrm{Li}_2(-e^{-2\pi})+6\pi\,\mathrm{Li}_3(-e^{-2\pi}) + 3\mathrm{Li}_4(-e^{-2\pi})+\pi^4\left(\frac{1687}{240}+\tanh(\pi)\right)\Big).$$

Thus, we obtain the estimates

$$\begin{aligned}-\frac{1}{6}\int_0^1\left(r^2+\frac{1}{4}\right)^{3/2}\operatorname{sech}^2(\pi r)dr &\le \frac{1}{2880\pi^5}\times\big(1680\pi^3\log(1+e^{2\pi})-1800\pi^2\mathrm{Li}_2(-e^{-2\pi})-1440\pi\,\mathrm{Li}_3(-e^{-2\pi})\\ &\quad -720\,\mathrm{Li}_4(-e^{-2\pi})-\pi^4(2797+660\tanh(\pi))\big)\\ &< -0.010\,289\end{aligned}$$

and

$$\begin{aligned}-\frac{1}{6}\int_0^1\left(r^2+\frac{1}{4}\right)^{3/2}\operatorname{sech}^2(\pi r)dr &\ge \frac{1}{1920\pi^5}\big(1440\pi^3\log(1+e^{2\pi})-1680\pi^2\mathrm{Li}_2(-e^{-2\pi})-1440\pi\,\mathrm{Li}_3(-e^{-2\pi})\\ &\quad -720\,\mathrm{Li}_4(-e^{-2\pi})-\pi^4(2427+520\tanh(\pi))\big)\\ &> -0.010\,575\,3.\end{aligned}$$

On the other hand,

$$r^3+\frac{3r}{8}\le\left(r^2+\frac{1}{4}\right)^{3/2}\le r^3+\frac{4r}{8},\qquad r\ge 1.$$

We calculate that

$$\int_1^\infty r^3\operatorname{sech}^2(\pi r)dr = \frac{1}{2\pi^4}\big(6\pi^2\log(1+e^{2\pi})-6\pi\mathrm{Li}_2(-e^{-2\pi}) -3\mathrm{Li}_3(-e^{-2\pi})-2\pi^3(5+\tanh(\pi))\big)$$

and

$$\int_1^\infty r\operatorname{sech}^2(\pi r)dr = \frac{\log(2\cosh(\pi))-\pi\tanh(\pi)}{\pi^2}.$$

Thus,

$$\begin{aligned}&-0.000\,441\,822\\ &< \frac{-6\pi^2\log(1+e^{2\pi})+6\pi\mathrm{Li}_2(-e^{-2\pi})+3\mathrm{Li}_3(-e^{-2\pi})+2\pi^3(5+\tanh(\pi))}{12\pi^4}\\ &-\frac{\log(2\cosh(\pi))-\pi\tanh(\pi)}{12\pi^2}\\ &\le -\frac{1}{6}\int_1^\infty\left(r^2+\frac{1}{4}\right)^{3/2}\operatorname{sech}^2(\pi r)dr\\ &\le \frac{-6\pi^2\log(1+e^{2\pi})+6\pi\mathrm{Li}_2(-e^{-2\pi})+3\mathrm{Li}_3(-e^{-2\pi})+2\pi^3(5+\tanh(\pi))}{12\pi^4}\\ &-\frac{\log(2\cosh(\pi))-\pi\tanh(\pi)}{16\pi^2}\\ &< -0.000\,413\,16.\end{aligned}$$

Consequently,

$$-0.011\,017\,1 < -\frac{1}{6}\int_0^\infty \left(r^2 + \frac{1}{4}\right)^{3/2} \operatorname{sech}^2(\pi r)dr < -0.010\,702\,2.$$

Combining with our upper and lower bounds for the integral from 0 to 1 completes the proof. □

Corollary 4.4. The identity contribution to the Casimir energy of the (2,3,7)-triangle group orbifold belongs to $[-0.001\,648\,16, -0.001\,601\,04]$.

Proof. We use Lemma 4.3 and the formula for the area of the surface, (1.2), to get the lower estimate

$$-2\pi\left(1 - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{7}\right)\right)0.011\,017\,1 \approx -0.001\,648\,15 > -0.001\,648\,16$$

and the upper estimate

$$-2\pi\left(1 - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{7}\right)\right)0.010\,702\,2 \approx -0.001\,601\,05 < -0.001\,601\,04.$$

□

V. CONTRIBUTION FROM HYPERBOLIC ELEMENTS

The hyperbolic contribution to $\zeta_\Gamma(s)$ [see (2.5)] is equal to

$$\frac{(4\pi)^{-1/2}}{\Gamma(s)} \sum_{\{\mathcal{P}\}_p} \sum_{n=1}^{\infty} (\ell_\gamma/n)^{1/2} \operatorname{csch}\left(\frac{n\ell_\gamma}{2}\right)(n\ell_\gamma)^s K_{1/2-s}\left(\frac{n\ell_\gamma}{2}\right).$$

We recall that $\sum_{\{\mathcal{P}\}_p}$ denotes the summation over all conjugacy classes of primitive hyperbolic elements. Specialized at $s = -1/2$, this reads

$$\begin{aligned}
&\frac{(4\pi)^{-1/2}}{\Gamma(-1/2)} \sum_{\{\mathcal{P}\}_p} \sum_{n=1}^{\infty} (\ell_\gamma/n)^{1/2} \operatorname{csch}\left(\frac{n\ell_\gamma}{2}\right)(n\ell_\gamma)^{-1/2} K_1\left(\frac{n\ell_\gamma}{2}\right)\\
&= -\frac{1}{4\pi} \sum_{\{\mathcal{P}\}_p} \sum_{n=1}^{\infty} (\ell_\gamma/n)^{1/2} \operatorname{csch}\left(\frac{n\ell_\gamma}{2}\right)(n\ell_\gamma)^{-1/2} K_1\left(\frac{n\ell_\gamma}{2}\right)\\
&= -\frac{1}{4\pi} \sum_{\{\mathcal{P}\}_p} \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{csch}\left(\frac{n\ell_\gamma}{2}\right) K_1\left(\frac{n\ell_\gamma}{2}\right).
\end{aligned} \tag{5.1}$$

To estimate the hyperbolic contribution, we will use the explicit expressions for the lengths of the first 50 primitive closed geodesics of the (2,3,7)-triangle group orbifold calculated by Vogeler in 2003 Ref. 42, p. 32. The enumeration of closed geodesics and the explicit calculation of their lengths is an interesting task and has been achieved for a handful of Riemann surfaces. For example, in 1988 Aurich and Steiner enumerated the first 2×10^6 closed geodesics for the simplest Riemann surface whose fundamental group is the octagon group.[53] Their method is based on symbolic dynamics and revealed "a strange arithmetical structure of chaos," in that it seemed that there was an exact formula for the lengths of primitive closed geodesics. Together with Bogomolny they rigorously proved that this formula holds.[54] A few years later, for the compact Riemann surface of genus two generated by Gutzwiller's arthimetical Fuchsian group, Ninnemann computed[34] the lengths of the shortest 4 369 202 closed geodesics. For the surfaced obtained as the quotient by the $\Gamma(2,3,8)$ triangle group, he computed the lengths of the shortest 120 000 000 closed geodesics. Here, we use Vogeler's work[42] for the (2,3,7)-triangle group orbifold to calculate the contribution of the first 50 primitive closed geodesics to the Casimir energy quite accurately.

Lemma 5.1. The contribution to the Casimir energy of $\Gamma(2,3,7)\backslash\mathbb{H}$ from the first 50 primitive geodesics rounded to six decimal places is equal to $-0.568\,085\,1$.

Proof. The proof of this lemma uses explicit formulas for the first geodesics calculated in Ref. 42, p. 32. The length spectrum is the non-decreasing ordered set of all positive numbers which occur as lengths of hyperbolic translations. The multiplicity of each such length is the number of distinct similarity classes whose elements have the same length. The rotations of order 3 and 7 that generate the group $\Gamma(2,3,7)$

are represented as elements of $PSL_2(\mathbb{R})$ by the matrices A and B below:

$$
\begin{aligned}
b &= \frac{1}{3}\left(\sqrt{3(\cot^2(\frac{\pi}{7})-3)}+\sqrt{3}\cot(\frac{\pi}{7})\right),\\
A &= \begin{pmatrix} \cos(\frac{\pi}{3}) & \sin(\frac{\pi}{3}) \\ -\sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{pmatrix}, \qquad B = \begin{pmatrix} \cos\left(\frac{\pi}{7}\right) & b\sin\left(\frac{\pi}{7}\right) \\ -b^{-1}\sin\left(\frac{\pi}{7}\right) & \cos\left(\frac{\pi}{7}\right) \end{pmatrix},\\
R &= A^{-1}B, \qquad L = B.
\end{aligned}
$$

Each hyperbolic element may therefore be represented as a product of R and L. For such a product, which is a matrix we may denote by M_γ, the length of the corresponding closed geodesic is

$$2\cosh^{-1}|\mathrm{Tr}\,(M_\gamma))/2|.$$

On Ref. 42, p. 32, the author calculates a finite portion of the length spectrum. To do this, he develops a combinatorial approach which leads to a classification of the conjugacy classes of hyperbolic elements of $\Gamma(2,3,7)$, arranged by length. For the convenience of the reader, we present the approximate lengths of primitive closed geodesics together with representatives of the corresponding conjugacy classes in Table I. We note that the lengths of closed geodesics can be expressed as a finite combination of elementary functions and thus may be calculated with an arbitrary precision. Moreover, we note Ref. 55, p. 95 that for each $\gamma\in\Gamma$,

$$\mathrm{Tr}\,(M_\gamma)\in\mathbb{Q}(2\cos(2\pi/7)).$$

For example,

$$
\begin{aligned}
\mathrm{Tr}\,(R.L) &= 1+2\cos(2\pi/7),\\
\mathrm{Tr}\,(R.R.L.L) &= 2+2\cos(2\pi/7)+2\cos(4\pi/7),\\
\mathrm{Tr}\,(R.L.R.L) &= 1+4\cos(2\pi/7)+2\cos(4\pi/7),\\
\mathrm{Tr}\,(R.L.R.L.L) &= 2+2\cos(2\pi/7),\\
\mathrm{Tr}\,(R.L.R.R.L.L) &= 2+4\cos(2\pi/7)+2\cos(4\pi/7),\\
\mathrm{Tr}\,(R.L.L.R.R.L.L) &= 2+4\cos(2\pi/7),\\
\mathrm{Tr}\,(R.L.R.L.R.L) &= 3+4\cos(2\pi/7)+4\cos(4\pi/7).
\end{aligned}
\tag{5.2}
$$

The situation with multiplicities is a bit subtle. Let $\gamma\in\Gamma(2,3,7)$ be a primitive hyperbolic element and denote the length of the corresponding closed geodesic by ℓ_γ. Then, γ^{-1} is also a hyperbolic element and $\ell_{\gamma^{-1}}=\ell_\gamma$. As described in Ref. 42, p. 24, by changing the R's and L's in the representation of γ, one obtains a hyperbolic element γ^* with $\ell_{\gamma^*}=\ell_\gamma$. We have, Ref. 42, p. 24,

$$\ell_\gamma=\ell_{\gamma^{-1}}=\ell_{\gamma^*}=\ell_{(\gamma^*)^{-1}}.$$

We let $s(\gamma)$ be the number of distinct conjugacy classes among $\{\gamma\}, \{\gamma^{-1}\}, \{\gamma^*\}$, and $\{(\gamma^*)^{-1}\}$. For example, let γ be a hyperbolic element and assume that $\{\gamma\}=\{\gamma^{-1}\}$ and $\{\gamma^*\}=\{(\gamma^*)^{-1}\}$, but $\{\gamma\}\neq\{\gamma^*\}$; in this case we say $s(\gamma)=2$. If, on the other hand, $\{\gamma\}=\{\gamma^{-1}\}=\{\gamma^*\}=\{(\gamma^*)^{-1}\}$, then $s(\gamma)=1$. If it turns out that the conjugacy classes $\{\gamma\}, \{\gamma^{-1}\}, \{\gamma^*\}$, and $\{(\gamma^*)^{-1}\}$ are pairwise different, then $s(\gamma)=4$.

We additionally note that $s(\gamma)$ is not necessarily a multiplicity of the geodesic length. It might happen that $\ell_\gamma=\ell_{\gamma'}$, but at the same time,

$$\gamma'\notin\{\gamma\}\cup\{\gamma^{-1}\}\cup\{\gamma^*\}\cup\{(\gamma^*)^{-1}\}.$$

In this case, the multiplicity of the geodesic length is bigger than $s(\gamma)$. Among the geodesics that we take into account, this situation happens exactly once: in Table I, one finds two geodesics of approximate lengths 5.2889, that are, however, listed separately.

To sum it up, for each hyperbolic element $\gamma\in\Gamma$, there are $s(\gamma)$ closed geodesics of length ℓ_γ corresponding to distinct conjugacy classes among $\{\gamma\}, \{\gamma^{-1}\}, \{\gamma^*\}$ and $\{(\gamma^*)^{-1}\}$. This implies that Table I contains the first 50 primitive closed geodesics of the (2,3,7)-triangle group orbifold. We denote by ℓ_n the length of the nth primitive closed geodesic; thus, $\ell_1\approx 0.98, \ell_3=\ell_4\approx 2.13$. We define

$$A(\gamma)=-\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{s(\gamma)}{n}\mathrm{csch}\left(\frac{n\ell_\gamma}{2}\right)K_1\left(\frac{n\ell_\gamma}{2}\right).$$

The total contribution of hyperbolic elements to the spectral zeta function is equal to the sum of $A(\gamma)$ where γ ranges over all primitive hyperbolic elements. Summing up all of $A(\gamma)$ from Table I, we obtain a value of $-0.568\,0851$, rounding to seven decimal places.

We denote by

$$\mathcal{N}_L=\#\{\ell_\gamma\leq L\}$$

the number of primitive hyperbolic geodesics γ (counted with multiplicity) of length ℓ_γ less or equal than L. The prime geodesic Theorem[56] states that

$$\lim_{L\to\infty}\frac{L\mathcal{N}_L}{e^L}=1.$$

Consequently, it follows that when we enumerate these lengths (counting multiplicity) as ℓ_j, we obtain

$$\lim_{j\to\infty}\frac{\ell_j}{\log(j)+\log\log(j)}=1.$$

We propose that it is reasonable to assume, and we note that this inequality holds for $j < 51$,

$$\ell_j \geq \log(j)+\log\log(j),\qquad \forall j\geq 51. \tag{5.3}$$

Lemma 5.2. Under the assumption that (5.3) holds, the contribution from all but the first 50 hyperbolic elements is greater than or equal to $-0.293\,892$.

We need a small technical lemma before proceeding to the Proof of Lemma 5.2.

Lemma 5.3. For $j \geq 16$ and $n\in\mathbb{N}$,

$$j^n\log^{n+1/2}j \leq (j^n\log^n j-1)(\log j+\log\log j)^{1/2}.$$

Proof. We note that the statement of the Lemma follows (after the change of variables $x = \log j$) from

$$\begin{aligned}\sqrt{x+\log x} &\leq e^{nx}x^n\left(\sqrt{x+\log x}-\sqrt{x}\right) \iff\\ \frac{\sqrt{x+\log x}}{\sqrt{x+\log x}-\sqrt{x}} &\leq e^{nx}x^n \iff\\ \frac{x+\log x+\sqrt{x}\sqrt{x+\log x}}{\log x} &\leq e^{nx}x^n.\end{aligned}$$

Since $n\geq 1$, $xe^x \leq x^n e^{nx}$ for $x > 0$. Moreover, for $\log x \geq 1$, which further guarantees that $x \geq 1$, and for $e^x \geq 2+\sqrt{2}$, we have

$$\frac{x+\log x+\sqrt{x}\sqrt{x+\log x}}{\log x}\leq x+1+\sqrt{x}\sqrt{2x}\leq(2+\sqrt{2})x\leq xe^x.$$

Recalling that $x = \log j$, it is enough to assume that $j \geq e^e$. □

Proof of Lemma 5.2. We split the sum in three parts:

1. $n = 1, j\in[51, 10^7]$,
2. $n = 1, j \geq 10^7+1$,
3. $n\geq 2$.

Since *csch* and K_1 are decreasing functions on $(0,\infty)$, Ref. 31, 10.37, then under the assumption (5.3) we obtain

$$\begin{aligned}&\operatorname{csch}\left(\frac{n\ell_j}{2}\right)K_1\left(\frac{n\ell_j}{2}\right)\\ &\quad\leq \operatorname{csch}\left(\frac{n(\log j+\log\log j)}{2}\right)K_1\left(\frac{n(\log j+\log\log j)}{2}\right).\end{aligned}$$

We use this to obtain upper bounds for the sums

$$\begin{aligned}B_1 &= \sum_{j=51}^{10^7}\frac{1}{4\pi}\operatorname{csch}\left(\frac{\ell_j}{2}\right)K_1\left(\frac{\ell_j}{2}\right),\\ B_2 &= \sum_{j=10^7+1}^{\infty}\frac{1}{4\pi}\operatorname{csch}\left(\frac{\ell_j}{2}\right)K_1\left(\frac{\ell_j}{2}\right),\\ B_3 &= \sum_{n=2}^{\infty}\sum_{j=51}^{\infty}\frac{1}{4\pi n}\operatorname{csch}\left(\frac{n\ell_j}{2}\right)K_1\left(\frac{n\ell_j}{2}\right).\end{aligned}$$

For B_1, we obtain by explicit calculation the estimate

$$B_1 \leq \sum_{j=51}^{10^7} \frac{\operatorname{csch}\left[\frac{\log(j)+\log\log(j)}{2}\right] K_1\left[\frac{\log(j)+\log\log(j)}{2}\right]}{4\pi} \approx 0.138\,415. \tag{5.4}$$

For B_2 and B_3, we note that for any $j \geq 10^7$, Ref. 31, (10.37.1) implies

$$\frac{\operatorname{csch}\left[\frac{\log(j)+\log\log(j)}{2} n\right] K_1\left[\frac{\log(j)+\log\log(j)}{2} n\right]}{4\pi n} \leq \frac{\operatorname{csch}\left[\frac{\log(j)+\log\log(j)}{2} n\right] K_{\frac{3}{2}}\left[\frac{\log(j)+\log\log(j)}{2} n\right]}{4\pi n}.$$

By Ref. 31, 10.39.2,

$$K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$

By Ref. 31, 10.25(ii), 10.29(i)

$$e^{3\pi i/2} K_{3/2}(z) = -e^{-i\pi/2} K_{-1/2}(z) + 2e^{i\pi/2} \frac{\partial}{\partial z} K_{1/2}(z).$$

We therefore obtain

$$-iK_{3/2}(z) = i\sqrt{\frac{\pi}{2z}} e^{-z} - i\sqrt{\frac{\pi}{2z}} z^{-1} e^{-z} - 2i\sqrt{\frac{\pi}{2z}} e^{-z}$$
$$\Rightarrow K_{3/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}(z^{-1} + 1).$$

Consequently, since $\operatorname{csch}(z) = \sinh(z)^{-1} = \frac{2}{e^z - e^{-z}}$,

$$\operatorname{csch}(z) K_{3/2}(z) = \sqrt{\frac{2\pi}{z}} \frac{z^{-1} + 1}{e^{2z} - 1} = \sqrt{\frac{2\pi}{z}} \frac{1 + z}{ze^{2z} - z}.$$

We then obtain by setting $z = n(\log j + \log\log j)/2$ and dividing by $4\pi n$

$$\frac{\operatorname{csch}\left[\frac{\log(j)+\log\log(j)}{2} n\right] K_{\frac{3}{2}}\left[\frac{\log(j)+\log\log(j)}{2} n\right]}{4\pi n} = \frac{n(\log(j) + \log\log(j)) + 2}{2\sqrt{\pi} n(j^n \log^n(j) - 1) n^{3/2} (\log(j) + \log\log(j))^{3/2}}. \tag{5.5}$$

We recall that $n \geq 1$ and note that for $j \geq j_N$ we can estimate

$$n(\log(j) + \log\log(j)) + 2 \leq A_{n,j_N} n(\log(j) + \log\log(j)),$$

where

$$A_{n,j_N} = 1 + \frac{2}{n(\log(j_N) + \log\log(j_N))}.$$

Thus, we can estimate (5.5) from above by

$$A_{n,j_N} n \frac{\log(j) + \log\log(j)}{2\sqrt{\pi} n(j^n \log^n(j) - 1)[n(\log(j) + \log\log(j))]^{3/2}}$$
$$\leq \frac{A_{n,j_N}}{2\sqrt{\pi} n^{3/2} (j^n \log^n(j) - 1)(\log(j) + \log\log(j))^{1/2}}$$
$$\leq \frac{A_{n,j_N}}{2\sqrt{\pi}} \frac{j^{-n} \log^{-n-1/2}(j)}{n^{3/2}}.$$

In the last step we used Lemma 5.3. We therefore obtain the estimate for B_2,

$$\sum_{j=10^7+1}^{\infty} \operatorname{csch}\left(\frac{\ell_j}{2}\right)K_1\left(\frac{\ell_j}{2}\right) \leq \frac{1+\frac{2}{\log(10^7)+\log(\log(10^7))}}{2\sqrt{\pi}} \sum_{j=10^7+1} j^{-1}\log^{-3/2} j.$$

For $n = 1$ and $j_N = 10^7$, we evaluate

$$\frac{A_{1,10^7}}{2\sqrt{\pi}} = \frac{1+\frac{2}{\log(10^7)+\log(\log(10^7))}}{2\sqrt{\pi}} \approx 0.311\,949$$

and

$$\sum_{j=10^7+1}^{\infty} j^{-1}\log^{-1-\frac{1}{2}}(j) \leq \int_{10^7}^{\infty} \frac{\log^{-1-\frac{1}{2}}(j)}{j}dj = \frac{2}{\sqrt{\log(10^7)}} \approx 0.498\,165.$$

This gives the estimate

$$B_2 \leq 0.155\,402. \tag{5.6}$$

For $n \geq 2$ and $j_N = 51$, we obtain

$$\sum_{n=2}^{\infty}\sum_{j=51}^{\infty} \frac{1}{4\pi n}\operatorname{csch}\left(\frac{n\ell_j}{2}\right)K_1\left(\frac{n\ell_j}{2}\right) \leq \sum_{n=2}^{\infty}\sum_{j=51}^{\infty} \frac{A_{n,51}j^{-n}\log^{-n-1/2}(j)}{2\sqrt{\pi}n^{3/2}}.$$

We evaluate for $n \geq 2$

$$\frac{A_{n,51}}{2\sqrt{\pi}} \leq \frac{1+\frac{1}{\log(51)+\log\,\log(51)}}{2\sqrt{\pi}} \approx 0.335\,311.$$

Using the definition of the polylogarithm $\mathrm{Li}_{3/2}$ of order $3/2$, we obtain

$$\mathrm{Li}_{\frac{3}{2}}\left(\frac{1}{j\log(j)}\right) = \sum_{n=1}^{\infty} \frac{j^{-n}\log(j)^{-n}}{n^{3/2}},$$

and thus for each $j \geq 51$,

$$\sum_{n=2}^{\infty} \frac{j^{-n}\log(j)^{-n-1/2}}{n^{3/2}} = \log^{-1/2}(j)\left(\mathrm{Li}_{\frac{3}{2}}\left(\frac{1}{j\log(j)}\right) - j^{-1}\log^{-1}(j)\right).$$

We calculate

$$\begin{aligned}
\sum_{j=51}^{\infty}\sum_{n=2}^{\infty} \frac{j^{-n}\log^{-n-\frac{1}{2}}(j)}{n^{3/2}} &= \sum_{j=51}^{\infty} \frac{\left(\mathrm{Li}_{\frac{3}{2}}\left(\frac{1}{j\log(j)}\right) - j^{-1}\log^{-1}(j)\right)}{\log^{1/2}(j)} \\
&= \sum_{j=51}^{\infty} \frac{j\log(j)\mathrm{Li}_{\frac{3}{2}}\left(\frac{1}{j\log(j)}\right) - 1}{j\log^{\frac{3}{2}}(j)} \\
&\leq \sum_{j=51}^{\infty} \frac{j\log(j)\mathrm{Li}_1\left(\frac{1}{j\log(j)}\right) - 1}{j\log^{\frac{3}{2}}(j)} \\
&= \sum_{j=51}^{\infty} \frac{-j\log(j)\log\left(1-\frac{1}{j\log(j)}\right) - 1}{j\log^{\frac{3}{2}}(j)}.
\end{aligned} \tag{5.7}$$

Above, both the first and the third equality follow from the definitions of Li_1 and $\mathrm{Li}_{3/2}$. We note that for $x \geq 50$, the following inequality holds:

$$0 < -\log\left(1-\frac{1}{x}\right) \leq \frac{1}{x} + \frac{1}{1.9x^2},$$

thus for $j > 50$,

$$0 < -\log\left(1-\frac{1}{j\log(j)}\right) \leq \frac{1}{j\log(j)} + \frac{1}{1.9j^2\log(j)^2}.$$

With this we estimate (5.7) from above by

$$\sum_{j=51}^{\infty} \frac{1}{1.9j^2 \log^{\frac{5}{2}}(j)} \le \frac{1}{1.9}\int_{50}^{\infty} \frac{1}{j^2 \log(j)^{5/2}} dj$$
$$= \frac{1}{1.9}\left(\frac{2}{150\log^{3/2}(50)} - \frac{4}{150\sqrt{\log(50)}} + \frac{4\Gamma(\log(50), \frac{1}{2})}{3}\right)$$
$$\approx 0.000\,224.$$

Above $\Gamma(a,s) = \int_a^\infty t^{s-1}e^{-t}\,dt$ in the incomplete Gamma function. We further note that the first equality follows from the calculation

$$\int \frac{1}{j^2\log^{\frac{5}{2}}(j)}\,dj = -\frac{2}{3j\log^{\frac{3}{2}}(j)} + \frac{4}{3j\sqrt{\log(j)}} - \frac{4\Gamma\big(\log(j), \frac{1}{2}\big)}{3} + C.$$

Thus, we obtain the estimate

$$B_3 \le 0.335\,311 \cdot 0.000\,224 \approx 0.000\,075. \tag{5.8}$$

Consequently, summing (5.4), (5.6), and (5.8) we obtain that

$$B_1 + B_2 + B_3 \le 0.293\,892.$$

Recalling the minus sign in front of the hyperbolic contribution thereby completes the proof of its lower bound. □

Proof of Conjecture 1 under the assumption (5.3). By Corollary 4.4, the identity contribution to the Casimir energy is at least −0.001 648 16. By Lemma 5.1 the contribution from the first 50 primitive hyperbolic geodesics is, up to six decimal places, −0.568 085 1. By Lemma 3.5 the contribution of the elliptic elements is, up to six decimal places, 0.875 676. By Lemma 5.2 the contribution from all but the first 50 hyperbolic elements is at least 0.293 892. We therefore obtain a lower bound of the Casimir energy

$$\zeta_\Gamma(-1/2) \ge 0.875\,676 - 0.001\,648\,16 - 0.568\,085\,1 - 0.293\,892 = 0.012\,050\,7.$$

□

VI. CONCLUDING REMARKS

It is well known that the Casimir energy of a hyperbolic orbifold surface depends on the geometry of the surface as this follows from the representation of the spectral zeta function through the Selberg trace formula.[43] Physically, the surface may be used to represent a quantum field theory. Conjecture 1 indicates that the Casimir energy can be attractive or repulsive depending on the geometry of the orbifold. In particular, without conical singularities, the energy is negative (attractive), and with singularities it may in fact be positive (repulsive). We reasonably expect to be able to prove the conjecture, but this will require not only the asymptotic behavior of the lengths, which is well known,[56] but also explicit lower bounds for the lengths. One can obtain a crude lower bound via volume growth considerations, but we reasonably expect it is possible to obtain a bound that would be sufficient to prove the conjecture. Moreover, we expect that the hyperbolic elements in other (p,q,r)-triangle groups may admit a description in the spirit of Ref. 42, so that we may be able to prove that for many corresponding orbifold surfaces, the Casimir energy is also positive (repulsive). Explicit expressions for the lengths of closed geodesics in certain Riemann surfaces have been obtained,[34,53,54] but these examples do not have conical singularities. Hence they may be interesting to use to calculate the Casimir energy to high accuracy, but it will be strictly negative. We would need to expand these techniques in the spirit of Vogeler[42] to allow for conical singularities to generalize the results obtained here to show that other orbifold surfaces also have positive Casimir energy. Obtaining further results of this type would help to show that the conical singularities profoundly influence the Casimir energy and Casimir effect. If the orbifold represents a certain quantum field theory, what are the physical implications of such a repulsive Casimir effect? Perhaps this would be interesting for physicists to consider further and develop experimental tests.[57]

ACKNOWLEDGMENTS

J.R. is grateful to Peter Sarnak for inspiring discussions and to Roger Vogeler for elucidating correspondence. G.Z.'s research is partially supported by the Swedish Research Council (VR). K.F. is partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy Grant No. EXC 2044-390685587, Mathematics Münster: Dynamics-Geometry-Structure.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Ksenia Fedosova: Conceptualization (equal); Data curation (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Resources (equal); Software (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Julie Rowlett**: Conceptualization (equal); Data curation (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Resources (equal); Software (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Genkai Zhang**: Conceptualization (equal); Data curation (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Resources (equal); Software (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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