Pure [⁎]-homomorphisms

Citation for the original published paper (version of record):

Bosa, J., Vilalta Vila, E. (2025). Pure [⁎]-homomorphisms. Journal of Functional Analysis, 288(3). http://dx.doi.org/10.1016/j.jfa.2024.110739

Journal of Functional Analysis 288 (2025) 110739

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Pure *-homomorphisms ☆

Joan Bosa [a,*], Eduard Vilalta [b]

[a] *Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Zaragoza, Spain*
[b] *Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Chalmers Tvärgata 3, SE-412 96 Gothenburg, Sweden*

A R T I C L E I N F O

Article history:
Received 30 June 2024
Accepted 29 October 2024
Available online 6 November 2024
Communicated by Stefaan Vaes

MSC:
primary 46L05
secondary 19K14, 46L80

Keywords:
C^*-algebras
Cuntz semigroups
Factorization of maps
Pureness

A B S T R A C T

We introduce and study the notion of pureness for *-homomorphisms and, more generally, for cpc order-zero maps. After providing various important examples of pureness, we show our main result: Any composition of two pure maps factors through a pure object up to Cuntz equivalence. This is used to obtain several factorization results at the level of C*-algebras.

☆ All authors were partially supported by Ministerio de Ciencia, Innovación y Universidades (Gobierno de España) (grant No. PID2020-113047GB-I00/AEI/10.13039/501100011033), and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya (grant No. 2021SGR01015). The first author was also partially supported by Ministerio de Ciencia, Innovación y Universidades (Gobierno de España) (Consolidación Investigadora grant No. CNS2022-135340) and by the Dep. Ciencia, Universidad y sociedad del conocimiento del Gobierno de Aragón (grupo E22_23R). The second author was also supported by The Fields Institute for Research in Mathematical Sciences and by the Knut and Alice Wallenberg Foundation (KAW 2021.0140).

* Corresponding author.
E-mail addresses: jbosa@unizar.es (J. Bosa), vilalta@chalmers.se (E. Vilalta).
URLs: https://personal.unizar.es/jbosa/ (J. Bosa), https://www.eduardvilalta.com (E. Vilalta).

https://doi.org/10.1016/j.jfa.2024.110739
0022-1236/

1. Introduction

The classification of unital separable simple nuclear non-elementary C*-algebras by their K-theory and tracial data, also known as the Elliott classification program, has been one of the major developments in the theory of operator algebras in the past 40 years. A landmark achievement, obtained as the collaborative effort of many hands and decades of work (see, among many others, [31,13,29]), states that any such C*-algebra A can be classified by the so-called Elliott invariant as long as A is $\mathcal{Z}$-stable (i.e. it absorbs the Jiang-Su algebra tensorially) and satisfies the universal coefficient theorem (UCT). Some of the first results in the Elliott classification program relied heavily on the inductive limit presentation of the algebras under consideration; however, the modern approach to classification exploits conditions that are more abstract in nature. One such condition is Winter's notion of *pure C*-algebras*, which were defined in [31] as those algebras that are both almost divisible and almost unperforated. This notion is deeply connected to $\mathcal{Z}$-stability, as stated explicitly in the famous Toms-Winter conjecture: For unital separable simple nuclear non-elementary C*-algebras, the conditions of almost unperforation, $\mathcal{Z}$-stability and finite nuclear dimension should all coincide. The conjecture is by now largely a theorem [24,31,6,9], with the only remaining implication being if almost unperforation implies $\mathcal{Z}$-stability. If true, pureness and $\mathcal{Z}$-stability agree for all unital separable simple nuclear C*-algebras. However, it should be noted that $\mathcal{Z}$-stability and pureness do not agree in general, with the latter then becoming an important regularity property on its own right; see [3].

The current approaches to the classification program classify C*-algebras by classifying maps. In such results, strong conditions are only imposed either on the domain or codomain, while the assumptions on the other side tend to be milder. One important example of this phenomenon is Robert's classification of *-homomorphisms from 1-dimensional NCCW-complexes with trivial K_1-group (and their inductive limits) to stable rank one C*-algebras via the *Cuntz semigroup* [19]. This semigroup is a rich invariant for C*-algebras, which plays a crucial role in both Elliott's program and the Toms-Winter conjecture. Robert's theorem has been an important tool in recent classification results, and can be regarded as an example where the result for *-homomorphisms is much more powerful than its induced result for C*-algebras. A recent groundbreaking development along these lines can be found in [8], where unital embeddings from unital separable nuclear C*-algebras satisfying the UCT to unital simple separable nuclear $\mathcal{Z}$-stable C*-algebras are classified. Further, a current trend in such results is to move the conditions on the domain or codomain to the maps themselves. This can be seen, for example, in the definition of $\mathcal{O}_2$-stable *-homomorphisms [14], in the study of $\mathcal{O}_\infty$-stable *-homomorphisms run in [7], in the introduction of real rank zero inclusions [15], and many others.

Inspired by this modern approach to classification —as well as by its importance as a regularity property— in this paper we generalize the notion of pureness to maps between C*-algebras. Extending the original definition, we say that a cpc order-zero map (in particular, a *-homomorphism) $\theta\colon A \to B$ is *pure* if it is both almost unperforated and almost divisible, in their suitable versions (Definition 3.2).

Our first examples of pureness arise from the study of maps akin to $\mathcal{Z}$-stable C*-algebras. Recall from [16, Proposition 4.4] that a unital separable C*-algebra A is $\mathcal{Z}$-stable if and only if $\mathcal{Z}$ embeds unitally into $A_\omega \cap A'$ (for a free ultrafilter ω), which is equivalent to $\mathcal{Z}$ embedding unitally to $A_\omega \cap A' \cap S'$ for any separable sub-C*-algebra $S \subseteq A_\omega \cap A'$. In particular, the unit in $A_\omega \cap A' \cap S'$ is almost divisible for each S. In our setting, we get the following:

Theorem 1.1 *(cf. 3.9). Let A be σ-unital, and let $\theta\colon A \to B$ be a *-homomorphism. Assume that $1 \in B_\omega \cap \theta(A)'/\mathrm{Ann}(\theta(A))$ is almost divisible. Then, θ is pure.*

Further, if $1 \in B_\omega \cap (\theta(A) \cup S)'/\mathrm{Ann}(\theta(A) \cup S)$ is almost divisible for every separable sub-C*-algebra $S \subseteq B_\omega \cap \theta(A)'$, there exists a pure sub-C*-algebra $C \subseteq B_\omega$ such that $\theta(A) \subseteq C \subseteq B_\omega$.

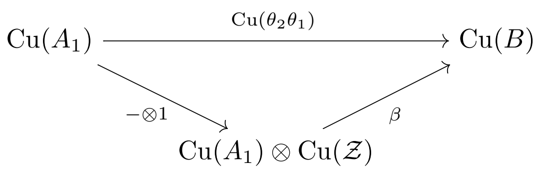
The second part of Theorem 1.1 says that, up to passing to the ultraproduct, certain pure *-homomorphisms factor through a pure C*-algebra. In the general setting, a central question in the study of regularity properties for maps is which *-homomorphisms with a certain property factor, up to Murray-von Neumann equivalence (Lemma 4.5), through a C*-algebra with said property; see the comments before Question 3.10 for a more in-depth discussion. Restricted to our case, the question is:

Question 1.2 *(3.10-3.11).* Let $\theta\colon A \to B$ be a *-homomorphism. Is θ pure if and only if the map $\iota_B \circ \theta\colon A \to B_\omega$ factors, up to Murray-von Neumann equivalence, through a pure C*-algebra?

More generally, does there exist $n \in \mathbb{N}$ such that, for any tuple $\theta_1, \ldots, \theta_n$ of pairwise composable pure *-homomorphisms, the composition $\iota_B \circ \theta_n \circ \cdots \circ \theta_1$ factors up to Murray-von Neumann equivalence through a pure C*-algebra?

We start our study defining the notion of pureness and asserting Theorem 1.1 in Section 3, where we also provide a number of examples. In Section 4 we establish several permanence properties of pureness that are used throughout the paper. In Section 5, the main section of the paper, we combine all the previous results to provide an answer to Question 1.2. To do so, we exploit the structure of (abstract) Cuntz semigroup morphisms that are both almost divisible and almost unperforated. Our main technical result (Theorem 5.5) says that, at the level of Cuntz semigroups, any composition of two pure maps factors through a pure object. Restricted to C*-algebras, the result reads as follows:

Theorem 1.3 *(cf. 5.9). Let $\theta_1\colon A_1 \to A_2$ and $\theta_2\colon A_2 \to B$ be pure *-homomorphisms. Then, there exists a* Cu*-morphism β such that the following diagram commutes*



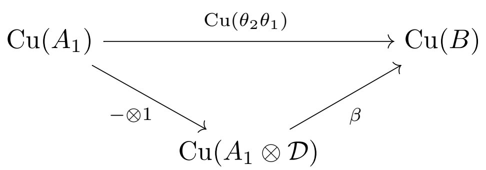
Theorem 5.9 is in fact more general. For the previous result to hold, one only needs θ_1, θ_2 to be cpc order-zero maps, with θ_1 almost divisible and θ_2 almost unperforated; see Definition 3.2 for details. This generality provides us with many applications where the above statement can be used. Indeed, one obtains a complete answer to Question 1.2 when A_1 is AF and B has stable rank one.

Corollary 1.4 *(cf. 5.10). Let A_1 be a unital AF-algebra, and let B be a unital* C**-algebra of stable rank one. Let $\theta_1 \colon A_1 \to A_2$ and $\theta_2 \colon A_2 \to B$ be unital, pure* **-homomorphisms. Then, $\theta_2\theta_1$ factors, up to approximate unitary equivalence, through $A_1 \otimes \mathcal{Z}$.*

Further, if B has strict comparison, one can set $\theta_2 = id_B$ to answer Question 1.2 for a single *-homomorphism in the above situation. We state this in Corollary 5.13.

The last section of the paper, Section 6, is devoted to two types of pure maps: q-rational *-homomorphisms (Definition 6.1) and soft, pure *-homomorphisms (see Definition 6.6). Here, we deduce analogues of Theorem 5.9 for other types of tensorial absorption. Again, such results are also valid for cpc order-zero maps, at the expense of β being only a generalized Cu-morphism.

Theorem 1.5 *(cf. 6.3, 6.10). Let $\theta_1 \colon A_1 \to A_2$ and $\theta_2 \colon A_2 \to B$ be q-rational (resp. soft, pure)* **-homomorphisms. Then, there exists a* Cu*-morphism β such that the following diagram commutes*



where $\mathcal{D}$ is the UHF-algebra M_q (resp. the Jacelon-Razak algebra $\mathcal{W}$).

Combining the results above with Robert's classification result, we obtain:

Corollary 1.6 *(cf. 6.4). Let $\theta_1 \colon A_1 \to A_2$ and $\theta_2 \colon A_2 \to B$ be two unital q-rational* **-homomorphisms. Assume that A_1 is stably isomorphic to an inductive limit of* 1*-dimensional NCCW-complexes with trivial K_1-group, and that B is of stable rank one. Then, $\theta_2\theta_1$ factors, up to approximate unitary equivalence, through $A_1 \otimes M_q$.*

Acknowledgments

This work was carried out during two visits of EV to JB at Universidad de Zaragoza. He is grateful to the institution for its hospitality.

Both authors would also like to thank Hannes Thiel for comments on a first draft of the paper. They would also like to thank the anonymous referee for their valuable comments and suggestions.

2. Preliminaries

2.1 *(The Cuntz semigroup).* Given two positive elements a, b in a C*-algebra A, we write $a \precsim b$ whenever a is *Cuntz subequivalent* to b, that is, whenever there exists $(r_n)_n \subseteq A$ such that $a = \lim_n r_n b r_n^*$. Further, we say that a is *Cuntz equivalent* to b, and write $a \sim b$, whenever $a \precsim b$ and $b \precsim a$.

The *Cuntz semigroup* of A is defined to be the set $(A \otimes \mathcal{K})_+/\sim$, equipped with the addition induced by diagonal addition and the partial order induced by $\precsim$. We denote this monoid by $\mathrm{Cu}(A)$; see [11].

As shown in [32], any completely positive, contractive (cpc), order-zero map $\varphi\colon A \to B$ (for example, any *-homomorphism) induces a well-defined, partially ordered, monoid morphism $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ given by $\mathrm{Cu}(\varphi)([a]) = [\varphi(a)]$.

2.2 *(Abstract Cuntz semigroups).* As defined in [11], a positively ordered monoid S is a Cu-*semigroup* if it satisfies the following four conditions:

(O1) every increasing sequence has a supremum;
(O2) every element is the supremum of a $\ll$-increasing sequence;
(O3) $x' + y' \ll x + y$ whenever $x' \ll x$ and $y' \ll y$;
(O4) $\sup_n(x_n + y_n) = \sup_n x_n + \sup_n y_n$ for any pair of increasing sequences $(x_n), (y_n)$ in S,

where one writes $x \ll y$ if, for any increasing sequence $(z_n)_n$ with supremum greater than or equal to y, there exists $n \in \mathbb{N}$ such that $x \leq z_n$.

A monoid morphism $S \to T$ between Cu-semigroups is said to be a *generalized* Cu-*morphism* if it preserves both the order and suprema of increasing sequences. A Cu-*morphism* is any generalized Cu-morphism that also preserves the $\ll$-relation. Any cpc order-zero map induces a generalized Cu-morphism, while any *-homomorphism induces a Cu-morphism; see [32] and [11] respectively.

The reader is referred to [2] for an in-depth introduction to Cu-semigroups.

3. Pure *-homomorphisms

We introduce in Definition 3.2 a notion of $\mathrm{Cu}(\mathcal{Z})$-*multiplication* for Cu-morphisms, and we say that a *-homomorphism is *pure* if its induced Cu-morphism has $\mathrm{Cu}(\mathcal{Z})$-

multiplication. We then provide a number of examples (3.3-3.5), and exhibit a class of pure maps whose composition with the canonical diagonal embedding always factorizes through a pure C*-algebra; see Theorem 3.9.

The section ends with Question 3.10 and its weakening Question 3.11. A satisfactory general answer to the second question is given in Section 5.

3.1. Recall from [31, Definitions 3.1, 3.5, 3.6] (see [22] for the concrete definition displayed here) that a Cu-semigroup S is said to be

- *almost unperforated* if, whenever $x, y \in S$ are such that $(m+1)x \leq my$ for some $m \in \mathbb{N}$, one has $x \leq y$.
- *almost divisible* if for every $k \in \mathbb{N}$ and $x', x \in S$ such that $x' \ll x$ there exists $z \in S$ such that $kz \leq x$ and $x' \leq (k+1)z$.

One says that a C*-algebra is *pure* if its Cuntz semigroup is almost unperforated and almost divisible.

In [2], a theory of tensor products and *multiplication* for Cuntz semigroups was developed. Amongst other results, the authors showed in [2, Theorem 7.3.11] that a Cu-semigroup S is pure if and only if $S \cong S \otimes \mathrm{Cu}(\mathcal{Z})$. Further, one can see that the Cuntz semigroup of any $\mathcal{Z}$-stable C*-algebra has $\mathrm{Cu}(\mathcal{Z})$-multiplication (i.e. it tensorially absorbs $\mathrm{Cu}(\mathcal{Z})$).

Definition 3.2. Let $\varphi\colon S \to T$ be a generalized Cu-morphism. We will say that φ is

(i) *almost unperforated* if $\varphi(x) \leq \varphi(y)$ whenever $(m+1)x \leq my$ for some $m \in \mathbb{N}$.
(ii) *almost divisible* if for every $k \in \mathbb{N}$ and $x', x \in S$ such that $x' \ll x$ there exists $z \in T$ such that $kz \leq \varphi(x)$ and $\varphi(x') \leq (k+1)z$.

The generalized Cu-morphism φ will be said to have $\mathrm{Cu}(\mathcal{Z})$-*multiplication* if it is almost unperforated and almost divisible, and a cpc order-zero map $\theta\colon A \to B$ will be called *pure* if $\mathrm{Cu}(\theta)$ has $\mathrm{Cu}(\mathcal{Z})$-multiplication.

Let us begin the section with some examples of pure maps:

Example 3.3. It is readily checked that any *-homomorphism $A \to B$ is pure whenever A or B is pure. More generally, the same holds if $\mathrm{Cu}(A \to B)$ factors through the Cuntz semigroup of a pure C*-algebra; see Proposition 4.2 for details and more general statements.

As a concrete example, any *-homomorphism that factors through an infinite reduced free product $*_{i=1}^{\infty}(A,\tau)$, with A unital and τ a faithful trace, is pure. These products are always simple and monotracial ([5, Corollary, page 431]), have stable rank one ([12, Theorem 3.8]), and strict comparison ([19, Proposition 6.3.2]). A standard argument, using the positive solution to the ranks problem for simple, stable rank one C*-algebras from [25], proves that the Cuntz semigroup is almost divisible; see, for example, [30, Remark 4.4], or its generalization Proposition 3.5 below.

Example 3.4. Let A, B be unital C*-algebras, and assume that B is almost divisible. Then, it follows from [21, Lemma 6.1(i)] that the first factor embedding $A \to A \otimes B$ is almost unperforated and, therefore, pure.

Another example of pure maps is given by the following proposition. Recall that a C*-algebra A is said to be *nowhere scattered* if it has no nonzero elementary ideal-quotients; see [27].

Proposition 3.5. *Let A be a separable nowhere scattered* C*-*algebra of stable rank one, and let $\theta\colon A \to B$ be a cpc order-zero, almost unperforated map. Then, θ is pure.*

Proof. We need to show that $\mathrm{Cu}(\theta)$ is almost divisible. To see this, let $x \in \mathrm{Cu}(A)$ and take $k \in \mathbb{N}$. Then, it follows from [1, Theorem 7.13] that there exists $y \in \mathrm{Cu}(A)$ such that $\widehat{y} = \frac{2}{2k+1}\widehat{x}$ in $L(F(\mathrm{Cu}(A)))$; see [1, Section 7] for the appropriate definitions.

Thus, we have

$$\infty x = \infty y, \quad \lambda(ky) < \lambda(x), \quad \text{and} \quad \lambda(x) < \lambda((k+1)y)$$

for every normalized functional $\lambda \in F(\mathrm{Cu}(A))$.

By [18, Proposition 2.1], there exist $n, m \in \mathbb{N}$ such that

$$(n+1)(ky) \leq nx, \quad \text{and} \quad (m+1)x \leq m(k+1)y.$$

Using that $\mathrm{Cu}(\theta)$ is almost unperforated, we obtain $k\,\mathrm{Cu}(\theta)(y) \leq \mathrm{Cu}(\theta)(x) \leq (k+1)\,\mathrm{Cu}(\theta)(y)$, as desired. $\square$

Remark 3.6. The assumption of separability is not needed in Proposition 3.5. This will be explained in detail in Remark 4.4.

Recall that a separable unital C*-algebra is $\mathcal{Z}$-stable if and only if $\mathcal{Z}$ embeds unitally into $A_\omega \cap A'$; see, for example, [16, Proposition 4.4]. A reindexing argument shows that this is equivalent to $\mathcal{Z}$ embedding unitally to $A_\omega \cap A' \cap C'$ for any separable sub-C*-algebra $C \subseteq A_\omega \cap A'$. In particular, any such C*-algebra satisfies that the unit $1 \in A_\omega \cap A' \cap C'$ is almost divisible.

In what follows (Theorem 3.9), we show that any *-homomorphism satisfying the analogue of this property is pure. For that, recall that given a *-homomorphism $\theta\colon A \to B$, the annihilator of θ, in symbols $\mathrm{Ann}(\theta)$, is defined as the set of elements x in B such that $x\theta(A) = \theta(A)x = \{0\}$.

Proposition 3.7. *Let $\theta\colon A \to B$ be a *-homomorphism. Assume that there exists a net of contractive, positive elements $(e_\lambda)_{\lambda\in\Lambda}$ in $\theta(A)'\cap B$ such that $(e_\lambda)_\lambda$ acts as an approximate unit for $\theta(A)$ and such that $\overline{e_\lambda}$ is almost divisible in $\theta(A)'\cap B/\mathrm{Ann}(\theta(A))$ for each λ. Then, there exists a sub-C*-algebra $C \subseteq \theta(A)'\cap B$ such that the map $A \to C^*(\theta(A), C)$ is pure. If $\Lambda = \mathbb{N}$, C can be taken to be separable.*

Proof. For each triple $\mu = (\lambda, k, p) \in \Lambda \times \mathbb{N} \times \mathbb{Q}_{+,\leq 1}$, let $f_\mu \in (\theta(A)'\cap B)_+$ be such that

$$\overline{f_\mu^{\oplus k}} \precsim \overline{e_\lambda}, \quad \text{and} \quad \overline{(e_\lambda - p)_+} \precsim \overline{f_\mu^{\oplus(k+1)}}$$

in $\mathrm{Cu}(\theta(A)'\cap B/\mathrm{Ann}(\theta(A)))$, where the superscript $\oplus k$ denotes the diagonal formed by k copies of the element.

Thus, up to a cut-down of f_μ, there exists a finite matrix h_μ with entries in $\mathrm{Ann}(\theta(A))$ such that

$$f_\mu^{\oplus k} \oplus 0 \precsim e_\lambda \oplus h_\mu, \quad \text{and} \quad (e_\lambda - p)_+ \oplus 0 \precsim f_\mu^{\oplus(k+1)} \oplus h_\mu$$

in $(\theta(A)'\cap B)\otimes M_l$ for some l, where the 0's denote zero matrices of appropriate sizes.

Now, for any $q \in \mathbb{Q}_{+,\leq 1}$, it follows from [23, Proposition 2.4] that there exist finite matrices $r_{\mu,q}$ and $s_{\mu,q}$ over $\theta(A)'\cap B$ such that

$$(f_\mu^{\oplus k} - q)_+ \oplus 0 = r_{\mu,q}(e_\lambda \oplus h_\mu) r_{\mu,q}^* \tag{1}$$

and

$$(e_\lambda - p - q)_+ \oplus 0 = s_{\mu,q}(f_\mu^{\oplus(k+1)} \oplus h_\mu) s_{\mu,q}^*. \tag{2}$$

Set $C = C^*(\{e_\lambda, f_\mu, r_{\mu,q}(i,j), s_{\mu,q}(i,j)\}_{\mu,q,i,j})$. If $\Lambda = \mathbb{N}$, then C is separable by construction.

Let us show that $A \to C^*(\theta(A), C)$ is pure. For any $m \in \mathbb{N}$ and C*-algebra E, let $\iota_m\colon E \to M_m(E)$ denote the map $e \mapsto e^{\oplus m}$. We will also use this notation for the matrix amplification $M_k(E) \to M_{km}(E)$ defined entry-wise. Note that, for a matrix $e \in M_n(E)$, $e^{\oplus m}$ denotes the $mn \times mn$ matrix $e \oplus \ldots \oplus e$, while $\iota_m(e)$ is the $mn \times mn$ where each entry of e has been replaced by a diagonal $m \times m$ matrix. An important fact that we will use is: Given $a \in M_n(A)$ and $b \in M_m(\theta(A)'\cap B)$, then $\iota_n(b)$ commutes with $\theta(a)^{\oplus m}$ (and the roles of a and b can be reversed).

First, given a contraction $a \in M_n(A)_+$, note that applying ι_n to Equation (1) and multiplying by an $l = l(\mu, q)$ diagonal matrix of $\theta(a)$'s, one gets

$$\begin{aligned}(\theta(a)(f_\mu - q)_+^{\oplus n})^{\oplus k} \oplus 0 &= \theta(a)^{\oplus l}\iota_n((f_\mu^{\oplus k} - q)_+ \oplus 0)\\ &= \theta(a)^{\oplus l}\iota_n(r_{\mu,q})\iota_n(e_\lambda \oplus h_\mu)\iota_n(r_{\mu,q})^*\\ &= \iota_n(r_{\mu,q})\theta(a)^{\oplus l}\iota_n(e_\lambda \oplus h_\mu)\iota_n(r_{\mu,q})^*\end{aligned}$$

$$= \iota_n(r_{\mu,q})(\theta(a)\iota_n(e_\lambda) \oplus 0)\iota_n(r_{\mu,q})^*$$
$$\precsim \theta(a)\iota_n(e_\lambda)$$

and, by multiplying an $m = m(\mu, q)$ diagonal matrix of a's to an enlarged Equation (2), we obtain

$$\begin{aligned}\theta(a)(e_\lambda - p - q)_+^{\oplus n} \oplus 0 &= \theta(a)^{\oplus m}\iota_n(s_{\mu,q})\iota_n(f_\mu^{\oplus(k+1)} \oplus h_\mu)\iota_n(s_{\mu,q})^* \\ &= \iota_n(s_{\mu,q})((\theta(a)\iota_n f_\mu^{\oplus(k+1)} \oplus 0)\iota_n(s_{\mu,q})^* \\ &= \iota_n(s_{\mu,q})((\theta(a)f_\mu^{\oplus n})^{\oplus(k+1)} \oplus 0)\iota_n(s_{\mu,q})^* \\ &\precsim (\theta(a)f_\mu^{\oplus n})^{\oplus(k+1)}.\end{aligned}$$

Thus,

$$((\theta(a)f_\mu^{\oplus n} - q)_+)^{\oplus k} \precsim \theta(a)e_\lambda^{\oplus n} \precsim \theta(a), \text{ and } (\theta(a)(e_\lambda^{\oplus n} - p - q)_+) \precsim (\theta(a)f_\mu^{\oplus n})^{\oplus(k+1)}$$

in $C^*(\theta(A), C) \otimes \mathcal{K}$.

Since this holds for any choice of p and q, taking suprema gives $(\theta(a)f_\mu^{\oplus n})^{\oplus k} \precsim \theta(a)$ and $(\theta(a)e_\lambda^{\oplus n}) \precsim (\theta(a)f_\mu^{\oplus n})^{\oplus(k+1)}$. Further, since the $e_\lambda^{\oplus n}$'s are an approximate unit for $\theta(a)$, it follows that a is almost divisible in $C^*(\theta(A), C) \otimes \mathcal{K}$. Since this holds for any finite matrix over $\theta(A)$, we get that $A \to C^*(\theta(A), C)$ is almost divisible.

To show that $A \to C^*(\theta(A), C)$ is almost unperforated, we follow the same strategy as in [24, Lemma 4.3]. Thus, let $a, b \in M_n(A)_+$ be contractions such that $a^{\oplus(k+1)} \precsim b^{\oplus k}$ in $A \otimes \mathcal{K}$ for some $k \in \mathbb{N}$. For any $\varepsilon > 0$, there exists a finite matrix v with entries in A such that $(a - \varepsilon)_+^{\oplus(k+1)} = v(b^{\oplus k} \oplus 0)v^*$. Given $\mu = (\lambda, k, p)$, we can use that f_μ is central to obtain

$$(f_\mu^{\oplus n}\theta((a - \varepsilon)_+))^{\oplus(k+1)} = \theta(v)((f_\mu^{\oplus n}\theta(b))^{\oplus k} \oplus 0)\theta(v)^*.$$

In particular, $(f_\mu^{\oplus n}\theta((a-\varepsilon)_+))^{\oplus(k+1)} \precsim (f_\mu^{\oplus n}\theta(b))^{\oplus k}$ in $\mathrm{Cu}(C^*(\theta(A), C))$. Recall from the above computations that we have

$$(\theta((a - \varepsilon)_+)e_\lambda^{\oplus n}) \precsim (\theta((a - \varepsilon)_+)f_\mu^{\oplus n})^{\oplus(k+1)}, \quad \text{and} \quad (\theta(b)f_\mu^{\oplus n})^{\oplus k} \precsim \theta(b).$$

Chaining these three $\precsim$-relations together, one obtains

$$(\theta((a - \varepsilon)_+)e_\lambda^{\oplus n}) \precsim (\theta((a - \varepsilon)_+)f_\mu^{\oplus n})^{\oplus(k+1)} \precsim (f_\mu^{\oplus n}\theta(b))^{\oplus k} \precsim \theta(b).$$

Using once again that the $e_\lambda^{\oplus n}$'s are an approximate unit for $\theta((a - \varepsilon)_+)$, we get $(\theta(a) - 2\varepsilon)_+ \precsim \theta(b)$. This proves that $A \to C^*(\theta(A), C)$ is almost unperforated, as required. □

Lemma 3.8 below shows that, when studying pureness of maps, one can restrict to those with an ultraproduct for a codomain.

Lemma 3.8. *Let $\theta\colon A \to B$ be a cpc order-zero map, and let $\iota_B\colon B \to B_\omega$ be the natural inclusion. Then, θ is pure if and only if $\iota_B\theta$ is pure.*

Proof. If θ is pure, the composition $\iota_B\theta$ is pure; see Proposition 4.2 for details.

Conversely, assume that $\iota_B\theta$ is pure. We need to show that $\mathrm{Cu}(\theta)$ is both almost divisible and almost unperforated. First, let $n \in \mathbb{N}$, $[a] \in \mathrm{Cu}(A)$ and $\varepsilon > 0$. Since any element in $\mathrm{Cu}(A)$ can be written as the supremum of classes in $M_\infty(A)_+$ (and since $\mathrm{Cu}(\theta)$ preserves suprema), we may assume $a \in M_k(A)_+$ for some $k \in \mathbb{N}$. Using that $\iota_B\theta$ is pure, we find $[b] \in \mathrm{Cu}(B_\omega)$ such that $[\iota_B\theta((a-\varepsilon/3)_+)] \leq (n+1)[b]$ and $n[b] \leq [\iota_B\theta(a)]$. Upon cutting-down a and b if needed (say, to $(a-\varepsilon/2)_+$), we may assume $b \in B_\omega \otimes M_k$ and that there exist finite matrices r, s over B_ω such that $\iota_B\theta((a-\varepsilon/2)_+) = rb^{\oplus(n+1)}r^*$ and $b^{\oplus n} = s\iota_B\theta(a)s^*$.

Further, note that $B_\omega \otimes M_k \cong (B \otimes M_k)_\omega$. Thus, by going sufficiently far in the sequences corresponding to b and the entries of r and s, we can find elements $b_0 \in M_k(B)$ and $r_0, s_0 \in M_\infty(B)$ such that $[\theta((a-\varepsilon)_+)] \leq (n+1)[b_0]$ and $n[b_0] \leq [\theta(a)]$. This shows that $\mathrm{Cu}(\theta)$ is almost divisible.

The proof of almost unperforation is analoguous. □

Theorem 3.9. *Let A be σ-unital, and let $\theta\colon A \to B$ be a *-homomorphism. Assume that $1 \in B_\omega \cap \theta(A)'/\mathrm{Ann}(\theta(A))$ is almost divisible. Then, θ is pure.*

Further, if $1 \in B_\omega \cap (\theta(A) \cup S)'/\mathrm{Ann}(\theta(A) \cup S)$ is almost divisible for every separable sub-C-algebra $S \subseteq B_\omega \cap \theta(A)'$, there exists a pure sub-C*-algebra $C \subseteq B_\omega$ such that $\theta(A) \subseteq C \subseteq B_\omega$.*

Proof. As shown in Lemma 3.8 above, a *-homomorphism $A \to B$ is pure if and only if the induced map $A \to B_\omega$ is pure. Further, since $1 \in B_\omega \cap \theta(A)'/\mathrm{Ann}(\theta(A))$ is almost divisible, it follows from Proposition 3.7 above that $A \to B_\omega$ is pure. This gives the first part of the statement.

For the second part, note that it follows from Proposition 3.7 that there exists a separable sub-C*-algebra $C_1 \subseteq B_\omega \cap \theta(A)'$ such that inclusion $\theta(A) \subseteq C^*(\theta(A), C_1)$ is pure. Since C_1 is separable, we know by our assumption that $1 \in B_\omega \cap (\theta(A) \cup C_1)'/\mathrm{Ann}(\theta(A) \cup C_1)$ is almost divisible. Proposition 3.7 proves the existence of a separable $C_2 \subseteq B_\omega \cap \theta(A)' \cap C_1'$ such that $C^*(\theta(A), C_1) \subseteq C^*(\theta(A), C_1, C_2)$ is pure.

Proceeding inductively, one obtains a sequence of pure inclusions in B_ω. Their limit, denoted by C, is pure. □

The previous result justifies the following question:

Question 3.10. Let $\theta\colon A \to B$ be a *-homomorphism. Is θ pure if and only if the composition $\iota_B\theta\colon A \to B_\omega$ factors, up to Murray-von Neumann equivalence (see Lemma 4.5), through a pure C*-algebra?

Question 3.10 is a particular instance of a general question that one can ask for any property P. Namely, does any *-homomorphism with P 'come' from a C*-algebra with P? In other words, does P admit a McDuff type characterization?

This question has been posed for: real rank zero inclusions [15], where it remains open; for $\mathcal{O}_2$-stable morphisms, answered in [14, Corollary 4.5]; and for morphisms of nuclear dimension 0, with a partial answer provided in [10].

For any of the properties P listed above, one has that an inductive system where each map satisfies P has a limit with P. Loosely, one can interpret this as saying that any infinite composition of maps with P always factorizes through a C*-algebra with P. In this sense, one may also ask if there exists a natural number n_P such that the composition of n_P morphisms with P always factorizes through a C*-algebra with P. Specialized to our setting, the question is:

Question 3.11. Does there exist $n \in \mathbb{N}$ such that, for any tuple $\theta_1, \ldots, \theta_n$ of pairwise composable pure *-homomorphisms, the composition $\iota_B \theta_n \cdots \theta_1$ factors up to Murray-von Neumann equivalence through a pure C*-algebra?

To our knowledge, an answer to Question 3.11 is not known for real rank zero inclusions. In what follows, we investigate the question for our notion of pureness.

4. Pureness and Cu($\mathcal{Z}$)-multiplication

This section compiles permanence properties of Cu($\mathcal{Z}$)-multiplication for generalized Cu-morphisms. We state some of these results in the language of abstract Cuntz semigroups to highlight when the $\ll$-relation needs to be preserved.

Propositions 4.1 and 4.2 below are in analogy to Proposition 3.19 and Lemma 3.20 from [14].

Proposition 4.1. *Let S be a* Cu*-semigroup. Then $S \cong S \otimes \mathrm{Cu}(\mathcal{Z})$ if and only if id_S has* Cu($\mathcal{Z}$)*-multiplication.*

Proof. It follows from Theorems 7.3.11 and 7.5.4 in [2] that $S \cong S \otimes \mathrm{Cu}(\mathcal{Z})$ if and only if S is almost divisible and almost unperforated. The statement now follows from the definitions. □

Proposition 4.2. *Let $\varphi_1 \colon S_1 \to S_2$ and $\varphi_2 \colon S_2 \to T$ be generalized* Cu*-morphisms. Then,*

(1) *if φ_1 has* Cu($\mathcal{Z}$)*-multiplication, so has $\varphi_2\varphi_1$.*
(2) *if φ_1 is a* Cu*-morphism and φ_2 has* Cu($\mathcal{Z}$)*-multiplication, the composition $\varphi_2\varphi_1$ also has* Cu($\mathcal{Z}$)*-multiplication.*
(3) *if S_2 has* Cu($\mathcal{Z}$)*-multiplication, so have both φ_2 and φ_1.*

Proof. (1) Given $x' \ll x$ in S_1 and $k \in \mathbb{N}$, there exists $z \in S_2$ such that $\varphi_1(x') \leq (k+1)z$ and $kz \leq \varphi_1(x)$. Thus, one has $\varphi_2\varphi_1(x') \leq (k+1)\varphi_2(z)$ and $k\varphi_2(z) \leq \varphi_2\varphi_1(x)$.

Similarly, if $(m+1)x \leq my$ for some $m \in \mathbb{N}$ in S_1, one gets $\varphi_1(x) \leq \varphi_1(y)$ and, consequently, $\varphi_2\varphi_1(x) \leq \varphi_2\varphi_1(y)$.

(2) If φ_1 is now a Cu-morphism and φ_2 is the map that has Cu($\mathcal{Z}$)-multiplication, one can take $x' \ll x$ in S_1 and consider the induced relation $\varphi_1(x') \ll \varphi_1(x)$ in S_2. Since φ_2 has Cu($\mathcal{Z}$)-multiplication, one obtains the element needed for the almost division in T.

The same argument shows that if $(m+1)x \leq my$ in S, then $\varphi_2\varphi_1(x) \leq \varphi_2\varphi_1(y)$ in T, as desired.

(3) If S_2 has Cu($\mathcal{Z}$)-multiplication, [2, Proposition 7.3.8] implies that for any element $x \in S_2$ and $k \in \mathbb{N}$, there exists $y \in S_2$ such that $ky \leq x \leq (k+1)y$. With this stronger property, it is routine to check that both φ_1 and φ_2 have Cu($\mathcal{Z}$)-multiplication. □

Lemma 4.3. *Let $\varphi\colon S \to T$ be a generalized* Cu*-morphism. Then φ has* Cu($\mathcal{Z}$)*-multiplication if and only if $\varphi\iota_H$ has* Cu($\mathcal{Z}$)*-multiplication for every inclusion ι_H from a countably based sub-*Cu*-semigroup H to S.*

Proof. The proof uses standard model theoretic techniques applied to Cu-semigroups. We sketch it here for the convenience of the reader.

First, note that the forward implication follows from Proposition 4.2 (2). For the backwards implication, let $x, y \in S$ and let $n \in \mathbb{N}$. Using [26, Lemma 5.1], find a countably based sub-Cu-semigroup H of S containing x and y.

By our assumptions, the composition $\varphi\iota_H$ has Cu($\mathcal{Z}$)-multiplication. Thus, if $(n+1)x \leq ny$ in S (equivalently in H) we obtain $\varphi(x) = \varphi\iota_H(x) \leq \varphi\iota_H(y) = \varphi(y)$. This shows that φ is almost unperforated. A similar proof shows that φ is also almost divisible. □

Remark 4.4. As mentioned in Remark 3.6, the assumption of separability is not needed in Proposition 3.5. Indeed, let A be a nowhere scattered C*-algebra of stable rank one (not necessarily separable), and let $\theta\colon A \to B$ be a cpc order-zero almost unperforated map.

Then, for any countably based sub-Cu-semigroup $H \subseteq \mathrm{Cu}(A)$, there exists a separable sub-C*-algebra $C \subseteq A$ such that the Cuntz morphism induced by the inclusion $\iota_C\colon C \to A$ is an order-embedding and contains H in its image; see [26, Proposition 6.1]. Further, one can choose such a C to be nowhere scattered and of stable rank one, since separable nowhere scattered stable rank one sub-C*-algebras of A form a σ-complete and cofinal family amongst all separable sub-C*-algebras of A (for nowhere scatteredness see [27, Proposition 4.11]).

Now, by Proposition 3.5, the composition $\theta\iota_C\colon C \to B$ is pure, which implies that $\mathrm{Cu}(\theta)\iota_H$ has Cu($\mathcal{Z}$)-multiplication. Since this holds for each countably based sub-Cu-semigroup, we deduce from Lemma 4.3 above that θ is pure.

Recall from [14, Definition 3.4] that a pair of *-homomorphisms $\theta, \eta \colon A \to B$ are *approximately Murray-von Neumann equivalent* if, for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$, there exists $u \in B$ such that

$$\|u\theta(a)u^* - \eta(a)\| < \varepsilon, \quad \text{and} \quad \|u^*\eta(a)u - \theta(a)\| < \varepsilon,$$

for every $a \in \mathcal{F}$.

The following is essentially [14, Corollary 3.11] applied to the category Cu, with the only difference that we drop the assumption of A being separable.

Lemma 4.5. *Let θ, $\eta \colon A \to B$ be two approximately Murray-von Neumann equivalent *-homomorphisms. Then,* $\mathrm{Cu}(\theta) = \mathrm{Cu}(\eta)$.

Proof. Let $x \in \mathrm{Cu}(A)$. By [26, Proposition 6.1], there exists a separable sub-C*-algebra $A' \subseteq A$ such that $\mathrm{Cu}(\iota_{A'})(\mathrm{Cu}(A'))$ is a sub-Cu-semigroup of $\mathrm{Cu}(A)$ containing x. Here, $\iota_{A'}$ denotes the inclusion from A' to A.

Since the functor $\mathrm{Cu}(\cdot)$ is M_2-stable and invariant under approximate unitary equivalence, [14, Corollary 3.11] implies that

$$\mathrm{Cu}(\theta)(x) = \mathrm{Cu}(\theta\iota_{A'})(x) = \mathrm{Cu}(\eta\iota_{A'})(x) = \mathrm{Cu}(\eta)(x),$$

as required. □

The following result gives one of the implications of Question 3.10.

Proposition 4.6. *Let $\theta \colon A \to B$ be a *-homomorphism. Assume that, for any separable sub-*C**-algebra $A' \subseteq A$, there exist cpc order-zero maps $\eta \colon A' \to C$ and $\rho \colon C \to B$ such that C is pure and $\rho\eta$ is approximately Murray-von Neumann equivalent to $\theta\iota_{A'}$.*

Then, θ is pure.

Proof. Assume first that A is separable. Then, θ is Murray-von Neumann equivalent to a map that factorizes through C. Thus, by Lemma 4.5, $\mathrm{Cu}(\theta)$ itself factorizes through $\mathrm{Cu}(C)$, and so θ is pure by Proposition 4.2.

If A is not separable, we know from [26, Proposition 6.1] that any countably based sub-Cu-semigroup H in $\mathrm{Cu}(A)$ is contained in $\mathrm{Cu}(A')$ for some separable sub-C*-algebra A' of A. Thus, the argument above shows that $\mathrm{Cu}(\theta\iota_{A'})$ has $\mathrm{Cu}(\mathcal{Z})$-multiplication.

Consequently, since any inclusion gives rise to a Cu-morphism, Proposition 4.2 shows that $\mathrm{Cu}(\varphi)\iota_H$ has $\mathrm{Cu}(\mathcal{Z})$-multiplication. Lemma 4.3 gives the required result. □

4.7 *(Approximations of *-homomorphisms).* Recall that a C*-algebra A is said to be *approximated* by a family of sub-C*-algebras $(A_\lambda)_{\lambda\in\Lambda}$ if, for each $\varepsilon > 0$ and every choice of finitely many elements $a_1, \ldots, a_n \in A$, there exist $\lambda \in \Lambda$ and $b_1, \ldots, b_n \in A_\lambda$ such that $\|b_j - a_j\| < \varepsilon$ for each j.

Let $\theta\colon A \to B$ be a cpc order-zero map. We will say that a tuple $(A_\lambda, \theta_\lambda\colon A_\lambda \to B)_{\lambda\in\Lambda}$ *approximates* θ if each θ_λ is cpc order-zero and the following condition holds:

For every $\varepsilon > 0$ and every finite tuple $a_1, \dots, a_n \in A$, there exist $\lambda \in \Lambda$ and $b_1, \dots, b_n \in A_\lambda$ such that

$$\|b_j - a_j\| < \varepsilon, \quad \text{and} \quad \|\theta_\lambda(b_j) - \theta(a_j)\| < \varepsilon$$

for each j.

Note that this notion of approximation naturally includes the notion of limit morphism (i.e. when $A_\lambda = A$ for each λ).

We will now show that pureness is preserved under approximations. We do this by proving a much more general result, which we expect to find other uses elsewhere. Informally, Proposition 4.8 below says that any formula of the Cuntz semigroup is inherited by the approximated map. This generalizes [26, Proposition 3.7].

Proposition 4.8. *Let $\theta\colon A \to B$ be a cpc order-zero map, and let $(A_\lambda, \theta_\lambda)_{\lambda\in\Lambda}$ approximate θ. Then, for any pair of finite index sets $J, K \subseteq \mathbb{N}$, any family of pairs $[a_j'], [a_j] \in \mathrm{Cu}(A)$ such that $[a_j'] \ll [a_j]$ for each $j \in J$, and any pair of functions $m_k, n_k\colon J \to \mathbb{N}$ such that*

$$\sum_{j\in J} m_k(j)[a_j] \ll \sum_{j\in J} n_k(j)[a_j']$$

for all $k \in K$, there exists $\lambda \in \Lambda$, and $c_j \in (A_\lambda \otimes \mathcal{K})_\lambda$ for each j, such that $[\theta(a_j')] \ll [\theta_\lambda(c_j)] \ll [\theta(a_j)]$, and $[a_j'] \ll [c_j] \ll [a_j]$ in $\mathrm{Cu}(A)$, and

$$\sum_{j\in J} m_k(j)[c_j] \ll \sum_{j\in J} n_k(j)[c_j]$$

in $\mathrm{Cu}(A_\lambda)$.

Proof. Let $\varepsilon > 0$ be such that $[a_j'] \leq [(a_j - 2\varepsilon)_+]$ for each j. Note that, by definition, the A_λ's approximate A. Thus, it follows from (the proof of) [26, Proposition 3.7] that, for every sufficiently small positive $\sigma > 0$ with $\sigma < \varepsilon$, one can find $\lambda \in \Lambda$ and $b_j \in (A_\lambda \otimes \mathcal{K})_\lambda$ such that $[a_j'] \ll [(b_j - \varepsilon)_+] \ll [a_j]$ in $\mathrm{Cu}(A)$ and

$$\sum_{j\in J} m_k(j)[(b_j - \varepsilon)_+] \ll \sum_{j\in J} n_k(j)[(b_j - \varepsilon)_+]$$

in $\mathrm{Cu}(A_\lambda)$ for every k and, additionally, such that $\|\theta_\lambda(b_j) - \theta(a_j)\| \leq \sigma$ for every j.

Set $c_j := (b_j - \varepsilon)_+$. Then, since $\sigma < \varepsilon$, it follows that $\theta((a_j - 2\varepsilon)_+) \precsim \theta_\lambda(c_j) \precsim \theta(a_j)$. Using that θ is cpc order zero, we see that $\theta(a_j') \precsim \theta((a_j - 2\varepsilon)_+)$ and, consequently, $\theta(a_j') \precsim \theta_\lambda(c_j)$. $\square$

Corollary 4.9. *Let $\theta\colon A \to B$ be a cpc order-zero map, and let $(A_\lambda, \theta_\lambda)_{\lambda\in\Lambda}$ approximate θ. Assume that θ_λ is pure for each λ. Then, θ is pure.*

Proof. First, let $n \in \mathbb{N}$. Given $[a] \in \mathrm{Cu}(A)$ and $\varepsilon > 0$, use Proposition 4.8 (with $K = \emptyset$) to find $\lambda \in \Lambda$ and $c \in (A_\lambda \otimes \mathcal{K})_+$ such that $[\theta((a-\varepsilon)_+)] \ll [\theta_\lambda(c)] \ll [\theta(a)]$ and $[(a-\varepsilon)_+] \ll [c] \ll [a]$. Take $\delta > 0$ such that $[\theta((a-\varepsilon)_+)] \leq [\theta_\lambda((c-\delta)_+)]$. Then, since θ_λ is pure, there exists $[d] \in \mathrm{Cu}(B)$ such that $n[d] \leq [\theta_\lambda(c)]$ and $[\theta_\lambda((c-\delta)_+)] \leq (n+1)[d]$ in $\mathrm{Cu}(B)$. This implies

$$n[d] \leq [\theta_\lambda(c)] \leq [\theta(a)], \quad \text{and} \quad [\theta((a-\varepsilon)_+)] \leq [\theta_\lambda((c-\delta)_+)] \leq (n+1)[d],$$

which shows that θ is almost divisible.

Now assume that $[a_1], [a_2] \in \mathrm{Cu}(A)$ are such that $(m+1)[a_1] \leq m[a_2]$ for some $m \in \mathbb{N}$. Take any pair of elements a_1', a_1'' such that $[a_1'] \ll [a_1''] \ll [a_1]$, and find a_2' such that $[a_2'] \ll [a_2]$ and $(m+1)[a_1''] \ll m[a_2']$. Apply Proposition 4.8 for the pairs a_1', a_1'' and a_2', a_2 and the formula $(m+1)[a_1''] \ll m[a_2']$ to find $\lambda \in \Lambda$ and $c_1, c_2 \in (A_\lambda \otimes \mathcal{K})_+$ such that

$$[\theta(a_1')] \ll [\theta_\lambda(c_1)] \ll [\theta(a_1'')], \quad [\theta(a_2')] \ll [\theta_\lambda(c_2)] \ll [\theta(a_2)]$$

in $\mathrm{Cu}(A)$ and $(m+1)[c_1] \ll m[c_2]$ in $\mathrm{Cu}(A_\lambda)$.

Thus, since $\mathrm{Cu}(\theta_\lambda)$ is almost unperforated, $[\theta_\lambda(c_1)] \leq [\theta_\lambda(c_2)]$. This shows that $[\theta(a_1')] \leq [\theta(a_2)]$ and, since the choice of a_1' was arbitrary, we obtain $[\theta(a_1)] \leq [\theta(a_2)]$. □

5. Factorizing compositions of pure *-homomorphisms

The aim of this section is to provide a partial answer to Question 3.11. Namely, we show that —at the Cuntz semigroup level— the composition of any two pure *-homomorphisms (in fact, two cpc order-zero maps) factors through a pure Cu-semigroup; see Theorem 5.9. In order to prove such a result, we start with a study of pureness in the category Cu, which we use to prove the technical result Theorem 5.5.

Recall that the Cuntz semigroup of the Jiang-Su algebra $\mathcal{Z}$ is isomorphic to $Z = \mathbb{N} \cup (0, \infty]$ (see e.g. [2]). Throughout this subsection, we will write Z as the union $Z = Z_c \cup Z_{\mathrm{soft}}$, where $Z_c = \mathbb{N}$ and $Z_{\mathrm{soft}} = [0, \infty]$ with $Z_c \cap Z_{\mathrm{soft}} = \{0\}$. Denote by $\sigma\colon Z \to [0, \infty]$ the soft retraction, that is, the map that sends each compact element $n \in Z_c$ to its soft counterpart $n \in Z_{\mathrm{soft}} = [0, \infty]$ and leaves the soft part invariant.

Lemma 5.1. *Let S be a* Cu*-semigroup, and let $\gamma\colon Z \to S$ be a map such that $\gamma|_{Z_c}$ is an order-preserving monoid morphism. Then, γ is a generalized* Cu*-morphism if and only if the following two conditions are satisfied:*

(i) $\gamma(\sigma(1)) \leq \gamma(1) \leq \gamma(1+\varepsilon)$ *for every $\varepsilon > 0$; and*
(ii) $\gamma|_{Z_{\mathrm{soft}}}$ *is a generalized* Cu*-morphism.*

Proof. The forward implication is trivial. For the reverse implication, assume that $\gamma|_{Z_{\text{soft}}}$ is a generalized Cu-morphism and that $\gamma(\sigma(1)) \leq \gamma(1) \leq \gamma(1+\varepsilon)$ for each $\varepsilon > 0$. Note, in particular, that we have $\gamma(\sigma(n)) \leq \gamma(n) \leq \gamma(n+\varepsilon)$ for each $n \in Z_c$. To show that γ is order-preserving, take $n \in Z_c$ and $t \in Z_{\text{soft}}$ such that $t \leq n$. Then, one has $t \leq \sigma(n)$ and, consequently, $\gamma(t) \leq \gamma(\sigma(n)) \leq \gamma(n)$. Conversely, if $n < t$, we know that $\sigma(n) < t$ in Z_{soft}. Let $\varepsilon > 0$ be such that $\sigma(n) + \varepsilon < t$. Then,

$$\gamma(n) \leq \gamma(n+\varepsilon) = \gamma(\sigma(n)+\varepsilon) \leq \gamma(t).$$

To show that it preserves suprema, note that any increasing sequence in Z has a cofinal subsequence either in Z_c or Z_{soft}. Thus, we may assume that we are in one of these two cases. If the increasing sequence $(t_d)_d$ is in $Z_{\text{soft}} = [0, \infty]$, γ preserves its supremum by assumption. Else, if $(t_d)_d$ is in Z_c, it either stabilizes (in which case γ trivially preserves its supremum) or it tends to $\infty \in Z_{\text{soft}}$. In this situation, one can take the sequence $(\sigma(t_d))_d$ induced by the soft elements corresponding to our compact sequence. These two sequences share ∞ as their supremum. One has

$$\gamma(\sigma(t_d)) \leq \gamma(t_d) \leq \sup_d \gamma(t_d), \quad \text{and} \quad \gamma(t_d) \leq \gamma\left(t_d + \frac{1}{d}\right)$$

for every $d \geq 2$. This implies $\gamma(\infty) = \sup_d \gamma(\sigma(t_d)) \leq \sup_d \gamma(t_d)$ and $\sup_d \gamma(t_d) \leq \sup_d \gamma(t_d + \frac{1}{d}) = \gamma(\infty)$. This shows $\sup_d \gamma(t_d) = \gamma(\infty)$, as desired.

Finally, to see that the map is additive, take $n \in Z_c$ and $t \in Z_{\text{soft}}$. Then,

$$\gamma(n+t) = \gamma(\sigma(n)+t) = \gamma(\sigma(n)) + \gamma(t) \leq \gamma(n) + \gamma(t).$$

Conversely, if $t \neq 0$, let $\varepsilon > 0$ such that $t - \varepsilon > 0$. Then, $n + t = (\sigma(n)+\varepsilon) + (t-\varepsilon)$. This implies

$$\gamma(n) + \gamma(t-\varepsilon) \leq \gamma(\sigma(n)+\varepsilon) + \gamma(t-\varepsilon) = \gamma(n+t)$$

and, letting ε tend to 0, we obtain $\gamma(n) + \gamma(t) \leq \gamma(n+t)$, as required. □

Let S be a Cu-semigroup. The following notation is inspired by [2, Theorem 6.3.3]: For any pair $x' \leq x$ and any $k, n \in \mathbb{N}$, set

$$\mu((k,n), x', x) := \{y \in S \mid ny \leq kx, \text{ and } kx' \leq (n+1)y\}.$$

Note that this set is not empty whenever $x' \ll x$ and x is almost divisible. Further, one has that

$$\mu((k,n), x'', x) \subseteq \mu((k,n), x', x) \subseteq \mu((k,n), 0, x)$$

whenever $x' \leq x'' \leq x$, and that $\mu((k,n), 0, x) = \{y \in S \mid ny \leq kx\}$.

What follows is a generalization of [2, Theorem 6.3.3] to our setting. In our case, the proof becomes more technical (for example, we cannot use unicity arguments) and so we proceed with additional care. As another difference between the methods, we will need to use the extension result (Lemma 5.1) proved above.

Lemma 5.2. *Let $\varphi\colon S \to T$ be an almost unperforated generalized* Cu*-morphism. Let $x_1, x_2 \in S$ and $k_1, k_2, n_1, n_2 \in \mathbb{N}$ such that $k_1/n_1 < k_2/(n_2+1)$. Assume that $x_1 \leq x_2$. Then,*

(1) *$\varphi(y_1) \leq \varphi(y_2)$ for every $y_1 \in \mu((k_1,n_1),0,x_1)$ and $y_2 \in \mu((k_2,n_2),x_1,x_2)$.*
(2) *If φ is a* Cu*-morphism, then $\varphi(y_1) \ll \varphi(y_2)$ whenever $y_1 \in \mu((k_1,n_1),0,x_1)$, $y_2 \in \mu((k_2,n_2),x_2',x_2)$ and $x_1 \ll x_2' \ll x_2$.*

Proof. One has $n_1 y_1 \leq k_1 x_1$ and $k_2 x_1 \leq (n_2+1) y_2$. In particular,

$$n_1 k_2 y_1 \leq k_1 k_2 x_1 \leq (n_2+1) k_1 y_2.$$

It follows from almost unperforation of φ that $\varphi(y_1) \leq \varphi(y_2)$, which shows (1).

For (2), simply note that one gets

$$n_1 k_2 y_1 \leq k_1 k_2 x_1 \ll k_1 k_2 x_2' \leq (n_2+1) k_1 y_2.$$

Thus, we can find y_2' such that $y_2' \ll y_2$ and $n_1 k_2 y_1 \leq (n_2+1) k_1 y_2'$. This implies $\varphi(y_1) \leq \varphi(y_2') \ll \varphi(y_2)$, as required. □

Lemma 5.3. *Let $\varphi_1\colon S_1 \to S_2$ and $\varphi_2\colon S_2 \to T$ be generalized* Cu*-morphisms. Assume that φ_1 is almost divisible, and that φ_2 is almost unperforated. Then, for any $x \in S_1$ and $t \in (0,\infty]$, the set*

$$\Phi(t,\varphi_1(x)) := \left\{ \varphi_2(y) \mid y \in \mu((k,n),0,\varphi_1(x)) \text{ for some } k,n \in \mathbb{N} \text{ such that } \frac{k}{n} < t \right\}$$

has a supremum, bounded by $\lceil t \rceil \varphi_2\varphi_1(x)$ (here, $\lceil \infty \rceil := \infty$).

Proof. For every $d \in \mathbb{N}$, take $k_d, n_d \in \mathbb{N}$ and $x_d \in S_1$ such that

$$\frac{k_d}{n_d} < \frac{k_{d+1}}{n_{d+1}+1}, \quad \sup_d \left(\frac{k_d}{n_d}\right) = t, \quad x_d \ll x_{d+1}, \quad \text{and} \quad \sup_d x_d = x.$$

Set $x_0 = 0$. For each d, take $y_d \in \mu((k_d,n_d),\varphi_1(x_{d-1}),\varphi_1(x_d))$, which exists by almost divisibility of φ_1. By Lemma 5.2 (1), we see that the sequence $(\varphi_2(y_d))_d$ is increasing in T. Consider $z = \sup_d \varphi_2(y_d)$. We will prove that z is the supremum of $\Phi(t,\varphi_1(x))$.

Take $y \in \mu((k,n),0,\varphi_1(x))$ for some $k,n \in \mathbb{N}$ such that $k/n < t$. Take $y' \in S_2$ such that $y' \ll y$, and find $d \in \mathbb{N}$ such that

$$\frac{k}{n} < \frac{k_{d+1}}{n_{d+1}+1}, \quad \text{and} \quad y' \in \mu((k,n),0,\varphi_1(x_d)).$$

Since $y_{d+1} \in \mu((k_{d+1},n_{d+1}),\varphi_1(x_d),\varphi_1(x_{d+1}))$, it follows from Lemma 5.2 (1) that $\varphi_2(y') \le \varphi_2(y_{d+1}) \le z$. As this holds for every y' $\ll$-below y, we get $\varphi_2(y) \le z$. This shows that z is the supremum of $\Phi(t,\varphi_1(x))$, as desired.

To see that z is bounded by $\lceil t\rceil\varphi_2\varphi_1(x)$, simply note that for any pair k,n such that $k/n < t$ we have $k+1 \le \lceil t\rceil n$. Thus, one gets $(k+1)y \le \lceil t\rceil ny \le k(\lceil t\rceil x)$. Consequently, we obtain $\varphi_2(y) \le \lceil t\rceil\varphi_2\varphi_1(x)$, as desired. □

Proposition 5.4. *Let $\varphi_1\colon S_1 \to S_2$ and $\varphi_2\colon S_2 \to T$ be generalized* Cu*-morphisms. Assume that φ_1 is almost divisible, and that φ_2 is almost unperforated. Then, for any $x \in S_1$, there exists a generalized* Cu*-morphism $\alpha_x\colon Z \to T$ such that $\alpha_x(1) = \varphi_2\varphi_1(x)$.*

Proof. For every $n \in Z_c$, set $\alpha_x(n) := n\varphi_2\varphi_1(x)$. For each $t \in Z_{\rm soft}$, define

$$\alpha_x(t) := \sup\Phi(t,\varphi_1(x)),$$

which exists by Lemma 5.3.

Note that, for any $s \ge t$ in $Z_{\rm soft}$, one has $\Phi(t,\varphi_1(x)) \subseteq \Phi(s,\varphi_1(x))$. This implies $\alpha_x(t) \le \alpha_x(s)$. Further, it follows from Lemma 5.3 that $\alpha_x(\sigma(1)) \le \varphi_2\varphi_1(1) = \alpha_x(1)$. Additionally, for any $\varepsilon > 0$, take any $x' \in S_1$ such that $x' \ll x$ and let $n \in \mathbb{N}$ be such that $1 < (n+2)/n < 1+\varepsilon$. Using almost divisibility, there exists y in $\mu(((n+2),n),\varphi_1(x'),\varphi_1(x))$. One has $(n+2)\varphi_1(x') \le (n+1)y$. Using almost unperforation, $\varphi_2\varphi_1(x') \le \varphi_2(y) \le \alpha_x(1+\varepsilon)$ and, by taking suprema on x', we deduce $\alpha_x(1) = \varphi_2\varphi_1(x) \le \alpha_x(1+\varepsilon)$.

The arguments above show that $\alpha_x(\sigma(1)) \le \alpha_x(1) \le \alpha_x(1+\varepsilon)$ for every $\varepsilon > 0$. Further, note that $\alpha_x|_{Z_c}$ is trivially an order-preserving monoid morphism. We will now prove that $\alpha_x|_{Z_{\rm soft}}$ is a generalized Cu-morphism. Lemma 5.1 will then imply that α_x is a generalized Cu-morphism. We have already shown that $\alpha_x|_{Z_{\rm soft}}$ is order-preserving, so it suffices to prove that the map preserves suprema and addition.

To show that it preserves suprema, note that any increasing sequence $(t_d)_d$ in $Z_{\rm soft}$ satisfies $\cup_d\Phi(t_d,\varphi_1(x)) = \Phi(\sup_d t_d,\varphi_1(x))$. This proves that α_x preserves suprema in $Z_{\rm soft}$.

To see that the map is additive, let $t_1,t_2 \in Z_{\rm soft} = [0,\infty]$. First, for each $i=1,2$, let $y_i \in \mu((k_i,n_i),0,\varphi_1(x))$ for some $k_i,n_i \in \mathbb{N}$ such that

$$\frac{k_i}{n_i} < t_i.$$

Take $y_i' \in S_2$ such that $y_i' \ll y_i$ for $i=1,2$, and let $x_0 \in S_1$ be such that

$$x_0 \ll x, \quad \text{and} \quad y_i' \in \mu((k_i,n_i),0,\varphi_1(x_0)).$$

Choose $k, n \in \mathbb{N}$ such that

$$\frac{k_1}{n_1} + \frac{k_2}{n_2} = \frac{k_1 n_2 + k_2 n_1}{n_1 n_2} < \frac{k}{n+1}, \quad \text{and} \quad \frac{k}{n} < t_1 + t_2.$$

Using that φ_1 is almost divisible, find $y \in \mu((k,n), \varphi_1(x_0), \varphi_1(x))$. Note that $y_1' + y_2' \in \mu((k_1 n_2 + k_2 n_1, n_1 n_2), 0, \varphi_1(x_0))$. By Lemma 5.2 (1), one gets $\varphi_2(y_1' + y_2') \leq \varphi_2(y)$. Thus, we get

$$\varphi_2(y_1') + \varphi_2(y_2') \leq \varphi_2(y) \leq \alpha_x(t_1 + t_2)$$

and, since this holds for every choice of y_1', y_2', one obtains $\varphi_2(y_1) + \varphi_2(y_2) \leq \alpha_x(t_1 + t_2)$. Taking suprema now on y_1, y_2, we have $\alpha_x(t_1) + \alpha_x(t_2) \leq \alpha_x(t_1 + t_2)$.

Conversely, take $y \in \mu((k,n), 0, \varphi_1(x))$ for k, n with $k/n < t_1 + t_2$ and $x' \ll x$. Find $k_i, t_i \in \mathbb{N}$ such that $\frac{k_i}{n_i} < t_i$ and

$$\frac{k}{n} < \frac{k_1}{n_1 + 1} + \frac{k_2}{n_2 + 1} = \frac{k_1(n_2+1) + k_2(n_1+1)}{(n_1+1)(n_2+1)}.$$

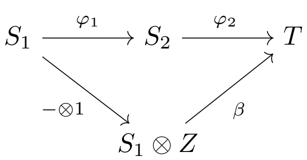
Proceeding as before, take $y' \ll y$ and find x' such that $x' \ll x$ and $y' \in \mu((k,n), 0, \varphi_1(x'))$. Find $y_i \in \mu((k_i, n_i), \varphi_1(x'), \varphi_1(x))$. In particular, we have

$$\begin{aligned}(k_1(n_2+1) + k_2(n_1+1))ny' &\leq (k_1(n_2+1) + k_2(n_1+1))k\varphi_1(x') \\ &\leq (n_1+1)(n_2+1)k(y_1 + y_2).\end{aligned}$$

Since φ_2 is almost unperforated, we obtain $\varphi_2(y') \leq \varphi_2(y_1) + \varphi_2(y_2) \leq \alpha_x(t_1) + \alpha_x(t_2)$. Taking suprema on y', and then on y, we deduce $\alpha_x(t_1 + t_2) \leq \alpha_x(t_1) + \alpha_x(t_2)$, as desired. This proves that α_x is additive in Z_{soft}.

Lemma 5.1 shows that $\alpha_x \colon Z \to T$ is a generalized Cu-morphism. $\square$

Theorem 5.5. *Let $\varphi_1 \colon S_1 \to S_2$ and $\varphi_2 \colon S_2 \to T$ be (generalized)* Cu*-morphisms. Assume that φ_1 is almost divisible, and that φ_2 is almost unperforated. Then, there exists a (generalized)* Cu*-morphism $\beta \colon S_1 \otimes Z \to T$ such that the following diagram commutes*



Proof. We define the map $\alpha \colon S_1 \times Z \to T$ by

$$\alpha(x,t) := \alpha_x(t),$$

which satisfies $\alpha(x, 1) = \varphi_2 \varphi_1(x)$.

We will now prove that α is a generalized Cu-bimorphism. Using [2, Theorem 6.3.3 (1)], this will imply the existence of $\beta\colon S\otimes Z\to T$ with the required properties.

Note that $\alpha(x,\cdot)=\alpha_x$ is a generalized Cu-morphism by the results above. Further, $\alpha(\cdot,n)$ is trivially a generalized Cu-morphism for every $n\in Z_c$. Set $\gamma_t(x):=\alpha(x,t)$ for every $t\in Z_{\rm soft}$, and let us show that γ_t is a generalized Cu-morphism.

To see that γ_t preserves order, take $x_0,x\in S_1$ such that $x_0\leq x$. Clearly, one has

$$\Phi(t,\varphi_1(x_0))\subseteq\Phi(t,\varphi_1(x))$$

and thus $\gamma_t(x_0)=\alpha_{x_0}(t)\leq\alpha_x(t)=\gamma_t(x)$.

Now take $(x_d)_d$ in S_1 increasing with supremum x. Then, for any element $y\in\mu((k,n),0,\varphi_1(x))$ with $k/n<t$, take $y'\ll y$ and find $d\in\mathbb{N}$ such that $y'\in\mu((k,n),0,\varphi_1(x_d))$. This shows that $\varphi_2(y')\leq\gamma_t(x_d)$ and, consequently, $\varphi_2(y)\leq\sup_d\gamma_t(x_d)$. Taking suprema, one gets $\gamma_t(x)\leq\sup_d\gamma_t(x_d)$. Since γ_t is order-preserving, we also obtain $\sup_d\gamma_t(x_d)\leq\gamma_t(x)$. In other words, γ_t preserves suprema of increasing sequences.

To prove that γ_t is superadditive, take $x_1,x_2\in S_1$ and let $y_1,y_2\in S_2$ be such that $y_i\in\mu((k_i,n_i),0,\varphi_1(x_i))$ for some k_i,n_i's such that $k_i/n_i<t$. Find $k,n\in\mathbb{N}$ such that $k_i/n_i<k/(n+1)$ and $k/n<t$. Take $y_i'\in S_2$ such that $y_i'\ll y_i$, and let $x_i'\in S_1$ be such that $x_i'\ll x_i$ and $y_i'\in\mu((k_i,n_i),0,\varphi_1(x_i'))$. Using almost divisibility of φ_1, find $z_i\in\mu((k,n),\varphi_1(x_i'),\varphi_1(x_i))$. By Lemma 5.2 (1), we get $\varphi_2(y_i')\leq\varphi_2(z_i)$. Note that we have $z_1+z_2\in\mu((k,n),0,\varphi_1(x_1+x_2))$. Thus, one gets

$$\varphi_2(y_1')+\varphi_2(y_2')\leq\varphi_2(z_1)+\varphi_2(z_2)=\varphi_2(z_1+z_2)\leq\alpha_{x_1+x_2}(t)=\gamma_t(x_1+x_2).$$

Taking suprema on y_1' and y_2', this implies $\varphi_2(y_1)+\varphi_2(y_2)\leq\gamma_t(x_1+x_2)$. Taking now suprema on y_1,y_2,k and n gives $\gamma_t(x_1)+\gamma_t(x_2)\leq\gamma_t(x_1+x_2)$.

Conversely, to prove subadditivity, let $y\in\mu((k,n),0,\varphi_1(x_1+x_2))$. Take $y'\ll y$ and let $x_i'\ll x_i$ be such that $y'\in\mu((k,n),0,\varphi_1(x_1'+x_2'))$. Find $l,m\in\mathbb{N}$ such that $k/n<l/(m+1)$ and $l/m<t$. Find $y_i\in\mu((l,m),\varphi_1(x_i'),\varphi_1(x_i))$. Then, $y_1+y_2\in\mu((l,m),\varphi_1(x_1'+x_2'),\varphi_1(x_1+x_2))$. By Lemma 5.2, one obtains $\varphi_2(y')\leq\varphi_2(y_1)+\varphi_2(y_2)$. Again, this implies $\varphi_2(y)\leq\gamma_t(x_1)+\gamma_t(x_2)$ and, consequently, $\gamma_t(x_1+x_2)\leq\gamma_t(x_1)+\gamma_t(x_2)$.

We have shown that each coordinate of α is a generalized Cu-morphism. In particular, we know that there exists a generalized Cu-morphism $\beta\colon S_1\otimes Z\to T$ with the desired properties; see [2, Lemma 6.3.2, Theorem 6.3.3].

Now assume that φ_1 and φ_2 are Cu-morphisms. To prove that α is in fact a Cu-bimorphism, take $t',t\in Z$ and $x',x\in S$ such that $t'\ll t$ and $x'\ll x$. We have to show that $\alpha(x',t')\ll\alpha(x,t)$. If t' or t are in $\mathbb{N}$, we may assume $t=t'$. In this case, one has $\alpha(x',t)=t\varphi_2\varphi_1(x')\ll t\varphi_2\varphi_1(x)=\alpha(x,t)$ because both φ_1 and φ_2 are Cu-morphisms. Finally, assume $t',t\in(0,\infty]$. Take $x_1,x_2\in S_1$ such that $x'\ll x_1\ll x_2\ll x$.

Find $l_1, l_2, m_1, m_2 \in \mathbb{N}$ such that

$$t' < \frac{l_1}{m_1+1}, \quad \frac{l_1}{m_1} < \frac{l_2}{m_2+1}, \quad \frac{l_2}{m_2} < t.$$

By almost divisibility of φ_1, there exist elements $y_1, y_2 \in S_2$ such that $y_1 \in \mu((l_1, m_1), \varphi_1(x'), \varphi_1(x_1))$ and $y_2 \in \mu((l_2, m_2), \varphi_1(x_2), \varphi_1(x))$. By Lemma 5.2 (2), one gets $\varphi_2(y_1) \ll \varphi_2(y_2) \leq \alpha(x, t)$.

Now note that for every $y \in \mu((k, n), 0, \varphi_1(x'))$ with $k/n < t'$, one has $k/n < l_1/(m_1 + 1)$. Thus, another application of Lemma 5.2 (1) gives $\varphi_2(y) \leq \varphi_2(y_1)$. In other words, $\alpha(x', t') \leq \varphi_2(y_1)$. Since we already know that $\varphi_2(y_1) \ll \alpha(x, t)$, one gets $\alpha(x', t') \ll \alpha(x, t)$, as desired.

Now [2, Theorem 6.3.3] shows that $\beta\colon S_1 \otimes Z \to T$ is a Cu-morphism with the desired properties. □

Corollary 5.6. *Let $\varphi\colon S \to T$ be a* Cu*-morphism. Then,*

(i) *if S is almost divisible and φ is almost unperforated, φ factorizes through $S \otimes Z$.*
(ii) *if T is almost unperforated and φ is almost divisible, φ factorizes through $S \otimes Z$.*

Proof. For (i), consider the composition of maps $S \to S \to T$ and apply Theorem 5.5. For (ii), consider $S \to T \to T$ and apply Theorem 5.5. □

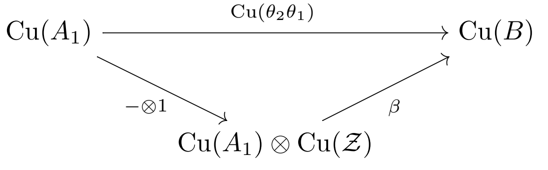
5.7. The proof of Theorem 5.5 requires the order of the conditions to be as stated (first almost divisibility and then almost unperforation). It is unclear to us if the same statement holds when the conditions are reversed.

The intuitive reason behind this is that one always wants to compare the divisors of two elements from S_1, which cannot be done if the conditions on the morphisms are exchanged.

Question 5.8. Can the roles of almost divisibility and almost unperforation be reversed in Theorem 5.5?

Theorem 5.5 above provides a partial answer to Question 3.11:

Theorem 5.9. *Let $\theta_1\colon A_1 \to A_2$ and $\theta_2\colon A_2 \to B$ be pure *-homomorphisms. Then, there exists a* Cu*-morphism β such that the following diagram commutes*



One also obtains the following proposition, which answers Question 3.11 completely when the initial domain has some extra assumptions and the codomain is of stable rank one. We believe that this result may be far more general, but new or tinkered techniques need to be developed in order to do so; see Question 5.11.

Proposition 5.10. *Let A_1 be a* C*-*algebra, and let B be a unital* C*-*algebra of stable rank one. Let $\theta_1\colon A_1 \to A_2$ and $\theta_2\colon A_2 \to B$ be unital, pure* *-*homomorphisms. Assume that A_1 satisfies both of the following conditions:*

(i) $\mathrm{Cu}(A_1 \otimes \mathcal{Z}) \cong \mathrm{Cu}(A_1) \otimes \mathrm{Cu}(\mathcal{Z})$*;*
(ii) *$A_1 \otimes \mathcal{Z}$ is an inductive limit of 1-dimensional NCCW complexes with trivial K_1-group.*

Then, $\theta_2\theta_1$ factors, up to approximate unitary equivalence, through $A_1 \otimes \mathcal{Z}$.
In particular, this holds if A_1 is a unital AF-algebra.

Proof. The induced composition of Cu-morphisms factorizes through $\mathrm{Cu}(A_1) \otimes \mathrm{Cu}(\mathcal{Z})$ by Theorem 5.5. By assumption, we have $\mathrm{Cu}(A_1) \otimes \mathrm{Cu}(\mathcal{Z}) \cong \mathrm{Cu}(A_1 \otimes \mathcal{Z})$.

Since the C*-algebra $A_1 \otimes \mathcal{Z}$ is an inductive limit of 1-dimensional NCCW complexes with trivial K_1-group, the result now follows from [19, Theorem 1.0.1].

We note that, if A_1 is a unital AF-algebra, [2, Proposition 6.4.13] implies that $\mathrm{Cu}(A_1) \otimes \mathrm{Cu}(\mathcal{Z}) \cong \mathrm{Cu}(A_1 \otimes \mathcal{Z})$. Further, it is readily checked that condition (ii) also holds. □

Note that there are two obstructions to generalizing Proposition 5.10. First, the Cuntz semigroup tensor product does not generally behave well with its C^*-algebraic counterpart. For example, it is known that $\mathrm{Cu}(C[0,1]) \otimes \mathrm{Cu}(\mathcal{Z}) \not\cong \mathrm{Cu}(C[0,1] \otimes \mathcal{Z})$; see [2, Proposition 6.4.4]. Secondly, the only currently available result for lifting *-homomorphisms is Robert's [19, Theorem 1.0.1]. However, it is conceivable that such result can be generalized whenever A, B are sufficiently noncommutative (say, if A, B are simple) and $\mathrm{Cu}(\theta)$ maps every Cuntz class to a strongly soft class. The following question is related to the first of the two obstructions.

Question 5.11. Let A, B be C*-algebras, and let $\psi\colon \mathrm{Cu}(A) \otimes \mathrm{Cu}(\mathcal{Z}) \to \mathrm{Cu}(B)$ be a Cu-morphism. When does there exist a Cu-morphism $\rho\colon \mathrm{Cu}(A \otimes \mathcal{Z}) \to \mathrm{Cu}(B)$ such that $\psi([a] \otimes 1) = \rho([a \otimes 1])$?

As shown by Winter in [31, Corollary 7.4], a separable, unital, simple, non-elementary C*-algebra of locally finite nuclear dimension is pure if and only if it is $\mathcal{Z}$-stable. In analogy to this result, one may ask:

Question 5.12. Let A, B be C*-algebras. Under which conditions on A and B does every pure *-homomorphism $\theta\colon A \to B$ factor (in a suitable sense) through a $\mathcal{Z}$-stable C*-algebra?

Restricting the codomain in Proposition 5.10 further, we obtain a first answer to Question 5.12.

Corollary 5.13. *Let A be a unital AF-algebra, and let B be a unital* C*-*algebra of stable rank one and with strict comparison. Let $\theta\colon A \to B$ be a unital, pure* *-*homomorphism. Then, θ factors up to approximate unitary equivalence through $A \otimes \mathcal{Z}$.*

Remark 5.14. Following the ideas from Definition 3.2, one could also define other Cu-like notions for morphisms, such as algebraicity and (weak) $(2, \omega)$-divisibility.

6. Soft and rational *-homomorphisms

In this last section we exploit Theorem 5.9 in two cases of interest: Pure maps with a soft image (Definition 6.6), and rational maps (Definition 6.1). These notions are meant to generalize tensorial absorption, at a Cuntz semigroup level, of the Jacelon-Razak algebra and UHF-algebras respectively. In contrast to Theorem 5.9, Cu-tensor products and C^*-tensor products of such algebras do behave nicely. This allows us to show that a composition of maps always factors (at the level of Cu) through $A \otimes M_q$ and $A \otimes \mathcal{W}$ respectively; see Theorems 6.3 and 6.10.

6.1. q-rational morphisms

Given a supernatural number q such that $q = q^2$ and $q \neq 1$, let M_q denote the UHF-algebra associated to q. As shown in [2, Section 7.4], $\mathrm{Cu}(M_q) \cong K_q \sqcup (0, \infty]$ where K_q is the subset of $\mathbb{Q}_+$ formed by the elements of the form $\frac{k}{n}$ with k, n coprime and n a divisor of q.

Adapting [2, Definition 7.4.6] to our setting, we define:

Definition 6.1. Let $\varphi\colon S \to T$ be a generalized Cu-morphism, and let q be a supernatural number as above. We will say that φ is *q-rational* if it is both

(i) *q-divisible*, that is, if for every $x \in S$ and every finite divisor n of q there exists $y \in T$ such that $\varphi(x) = ny$.
(ii) *q-unperforated*, that is, if whenever $nx \leq ny$, for some finite divisor n of q, one has $\varphi(x) \leq \varphi(y)$.

As shown in [2, Theorem 7.4.10], a Cu-semigroup S tensorially absorbs $\mathrm{Cu}(M_q)$ if and only if S is q-divisible and q-unperforated. Thus, examples of morphisms satisfying the two conditions above include all morphisms whose domain or codomain absorbs $\mathrm{Cu}(M_q)$ tensorially.

Proposition 6.2. *Let q be a supernatural number such that $q = q^2$ and $q \neq 1$. Let $\varphi_1 \colon S_1 \to S_2$ be a q-divisible (generalized)* Cu*-morphism and let $\varphi_2 \colon S_2 \to T$ be a q-unperforated (generalized)* Cu*-morphism. Then, there exists a (generalized)* Cu*-morphism $\gamma \colon S_1 \otimes \mathrm{Cu}(M_q) \to T$ such that $\gamma(x \otimes 1) = \varphi_2\varphi_1(x)$.*

Proof. We mimic the approach of [2, Theorem 7.4.10].

First, note that for every $x \in S_1$ and every n divisor of q there exists a unique element $\omega_n(x) \in \varphi_2(S_2)$ such that $\omega_n(x) = \varphi_2(z)$ where z is such that $\varphi_1(x) = nz$.

Indeed, existence follows from q-divisibility of φ_1, while uniqueness is given by φ_2. Thus, we can define the map $\omega_n \colon S_1 \to T$ as the assignment $x \mapsto \omega_n(x)$. It is readily checked that ω_n is a (generalized) Cu-morphism whenever φ_1 and φ_2 are.

Further, it follows from Theorem 5.5 that there exists a (generalized) Cu-bimorphism $\alpha \colon S_1 \times Z \to T$ such that $\alpha(x, 1) = \varphi_2\varphi_1(x)$. Now, define a (generalized) Cu-bimorphism $\alpha_q \colon S_1 \times (K_q \sqcup (0, \infty]) \to T$ as follows: Given $t \in (0, \infty]$, simply set $\alpha_q(x, t) := \alpha(x, t)$. Else, if $t = \frac{k}{n}$ for some (unique) coprime pair k, n with n a divisor of q, set $\alpha_q(x, t) := k\omega_n(x)$. By construction, $\alpha_q(\cdot, t)$ is a generalized Cu-morphism.

A proof analoguous to that of Lemma 5.1 shows that $\alpha_q(x, \cdot)$ is a generalized Cu-morphism if and only if $\alpha_q(x, \cdot)|_{(0,\infty]}$ is a generalized Cu-morphism and $\alpha_q(x, \sigma(\frac{1}{n})) \leq \alpha_q(x, \frac{1}{n}) \leq \alpha_q(x, \sigma(\frac{1}{n}) + \varepsilon)$ for every $\varepsilon > 0$ and any n dividing q. Note that the first condition is satisfied by construction of α, while the second condition is readily checked after a careful examination of α_q. Thus, [2, Theorem 6.3.3 (1)] implies the existence of the map γ with the required properties. □

We know from [2, Proposition 7.6.3] that $\mathrm{Cu}(A \otimes M_q) \cong \mathrm{Cu}(A) \otimes \mathrm{Cu}(M_q)$ always. Thus, one gets the following result, where *-homomorphism and Cu-morphism can be changed to cpc order-zero map and generalized Cu-morphism respectively.

Theorem 6.3. *Let $\theta_1 \colon A_1 \to A_2$ and $\theta_2 \colon A_2 \to B$ be *-homomorphisms. Assume that θ_1 is q-divisible and that θ_2 is q-unperforated. Then, there exists a* Cu*-morphism $\beta \colon \mathrm{Cu}(A \otimes M_q) \to \mathrm{Cu}(B)$ such that $\mathrm{Cu}(\theta_2\theta_1)[a] = \beta([a \otimes 1])$ for each $[a] \in \mathrm{Cu}(A)$.*

Applying Robert's classification result ([19, Theorem 1.0.1]), one obtains:

Corollary 6.4. *Retain the above assumptions. Assume that A_1 is a unital* C*-algebra stably isomorphic to an inductive limit of* 1*-dimensional NCCW-complexes with trivial K_1-group, that B is unital and of stable rank one, and that θ_1 and θ_2 are unital. Then, $\theta_2\theta_1$ factors, up to approximate unitary equivalence, through $A \otimes M_q$.*

6.2. Soft maps

Recall that the Cuntz semigroup of the Jacelon-Razak algebra $\mathcal{W}$ is isomorphic to the monoid $[0, \infty]$. As shown in [2, Theorem 7.5.4], every element in a pure Cu-semigroup S is strongly soft (see Paragraph 6.5 below) if and only if $S \cong S \otimes \mathrm{Cu}(\mathcal{W})$. We show the analogue of Theorem 5.5 in Theorem 6.9.

6.5 *(Soft elements and strongly soft Cuntz classes).* Let S be a Cu-semigroup, and let $x \in S$. Recall from [28, Definition 4.3, Proposition 4.14] that x is *strongly soft* if, for any $x' \in S$ such that $x' \ll x$, there exists $t \in S$ such that $x' + t \leq x \leq \infty t$. Making an abuse of notation, the set of strongly soft elements is usually denoted by S_{soft}.

Let A be a stable C*-algebra. Under the presence of sufficient non-commutativity on A (for example, if A has the Global Glimm Property), a positive element $a \in A_+$ has a strongly soft Cuntz class if and only if a is Cuntz equivalent to a *soft element*, that is, an element $b \in A_+$ such that no nontrivial quotient of $\overline{bAb}$ is unital; this equivalence is proved in [4, Corollary 3.4].

Definition 6.6. Let $\varphi\colon S \to T$ be a generalized Cu-morphism. We will say that φ is *soft* if $\varphi(S) \subseteq T_{\mathrm{soft}}$.

Further, φ will be said to have $\mathrm{Cu}(\mathcal{W})$-*multiplication* if it is soft and has $\mathrm{Cu}(\mathcal{Z})$-multiplication.

Remark 6.7.

(1) Let $\theta\colon A \to B$ be a *-homomorphism between stable C*-algebras, and assume that B satisfies the Global Glimm Property. Then, $\mathrm{Cu}(\theta)$ has $\mathrm{Cu}(\mathcal{W})$-multiplication if and only if θ is pure and $\theta(a)$ is Cuntz equivalent to a soft element for every $a \in A_+$.
(2) Note the analogue statements of Lemma 4.3 and Propositions 4.1, 4.2 and 4.6 also work with $\mathrm{Cu}(\mathcal{W})$-multiplication instead of $\mathrm{Cu}(\mathcal{Z})$-multiplication.

Example 6.8. By (the analogue of) Proposition 4.2, any *-homomorphism $A \to B$ that factorizes through $A \otimes \mathcal{W}$ induces a Cu-morphism with $\mathrm{Cu}(\mathcal{W})$-multiplication.

As noted in [14, Remark 3.21], the infinite repeat $\phi \otimes 1_{\mathcal{M}(\mathcal{K})}\colon A \to \mathcal{M}(B \otimes \mathcal{K})$ of any *-homomorphism of the form $A \to \mathcal{M}(B)$ factorizes through $\mathcal{M}(B) \otimes \mathcal{O}_2$.

Since every $\mathcal{O}_2$-stable C*-algebra is purely infinite by [17, Theorem 5.11], its Cuntz semigroup has $\{0, \infty\}$-multiplication and, also, $[0, \infty]$-multiplication. Thus, it follows that $\mathrm{Cu}(\phi \otimes 1_{\mathcal{M}(\mathcal{K})})$ always has $\mathrm{Cu}(\mathcal{W})$-multiplication regardless of our choice of A and B.

Theorem 6.9. *Let $\varphi_1\colon S_1 \to S_2$ and $\varphi_2\colon S_2 \to T$ be (generalized)* Cu*-morphisms. Assume that φ_1 is soft and almost divisible, and that φ_2 is almost unperforated. Then, there exists a (generalized)* Cu*-morphism $\gamma\colon S_1 \otimes [0, \infty] \to T$ such that $\gamma(x \otimes 1) = \varphi_2\varphi_1(x)$.*

Proof. Let $\beta\colon S_1 \otimes Z \to T$ be the map constructed in Theorem 5.5, and denote by γ the restriction of β to $(S \otimes Z)_{\rm soft} \cong S \otimes [0, \infty]$.

Let $x \in S_1$ and take $x' \in S_1$ such that $x' \ll x$. Then, since $\varphi_1(x)$ is soft, there exists $n \in \mathbb{N}$ such that $(n+1)\varphi_1(x') \leq n\varphi_1(x)$; see, for example, [28, Proposition 4.6]. This implies that $\varphi_1(x') \in \mu((n, n+1), 0, \varphi_1(x))$ and, in particular, that $\varphi_2\varphi_1(x') \in \Phi(1, \varphi_1(x))$. Thus, one has $\varphi_2\varphi_1(x') \leq \gamma(x \otimes 1)$ for every x'. Consequently, $\varphi_2\varphi_1(x) \leq \gamma(x \otimes 1)$. Further, note that one gets

$$\gamma(x \otimes 1) = \beta(x \otimes 1_{Z_{\rm soft}}) \leq \beta(x \otimes 1_Z) = \varphi_2\varphi_1(x),$$

that is, $\gamma(x \otimes 1) = \varphi_2\varphi_1(x)$, as desired. $\square$

Combining [20, Theorem 5.1.2] and [2, Proposition 7.6.3], one has that $\mathrm{Cu}(A \otimes \mathcal{W}) \cong \mathrm{Cu}(A) \otimes [0, \infty]$; hence, we obtain the following result.

Theorem 6.10. *Let $\theta_1\colon A_1 \to A_2$ be a soft and pure *-homomorphism, and let $\theta_2\colon A_2 \to B$ be a pure *-homomorphism. Then, there exists a* Cu*-morphism $\beta\colon \mathrm{Cu}(A \otimes \mathcal{W}) \to \mathrm{Cu}(B)$ such that $\mathrm{Cu}(\theta_2\theta_1)[a] = \beta([a \otimes 1])$ for each $[a] \in \mathrm{Cu}(A)$.*

Data availability

No data was used for the research described in the article.

References

[1] R. Antoine, F. Perera, L. Robert, H. Thiel, C*-algebras of stable rank one and their Cuntz semigroups, Duke Math. J. 171 (2022) 33–99.
[2] R. Antoine, F. Perera, H. Thiel, Tensor products and regularity properties of Cuntz semigroups, Mem. Am. Math. Soc. 251 (2018), viii+191.
[3] R. Antoine, F. Perera, H. Thiel, E. Vilalta, Pure C*-algebras, arXiv:2406.11052 [math.OA], 2024.
[4] A. Asadi-Vasfi, H. Thiel, E. Vilalta, Ranks of soft operators in nowhere scattered C*-algebras, J. Inst. Math. Jussieu (2024), in press.
[5] D. Avitzour, Free products of C*-algebras, Trans. Am. Math. Soc. 271 (1982) 423–435.
[6] J. Bosa, N. Brown, Y. Sato, A. Tikuisis, S. White, W. Winter, Covering dimension of C*-algebras and 2-coloured classification, Mem. Am. Math. Soc. 257 (2019) 1223.
[7] J. Bosa, J. Gabe, A. Sims, S. White, The nuclear dimension of $\mathcal{O}_\infty$-stable C*-algebras, Adv. Math. 401 (2022) 108250.
[8] J.R. Carrión, J. Gabe, C. Schafhauser, A. Tikuisis, S. White, Classifying *-homomorphisms I: unital simple nuclear C*-algebras, preprint, arXiv:2307.06480 [math.OA], 2023.
[9] J. Castillejos, S. Evington, A. Tikuisis, S. White, W. Winter, Nuclear dimension of simple C*-algebras, Invent. Math. 224 (2021) 245–290.
[10] J. Castillejos, R. Neagu, On topologically zero-dimensional morphisms, J. Funct. Anal. 286 (2024) 110368.
[11] K.T. Coward, G.A. Elliott, C. Ivanescu, The Cuntz semigroup as an invariant for C*-algebras, J. Reine Angew. Math. 623 (2008) 161–193.
[12] K. Dykema, U. Haagerup, M. Rørdam, The stable rank of some free product C*-algebras, Duke Math. J. 90 (1997) 95–121.
[13] G.A. Elliott, G. Gong, H. Lin, Z. Niu, On the classification of simple amenable C*-algebras with finite decomposition rank, II, J. Noncommut. Geom. (2024), https://doi.org/10.4171/JNCG/560, published online first.

[14] J. Gabe, A new proof of Kirchberg's $\mathcal{O}_2$-stable classification, J. Reine Angew. Math. 2020 (2020) 247–289.
[15] J. Gabe, R. Neagu, Inclusions of real rank zero, preprint, arXiv:2312.03622 [math.OA], 2023.
[16] E. Kirchberg, Central sequences in C*-algebras and strongly purely infinite algebras, in: Operator Algebras: The Abel Symposium 2004, in: Abel Symp., vol. 1, Springer, Berlin, 2006, pp. 175–231.
[17] E. Kirchberg, M. Rørdam, Non-simple purely infinite C*-algebras, Am. J. Math. 122 (2000) 637–666.
[18] E. Ortega, F. Perera, M. Rørdam, The corona factorization property, stability, and the Cuntz semigroup of a C*-algebra, Int. Math. Res. Not. (2012) 34–66.
[19] L. Robert, Classification of inductive limits of 1-dimensional NCCW complexes, Adv. Math. 231 (2012) 2802–2836.
[20] L. Robert, The cone of functionals on the Cuntz semigroup, Math. Scand. 113 (2013) 161–186.
[21] L. Robert, M. Rørdam, Divisibility properties for C*-algebras, Proc. Lond. Math. Soc. (3) 106 (2013) 1330–1370.
[22] L. Robert, A. Tikuisis, Nuclear dimension and $\mathcal{Z}$-stability of non-simple C*-algebras, Trans. Am. Math. Soc. 369 (2017) 4631–4670.
[23] M. Rørdam, On the structure of simple C*-algebras tensored with a UHF-algebra. II, J. Funct. Anal. 107 (1992) 255–269.
[24] M. Rørdam, The stable and the real rank of $\mathcal{Z}$-absorbing C*-algebras, Int. J. Math. 15 (2004) 1065–1084.
[25] H. Thiel, Ranks of operators in simple C*-algebras with stable rank one, Commun. Math. Phys. 377 (2020) 37–76.
[26] H. Thiel, E. Vilalta, Covering dimension of Cuntz semigroups II, Int. J. Math. 32 (2021) 2150100.
[27] H. Thiel, E. Vilalta, Nowhere scattered C*-algebras, J. Noncommut. Geom. 18 (2024) 231–263.
[28] H. Thiel, E. Vilalta, Soft operators in C*-algebras, J. Math. Anal. Appl. 536 (2024) 128167.
[29] A. Tikuisis, S. White, W. Winter, Quasidiagonality of nuclear C*-algebras, Ann. Math. (2) 185 (2017) 229–284.
[30] E. Vilalta, Nowhere scattered multiplier algebras, Proc. R. Soc. Edinb., Sect. A (First view) (2023), https://doi.org/10.1017/prm.2023.123.
[31] W. Winter, Nuclear dimension and $\mathcal{Z}$-stability of pure C*-algebras, Invent. Math. 187 (2012) 259–342.
[32] W. Winter, J. Zacharias, Completely positive maps of order zero, Münster J. Math. 2 (2009) 311–324.