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The Journal of Geometric Analysis (2025) 35:25
https://doi.org/10.1007/s12220-024-01847-8

Poincaré-Lelong Type Formulas and Segre Numbers

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Received: 21 May 2024 / Accepted: 27 October 2024

Abstract

Let E and F be Hermitian vector bundles over a complex manifold X and let $g\colon E \to F$ be a holomorphic morphism. We prove a Poincaré-Lelong type formula with a residue term M^g. The currents M^g so obtained have an expected functorial property. We discuss various applications: If F has a trivial holomorphic subbundle of rank r outside the analytic set Z, then we get currents with support on Z that represent the Bott-Chern classes $\hat{c}_k(E)$ for $k >$ rank $E - r$. We also consider Segre and Chern forms associated with certain singular metrics on E. The multiplicities (Lelong numbers) of the various components of M^g only depend on the cokernel of the adjoint sheaf morphism g^*. This leads to a notion of distinguished varieties and Segre numbers of an arbitrary coherent sheaf, generalizing these notions, in particular the Hilbert-Samuel multiplicity, in case of an ideal sheaf.

Keywords Poincaré–Lelong formula · Segre numbers · Segre form · Chern form

Mathematics Subject Classification 32C30 · 32L10 · 32S05 · 32S20

1 Introduction

Let g be a non-trivial holomorphic (or meromorphic) section of a Hermitian line bundle $L \to X$, X a complex manifold of dimension n, and let $[\mathrm{div} g]$ be the current of integration associated with the divisor defined by g. The Poincaré-Lelong formula states that

$$dd^c \log |g|^2 = [\mathrm{div} g] - c_1(L), \tag{1.1}$$

The author was partially supported by the Swedish Natural Science Research Council.

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Published online: 22 November 2024

where $c_1(L)$ is the first Chern form associated with the Chern connection on L, i.e., $c_1(L) = (i/2\pi)\Theta_L$, where Θ_L is the curvature form. Here and throughout this paper $d^c = (i/2\pi)(\bar\partial - \partial)$; the constant varies in the literature and is chosen here so that $dd^c \log|\zeta_1|^2 = [\zeta_1 = 0]$. Thus the Bott-Chern class $\hat{c}_1(L)$ determined by L has the current representative $[\mathrm{div} g]$ with support on the zero set Z of g. This reflects the fact that L is trivial in the set $X \setminus Z$.

Various generalizations to sections of, not necessarily holomorphic, higher rank bundles are found in, e.g., [21, 34], and [5]. In [22, 23] is developed, for a quite general class of smooth bundle morphisms $g\colon E \to F$, a technique to express any characteristic form of E or of F as a sum $L + dT$, where L is a current with support on the singular set Z of g and T is locally integrable. It is based on a transgression that roughly speaking deforms the given connection on E or F so that the associated characteristic form concentrates on Z.

The aim of this paper is to present variants of (1.1) when $g\colon E \to F$ is a holomorphic morphism with equalities modulo dd^c-exact terms, and to give some applictions. In case when E is a trivial line bundle such a result was obtained in [5], using transgression relying on the ideas and results in [13].

In this paper we use a completely different approach that works for E of higher rank. Let us first consider again a section g of the line bundle $L \to X$. Recall that $c(L) = 1 + c_1(L)$ is the full Chern form and that the Segre form of L is $s(L) = 1/c(L) = 1 - c_1(L) + c_1(L)^2 - \cdots$. If

$$M^g = s(L)\wedge[\mathrm{div} g] \tag{1.2}$$

and $W^g = s(L)\log|g|^2$, then (1.1) can be reformulated as

$$dd^c W^g = M^g + s(L) - 1. \tag{1.3}$$

For a section g of a Hermitian vector bundle $F \to X$ with zero set Z, we introduced in [3, 4] the closed current current

$$M^g := \sum_{k=0}^{\infty} M_k^g \tag{1.4}$$

where M_k^g are the residues $M_k^g := \mathbf{1}_Z[dd^c \log|g|^2]^k,\ k = 0, 1, 2, \cdots$, of the generalized Monge-Ampère products $[dd^c \log|g|^2]^k$, see Sect. 2.4.

The currents M_k^g are *generalized cycles*, a notion introduced in [7], see Sect. 2.3. A generalized cycle μ of codimension k has well-defined integer multiplicities $\mathrm{mult}_x\mu$ at each point x and a unique global decomposition into a (Lelong current of a) cycle of codimension k, the *fixed part*, and the *moving part*; the multiplicities of the latter one vanish outside an analytic set of codimension $\geq k + 1$. In case μ is positive this is the Lelong numbers and its Siu decomposition of μ, respectively. It was proved in [6, 7] that $\mathrm{mult}_x M_k^g$ coincide with the so-called *Segre numbers* of the ideal $\mathcal{J}_x$ at x generated by g, generalizing the Hilbert-Samuel multiplicity of $\mathcal{J}_x$, and that the fixed part of M_k^g is the sum (with multiplicities) of the *distinguished varieties* of the ideal. See Sect. 10.

Notice that a section g of F is can be considered as a morphism $X \times \mathbb{C} \to F$. For an arbitrary holomorphic morphism $g\colon E \to F$, where E and F are Hermitian vector bundles over X, we introduce in this paper a current $M^g = M_0^g + \cdots + M_n^g$, which coincides with M^g above when E is trivial. The current M^g has support on the analytic set Z where g is not injective. Here M_k^g are closed currents of bidegree (k, k) and in fact generalized cycles. Notice that $\mathrm{Im}\, g$ is a subbundle of F over $X \setminus Z$ and thus the associated Segre form $s(\mathrm{Im}\, g)$ is defined there, cf. Sect. 2. Our first main result is

Theorem 1.1 *With the notation above $\mathbf{1}_{X\setminus Z}s(\mathrm{Im}\, g)$ is locally integrable in X and there is a current W^g with singularities along Z such that*

$$dd^c W^g = M^g + \mathbf{1}_{X\setminus Z}s(\mathrm{Im}\, g) - s(E). \tag{1.5}$$

If E is trivial and F is a line bundle, then (1.5) is precisely (1.3). In case E is a line bundle Theorem 1.1 as well as other results in this paper are readily deduced from [5] combined with [7], cf. Remark 11.12. The substantial novelty therefore is when E has higher rank.

Further properties of W^g and M^g are stated in Theorem 4.4. For instance, the multiplicities $\mathrm{mult}_x M_k$ are non-negative, and independent of the metrics on E and F. Moreover, M^g satisfy a certain functorial property so that its definition is determined by the case when g is generically an isomorphism. Then Z has positive codimension (unless X is a point) and thus $\mathbf{1}_{X\setminus Z}s(\mathrm{Im}\, g) = s(F)$. We have the following direct generalization of (1.3).

Corollary 1.2 *If $g\colon E \to F$ is generically an isomorphism, then*

$$dd^c W^g = M^g + s(F) - s(E). \tag{1.6}$$

We have variants of (1.1); notice that $c(F)\wedge M^g$ is a current with support on Z.

Proposition 1.3 *Assume that E is trivial with trivial metric and g is generically injective. Then $\mathbf{1}_{X\setminus Z}c(F/\mathrm{Im}\, g)$ is a locally integrable closed current in X and there is a current V^g with singularities along Z such that*

$$dd^c V^g = c(F)\wedge M^g + \mathbf{1}_{X\setminus Z}c(F/\mathrm{Im}\, g) - c(F). \tag{1.7}$$

Since $c_k(F/\mathrm{Im}\, g) = 0$ in $X \setminus Z$ for $k > m - r$, $r = \mathrm{rank}\, E$, we get from (1.7):

Corollary 1.4 *If $g_1, \ldots, g_r$ are sections of F that are linearly independent outside Z, then there are currents V_k^g such that*

$$dd^c V_{k-1}^g = (c(F)\wedge M^g)_k - c_k(F), \quad k > m - r. \tag{1.8}$$

The g_i define a trivial subbundle F of rank k in $X \backslash Z$. It is therefore expected that the Bott-Chern classes $\hat{c}_k(F)$, $k > m - r$, can be represented by currents that have support on on Z. In case $r = 1$, Corollary 1.4 appeared in [5], cf. Remark 11.12 below.

We now turn our attention to a slightly different generalization of the Poincaré-Lelong formula. Assume that $g\colon E \to F$ is a morphism as before and that g has optimal rank on $X \backslash Z_0$. In this open set we have the short exact sequence $0 \to \operatorname{Ker} g \to E \to \operatorname{Im} g \to 0$ and hence the (non-isometric) isomorphism $a\colon E/\operatorname{Ker} g \simeq \operatorname{Im} g$. Therefore there is a smooth form w in $X \setminus Z_0$ such that $dd^c w = s(\operatorname{Im} g) - s(E/\operatorname{Ker} g)$. We have an extension across Z_0:

Theorem 1.5 *The natural extensions* $\mathbf{1}_{X\backslash Z_0} s(E/\operatorname{Ker} g)$ *and* $\mathbf{1}_{X\backslash Z_0} s(\operatorname{Im} g)$ *are locally integrable and closed. There is a current* M^a *with support on* Z_0 *and a current* W^a *such that*

$$dd^c W^a = M^a + \mathbf{1}_{X\backslash Z_0} s(\operatorname{Im} g) - \mathbf{1}_{X\backslash Z_0} s(E/\operatorname{Ker} g). \tag{1.9}$$

In Sect. 8 we give an extended version (Theorem 8.1). The current M^a is (at least locally) a generalized cycle and it turns out that $\operatorname{mult}_x M_k^a$ may be negative.

In Sect. 9 we discuss Chern and Segre forms associated with some singular metrics on a vector bundle. A notion of distinguished varieties and Segre type numbers of a general coherent sheaf are discussed in Sect. 10. In case $Z = \{x\}$ is a point the number $\operatorname{mult}_x M_n^g$ is equal to the so-called Buchsbaum-Rim multiplicity, [14], see Remark 10.3.

The plan for the rest of the paper is as follows. In Sect. 2 we have collected material that is known, except for the regularization in Propositions 2.3 and 2.5. Then we discuss modifications that admit extensions of certain generically defined subbundles in Sect. 3. In Sects. 4, 5 and 6 we define M^g and state and prove the main results. The proofs rely on results from [7] and [37], and are inspired by [31, 32]. A new Siu type result for generalized cycles, proved in Sect. 7, is crucial for the proof of Theorem 8.1. In the last section, Sect. 11, we compute various examples that aim to shed light on the notions and results.

2 Preliminaries

Throughout this paper X is a connected complex manifold of dimension n.

2.1 Singularities of Logarithmic Type

A current W on X is of logarithmic type along the subvariety Z, cf. [11], if W is smooth in $X \setminus Z$, locally integrable in X, and so that the following holds: Each point on Z has a neighborhood U such that $W|_U$ is the direct image under a proper mapping $h\colon \widetilde{U} \to U$

of a smooth form γ in $h^{-1}(U \setminus \mathcal{Z})$ that locally in $\widetilde{U}$ has the form $\gamma = \sum_j \alpha_j \log|\tau_j|^2 + \beta$, where α_j, β are smooth forms, α_j are closed, and τ_j are local coordinates.

This requirement is imposed, see, e.g., [11, 38], to make it possible to define multiplication of v and the Lelong current of another variety intersecting Z properly. In this paper we use this notion merely to point out that the current in question has in a certain sense simple singularities.

2.2 Segre and Chern Classes

Assume that $\pi : E \to X$ is a holomorphic vector bundle, let $\mathbb{P}(E)$ be its projectivization (so that at each point $x \in X$ the fiber consists of all lines through the origin in E_x), and let $p\colon \mathbb{P}(E) \to X$ be the natural submersion. Consider the pullback $p^*E \to \mathbb{P}(E)$ and let $L = \mathcal{O}(-1) \subset p^*E$ be the tautological line bundle, equipped with the induced Hermitian metric, and Chern form $c(L) = 1 + c_1(L)$. Then

$$s(E) = p_*(1/c(L)) = \sum_{k=0}^{\infty}(-1)^k p_* c_1(L)^k \tag{2.1}$$

and

$$c(E) = \frac{1}{s(E)}. \tag{2.2}$$

Since p is a submersion, $s(E)$ and $c(E)$ are smooth closed forms. It is proved in [35] that this definition of Chern form of E coincides with the differential-geometric definition

$$c(E) = \det\big(I_E + \frac{i}{2\pi}\Theta_E\big), \tag{2.3}$$

where Θ_E is the curvature tensor associated with the Chern connection.

If $h\colon X' \to X$ is a holomorphic mapping, then

$$s(h^*E) = h^*s(E), \quad c(h^*E) = h^*c(E). \tag{2.4}$$

If $g\colon E \to E'$ is a holomorphic vector bundle isomorphism, then we have an induced biholomorphic mapping $\tilde{g}\colon \mathbb{P}(E) \to \mathbb{P}(E')$. If L' is the tautological line bundle over $\mathbb{P}(E')$, then $L = \tilde{g}^*L'$. If E and E' are Hermitian, then there is a smooth form w such that $dd^c w = s(E') - s(E)$.

More generally, $0 \to S \to E \to Q \to 0$ is a short exact sequence of holomorphic Hermitian vector bundles over X, then, see [13], there is a smooth form v so that

$$dd^c v = c(E) - c(Q)c(S). \tag{2.5}$$

It follows from (2.2) that we have a similar relation for the Segre forms. In fact, if $w = -s(E)s(Q)s(S)v$, then $dd^c w = s(E) - s(Q)s(S)$.

2.3 Generalized Cycles

Let $\mathcal{Z}(X)$ be the $\mathbb{Z}$-module of analytic cycles on X; i.e., locally finite sums

$$\sum a_j Z_j,$$

where Z_j are irreducible subvarieties Z of X. Such a sum can be identified with its Lelong current

$$\sum a_j [Z_j].$$

Let $\tau: W \to X$ be a proper holomorphic mapping, and let $\gamma = c_{k_1}(E_1)\cdots c_{k_\rho}(E_\rho)$ be a product of components of Chern forms of Hermitian vector bundles $E_1, \ldots, E_\rho$ over W. Then $\tau_*\gamma$ is a closed current of order 0 on X. Let $\mathcal{GZ}(X)$ be the $\mathbb{Z}$-module of all locally finite sums of such currents. If we identify cycles with their Lelong currents we get a natural inclusion $\mathcal{Z}(X) \subset \mathcal{GZ}(X)$. This module was introduced in [7] and all properties stated here can be found there with proofs.

We have a natural decomposition

$$\mathcal{GZ}(X) = \sum_{k=0}^{\dim X} \mathcal{GZ}_k(X),$$

where $\mathcal{GZ}_k(X)$ is the elements of dimension k; that is, of bidegree $(n-k, n-k)$. Each generalized cycle has a well-defined Zariski-support. However the support of μ can have strictly larger dimension than the dimension of μ, cf. Example 2.1.

Given any analytic variety in X we have the natural restriction operator

$$\mathbf{1}_V : \mathcal{GZ}_k(X) \to \mathcal{GZ}_k(X), \quad \mu \mapsto \mathbf{1}_V \mu.$$

There is a notion of irreducibility and any $\mu \in \mathcal{GZ}_k(X)$ has a unique decomposition into irreducible terms. Moreover, $\mathbf{1}_V\mu$ is precisely the sum of the irreducible components of μ whose Zariski-supports are contained in V.

If γ is a component of a Chern form on X, then we have the mapping

$$\mu \mapsto \gamma \wedge \mu \tag{2.6}$$

on $\mathcal{GZ}(X)$.

If $h: X \to Y$ is a proper mapping, then we have a natural mapping $h_*: \mathcal{GZ}(X) \to \mathcal{GZ}(Y)$, which is consistent with the usual push-forward mapping of cycles. One can define $\mathcal{GZ}(Z)$ just as well for a non-smooth reduced analytic space Z. If $i: Z \to X$ is an inclusion, then the image of i_* is precisely the elements in $\mathcal{GZ}(X)$ that has support on i_*Z. It is therefore often natural to think of generalized cycles as purely geometric objects on X and suppress the fact that they formally are currents.

If $\mu \in \mathcal{GZ}_k(X)$, then for each point $x \in X$ there is a well-defined integer $\mathrm{mult}_x\mu$, the multiplicity of μ at x. If μ is effective, i.e., a positive current, it is precisely the Lelong number of μ at x. It coincides with the usual notion of multiplicity if μ is an analytic cycle. If μ is in $\mathcal{GZ}(Z)$ and $i\colon Z \to X$ is an inclusion, then $\operatorname{mult}_x\mu = \operatorname{mult}_{i(x)} i_*\mu$.

Example 2.1 If $X = \mathbb{P}^2_{[x_0,x_1,x_2]}$ then $\mu = dd^c \log(|x_1|^2 + |x_2|^2)$ is in $\mathcal{GZ}(\mathbb{P}^2)$. It is smooth except at $p = [1, 0, 0]$, and $\operatorname{mult}_x\mu = 1$ at $x = p$ and 0 elsewhere. Moreover, μ is irreducible, has dimension 1, and its Zariski-support is X.

We say that β is a B-form on W if it is a component of the form $c(E) - c(S)c(Q)$, where $0 \to S \to E \to Q$ is a short exact sequence of Hermitian vector bundles on W. We say that $\mu \sim 0$ in $\mathcal{GZ}_k(X)$ if it is a locally finite sum of currents of the form $\tau_*(\beta\wedge\gamma)$, where $\tau\colon W \to X$ is proper, β is a B-form and γ is a product of components of Chern forms on W.

We let $\mathcal{B}_k(X) = \mathcal{GZ}_k(X)/\sim$ and $\mathcal{B}(X) = \oplus_{k=0}^{\infty}\mathcal{B}_k(X)$. It turns out that $\mathcal{Z}(X)$ is a submodule of $\mathcal{B}(X)$ as well. The other properties mentioned above regarding $\mathcal{GZ}(X)$ still hold for $\mathcal{B}(X)$. The most important one in this paper is that the multiplicity $\operatorname{mult}_x\mu$ of $\mu \in \mathcal{GZ}_k(X)$ only depends on the class of μ in $\mathcal{B}_k(X)$.

Lemma 2.2 *If γ has positive bidegree, then, cf. (2.6), $\operatorname{mult}_x(\gamma\wedge\mu) = 0$.*

Any μ is in $\mathcal{GZ}_{n-k}(X)$ has a unique decomposition

$$\mu = \sum_j \beta_j[Z_j] + N, \tag{2.7}$$

where Z_j have codimension k and $\operatorname{mult}_x N$ vanishes outside an analytic set of codimension $\geq k + 1$. In case μ is effective, i.e., the (k, k)-current μ is a positive, then (2.7) is the Siu decomposition of μ. For a general μ, see Theorem 7.1 below. If μ' is another representative of the same class in $\mathcal{B}_{n-k}(X)$, then the Lelong current in its decomposition (2.7) is the same whereas the term N may be different. As already mentioned in the introduction the Lelong current and N are referred to as the fixed and moving part, respectively, of μ.

2.4 Generalized Monge-Ampère Products

Let us assume that X is connected and let ϕ be a section, with zero set Z, of the Hermitian vector bundle $F \to X$. One can recursively define closed currents of order zero,

$$\begin{aligned}&[dd^c \log|\phi|^2]^0 = 1, \quad [dd^c \log|\phi|^2]^k \\ &\quad = dd^c\big(\log|\phi|^2 \mathbf{1}_{X\setminus Z}[dd^c \log|\phi|^2]^{k-1}\big), \ k = 0, 1, 2, \ldots\end{aligned} \tag{2.8}$$

For each $k \geq 0$,

$$M_k^\phi := \mathbf{1}_Z[dd^c \log|\phi|^2]^k$$

is a closed current of order 0 of bidegree (k, k) with support on Z so it vanishes if $k \leq \operatorname{codim} Z$. Thus (2.8) is the classical Bedford-Taylor-Demailly product for

$k \le \operatorname{codim} Z$. The definition for larger k might look artificial, but indeed, e.g., [4, Proposition 4.4],

$$[dd^c \log|\phi|^2]^k = \lim_{\epsilon\to 0} \big(dd^c \log(|\phi|^2+\epsilon)\big)^k, \quad k = 0, 1, 2, \ldots. \tag{2.9}$$

For future reference we sketch a proof for that this definition makes sense: If ϕ is identically 0 then $M^\phi = M_0^\phi = \mathbf{1}_Z = 1$. Let us assume that Z has positive codimension. Let $\pi : \widetilde{X} \to X$ be a smooth modification such that the sheaf generated by the section $\pi^*\phi$ of $\pi^*F \to \widetilde{X}$ is principal, and generated by the section ϕ^0 of a line bundle $\mathcal{L} \to \widetilde{X}$. Then[1]

$$\pi^*\phi = \phi^0\phi',$$

where ϕ' is a section of $\mathcal{L}^* \otimes \pi^*F$. Since $\mathcal{L} \to \pi^*F, \quad v \mapsto v\phi'$, is injective, $\mathcal{L}$ is a subbundle of π^*F. If we equip $\mathcal{L}$ with the induced metric, then $|\phi^0| = |\pi^*\phi|$ and

$$dd^c \log|\pi^*\phi|^2 = dd^c \log|\phi^0|^2 = [D] - c_1(\mathcal{L}) = [D] + s_1(\mathcal{L}) \tag{2.10}$$

by (1.1). In particular,

$$\mathbf{1}_{X'\setminus|D|} dd^c \log|\pi^*\phi|^2 = s_1(\mathcal{L}).$$

Let

$$\langle dd^c \log|\phi|^2\rangle^\ell := \mathbf{1}_{X\setminus Z}[dd^c \log|\phi|^2]^\ell = \pi_* s_1(\mathcal{L})^\ell, \tag{2.11}$$

$$\log|\phi|^2 \langle dd^c \log|\phi|^2\rangle^\ell = \pi_*\big(\log|\pi^*\phi|^2 s_1(\mathcal{L})^\ell\big), \tag{2.12}$$

$$[dd^c \log|\phi|^2]^\ell = \pi_*\big([D]\wedge s_1(\mathcal{L})^{\ell-1} + s_1(\mathcal{L})^\ell\big) \tag{2.13}$$

and

$$M_\ell^\phi = \mathbf{1}_Z[dd^c \log|\phi|^2]^\ell = \pi_*\big([D]\wedge s_1(\mathcal{L})^{\ell-1}\big). \tag{2.14}$$

It follows that the currents in (2.11) and (2.12) are locally integrable. Moreover, since $|D| = \pi^{-1}Z$ ($|D|$ is the Zariski-support of D), it follows that

$$dd^c\big(\log|\phi|^2 \langle dd^c \log|\phi|^2\rangle^{\ell-1}\big) = [dd^c \log|\phi|^2]^\ell$$
$$= M_\ell^\phi + \langle dd^c \log|\phi|^2\rangle^\ell, \quad \ell = 1, 2, \ldots. \tag{2.15}$$

Thus the recursion (2.8) makes sense and produces precisely the currents $\langle dd^c \log|\phi|^2\rangle^\ell$, $[dd^c \log|\phi|^2]^\ell$ and M_ℓ^ϕ. From (2.11), (2.13) and (2.14) we see that they are generalized cycles on X.

[1] Such a smooth modification exists by Hironaka's theorem. The argument here works just as well if one takes π as the normalization of the blow up of X along the ideal sheaf defined by ϕ.

We let $M^\phi = M_0^\phi + M_1^\phi + \cdots$. If $\pi : \widetilde{X} \to X$ is any modification, then

$$\pi_* M^{\pi^*\phi} = M^\phi, \tag{2.16}$$

see [7]. Furthermore, if $\hat{\phi}$ is a section of a Hermitian bundle $\hat{F} \to X$ such that $|\hat{\phi}| \sim |\phi|$ locally on X, then $M^{\hat{\phi}}$ and M^ϕ define the same element in $\mathcal{B}(X)$. In particular, $\hat{F}$ can be F but with another Hermitian metric.

For a thorough discussion of regularizations of generalized Monge-Ampère products, see, e.g., [33]. We will need the following variant that, as far as we know, has not appeared before.

Proposition 2.3 *Let $\chi(t)$ be a smooth function on $\mathbb{R}$ that is 0 for $t < 1/2$ and 1 for $t > 3/4$ and let $\chi_\epsilon = \chi(|\varphi|^2/\epsilon)$, where φ is a section of a vector bundle (tuple of holomorphic functions) with zero set V of positive codimension in X. Then the currents*[2]

$$T_V^{\phi,\epsilon} = (1-\chi_\epsilon)\mathbf{1}_Z + \bar\partial\chi_\epsilon \wedge \frac{\partial \log|\phi|^2}{2\pi i} \wedge \sum_{\ell=0}^{\infty} \langle dd^c \log|\phi|^2\rangle^\ell \tag{2.17}$$

tend to $\mathbf{1}_V M^\phi$ when $\epsilon \to 0$.

If V contains Z then $T_V^{\phi,\epsilon}$ are smooth and tend to M^ϕ.

Proof It is clear that $(1-\chi_\epsilon)\mathbf{1}_Z \to \mathbf{1}_V\mathbf{1}_Z = \mathbf{1}_V M_0^\phi$. Let

$$T = \sum_{\ell=0}^{\infty} \langle dd^c \log|\phi|^2\rangle^\ell.$$

We have

$$\begin{aligned}\bar\partial\chi_\epsilon \wedge \partial \log|\phi|^2 (2\pi i)^{-1} \wedge T &= \bar\partial\big(\chi_\epsilon \partial \log|\phi|^2 (2\pi i)^{-1} \wedge T\big) \\ &\quad -\chi_\epsilon \wedge \bar\partial\partial \log|\phi|^2 (2\pi i)^{-1} \wedge T.\end{aligned} \tag{2.18}$$

In view of (2.12)

$$\chi_\epsilon \partial \log|\phi|^2 \wedge T \to \partial \log|\phi|^2 \wedge T,$$

and hence the first term on the right hand side of (2.18), cf. (2.15), tends to

$$\sum_{\ell=1}^{\infty} [dd^c \log|\phi|^2]^\ell,$$

[2] The first term on the right hand side of (2.17) vanishes unless $\phi \equiv 0$.

whereas the second term tends to

$$\mathbf{1}_{X\setminus V}\sum_{\ell=1}^{\infty}[dd^c\log|\phi|^2]^\ell.$$

Now (2.17) follows since codim $V>0$ and $\langle dd^c\log|\phi|^2\rangle^\ell$ is locally integrable, so that

$$\mathbf{1}_V[dd^c\log|\phi|^2]^\ell=\mathbf{1}_V M_\ell^\phi+\mathbf{1}_V\langle dd^c\log|\phi|^2\rangle^\ell=\mathbf{1}_V M_\ell^\phi.$$

□

2.5 Twisting with a Line Bundle

We keep the notation from the previous subsection. Let $S\to X$ be a line bundle (with no specified metric) and assume that ψ is a section of $F\otimes S^*$. If s is a local non-vanishing section of S we let $|\psi|_\circ=|s\psi|$. Then $dd^c\log|\psi|_\circ:=dd^c\log|s\psi|$ is independent of the choice of s and hence a global current on X. In this way we define the global currents $[dd^c\log|\psi|_\circ^2]^\ell:=[dd^c\log|s\psi|^2]^\ell$, cf. Remark 2.6 below, and $\mathring{M}^\psi:=M^{s\psi}$.

Lemma 2.4 *If $\pi:\widetilde{X}\to X$ is a modification, then*

$$\pi_*\mathring{M}^{\pi^*\psi}=\mathring{M}^\psi. \tag{2.19}$$

The current $\mathring{M}^\psi$ is an element in $\mathcal{GZ}(X)$. If $\hat{\psi}$ is a section of $\hat{F}\otimes S^*$, where $\hat{F}\to X$ is another Hermitian vector bundle and $|\hat{\psi}|\sim|\psi|$, then $\mathring{M}^{\hat{\psi}}$ defines the same class in $\mathcal{B}(X)$.

Notice that $\pi^*\psi$ is a section of $\pi^*F\otimes\pi^*S^*$; we define $\mathring{M}^{\pi^*\psi}$ by suppressing $(\pi^*S)^*$.

Proof Since locally $\mathring{M}^\psi=M^{s\psi}$, where s is a local non-vanishing section of S, by (2.16)

$$\pi_*\mathring{M}^{\pi^*\psi}=\pi_*M^{\pi^*s\psi}=M^{s\psi}=\mathring{M}^\psi,$$

and thus (2.19) holds.

We now choose[3] $\pi:\widetilde{X}\to X$ such that $\pi^*\psi$ is principal, as in Sect. 2.4. Then $\pi^*\psi=\psi^0\psi'$, where ψ' is a non-vanishing section of $\pi^*F\otimes\mathcal{L}^*\otimes\pi^*S^*$. Then $\pi^*(s\psi)=(\pi^*s)\psi^0\psi'$. As in Sect. 2.4 we see that $s_1(\mathcal{L})=dd^c\log|(\pi^*s)\psi'|^2=dd^c\log|\psi'|_\circ$, where the $\circ$ means that π^*S^* is suppressed, so that

$$\mathring{M}^{\pi^*\psi}=[D]\wedge s(\mathcal{L}). \tag{2.20}$$

[3] Since the case $\psi\equiv0$ is trivial, we may assume that ψ is not vanishing identically.

Hence

$$\mathring{M}^{\psi} = \pi_* \mathring{M}^{\pi^*\psi} = \pi_*([D]\wedge s(\mathcal{L})) \tag{2.21}$$

is an element in $\mathcal{GZ}(X)$.

If $|\hat{\psi}| \sim |\psi|$, then $\pi^*\hat{\psi} = \psi^0\hat{\psi}'$ and therefore, cf. (2.20), $\mathring{M}^{\pi^*\hat{\psi}} = [D]\wedge\hat{s}(\mathcal{L})$, where $\hat{s}(\mathcal{L})$ denotes the Segre form of $\mathcal{L}$ with respect to the metric induced by $\hat{F}$. Thus $\mathring{M}^{\pi^*\hat{\psi}}$ and $\mathring{M}^{\pi^*\psi}$ are in the same class in $\mathcal{B}(\widetilde{X})$, and so the last part follows. □

We have the following variant of Proposition 2.3. Let χ_ϵ be a sequence as in this proposition that tends to $\mathbf{1}_{X\setminus V}$.

Proposition 2.5 *Assume that ϕ is a section of $F\otimes S^*$ and that α is a non-vanishing section of $H\otimes S^*$ for some Hermitian vector bundle $H\to X$. Then the currents*

$$\mathring{T}_V^{\phi,\epsilon} = (1-\chi_\epsilon)\mathbf{1}_Z + \bar\partial\chi_\epsilon\wedge\frac{\partial\log(|\phi|/|\alpha|)^2}{2\pi i}\wedge\sum_{\ell=0}^{\infty}\langle dd^c\log|\phi|_\circ^2\rangle^\ell \tag{2.22}$$

tend to $\mathbf{1}_V\mathring{M}^\phi$ when $\epsilon\to 0$.

Here $|\phi|/|\alpha|$ is the global function defined locally as $|s\phi|/|s\alpha|$, where s is any local non-vanishing section of S^*.

Proof Given a local section s we have, with the notation in Proposition 2.3, that

$$2\pi i\mathring{T}_V^{\phi,\epsilon} = 2\pi i T_V^{s\phi,\epsilon} - \bar\partial\chi_\epsilon\wedge\partial\log|s\alpha|^2\wedge\sum_{\ell=0}^{\infty}\langle dd^c\log|\phi|_\circ^2\rangle^\ell. \tag{2.23}$$

Since $\partial\log|s\alpha|^2$ is smooth, letting T denote the last sum, the last term in (2.23) is equal to

$$\bar\partial\big(\chi_\epsilon\partial\log|s\alpha|^2\wedge T\big) - \chi_\epsilon\bar\partial\partial\log|s\alpha|^2\wedge T$$

which tends to $\bar\partial\partial\log|s\alpha|^2\wedge T - \bar\partial\partial\log|s\alpha|^2\wedge T = 0$, since V has positive codimension so that $\mathbf{1}_V T = 0$. By Proposition 2.3 thus $\mathring{T}_V^{\phi,\epsilon} = T_V^{s\phi,\epsilon} + o(1) \to \mathbf{1}_V M^{s\phi} = \mathbf{1}_V\mathring{M}^\phi$. □

Remark 2.6 Assume that we have a (strictly positive) Hermitian metric on S^* with metric form ω. Then $\omega = dd^c\log|s|^2$ for any non-vanishing local section of S. Now $|\psi|$ has a global meaning, $|s\psi|^2 = |s|^2|\psi|^2$, and $dd^c\log|\psi|_\circ^2 = dd^c\log|s\psi|^2 = dd^c\log|\psi|^2 + \omega$. Thus

$$dd^c\log|\psi|_\circ^2 = dd^c u + \omega, \quad u = \log|\psi|^2. \tag{2.24}$$

If we assume that E is a trivial bundle with a trivial metric, then $dd^c\log|\psi|_\circ^2 \ge 0$ and by (2.24) therefore u is quasi-psh with respect to ω. The currents $[dd^c u+\omega]^k$

and their residues $\mathbf{1}_Z[dd^c u + \omega]^k$ were introduced for arbitrary k and studied in [8], and further in, e.g., [12]. Here u can be any ω-psh function with analytic singularities. Analogues for other classes of ω-psh functions are studied in [9].

2.6 Regular Embeddings

Let g be a section of $F \to X$ and let $\mathcal{J}$ be the ideal sheaf generated by g. We have a non-reduced subspace $\iota\colon Z_{\mathcal{J}} \to X$ with structure sheaf $\mathcal{O}_{Z_{\mathcal{J}}} = X_{\mathcal{O}}/\mathcal{J}$. If the zero set of $\mathcal{J}$ has codimension κ, and in addition $\mathcal{J}$ is locally generated by κ holomorphic functions, then one says that ι is a regular embedding. In this case, $Z_{\mathcal{J}}$ has a well-defined normal bundle $\mathcal{N}$ over Z and g defines a canonical embedding of $\mathcal{N}$ in F. If we equip $\mathcal{N}$ with the induced metric, then we have a well-defined Segre form $s(\mathcal{N})$ over Z. Let $[Z_{\mathcal{J}}]$ denote the Lelong current of the fundamental class of $Z_{\mathcal{J}}$. Then $[Z_{\mathcal{J}}] = \sum_j a_j[Z_j]$, where Z_j are the irreducible components of Z and a_j are positive integers. We have the generalization

$$M^g = s(\mathcal{N})\wedge[Z_{\mathcal{J}}], \tag{2.25}$$

of (1.2), see [7, Proposition 1.5].

If ψ is a section of $F \otimes S^*$ as in Sect. 2.5 that defines a regular embedding, then we have an embedding $\mathcal{N} \otimes S \to F$ obtained from the embedding $\mathcal{N} \to F \otimes S^*$ induced by ψ. Now

$$\mathring{M}^{\psi} = s(\mathcal{N} \otimes S)\wedge[Z_{\mathcal{J}}]. \tag{2.26}$$

In fact, if s is a local non-vanishing section of S, then by (2.25), $\mathring{M}^{\psi} = M^{s\psi} = s(\mathcal{N} \otimes S)\wedge[Z_{\mathcal{J}}]$, and so (2.26) follows.

2.7 Rank of a Holomorphic Mapping

Assume that W is irreducible and $f\colon W \to Z$ is any holomorphic mapping. Then the rank of f at y, $\dim W - \dim f^{-1}(f(y))$, is lower semi-continuous on W and its maximum, rank f, is attained on a dense open subset of W_{reg}, see, e.g., [19, II, Sect. 8.1]. We have, [19, Corollary II.8.6],

Proposition 2.7 *If f is surjective, then* rank $f = \dim Z$.

3 Extensions of Subbundles

If $g\colon E \to F$ is a morphism on X, then outside an analytic subvariety Z_0 of positive codimension g has constant and optimal rank, and thus Im g and Ker g are subbundles of F and E, respectively, in $X \setminus Z_0$ (recall that X is always assumed to be connected).

Lemma 3.1 *Let $S = \oplus_{j=1}^r S_j$ be a direct sum of line bundles $S_j \to X$. If $g\colon E \to S$ is a morphism that has optimal rank in $X \setminus Z_0$, then there is a modification $\pi\colon \widetilde{X} \to X$*

such that $\mathrm{Ker}\,\pi^*g$ *has an extension across* $\pi^{-1}Z_0$ *as a holomorphic subbundle of* π^*E.

Since $\mathrm{Im}\, g^* = (\mathrm{Ker}\, g)^{\perp}$ the lemma can be rephrased: If $g^*\colon S^* \to E^*$ has optimal rank in $X \setminus Z_0$, then the pullback to $\widetilde{X} \setminus \pi^{-1}Z_0$ of the subbundle $\mathrm{Im}\, g^*$ has an extension to $\widetilde{X}$.

Proof Let us assume that the optimal rank is ρ. Let $g_j\colon E \to S_j$, $j = 1, 2, \dots$ and let i_1 be the first index such that g_{i_1} is not identically 0. Let $\pi_1\colon X_1 \to X$ be a modification such that $\pi_1^* g_{i_1} = g_1^0 g_1'$, where g_1^0 is a section of a line bundle $L_1 \to X_1$ and g_1' is a non-vanishing section of $\pi_1^*E \otimes L_1^*$. Then $N_1 := \mathrm{Ker}\, g_1'$ is a subbundle of π_1^*E of codimension 1 over X_1. Let now $i_2 > i_1$ be the first index so that $\pi_1^* g_{i_2}|_{N_1}\colon N_1 \to S_{i_2}$ does not vanishing identically. Then there is a a modification $\pi_2\colon X_2 \to X_1$ such that $\pi_2^*\pi_1^* g_{i_2} = g_2^0 g_2'$, where g_2' is non-vanishing. Hence $N_2 := \mathrm{Ker}\, g_2'$ is a subbundle of $\pi_2^*\pi_1^*E$ of codimension 2 over X_2. Proceeding in this way we end up with a subbundle N_ρ of π^*E over $\tilde{X} = X_\rho$, where $\pi = \pi_1 \circ \cdots \pi_\rho\colon \widetilde{X} \to X$. In the Zariski-open subset of $\widetilde{X}$ where π is a biholomorphism, $N_\rho = \cap_j \mathrm{Ker}\, \pi^* g_j = \mathrm{Ker}\, \pi^* g$ and hence N_ρ is the desired extension to $\widetilde{X}$. □

Proposition 3.2 *Assume that* E, F *are Hermitian bundles and* $g\colon E \to F$ *has optimal rank in* $X \setminus Z_0$. *Then the natural extensions from* $X \setminus Z_0$ *to* X *of* $s(E/\mathrm{Ker}\, g)$ *and* $s(\mathrm{Im}\, g)$ *as well as of* $c(E/\mathrm{Ker}\, g)$ *and* $c(\mathrm{Im}\, g)$ *are locally integrable in* X.

If $\pi\colon \widetilde{X} \to X$ is a modification, then it is generically one-to-one and hence $\pi_* 1 = 1$. It follows that $\pi_*\pi^* a = a$ if a is a smooth form on X.

Proof In a neighborhood U of any given point $x \in X$ both E and F are trivial and by Lemma 3.1 there is a modification $\pi\colon \widetilde{U} \to U$ such that $\mathrm{Im}\,\pi^*g$ and $\mathrm{Ker}\,\pi^*g$ have extensions from $\widetilde{U}\setminus\pi^{-1}Z_0$ to $\widetilde{U}$. Since these extensions are subbundles of p^*F and p^*E, respectively, they inherit Hermitian metrics. In $\widetilde{U} \setminus \pi^{-1}Z_0$ we have $\pi^* s(E/\mathrm{Ker}\, g) = s(\pi^*E/\mathrm{Ker}\,\pi^*g)$, and thus

$$s(E/\mathrm{Ker}\, g) = \pi_* s(\pi^*E/\mathrm{Ker}\,\pi^*g) \tag{3.1}$$

in $U \setminus Z_0$. Since the Hermitian bundle $\pi^*E/\mathrm{Ker}\,\pi^*g$ has an extension to $\widetilde{U}$, $s(\pi^*E/\mathrm{Ker}\,\pi^*g)$ has a smooth extension to $\widetilde{U}$, in particular it is locally integrable, and hence $\pi_* s(\pi^*E/\mathrm{Ker}\,\pi^*g)$ is locally integrable in U. In view of (3.1) it coincides with $s(E/\mathrm{Ker}\, g)$ in $U \setminus Z_0$ and since Z_0 is a set of measure zero, thus $\mathbf{1}_{U\setminus Z_0} s(E/\mathrm{Ker}\, g)$ is locally integrable. The other statements are proved in the same way. □

Lemma 3.3 *If* X *is compact and projective and* $g\colon E \to F$ *is any morphism, then there is a modification* $\pi\colon \widetilde{X} \to X$ *such that both* $\mathrm{Ker}\,\pi^*g$ *and* $\mathrm{Im}\,\pi^*g$ *have bundle extensions to* $\widetilde{X}$.

Proof Let $L \to X$ be an ample line. Since $\mathcal{F} = \mathcal{K}er(\mathcal{O}(E) \xrightarrow{g} \mathcal{O}(F))$ is a coherent sheaf, and X is compact, $\mathcal{F} \otimes L^\kappa$ is generated by a finite number of global sections if κ is large enough, see, e.g., [30, Theorem 1.2.6]. If $S_j = L^{-\kappa}$ and $S = \oplus_1^r S_j$, we therefore have a morphism h so that $\mathcal{O}(S) \xrightarrow{h} \mathcal{O}(E) \xrightarrow{g} \mathcal{O}(F)$ is an exact sequence of sheaves. It follows that $S \xrightarrow{h} E \xrightarrow{g} F$ is a generically exact complex of vector bundles. By Lemma 3.1 there is a modification such that $\operatorname{Im} \pi^* h$ has a bundle extension to $\widetilde{X}$. Since it coincides generically with $\operatorname{Ker} \pi^* g$, therefore $\operatorname{Ker} \pi^* g$ has the same extension to $\widetilde{X}$.

In the same way we can find a similar bundle S^* and a homomorfism f such that $S^* \xrightarrow{f} F^* \xrightarrow{g} E^*$ is generically exact. Hence $E \xrightarrow{g} F \xrightarrow{f} S$ is generically exact and it follows from Lemma 3.1 that there is a further modification such that $\operatorname{Ker} \pi^* f$ and hence $\operatorname{Im} \pi^* g$ have bundle extensions to $\widetilde{X}$. □

Remark 3.4 Following the proof of Lemma 3.1 we can produce a local holomorphic frame for the extension of $\operatorname{Ker} g$. To simplify notation we suppress all π_j. We can assume that all S_j are trivial so that g_j are just sections of E^*. Moreover, we can assume that $r = \rho$, since otherwise we delete 'unnecessary' g_j^* from the beginning. Now $g_1 = g_1^0 g_1'$, where g_1' is non-vanishing and hence defines a subbundle of E^* of rank 1, or equivalently a subbundle N_1 of E of codimension 1. Locally we can find a section e_1^* of E^* that is parallell with g_1' so that $g_1 = \alpha_{11} e_1^*$. By assumption the restriction of g_2 to N_1 does not vanish identically. Thus after a further modification $g_2 = g_2^0 g_2'$ where g_2' is non-vanishing on N_1. We can choose a local section e_2^* of E^* such that its image in N_1 is parallell with g_2'. It follows that $g_2 = \alpha_{21} e_1^* + \alpha_{22} e_2^*$. Proceeding in this way we get linearly independent sections $e_1^*, \ldots, e_r^*$ of E^* such that N is subbundle of E that annihilates all of them. Moreover, for $\ell = 1, \ldots, r$,

$$g_\ell = \alpha_{\ell 1} e_1^* + \cdots + \alpha_{\ell\ell} e_\ell^*,$$

where $\alpha_{\ell\ell}$ does not vanish identically. Notice that $\det g = g_1 \wedge \ldots \wedge g_r = \alpha_{11} \cdots \alpha_{rr} e_1^* \wedge \ldots \wedge e_r^*$. If we extend e_j^* to a local frame $e_1^*, \ldots, e_m^*$ for E^* and let $e_1, \ldots, e_m$ be he dual frame for E, then N is spanned by $e_{r+1}, \ldots, e_n$.

4 Definition of M^g and the Main Result Theorem 4.4

First assume that E is a line bundle so that g is a section of $F \otimes E^*$. We define

$$M^g = s(E) \wedge \sum_{\ell=0}^{\infty} \mathbf{1}_Z [dd^c \log |g|_\circ^2]^\ell, \tag{4.1}$$

where $|g|_\circ$ means that we suppress E^* so that locally $dd^c \log |g|_\circ^2 = dd^c \log |ag|^2$ for any non-vanishing section a of E, cf. Sect. 2.5.

From now on we assume that $r = \operatorname{rank} E \geq 2$. Let $\mathbb{P}(E)$ be the projectivization of E, let $p \colon \mathbb{P}(E) \to X$ be the natural projection, and let $L \subset p^* E$ be the tautological bundle, cf. Sect. 2.2. Notice that a local section σ of L has the form

$$\sigma = s(x, \alpha)\alpha \tag{4.2}$$

at $(x, [\alpha])$, $\alpha \in E_x$, where $s(x, \alpha)$ is a holomorphic function on $E \setminus \{\mathbf{0}\}$, $\mathbf{0}$ denoting the zero section, that is -1-homogeneous in $\alpha \in E_x \setminus \{0\}$. By (4.2) we can identify sections σ of L with such $s(x, \alpha)$, and thus consider α as a section of $p^*E \otimes L^*$. Therefore, cf. Sect. 2.5, $dd^c \log |\alpha|^2_\circ := dd^c \log |s\alpha|^2$ is a global form on $\mathbb{P}(E)$, and in fact equal to $dd^c \log |\sigma|^2 = -c_1(L)$, cf. (1.1). Thus $c(L) = 1 - dd^c \log |\alpha|^2_\circ$ so that

$$s(L) = \sum_{\ell=0}^{\infty} \omega_\alpha^\ell, \quad \omega_\alpha = dd^c \log |\alpha|^2_\circ. \tag{4.3}$$

Since g induces a morphism $p^*E \to p^*F$, in particular it defines a morphism $L \to p^*F$. A local section of L, represented by the -1-homogeneous function $s(x, \alpha)$ as above, is mapped to the well-defined section $s(x, \alpha)g(x)\alpha$ of p^*F. Thus

$$G(x, \alpha) := g(x)\alpha \tag{4.4}$$

is a holomorphic section of $p^*F \otimes L^* \to \mathbb{P}(E)$.

Let Z' be the zero set of G on $\mathbb{P}(E)$. As before, let Z be the set where g is not injective and let Z_0 be the set where g does not have optimal rank. If g is generically injective, then $Z = Z_0$ and $Z' \subset p^{-1}Z_0$. If g is not generically injective, then $Z = X$ and $p(Z') = X$. If $N = \operatorname{Ker} g$, then $\mathbb{P}(N)$ is a submanifold of $\mathbb{P}(E)$ in $p^{-1}(X \setminus Z_0)$ and

$$Z' \cap p^{-1}(X \setminus Z_0) = \mathbb{P}(N) \cap p^{-1}(X \setminus Z_0).$$

Letting $|g\alpha|_\circ = |G|_\circ = |sG| = |sg\alpha|$, where s is a local non-vanishing section of L, we have, following Sect. 2.5, the generalized Monge-Ampère powers

$$[dd^c \log |g\alpha|^2_\circ]^\ell, \quad \ell = 0, 1, 2, \ldots$$

and their residues $\mathring{M}^{g\alpha}_\ell = \mathbf{1}_{Z'}[dd^c \log |g\alpha|^2_\circ]^\ell$, $\ell = 0, 1, 2, \ldots$. Locally on $\mathbb{P}(E)$ thus

$$\mathring{M}^{g\alpha} = M^{sg\alpha}. \tag{4.5}$$

Definition 4.1 We define $M^g = p_*\big(s(L) \wedge \mathring{M}^{g\alpha}\big)$.

Thus $M^g = M^g_0 + M^g_1 + \cdots + M^g_n$, where $M^g_k = p_*(s(L) \wedge \mathring{M}^{g\alpha})_{k+r-1}$ are closed (k, k)-currents with support on Z. Notice that $\mathring{M}^{g\alpha}$ and $s(L)$ only depend on the metrics on F and E, respectively.

Example 4.2 Assume that E and F are trivial and have trivial metrics. We can assume that $E = \mathbb{C}^r_\alpha \times X$, $F = X \times \mathbb{C}^m$, with the Euclidian metric on $\mathbb{C}^r_\alpha$ and $\mathbb{C}^m$. Then $\mathbb{P}(E) = X \times \mathbb{P}(\mathbb{C}^r_\alpha)$ and $\omega_\alpha = dd^c \log |\alpha|^2_\circ$ is the usual Fubini-Study metric form on $\mathbb{P}(\mathbb{C}^r_\alpha)$; in particular it is a positive form. Thus $s(L)$, cf. (4.3), is independent of

x. Moreover, locally, for any non-vanishing holomorphic -1-homogeneous s, e.g., $s = 1/\alpha_j$ in the open set $\alpha_j \neq 0$,

$$[dd^c \log |g\alpha|_\circ^2]^\ell = [dd^c \log |sg\alpha|^2]^\ell$$

is a positive current. Since ω_α is a positive $(1,1)$-form therefore, cf. (4.3), $s(L)\wedge \mathring{M}^{g\alpha}$ is a positive current on $\mathbb{P}(E)$, and thus M^g is a positive current on X.

Definition 4.3 We say that the morphisms $g\colon E \to F$ and $g'\colon E \to F'$ are *comparable* if locally in X

$$|g(x)\alpha| \sim |g'(x)\alpha|, \quad \alpha \in E_x.$$

In case $r = 1$, comparability means that the entries in g and g', respectively, generate ideal sheaves with the same integral closure,

Theorem 4.4 *Let E and F be Hermitian vector bundles over X and $g\colon E \to F$ a holomorphic morphism. The following holds:*

(o) *The currents M_k^g are generalized cycles, smooth in the Zariski-open set $X \setminus Z_0$ where g has optimal rank, and positive on X if E and F have trivial metrics.*

(i) *The natural extension $\mathbf{1}_{X\setminus Z} s(\operatorname{Im} g)$ to X of $s(\operatorname{Im} g)$ is locally integrable and closed, and there is a current W^g with singularities of logarithmic type along Z_0 such that*

$$dd^c W^g = M^g + \mathbf{1}_{X\setminus Z} s(\operatorname{Im} g) - s(E). \tag{4.6}$$

(ii) *If $i\colon X' \to X$ is an open subset, then M^{i^*g} is the restriction of M^g to X'.*

(iii) *If $\pi\colon \widetilde{X} \to X$ is a modification, then $\pi_* M^{\pi^* g} = M^g$.*

(iv) *If $i\colon F \to F'$ is a subbundle with the metric inherited from F', then*

$$M^{i\circ g} = M^g. \tag{4.7}$$

(v) *If $g'\colon E' \to F'$ is pointwise injective, then*

$$M^{g\oplus g'} = s(\operatorname{Im} g')\wedge M^g. \tag{4.8}$$

(vi) *The multiplicities $\operatorname{mult}_x M_k^g$ are non-negative integers.*

(vii) *If g and g' are comparable, then $\operatorname{mult}_x M_k^g = \operatorname{mult}_x M_k^{g'}$ for each k and each point x.*

(viii) *For each k we have a unique decomposition*

$$M_k^g = \sum_j \beta_j^k [Z_j^k] + N_k^g =: S_k^g + N_k^g, \tag{4.9}$$

where Z_j^k are irreducible subvarieties of codimension k, β_k^j are positive integers, and N_k^g is a closed (k,k)-current with support on Z whose multiplicities vanish outside a variety of codimension $\geq k+1$. Moreover, $\cup_{jk} Z_j^k = Z$.

Notice that Z has positive codimension if and only if g is generically injective. If g is not generically injective, thus $dd^c W^g = M^g - s(E)$, and M^g is smooth in the open set $X \setminus Z_0$ where g has optimal rank, see part (o) and Proposition 5.3.
By the dimension principle for normal currents, $M_k^g = 0$ if $k < \text{codim}\, Z$.
If g is generically injective, then E and $\text{Im}\, g$ are isomorphic in $X \setminus Z$ so that $s(E)$ and $s(\text{Im}\, g)$ define the same Bott-Chern (cohomology) class there. Equality (4.6) is an extension across Z. If g is not generically injective, then M^g is a representative of the Bott-Chern cohomology class $\hat{s}(E)$.
A variant of (iii) holds for a general proper mapping h, see Proposition 6.1. Regarding (v), notice that $\text{Im}\,(g \oplus g') = \text{Im}\, g \oplus \text{Im}\, g'$ in $X \setminus Z$ and $s(\text{Im}\, g \oplus \text{Im}\, g') = s(\text{Im}\, g) \wedge s(\text{Im}\, g')$, and thus (4.8) is consistent with (4.6).
Parts (vii) and (viii) of Theorem 4.4 imply that if g and g' are comparable, then M_k^g and $M_k^{g'}$ have the same fixed part.

5 Proofs of Theorem 4.4 and Proposition 1.3

First we need some preparations. We keep the notation from Sect. 4. In particular, recall that $N = \text{Ker}\, g$ over $X \setminus Z_0$.

Lemma 5.1 *Assume that g is not generically injective. Then the section G generates the ideal defining $\mathbb{P}(N)$ in $\mathbb{P}(E) \setminus p^{-1} Z_0$.*

Proof Locally in $X \setminus Z_0$ we can choose a trivialization $E = \mathcal{U} \times \mathbb{C}_\alpha^r$ such that $N = \{\alpha_1 = \cdots = \alpha_\rho = 0\}$. Let $\alpha = (\alpha', \alpha'') = (\alpha_1, \ldots \alpha_\rho, \alpha'')$. Then $\alpha' \mapsto g(\alpha', \alpha'') = g(\alpha', 0) = g'\alpha'$ is injective, and hence there is h such that $hg'\alpha' = \alpha'$. Thus $\langle g'\alpha' \rangle = \langle \alpha' \rangle$ so that $g\alpha = g'\alpha'$ generates N. Now the lemma follows. □

By the lemma G defines a regular embedding in $\mathbb{P}(E) \setminus p^{-1} Z_0$ and thus, cf. Sect. 2.6, it induces an embedding $\iota \colon \mathcal{N}_{\mathbb{P}(N)} \to p^* F \otimes L^*$ and hence a mapping $\iota \colon \mathcal{N}_{\mathbb{P}(N)} \to p^* \text{Im}\, g \otimes L^*$ on $\mathbb{P}(N)$. For dimension reasons ι and hence the induced mapping

$$\mathcal{N}_{\mathbb{P}(N)} \otimes L \simeq p^* \text{Im}\, g \tag{5.1}$$

must be isomorphisms on $\mathbb{P}(N) \setminus p^{-1} Z_0$.

Remark 5.2 One can establish the isomorphism (5.1) in a more direct way. Recalling that $T\mathbb{P}(E) = p^* E/[\alpha]$ and similarly $T\mathbb{P}(N) = p^* N/[\alpha]$ we have

$$\mathcal{N}_{\mathbb{P}(N)} = T\mathbb{P}(E)/T\mathbb{P}(N) = p^* E/[\alpha] \big/ p^* N/[\alpha].$$

Since $(x, \gamma) \mapsto g(x)\gamma \in F \otimes L^*$ is injective on $p^* E/[\alpha] \big/ p^* N/[\alpha]$, the isomorphism follows.

We conclude that

$$s(\mathcal{N}_{\mathbb{P}(N)} \otimes L) = s(p^*\mathrm{Im}\, g) = p^* s(\mathrm{Im}\, g)$$

on $\mathbb{P}(N) \setminus p^{-1} Z_0$. From (2.26) we have the representation

$$\mathring{M}^G = p^* s(\mathrm{Im}\, g) \wedge [\mathbb{P}(N)]. \tag{5.2}$$

Let $p' \colon \mathbb{P}(N) \to X$ be the natural projection. Then by (5.2),

$$\begin{aligned} M^g &= p_*\big(s(L)\wedge \mathring{M}^G\big) = p_*(s(L)\wedge p^* s(\mathrm{Im}\, g)\wedge[\mathbb{P}(N)]\big) \\ &= s(\mathrm{Im}\, g)\wedge p_*\big(s(L)\wedge[\mathbb{P}(N)]\big) = s(\mathrm{Im}\, g)\wedge p'_* s(L) = s(\mathrm{Im}\, g)\wedge s(N) \end{aligned}$$

on $X \setminus Z_0$. The last equality holds since the restriction of $s(L)$ to $\mathbb{P}(N)$ is equal to $s(L')$ where L' is the tautological line bundle on $\mathbb{P}(N)$ with the metric inherited from E.

Proposition 5.3 *Assume that g is not generically injective. In $X \setminus Z_0$ we have that $M^g = s(\mathrm{Im}\, g)\wedge s(N)$.*

We now turn our attention to regularizations of M^g. To begin with we apply Proposition 2.5 to $\mathring{M}^{g\alpha}$ with $V = Z'$. Notice that $|g\alpha|^2/|\alpha|^2$ is a global function on $\mathbb{P}(E)$ with zero set Z' so we can take $\chi_\epsilon = \chi(|g\alpha|^2/\epsilon|\alpha|^2)$. Also notice that $g\alpha$ is a section of $p^*F \otimes L^*$ and that α is a non-vanishing section α of $p^*E \otimes L^*$. By Proposition 2.5 the smooth forms

$$1 - \chi_\epsilon + \bar\partial \chi_\epsilon \wedge \frac{\partial \log(|g\alpha|/|\alpha|)^2}{2\pi i} \wedge \sum_{\ell=0}^{\infty} \langle dd^c \log |g\alpha|_\circ^2\rangle^\ell$$

on $\mathbb{P}(E)$ tend to $\mathring{M}^{g\alpha}$. Since $p\colon \mathbb{P}(E) \to X$ is a submersion we get

Proposition 5.4 *With the notation above the forms*[4]

$$M^{g,\epsilon} := p_*\Big(s(L)\wedge\big(1 - \chi_\epsilon + \bar\partial \chi_\epsilon \wedge \frac{\partial \log(|g\alpha|/|\alpha|)^2}{2\pi i} \wedge \sum_{\ell=0}^{\infty} \langle dd^c \log |g\alpha|_\circ^2\rangle^\ell\big)\Big) \tag{5.3}$$

are smooth on X and tend to M^g when $\epsilon \to 0$.

Let us now assume that $g\colon E \to F$ is generically injective. That is, $Z = Z_0$ and $p(Z') = Z_0$. Then $p^{-1}Z$ has positive codimension in $\mathbb{P}(E)$ so we can take $V = p^{-1}Z$ in Proposition 2.5. Moreover, Z is the zero set of the global section $\varphi = \det g$ of $\Lambda^r E^* \otimes \Lambda^r F$, where $r = \dim E$. In local frames for E and F it is the tuple of all $r \times r$ minors of the associated matrix. Let $\chi_\epsilon = \chi(|\det g|^2/\epsilon)$, and for simplicity let us write χ_ϵ also for $p^*\chi_\epsilon$.

[4] The term $1 - \chi_\epsilon$ can be omitted unless $g \equiv 0$.

Proposition 5.5 *Assume that g is generically injective and $\chi_\epsilon = \chi(|\det g|^2/\epsilon)$. For each $\epsilon > 0$*

$$M^{g,\epsilon} := p_*\Big(s(L)\wedge\bar\partial\chi_\epsilon\wedge\frac{\partial\log(|g\alpha|/|\alpha|)^2}{2\pi i}\wedge\sum_{\ell=0}^{\infty}\langle dd^c\log|g\alpha|^2_\circ\rangle^\ell\Big) \tag{5.4}$$

is a smooth forms on X that vanishes in a neighborhood of $Z = Z_0$, and the sequence tends to M^g when $\epsilon \to 0$.

Remark 5.6 It follows from Proposition 2.5 that the currents in (5.4) tend to $\mathbf{1}_{Z_0}M^g$, provided that g is not identically 0, if we choose χ_ϵ that converges to $\mathbf{1}_{X\setminus Z_0}$. If the optimal rank of g is ρ we can take φ as the ρ-determinant of g. Still each $M^{g,\epsilon}$ vanishes in a neighborhood of Z_0 but it is not smooth in general.

5.1 Proof of (o) and (i) of Theorem 4.4

By Proposition 5.3 M^g is smooth in $X \setminus Z_0$. Lemma 2.4 claims that $\mathring{M}^{g\alpha}$ is an element in $\mathcal{GZ}(\mathbb{P}(E))$ and, cf. Sect. 2.3, therefore $s(L)\wedge\mathring{M}^{g\alpha}$ is in $\mathcal{GZ}(\mathbb{P}(E))$. Since $p\colon \mathbb{P}(E) \to X$ is proper, cf. Definition 4.1, $M^g_k = p_*\big((s(L)\wedge\mathring{M}^{g\alpha})_{k+r-1}\big)$ is in $\mathcal{GZ}_{n-k}(X)$ for $k = 0, 1, 2, \ldots$. If the metrics on E and F are trivial, then M^g is positive, cf. Example 4.2. Thus (o) holds.

If $g \equiv 0$, then $Z = X$ and $M^g = s(E)$, and so (i) is trivial. Let us therefore assume that g is not identically 0. We first consider the case when E is a line bundle so that g is a section of $F \otimes E^*$. Let a be a local non-vanishing section of E. In $X \setminus Z$ then ga is a non-vanishing section of the line bundle $\operatorname{Im} g$. Therefore the locally integrable currents, cf. Sect. 4,

$$\langle dd^c\log|ga|^2\rangle, \quad \sum_{\ell=0}^{\infty}\langle dd^c\log|ga|^2\rangle^\ell,$$

are equal to $s_1(\operatorname{Im} g)$ and $s(\operatorname{Im} g)$, respectively in $X \setminus Z$. Moreover, cf. Sect. 4,

$$w^g := \log(|ga|^2/|a|^2)s(E)\wedge\mathbf{1}_{X\setminus Z}s(\operatorname{Im} g)$$

is locally integrable in X. In $X \setminus Z$ we have $dd^c\log(|ga|^2/|a|^2) = s_1(\operatorname{Im} g) - s_1(E)$. Thus

$$\mathbf{1}_{X\setminus Z}dd^c w^g = \mathbf{1}_{X\setminus Z}(s_1(\operatorname{Im} g) - s_1(E))\frac{1}{1-s_1(E)}\frac{1}{1-s_1(\operatorname{Im} g)} =$$

$$\mathbf{1}_{X\setminus Z}\Big(\frac{1}{1-s_1(\operatorname{Im} g)} - \frac{1}{1-s_1(E)}\Big) = \mathbf{1}_{X\setminus Z}s(\operatorname{Im} g) - s(E) \tag{5.5}$$

whereas

$$\mathbf{1}_Z dd^c w^g =$$

$$s(E)\wedge\sum_{\ell=0}^{\infty}\mathbf{1}_Z dd^c(\log(|ga|^2\langle dd^c\log|ga|^2\rangle^\ell)=s(E)\wedge\mathbf{1}_Z\sum_{\ell=1}^{\infty}[dd^c\log|ga|^2]^\ell=M^g, \tag{5.6}$$

cf. (4.1), since $s(E)$ is smooth and closed and

$$\mathbf{1}_Z dd^c\big(\log|a|^2\langle dd^c\log|ga|^2\rangle^\ell\big)=s_1(E)\mathbf{1}_Z\langle dd^c\log|ga|^2\rangle^\ell=0.$$

Part (i) of Theorem 4.4 now follows from (5.5) and (5.6) in case rank $E=1$.

Let us now assume that $r=\text{rank}\,E\geq 2$. We keep the notation from Sect. 4. Let $p'\colon\mathbb{P}(F)\to X$ and let L' be the tautological line bundle in $(p')^*F\to\mathbb{P}(F)$. Notice that g induces a holomorphic mapping $\tilde{g}\colon(\mathbb{P}(E)\backslash Z')\to\mathbb{P}(F)$ and so $\tilde{g}^*L'$ is a well-defined line bundle over $\mathbb{P}(E)\setminus Z'$. Moreover $p=p'\circ\tilde{g}$. From now on we write g rather that $\tilde{g}$ for notational simplicity. If s' is a section of L' then $s=g^*s'$ is a section of L. Therefore, since $g(x,[\alpha])=(x,[g(x)\alpha])$, letting if β denote elements in F

$$g^*s_1(L')=g^*dd^c\log|\beta|^2_\circ=g^*dd^c\log|\beta s'|^2=dd^c\log|g\alpha s|^2=dd^c\log|g\alpha|^2_\circ.$$

In view of (4.3) it is natural to introduce the locally integrable form

$$g^*s(L'):=\sum_{\ell=0}^{\infty}\mathbf{1}_{\mathbb{P}(E)\backslash Z'}[dd^c\log|g\alpha|^2_\circ]^\ell=\sum_{\ell=0}^{\infty}\langle dd^c\log|g\alpha|^2_\circ\rangle^\ell. \tag{5.7}$$

on $\mathbb{P}(E)$.

Lemma 5.7 *We have that*

$$p_*g^*s(L')=\mathbf{1}_{X\backslash Z}s(\text{Im}\,g). \tag{5.8}$$

Proof First assume that g is generically injective. Then $p^{-1}Z$ has positive codimension in $\mathbb{P}(E)$ and therefore $\mathbf{1}_{p^{-1}Z}g^*s(L')=0$. Thus it is enough to prove (5.8) in $X\setminus Z$. There $g\colon E\to\text{Im}\,g$ is an isomorphism and hence $g\colon\mathbb{P}(E)\to\mathbb{P}(\text{Im}\,g)$ is a biholomorphism and so $g^*=g_*^{-1}$. Moreover, the restriction of $L'\to\mathbb{P}(F)$ to $\mathbb{P}(\text{Im}\,g)$ is the tautological line bundle over $\mathbb{P}(\text{Im}\,g)$; let us denote this restriction by L'. Noticing that $p'=pg^{-1}$ we get

$$p_*g^*s(L')=p_*g_*^{-1}s(L')=(pg^{-1})_*s(L')=(p')_*s(L')=s(\text{Im}\,g).$$

We now assume that the generic rank of g is $<\text{rank}\,E$. Then $Z=X$ and so the right hand side of (5.8) vanishes on X. We must ensure that the left hand side vanishes as well. Since $g^*s(L')$ is locally integrable on $\mathbb{P}(E)$, $p_*g^*s(L')$ is locally integrable on X and thus it is enough to see that it vanishes on $X\setminus Z_0$. There Ker g is a subbundle of E of positive dimension. Let us choose a local frame $e_1,\ldots,e_{r-1},e_r$ for E so that e_r belongs to Ker g. Then $E=X\times\mathbb{C}^r_\alpha$, where $\alpha=\alpha_1e_1+\cdots+\alpha_{r-1}e_{r-1}+\alpha_re_r$.

Clearly $g\alpha = g(\alpha_1 e_1 + \cdots + \alpha_{r-1}e_{r-1})$. In a neighborhood of a point on $\mathbb{P}(E)$ where, say, $\alpha_{r-1} \neq 0$, and $g\alpha \neq 0$, we have

$$dd^c \log |g\alpha|^2 = dd^c \log |g((\alpha_1/\alpha_{r-1})e_1 + \cdots + (\alpha_{r-2}/\alpha_{r-1})e_{r-2} + e_{r-1})|^2.$$

Locally on $\mathbb{P}(E)$, $\alpha'_j = \alpha_j/\alpha_{r-1}$, $j \neq r-1$, together with x form a local system of coordinates, and we see that $(dd^c \log |g\alpha|^2)^\ell$ has at most bidegree $(r-2, r-2)$ in α'. Since p is $(x, [\alpha]) \mapsto x$ it follows that the left hand side of (5.8) vanishes. □

Notice that

$$w = \log\left(|g\alpha|^2/|\alpha|^2\right)$$

is a global function on $\mathbb{P}(E)$ which has singularities of logarithmic type along Z'. We claim that

$$dd^c\big(ws(L)g^*s(L')\big) = s(L)\wedge \mathring{M}^{g\alpha} + g^*s(L') - s(L). \tag{5.9}$$

Outside Z' the left hand side of (5.9) is

$$dd^c\left(w\frac{1}{(1-s_1(L))(1-g^*s_1(L'))}\right) = \frac{g^*s_1(L') - s_1(L)}{(1-s_1(L))(1-g^*s_1(L'))}$$
$$= \frac{1}{1-g^*s_1(L')} - \frac{1}{1-s_1(L)} = g^*s(L') - s(L).$$

The only contribution at Z' comes from the residue term which is

$$s(L)\wedge \mathbf{1}_{Z'}dd^c\frac{\log|g\alpha|^2_\circ}{1-\langle dd^c \log|g\alpha|^2_\circ\rangle} = s(L)\wedge\mathbf{1}_{Z'}\sum_{\ell=1}^{\infty}[dd^c\log|g\alpha|^2_\circ]^\ell = s(L)\wedge\mathring{M}^{g\alpha},$$

cf. (5.7). For the last equality we have used that Z has positive codimension so that $\mathring{M}_0^{g\alpha} = 0$. Thus (5.9) holds.

With a modification $\pi : Y \to \mathbb{P}(E)$ as in the proof of Lemma 2.4 we see that $g^*s(L') = \pi_*s(\mathcal{L})$ and that π^*w locally has the form $\log|\psi^0|^2 + smooth$ on Y. Thus $ws(L)g^*s(L')$ has singularities of logarithmic type along Z' and hence along $p^{-1}Z$. Therefore

$$W^g := p_*(ws(L)g^*s(L')) \tag{5.10}$$

has singularities of logarithmic type along Z. From (5.9), (5.8) and Definition 4.1 we have

$$dd^cW^g = p_*(s(L)\wedge\mathring{M}^{g\alpha}) + p_*g^*s(L') - p_*s(L) = M^g + \mathbf{1}_{X\setminus Z}s(\operatorname{Im} g) - s(E).$$

Summing up we have proved part (i) of Theorem 4.4.

5.2 Proof of (ii), (iii) and (iv)

Part (ii) is clear since all definitions and arguments we use are local on X. Part (iii) is precisely Lemma 5.8.

Lemma 5.8 *Assume that $g\colon E \to F$ is a morphism and $\pi : \widetilde{X} \to X$ is a modification. Then we have an induced mapping $\pi^* g\colon \pi^* E \to \pi^* F$ on $\widetilde{X}$ and $\pi_* M^{\pi^* g} = M^g$.*

Proof Let $\widetilde{E} = \pi^* E$. There is a natural mapping $\hat{\pi} : \hat{\mathbb{P}}(\widetilde{E}) \to \mathbb{P}(E)$ so that

$$\begin{array}{ccc} \mathbb{P}(\widetilde{E}) & \xrightarrow{\hat{\pi}} & \mathbb{P}(E) \\ \downarrow \tilde{p} & & \downarrow p \\ \widetilde{X} & \xrightarrow{\pi} & X \end{array} \tag{5.11}$$

commutes, and similarly for F. The morphism $g\colon E \to F$ induces a morphism $\pi^* g\colon \widetilde{E} \to \widetilde{F}$ such that, for $y \in \widetilde{X}$ and $\alpha \in E_{\pi(y)}$,

$$\pi^* g(y)\alpha = g(\pi(y))\alpha, \quad y \in \widetilde{X}, \quad \alpha \in E_{\pi(y)}, \tag{5.12}$$

and

$$\begin{array}{ccc} p^* E & \xrightarrow{g} & p^* F \\ \downarrow \hat{\pi}^* & & \downarrow \hat{\pi}^* \\ \tilde{p}^* \widetilde{E} & \xrightarrow{\pi^* g} & \tilde{p}^* \widetilde{F} \end{array} \tag{5.13}$$

commutes. If $L \to \mathbb{P}(E)$ is the tautological line subbundle of $p^* E$, then $\widetilde{L} := \hat{\pi}^* L$ is the tautological subbundle of $\hat{\pi}^* p^* E = \tilde{p}^* \pi^* E$, cf. (5.11). In particular,

$$s(\widetilde{L}) = \hat{\pi}^* s(L). \tag{5.14}$$

Let s be a local non-vanishing holomorphic section of L on $\mathbb{P}(E)$. If in addition $g(x)\alpha \neq 0$ and $\tilde{s} = \hat{\pi}^* s$, then by (5.12),

$$\hat{\pi}^*(sg\alpha) = \tilde{s}\hat{\pi}^*(g\alpha), \quad \hat{\pi}^*(g\alpha) = \pi^* g\alpha. \tag{5.15}$$

Since $\hat{\pi}$ is generically $1-1$ it follows from [7, Example 5.3] that $\hat{\pi}_* M^{\hat{\pi}^*(sg\alpha)} = M^{sg\alpha}$ where s is defined. From (5.14), (5.15) and the definition of $\mathring{M}$, cf. Sect. 2.5, we conclude that

$$\hat{\pi}_* \mathring{M}^{\hat{\pi}^*(g\alpha)} = \mathring{M}^{g\alpha}. \tag{5.16}$$

By (5.11), (5.14), (5.15), and (5.16) thus

$$\pi_* M^{\pi^* g} = \pi_* \tilde{p}_* \big(s(\widetilde{L}) \wedge \mathring{M}^{\pi^* g\alpha} \big) =$$
$$p_* \hat{\pi}_* \big(\hat{\pi}^* s(L) \wedge \mathring{M}^{\pi^*(g\alpha)} \big) = p_* \big(s(L) \wedge \mathring{M}^{g\alpha} \big) = M^g.$$

Thus the lemma is proved. □

The definitions and arguments are not affected if we consider g as a morphism $E \to F'$ rather than $E \to F$. Thus (iv) follows.

5.3 Proof of (v)

Assume that $g'\colon E' \to F'$ is pointwise injective on X. Let $p\colon \mathbb{P}(E) \to X$ and $\hat{p}\colon \mathbb{P}(E \oplus E') \to X$ be the natural mappings. Moreover, let

$$j\colon \mathbb{P}(E) \to \mathbb{P}(E \oplus E'), \quad [\alpha] \mapsto [\alpha, 0].$$

We claim that

$$\mathring{M}^{g\alpha \oplus g'\alpha'} = j_*\big(p^* s(\operatorname{Im} g') \wedge \mathring{M}^{g\alpha}\big). \tag{5.17}$$

To see (5.17), assume that $U \subset X$ is an open set where $E = U \times \mathbb{C}^r_\alpha$ and $E' = U \times \mathbb{C}^{r'}_{\alpha'}$. It is enough to prove (5.17) in each set $\mathcal{U}_i = \hat{p}^{-1}U \cap \{[\alpha, \alpha'],\ \alpha_i \neq 0\}$. Let $i = 1$. Then $[\alpha, \alpha']$ is represented by

$$(1, \alpha_2/\alpha_1, \cdots, \alpha_r/\alpha_1, \alpha'_1/\alpha_1, \ldots \alpha'_{r'}/\alpha_1).$$

The image of $j\colon \mathbb{P}(E) \to \mathbb{P}(E \oplus E')$ is cut out by the section $g'(x)\alpha'/\alpha_1$ of $\hat{p}^*\operatorname{Im} g'$ over $\mathcal{U}_1$. Since $\operatorname{Im} g'$ has the same rank as the codimension, the normal bundle of the image of j is precisely $\hat{p}^*\operatorname{Im} g'$. From [37, Lemma 5.9] we have that

$$\hat{p}^* c(\operatorname{Im} g') \wedge M^{g\alpha \oplus g'\alpha'} = j_* M^{g\alpha}.$$

Now $\hat{p}^* s(\operatorname{Im} g') \wedge j_* M^{g\alpha} = j_*(j^*\hat{p}^* s(\operatorname{Im} g') \wedge M^{g\alpha})$ and thus (5.17) holds in $\mathcal{U}_1$ since $p^* = j^*\hat{p}^*$ In the same way it holds in any $\mathcal{U}_i$, $i = 1, \ldots, r$, and so (5.17) is proved.

Let $\hat{L}$ be the tautological line bundle in $\hat{p}^*(E \oplus E') \to \mathbb{P}(E \oplus E')$, and recall that

$$s(\hat{L}) = \sum_{\ell=0}^{\infty} (dd^c \log(|\alpha|^2 + |\alpha'|^2)_\circ)^\ell.$$

Since the pullback to $\{\alpha' = 0\}$ of $dd^c \log(|\alpha|^2 + |\alpha'|^2)_\circ$ is $dd^c \log |\alpha|^2_\circ$, (5.17) implies that

$$s(\hat{L}) \wedge \mathring{M}^{g\alpha \oplus g'\alpha'} = j_*\big(j^* s(\hat{L}) \wedge p^* s(\operatorname{Im} g') \wedge \mathring{M}^{g\alpha}\big) = j_*\big(s(L) \wedge p^* s(\operatorname{Im} g') \wedge \mathring{M}^{g\alpha}\big) \tag{5.18}$$

Since $p_* = \hat{p}_* j_*$ we get from (5.18) that

$$M^{g \oplus g'} = \hat{p}_*\big(s(\hat{L}) \wedge \mathring{M}^{g\alpha \oplus g'\alpha'}\big) = p_*\big(p^* s(\operatorname{Im} g') \wedge s(L) \wedge \mathring{M}^{g\alpha}\big) = s(\operatorname{Im} g') \wedge M^g.$$

Thus (v) is proved.

5.4 Proof of (vi) and (vii)

If g' is a morphism such that $|g'\alpha| \sim |g\alpha|$, then by Lemma 2.4 $\mathring{M}^{g\alpha}$ and $\mathring{M}^{g'\alpha}$ define the same class in $\mathcal{B}(\mathbb{P}(E))$. It follows that M^g and $M^{g'}$ define the same class in $\mathcal{B}(X)$. Therefore the multiplicities of M^g and $M^{g'}$ at each point $x \in X$ coincide, and are integers. Locally, cf. Example 4.2, we can choose metrics so that M^g is a positive current. We conclude that the multiplicities are non-negative integers. Thus (vi) and (vii) are proved.

5.5 Proof of (viii)

Since M_k^g is in $\mathcal{GZ}_{n-k}(X)$, the decomposition (4.9) follows from (2.7). The last statement in (viii) requires an additional argument: Let Z_i' be the subvarieties of $\mathbb{P}(E)$ that appear in the decomposition (2.7) of M_ℓ^G for various ℓ. It is well-known, and follows from Sect. 2.4, that their union is precisely the zero set Z' of G. It is clear that $p(Z') = Z$. Thus it is enough to prove, for each Z_i', that $[p(Z_i')]$ appears in the fixed part in (4.9) if $p(Z_i')$ has codimension k in X. It is enough to prove this locally on X, so we can assume that the metrics are trivial, keeping in mind that the fixed part only depends on the class of M_k^g in $\mathcal{B}(X)$. If $p|_{Z_i'}$ has generic rank $\rho = n - k$, cf. Sect. 2.7, then the generic dimension of the fibers $(p|_{Z_i'})^{-1}x$, $x \in p(Z_i')$, is $\nu = \dim Z_i' - \rho$. If locally $E = X \times \mathbb{C}_\alpha^r$, then p is $([\alpha], x) \mapsto x$ and $\omega_\alpha = dd^c \log|\alpha|_\circ$ is strictly positive on each fiber. Therefore $p_*(\omega_\alpha^\nu \wedge [Z_i'])$ has support on $p(Z_i')$, is non-zero, and has bidegree (k, k). Hence it is $c[p(Z_i')]$ for some integer $c \geq 1$. It follows that

$$M_k^g = p_*((s(L) \wedge \mathring{M}^{g\alpha})_{k+r-1}) = c[p(Z_i)'] + \cdots$$

where all terms in $\cdots$ are non-negative, cf. Example 4.2. The proof of Theorem 4.4 is complete.

Proof of Proposition 1.3 Assume that $g\colon E \to F$ and W^g are as in (4.6). By (4.6) and (2.2),

$$dd^c(c(F) \wedge W^g) = c(F) \wedge M^g + c(F) \wedge \mathbf{1}_{X \setminus Z} s(\operatorname{Im} g) - c(F) \tag{5.19}$$

since E is trivial so that $s(E) = 1$. By Lemma 3.1 there is a modification $\pi\colon \widetilde{X} \to X$ such that $\operatorname{Im} \pi^* g$ has an extension to a subbundle H of $\pi^* F$. In $\widetilde{X}$ we thus have the pointwise exact sequence

$$0 \to H \to \pi^* F \to \pi^* F/H \to 0.$$

By (2.5) there is a smooth form v such that $dd^c v = c(\pi^* F) - c(\pi^* F/H) \wedge c(H)$. Hence

$$dd^c(s(H) \wedge v) = c(\pi^* F) \wedge s(H) - c(\pi^* F/H).$$

Applying π_* we see that $\mathbf{1}_{X\setminus Z} c(F/\mathrm{Im}\, g)$ is locally integrable and closed, and

$$dd^c \pi_*(s(H)\wedge v) = c(F)\wedge \mathbf{1}_{X\setminus Z} s(\mathrm{Im}\, g) - \mathbf{1}_{X\setminus Z} c(F/\mathrm{Im}\, g). \tag{5.20}$$

From (5.19) and (5.20) we see that (1.7) holds with $V^g = c(F)\wedge W^g - \pi_*(s(H)\wedge v)$. □

5.6 A remark

Here is an alternative way to find regularizations of M^g. Let us introduce the Hermitian norm on $p^*F \otimes L^* \to \mathbb{P}(E)$ so that $|G| = |g\alpha|/|\alpha|$ and consider the current M^G, cf. Remark 2.6 above.

Lemma 5.9 *For $k = 0, 1, 2, \ldots$ we have the relations*

$$\langle dd^c \log |g\alpha|_\circ^2\rangle^k = \sum_{j=0}^{k} \binom{k}{j} \langle dd^c \log |G|^2\rangle^j \wedge \omega_\alpha^{k-j} \tag{5.21}$$

and

$$\mathring{M}^G_{k+1} = \sum_{j=0}^{k} \binom{k}{j} M^G_{j+1} \wedge \omega_\alpha^{k-j}. \tag{5.22}$$

Proof Notice that

$$\log |G|^2 = \log |sg\alpha| - \log |s\alpha| = \log |g\alpha|_\circ - \log |\alpha|_\circ^2 \tag{5.23}$$

We proceed by induction. Notice that the case $k = 0$ of (5.21) is trivial. Assume that it is proved for some k. Together with (5.23) and the recursion formula for $[dd^c \log |G|^2]^\ell$, cf. (2.8),

$$[dd^c \log |g\alpha|_\circ^2]^{k+1} = dd^c\big((\log |G|^2 + \log |\alpha|_\circ^2)\langle dd^c \log |g\alpha|_\circ^2\rangle^k\big) =$$
$$\sum_{j=0}^{k} \binom{k}{j} [dd^c \log |G|^2]^{j+1} \wedge \omega_\alpha^{k-j} + \sum_{j=0}^{k} \binom{k}{j} \langle dd^c \log |G|^2\rangle^j \wedge \omega_\alpha^{k-j+1}.$$

If we apply $\mathbf{1}_{Z'}$ to this relation we get (5.22) for $k+1$. If we apply $\mathbf{1}_{\mathbb{P}(E)\setminus Z'}$ we get (5.21) for $k+1$. Thus the lemma is proved. □

There are several formulas for regularization of M^G_k. For instance, see [7, Proposition 5.7],

$$M^G_{k,\epsilon} = \frac{\epsilon}{([G|^2 + \epsilon)^{k+1}} (dd^c |G|^2)^k, \quad k = 0, 1, 2, \cdots .$$

By (5.22) we therefore get global smooth $M^{g,\epsilon}$ such that $M^{g,\epsilon} \to M^g$. Clearly, $\mathbf{1}_{Z'}[dd^c \log|g\alpha|_\circ^2]^0 = \mathbf{1}_{Z'} = M_0^G$. In view of (4.3), Definition 4.1 and Lemma 5.9 there are non-negative integers $c_{j,k}$ such that

$$M_\epsilon^g = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{j,k}\, p_*\Big(M_{k,\epsilon}^G \wedge \omega_\alpha^j\Big) \tag{5.24}$$

is a sequence of smooth forms that tends to M^g.

6 Behaviour of M^g Under General Proper Mappings

We have the following extension of Theorem 4.4 (iii).

Proposition 6.1 *Let $g\colon E \to F$ be a morphism on X. Then M^g induces a mapping $\mu \mapsto M^g \wedge \mu$ on $\mathcal{GZ}(X)$ and $\mathcal{B}(X)$ and if $h\colon X' \to X$ is any proper holomorphic mapping, then*

$$h_*(M^{h^*g} \wedge \mu) = M^g \wedge h_*\mu \tag{6.1}$$

for all $\mu \in \mathcal{GZ}(X')$ and $\mu \in \mathcal{B}(X')$.

Example 6.2 If h is a finite mapping, say generically m to 1, then we can apply (6.1) to the function $\mu = 1$. It follows that $h_* M^{h^*g} = m M^g$.

Proof of Proposition 6.1 If $\tau\colon W \to X$ is proper, then we have the commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\tau^* E) & \xrightarrow{\tilde{\tau}} & \mathbb{P}(E) \\ \downarrow \tilde{p} & & \downarrow p \\ W & \xrightarrow{\tau} & X \end{array} \tag{6.2}$$

In fact, in a local trivialization $E = X \times \mathbb{C}_\alpha^r$ and $\tau^* E = W \times \mathbb{C}_\alpha^r$, so that $\mathbb{P}(E) = X \times \mathbb{P}(\mathbb{C}_\alpha^r)$ and $\mathbb{P}(\tau^* E) = W \times \mathbb{P}(\mathbb{C}_\alpha^r)$. Assume that γ is a product of first Chern forms and let $\mu = \tau_*\gamma$. Since p is a proper submersion the pullback $p^*\mu$ exists. We claim that

$$p^*\mu = \tilde{\tau}_* \tilde{p}^* \gamma. \tag{6.3}$$

The equality (6.3) means that

$$p^*\mu.\xi = \tilde{\tau}_* \tilde{p}^* \gamma.\xi, \quad \xi \in \mathcal{E}(\mathbb{P}(E)). \tag{6.4}$$

The left hand side of (6.4) is, by definition, $\mu.p_*\xi$ which in turn is

$$\mu(x).\int_\alpha \xi(x,\alpha) = \gamma(w).\int_\alpha \xi(\tau(w),\alpha)$$

The right hand side is

$$\gamma.\tilde{p}_*\tilde{\tau}^*\xi = \gamma(w).\int_\alpha \xi(\tau(w), \alpha)$$

as well. Thus (6.3) holds. In particular, $p^*\gamma$ is in $\mathcal{GZ}(\mathbb{P}(E))$. Since $M^{g,\epsilon} = p_*\big(s(L)\wedge\mathring{M}^{g\alpha,\epsilon}\big)$, cf. Proposition 5.4,

$$M^{g,\epsilon}\wedge\mu = p_*\big(s(L)\wedge\mathring{M}^{g\alpha,\epsilon}\wedge p^*\mu\big).$$

Since $p^*\mu$ is in $\mathcal{GZ}(\mathbb{P}(E))$ we can take limits, following the proof of [7, Theorem 5.2], and get

$$M^g\wedge\mu := p_*\big(s(L)\wedge\mathring{M}^{g\alpha}\wedge p^*\mu\big). \tag{6.5}$$

We can extend by linearity to a general μ. It is clear from (6.5) that this definition only depends on μ and not on its representation. One must also check that if $\mu' \sim \mu$, then $M^g\wedge\mu \sim M^g\wedge\mu'$, but we omit the details. The equality (6.1) follows from the corresponding property for $\mathring{M}^{g\alpha}$, again see [7, Theorem 5.2], following the proof of Theorem 4.4 (iii) above. □

7 Vanishing of Multiplicities

Theorem 7.1 *Any $\mu \in \mathcal{GZ}_{n-k}(X)$ has a unique decomposition (2.7), where each irreducible component of N has Zariski support on a set of codimension $\leq k-1$. The multiplicities of N vanish outside an analytic set of codimension $\geq k+1$.*

Since μ has a unique decomposition in irreducible components, the theorem follows from:

Proposition 7.2 *If $\mu \in \mathcal{GZ}_{n-k}(X)$ is irreducible with Zariski support Z and* codim $Z \leq k-1$*, then* $\text{mult}_x\mu$ *vanish outside an analytic subset of Z of codimension* $\geq k+1$.

In view of [7, Remark 3.10], an irreducible μ as in Proposition 7.2 is a finite sum of (k,k)-currents $\tau_*\gamma$, where $\tau\colon W \to X$ and $\tau(W) = Z$. If $\tau = i\circ\tau'$, where $i\colon Z \to X$, then $\text{mult}_x\tau'_*\gamma = \text{mult}_{i(x)}\tau_*\gamma$, see Sect. 2.3. It is therefore enough to consider a surjective mapping $\tau\colon W \to Z$ and prove that if $\mu = \tau_*\gamma$ has bidegree (ℓ,ℓ) on Z, $\ell \geq 1$, then the subset of Z where $\text{mult}_x\mu \neq 0$ is contained in an analytic subset of codimension $\geq \ell+1$. Now Proposition 7.2 follows from Lemma 7.3 and Proposition 7.4 below.

Lemma 7.3 *Assume that $\tau\colon W \to Z$ is proper and surjective and $\mu = \tau_*\gamma$ has bidegree (ℓ,ℓ). Let $r = \dim W - \dim Z$. If $\text{mult}_x\mu \neq 0$, then $\dim\tau^{-1}(x) \geq r+\ell$.*

Proof Let $n = \dim Z$ and let ξ be a tuple that defines the maximal ideal at x. Then, [7, Sect. 6, Eq. (6.1)],

$$\text{mult}_x \mu[x] = M^{\xi}_{n-\ell} \wedge \tau_* \gamma = \tau_* \big(M^{\tau^* \xi}_{n-\ell} \wedge \gamma \big).$$

If this is non-vanishing, then since γ is smooth, $M^{\tau^*\xi}_{n-\ell}$ is non-vanishing. It has support on $\tau^{-1}(x)$ and therefore $n - \ell \geq \text{codim}_W \tau^{-1}(x) = n + r - \dim \tau^{-1}(x)$. □

The following proposition should be well-known but as we did not find a precise reference we provide a proof, cf. Remark 7.5.

Proposition 7.4 *If W is irreducible, $f : W \to Z$ is a proper surjective mapping and $r = \dim W - \dim Z$, then for each $\ell \geq 1$, the set*

$$A^f_{r+\ell} := \{x;\ \dim f^{-1}(x) \geq r + \ell\}$$

is contained in an analytic subset of codimension $\geq \ell + 1$ in Z.

Proof of Proposition 7.4 We can assume that W is smooth, because otherwise we take a regularization $\pi : W' \to W$ and consider $f' = f \circ \pi$, noticing that

$$\{x;\ \dim f^{-1}(x) \geq r + \ell\} \subset \{x;\ \dim (f \circ \pi)^{-1}(x) \geq r + \ell\}.$$

We proceed by induction over $\dim W$. Assume that the proposition holds for all W with dimension $\leq m$ and r such that $0 \leq r \leq \dim W$, and that our W has dimension $m + 1$. We first consider the case when $r = m + 1$. Then all the sets $A^f_{r+\ell}$ for $\ell \geq 1$ are empty. Thus we can assume from now on that $r \leq m$. Notice that the set $W' \subset W$ where $\partial f / \partial w$ does not have optimal rank is analytic of dimension $\leq m$.

Moreover, observe that if $w \in W \backslash W'$, then $\partial f / \partial w$ has the same rank in a neighborhood of w so by the constant rank theorem, there is a neighborhood U of w such that $f^{-1}(f(w)) \cap U$ has dimension r.

Let W'_j be the irreducible components of W' and let f'_j be the restriction of f to W'_j so that $f'_j : W'_j \to f(W'_j)$. Since f is proper, each $f(W'_j)$ is an analytic set. We claim that

$$A^f_{r+\ell} = \cup_j A^{f'_j}_{r+\ell}. \tag{7.1}$$

In fact, assume that $f^{-1}(x)$ has an irreducible component V of dimension $\geq r + \ell$. From the observation above it follows that a generic point on V belongs to W', and hence V is contained in W'. Thus $V = \cup_j V \cap W'_j$. It follows that at least one of the analytic sets $V \cap W'_j$ has dimension $\geq r + \ell$. Thus $(f'_j)^{-1}(x)$ has dimension $\geq r + \ell$ so that $x \in A^{f'_j}_{r+\ell}$. Now (7.1) follows.

In view of (7.1) it is enough to consider each $A^{f'_j}_{r+\ell}$. Assume that $(f'_j)^{-1}(x)$ has generic dimension $r + \ell'$. By definition then

$$\begin{aligned}\text{rank } f'_j &= \dim W'_j - r - \ell' \le \dim W - 1 - r - \ell' \\ &= \dim W - 1 - (\dim W - \dim Z) - \ell' = \dim Z - \ell' - 1.\end{aligned}$$

Proposition 2.7 implies that

$$\operatorname{codim} f'_j(W'_j) \ge \ell' + 1. \tag{7.2}$$

First assume that $\ell' \ge \ell$. Since $A^{f'_j}_{r+\ell} \subset f'_j(W'_j)$, by (7.2),

$$\operatorname{codim} A^{f'_j}_{r+\ell} \ge \ell' + 1 \ge \ell + 1$$

as desired. Now assume that $\ell' < \ell$. Since

$$A^{f'_j}_{r+\ell} = A^{f'_j}_{r+\ell'+\ell-\ell'}$$

it follows from the induction hypothesis that $A^{f'_j}_{r+\ell}$ is contained in an analytic subset of $f'_j(W'_j)$ of codimension $\ge \ell - \ell' + 1$. In view of (7.2) we conclude that this analytic set has at least codimension $\ell - \ell' + 1 + \ell' + 1 = \ell + 2$ in Z. Thus Proposition 7.4 is proved. □

Remark 7.5 If γ in the proof of Lemma 7.3 is strictly positive, then the multiplicity is strictly positive if and only if $\dim \tau^{-1}(x) \ge r + \ell$. If W in Proposition 7.4 has a Kähler form ω, then $\gamma_\ell = \omega^\ell$ are strictly positive closed forms for $1 \le \ell \le \dim W$. In this case therefore Proposition 7.4 follows from Siu's theorem applied to the positive closed currents $f_*\gamma_\ell$.

8 An Extension of Theorem 1.5

Let $g\colon E \to F$ be a morphism and let $a\colon s(E/\operatorname{Ker} g) \to \operatorname{Im} g$ be the induced isomorphism over $X \setminus Z_0$. Here is an extended version of Theorem 1.5.

Theorem 8.1 *The natural extensions $\mathbf{1}_{X\setminus Z_0} s(E/\operatorname{Ker} g)$ and $\mathbf{1}_{X\setminus Z_0} s(\operatorname{Im} g)$ are locally integrable and closed in X, and there is a current M^a, which is locally a generalized cycle, with support on Z_0 such that the following holds:*

(a) *There is a current W^a with singularities of logarithmic type along Z_0 such that*

$$dd^c W^a = M^a + \mathbf{1}_{X\setminus Z_0} s(\operatorname{Im} g) - \mathbf{1}_{X\setminus Z_0} s(E/\operatorname{Ker} g). \tag{8.1}$$

The analogue of Theorem 4.4 (ii) *holds for M^a.*

(b) *If $\pi\colon \widetilde{X} \to X$ is a modification, and $M^{\pi^* a}$ denotes the current obtained from $\pi^* g$, then*

$$\pi_* M^{\pi^* a} = M^a. \tag{8.2}$$

(c) *If* Ker g *has an extension to a subbundle* N *of* E*, and* a' *is the induced extension to a morphism* $a'\colon E/N \to F$*, then* $M^a = M^{a'}$*, where* $M^{a'}$ *is the current in Theorem* 4.4.

(d) *All multiplicities* $\text{mult}_x M_k^a$ *are integers. There is a unique decomposition of the form* (4.9)*, where* $\text{mult}_x N_k^a$ *vanishes outside an analytic set of codimension* $\geq k+1$*. All the coefficients* β_j^k *are integers. If* g *and* $\hat{g}$ *are comparable, then the associated* M_k^a *and* $M_k^{\hat{a}}$ *have the same multiplicities.*

Some of the multiplicities $\text{mult}_x M_k^a$ and coefficents β_j^k may be negative, see Example 11.10.
If $\pi : \widetilde{X} \to X$ is a modification such that $\pi^*\text{Ker}\, g$ has an extension as a subbundle $\widetilde{N}$ over $\widetilde{X}$, such a modification always exists at least locally, and $\tilde{a}$ denotes the induced morphism $\pi^*E/\widetilde{N} \to \pi^*F$, then $M^a = \pi_* M^{(\pi^*a)'}$ in view of (b) and (c) above. Thus M^a is determined by these properties.

Although a is only defined on $X \setminus Z_0$, we can define smooth forms $M^{a,\epsilon}$ on X by (5.4).

Proposition 8.2 *The limit*

$$M^a := \lim_{\epsilon\to 0} M^{a,\epsilon} \tag{8.3}$$

exists and is independent of the choice of χ *in* (5.4).

If the subbundle Ker $g \subset E$ defined in $X \setminus Z_0$ happens to have an extension as a subbundle N of E over X, then by continuity $N \subset \text{Ker}\, g$ and therefore g induces a morphism $a\colon E/N \to F$. By Proposition 5.5 then (8.3) is consistent with the previous definition of M^a.

Proof Assume that $\pi : X' \to X$ is a modification such that the subbundle $\pi^*N \subset \pi^*E$ on $X' \setminus \pi^{-1}Z_0$ extends to a subbundle N' of $E' = \pi^*E$ on X' Let $g' = \pi^*g\colon E' \to F' = \pi^*F$. Then $N' \subset \text{Ker}\, g'$ and so g' induces a generically injective mapping $a'\colon E'/N' \to F'$. Thus $M^{a'}$ is a well-defined current on X'. By Lemma 5.8, and its proof, $M^{a,\epsilon} = \pi_* M^{a',\epsilon}$ and so

$$M^a = \pi_* M^{a'}. \tag{8.4}$$

In particular, it is independent of the choice of χ. At least locally in X such a modification π exists, cf. Sect. 3, and thus the proposition is proved. □

Proof of Theorem 8.1 Let W^a be the form (5.10) but associated with a rather than g in $X \setminus Z_0$. Then $W^{a,\epsilon} := \chi_\epsilon W^a$ is well-defined in X for each $\epsilon > 0$. We claim that

$$W^a := \lim_{\epsilon\to 0} W^{a,\epsilon} \tag{8.5}$$

exists. To see this let $\pi : X' \to X$ be a modification as in the previous proof. If $\chi'_\epsilon = \pi^*\chi_\epsilon$, then $W^{a',\epsilon} = \chi'_\epsilon W^{a'} = \pi^* W^{a,\epsilon}$. Thus $W^{a,\epsilon} = \pi_* W^{a',\epsilon}$ and hence the limit (8.5) exists and

$$W^a = \pi_* W^{a'}. \tag{8.6}$$

By Theorem 4.4 (i) we have

$$dd^c W^{a'} = M^{a'} + s(\mathrm{Im}\, g') - s(E'/N'). \tag{8.7}$$

Notice that $\mathbf{1}_{X\setminus Z_0} s(E/\mathrm{Ker}\, g) = \mathbf{1}_{X\setminus Z_0} s(E/N) = \pi_* s(E'/N')$ and $\mathbf{1}_{X\setminus Z_0} s(\mathrm{Im}\, g) = \pi_* s(\mathrm{Im}\, a')$ are locally integrable and closed. Taking π_* now (8.1) follows from (8.4), (8.6) and (8.7).

Part (b) of the theorem follows in a standard way by choosing a π as above which in addition factorizes over $\widetilde{X}$. We omit the details. Part (c) follows from the proof of (a).

Since M^a, at least locally, is a generalized cycle, all its multiplicities are integers, and we have the unique decomposition (4.9), cf. Sect. 2.3.

If g and $\hat{g}$ are comparable, then $\pi^* g$ and $\pi^* \hat{g}$ are comparable in X' and hence the associated a' and $\hat{a}'$ are comparable in X'. It follows from the proof of Theorem 4.4 (vii) that $M^{a'}$ and $M^{\hat{a}'}$ belong to the same class in $\mathcal{B}(X')$. In view of (8.4) therefore M_k^a and $M_k^{\hat{a}}$ belong to the same $\mathcal{B}$-class and hence they have the same multiplicities. Thus Theorem 8.1 is proved. □

Remark 8.3 The non-negativity of the multiplicities in Theorem 4.4 was proved by locally choosing trivial metrics locally on X on E and F. This argument breaks down for M^a since it is the push-forward of $M^{a'}$ under a modification, and in general one cannot choose a metric locally on X so that $M^{a'}$ is non-negative on the exceptional divisor, cf. Example 11.11.

9 Chern and Segre Forms Associated with Certain Singular Metrics

Singular metrics on line bundles have played a fundamental role in algebraic geometry during the last decades, starting with [18]. Singular metrics on a higher rank bundle were introduced in [10], see also [17], and have been studied by several authors since then, e.g., in [25] and [40]. In [36] and later on in [26, 31, 32] are introduced associated Chern forms. In [31] quite general singular metrics are allowed, but there are restrictions on the degrees. In [32] the whole Chern forms for metrics with analytic singularities are defined; however in situations that go beyond [31] an a priori choice of a smooth metric form is needed. These Chern forms are as expected where the metric is non-singular and represent the de Rham cohomology classes. We will use Theorems 4.4 and 8.1 to provide Chern and Segre forms, that in addition represent the expected Bott-Chern classes, for two classes of singular metrics.

Definition 9.1 Let $\hat{F} \to X$ be a holomorphic vector bundle with a metric that is non-singular outside an analytic set Z of positive codimension. We say that a current $s(\hat{F})$ on X is a Segre form for $\hat{F}$ if it represents the Bott-Chern class $\hat{c}(F)$ and is equal to the Segre form defined by the metric where it is non-singular. We have the analogous definition of $c(\hat{F})$.

Example 9.2 Let E and F be Hermitian vector bundles and $g\colon E \to F$ a holomorphic morphism. Let $\hat{E}$ be E but equipped by the singular metric so that $|s|_{\hat{E}} = |gs|$. It was proved in [32] that, in our notation, the current

$$s(\hat{E}) = M^g + \mathbf{1}_{X\setminus Z} s(\mathrm{Im}\, g)$$

defines the same de Rham cohomology class as $s(E)$. Theorem 4.4 (i) states that it in fact defines the same Bott-Chern class, so that $s(\hat{E})$ is a Segre form for $\hat{E}$ in the sense of Definition 9.1. If g is generically surjective it follows from the proof of Proposition 1.3 that

$$c(\hat{E}) = -c(E)c(F)M^g + c(F) \tag{9.1}$$

is a Chern form for $\hat{E}$. Notice that the multiplicities of $s(\hat{E})$ and $-c(\hat{E})$ coincide and are independent of the smooth metrics on E and F.

Remark 9.3 One can obtain an analogue of (9.1) for an arbitrary g; for simplicity though we assume that Z has positive codimension. Using the ideas in the proof of Proposition 9.4 below one can define a current $M^{g,b}$ and a locally integrable V^g such that $dd^c V^g = -M^{g,b} + \mathbf{1}_{X\setminus Z} c(\mathrm{Im}\, g) - c(E)$, so that $c(\hat{E}) = -M^{g,b} + \mathbf{1}_{X\setminus Z} c(\mathrm{Im}\, g)$ is a Chern form for $\hat{E}$.

In our second example we assume that $g\colon E \to F$ is a generically surjective morphism, E and F Hermitian vector bundles, and we let $\hat{F}$ be F but equipped with the singular metric induced from E. That is, for $\beta \in F$ and $x \in X\setminus Z_0$, $|\beta|_{\hat{F}} = |g^{-1}\beta|_{E/\mathrm{Ker}\, g}$. Then clearly $\hat{F}$ is isometric to $E/\mathrm{Ker}\, g$ in $X \setminus Z_0$ so that $s(\hat{F}) = s(E/\mathrm{Ker}\, g)$ and $c(\hat{F}) = c(E/\mathrm{Ker}\, g)$ there.

Proposition 9.4 *With the notation in Theorem 8.1,*

$$s(\hat{F}) = \mathbf{1}_{X\setminus Z_0} s(E/\mathrm{Ker}\, g) - M^a \tag{9.2}$$

is a Segre form for $\hat{F}$. There is a related current $M^{a,c}$ with support on Z_0 such that

$$c(\hat{F}) = \mathbf{1}_{X\setminus Z_0} c(E/\mathrm{Ker}\, g) + M^{a,c} \tag{9.3}$$

is a Chern form for $\hat{F}$. The multiplicities of M^a and $M^{a,c}$ are independent of the smooth metrics on E and F.

Corollary 9.5 *If g is generically an isomorphism and E is trivial with a trivial metric, then*

$$s(\hat{F}) = 1 - M^g, \quad c(\hat{F}) = 1 + M^{g,c}. \tag{9.4}$$

Different trivial metrics on E may produce different M^g, see Examples 11.4 and 11.5. However, $-s_1(\hat{F}) = c_1(\hat{F}) = [\det g]$, see Proposition 11.1 and the definition of $M^{g,c}$ below.

Proof of Proposition 9.4 Clearly (9.2) is equal to $s(E/\text{Ker}\, g)$ in $X \setminus Z_0$. Theorem 8.1 (a) implies that (9.2) is Bott-Chern cohomologous with $s(F)$, and thus a Segre form for $\hat{F}$.

Let $\pi : X' \to X$ be a modification as in the proof of Theorem 8.1. Then we have, cf. (8.7), $dd^c W^{a'} = M^{a'} + s(F') - s(E'/N')$. Since $s(E'/N')$ and $c(E'/N')$ are smooth, we get

$$dd^c V^{a'} = M^{a',c} + c(E'/N') - c(F'), \tag{9.5}$$

where $M^{a',c} = c(F')c(E'/N')M^{a'}$ and $V^{a'} = c(F')c(E'/N')W^{a'}$. We define

$$M^{a,c} = \pi_* M^{a',c}, \quad V^a = \pi_* V^{a'}. \tag{9.6}$$

By regularization as in the proof of Theorem 8.1 one verifies that the definitions in (9.6) are independent of π. Thus $M^{a,c}$ and V^a are globally defined on X. Applying π_* to (9.5) we get

$$dd^c V = M^{a,c} + \mathbf{1}_{X \setminus Z_0} c(E/\text{Ker}\, g) - c(F).$$

Thus (9.3) is a Chern form for $\hat{F}$.

The class of the current M^a in $\mathcal{B}(X)$ is independent of the smooth metrics on E and F. The same holds for the class of $M^{a',c}$ in $\mathcal{B}(X')$ and hence for the class of $M^{a,c}$ in $\mathcal{B}(X)$. Thus the statements about multM^a and mult$M^{a,c}$ follow. □

10 Segre Numbers and Distinguished Varieties Associated with a Coherent Sheaf

These numbers, which generalize the Hilbert-Samuel multiplicity of $\mathcal{J}_x$, were introduced, with a geometric definition, in the '90 s, independently by Tworzewski, [39] and Gaffney-Gassler [20]. Later on a purely algebraic definition was given in Achilles-Manaresi [1], and Achilles-Rams, [2]. We can consider such a g as a morphism $E \to F$, where $E = X \times \mathbb{C}$ is a trivial line bundle with a trivial metric.

Assume that g is a holomorphic section of a vector bundle F, that is, E is trivial line bundle in our set-up. Then g generates an ideal sheaf $\mathcal{J} \subset \mathcal{O}$ which is precisely the image of the dual morphism $g^* : \mathcal{O}(F^*) \to \mathcal{O}(E^*) = \mathcal{O}$. The decomposition (4.9) is a generalization of the classical King formula, [27], and the analytic sets Z_j^k that appear in the fixed part are precisely the so-called distinguished varieties associated with $\mathcal{J}$. If $\pi : X' \to X$ is the blow-up of X along $\mathcal{J}$, then Z_j^k are precisely the images of the various irrreducible components of the exceptional divisor in X'. As mentioned in the introduction, the multiplicities $\text{mult}_x M_k^g$ are the so-called Segre numbers $e_k(\mathcal{J}_x)$ of $\mathcal{J}_x$.

We will discuss generalizations to arbitrary coherent (analytic) sheaves. As for notions like Cohen-Macaulay, dimension etc, we 'identify' an ideal sheaf $\mathcal{J}$ with the quotient sheaf $\mathcal{O}/\mathcal{J}$. By definition an arbitrary coherent sheaf $\mathcal{F}$ locally has a representation $\mathcal{F} = \mathcal{O}(E^*)/Im\, g^*$, where $g\colon E \to F$ is a holomorphic morphism.

Proposition 10.1 *Given a coherent sheaf $\mathcal{F} = \mathcal{O}(E^*)/Im\, g^*$, the multiplicities $mult_x M_k^g$ and the fixed part of the decomposition (4.9) only depends on $\mathcal{F}$.*

Taking this proposition for granted the following definitions may be reasonable.

Definition 10.2 If the coherent $\mathcal{F}$ has the local presentation $\mathcal{F} = \mathcal{O}(E^*)/Im\, g^*$, then we define its Segre numbers $e_k(\mathcal{F}_x) = \text{mult}_x M_k^g$, $k = 0, 1, \ldots$, and its distinguished varieties as the various components of the fixed part in (4.9) for various k.

It follows from Theorem 4.4 that the Segre numbers $e_k(\mathcal{F}_x)$ are non-negative integers that can be strictly positive only if $x \in Z$ and $k \geq \text{codim}\, Z$.

Remark 10.3 If $\mathcal{F}$ has zero set $\{x\}$, then its Buchsbaum-Rim multiplicity was introduced in [14]. This definition is algebraic, but a geometric description appeared in [28, 29] and [24]. One can verify that it indeed coincides with $\text{mult}_x M_n^g$. A detailed argument will be given in a forthcoming paper. If the singularity is not isolated, in [15] is defined algebraically a list of numbers, generalizing the description in [1] of Segre numbers in case of an ideal. One could guess that these numbers coincide with the numbers $\text{mult}_x M_k^g$.

Let $\pi : Y \to \mathbb{P}(E)$ be the blow-up of $\mathbb{P}(E)$ along $G = g\alpha$. In view of the discussion above and the proof of Theorem 4.4 (viii), the distinguished varieties of $\mathcal{F}$ are the images under $p \circ \pi$ of the various irreducible components of the exceptional divisor of π.

Proof of Proposition 10.1 A minimal free resolution of $\mathcal{F}$ at a point x is unique, up to biholomorphisms, and any resolution at x is the direct sum of a minimal resolution and a resolution of 0. The latter resolution ends with a pointwise surjective mapping $(g')^*\colon (F)^* \to (E')^*$. If g^* is the last mapping in a minimal resolution of $\mathcal{F}$ at x, then $\mathcal{F} = \mathcal{O}(E^*)/\text{Im}\, g^*$ and any other representation has the form

$$\mathcal{F} = \mathcal{O}(E^* \oplus (E')^*)/\text{Im}\,(g^* \oplus (g')^*),$$

where $g'\colon E' \to F'$ is pointwise injective. In view of Theorem 4.4 (v) and Lemma 2.2 thus

$$\text{mult}_x M_k^{g\oplus g'} = \text{mult}_x M_k^g, \quad k = 0, 1, 2, \ldots. \tag{10.1}$$

Thus these numbers are intrinsic for the sheaf $\mathcal{F}$ at x. Consider now the representation (4.9) for $M_k^{g\oplus g'}$ and M_k^g, respectively. Since $N_k^{g\oplus g'}$ and N_k^g only have non-zero multiplicities on sets of codimension $\geq k+1$, (10.1) implies that $M_k^{g\oplus g'}$ and M_k^g have the same fixed part. □

Example 10.4 The morphism $g^*(x)$ in Example 11.2 below gives the coherent sheaf $\mathcal{F} = \mathcal{O} \oplus \mathcal{O}/x_1\mathcal{O} \oplus x_2\mathcal{O}$, and it is shown that its distinguished varieties are the axes and the point $(0, 0)$. Moreover, it has non-zero multiplicities on both codimension 1 and 2. The morphism defined by the matrix

$$\begin{bmatrix} x_1x_2 & 0 \\ 0 & 1 \end{bmatrix}. \tag{10.2}$$

gives the sheaf $\mathcal{F} = \mathcal{O} \oplus \mathcal{O}/x_1x_2\mathcal{O} \oplus \mathcal{O} = \mathcal{O}/x_1x_2\mathcal{O}$, which we identify with the ideal sheaf $\langle x_1x_2\rangle$. It has the coordinate axes as distinguished varieties and non-zero multiplicities only on codimension 1. However, the determinant ideals in both cases are $\langle x_1x_2\rangle$. Thus neither distinguished varieties nor multiplicities can be computed from the determinant ideal.

11 Some Examples and Remarks

We will use the notation introduced in Sect. 4. We present our first example as a proposition.

Proposition 11.1 *If $g : E \to F$ is generically an isomorphism, then*

$$M_1^g = [\mathrm{div}(\det g)]. \tag{11.1}$$

Proof Let Z be the zero set of $\det g$. Since M_1^g is a $(1, 1)$-current with support on the hypersurface Z it must be (the Lelong current of) a cycle with support on Z. It is therefore enough to check, for any regular point $x \in Z$, that $\mathrm{mult}_x M^g = \mathrm{mult}_x[\mathrm{div}(\det g)]$.

Let us first assume that $n = 1$, that the base space X is a neighborhood $\mathcal{U}$ of the closed unit disk, $E = \mathcal{U} \times \mathbb{C}^r$, and $F = \mathcal{U} \times \mathbb{C}^r$ and $\det g(x) = x^\nu a$ in $\mathcal{U}$, where a is non-vanishing. Since the multiplicities are independent of the metrics on E and F we can assume that they are trivial in $\mathcal{U}$. If $\nu = 1$, then $g(0)$ has a simple eigenvalue and hence a one-dimensional kernel. Thus $\mathring{M}_r^{g\alpha}$ is a point mass in $\mathbb{P}(E)$ and hence

$$M_1^g = p_*(s(L)\wedge \mathring{M}_r^{g\alpha}) = p_*\mathring{M}_r^{g\alpha} = [0].$$

Now assume that $\nu > 1$. We can choose a continuous perturbation g_t of g such that $g_0 = g$ and $\det g_1$ has ν distinct simple zeros $x^1, \ldots, x^\nu$ close to $x = 0$. Then the kernel of each $g(x^j)$ is one-dimensional, so that $M^{g_1} = [x^1] + \cdots + [x^\nu]$ and so its total mass is ν. Since we have trivial metrics $s_1(E) = 0$ and $s_1(F) = 0$, so by (1.6),

$$\int_{|x|<1} M_1^{g_t} = \int_{|x|<1} dd^c W_0^{g_t} = \int_{|x|=1} d^c W_0^{g_t}.$$

For each t the integral is a sum of the Lelong numbers (multiplicities) of $M_1^{g_t}$ so by Theorem 4.4 it is a positive integer. From formula (5.10) we see that w^{g_t} depends continuously on t on $|x| = 1$. Thus the integral is ν also for $g = g_0$, so the proposition holds when $n = 1$.

Now assume that $n > 1$, 0 is a regular point on Z, and locally $Z = \{x_1 = 0\}$. Then $\det g = x_1^\nu a$ for some ν and non-vanishing holomorphic function a. From the discussion above we know that $M_1^g = \mu[x_1 = 0]$ so we have to prove that $\mu = \nu$. For a generic choice of complementary coordinate functions $x_2, \ldots, x_n$,

$$\mu = \text{mult}_0 M_1^g = \text{mult}_0\big([x_2 = \cdots = x_n = 0]\wedge M_1^g\big).$$

Let $i\colon \mathbb{C}_{x_1} \to \mathbb{C}_x^n,\ x_1 \mapsto (x_1, 0, \ldots, 0)$. By Proposition 6.1 thus

$$i_* M_1^{i^* g} = [x_2 = \cdots = x_n = 0]\wedge M_1^g = \mu[0]. \tag{11.2}$$

Now $\det i^* g(x_1) = a(x_1, 0)x_1^\nu$ so from the case $n = 1$ we have $M_1^{i^* g} = \nu[0]$ in $\mathbb{C}$ so that $i_* M_1^{i^* g} = \nu[0]$ in $\mathbb{C}^n$. In view of (11.2) thus $\mu = \nu$. □

We will use the following form of Crofton's formula, see, e.g., [6, Lemma 6.3]: If $(f_1, \ldots, f_m)$ is a tuple of holomorphic functions and $[\gamma_1, \ldots, \gamma_m] \in \mathbb{P}(\mathbb{C}_\gamma^m)$, then

$$\int_\gamma [\text{div}(\gamma_1 f_1 + \cdots + \gamma_m f_m)]d\sigma(\gamma) = dd^c \log(|f_1|^2 + \cdots + |f_m|^2). \tag{11.3}$$

Here $d\sigma$ is the normalized volume form associated with the Fubini-Study metric on $\mathbb{P}(\mathbb{C}^m)$. If in addition $\text{div} f_1, \ldots, \text{div} f_m$ intersect properly, i.e., the codimension of their intersection is m, then

$$\big(dd^c \log(|f_1|^2 + \cdots + |f_m|^2)\big)^k = \big[dd^c \log(|f_1|^2 + \cdots + |f_m|^2)\big]^k$$

is locally integrable for $k < m$ and

$$M_m^f = [dd^c \log(|f_1|^2 + \cdots + |f_m|^2)]^m = [\text{div} f_1]\wedge\cdots\wedge[\text{div} f_m]. \tag{11.4}$$

The right hand side is the (Lelong current of the) of the intersection product of the divisors and can be defined by any reasonable regularizations of the $[f_j]$, see [16, 2.12.3]. It is well-known that this product is unchanged if f_j are replaced by $\gamma^j \cdot f = f_1\gamma_1^j + \cdots + \gamma_m^j f_m$ for generic choices of $\gamma^j \in \mathbb{P}(\mathbb{C}^m)$. Therefore one can deduce (11.4) from (11.3). In the examples below we often write $[f = 0]$ for $[\text{div} f]$.

Example 11.2 Let $X = \mathbb{C}_x^2$, $E = X \times \mathbb{C}_\alpha^2$, $F = X \times \mathbb{C}^2$, both with trivial metric. and $g\colon E \to F$ defined by

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}. \tag{11.5}$$

Then $g\alpha = (x_1\alpha_1, x_2\alpha_2)$ defines a proper intersection in $\mathbb{C}_x^2 \times \mathbb{P}(\mathbb{C}_\alpha^2)$ so by (11.4)

$$\mathring{M}^{g\alpha} = \mathring{M}_2^{g\alpha} = [dd^c \log |g\alpha|_\circ^2]^2 = [x_1 = \alpha_2 = 0] + [x_2 = \alpha_1 = 0] + [x_1 = x_2 = 0].$$

(11.6)

Since $s(L) = 1 + \omega_\alpha$ we see that $M^g = M_1^g + M_2^g$, where

$$M_1^g = [x_1 = 0] + [x_2 = 0] = [x_1 x_2 = 0], \quad M_2^g = [x_1 = x_2 = 0].$$

Notice that $M_1^g = [\mathrm{div}(\det g)]$ in accordance with Proposition 11.1.

The next example shows that in general several components of $\mathring{M}^{g\alpha}$ come into play to produce M_1^g.

Example 11.3 Let $X = \mathbb{C}$, $E = X \times \mathbb{C}^2_\alpha$ and $F = X \times \mathbb{C}^2$, and let both E and F be equipped with the trivial metric. Let $g\colon E \to F$ be the morphism defined by the matrix

$$\begin{bmatrix} x^2 & 0 \\ 0 & x \end{bmatrix}.$$

Notice that $\det g = x^3$, $Z = \{0\}$, and $Z' = \{0\} \times \mathbb{P}(\mathbb{C}^2_\alpha)$. Now $dd^c \log |g\alpha|^2_\circ = [x = 0] + dd^c \log([x\alpha_1|^2 + |\alpha_2|^2)_\circ$ so that

$$\mathring{M}_1^{g\alpha} = \mathbf{1}_{Z'} dd^c \log |g\alpha|^2_\circ = [x = 0].$$

Furthermore, a computation using (11.4), yields that $dd^c\big(\log |g\alpha|^2_\circ \mathbf{1}_{\mathbb{P}(E)\setminus Z'} dd^c \log |g\alpha|^2_\circ\big) = [x = 0]\wedge[\alpha_2] + [x\alpha_1 = 0]\wedge[\alpha_2 = 0]$ and hence

$$\mathring{M}_2^{g\alpha} = 2[x = 0]\wedge[\alpha_2 = 0].$$

Altogether, as expected from Proposition 11.1

$$M_1^g = p_*\big(s(L)\wedge\mathring{M}^{g\alpha}\big) = p_*\big(\omega_\alpha\wedge[x = 0] + 2[x = 0]\wedge[\alpha_2 = 0]\big) = 3[x = 0].$$

Example 11.4 Let $X = \mathbb{C}^2$, $E = X \times \mathbb{C}^2_\alpha$, $F = X \times \mathbb{C}$ with trivial metrics, and g the morphism given by $[x_1\ x_2]$. Since g is not generically injective, $Z = X$. Moreover, $Z' = \{(x, [\alpha]);\ x_1\alpha_1 + x_2\alpha_2 = 0\}$. We have $\mathring{M}^{g\alpha} = \mathring{M}_1^{g\alpha} = [x_1\alpha_1 + x_2\alpha_2 = 0]$. Since $s(L) = 1 + \omega_\alpha$ we get, using (11.3),

$$M^g = M_0^g + M_1^g = \mathbf{1}_X + dd^c \log(|x_1|^2 + |x_2|^2).$$

Here M_0^g is the fixed part, and it consists of the single distinguished variety X. The term M_1^g has dimension 1 and is geometrically the mean value of lines through the origin in X, so it is a moving term. It follows that $\mathrm{mult}_{(0,0)} M_1^g = 1$ but $\mathrm{mult}_{(x_1,x_2)} M_1^g = 0$ for $(x_1, x_2) \neq (0, 0)$.

If we change the trivial metric on E, e.g., by letting $|\alpha|^2 := |\alpha_1|^2 + 2|\alpha_2|^2$, then $\omega_\alpha = dd^c \log(|\alpha_1|^2 + 2|\alpha_2|^2)$ and one can verify that then $M_1^g = dd^c \log(2|x_1|^2 + |x_2|^2)$.

Here is a similar example but where g is generically injective.

Example 11.5 Let $X = \mathbb{C}^3$, $E = F = X \times \mathbb{C}^3$ with trivial metrics, and g given by

$$\begin{bmatrix} x_1x_3 & 0 & 0 \\ 0 & x_2x_3 & 0 \\ 0 & 0 & x_3^2 \end{bmatrix}.$$

Then $Z = \{x_1x_2x_3 = 0\}$, and $g\alpha = x_3(x_1\alpha_1, x_2\alpha_2, x_3\alpha_3)$ so that

$$dd^c \log |g\alpha|_\circ^2 = [x_3 = 0] + dd^c \log(|x_1\alpha_1|^2 + |x_2\alpha_2|^2 + |x_3\alpha_3|^2).$$

Thus $\mathring{M}_1^{g\alpha} = [x_3 = 0]$. Next we have

$$[dd^c \log |g\alpha|_\circ^2]^2 =$$
$$dd^c\big(\log |x_3|^2 + \log(|x_1\alpha_1|^2 + |x_2\alpha_2|^2 + |x_3\alpha_3|^2)\wedge dd^c \log(|x_1\alpha_1|^2 + |x_2\alpha_2|^2 + |x_3\alpha_3|^2)\big) =$$
$$[x_3 = 0]\wedge dd^c \log(|x_1\alpha_1|^2 + |x_2\alpha_2|^2) + \big(dd^c \log(|x_1\alpha_1|^2 + |x_2\alpha_2|^2 + |x_3\alpha_3|^2)\big)^2.$$

Thus

$$\mathring{M}_2^{g\alpha} = [x_3 = 0]\wedge dd^c \log(|x_1\alpha_1|^2 + |x_2\alpha_2|^2).$$

Furthermore we get, using (11.4),

$$\mathring{M}_3^{g\alpha} = [x_3 = 0]\wedge\big(dd^c \log(|x_1\alpha_1|^2 + |x_2\alpha_2|^2)\big)^2 + [x_1\alpha_1 = 0]\wedge[x_2\alpha_2 = 0]\wedge[x_3\alpha_3 = 0].$$

We do not compute all terms of M^g but notice that, e.g.,

$$\int_\alpha [x_3 = 0]\wedge dd^c \log(|x_1\alpha_1|^2 + |x_2\alpha_2|^2)\wedge\omega_\alpha$$

is a non-zero term in M_2^g that has support on the hyperplane $[x_3 = 0]$. As in Example 11.4 one can verify that M_2^g depends on the choice of trivial metric on E.

Assume that $g\colon E \to F$ is generically an isomorphism. Then $g^*\colon F^* \to E^*$ is as well. In view of (1.6) and the fact that

$$s_k(E^*) = (-1)^k s_k(E) \tag{11.7}$$

it follows that $M_k^{g^*}$ and $(-1)^{k+1}M_k^g$ define the same Bott-Chern class.

Remark 11.6 It is *not* true in general that $M_k^{g^*} = (-1)^{k+1}M_k^g$. In fact, given trivial metrics on E and F we know that both M^g and M^{g^*} are positive currents. Therefore (11.7) fails as soon as M_k^g is non-zero for an even k. See, e.g., M_2^g in Example 11.2.

Let us now consider a global version of Example 11.2.

Example 11.7 Let $X = \mathbb{P}^2 = \mathbb{P}(\mathbb{C}_{x_0,x_1,x_2})$. Then x_j is a section of $\mathcal{O}(1) \to X$ and thus defines a morphism $\mathcal{O}(-1) \to X \times \mathbb{C}$. If $E = X \times \mathcal{O}(-1) \otimes \mathbb{C}^2_\alpha$ and $F = X \times \mathbb{C}^2_\alpha$ thus (11.5) defines a morphism $g\colon E \to F$. We choose the natural metric on $\mathcal{O}(-1)$ so that $s_1(\mathcal{O}(-1)) = dd^c \log |x|^2 = \omega_x$ on X. It follows that L then is the tautological line bundle with respect to trivial metric on $\mathbb{C}^2_\alpha$ tensored by $\mathcal{O}(-1)$, so that $s_1(L) = \omega_\alpha + \omega_x$. Noting that $\mathbb{P}(E)$ has dimension 1 in α, therefore

$$s(L) = 1 + \omega_\alpha + \omega_x + 2\omega_\alpha \wedge \omega_x + \omega_x^2 + 3\omega_x^2 \wedge \omega_\alpha. \tag{11.8}$$

Applying p_* to (11.8) we get

$$s(E) = 1 + 2\omega_x + 3\omega_x^2. \tag{11.9}$$

Since the metric on F is trivial we see that (11.6) still holds in this case (but interpreted on $\mathbb{P}(E)$). Combined with (11.8) we can compute $M^g = p_*(s(L)\wedge \mathring{M}^{g\alpha})$ and find that

$$\begin{aligned} M_1^g &= [x_1 = 0] + [x_2 = 0], \\ M_2^g &= \omega_x \wedge [x_1 = 0] + \omega_x \wedge [x_2 = 0] + [x_1 = x_2 = 0]. \end{aligned} \tag{11.10}$$

Notice that (11.9) and (11.10) are in accordance with (1.6) since M_1^g and M_2^g are Bott-Chern cohomologous with $2\omega_x$ and $3\omega_x^2$, respectively, on $X = \mathbb{P}^2$ and $s(F) = 1$.

Example 11.8 Let us consider the adjoint mapping $g\colon X \times \mathbb{C}^2_\alpha \to X \times \mathcal{O}(1) \otimes \mathbb{C}^2$. In this case $s(L) = 1 + \omega_\alpha$ and $s(E) = 1$. Now

$$|g\alpha|^2_\circ = (|x_1\alpha_1|^2 + |x_2\alpha_2|^2)_\circ / |x|^2$$

and so

$$dd^c \log |g\alpha|^2_\circ = dd^c \log((|x_1\alpha_1|^2 + |x_2\alpha_2|^2)_\circ - \omega_x.$$

We see that

$$\mathring{M}_2^{g\alpha} = \mathbf{1}_{Z'}[dd^c \log |g\alpha|^2_\circ]^2 = [x_1 = \alpha_2 = 0] + [x_2 = \alpha_1 = 0] + [x_1 = x_2 = 0]$$

as before, whereas

$$\mathbf{1}_{\mathbb{P}(E)\setminus Z'}[dd^c \log |g\alpha|^2_\circ]^2 = -2dd^c \log(|x_1\alpha_1|^2 + |x_2\alpha_2|^2)_\circ \wedge \omega_x + \omega_x^2.$$

Therefore

$$\begin{aligned} \mathring{M}_3^{g\alpha} &= \mathbf{1}_{Z'}[dd^c \log |g\alpha|^2_\circ]^3 \\ &= -2\big([x_1 = \alpha_2 = 0] + [x_2 = \alpha_1 = 0] + [x_1 = x_2 = 0]\big)\wedge \omega_x \\ &= -2[x_1 = \alpha_2 = 0]\wedge \omega_x - 2[x_2 = \alpha_1 = 0]\wedge \omega_x. \end{aligned}$$

Recalling that $s(L) = 1 + \omega_\alpha$ we get

$$M_1^g = [x_1 = 0] + [x_2 = 0],$$
$$M_2^g = [x_1 = x_2 = 0] - 2[x_1 = 0]\wedge\omega_x - 2[x_2 = 0]\wedge\omega_x. \qquad (11.11)$$

In view of (11.7) and (11.9), $s(F) = 1 - 2\omega_x + 3\omega_x^2$. Thus, cf. (11.11), (1.6) is respected.

Example 11.9 Let $X = \mathbb{P}_x^2$ and consider the morphism $g\colon \mathcal{O}(-1) \to X \times \mathbb{C}_\alpha^2$, where $g = [x_1\ x_2]$, so that g is singular at the point $p = [1, x_1, x_2]$. We see that

$$s_1(\mathrm{Im}\, g) = dd^c \log(|x_1|^2 + |x_2|^2) =: \omega_p$$

in $X \setminus \{p\}$. It follows that $s_2(\mathrm{Im}\, g) = \omega_p^2 = 0$ in $X\setminus\{p\}$. Since M_2^g has support at p it must be $\alpha[p]$ for some integer α. By (4.6),

$$dd^c W_1^g = \mathbf{1}_{X\setminus\{p\}} s_2(\mathrm{Im}\, g) - s_2(\mathcal{O}(-1))^2 + M_2^g = -\omega^2 + M_2^g$$

so we conclude that $M_2^g = [p]$. It also follows directly, cf. (11.4), that

$$M_2^g = \mathbf{1}_{\{p\}} dd^c\big(\log(|x_1|^2 + |x_2|^2)\mathbf{1}_{X\setminus\{p\}} dd^c \log(|x_1|^2 + |x_2|^2)\big) = [p].$$

We shall now see that the morphism a in Theorem 8.1 can have negative multiplicities.

Example 11.10 Let $X = \mathbb{P}^2$ and consider the morphism $g'\colon X \times \mathbb{C}_\alpha^2 \to \mathcal{O}(1)$; it is the dual of the morphism in Example 11.9. Consider the induced morphism

$$a\colon \mathbb{C}^2 \times \mathbb{P}^2/\mathrm{Ker}\, g' \to \mathcal{O}(1).$$

From (11.7) we see that $s_2(E/\mathrm{Ker}\, g') = 0$. By (8.1) in Theorem 8.1 therefore

$$dd^c W_1^a = \omega^2 + M_2^a,$$

so we can conclude that $M_2^a = -[p]$.

Let us now make a direct computation that reveals how the minus sign in the previous example appears, without relying on the global formula (8.1). We consider a somewhat more general mapping, but restrict to the local situation.

Example 11.11 Let $X = \mathcal{U} \subset \mathbb{C}^2$, $E = X \times \mathbb{C}_\alpha^2$, $F = X \times \mathbb{C}$ (with trivial metrics) and $g = (g_1, g_2)$ with an isolated zero at $0 \in \mathcal{U}$. Let $\pi : X' \to X$ be a modification such that $\pi^* g = g^0 g'$, where g^0 is a section of the line bundle $\mathcal{L} \to X'$ and $g' = (g_1', g_2')$ is a non-vanishing section of $\mathcal{L}^* \otimes \mathbb{C}^2$. The kernel of $\pi^* g$ is generated by $(-g_2', g_1')$ in $X' \setminus \pi^*(0)$ and it thus has a holomorphic extension to a subbundle of $E' = \pi^* E$ over X'. Notice that the image in E'/N' of the holomorphic section $u_1 = (1, 0)$ is non-vanishing in the open subset of X' where $g'_1 \neq 0$. The norm of the image of u_1 in E'/N' is the E'-norm of

$$\hat{u}_1 = u_1 - \frac{u_1 \cdot (-\bar{g}'_2, \bar{g}'_1)}{|g'|^2}(-g'_2, g'_1).$$

A straight forward computation reveals that $|\hat{u}_1|^2 = |g'_2|^2/|g'|^2$, and thus $dd^c \log|\hat{u}_1|^2 = -dd^c \log|g'|^2$ in the set where $g'_1 \neq 0$. An analogous formula holds where $g'_2 \neq 0$. Since E'/N' is a line bundle we conclude that

$$s_1(E'/N') = -dd^c \log|g'|^2, \quad s_2(E'/N') = (-dd^c \log|g'|^2)^2 = 0.$$

Notice that $a' \colon E'/N' \to \pi^*F$ is defined by $a' = g^0(g'_1, g'_2)$ so that $\mathrm{div} a' = [g^0 = 0]$. Recalling, cf. (4.1), that $M^{a'} = s(E'/N') \wedge [\mathrm{div} a']$ we thus have $M_1^{a'} = [g^0 = 0]$ and $M_2^{a'} = s_1(E'/N') \wedge [g^0 = 0]$. We conclude that

$$M_2^a = \pi_* M_2^{a'} = -c[0],$$

where c is a positive integer. In fact, $M_2^{g^*} = c[0]$ so c is the multiplicity of the zero of g at 0.

Remark 11.12 Let E be a trivial line bundle (with trivial metric) and let $g \colon E \to F$ be generically injective morphism, i.e., a non-trivial holomorphic section of F. With the notation in this paper a residue current, here denoted by $M^{g,a}$, was defined in [5] in the following way. Let S denote E but with the singular metric inherited from F. Then, writing $c(F/S)$ is locally integrable in X and

$$M^{g,a} := \mathbf{1}_Z dd^c(\log|g|^2 c(F/S)).$$

If $\pi : X' \to X$ is a suitable modification, then $\pi^* c(S)$ and $\pi^* c(F/S)$ are smooth in X' and so there is a smooth form v such that $dd^c v = \pi^* c(F) - \pi^* c(S) \pi^* c(F/S)$. By arguments as in the proof of Proposition 1.3 it follows that $M^{g,a}$ is in the same class in $\mathcal{B}(X)$ as

$$\mathbf{1}_Z dd^c(\log|g|^2 c(F)/c(S)) = c(F)\mathbf{1}_Z dd^c(\log|g|^2 \sum_{\ell=0}^{\infty} \langle dd^c \log|g|^2 \rangle^\ell)$$
$$= c(F) \wedge M^g =: M^{g,b}.$$

In particular, $M^{g,a}$ and M^g, as well as $M^{g,b}$, have the same multiplicities and fixed part. In case F has a trivial metric, these three currents coincide.

Let us conclude by mentioning two natural question that are not discussed in this paper. The classical Poincaré-Lelong formula sometimes occurs in the form $\bar\partial(1/g) \wedge Dg/2\pi i = [\mathrm{div} g]$, where D is the Chern connection, which means that

$$\bar{\partial}\frac{1}{g}\wedge s(F)Dg/2\pi i = M^g.$$

Thus M^g is a product of a residue current and a smooth form. In a similar way the current $M^{g,a}$ in Remark 11.12, see [5, (6.4)], can be written $M^{g,a} = R^g \cdot \varphi$, where R^g is the Bochner-Martinelli residue current and φ is a matrix of smooth forms involving both Dg and the curvature tensor. We do not know whether there are analogues for M^g even when E is a line bundle. Another natural question is whether some assumptions of positivity/negativity on F and/or E will imply positivity of M^g; see [5] for some results of this kind for $M^{g,a}$.

Acknowledgements The author is grateful to Richard Lärkäng, Håkan Samuelsson Kalm and Elizabeth Wulcan for valuable discussions related to this paper, as well as to the referees for pointing out several mistakes and obscurities.

Funding Open access funding provided by Chalmers University of Technology.

Data Availability Not applicable.

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