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PAPER

Synthetic G-jet-structures in modal homotopy type theory

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Abstract

This article constructs the moduli stack of torsion-free G -jet-structures in homotopy type theory with one monadic modality. This yields a construction of this moduli stack for any ∞ -topos equipped with any stable factorization systems.

In the intended applications of this theory, the factorization systems are given by the deRham-Stack construction. Homotopy type theory allows a formulation of this abstract theory with surprisingly low complexity. This is witnessed by the accompanying formalization of large parts of this work.

Keywords: Homotopy type theory; higher differential geometry; synthetic mathematics

1. Introduction

The constructions and theorems in this article are formulated in homotopy type theory. In (Shulman, 2019), Michael Shulman has shown that homotopy type theory can be interpreted in any Grothendieck $(\infty, 1)$ -topos as defined in (Lurie, 2009) [Definition 6.1.0.4]. Throughout the article, we assume a fixed monadic modality. By (*monadic modality*) we mean the same as “modality” defined in (The Univalent Foundations Program, 2013) [Definition 7.7.5] or the “higher modalities” (Rijke, Shulman and Spitters, 2020) [Definition 1.1] or the equivalent notion of “uniquely eliminating modalities” (Rijke, Shulman and Spitters, 2020) [Definition 1.2].

A modality may be described as an operation \mathfrak{S} together with a map $\iota_X : X \rightarrow \mathfrak{S}X$ for any type X , such that a dependent version of the following commonly known property of a reflector holds:

For all Y such that $\iota_Y : Y \rightarrow \mathfrak{S}Y$ is an equivalence and all maps $f : X \rightarrow Y$, there is a unique $\psi : \mathfrak{S}X \rightarrow Y$, such that the diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\iota_X} & \mathfrak{S}X \\ & \searrow f & \downarrow \exists! \psi \\ & & Y \end{array}$$

The dependent version of this universal property will be axiom 2.5 – which we assume throughout this article for convenience. Externally, a monadic modality in homotopy type theory corresponds to a stable factorization system on an $(\infty, 1)$ -topos (Rijke, Shulman and Spitters, 2020) [Appendix A, in particular, p. 76].



The examples of modalities (\mathfrak{S}, ι) we had in mind when writing this article should be thought of as providing a notion of *infinitesimally close*. More specifically, two points $x, y : X$ are infinitesimally close if their images under ι coincide. This makes only sense in a context where there are infinitesimals in the first place.

As far as the author knows, all relevant examples of such particular modalities are constructed by passing from spaces to algebras of functions on spaces and by introducing infinitesimals via nilpotent elements in those algebras. A good intuition is that the functions are coordinate functions, and in an infinitesimal space, the coordinates can be so small that taking a power of them actually turns them into zero. If these infinitesimal spaces are around, macroscopic spaces X can be probed by them.

The information, that can be probed in this way, may be collapsed by passing to $\mathfrak{S}X$. It is important to note that this collapse almost never preserves structured spaces like manifolds or schemes – they are replaced by macroscopically similar spaces, which have trivial infinitesimal structure and therefore trivial tangent spaces. Spaces that are only of infinitesimal extent, like the formal or k -th order disks of algebraic geometry, are mapped to the one point space by \mathfrak{S} . We will sketch an easy way of constructing \mathfrak{S} below, which works for a class of examples. It turns out to be more natural to have a modality \mathfrak{S} which collapses infinitesimals of all orders at once – it is possible to construct models which capture the notion of a similar modality collapsing only first-order infinitesimals, but the models the author came up with lacked other desirable properties. The use of G -jet-structures in this article instead of G -structures stems from this decision to use general infinitesimals as a primary notion. In Subsection 4.3, we will briefly explain why we do expect that we can also cover the case of G -structures with G -jet-structures in the case of $G = \text{GL}(n, \mathbb{R})$.

Urs Schreiber and Igor Khavkine define basic notions of differential geometry as well as generalized partial differential equations in (Khavkine and Schreiber, 2017), and most of their constructions, as they note, do not depend on the particular topos and the particular modality “ \mathfrak{S} ” they use. Crucially, they show that in the topos they use, the abstract definitions of formal disks and formally étale maps, analogous to Definition 3.3 and Definition 3.14 in this article, coincide with formal disks in manifolds and local diffeomorphisms of manifolds.

In the appendix of (Cherubini and Rijke, 2021), it is shown that in the Zariski topos, the definitions of formal disks and formally étale maps Definition 3.14, Definition 3.3 correspond to usual formal neighborhoods and formally étale maps of algebraic geometry. It is certainly noteworthy that the abstract theory in this article combines quite well with synthetic differential geometry, which is used extensively in the preprint (Myers, 2022b).

In (Schreiber, 2015), Urs Schreiber presented a couple of problems together with proposals for their solution to the homotopy type theory community. This article solves one of these problems, which is the construction of the moduli space of torsion-free G -jet-structures Definition 4.33, where Theorem 3.12 is an important step also mentioned by Schreiber. The proof of the latter theorem in this article is a vast simplification of Schreiber’s proof, which relied on pasting of homotopy pullbacks, whereas the proof in this article uses simpler reasoning with dependent types. A solution to Schreiber’s problem was already given in the PhD thesis of the author (Wellen, 2017), but not published under peer review.

A minor difference to the construction of the moduli space proposed in (Schreiber, 2015) is that the G -jet-structures are checked for triviality on first-order infinitesimal disks, while for this article, after discussing with Schreiber, full formal disks are used everywhere. It is left to check in future work that the construction given here type-theoretically yields the same space as the classical construction.

Important advantages of homotopy type theory for this work include the unusual conciseness for a higher categorical framework. Furthermore, a proof-assistant software, in this case *Agda*, can be used to check definitions and proofs written out in homotopy type theory. This was of great help to the author during the development of the theory in this article and while learning the subject. The formalization can be viewed at <https://github.com/felixwellen/DCHoTT-Agda>,

where Theorem 3.12 is to be found in the file `FormalDiskBundle.agda` and the central construction Definition 4.33 in `G-structures.agda`.

We will conclude this introduction by giving more intuition for the intended models. This part is aimed, in particular, at readers not familiar with higher stacks or synthetic differential geometry.

An important thing to note is that manifolds and other simple spaces of interest in differential geometry are, maybe to the surprise of some readers, *not* to be thought of as *higher* types. Note that this is also the case for the topological spaces in (Shulman, 2018). Instead, in the applications of interest, a manifold is usually a 0-truncated type. The higher types in this context are given by passing from the ordinary, 1-categorical notion of Grothendieck toposes, to their higher categorical version. The latter includes the former, as the subcategory of 0-truncated objects. Thus, the spaces of interest, which already exist in the 1-categorical topos, are included in the 0-truncated types.

The theory in this article may, however, also be applied to objects more general than manifolds, which are not 0-truncated. One important example is quotient stacks. In addition, it is also possible to consider spaces, which are not locally modeled on 0-truncated types. Both cases are not ruled out by Definition 4.14.

Furthermore, the ambient higher types admit the construction of classifying morphisms (see Definition 4.9) of fiber bundles, which is crucial for the goals of the article. In addition to that, there are exceptionally easy ways to describe homotopy theoretic quotients of spaces by simple type theoretic constructions, which rely on higher identity types as well. This will be explained in the discussion preceding Definition 4.33.

The kind of modality that can be used to access the differential geometric structure of the objects of a topos from within type theory is in some fortunate cases generated by reducing algebras. More precisely, in one of the most basic models, namely simplicial sheaves on the category of $k\text{-Alg}_{\mathcal{G}_p}^{\text{op}}$ finitely presented algebras over a field k ,¹ there is an endofunctor \mathfrak{S} given by:

$$(\mathfrak{S}X)(A) := X(A/\sqrt{0})$$

for any sheaf X . If reduction preserves covers, as it does for the Zariski topology, this is an idempotent left and right adjoint functor, which is enough to generate a modality on the topos. The same approach yields modalities on toposes suitable for differential geometry. Roughly, this is achieved by passing to algebras of smooth functions and taking tensor products with nilpotent algebras, to add infinitesimals to the theory (see (Khavkine and Schreiber, 2017)).

It is also possible to only add square-zero algebras instead of general nilpotent algebras, which makes the definition of formal disks (Definition 3.3) collapse to first-order neighborhoods – something very close to a tangent space. This leads to a simpler theory, which is easier to compare with differential geometry, but it also yields a category that doesn't have the right limits, so we will not consider it any further.

The functor \mathfrak{S} appears in the differential part of a *Differential Cohesive Topos*, a notion due to Urs Schreiber (Schreiber) [Definition 4.2.1], extending Lawvere's Axiomatic Cohesion (Lawvere, 2007). The differential structure is also used on toposes of set-valued sheaves (Khavkine and Schreiber, 2017), where it is applied to a site suitable for differential geometry and therefore spaces modeled on vector spaces over the reals.

Since this modality \mathfrak{S} , which we will use in our type theory, allows us to build at least some abstract differential geometry relative to it, one might ask what role the external functor \mathfrak{S} from above plays in conventional geometry. The answer is that concepts very close to it appear very early in the Grothendieck school of algebraic geometry, which is no surprise, since algebras with nilpotent elements were specifically used to admit reasoning with this kind of infinitesimals. However, the functor itself leaves the impression of a rather exotic concept under the names of deRham prestack (Gaitsgory and Rozenblyum, 2014), deRham stack, deRham space, or infinitesimal shape, and is usually used to represent D-modules over a smooth scheme or algebraic stack X as quasi-coherent sheaves over $\mathfrak{S}X$. A functor \mathfrak{S} also exists in meaningful ways in

non-commutative geometry (Kontsevich and Rosenberg). In the face of these rather advanced use cases of \mathfrak{S} , it might be irritating that we use it as a basis for differential geometry. One reason $\mathfrak{S}X$ appears so infrequent in geometry might be that it is quite hard to build intuition for what it is like as a space. If X is a structured space like a manifold or a scheme, $\mathfrak{S}X$ will only be a manifold or scheme in degenerate cases. On the other hand, the relation provided by the map $\iota_X : X \rightarrow \mathfrak{S}X$ can be understood quite intuitively as “*infinitesimally close*.” This is how we will start to develop differential geometry based on \mathfrak{S} .

Content

- We define the formal disk at a point in a type in 3.3. These disks contain roughly similar information as the tangent and jet spaces in differential geometry. The definition is relative to a modality and for the n -truncation modality known as the connected cover of a homotopy type.
- We introduce a notion of homogeneous type in 3.9, which is tailored to our application as a basic building block for manifolds. It is proven that the formal disk bundle of a homogeneous type is trivial.
- Formally étale maps are defined in 3.14. Between manifolds, formally étale maps are known to correspond to local diffeomorphisms. We show stability properties of the class of formally étale maps, for example, closure under arbitrary pullbacks. This definition is again relative to a modality.
- Multiple definitions of fiber bundle are shown to be equivalent in 4.11. Notably, we show that if all fibers of a map are merely equal to a fixed type, then there is a trivializing cover.
- For homogeneous types V , we define V -manifolds in 4.14. They are spaces infinitesimally modeled on V .
- Finally, we define G -jet-structures in 4.22 and their moduli space for a given manifold. We also define torsion-free G -jet-structures and show that the trivial 1-jet-structure of a 1-group is torsion-free.

This project was suggested by Urs Schreiber in 2015 as a PhD thesis project for the author. The (external) definitions of formally étale maps, V -manifolds, and G -jet-structures have been used by Urs Schreiber and others. Our contribution is the formulation in homotopy type theory and type-theoretic solution of the proposed problems,² which allowed us to produce a theory of low complexity and high clarity, which is hard to imagine to be possible in a more classical framework like higher category theory in its simplicial incarnation.

Formalization

The formalization located here:

<https://github.com/felixwellen/DCHoTT-Agda>

covers everything up to and including the definition of G -jet-structures, but not definitions building on top of that. However, crucial ingredients for the construction of the moduli space of G -jet-structures and torsion-free G -jet-structures, like the chain rule, are checked. It turned out that the necessary engineering work to actually combine those ingredients is not justified by the gain in understanding. Furthermore, before the code is used as a basis for future work, it should be ported to a suitable library.

2. Modal Homotopy Type Theory

2.1 Terminology and notation

Mostly, we use the same terminology and notation as the HoTT-Book (The Univalent Foundations Program, 2013). However, there are a few exceptions. To denote terms of type $\prod_{x:A} B(x)$, we use the notation for λ -expressions from pure mathematics, i.e. $x \mapsto f(x)$. There are no implicit propositional truncations. If the propositional truncation of a statement is used, it is indicated by the word “merely.” Phrases like “for all” and “there is” are to be interpreted as \prod - and \sum -types. For example, the sentence

For all $x:A$, we have $t:B(x)$.

is to be read as the statement describing the term $(x:A) \mapsto t$ of type $\prod_{x:A} B(x)$. We sometimes write f_a for the application of a dependent function $f: \prod_{x:A} B(x)$ to $a:A$, instead of $f(a)$.

Furthermore, similar to (Shulman, 2018), when dealing with identity types, we avoid topology and geometry-related words. For example, we write “equality” instead of “path” and “2-cell” instead of “homotopy,” to avoid confusion with the notions of paths and homotopies for the classical geometric objects we like to study by including them in our theory as 0-types. We use $p \bullet q$ to denote the concatenation of equalities p and q . We say that x is unique with some properties if the type of all x with these properties is contractible.

2.2 Preliminaries from homotopy type theory

We use a fragment of the Type Theory from (The Univalent Foundations Program, 2013). Function extensionality is always assumed to hold. Furthermore, we assume a propositional truncation modality “ $\|_-\|$ ” and univalent universes.

In the next section, we will give axioms for a modality “ \Im ,” which will be assumed throughout the article. Some knowledge of the basic concepts in (The Univalent Foundations Program, 2013) is assumed. In addition, we will use more facts about pullbacks than presented in (The Univalent Foundations Program, 2013), which we will list in this section.

It is very useful to switch between pullback squares and equivalences over a morphism. We start with the latter concept.

Definition 2.1. Let $f:A \rightarrow B$ be a map and $P:A \rightarrow \mathcal{U}$, $Q:B \rightarrow \mathcal{U}$ be dependent types.

(a) A morphism over f or fibered morphism is a:

$$\varphi : \prod_{x:A} P(x) \rightarrow Q(f(x)).$$

(b) An equivalence over f or fibered equivalence is a:

$$\varphi : \prod_{x:A} P(x) \simeq Q(f(x)).$$

For every morphism over $f:A \rightarrow B$ as above, we can construct a square³

$$\begin{array}{ccc} \sum_{x:A} P(x) & \longrightarrow & \sum_{x:B} Q(x) \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ A & \xrightarrow{f} & B \end{array}$$

where the top map is given as $(a, p_a) \mapsto (f(a), \varphi_a(p_a))$. This square will turn out to be a pullback in the sense we are going to describe now, if and only if φ is an equivalence over f .

For a cospan given by the maps $f : A \rightarrow C$ and $g : B \rightarrow C$, we can construct a pullback square:

$$\begin{array}{ccc} \sum_{x:A,y:B} f(x) = g(y) & \xrightarrow{\pi_2} & B \\ \downarrow \pi_1 & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

Then, for any other completion of the cospan to a square

$$\begin{array}{ccc} X & \xrightarrow{\varphi_B} & B \\ \varphi_A \downarrow & \eta \nearrow & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

where $\eta : \prod_{x:X} g(x) = f(x)$ is a 2-cell letting it commute, an induced map to the pullback is given by $x \mapsto (\varphi_A(x), \varphi_B(x), \eta_x)$.

Definition 2.2. A square is given by four maps as above and a 2-cell like η . A square is a pullback square if the induced map described above is an equivalence.

To reverse the construction of a square for a morphism over “ f ” above, we can start with a general square:

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ p_A \downarrow & \swarrow \eta & \downarrow p_B \\ A & \xrightarrow{f} & B \end{array}$$

Let $P : A \rightarrow \mathcal{U}$ and $Q : B \rightarrow \mathcal{U}$ be the fiber types of the vertical maps, i.e.:

$$P(a : A) := \sum_{x:X} p_A(x) = a$$

$$Q(b : B) := \sum_{y:Y} p_B(y) = b$$

Then, for all $a : A$, a morphism $\varphi_a : P(a) \rightarrow Q(a)$ is given as:

$$\varphi_a((a, (x, p))) := (f(a), (g(x), \eta_x \bullet f(p))).$$

So φ is a morphism from P to Q over f . The following statement is quite useful and will be used frequently in this article:

Lemma 2.3.

- (a) A square is a pullback if and only if the induced fibered morphism is an equivalence.
- (b) A fibered morphism is an equivalence if and only if the corresponding square is a pullback.

Now, the following corollary can be derived by using the fact that equivalences are stable under pullback:

Corollary 2.4. Let $f : A \rightarrow B$ be an equivalence, $P : A \rightarrow \mathcal{U}$, $Q : B \rightarrow \mathcal{U}$ dependent types, and $\varphi : \prod_{x:A} P(x) \rightarrow Q(f(x))$ an equivalence over f . Then the induced map:

$$\left(\sum_{x:A} P(x)\right) \rightarrow \left(\sum_{x:B} Q(x)\right)$$

is an equivalence.

2.3 Modalities

From this section on, we will always assume a *modality* \mathfrak{S} . We use the definition of a uniquely eliminating modality from (Rijke, Shulman and Spitters, 2020), which is equivalent to the definition given in (The Univalent Foundations Program, 2013, Section 7.7). More on modalities and their relation to concepts in category theory can be found in (Rijke, Shulman and Spitters, 2020). We deviate from the usual symbol for modalities, which would be “ \bigcirc ” to remind us that while we will technically work with a general modality, we have some particular kind of modality in mind. Furthermore, the work in this article could be reused in a type theory that provides more modal operators from differential cohesion, for example, in the work in progress (Myers, 2022b), which also uses homotopy type theory as a basis and where \mathfrak{S} is called *crystalline modality*.

The modality \mathfrak{S} is also used in category theory based differential cohesion⁴ and is called infinitesimal shape.⁵

Axiom 2.5. *From this point on, we assume existence of a map $\mathfrak{S} : \mathcal{U} \rightarrow \mathcal{U}$ and maps $\iota_A : A \rightarrow \mathfrak{S}A$ for all types A , subject to this condition: For any $B : \mathfrak{S}A \rightarrow \mathcal{U}$, the map*

$$- \circ \iota_A : \left(\prod_{a:\mathfrak{S}A} \mathfrak{S}B(a)\right) \rightarrow \left(\prod_{a:A} \mathfrak{S}B(\iota_A(a))\right)$$

is an equivalence.

We call the inverse of the equivalence \mathfrak{S} -*elimination*. Elimination in type theory is a principle that lets us define maps starting in an inductive type like the natural numbers. For example, eliminating from the natural numbers \mathbb{N} to a dependent proposition $P : \mathbb{N} \rightarrow \mathcal{U}$ means essentially to prove the proposition for each possible way to construct a natural number, which is either to take it to be the constant 0 or the successor $s(n)$ of another natural number n .

The analogy to \mathfrak{S} -elimination is that to eliminate from $\mathfrak{S}A$ into the dependent modal type $\mathfrak{S}B(_)$, we only need to provide a value for the case that $x : \mathfrak{S}A$ is of the form $\iota_A(y)$. This is exactly what the inverse of the map in axiom 2.5 allows us to do. A different way to put this is that $\mathfrak{S}A$ has the same elimination principle as an inductive type with constructor $\iota_A : A \rightarrow \mathfrak{S}A$ would have, except that it can only be used to construct functions with modal codomain.

Note that it is possible to conclude a variant of \mathfrak{S} -elimination from axiom 2.5, where \mathfrak{S} is not applied to the type family B , but the type family is required to have values in \mathfrak{S} -modal types.⁶

Note that the equivalence in axiom 2.5 specializes to the universal property of a reflection if the family B is constant:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \mathfrak{S}A \\ & \searrow f & \downarrow \exists! \psi \\ & & \mathfrak{S}B \end{array}$$

i.e. for all types B and all $f : A \rightarrow \mathfrak{S}B$, we get a unique ψ letting the triangle commute up to a 2-cell. Unique means here that there is a contractible type of maps with 2-cells letting the triangle commute. That type is also a fiber of the equivalence “ $- \circ \iota_A$,” so we do know that it is contractible.

We will make use of this in showing that \mathfrak{S} is idempotent in the following sense:

Proposition 2.6. *For all types A , the map $\iota_{\mathfrak{S}A} : \mathfrak{S}A \rightarrow \mathfrak{S}(\mathfrak{S}A)$ is an equivalence.*

Proof. By the universal property we just discussed, we get a candidate for an inverse to $\iota_{\mathfrak{S}A}$, which we call φ :

$$\begin{array}{ccc} \mathfrak{S}A & \xrightarrow{\iota_{\mathfrak{S}A}} & \mathfrak{S}(\mathfrak{S}A) \\ & \searrow \text{id} & \downarrow \exists! \varphi \\ & & \mathfrak{S}A \end{array}$$

By construction, φ is already a left inverse of $\iota_{\mathfrak{S}A}$. We consider the diagram

$$\begin{array}{ccc} & \mathfrak{S}(\mathfrak{S}A) & \\ \iota_{\mathfrak{S}A} \nearrow & \downarrow \varphi & \searrow \text{id} \\ \mathfrak{S}A & \xrightarrow{\text{id}} & \mathfrak{S}A \\ \iota_{\mathfrak{S}A} \searrow & \downarrow \iota_{\mathfrak{S}A} & \nearrow \text{id} \\ & \mathfrak{S}(\mathfrak{S}A) & \end{array}$$

and conclude that φ is also a right inverse by uniqueness. □

Like reflections determine a subcategory, \mathfrak{S} determines a subuniverse of the universe \mathcal{U} of all types.⁷

Definition 2.7.

- (a) A type A is \mathfrak{S} -modal if ι_A is an equivalence.
- (b) The universe of \mathfrak{S} -modal types is

$$\mathcal{U}_{\mathfrak{S}} := \sum_{A:\mathcal{U}} (A \text{ is } \mathfrak{S}\text{-modal})$$

From what we proved above, all types $\mathfrak{S}A$ will be modal.

As we explained in the introduction, we will not be very interested in spaces of the form “ $\mathfrak{S}X$,” but more in the “quotient map” $\iota_X : X \rightarrow \mathfrak{S}X$, which we will view as identifying infinitesimally close points.

Like a functor, \mathfrak{S} extends to maps and we get a naturality square for ι :

Definition 2.8.

- (i) For any function $f : A \rightarrow B$ between arbitrary types A and B , we have a function:

$$\mathfrak{S}f : \mathfrak{S}A \rightarrow \mathfrak{S}B$$

given by \mathfrak{S} -elimination.

- (ii) For any function $f : A \rightarrow B$ between arbitrary types A and B , there is a 2-cell η witnessing that the following commutes:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \mathfrak{S}A \\ f \downarrow & \swarrow \eta_f & \downarrow \mathfrak{S}f \\ B & \xrightarrow{\iota_B} & \mathfrak{S}B \end{array}$$

It is also straightforward to prove that the application of \mathfrak{S} to maps commutes with composition of maps up to equality and preserves identities up to equality. In general, we expect that any coherence between these equalities needed in practice can be constructed.

Remark 2.9. For any 2-cell $\eta : f \Rightarrow g$, we have a 2-cell between the images:

$$\mathfrak{S}\eta : \mathfrak{S}f \Rightarrow \mathfrak{S}g.$$

\mathfrak{S} -Modal types have various closedness properties, which we review in the following lemma.

Proposition 2.10. Let A be any type and $B : A \rightarrow \mathcal{U}$ be such that for all $a : A$ the type $B(a)$ is \mathfrak{S} -modal.

- (a) Retracts of \mathfrak{S} -modal types are \mathfrak{S} -modal.
- (b) The dependent product

$$\prod_{a:A} B(a)$$

is \mathfrak{S} -modal. Note that A is not required to be \mathfrak{S} -modal here and this implies all function spaces with \mathfrak{S} -modal codomain are \mathfrak{S} -modal.

- (c) If A is \mathfrak{S} -modal, the sum

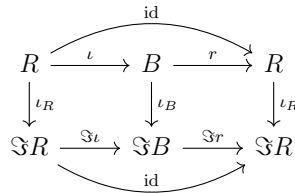
$$\sum_{a:A} B(a)$$

is \mathfrak{S} -modal.

- (d) \mathfrak{S} -modal types have \mathfrak{S} -modal identity types.

Proof.

- (a) A type R is a retract of B if there are maps $r : B \rightarrow R$ and $\iota : R \rightarrow B$, such that $r \circ \iota$ is equal to the identity. For all \mathfrak{S} -modal B and retracts R of B , we have the following diagram:



Since ι_B is an equivalence, it has an inverse, and by the diagram, $r \circ \iota_B^{-1} \circ \mathfrak{S}\iota$ is a biinverse to ι_R .

- (b) This is proved, up to equivalence, in (The Univalent Foundations Program, 2013, Theorem 7.7.7).
- (c) This is (The Univalent Foundations Program, 2013, Theorem 7.7.4). □
- (d) This is (Rijke, Shulman and Spitters, 2020, Lemma 1.25).

One immediate consequence is $\mathfrak{S}1 \simeq 1$ – this is the only provably \mathfrak{S} -modal type. We cannot expect to prove more types to be \mathfrak{S} -modal, since there is always the modality that maps all types to 1, so 1 *could* be the only \mathfrak{S} -modal type.

The following is a slight variation of (Rijke, Shulman, and Spitters, 2020) [Lemma 1.24], and plays a central role in the abstract (Wellen, 2018), which was the beginning of (Cherubini and Rijke, 2021):

Proposition 2.11. *Let A be a type and $B: \mathfrak{S}A \rightarrow \mathcal{U}$ a dependent type. Then the induced map is an equivalence:*

$$\mathfrak{S} \left(\sum_{x:A} B(\iota_A(x)) \right) \simeq \left(\sum_{x:\mathfrak{S}A} \mathfrak{S}(B(x)) \right).$$

A more category theoretic implication of this proposition is that for the map

$$\pi_1 : \left(\sum_{x:A} B(\iota_A(x)) \right) \rightarrow A$$

taking fibers commutes with application of \mathfrak{S} . Here, π_1 is an example of a formally étale map, which we will introduce in the next section. More abstractly, this relates to the principle in algebraic topology that homotopy fibers coincide with ordinary fibers of certain fibrations. This point is highlighted and used in (Myers, 2022a).

3. A basis for differential geometry

3.1 Formal disks

We will start to build geometric notions on top of the modality \mathfrak{S} and its unit ι . In the intended applications, the modality \mathfrak{S} provides us with a notion of infinitesimal proximity. To see if two points x, y in some type A are infinitesimally close to each other, we map them to $\mathfrak{S}A$ and ask if the images are equal.

Definition 3.1. *Let $x, y: A$. Then we have a type that could be read “ x is infinitesimally close to y ” and is given as:*

$$x \sim y \equiv (\iota_A(x) = \iota_A(y)).$$

Of course, this is in general not a proposition, but it is useful to think about $\iota_A(x) = \iota_A(y)$ in this way. The name “infinitesimally close” is a poor choice for a general modality,⁸ so the reader should keep in mind from now on that the terminology is adapted to a modality in the intended applications.⁹

It turns out that all morphisms of types already respect this notion of infinitesimal closedness, i.e., if two points are infinitesimally close to each other, their images are close as well.

Remark 3.2. *If $x, y: A$ are infinitesimally close, then for any map $f: A \rightarrow B$, the images $f(x)$ and $f(y)$ are infinitesimally close. More precisely, we have an induced function:*

$$\tilde{f} : (x \sim y) \rightarrow (f(x) \sim f(y))$$

Proof. We construct a map between the two types $\iota_A(x) = \iota_A(y)$ and $\iota_B(f(x)) = \iota_B(f(y))$. By 2.8, we can apply \mathfrak{S} to maps and get a map $\mathfrak{S}f: \mathfrak{S}A \rightarrow \mathfrak{S}B$. So we can apply $\mathfrak{S}f$ to an equality $\gamma: \iota_A(x) = \iota_A(y)$ to get an equality:

$$\mathfrak{S}f(\gamma) : \mathfrak{S}f(\iota_A(x)) = \mathfrak{S}f(\iota_A(y))$$

Again by 2.8, we know that we have a naturality square:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \mathfrak{S}A \\ f \downarrow & \swarrow \eta_f & \downarrow \mathfrak{S}f \\ B & \xrightarrow{\iota_B} & \mathfrak{S}B \end{array}$$

and hence equalities $\eta_f(x) : \mathfrak{S}f(\iota_A(x)) = \iota_B(f(x))$ and $\eta_f(y) : \mathfrak{S}f(\iota_A(y)) = \iota_B(f(y))$. This yields an equality of the desired type:

$$\eta_f(x)^{-1} \bullet \mathfrak{S}f(\gamma) \bullet \eta_f(y) \quad \square$$

A formal disk at a point is the “collection” of all other points infinitesimally close to it:

Definition 3.3. *Let A be a type and $a : A$. The type \mathbb{D}_a defined below in three equivalent ways is called the formal disk at a .*

(i) \mathbb{D}_a is the sum of all points infinitesimally close to a , i.e.:

$$\mathbb{D}_a \equiv \sum_{x:A} \iota_A(x) = \iota_A(a)$$

(ii) \mathbb{D}_a is the fiber of ι_A at $\iota_A(a)$.

(iii) \mathbb{D}_a is defined by the following pullback square:

$$\begin{array}{ccc} \mathbb{D}_a & \longrightarrow & 1 \\ \downarrow & \text{(pb)} & \downarrow \text{*\to} \iota_A(a) \\ A & \xrightarrow{\iota_A} & \mathfrak{S}A \end{array}$$

The characterization (iii) is a verbatim translation of its topos-theoretic analog (Schreiber [Definition 5.3.50]) to homotopy type theory. Therefore, in the model from (Schreiber), composing a function on a manifold M with $\iota_M(x)$ would yield an ∞ -order jet of that function. Jets are higher order analogs of tangent vectors and the derivatives of a function at a point x can still be observed on a formal disk around x . To say that formal disks are just infinitesimal extensions of the point is supported by the following observation.

Proposition 3.4. *For any $x : X$, we have $\mathfrak{S}(\mathbb{D}_x) = 1$.*

Proof. Using Proposition 2.11 and Proposition 2.10 (d), we compute:

$$\begin{aligned} \mathbb{D}_x &\equiv \sum_{y:X} \iota_X(x) = \iota_X(y) \\ &= \sum_{y:X} \mathfrak{S}(\iota_X(x) = (\iota_X(y))) \\ &= \mathfrak{S} \left(\sum_{z:\mathfrak{S}X} \iota_X(x) = z \right) \\ &= 1. \quad \square \end{aligned}$$

As morphisms of manifolds induce maps on tangent spaces, maps of types induce morphisms on formal disks, containing information on the derivatives of a morphism of all orders:

Remark 3.5. *If $f : A \rightarrow B$ is a map, there is a dependent function:*

$$df : \prod_{x:A} \mathbb{D}_x \rightarrow \mathbb{D}_{f(x)}$$

We denote the evaluation at $a : A$ with

$$df_a : \mathbb{D}_a \rightarrow \mathbb{D}_{f(a)}$$

and call it the (generalized) differential of f at a .

Proof. To define df , we take the sum over the map from 3.2:

$$df_a \equiv (x, \epsilon) \mapsto (f(x), \eta_f^{-1}(x) \bullet \Im f(\epsilon) \bullet \eta_f(x))$$

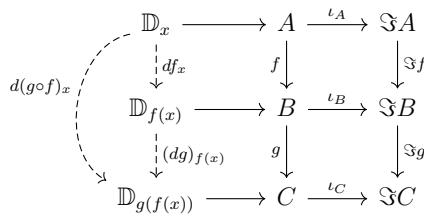
– where $\eta_f(x)$ is the equality from the naturality of ι . □

Some of the familiar rules for differentiation can be derived from this generality. We will need only the chain rule:

Lemma 3.6. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be maps. Then the following holds for all $x : A$*

$$d(g \circ f)_x = (dg)_{f(x)} \circ df_x.$$

Proof. Note that, in general, the differential df_x is equal to the map induced by the universal property of $\mathbb{D}_{f(x)}$ as a pullback. We can use this to get the desired “functoriality”:



– the induced map $d(g \circ f)_x$ and the composition $(dg)_{f(x)} \circ df_x$ solve the same factorization problem, so they are equal. □

In differential geometry, the tangent bundle is an important basic construction consisting of all the tangent spaces in a manifold, capturing first-order infinitesimal information. In this abstract all-order setting, we can mimic the construction by combining all the formal disks of a space in a bundle, capturing infinitesimal information of all orders at once.

Definition 3.7. *Let A be a type. The type $T_\infty A$ defined in one of the equivalent ways below is called the formal disk bundle of A .*

(i) $T_\infty A$ is the sum over all the formal disks in A :

$$T_\infty A \equiv \sum_{x:A} \mathbb{D}_x$$

(ii) $T_\infty A$ is defined by the following pullback square:

$$\begin{array}{ccc} T_\infty A & \longrightarrow & A \\ \downarrow & \text{(pb)} & \downarrow \iota_A \\ A & \xrightarrow{\iota_A} & \Im A \end{array}$$

Note that despite the seemingly symmetric second definition, we want $T_\infty A$ to be a bundle having formal disks as its fibers, so it is important to distinguish between the two projections and their meaning. If we look at $T_\infty A$ as a bundle, meaning a morphism $p : T_\infty A \rightarrow A$, we always take p to be the first projection in both cases. This convention agrees with the first definition – taking the sum yields a bundle with fibers of the first projection equivalent to the formal disks.

For any $f: A \rightarrow B$, we defined the induced map df on formal disks. This extends to formal disk bundles.

Definition 3.8. For a map $f: A \rightarrow B$, there is an induced map on the formal disk bundles, given as:

$$T_\infty f := (a, \epsilon) \mapsto (f(a), df_a(\epsilon))$$

In differential geometry, the tangent bundle may or may not be trivial. This is some interesting information about a space. If we have a smooth group structure on a manifold G , i.e. a Lie-group, we may consistently translate the tangent space at the unit to any other point. This may be used to construct an isomorphism of the tangent bundle with the projection from the product of G with the tangent space at the unit.

It turns out that this generalizes to formal disk bundles, and the group structure may be replaced by the weaker notion of a homogeneous type.

The notion of homogeneous type was developed by the author to satisfy two needs. The first is to match the intuition of a pointed space, that is equipped with a continuous family of translations that map the base point to any given point. The second need is to have just the right amount of data in all the proofs and constructions concerning homogeneous types. It has not been investigated in what circumstances this definition of homogeneous spaces coincides with the various notions of homogeneous spaces in geometry – apart from the obvious examples given below.

Definition 3.9. A type A is homogeneous if there are terms of the following types:

- (i) $e: A$
- (ii) $t: \prod_{x:A} A \simeq A$
- (iii) $p: \prod_{x:A} t_x(e) = x$

where t is called the family of translations and e is called the unit of A .

Examples 3.10.

- (a) Let G be a group in the sense of (The Univalent Foundations Program, 2013)[6.11], then G is a homogeneous type with $x \bullet _$ or $_ \bullet x$ as its family of translations.
- (b) Let G be an h -group, i.e. a type with a unit, operation, and inversion that satisfy the group axioms up to a 2-cell. Then G is a homogeneous type in the same two ways as above.
- (c) As a notable special case, for any type A and $*$: A , the loop space $* =_A *$ is homogeneous.
- (d) Let X be a connected H -space, then X is homogeneous, again in two ways. See (The Univalent Foundations Program, 2013)[8.5.2] and (Licata and Finster, 2014)[Section 4].
- (e) Let Q be a type with a quasigroup-structure, i.e. a binary operation $_ \bullet _$ such that all equations $a \bullet x = b$ and $x \bullet a = b$ have a contractible space of solutions, then Q is homogeneous if it has a left or right unit.

In the following, we will build a family of equivalences from one formal disk of a homogeneous type to any other formal disk of the space. We start by observing how equivalences and equalities act on formal disks.

Lemma 3.11.

- (a) If $f: A \rightarrow B$ is an equivalence, then

$$df_x: \mathbb{D}_x \rightarrow \mathbb{D}_{f(x)}$$

is an equivalence for all $x: A$.

(b) Let A be a type and $x, y: A$ two points. For any equality $\gamma: x = y$, we get an equivalence $\mathbb{D}_x \simeq \mathbb{D}_y$.

Proof.

(a) Let us first observe that for any $x, y: A$, the map $\iota_A(x) = \iota_A(y) \rightarrow \iota_B(f(x)) = \iota_B(f(y))$ is an equivalence. This follows from the fact that it is equal to the composition of two equivalences. One is the conjugation with the equalities from naturality of ι , the other is the equivalence of equalities induced by the equivalence $\mathfrak{S}f$.

Now, for a fixed $a: A$, we have two dependent types, $\iota_A(a) = \iota_A(x)$ and $\iota_B(f(a)) = \iota_B(f(x))$, and an equivalence over f between them. The sum of this equivalence over f is by definition df , and by 2.4, a sum of a fibered equivalence is an equivalence.

(b) The equivalence is just the transport in the dependent type $x \mapsto \mathbb{D}_x$. □

We are now ready to state and prove the triviality theorem.

Theorem 3.12. *Let V be a homogeneous type and \mathbb{D}_e the formal disk at its unit. Then the following is true:*

(a) *For all $x: V$, there is an equivalence*

$$\psi_x: \mathbb{D}_x \rightarrow \mathbb{D}_e$$

(b) *$T_\infty V$ is a trivial bundle with fiber \mathbb{D}_e , i.e., we have an equivalence $T_\infty V \rightarrow V \times \mathbb{D}_e$ and a homotopy commutative triangle*

$$\begin{array}{ccc} T_\infty V & \xrightarrow{\simeq} & V \times \mathbb{D}_e \\ \pi_1 \searrow & & \swarrow \pi_1 \\ & V & \end{array}$$

Proof.

(a) Let $x: V$ be any point in V . The translation t_x given by the homogeneous structure on V is an equivalence. Therefore, we have an equivalence $\psi'_x: \mathbb{D}_e \rightarrow \mathbb{D}_{t_x(e)}$ by 3.11. Also, directly from the homogeneous structure, we get an equality $t_x(e) = x$, and transporting along it yields an equivalence $\mathbb{D}_{t_x(e)} \rightarrow \mathbb{D}_x$. So we can compose and invert to get the desired ψ_x .

(b) By Definition 3.7 of the formal disk bundle, we have:

$$T_\infty V \equiv \sum_{x: V} \mathbb{D}_x$$

We define a morphism $\varphi: T_\infty V \rightarrow V \times \mathbb{D}_e$ by:

$$\varphi((x, \epsilon_x)) \equiv (x, \psi_x(\epsilon_x))$$

and its inverse by:

$$\varphi^{-1}((x, \epsilon_x)) \equiv (x, \psi_x^{-1}(\epsilon_x)).$$

Now, to see φ is an equivalence with inverse φ^{-1} , one has to provide equalities of types

$$\begin{aligned} (x, \epsilon_x) &= \varphi^{-1}(\varphi(x, \epsilon_x)) = (x, \psi^{-1}(\psi(\epsilon_x))) \\ \text{and } (x, \epsilon_e) &= \varphi(\varphi^{-1}(x, \epsilon_e)) = (x, \psi(\psi^{-1}(\epsilon_e))) \end{aligned}$$

– which exist since the ψ_x are equivalences by (a). □

In geometry, it is usually possible to add tangent vectors. Our formal disks can at least inherit the group-like properties of a homogeneous type:

Theorem 3.13. *Let A be homogeneous with unit $e : A$. Then \mathbb{D}_e is homogeneous.*

Proof. We look at the sequence

$$\mathbb{D}_e \xrightarrow{u_e} A \xrightarrow{\iota_A} \mathfrak{S}A$$

where u_e is the inclusion of the formal disk, given as the first projection. We will proceed by constructing a homogeneous structure on $\mathfrak{S}A$, note some properties of ι_A , which could be part of a definition of *morphism of homogeneous types*, and finally give some “kernel”-like construction of the structure on \mathbb{D}_e .

For $x : A$, there is a translation $t_x : A \simeq A$, since \mathfrak{S} preserves equivalences, this yields a $\mathfrak{S}t_x : \mathfrak{S}A \simeq \mathfrak{S}A$. By \mathfrak{S} -elimination, this extends to a family of translations

$$t' : \prod_{y:\mathfrak{S}A} \mathfrak{S}A \simeq \mathfrak{S}A, \text{ with } t'_{\iota_A(x)} = \mathfrak{S}t_x.$$

Let $e' := \iota_A(e)$, then $\mathfrak{S}A$ is homogeneous if we can produce a

$$p' : \prod_{y:\mathfrak{S}A} t'_y(e') = y.$$

By \mathfrak{S} -eliminating on y , we reduce the problem to

$$\prod_{x:A} t'_{\iota_A(x)}(\iota_A(e)) = \iota_A(x)$$

By definition, the left hand side is $\mathfrak{S}(t_x)(\iota_A(e))$ and by naturality of ι , we have an equality $\mathfrak{S}(t_x)(\iota_A(e)) = \iota_A(t_x(e))$. So by applying ι_A to the equality $p_x : t_x(e) = x$, we get a solution.

We start to construct the homogeneous structure on \mathbb{D}_e by letting $e'' := (e, \text{refl})$ be the unit. For the translations, we look at the dependent type $(x : A) \mapsto \iota_A(e) = \iota_A(x)$ and establish the following chain of equivalences for $y : A$ with $\iota_A(e) = \iota_A(y)$:

$$\begin{aligned} \iota_A(e) &= \iota_A(x) \\ &\simeq t'_{\iota_A(y)}\iota_A(e) = t'_{\iota_A(y)}\iota_A(x) \\ &\simeq t'_{\iota_A(y)}\iota_A(e) = \iota_A(t_y(x)) \\ &\simeq \iota_A(y) = \iota_A(t_y(x)) \\ &\simeq \iota_A(e) = \iota_A(t_y(x)) \end{aligned}$$

The resulting equivalence is an equivalence over t_y . So by 2.4, this induces an equivalence on the sum, which is \mathbb{D}_e .

This construction yields a family of equivalences $t'' : \prod_{y:\mathbb{D}_e} \mathbb{D}_e \simeq \mathbb{D}_e$. To finish the prove of the theorem, we need to construct a family of equalities $\prod_{x:\mathbb{D}_e} t''_x(e'') = x$. This is another computation using the same methods we have seen so far and we refer to the [formalization¹⁰](#) instead of giving the details here. □

3.2 Formally étale maps

The approach to formally étale maps presented here has been developed further in the ongoing synthetic algebraic geometry project.¹¹

In algebraic geometry, formally étale maps are supposed to be analogous to local diffeomorphisms in differential geometry. Below, we will give a not-so-well-known definition that matches the notion of algebraic geometry for finitely presented morphisms of schemes¹² and coincides with the local diffeomorphisms between manifolds in the case of differential geometry.¹³

Definition 3.14. A map $f : A \rightarrow B$ is formally étale if its naturality square is a pullback:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \mathfrak{S}A \\ f \downarrow & & \downarrow \mathfrak{S}f \\ B & \xrightarrow{\iota_B} & \mathfrak{S}B \end{array}$$

To see why this definition expresses that a map is an isomorphism on a infinitesimal scale, we can look at the following situation:

$$\begin{array}{ccccc} 1 & \longrightarrow & A & \xrightarrow{\iota_A} & \mathfrak{S}A \\ \downarrow & \nearrow \text{dashed} & \downarrow f & & \downarrow \mathfrak{S}f \\ \mathbb{D}_b & \longrightarrow & B & \xrightarrow{\iota_B} & \mathfrak{S}B \end{array}$$

– whenever we have a point $b : B$ that we can lift to A , there will be a unique lift of the whole formal disk around b to A by the universal property of the pullback A and the naturality of \mathfrak{S} .

This definition of formally étale maps was used extensively in (Schreiber) and (Khavkine and Schreiber, 2017).¹⁴ The same definition under different names was also used in category theory to study the relation between reflective subcategories and factorization systems (Cassidy, Hébert and Kelly, 1985). Here, the maps with a cartesian naturality square for the reflector are the right maps of a factorization system, whereas the left maps are those mapped to isomorphisms by the reflector. The factorization system can also be defined as a modality and studied internally (Cherubini and Rijke, 2021).

We will continue with some basic observations:

Lemma 3.15.

- (a) If $f : A \rightarrow B$ and $g : B \rightarrow C$ are formally étale, their composition $g \circ f$ is formally étale. If the composition $g \circ f$ and g are formally étale, then f is formally étale.
- (b) Equivalences are formally étale.
- (c) Maps between \mathfrak{S} -modal types are formally étale.
- (d) All fibers of a formally étale map are \mathfrak{S} -modal.

Proof.

- (a) By pullback pasting.
- (b) The naturality square for an equivalence is a commutative square with equivalences on opposite sides. Those squares are always pullback squares.
- (c) This is, again, a square with equivalences on opposite sides.
- (d) The pullback square witnessing $f : A \rightarrow B$ being formally étale yields an equivalence over ι_B . So, each fiber of f is equivalent to some fiber of $\mathfrak{S}f$. But fibers of maps between \mathfrak{S} -modal types are always \mathfrak{S} -modal by 2.10 (c), hence each fiber of f is equivalent to a \mathfrak{S} -modal type, thus itself \mathfrak{S} -modal. □

Together with the following, we have all the properties of formally étale maps needed in this article:

Lemma 3.16. *Let $f : A \rightarrow B$ be formally étale, then the following is true:*

- (a) *For all $x : A$, the differential df_x is an equivalence.*
- (b) *There is a pullback square of the following form:*

$$\begin{array}{ccc}
 T_\infty A & \longrightarrow & T_\infty B \\
 \downarrow & \text{(pb)} & \downarrow \\
 A & \xrightarrow{f} & B
 \end{array}$$

Proof.

- (a) The pullback square witnessing that f is formally étale can be reformulated as:
 For all $x : \mathfrak{S}A$, the induced map between the fibers of ι_A and ι_B is an equivalence. But these fibers are just the formal disks, so this can be applied to any $\iota_A(y)$ to see that df_y is an equivalence.
- (b) This is just a reformulation. □

One might wonder if the converse of this statement holds. With a mild condition on A , which is related to the concept of formal smoothness in algebraic geometry, this is the case:¹⁵

Remark 3.17. *Let A be a type such that $\iota_A : A \rightarrow \mathfrak{S}A$ is surjective and $f : A \rightarrow B$ any map. Then f is formally étale if df_x is an equivalence for all $x : A$*

Proof. As in the lemma, we use the equivalence of pullback squares and fibered equivalences. So to show that the ι -naturality square for f is a pullback, we have to show that for all $x : \mathfrak{S}A$, the induced map on fibers

$$\psi_x : \iota_A^{-1}(x) \rightarrow \iota_B^{-1}((\mathfrak{S}f)(x))$$

is an equivalence.

By surjectivity of ι_A , there is merely a \tilde{x} and $p : \iota_A(\tilde{x}) = x$. Since we show a proposition, we can use \tilde{x} and the equivalence $e_1 : \mathbb{D}_{\tilde{x}} \simeq \iota_A^{-1}(x)$. By naturality, we also have $e_2 : \mathbb{D}_{f(\tilde{x})} \simeq \iota_B^{-1}((\mathfrak{S}f)(x))$.

It remains to show that $\psi_x = e_2 \circ df_{\tilde{x}} \circ e_1^{-1}$. Induction on p simplifies e_1 and e_2 to the identity and we just have to note that $df_{\tilde{x}}$ was defined as a induced map on the fibers $\mathbb{D}_{\tilde{x}}$ and $\mathbb{D}_{f(\tilde{x})}$. □

The following is also proven in a different way in (Cherubini and Rijke, 2021) as corollary 5.2 (b).

Theorem 3.18. *Let $f : A \rightarrow B$ be formally étale and*

$$\begin{array}{ccc}
 A' & \longrightarrow & A \\
 f' \downarrow & \text{(pb)} & \downarrow f \\
 B' & \longrightarrow & B
 \end{array}$$

a pullback square. Then f' is formally étale.

Proof. Let us denote the bottom map with $\psi : B' \rightarrow B$. We start by describing A' as a pullback:

$$\begin{aligned} A' &\simeq \left(\sum f'^{-1} \right) \simeq \left(\sum f^{-1} \circ \psi \right) \simeq \left(\sum (\mathfrak{S}f)^{-1} \circ \iota_B \circ \psi \right) \\ &\simeq \left(\sum (\mathfrak{S}f)^{-1} \circ \mathfrak{S}\psi \circ \iota_{B'} \right) \end{aligned}$$

Now we can apply 2.11 to compute $\mathfrak{S}A'$:

$$\mathfrak{S}A' \simeq \mathfrak{S} \left(\sum (\mathfrak{S}f)^{-1} \circ \mathfrak{S}\psi \circ \iota_{B'} \right) \simeq \left(\sum (\mathfrak{S}f)^{-1} \circ \mathfrak{S}\psi \right)$$

Note that the right-hand side is the pullback of $\mathfrak{S}A$ along $\mathfrak{S}\psi$. This means that applying \mathfrak{S} to the pullback square given in the statement of the theorem is again a pullback, and by pullback pasting, the naturality square of f' is a pullback. \square

Corollary 3.19.

- (a) Let X be a type and $x : X$. The inclusion $\iota_x : \mathbb{D}_x \rightarrow X$ of the formal disk at x is a formally étale map.
- (b) Any pullback of a map between \mathfrak{S} -modal types is formally étale.

Proof. All maps between \mathfrak{S} -modal types are formally étale. Hence, the second statement follows from the theorem and the first follows as the special case for the map $\iota_X(x) : 1 \rightarrow \mathfrak{S}X$. \square

There is much more to be said about formally étale maps that is very useful, but not used in this article. One example that is interesting from a geometric perspective is that formally étale maps are the right class of a factorization system, whose left class are the \mathfrak{S} -equivalences. A consequence is that all maps can be factored into an \mathfrak{S} -equivalence followed by a formally étale map:

Remark 3.20. Let $f : A \rightarrow B$ be a map. The map f factors over

$$C_f := \sum_{x : \mathfrak{S}A, y : B} (\mathfrak{S}f)(x) = \iota_B(y)$$

by $l_f := (u : A) \mapsto (\iota_A(u), f(u), \eta_f)$ and $r_f := ((x, y, p) \mapsto y)$ to B . Furthermore, $\mathfrak{S}(l_f)$ is an equivalence and r_f is formally étale.

We sketch a proof – a full analysis of the factorization system can be found in (Cherubini and Rijke, 2021, Section 7).

Proof. Applying Proposition 2.11 twice on C_f shows that $\mathfrak{S}(l_f)$ is an equivalence. Additionally, r_f is formally étale since it is the pullback of the formally étale $\mathfrak{S}f$ along ι_B . \square

We will put formally étale maps to use in Section 4.2. The next section makes no reference to \mathfrak{S} .

4. Structures on V -manifolds

4.1 Fiber bundles

As mentioned in the introduction, the spaces we have in mind might have both a differential geometric structure and higher identity types. This section is about maps that correspond to fiber bundles, which are by definition locally trivial with respect to the higher identity or homotopical structure, but are expected to be locally trivial also with respect to a geometric structure that might be present in an application. By local triviality, we just mean that there is a surjection¹⁶ into the

base of the fiber bundle $p : E \rightarrow B$ such that pulling back along the surjection yields a projection from a product with a fixed given type F . It will turn out to be logically equivalent to ask all fibers of p to be merely equivalent to F .

In a basic intended applications, E and B might just be manifolds given by 0-types, and the reader might wonder if this notion of fiber bundle is too unrestrictive. However, it turns out that asking this *internally* turns into a surprisingly strong statement *externally*. In (Cherubini et al., 2023), it was discovered for a model based on a Grothendieck topos relevant to algebraic geometry that internal surjections have local sections with respect to the Grothendieck topology. It is reasonable to assume that similar principles work in differential geometry, and applying this to the surjective projection

$$\left(\sum_{x:B} \|p^{-1}(x) = F\| \right) \rightarrow B$$

would show that the fiber bundles defined in this section are actually locally trivial in the sense of classical definitions of fiber bundles. A ∞ -topos-theoretic version of this approach to fiber bundles may be found in (Nikolaus, Schreiber and Stevenson, 2015).

In this section, we will give four definitions of these fiber bundles¹⁷ and prove that they are equivalent. It will be useful in Section 4.3 to switch between the different definitions.

For the following statements about fiber bundles, we will make a lot of unavoidable use of a univalent universe \mathcal{U} and propositional truncation. We will frequently use that all maps of types $p : E \rightarrow B$ appear in a pullback square

$$\begin{array}{ccc} E & \longrightarrow & \tilde{\mathcal{U}} \\ p \downarrow & \text{(pb)} & \downarrow \\ B & \xrightarrow{p^{-1}} & \mathcal{U}, \end{array}$$

where $\tilde{\mathcal{U}}$ is called the *universal family* and obtained by summing over the dependent type $(A : \mathcal{U}) \mapsto A$. The bottom map p^{-1} determines p up to canonical equivalence over B and is called the *classifying map* of p . If E is a sum over a dependent type $q : B \rightarrow \mathcal{U}$, and p the projection to B , then q is the classifying map.

This way of using a univalent universe corresponds to looking at it as a *moduli space* or *classifying space* of types. We could replace the \mathcal{U} with some other moduli space to get specialized notions of fiber bundles with additional structure on the fibers.

Before we start, we will look at some preliminaries about surjective and injective maps. A surjective map is a map with merely inhabited fibers, or in other words, a $\|_-\|$ -connected map. An injective map has $\|_-\|$ -truncated fibers.¹⁸

Definition 4.1. Let $f : A \rightarrow B$ be a map of types.

(a) The map f is surjective if

$$\prod_{b:B} (\|f^{-1}(b)\| \simeq 1).$$

We write $f : A \twoheadrightarrow B$ in this case.

(b) The map f is injective if

$$\prod_{b:B} (f^{-1}(b) \text{ is a proposition}).$$

We write $f : A \hookrightarrow B$ in this case.

Lemma 4.2. *Surjective and injective maps are preserved by pullbacks.*

Proof. This is immediate by passing from pullback squares to fibered equivalences. □

Examples 4.3.

(a) *Let $f : A \rightarrow B$ be an equivalence of types. Then f is surjective and injective since all fibers of f are contractible.*

(b) *Let $P : A \rightarrow \mathcal{U}$ be a proposition. Then the projection*

$$\pi_1 : \sum_{a:A} P(a) \rightarrow A$$

is injective.

(c) *For the higher inductive type S^1 , the inclusion of the base point is a surjection.*

Lemma 4.4. *For any map $f : A \rightarrow B$ there is a unique triangle:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow e & \nearrow m \\ & \text{image}(f) & \end{array}$$

where e is surjective, m injective and $\text{image}(f)$ is given by

$$\text{image}(f) := \sum_{b:B} \left\| \sum_{a:A} f(a) = b \right\|.$$

A proof of the general case for $\|_n$ may be found in (The Univalent Foundations Program, 2013, chapter 7.6).

In Topology, an F -fiber bundle is a map $p : E \rightarrow B$ that is locally trivial with all its fibers are isomorphic to F . Local triviality means that B may be covered by open sets U_i , such that on each U_i the restricted map $p|_{p^{-1}(U_i)}$ is isomorphic to the projection $F \times U_i \rightarrow U_i$. We may rephrase this in a more economical way: From our cover, we construct a surjective map $w : \coprod_{i \in I} U_i \rightarrow B$. Then the local triviality translate to the pullback of p along w being isomorphic to the product projection $F \times \coprod_{i \in I} U_i \rightarrow \coprod_{i \in I} U_i$.

For fiber bundles in geometry, we would require more from a general surjective map, or *cover*, $w : W \rightarrow B$ than that pulling back along it turns p into a product projection. However, for the notion we discuss in this section, this turns out to be already enough.

Definition 4.5. *Let $p : E \rightarrow B$ be a map of types. For another map $w : W \rightarrow B$ we say w is a trivializing cover for p if w is a surjective map and there is a pullback square:*

$$\begin{array}{ccc} W \times F & \longrightarrow & E \\ \pi_1 \downarrow & \text{(pb)} & \downarrow p \\ W & \xrightarrow{w} & B \end{array}$$

The map p is called an F -fiber bundle if there merely is such a trivializing p .

We give an equivalent dependent version of this definition,¹⁹ which will be a lot easier to work with:

Definition 4.6. Let $E: B \rightarrow \mathcal{U}$ be a dependent type. We say that a surjection $w: W \rightarrow B$ is a trivializing cover for E if

$$\prod_{x: W} E(w(x)) \simeq F.$$

The dependent type E is called an F -fiber bundle if there merely is such a trivializing cover.

We can switch between the two definitions in the usual way: Given an F -fiber bundle $p: E \rightarrow B$ in the first sense, the dependent type of its fibers $p^{-1}: B \rightarrow \mathcal{U}$ will be an F -fiber bundle in the second sense, by direct application of 2.3. To go back, we take the projection from the sum of an F -fiber bundle $E: B \rightarrow \mathcal{U}$.

Note that in both cases, the propositional truncation of the trivializing datum is necessary to turn the definition into a proposition. In the following, we will see that we could have defined F -fiber bundles more easily with their classifying maps to a type called $\text{BAut}(F)$, providing us with a notion of F -fiber bundles, which is directly a proposition. However, in those definitions, while it is possible to construct a surjective trivializing map, it is unclear how we may require that this map has additional properties. One example, where we are interested in special surjections, is the definition of a V -manifold, where we will use formally étale surjections.

We review the type $\text{BAut}(F)$ now, which will be used to give the alternative definition of fiber bundles mentioned above:

Definition 4.7. Let F be a type and $t_F: 1 \rightarrow \mathcal{U}$ the map given by $* \mapsto F$.

- (a) Let $\text{BAut}(F) \equiv \text{image}(t_F)$.
- (b) We also have the injection $v_{\text{BAut}(F)}: \text{BAut}(F) \rightarrow \mathcal{U}$.
- (c) We use the notation $F//\text{Aut}(F) \equiv \sum_{(F', |\varphi|): \text{BAut}(F)} F'$, which is justified by the general fact that dependent sums over a map $\rho: BG \rightarrow \mathcal{U}$ are the homotopy quotient of $\rho(*)$ by the action of loops in BG via transport in ρ .

Remark 4.8. The first projection $\pi: F//\text{Aut}(F) \rightarrow \text{BAut}(F)$ is a pullback of $\tilde{\mathcal{U}} \rightarrow \mathcal{U}$ along $v_{\text{BAut}(F)}$. The map $\pi: F//\text{Aut}(F) \rightarrow \text{BAut}(F)$ is the universal F -fiber bundle, meaning all F -fiber bundles with any base will turn out to be pullbacks of this map.

We are now ready to give yet another definition of fiber bundles:

Definition 4.9. A map $p: E \rightarrow B$ is an F -fiber bundle if and only if there is a map $\chi: B \rightarrow \text{BAut}(F)$, such that there is a pullback square

$$\begin{array}{ccc} E & \longrightarrow & F//\text{Aut}(F) \\ p \downarrow & \text{(pb)} & \downarrow \pi \\ B & \xrightarrow{\chi} & \text{BAut}(F). \end{array}$$

In this case, χ is called the classifying map of p .

This definition also has a surprisingly easy dependent variant, which is obviously a mere proposition:

Definition 4.10. Let $E: B \rightarrow \mathcal{U}$ be a dependent type. We say E is an F -fiber bundle if

$$\prod_{b: B} \|E(b) \simeq F\|.$$

Again, we will switch between the dependent and non-dependent versions by taking fibers of p and the sum, respectively. To arrive at the dependent version, we can directly use the classifying morphism χ of an F -fiber bundle $p: E \rightarrow B$ to construct a term of

$$\prod_{b: B} \|p^{-1}(b) \simeq F\|,$$

since all points $\chi(b): \text{BAut}(F)$ are of the form (F', γ) , with $F' \simeq p^{-1}(b)$ by the pullback square and γ a proof that F' is merely equivalent to F .

Now, for the converse, let

$$E: B \rightarrow \mathcal{U}$$

be an F -fiber bundle by $t: \prod_{b: B} \|E(b) \simeq F\|$. Then the classifying map is given by $(x: B) \mapsto (E(b), t_x)$ and the pullback square is given by pasting:²⁰

$$\begin{array}{ccccc} & & & & \tilde{\mathcal{U}} \\ & & \text{---} & \text{---} & \downarrow \\ \sum E & \longrightarrow & F // \text{Aut}(F) & \longrightarrow & \tilde{\mathcal{U}} \\ \pi_1 \downarrow & & \pi \downarrow & \text{(pb)} & \downarrow \\ B & \xrightarrow{\chi} & \text{BAut}(F) & \longrightarrow & \mathcal{U}. \end{array}$$

We will conclude this section by showing that all our definitions of fiber bundles are equivalent and discuss some examples. The equivalence is most efficiently proven by establishing the equivalence of the two dependent definitions first:

Theorem 4.11. Let F be a type and $E: B \rightarrow \mathcal{U}$ be a dependent type, then

$$\prod_{b: B} \|E(b) \simeq F\|$$

if and only if there is a type W and a surjective $w: W \rightarrow B$ such that

$$\prod_{x: W} E(w(x)) \simeq F.$$

For the proof, we need to construct a trivializing cover at some point.²¹ The construction we use is similar to the universal cover and interesting on its own:

Definition 4.12. Let $E: B \rightarrow \mathcal{U}$ be an F -fiber bundle by $t: \prod_{b: B} \|E(b) \simeq F\|$, then

$$W: \equiv \sum_{b: B} E(b) \simeq F$$

together with its projection to B is the canonical trivializing cover of p .

The given t directly proves that this projection is surjective. Let us denote this projection by $w: W \rightarrow B$, then for all $x: W$, with $x = (b, e)$, we have

$$E(w(x)) \simeq E(\pi_1(b, e)) \simeq F$$

by transport and $e: E(b) \simeq F$ itself.

Proof (of 4.11). With the definition and remark above, it remains to show the converse. Let $E: B \rightarrow \mathcal{U}$ and $w: W \rightarrow B$ such that $t: \prod_{x: W} E(w(x)) \simeq F$. Now, for any $b: B$ and $x_b: w^{-1}(b)$, we get an equivalence $t_{\pi_1(x_b)}: E(w(\pi_1(x_b))) \simeq F$. By general properties of fibers, we have $w(\pi_1(x_b)) = b$ yielding $E(b) \simeq F$. By surjectivity of w , we merely have a $x_b: w^{-1}(b)$ for any $b: B$, therefore, we merely have an equivalence $E(b) \simeq F$. \square

Examples 4.13.

- (a) Let A be a pointed connected type, then any $E: A \rightarrow \mathcal{U}$ is an $E(*)$ -fiber bundle.²²
- (b) The map $1 \rightarrow S^1$ is a \mathbb{Z} -fiber bundle.
- (c) More generally, for a pointed connected type A , projection from the homotopical universal cover $\sum_{x:A} x = *$ to A is an ΩA -fiber bundle and the projection from $\sum_{x:A} \|x = *\|_1$ to A is a $\pi_1(A, *)$ -fiber bundle.
- (d) As $w: W \rightarrow B$ is a first projection, its fiber over any $b: B$ is equivalent to $E(b) \simeq F$. The latter type is merely equivalent to $\text{Aut}(F)$, since $E(b)$ is merely equivalent to F . This means w is an $\text{Aut}(F)$ -fiber bundle.

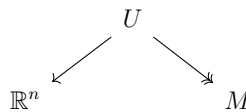
4.2 V-Manifolds

A smooth n -manifold is a space that is locally diffeomorphic to \mathbb{R}^n , Hausdorff, and second countable. A detailed comparison between the notion of V -manifold, which will be introduced below, and other notions of manifold may be found in (Khavkine and Schreiber, 2017)[3.3, 3.4] and (Myers, 2022b)[Section 5, p. 40 ff].

The definition of V -manifolds just mimics the property of being locally diffeomorphic to a fixed space, which we will only require to be homogenous (as defined in Definition 3.9). A covering $(U_i)_{i \in I}$ with $U_i \simeq \mathbb{R}^n$ of an n -manifold M yields a surjective local diffeomorphism

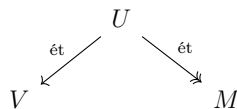
$$\coprod_{i \in I} U_i \rightarrow M.$$

By projecting, there is also a local diffeomorphism from $U := \coprod_{i \in I} U_i$ to \mathbb{R}^n . So in total, we have a span of local diffeomorphisms where the right one is surjective.



In applications, more general vector spaces might take the role of \mathbb{R}^n – so we will follow the literature and use the letter V in the more abstract definition of a V -manifold below. Instead of local diffeomorphisms, we will use formally étale maps. This is justified by the external calculation (Khavkine and Schreiber, 2017) [Proposition 3.2], which shows that formally étale maps between two smooth manifolds are exactly the local diffeomorphisms.

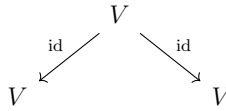
Definition 4.14. Let V be a homogeneous type. A type M is a V -manifold if there is a span



where the left map is formally étale and the right map is formally étale and surjective.

There is one trivial example:

Examples 4.15. Let V be a homogeneous type, then V is a V -manifold witnessed by the span:



Less obvious are the following two ways of producing new V -manifolds. However, without adding anything to our type theory to make the modality \mathfrak{S} more specific, we cannot hope for examples that are not given as homogeneous types. What could be added will be discussed at the beginning of the next section.

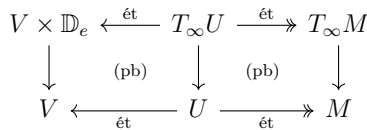
The statement in (a) is a variant of the classical fact that the tangent bundle of a manifold is a manifold, but in our case, the infinitesimal information is kept separate.

Lemma 4.16. Let V be homogeneous and M be a V -manifold.

- (a) The formal disk bundle $T_\infty M$ of M is a $(V \times \mathbb{D}_e)$ -manifold.
- (b) For any formally étale map $\varphi : N \rightarrow M$, N is a V -manifold.
- (c) If V' is a homogeneous V -manifold and N a V' -manifold, then N is also a V -manifold.²³

Proof.

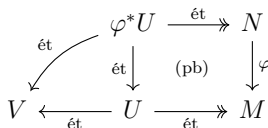
- (a) We can pull back the span witnessing that M is a V -manifold along the projection $T_\infty M \rightarrow M$:



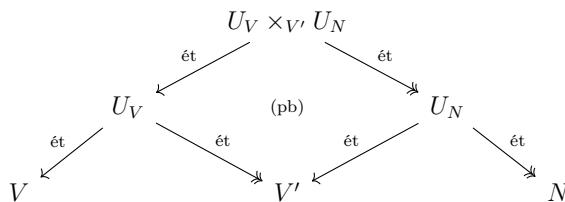
By 3.16 (b), we know that the pullback of the map $T_\infty U \rightarrow U$ is the projection from the formal disk bundle of U . Formally étale maps are preserved by pullbacks by 3.18 and surjective maps by 4.2, so the induced map $T_\infty U \rightarrow T_\infty M$ is formally étale surjective again.

By 3.12, we know that $T_\infty V = V \times \mathbb{D}_e$. So, again by 3.16 (b), we have the left pullback square. In 3.13, we showed that \mathbb{D}_e is homogeneous, so $V \times \mathbb{D}_e$ is homogeneous by giving it a componentwise structure.

- (b) Pullback along φ and composition gives us the following:



- (c) That N is a V -manifold is witnessed by the following diagram using preservation of surjections and formally étale maps under pullbacks:



□

One important special case of part (b) of the lemma is that any formal disk \mathbb{D}_x of M is a V -manifold.

In the following, let V be homogeneous and M be a fixed V -manifold. The definition of V -manifolds entails a stronger local triviality condition on the formal disk bundle of M than was discussed in the last section about F -fiber bundles since there has to be a formally étale trivializing cover.²⁴ This property of the trivializing cover will *not* be used in the following lemma.

Lemma 4.17.

(a) *The formal disk bundle of the covering U is trivial and there is a pullback square:*

$$\begin{array}{ccc} U \times \mathbb{D}_e & \longrightarrow & T_\infty M \\ \downarrow & \text{(pb)} & \downarrow \\ U & \longrightarrow & M \end{array}$$

(b) *The formal disk bundle of M has a classifying morphism $\tau : M \rightarrow \text{BAut}(\mathbb{D}_e)$, i.e., there is a pullback square: -*

$$\begin{array}{ccc} T_\infty M & \longrightarrow & \mathbb{D}_e // \text{Aut}(\mathbb{D}_e) \\ \downarrow & \text{(pb)} & \downarrow \pi \\ M & \xrightarrow{\tau_M} & \text{BAut}(\mathbb{D}_e) \end{array}$$

Proof.

1. By 3.16, there is a pullback square for the formally étale map to V :

$$\begin{array}{ccc} T_\infty U & \longrightarrow & T_\infty V \\ \downarrow & \text{(pb)} & \downarrow \\ U & \longrightarrow & V \end{array}$$

Since V is homogeneous, by 3.12, its formal disk bundle is trivial. This is preserved by pullback so $T_\infty U$ is trivial. The pullback square in the proposition is again given by 3.16.

2. The statement (a) tells us that $T_\infty M$ is a \mathbb{D}_e -fiber bundle by Definition 4.5. And (b) is just another way to state that fact, namely Definition 4.9. □

The classifying morphism τ_M is compatible with formally étale maps in the sense of the following remark.

Remark 4.18. *Let $\varphi : N \rightarrow M$ be formally étale, then N is also a V -manifold by 4.16. There is a 2-cell given by the differential of φ :*

$$\begin{array}{ccc} M & \xrightarrow{\tau_M} & \text{BAut}(\mathbb{D}_e) \\ \varphi \uparrow & \swarrow d\varphi & \nearrow \\ N & \xrightarrow{\tau_N} & \end{array}$$

Proof. We proved in Lemma 3.16 (a) that the differential of a formally étale map is an equivalence at all points. Applied to φ , this fact may be expressed in the following way:

$$d\varphi : \prod_{x:N} \mathbb{D}_x \simeq \mathbb{D}_{\varphi(x)}$$

$G \rightarrow GL(n)$	G -structure
$O(n) \rightarrow GL(n)$	Riemannian metric
$GL^+(n) \rightarrow GL(n)$	orientation
$O(n-1, 1) \rightarrow GL(n)$	pseudo-Riemannian metric
$SO(n, 2) \rightarrow GL(n)$	conformal structure
$GL(d, \mathbb{C}) \rightarrow GL(2d, \mathbb{R})$	almost complex structure
$U(d) \rightarrow GL(2d, \mathbb{R})$	almost Hermitian structure
$Sp(d) \rightarrow GL(2d, \mathbb{R})$	almost symplectic structure
$Spin(n) \rightarrow GL(n)$	spin structure

This yields a 2-cell of the desired type since the formal disks \mathbb{D}_x and $\mathbb{D}_{\varphi(x)}$ are merely equivalent to \mathbb{D}_e for all x . □

This will be useful when we work with G -jet-structures in the next section.

4.3 G-Jet-structures

Intuitively, the classifying morphism $\tau_M : M \rightarrow \text{BAut}(\mathbb{D}_e)$ of a V -manifold M describes how the formal disk bundle is glued together using automorphisms of \mathbb{D}_e . Lifts of τ_M along the delooping $BG \rightarrow \text{BAut}(\mathbb{D}_e)$ of a morphism from a group G will be called G -jet-structures.

Some simple, classical G -structures on \mathbb{R}^n -manifolds (or n -manifolds) only consider automorphisms of the tangent space, so their delooped automorphism group $\text{BGL}_n(\mathbb{R})$ ²⁵ takes the role of $\text{BAut}(\mathbb{D}_e)$ in our G -jet-structures. For an \mathbb{R}^n -manifold, the type $\text{BAut}(\mathbb{D}_e)$ will be a delooping of the infinite jet group $J_n^\infty(\mathbb{R})$. It is known (see e.g., (Kolař and Slovák, 1993, p. 131)), that the kernel of the projection $J_n^\infty(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$ is contractible. The projection also has a section given by extending linear automorphisms to the formal disk. This situation is nice enough, that we expect no problems with lifting our general classifying map $\tau_M : M \rightarrow \text{BAut}(\mathbb{D}_e)$, to a classical classifying map $M \rightarrow \text{BGL}_n(\mathbb{R})$ in the case of \mathbb{R}^n -manifolds – which would admit reusing the classical examples.

There are lots of interesting classical examples of structures on manifolds that can be encoded as G -structures. We give a list of examples of what group morphisms – which are almost always inclusions of subgroups – encode structures on a smooth n -manifold as G -structures. Some of the examples assume $n = 2d$.

For a definition of $O(n)$ - and $GL(d, \mathbb{C})$ -structures, see (Chern, 1966). Note that in all of the above examples, G is a 1-group,²⁶ yet our theory also supports higher groups. The string 2-group and the fivebrane 6-group are examples of higher G -structures of interest in physics. See (Sati et al., 2009) for details and references. In the classical theory, torsion-free G -structures are to G -structures what symplectic structures are to almost symplectic structures. We will give a candidate analog of torsion-freeness for G -jet-structures at the end of this section.

We will now turn to the formal treatment of G -jet-structures on V -manifolds and the construction of the moduli spaces of these structures. From now on, let V be a homogeneous type. As we learned in the last section in 4.17, the formal disk bundle of a V -manifold M is always classified by a morphism $\tau_M : M \rightarrow \text{BAut}(\mathbb{D}_e)$, where \mathbb{D}_e is the formal disk at the unit $e : V$. Since this is the only feature of a V -manifold that we need for the constructions in this section, we will work

with the following more general class of spaces, where D is an arbitrary type which takes the role of \mathbb{D}_e .

Definition 4.19. A type M is called formal D -space²⁷ if its formal disk bundle is a D -fiber bundle.

Remark 4.20.

- (a) Any V -manifold M is a formal \mathbb{D}_e -space.
- (b) Being a formal D -space is a proposition.

Proof.

- (a) This is 4.17.
- (b) One of the equivalent definitions of D -fiber bundle, 4.10, was directly a proposition:

$$(P : A \rightarrow \mathcal{U} \text{ is a } D\text{-fiber bundle}) \equiv \prod_{x:A} \|P(x) \simeq D\| \quad \square$$

We are interested in the case $D \equiv \mathbb{D}_e$ for $e : V$, meaning that M is a formal \mathbb{D}_e -space if $\prod_{x:M} \|\mathbb{D}_x \simeq \mathbb{D}_e\|$. In 4.16, we saw that we can “pullback” the structure of a V -manifold along a formally étale map. Formal \mathbb{D}_e -spaces behave the same way by virtue of the 2-cell we already saw in 4.18.

Lemma 4.21. Let M be a formal \mathbb{D}_e -space. For any formally étale $\varphi : N \rightarrow M$, N is also a formal \mathbb{D}_e -space and there is the triangle:

$$\begin{array}{ccc} M & \xrightarrow{\tau_M} & \text{BAut}(\mathbb{D}_e) \\ \varphi \uparrow & \swarrow d\varphi & \nearrow \\ N & \xrightarrow{\tau_N} & \end{array}$$

Proof. First, the triangle in the statement exists for a formally étale map between any types if $\text{BAut}(\mathbb{D}_e)$ is replaced with the universe:

$$\begin{array}{ccc} M & \xrightarrow{x \mapsto \mathbb{D}_x} & \mathcal{U} \\ \varphi \uparrow & \swarrow d\varphi & \nearrow \\ N & \xrightarrow{x \mapsto \mathbb{D}_x} & \end{array}$$

By assumption, we know that $(x : M) \mapsto \mathbb{D}_x$ lands in $\text{BAut}(\mathbb{D}_e)$. But φ is formally étale, so we have $d\varphi : \prod_{x:N} \mathbb{D}_x \simeq \mathbb{D}_{f(x)}$. The latter may be truncated and composed with $\tau_M : \prod_{x:M} \|\mathbb{D}_x \simeq \mathbb{D}_e\|$ to get $\tau_N : \prod_{x:N} \|\mathbb{D}_x \simeq \mathbb{D}_e\|$. So both maps to \mathcal{U} factor over $\text{BAut}(\mathbb{D}_e)$. \square

From now on, we assume that BG is a connected, pointed type and $(* =_{\text{BG}} *) \simeq G$. We will define G -jet-structures²⁸ or *reductions of the structure group*, a synonym hinting that in a lot of cases, G is a subgroup of $\text{Aut}(\mathbb{D}_e)$. We will not restrict ourselves to reductions to subgroups and look at general pointed maps $\text{BG} \rightarrow \text{BAut}(\mathbb{D}_e)$, which correspond to general group homomorphisms $G \rightarrow \text{Aut}(\mathbb{D}_e)$.

Definition 4.22. Let $\chi : \text{BG} \rightarrow \text{BAut}(\mathbb{D}_e)$ be a pointed map and M be a formal \mathbb{D}_e -space. A G -jet-structure on M is a map $\varphi : M \rightarrow \text{BG}$ together with a 2-cell $\eta : \chi \circ \varphi \Rightarrow \tau_M$:

$$\begin{array}{ccc}
 & & BG \\
 & \curvearrowright \varphi & \downarrow \chi \\
 M & \xrightarrow{\tau_M} & \text{BAut}(\mathbb{D}_e)
 \end{array}$$

We write

$$G\text{-str}(M) := \sum_{\varphi: M \rightarrow BG} (\chi \circ \varphi \Rightarrow \tau_M)$$

for the type of G -jet-structures on M .

The special case $G = 1$ turns out to be interesting – a 1-jet-structure on a formal \mathbb{D}_e -space is nothing else than a trivialization of the formal disk bundle, like we produced in 3.12 for any homogeneous type. This provides us with an example of a 1-jet-structure, whose construction is, in spite of the name we will give below, not entirely trivial.

Definition 4.23. *The trivial 1-jet-structure on V is the trivialization $\psi : \prod_{x:V} \mathbb{D}_e \simeq \mathbb{D}_x$ constructed in 3.12:*

$$\begin{array}{ccc}
 - \mapsto * & \rightarrow & B1 := 1 \\
 & \searrow \psi & \downarrow * \mapsto \mathbb{D}_e \\
 V & \xrightarrow{\tau_V} & \text{BAut}(\mathbb{D}_e)
 \end{array}$$

Since we have pointed maps, there is a triangle for any $\chi : BG \rightarrow \text{BAut}(\mathbb{D}_e)$:

$$\begin{array}{ccc}
 B1 & \xrightarrow{* \mapsto *} & BG \\
 * \mapsto \mathbb{D}_e \downarrow & \rightleftarrows & \downarrow \chi \\
 \text{BAut}(\mathbb{D}_e) & &
 \end{array}$$

So we can define a trivial structure in the same way as above for arbitrary G . Let us fix a pointed map $\chi : BG \rightarrow \text{BAut}(\mathbb{D}_e)$ from now on.

Definition 4.24. *Let $\mathcal{T} : \mathbb{D}_e \simeq \chi(*)$ be the transport along the equality witnessing that χ is pointed. The trivial G -jet-structure on V is given by $\psi'_x := \psi_x \circ \mathcal{T}$:*

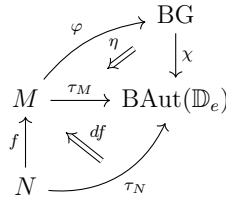
$$\begin{array}{ccc}
 - \mapsto * & \rightarrow & BG \\
 & \searrow \psi' & \downarrow \chi \\
 V & \xrightarrow{\tau_V} & \text{BAut}(\mathbb{D}_e)
 \end{array}$$

An important notion that we will introduce at the end of this section is a *torsion-free* G -structure. In some sense to be precise, these G -jet-structures will be trivial on all formal disks. Before we can do this, we need to be able to restrict G -jet-structures to formal disks, or more generally, to pull them back along formally étale maps.

Definition 4.25.

- (a) For M , a formal \mathbb{D}_e -space, and $f : N \rightarrow M$ a formally étale map from some type N , there is a map $f^* : G\text{-str}(M) \rightarrow G\text{-str}(N)$.
- (b) For the special case of formal disk inclusions $u_x : \mathbb{D}_x \rightarrow M$ and $\Theta : G\text{-str}(M)$, we call $u_x^* \Theta$ the restriction of Θ to the formal disk at x .

Construction 1 (of f^*). Let $\Theta \equiv (\varphi, \eta) : G\text{-str}(M)$. Then we can paste the triangle constructed in 4.18 to the triangle given by (φ, η) :

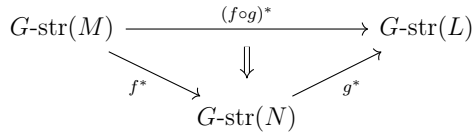


We define the result of the pasting to be $f^*(\varphi, \eta) : G\text{-str}(N)$. Or, put differently:

$$f^*(\varphi, \eta) \equiv (\varphi \circ f, (y : N) \mapsto \eta_{f(y)} \bullet df_y^{-1}). \quad \square$$

Pulling back G -jet-structures is 1-functorial in the following sense.

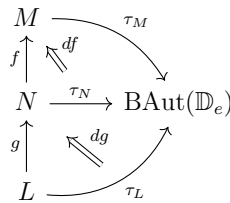
Remark 4.26. Let $f : N \rightarrow M, g : L \rightarrow N$ be formally étale, and M a formal \mathbb{D}_e -space. Then there is a triangle



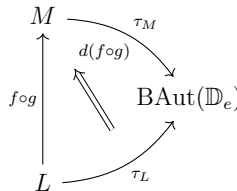
Proof. By 3.6, we have

$$d(f \circ g)_x = (df)_{g(x)} \circ dg_x.$$

In diagrams, this yields a 3-cell between the pasting of



and



This means the 2-cells we paste when applying $(f \circ g)^*$ or $g^* \circ f^*$ are equal, so the functions must be equal, too. \square

Let M be a fixed formal \mathbb{D}_e -space from now on. The final definition of this article is that of a torsion-free G -jet-structure. The aim is to ask if a G -jet-structure “looks like the trivial G -jet-structure everywhere on an infinitesimal scale.” To do this, we restrict a G -jet-structure to the

formal disk at a point and compare it to the trivial G -jet-structure on \mathbb{D}_e . So let us fix a notation for this structure:

Definition 4.27. Let $\xi : G\text{-str}(V)$ be the trivial G -jet-structure from 4.24 and $u_e : \mathbb{D}_e \rightarrow V$ the formal disk inclusion. Then

$$\xi_e := u_e^* \xi$$

is the trivial G -jet-structure on \mathbb{D}_e .

But a priori, we have no means of comparing G -jet-structures on formal disks with this trivial structure, so we need formally étale maps from all formal disks to \mathbb{D}_e . For formal \mathbb{D}_e -spaces, we merely have an equivalence from any formal disk to \mathbb{D}_e . More precisely, by 4.10, we have

$$\tau_M : \prod_{x:M} \|\mathbb{D}_x \simeq \mathbb{D}_e\|.$$

And by pulling back to the canonical cover $w : W \rightarrow M$ from 4.12, we get

$$\omega_M : \prod_{x:W} \mathbb{D}_{w(x)} \simeq \mathbb{D}_e$$

This is enough to make the indicated comparison.

Definition 4.28. A G -jet-structure Θ on M is torsion-free if

$$\prod_{x:W} \|(\omega_{M,w(x)}^{-1})^* u_{w(x)}^* \Theta = \xi_e\| \equiv: \text{torsion-free}(\Theta)$$

It turns out that even for the trivial 1-jet-structure on V , torsion-freeness is non-trivial. If the trivial 1-jet-structure is *left-invariant* as defined below, it is an example of a torsion-free 1-jet-structure. To match classic notions, we assume that the equivalences of the homogeneous structure are left translations.

Definition 4.29. The trivial G -jet-structure ξ on V is called *left-invariant*, if the following condition holds:

$$\prod_{x:V} t_x^* \xi = \xi$$

If our homogeneous space V is a Lie-Group, the trivial 1-jet-structure is constructed the same way as the Maurer–Cartan form, which satisfies the equation above. Turning this around, we get the following example:²⁹

Theorem 4.30. Let V be a 1-group and its homogeneous structure be given by left translations, then the trivial G -jet-structure given by this homogeneous structure is left-invariant.

Proof. We will use the following equation given by the group structure:

$$t_{t_x(y)} = t_{xy} = t_x \circ t_y$$

Evaluating at e and using the chain rule 3.6 yields:

$$d(t_{t_x(y)})_e = (dt_x)_{t_y(e)} \circ (dt_y)_e = (dt_x)_y \circ (dt_y)_e = (dt_y)_e \bullet (dt_x)_y$$

The latter equality is just moving our equation to $\text{BAut}(\mathbb{D}_e)$.

Now for the trivial G -jet-structure $\xi \equiv (_ \mapsto *, y \mapsto (dt_y)_e)$, we can calculate

$$\begin{aligned} t_x^* \xi &= (_ \mapsto *, y \mapsto (dt_{t_x(y)})_e \bullet (dt_x)_y^{-1}) \\ &= (_ \mapsto *, y \mapsto ((dt_y)_e \bullet (dt_x)_y) \bullet (dt_x)_y^{-1}) \\ &= (_ \mapsto *, y \mapsto (dt_y)_e) \\ &= \xi \end{aligned}$$

□

Theorem 4.31. *Let V be a homogeneous space such that the trivial G -jet-structure is left-invariant, then the trivial G -jet-structure on V is torsion-free.*

Proof. Let t_x be the translation to $x : V$ given by the homogeneous structure on V and $\xi \equiv (_ \mapsto *, x \mapsto (dt_x)_e)$ the trivial G -jet-structure on V . Then for all $x : V$, we have a square of formally étale maps:

$$\begin{array}{ccc} \mathbb{D}_x & \xrightarrow{u_x} & V \\ dt_x \uparrow & & \uparrow t_x \\ \mathbb{D}_e & \xrightarrow{u_e} & V \end{array}$$

By 4.26, we get the following formula:

$$u_e^* t_x^* \xi = dt_x^* u_x^* \xi$$

By 4.30, we can simplify the left-hand side:

$$u_e^* \xi = dt_x^* u_x^* \xi$$

The left-hand side is the trivial structure on \mathbb{D}_e and we have to identify the right-hand side with the term $(\omega_{M,x}^{-1})^* u_{\omega(x)}^* \xi$ from 4.28, where $\omega = \text{id}_V$. This amounts to an identification $dt_x^* u_x^* \xi = u_x^* \xi$, which is given by 3.12. □

Since torsion-freeness – as we defined it – is a proposition, the type of torsion-free G -jet-structures is a subtype of the type of G -jet-structures. The latter should be distinguished from the *moduli space* of G -jet-structures on M , which is the quotient of the type of G -jet-structures by the action of the automorphism group of M . If M is a 0-type, we could just build this quotient as a higher inductive type, but this is a bit unsatisfactory and not the most pleasant definition to work with. A more promising approach is to use that the quotient of an action given as a dependent type $\rho : \text{BG} \rightarrow \mathcal{U}$ is just $\sum_{x:\text{BG}} \rho(x)$. To make this approach work, the author reformulated a lot of the original theory in (Wellen, 2017). With the present version, we will see that this construction works without considerable effort.

To realize the construction of the moduli space as a dependent sum, we need to note that the definition of G -jet-structures is actually a dependent sum over $\text{BAut}(M)$.

Lemma 4.32. *There is a dependent type $G\text{-str} : \text{BAut}(M) \rightarrow \mathcal{U}$ with $G\text{-str}(M')$ being the G -jet-structures on M' .*

Proof. Since any $M' : \text{BAut}(M)$ is equivalent to M , it is merely a formal \mathbb{D}_e -space. Being a formal \mathbb{D}_e -space is a proposition, so $G\text{-str}(M')$ is defined as desired. □

This means that we can now construct the moduli spaces of G -jet-structures and torsion-free G -jet-structures in a nice way:

Definition 4.33. Let M be a formal \mathbb{D}_e -space and $\chi : \text{BG} \rightarrow \text{BAut}(\mathbb{D}_e)$ a pointed map.

(a) The moduli space of G -jet-structures on M is given as:

$$\sum_{M': \text{BAut}(M)} G\text{-str}(M').$$

(b) The moduli space of torsion-free G -jet-structures on M is given as:

$$\sum_{M': \text{BAut}(M)} \sum_{\Theta: G\text{-str}(M')} \text{torsion-free}(\Theta).$$

Conclusion

While we did not further discuss this, we expect that the homotopy type theory developed here has interpretation in suitable ∞ -toposes equipped with a fibered idempotent ∞ -monad. Our abstract construction of moduli spaces of torsion-free G -jet-structures should then have a translation to a corresponding construction internal to any of these ∞ -toposes. When written out in terms of traditional higher category theory, say as simplicial sheaves, these objects will look rather complicated and be cumbersome to work with. Our abstract language should hence serve to make the development of higher Cartan Geometry in ∞ -toposes tractable.

In addition to having little restrictions on models, the abstract way of working with just one monadic modality is also very clear and very suitable for formalization, since no axioms have to be postulated. Yet the author does not believe that this line of work should be continued on the level of abstraction used in this article. In the recent, more concrete framework from (Cherubini et al., 2023), admitting synthetic treatment of algebraic geometry, calculations were crucial in the advances made. The approach to cohomology developed there is likely to have a differential geometric analog. It should be fruitful to find extensions of the common axioms of synthetic differential geometry inspired by that research. In such an extension, it might be possible to show that the fiber bundles defined in Definition 4.5 are actually fiber bundles in the usual sense of local triviality with respect to a topology.

The idea for the approach to cohomology in (Cherubini et al., 2023) was to use the “higher-topos” approach³⁰ of just mapping into a higher type, usually an Eilenberg–MacLane space, and analyzing the 0-truncation of the resulting function type. This is quite easy to implement in homotopy type theory and was considered early in the history of the subject.³¹ One important insight from (Cherubini et al., 2023) is that the notion of local triviality, which comes from the topology of a (higher) sheaf topos, is internally accessible by a choice principle that is somewhat similar to the weakly initial set of covers axiom from (Berg and Moerdijk, 2013). Thus, finding the proposed extension of synthetic differential geometry amounts to checking if there are choice principles in differential geometry that can be used to make cohomological calculations. In the synthetic differential geometry, there already is an axiom called the “covering principle,” which would be a consequence of any reasonable choice axiom for synthetic differential geometry.³²

The connection to synthetic algebraic geometry does not stop at cohomology – there is also synthetic algebro geometric work on formally étale maps (Cherubini et al., 2024) which suggests an extension of the Kock–Lawvere axiom for synthetic differential geometry. If it can be shown that such an extension is supported by models, the theory in the last of a series of articles by Myers (Myers, 2022a), (Myers, 2021), and (Myers, 2022b) could be simplified.

The author believes that the best way to continue the work presented in this article is to find an extension of the usual axioms of synthetic differential geometry and use all of the recent advances to compute examples of V -manifolds, maps between them, cohomology groups, G -jet-structures, and their moduli spaces. We expect that when working internally, computations are a lot more

feasible and should be used to establish correspondences to the classical theory as well as to guide an expansion of the work in this article into a synthetic higher Cartan geometry.

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Schreiber explained a lot of mathematics important to this work to the author on his many visits in Bonn and answered countless questions via email.

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Competing interests. There are no competing interests.³³

Notes

- 1 This ensures that the nilradical is finitely generated – if this is not the case, the definition of \mathfrak{N} becomes more complicated.
- 2 These were the triviality of the formal disk bundle on a homogeneous type, local triviality of the formal disk bundle of a V -manifolds, and definition of G -structures and torsion-free G -structures.
- 3 By stating that it is a “square,” we implicitly assume that there is a 2-cell letting it commute, which is considered to be part of the square. In this particular case, the 2-cell is trivial.
- 4 For example in (Schreiber) and (Khavkine and Schreiber, 2017).
- 5 In the literature outside of differential cohesion, there is also the name “deRham stack.”
- 6 See Definition 2.7.
- 7 We implicitly assume a hierarchy of universes \mathcal{U}_i , but only mention indices if there is something interesting to say about them.
- 8 The concepts we will build up in this section are still of interest for other modalities, but this will be less and less true toward the end of this article.
- 9 Which we describe in the introduction.

- 10 <https://github.com/felixwellen/DCHoTT-Agda/blob/master/ImHomogeneousType.agda>
- 11 <https://github.com/felixwellen/synthetic-zariski/blob/main/README.md>
- 12 This is in the appendix of (Cherubini and Rijke, 2021)
- 13 See (Khavkine and Schreiber, 2017, Proposition 3.2) for a precise statement in an intended model.
- 14 One of the authors of these references, Schreiber, learned the definition from notes of Kontsevich and Rosenberg on Q -categories and communicated it to the author of this article.
- 15 Formulation and proof of this remark are a result of a discussion with Hugo Moeneclaey.
- 16 Internal surjections correspond to effective epimorphisms in a topos, so in a topos of sheaves, the topology *does* play a role in the internal surjections.
- 17 Some of the following definitions of fiber bundles were also used early in the short history of homotopy type theory at least by Mike Shulman, Ulrik Buchholtz, and Egbert Rijke.
- 18 Note that in a sheaf topos, this notion corresponds to epimorphisms and not to a pointwise surjective map. In (The Univalent Foundations Program, 2013, chapter 7), surjective maps are called (-1) -connected or also surjective if their domain and codomain are 0-types. Topos-theoretic analogs are defined in (Lurie, 2009, 6.5.1.10, 5.5.6.8) and are called 0-connective and (-1) -truncated. In the terminology of (Schreiber) or (The nLab, 2000) and in (Wellen, 2017) surjective maps would be 1-epimorphisms and injective maps 1-monomorphisms.
- 19 Following a suggestion from Max S. New <http://maxsnew.github.io/>.
- 20 Note that the outer rectangle is a pullback for all dependent types.
- 21 The author has to thank Ulrik Buchholtz for asking if such a cover always exists.
- 22 Thanks to Egbert Rijke for pointing this out.
- 23 This was a question by Ulrik Buchholtz.
- 24 “Trivializing cover” was defined in Definition 4.5 and Definition 4.6.
- 25 Note that $GL_n(\mathbb{R})$ and other groups appearing in the table below are *set-level* structures in the intended applications, so there is no problem with defining them.
- 26 We call a 0-type with a group structure a 1-group.
- 27 The name was invented by Urs Schreiber and the author for the present purpose.
- 28 See the introduction to this Section 4.3 for an explanation of the appearance of “jets” at this point.
- 29 This example and its presentation are a result of a discussion with Urs Schreiber.
- 30 The introduction (Lurie, 2009) explains how higher toposes offer a good perspective on cohomology.
- 31 Written down by Michael Shulman in a [blog post](#) that focuses on the connection with the axiom of choice.
- 32 Pointed out by David Jaz Myers in a discussion after an online talk. In his comment, Myers also implicitly suggests a choice principle based on countable covers for the metric topology.
- 33 This remark was added because the submission process of MCSC requires it.

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