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FORMAL SIEGEL MODULAR FORMS FOR ARITHMETIC SUBGROUPS

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ABSTRACT. The notion of formal Siegel modular forms for an arithmetic subgroup Γ of the symplectic group of genus n is a generalization of symmetric formal Fourier-Jacobi series. Assuming an upper bound on the affine covering number of the Siegel modular variety associated with Γ, we prove that all formal Siegel modular forms are given by Fourier-Jacobi expansions of classical holomorphic Siegel modular forms. We also show that the required upper bound is always met if $2 \leq n \leq 4$. As an application we consider the case of the paramodular group of squarefree level and genus 2.

1. INTRODUCTION

Let f be a holomorphic Siegel modular form of weight k for the full Siegel modular group $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ of genus n, that is, a holomorphic function on the Siegel upper half space $\mathbb{H}_n$ which transforms in weight k under the action of $\mathrm{Sp}_n(\mathbb{Z})$. For any integer l with $0 \leq l \leq n$, the function f has a Fourier-Jacobi expansion of cogenus l of the form

$$f(\tau) = \sum_{T_2} \phi_{T_2}(\tau_1, \tau_{12}) e(\mathrm{tr}(T_2 \tau_2)), \tag{1.1}$$

where we have written the variable $\tau \in \mathbb{H}_n$ in block form as

$$\tau = \begin{pmatrix} \tau_1 & \tau_{12} \\ {}^t\tau_{12} & \tau_2 \end{pmatrix}$$

with $\tau_1 \in \mathbb{H}_{n-l}$, $\tau_2 \in \mathbb{H}_l$, and $\tau_{12} \in \mathbb{C}^{(n-l)\times l}$. Moreover, T_2 runs over all positive semidefinite symmetric half-integral $l \times l$ matrices. The series (1.1) converges normally on $\mathbb{H}_n$. In the special case $l = n$ it reduces to the usual Fourier expansion of f.

The transformation law of f under the stabilizer in $\mathrm{Sp}_n(\mathbb{Z})$ of the standard boundary component of degree $n - l$ implies that the coefficients ϕ_{T_2} are Jacobi forms of weight k, index T_2, and genus $n - l$. In particular, they possess Fourier expansions of the form

$$\phi_{T_2}(\tau_1, \tau_{12}) = \sum_{T_1, T_{12}} a \begin{pmatrix} T_1 & T_{12} \\ {}^tT_{12} & T_2 \end{pmatrix} e\left(\mathrm{tr}(T_1\tau_1) + 2\,\mathrm{tr}(T_{12}{}^t\tau_{12})\right). \tag{1.2}$$

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Here T_1 runs through positive semidefinite symmetric half-integral $(n-l)\times(n-l)$ matrices, and T_{12} runs through $(n-l)\times l$ matrices with half-integral entries. Inserting these expansions into (1.1), one recovers the usual Fourier expansion of f, which explains our particular notation for the Fourier coefficients in (1.2). The transformation law of f under the Siegel parabolic subgroup (the stabilizer of the standard boundary component of degree 0) implies that the Fourier coefficients have the symmetry property

$$a({}^t uTu) = \det(u)^k a(T) \tag{1.3}$$

for all $T = \left(\begin{smallmatrix} T_1 & T_{12} \\ {}^tT_{12} & T_2 \end{smallmatrix}\right)$ and all $u \in \mathrm{GL}_n(\mathbb{Z})$.

In recent works, several authors considered formal analogues of Fourier-Jacobi expansions. A *formal* Fourier-Jacobi series f of weight k and cogenus l for the group $\mathrm{Sp}_n(\mathbb{Z})$ is a formal series as in (1.1), where the coefficients ϕ_{T_2} are holomorphic Jacobi forms of weight k, index T_2, and genus $n-l$, but where no convergence of the series is required. By considering the Fourier expansions of the ϕ_{T_2} as in (1.2) and inserting them into the series (1.1), one obtains a formal Fourier expansion of f. Recall that f is called *symmetric*, if the Fourier coefficients satisfy the symmetry condition (1.3) for all half-integral symmetric metrices T and all $u \in \mathrm{GL}_n(\mathbb{Z})$.

In particular, every holomorphic Siegel modular form of weight k for $\mathrm{Sp}_n(\mathbb{Z})$ defines a corresponding symmetric formal Fourier-Jacobi series of cogenus l. The main result of [BR] states that for $n > 1$ and for every l with $0 < l < n$, the converse also holds: Every symmetric formal Fourier-Jacobi series of weight k and cogenus l for $\mathrm{Sp}_n(\mathbb{Z})$ arises as the Fourier-Jacobi expansion of a classical holmorphic Siegel modular form.

In the special case when $n = 2$ this result was first proved by Aoki in [Ao]. It was generalized to vector valued symmetric formal Fourier-Jacobi series for $\mathrm{Sp}_2(\mathbb{Z})$ in [Br] and [Rau]. The case of the paramodular group of genus 2 and level ≤ 4 was considered in [IPY]. Part of the argument of [BR] is revisited in [Kr] using the arithmetic theory of Siegel modular forms of Faltings-Chai. Recent work of Xia deals with the somewhat analogous case of the unitary group $\mathrm{U}(n,n)$ over a norm-euclidian imaginary quadratic field [Xia]. Note that all these works deal with (vector valued) formal Fourier-Jacobi series for the full Siegel modular group of level 1. The proofs mainly rely on analytic techniques, such as bounds on the dimension of the space of symmetric formal Fourier-Jacobi series as the weight goes to infinity.

The purpose of the present paper is two-fold. First, we present a new approach to the problem. It is based on interpreting symmetric formal Fourier-Jacobi series as global sections of the line bundle of modular forms on the formal complex space given by the completion of the Satake compactification at its boundary. Second, we generalize the notion of symmetric formal Fourier-Jacobi series to arithmetic subgroups of the Siegel modular group and prove modularity results in this context. We now describe our results in more detail.

Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{Q})$ be an arithmetic subgroup, that is, a subgroup which is commensurable with $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$. Recall that the rational closure $\mathbb{H}_n^*$ of $\mathbb{H}_n$ is the disjoint union of $\mathbb{H}_n$ with all its proper rational boundary components. If $\mathbb{H}_n^*$ is equipped with the cylindrical topology, then Γ acts properly discontinuously on it, extending the action on $\mathbb{H}_n$ by fractional linear transformations. The Satake compactification of the Siegel modular variety $X_\Gamma = \Gamma\backslash\mathbb{H}_n$ is given by the quotient $X_\Gamma^* = \Gamma_n\backslash\mathbb{H}_n^*$, equipped with the Satake complex structure. It is a normal complex space, which has a natural structure as a projective algebraic variety over $\mathbb{C}$. The Satake boundary is a closed subspace of codimension n.

For $0 \leq l \leq n$, let I_l be the set of all rational boundary components of degree l. We define a *formal Siegel modular form* of weight k and cogenus l for the group Γ to be a family $(f_F)_{F \in I_{n-l}}$, where f_F is a formal Fourier-Jacobi series of weight k for the boundary component F and the group Γ satisfying the following conditions:

(i) for all $F \in I_{n-l}$ and all $\gamma \in \Gamma$ we have $f_F \mid_k \gamma = f_{\gamma^{-1} F}$;
(ii) for all pairs $F, F' \in I_{n-l}$ and all degree 0 boundary components $E \in I_0$ which are adjacent to F and F', the series f_F and $f_{F'}$ are compatible at E.

Here, the action of Γ on f_F in (i) is defined by the action on the coefficients of the formal Fourier-Jacobi series. The compatibility condition in (ii) means that f_F and $f_{F'}$ have the same formal Fourier expansion at E, see Section 3.1 and Definition 3.5 for details. We write $\mathrm{FM}_k^{(n,l)}(\Gamma)$ for the complex vector space of formal Siegel modular forms of weight k and cogenus l for Γ.

In the special case when $\Gamma = \Gamma_n$ is the full integral symplectic group and F_{n-l} is the standard rational boundary component of degree $n-l$, a formal Fourier-Jacobi series of weight k for the boundary component F_{n-l} and the group Γ_n is just a formal Fourier-Jacobi series of cogenus l as considered before. Using the fact that Γ_n acts transitively on I_{n-l}, it is easily seen that the compatibility condition (ii) implies the symmetry condition (1.3). Hence formal Siegel modular forms of weight k and cogenus l for Γ_n can be identified with symmetric formal Fourier-Jacobi series of the same type.

In Section 3.3 we give an algebraic geometric description of formal Siegel modular forms of cogenus 1. Let ω be the sheaf of modular forms of weight 1 on X_Γ^*. Let $\hat{X}_\Gamma^*$ be the formal complex space given by the completion of X_Γ^* at the Satake boundary $Y = X_\Gamma^* \setminus X_\Gamma$, and write

$$i : \hat{X}_\Gamma^* \to X_\Gamma^*$$

for the natural morphism of formal complex spaces. We denote by $\hat{\omega}^{\otimes k}$ the completion of the sheaf $\omega^{\otimes k}$ of modular forms of weight k with respect to Y. Since $\omega^{\otimes k}$ is coherent, the natural map $i^*(\omega^{\otimes k}) \to \hat{\omega}^{\otimes k}$ is an isomorphism. We provide an explicit description of the sections of $\hat{\omega}^{\otimes k}$ over a small open neighborhood of any rational boundary component F in Proposition 3.8. To this end we use the Grothendieck comparison theorem in the category of formal complex analytic spaces [Ba, Theorem 2] to reduce the computation to a smooth toroidal compactification. In particular, we find that for any rational boundary component F of degree $n-1$, the sections of $\hat{\omega}^{\otimes k}$ over a small open neighborhood of F can be identified with formal Fourier-Jacobi series of weight k for F. Moreover, these series satisfy the compatibility condition (ii) above. Hence, we obtain an injective linear map

$$\hat{\omega}^{\otimes k}(\hat{X}_\Gamma^*) \to \mathrm{FM}_k^{(n,1)}(\Gamma) \tag{1.4}$$

taking a global section to its formal Fourier-Jacobi expansions of cogenus 1. Our first result is a follows (see Theorem 3.13).

Theorem 1.1. *The map in* (1.4) *is an isomorphism.*

Recall that the *affine covering number* $\mathrm{acn}(S)$ of a scheme S is defined as one less than the smallest number of open affine sets required to cover S, see, e.g., [At], [RV]. It gives an upper bound for the cohomological dimension of S. If S is a quasi-projective scheme, then there is the trivial bound $\mathrm{acn}(S) \leq \dim(S)$. Using this notion, we may formulate our main result (see Theorem 4.9).

Theorem 1.2. *Assume that* $\mathrm{acn}(X_\Gamma) \leq \frac{n(n+1)}{2} - 2$. *Then the natural map*

$$H^0(X_\Gamma^*, \omega^{\otimes k}) \to \mathrm{FM}_k^{(n,1)}(\Gamma) \tag{1.5}$$

taking a holomorphic modular form to its cogenus 1 *formal Fourier-Jacobi expansions is an isomorphism.*

To prove this result, we use an algebraization theorem of Raynaud [Ray, Corollaire 2.8] to show that the natural map $H^0(X_\Gamma^*, \omega^{\otimes k}) \to H^0(\hat{X}_\Gamma^*, \hat{\omega}^{\otimes k})$ is an isomorphism. Then the assertion can be deduced by means of Theorem 1.1, see Section 4.2.

This raises the problem of computing (or bounding) the affine covering number of the Siegel modular variety X_Γ. By the theory of Baily-Borel, $\mathrm{acn}(X_\Gamma)$ is the smallest non-negative integer j, for which there exist cusp forms $F_0, \dots, F_j$ for Γ that have no common zero on $\mathbb{H}_n$.

According to [At, Theorem 4], we have $\mathrm{acn}(X_\Gamma) \geq n(n-1)/2$. It is believed that this lower bound is actually an equality. However, for general n not much is known in this direction. Here we employ results of Igusa, Salvati Manni, and Fontanari–Pascolutti on the reducible locus of X_{Γ_n} to prove the required upper bounds for small n, see Proposition 4.3 and Proposition 4.5. The following corollaries can be derived.

Corollary 1.3. *Assume that* $2 \leq n \leq 4$. *Then the natural map* (1.5) *is an isomorphism.*

Corollary 1.4. *Assume that* $2 \leq n \leq 4$. *Let* $U \subset X_\Gamma^*$ *be an open analytic neighborhood of the Satake boundary* Y. *Then the restriction map* $H^0(X_\Gamma^*, \omega^{\otimes k}) \to H^0(U, \omega^{\otimes k})$ *is an isomorphism.*

We remark that Theorem 1.2 and Corollary 1.3 have natural generalizations to vector valued modular forms transforming with a finite dimensional representation of Γ. Alternatively, one can derive such results for vector valued forms from the scalar case by means of the argument of [Br]. Using induction on the cogenus as in [BR, Lemma 5.2] one can also deduce an analogue for formal Siegel modular forms of higher cogenus $l < n$.

As an application, we consider the case of the paramodular group $K(N) \subset \mathrm{Sp}_2(\mathbb{Q})$ of level N and genus 2, see Section 4.4. We write $K(N)^*$ for the extension of $K(N)$ by all Atkin-Lehner type involutions. It contains $K(N)$ as a normal sugbroup, and

$$K(N)^*/K(N) \cong (\mathbb{Z}/2\mathbb{Z})^{\nu(N)},$$

where $\nu(N)$ denotes the number of prime divisors of N. Let f be a formal Fourier-Jacobi series of weight k for the standard boundary component F_1 and the group $K(N)$. Denote by

$$f(\tau) = \sum_T a(T)\, e(\mathrm{tr}\, T\tau)$$

its formal Fourier expansion at the boundary component F_0. Here T runs through all half integral positive semi-definite matrices 2×2-matrices with $N \mid T_2$. We call f *strongly symmetric* if there exists a character $\chi_f : K(N)^*/K(N) \to \{\pm 1\}$ such that

$$a(uT\,{}^t u) = \chi_f \begin{pmatrix} {}^t u^{-1} & 0 \\ 0 & u \end{pmatrix} \det(u)^k a(T) \tag{1.6}$$

for all T and all $u \in \Gamma_0(N)^*$, the extension of $\Gamma_0(N)$ by the Atkin-Lehner involutions (viewed as elements of $\mathrm{SL}_2(\mathbb{R})$).

Theorem 1.5. *Let N be a square-free positive integer. Let f be a strongly symmetric formal Fourier-Jacobi series of weight k for the boundary component F_1 and the group $K(N)$. Then f converges and defines a paramodular form in $M_k(K(N))$.*

For a discussion of the relationship of the symmetry condition in this result and the involution condition in the work of Ibukiyama, Poor, and Yuen [IPY] we refer to Section 4.4.

Part of the motivation to investigate the modularity of formal Fourier-Jacobi series comes from the Kudla program. Here certain generating series of classes of special cycles in Chow groups of orthogonal and unitary Shimura varieties play a central role, see, e.g., [Ku1], [Ku2], [Ku3]. The generating series of special cycles of codimension n is conjectured to be a Siegel (respectively Hermitian) modular form of genus n. By an argument of Wei Zhang [Zh] one can often show that the generating series is given by a symmetric formal Fourier-Jacobi series of genus n and cogenus $n-1$. Hence the conjectured modularity can be deduced from a suitable modularity result for symmetric formal Fourier-Jacobi series. In this way, Kudla's modularity conjecture was established for orthogonal Shimura varieties associated with quadratic spaces of signature $(m, 2)$ over $\mathbb{Q}$ in [BR] and for unitary Shimura varieties associated with hermitian spaces of signature $(m, 1)$ over norm-euclidian imaginary quadratic fields in [Xia], based on Liu's extension of Zhang's work [Liu]. An analogous result for special cycles on integral models of orthogonal Shimura varieties is proved in [HM]. Symmetric formal Fourier-Jacobi series can also be used for the computation of Siegel modular forms, see, e.g., [IPY].

We briefly describe the contents of this paper. In Section 2 we recall some facts on Siegel modular varieties, the Satake compactification, and on Siegel modular forms. This mainly serves to fix notation and to collect some important facts that will be used later. In Section 3 we introduce formal Siegel modular forms and provide the algebraic geometric description as formal sections of the line bundle of Siegel modular forms. In Section 4 we consider affine covering numbers of Siegel modular varieties, Raynaud's algebraization theorem, and its application to formal Siegel modular forms. Finally we discuss the case of the paramodular group in genus 2.

2. Siegel modular varieties

Here we recall some facts on Siegel modular varieties, the Satake compactification, and Siegel modular forms. This mainly serves to fix notation and to provide some background for the following sections.

Let n be a positive integer, and denote by $W = \mathbb{Q}^{2n}$ the standard symplectic space of dimension $2n$, equipped with the symplectic form given by $\langle x, y\rangle = xJ\,{}^t y$, where

$$J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

($x, y \in W$ are viewed as row vectors). Write $\mathrm{Sym}_n(\mathbb{C})$ for the space of symmetric complex $n \times n$-matrices. The real symplectic group $G := \mathrm{Sp}_n(\mathbb{R})$ acts on the Siegel upper half space $\mathbb{H}_n = \{\tau \in \mathrm{Sym}_n(\mathbb{C}) \mid \ \mathrm{Im}(\tau) > 0\}$ by fractional linear transformations

$$\tau \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = (a\tau + b)(c\tau + d)^{-1}.$$

The Cayley transformation $\tau \mapsto z = (\tau - i1_n)(\tau + i1_n)^{-1}$ maps $\mathbb{H}_n$ biholomorphically to the bounded symmetric domain

$$\mathcal{D}_n = \{z \in \mathrm{Sym}_n(\mathbb{C}) \mid \ 1_n - z\bar{z} > 0\}.$$

The action of G on $\mathbb{H}_n$ induces a compatible action on $\mathcal{D}_n$.

2.1. **Boundary components.** The action of G extends to the topological closure $\bar{\mathcal{D}}_n$ of $\mathcal{D}_n$ in $\mathrm{Sym}_n(\mathbb{C})$. Two points in $\bar{\mathcal{D}}_n$ are called equivalent if they can be connected by a finite chain of holomorphic curves $\xi_i : \{z \in \mathbb{C} \mid \ |z| < 1\} \to \bar{\mathcal{D}}_n$. It is easily seen that all points in $\mathcal{D}_n$ are equivalent. The equivalence classes in $\bar{\mathcal{D}}_n \setminus \mathcal{D}_n$ are called the proper *boundary components* of $\mathcal{D}_n$.

To any $z \in \bar{\mathcal{D}}_n$ we can associate a linear map

$$\psi_z : \mathbb{R}^{2n} \to \mathbb{C}^n, \quad \nu \mapsto \nu \begin{pmatrix} i(1_n + z) \\ 1_n - z \end{pmatrix},$$

where the elements of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ are viewed as row vectors. The subspace $U(z) = \ker \psi_z \subset \mathbb{R}^{2n} = W_{\mathbb{R}}$ is totally isotropic with respect to the symplectic form J. It is non-trivial if and only if $z \in \bar{\mathcal{D}}_n \setminus \mathcal{D}_n$ is a proper boundary point. Moreover, $U(z_1) = U(z_2)$ if and only if z_1 and z_2 are equivalent. Hence, we obtain a bijection $F \mapsto U(F)$ between the set of proper boundary components F of $\bar{\mathcal{D}}_n$ and the set of non-trivial isotropic subspaces $U \subset \mathbb{R}^{2n}$, see [HKW, Chapter I.3A] and [Na, Section 4]. The group G acts on isotropic subspaces $U \subset \mathbb{R}^{2n}$ by right translation $U \mapsto Ug^{-1}$ for $g \in G$. This action is compatible with the action on $\bar{\mathcal{D}}_n$, as we have $U(gz) = U(z)g^{-1}$.

Recall that a boundary component F is called *adjacent* to another boundary component F', if $\bar{F}' \supset F$ and $F' \neq F$. In this case we write $F' > F$. This is equivalent to the condition that the isotropic subspace $U(F)$ correponding to F strictly contains the subspace $U(F')$.

For $0 \leq m \leq n$, the subset

$$F_m = \left\{ \begin{pmatrix} z' & 0 \\ 0 & 1_{n-m} \end{pmatrix} \mid \ z' \in \mathcal{D}_m \right\} \subset \bar{\mathcal{D}}_n$$

is a boundary component of $\mathcal{D}_n$, called the *standard boundary component* of degree m. The corresponding isotropic subspace $U(F_m)$ has dimension $n - m$ and is given by

$$U(F_m) = \{(x_1, \dots, x_{2n}) \in \mathbb{R}^{2n} \mid \ x_1 = \cdots = x_{n+m} = 0\}. \tag{2.1}$$

In particular, we have $F_n = \mathcal{D}_n$, and F_0 is a point. The Cayley transformation $\mathbb{H}_m \to \mathcal{D}_m$ induces an isomorphism $\mathbb{H}_m \to F_m$.

The stabilizer

$$G_F = \{g \in G \mid \ g(F) = F\} \tag{2.2}$$

of a boundary component F is a maximal parabolic subgroup of G. We also consider the centralizer

$$G_F^0 = \{g \in G \mid \ g(z) = z \text{ for all } z \in F\} \tag{2.3}$$

of F, which is a normal subgroup of G_F. We denote by G'_F the center of the unipotent radical of G_F, and by

$$G_F^J = \{g \in G \mid \ ghg^{-1} = h \text{ for all } h \in G'_F\}$$

its centralizer in the group G. Recall from [Na, §7] and [AMRT, Chapter III.4] that here is a homomorphism

$$p_\ell : G_F \to \mathrm{Aut}(G'_F), \quad g \mapsto p_\ell(g) = (h \mapsto ghg^{-1}). \tag{2.4}$$

The image of p_ℓ preserves the quadratic form on G'_F induced by the Killing form and the cone of positive elements. Moreover, we have $G_F^J = \ker(p_\ell)$.

For the standard boundary component F_m we have

$$G_{F_m} = \left\{ \begin{pmatrix} a & 0 & b & * \\ * & u & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \right\}, \tag{2.5}$$

$$G^0_{F_m} = \left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ * & u & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \right\}, \tag{2.6}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_m(\mathbb{R})$ and $u \in \mathrm{GL}_{n-m}(\mathbb{R})$. The quotient group $G_{F_m}/G^0_{F_m}$ is isomorphic to $\mathrm{Sp}_m(\mathbb{R})$. Moreover, it is easily seen that

$$G'_{F_m} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

and the normalizer of G'_{F_m} in G is given by G_{F_m}. The group G^J_F consists of those matrices in G_{F_m} for which $u = \pm 1_{n-m}$. If we identify $G'_{F_m} \cong \mathrm{Sym}_{n-m}(\mathbb{R})$, then the action of $g \in G_{F_m}$ as in (2.5) on $s \in \mathrm{Sym}_{n-m}(\mathbb{R})$ is given by $p_\ell(g)(s) = s[{}^t u] = us\,{}^t u$. Let $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ and put

$$\Gamma_{n,m} = \Gamma_n \cap G_{F_m}, \tag{2.7}$$

$$\Gamma^0_{n,m} = \Gamma_n \cap G^0_{F_m}. \tag{2.8}$$

A boundary component F is called *rational* if G_F is defined over $\mathbb{Q}$. This is equivalent to the condition that $U(F) \subset W_{\mathbb{R}}$ is defined over $\mathbb{Q}$. Moreover, it is equivalent to the condition that F is a $\mathrm{Sp}_n(\mathbb{Q})$-translate of a standard boundary component. We define the *rational closure* of $\mathcal{D}_n$ by

$$\mathcal{D}_n^* = \bigcup_F F \subset \bar{\mathcal{D}}_n, \tag{2.9}$$

where the union extends over all rational boundary components (including the non-proper boundary component $\mathcal{D}_n$). The action of $\mathrm{Sp}_n(\mathbb{Q})$ on $\mathcal{D}_n$ extends to an action on $\mathcal{D}_n^*$.

2.2. **The cylindrical topology.** Let $0 \leq m \leq n$. Recall that there is a holomorphic map

$$\pi_{n,m} : \mathbb{H}_n \to \mathbb{H}_m \cong F_m, \quad \pi_{n,m}\begin{pmatrix} \tau_1 & \tau_{12} \\ {}^t\tau_{12} & \tau_2 \end{pmatrix} = \tau_1,$$

and a real analytic map

$$\rho_{n,m} : \mathbb{H}_n \to \mathrm{Sym}^+_{n-m}(\mathbb{R}), \quad \rho_{n,m}\begin{pmatrix} \tau_1 & \tau_{12} \\ {}^t\tau_{12} & \tau_2 \end{pmatrix} = v_2 - {}^t v_{12} v_1^{-1} v_{12},$$

where v denotes the imaginary part of τ. Both maps are equivariant for the action of the parabolic subgroup G_{F_m}.

We now recall the definition of the *cylindrical topology* on $\mathcal{D}_n^*$ following [Fr, Chapter II.6] (see also [Na, Section 5]). For a matrix $v \in \mathrm{Sym}^+_n(\mathbb{R})$ we define

$$\mathrm{m}(v) = \min_{\substack{x \in \mathbb{Z}^n \\ x \neq 0}} v[x],$$

i.e., the minimum of the quadratic form $x \mapsto v[x] = {}^t x v x$ on non-zero integral vectors. If $U \subset \mathbb{H}_m$ is open and $C > 0$, we consider the open subset

$$W_n(U, C) = \{\tau \in \mathbb{H}_n \mid \ \pi_{n,m}(\tau) \in U \text{ and } \mathrm{m}(\rho_{n,m}(\tau)) > C\} \tag{2.10}$$

of $\mathbb{H}_n$. Note that when $n = m$, we simply have $W_n(U, C) = U$. To define a basis of neighbourhoods of a point z in the standard boundary component $F_m \cong \mathbb{H}_m$, we consider the chain of standard boundary components

$$\mathcal{D}_n = F_n > F_{n-1} > \cdots > F_{m+1} > F_m$$

adjacent to F_m. Let $U \subset \mathbb{H}_m$ be an open neighbourhood of z. If $n \geq j \geq m$, we may view

$$W_j(U, C) \subset F_j$$

as a subset via the identification $\mathbb{H}_j \cong F_j$. We use this to define a subset of $\mathcal{D}_n^*$ by

$$\tilde{W}_n(U, C) = \Gamma^0_{n,m}\left(\bigcup_{n \geq j \geq m} W_j(U, C)\right). \tag{2.11}$$

Note that when $m = n$, then $\tilde{W}_n(U, C)$ simply reduces to U.

Definition 2.1. A set $V \subset \mathcal{D}_n^*$ is called open if for all $z \in V$ there exists a $g \in \mathrm{Sp}_n(\mathbb{Q})$ such that gz is contained in a standard boundary component F_m for some $0 \leq m \leq n$ and such that gV contains a set $\tilde{W}_n(U, C)$ for some open neighbourhood $U \subset F_m \cong \mathbb{H}_m$ of gz and some $C > 0$.

Proposition 2.2. *The cylindrical topology is the weakest topology on $\mathcal{D}_n^*$ in which all $\mathrm{Sp}_n(\mathbb{Q})$-translates of all the sets $\tilde{W}_n(U, C)$ are open for $U \subset F_m$ open, $0 \leq m \leq n$, and $C > 0$. The induced topology on the standard boundary components F_m agrees with the usual topology. The set $\mathcal{D}_n$ is open and dense in $\mathcal{D}_n^*$. Moreover, $\mathcal{D}_n^*$ is a Hausdorff space, and $\mathrm{Sp}_n(\mathbb{Q})$ acts on it by homeomorphisms.*

Remark 2.3. A sequence

$$\tau^{(\nu)} = \begin{pmatrix} \tau_1^{(\nu)} & * \\ * & * \end{pmatrix} \in \mathbb{H}_n \cong \mathcal{D}_n$$

with $\tau_1^{(\nu)} \in \mathbb{H}_m$ converges to a boundary point $\tau_1^* \in \mathbb{H}_m \cong F_m$, if and only if $\tau_1^{(\nu)} \to \tau_1^*$ in the usual sense and $\rho_{n,m}(\tau^{(\nu)}) \to \infty$. Here the latter condition means that for any $C > 0$ we have $\mathrm{m}(\rho_{n,m}(\tau^{(\nu)})) > C$ for all but finitely many ν. See, e.g., [Fr, Hilfssatz 6.18_1].

We write $z = x + iy$ for the decomposition of $z \in \mathbb{H}_n$ into its real and imaginary part. Moreover, we denote the Jacobi decomposition of y by

$$y = D[W] = {}^tWDW, \tag{2.12}$$

where D is a diagonal matrix with diagonal entries $d_1, \dots, d_n$ and $W = (w_{ij})$ is a unipotent upper triangular matrix. Recall that for $u > 0$ the *Siegel domain* $\mathcal{F}_n(u)$ is defined as the set of $z \in \mathbb{H}_n$ satisfying the following conditions:

(a) $|x_{ij}| < u$ for all $1 \leq i, j \leq n$,
(b) $|w_{ij}| < u$ for all $1 \leq i < j \leq n$,
(c) $d_i < ud_{i+1}$ for all $1 \leq i \leq n-1$,
(d) $1 < ud_1$,

see, e.g., [Fr, Chapter II, Definition 1.7]. The set of positive definite symmetric matrices $y \in \mathrm{Sym}_n^+(\mathbb{R})$ satisfying conditions (b) and (c) is denoted by $\mathcal{R}_n(u)$. If u is sufficiently large, then $\mathcal{F}_n(u)$ is a *fundamental set* for the action of Γ_n on $\mathbb{H}_n$, that is, it is a measurable subset of $\mathbb{H}_n$ which has non-trivial intersection with the Γ_n-orbit of any point in $\mathbb{H}_n$. Moreover, for every $u > 0$, the set $\mathcal{F}_n(u)$ has finite volume and there exist only finitely many $\gamma \in \Gamma_n$ such that $\gamma\mathcal{F}_n(u) \cap \mathcal{F}_n(u) \neq \emptyset$. We also define

$$\mathcal{F}_n^*(u) = \mathcal{F}_n(u) \cup \mathcal{F}_{n-1}(u) \cup \cdots \cup \mathcal{F}_0(u) \subset \mathcal{D}_n^* \tag{2.13}$$

as in [Fr, page 98]. Here $\mathcal{F}_m(u)$ is viewed as a subset of the standard boundary component F_m of degree m.

The following lemmas will be used to construct convenient neighborhoods of boundary components.

Lemma 2.4. *Let $U \subset \mathbb{H}_m$ be relatively compact.*

(i) *There exists a $C > 0$ such that every $\gamma \in \Gamma_n$ satisfying*

$$\gamma(W_n(U, C)) \cap W_n(U, C) \neq \emptyset$$

is contained in $\Gamma_{n,m}$.

(ii) *Moreover, there exists a $C > 0$ such that every $\gamma \in \Gamma_n$ satisfying*

$$\gamma(\tilde{W}_n(U, C)) \cap \tilde{W}_n(U, C) \neq \emptyset$$

is contained in $\Gamma_{n,m}$.

Proof. (i) The following argument is due to Eberhard Freitag. We decompose any $z \in \mathbb{H}_n$ as

$$z = \begin{pmatrix} z_1 & z_{12} \\ {}^tz_{12} & z_2 \end{pmatrix} \tag{2.14}$$

with $z_1 \in \mathbb{H}_m$. We argue indirectly. Assume that there exists a sequence $C_\nu \to \infty$ of positive real numbers such that for every $\nu \in \mathbb{Z}_{>0}$ there exists a $\gamma_\nu \in \Gamma_n \setminus \Gamma_{n,m}$ and points $z^{(\nu)}, w^{(\nu)} \in W_n(U, C_\nu)$ satisfying

$$\gamma_\nu z^{(\nu)} = w^{(\nu)}.$$

By taking a suitable subsequence, we may assume that $z_1^{(\nu)}$ converges to a point $z_1 \in \mathbb{H}_m$ for $\nu \to \infty$, and $w_1^{(\nu)}$ converges to a $w_1 \in \mathbb{H}_m$. Hence, with respect to the cylindrical topology we get convergent sequences

$$z^{(\nu)} \to z_1 \in F_m, \qquad w^{(\nu)} \to w_1 \in F_m.$$

According to [Fr, Hilfssatz 6.18$_2$] there exist elements α_ν in the stabilizer of z_1 inside Γ_n such that all

$$\tilde{z}^{(\nu)} = \alpha_\nu(z^{(\nu)})$$

are contained in some fixed Siegel domain $\mathcal{F}_n(u)$. Analogously, there are β_ν in the stabilizer of w_1 inside Γ_n such that all

$$\tilde{w}^{(\nu)} = \beta_\nu(w^{(\nu)})$$

lies in $\mathcal{F}_n(u)$. Both stabilizers are contained in $\Gamma_{n,m}$. The construction in loc. cit. shows that

$$z_1^{(\nu)} = \tilde{z}_1^{(\nu)}, \qquad \mathrm{m}(\rho_{n,m}(z^{(\nu)})) = \mathrm{m}(\rho_{n,m}(\tilde{z}^{(\nu)})),$$

and similarly for the $w^{(\nu)}$. Therefore, we may assume without loss of generality from the outset that the sequences $z^{(\nu)}$ and $w^{(\nu)}$ are contained in a fixed Siegel domain $\mathcal{F}_n(u)$.

The finiteness property of Siegel domains now implies that the γ_ν belong to a finite set. By taking a subsequence we may assume that $\gamma = \gamma_\nu$ is independent of ν. Since γ acts continuously on $\mathcal{D}_n^*$ we find that

$$\gamma(z_1) = w_1.$$

But this implies $\gamma \in \Gamma_{n,m}$, a contradiction.

(ii) Part (i) of the lemma immediately implies that there exists a $C > 0$ such that for every j with $n \leq j \leq m$ and every $\gamma \in \Gamma_{n,j}$ satisfying

$$\gamma(W_j(U, C)) \cap W_j(U, C) \neq \emptyset$$

we have $\gamma \in \Gamma_{n,m}$. We fix such a C and assume that $z, w \in \tilde{W}_n(U, C)$ and $\gamma \in \Gamma_n$ with the property that $\gamma z = w$.

By definition there exist $n \geq j, j' \geq m$ and $\gamma_1, \gamma_2 \in \Gamma_{n,m}^0$ and $z' \in W_j(U, C)$, $w' \in W_{j'}(U, C)$, such that

$$z = \gamma_1 z', \qquad w = \gamma_2 w', \qquad \gamma\gamma_1 z' = \gamma_2 w'.$$

Replacing γ by $\gamma_2^{-1}\gamma\gamma_1$ we may assume that $\gamma_1 = \gamma_2 = 1$ and $z = z'$, $w = w'$. Since the action of Γ_n preserves the degree of a boundary component, we may further assume that $j = j'$. But then, according to [Fr, Hilfssatz 2.5], the condition $\gamma z = w \in W_j(U, C) \subset F_j$ implies that $\gamma \in \Gamma_{n,j}$. Consequently, by our choice of C, we may conclude that $\gamma \in \Gamma_{n,m}$. □

Lemma 2.5. *Let $U \subset \mathbb{H}_m$ be relatively compact.*

(i) *There exists a $C > 0$ such that every $\gamma \in \Gamma_n$ satisfying*

$$\gamma(W_n(U, C)) \cap \mathcal{F}_n(u) \neq \emptyset$$

is contained in $\Gamma_{n,m}$.

(ii) *Moreover, there exists a $C > 0$ such that every $\gamma \in \Gamma_n$ satisfying*

$$\gamma(\tilde{W}_n(U, C)) \cap \mathcal{F}_n^*(u) \neq \emptyset$$

is contained in $\Gamma_{n,m}$.

Proof. This can be proved in the same way as Lemma 2.4. Therefore we omit the details. □

We now fix a $u > 0$ such that $\mathcal{F}_j(u)$ is a fundamental set for the action of $\mathrm{Sp}_j(\mathbb{Z})$ on $\mathbb{H}_j$ for every $0 \leq j \leq n$.

Lemma 2.6. *Let $U \subset \mathcal{F}_m(u) \subset F_m$ be relatively compact.*

(i) *There exists a $C > 0$ such that*

$$W_n(U, C) \subset \Gamma^0_{n,m}\mathcal{F}_n(u).$$

(ii) *There exists a $C > 0$ such that*

$$\tilde{W}_n(U, C) \subset \Gamma^0_{n,m}\mathcal{F}_n^*(u).$$

Proof. We write $z = x + iy \in \mathbb{H}_n$ for the decomposition in real and imaginary part. To prove the lemma, we use the block Jacobi decomposition

$$y = D[W] = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \left[\begin{pmatrix} 1_m & B \\ 0 & 1_{n-m} \end{pmatrix} \right],$$

and put

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \qquad W = \begin{pmatrix} W_1 & W_{12} \\ 0 & W_2 \end{pmatrix},$$

where D_1 is the diagonal matrix with diagonal entries $d_1, \dots, d_m$, and D_2 the diagonal matrix with entries $d_{m+1}, \dots, d_n$. The matrices W_1 and W_2 are unipotent upper triangular. It is easily seen that $Y_1 = D_1[W_1]$, $Y_2 = D_2[W_2]$, and $B = W_1^{-1}W_{12}$.

(i) Let $z \in W_n(U, C)$. We have to show that there exists a $\gamma \in \Gamma^0_{n,m}$ such that $\gamma z \in \mathcal{F}_n(u)$. We note that since $z_1 \in U$, we have $y_1 = Y_1 = D_1[W_1] \in \mathcal{R}_m(u)$. We may act with a matrix in $\Gamma^0_{n,m}$ of the form

$$\begin{pmatrix} {}^tA & 0 \\ 0 & A^{-1} \end{pmatrix}, \qquad A = \begin{pmatrix} 1_m & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_2 \in \mathrm{GL}_{n-m}(\mathbb{Z})$, to transform Y_2 to $Y_2[A_2] \in \mathcal{R}_{n-m}(u)$. Here Y_1 remains unchanged. Next we can act with a matrix in $\Gamma^0_{n,m}$ of the form

$$\begin{pmatrix} {}^tA & 0 \\ 0 & A^{-1} \end{pmatrix}, \qquad A = \begin{pmatrix} 1_m & A_{12} \\ 0 & 1_{n-m} \end{pmatrix},$$

where $A_{12} \in \mathbb{Z}^{m \times (n-m)}$, to transform W_{12} to $W'_{12} = W_1A_{12} + W_{12}$ with entries w'_{ij} satisfying $|w'_{ij}| < u$. Here D, W_1, W_2 remain unchanged.

Finally, we can act with a matrix in $\Gamma^0_{n,m}$ of the form

$$M = \begin{pmatrix} 1_n & T \\ 0 & 1_n \end{pmatrix}, \qquad T = \begin{pmatrix} 0_m & T_{12} \\ {}^tT_{12} & T_2 \end{pmatrix},$$

where $T_{12} \in \mathbb{Z}^{m\times(n-m)}$, $T_2 \in \mathbb{Z}^{(n-m)\times(n-m)}$, to transform x_{12} to $x_{12}+T_{12}$ and x_2 to x_2+T_2. Here x_1 and y remain unchanged.

In this way we can transform $z \in W_n(U,C)$ by an element of $\Gamma^0_{n,m}$ to a $z' \in \mathbb{H}_n$ meeting all conditions for $\mathcal{F}_n(u)$ except for possibly the condition

$$d'_m < u d'_{m+1}. \tag{2.15}$$

Here and throughout we have indicated the Jacobi coordinates of z' by d'_i, w'_{ij}, and x'_{ij}. We claim that the condition (2.15) is also met if we choose C sufficiently large. To see this we note that d'_m is bounded, since $z'_1 = z_1$ is contained in the bounded set U. Moreover, since $Y'_2 = Y_2[A_2] \in \mathcal{R}_{n-m}(u)$, we have

$$\mathrm{m}(Y'_2) \le d'_{m+1},$$

see, e.g., [Fr, Hilfssatz 1.2]. Hence, if

$$C > \frac{1}{u}\sup\{d_m \mid z_1 \in U\},$$

and $z \in W_n(U,C)$ then

$$\frac{1}{u}d'_m < C < \mathrm{m}(Y_2) = \mathrm{m}(Y'_2) \le d'_{m+1}.$$

This gives the remaining condition (2.15).

(ii) This assertion is an immediate consequence of (i). □

2.3. **The Satake compactification.** Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{Q})$ be an *arithmetic* subgroup, that is, a subgroup which is commensurable with $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$. Then Γ acts properly discontinuously on $\mathcal{D}^*_n$. The quotient $X^*_\Gamma := \Gamma\backslash\mathcal{D}^*_n$, equipped with the quotient topology, is a compact Hausdorff space, which contains $X_\Gamma := \Gamma\backslash\mathcal{D}_n$ as a dense open subset. The complex structure on X_Γ canonically extends to a complex structure on X^*_Γ, equipping it with the structure of a normal complex space. It is called the Satake compactification of X_Γ, see, e.g., [BB, Theorem 10.4], [Na, Section 5], or [Fr, Chapter II.6].

The boundary

$$\partial X^*_\Gamma = X^*_\Gamma \setminus X_\Gamma \tag{2.16}$$

is a closed analytic subset of codimension n. If F is a rational boundary component of $\mathcal{D}_n$ of degree m, we let

$$\begin{aligned}\Gamma_F &= \Gamma \cap G_F,\\ \Gamma^0_F &= \Gamma \cap G^0_F,\\ \Gamma'_F &= \Gamma \cap G'_F,\\ \Gamma^J_F &= \Gamma \cap G^J_F.\end{aligned}$$

We may view

$$\bar\Gamma_F = \Gamma_F/\Gamma^0_F$$

as an arithmetic subgroup of $\mathrm{Sp}_m(\mathbb{R})$. The quotient

$$X_{\Gamma,F} = \bar\Gamma_F\backslash F$$

is isomorphic to a Siegel modular variety of genus m. The boundary decomposes as a finite disjoint union

$$\partial X^*_\Gamma = \coprod_F X_{\Gamma,F}$$

of locally closed analytic subsets, where the union runs over the Γ-classes of proper rational boundary components, see [BB, Corollary 4.11]. The following two propositions ensure the existence of convenient neighborhoods of the boundary components.

Proposition 2.7. *For every rational boundary component F there exists an open neighborhood $V(F) \subset \mathcal{D}_n^*$ of F satisfying the following properties:*

(i) *$V(F)$ is invariant under the stabilizer $\Gamma_{n,F}$ of F in Γ_n.*

(ii) *The natural map*

$$\Gamma_{n,F}\backslash V(F) \to X^*_{\Gamma_n}$$

is injective.

(iii) *$V(\delta F) = \delta V(F)$ for all $\delta \in \Gamma_n$.*

(iv) *If F and F' are rational boundary components of degree m, we have*

$$V(F) \cap V(F') \neq \emptyset \quad \Rightarrow \quad F = F'.$$

Proof. We first prove (i) and (ii) for the standard boundary components. Let m be an integer with $0 \leq m < n$ and consider the standard boundary component F_m of degree m. Recall the notation $\Gamma_{n,F_m} = \Gamma_{n,m}$.

Let $u > 0$ be such that $\mathcal{F}_j(u)$ is a fundamental set for the group Γ_j for all $0 \leq j \leq n$. Choose a sequence $U_\nu \subset \mathcal{F}_m(u)$ of relatively compact open sets (for $\nu \in \mathbb{Z}_{>0}$) such that

$$\mathcal{F}_m(u) = \bigcup_{\nu \geq 1} U_\nu.$$

According to Lemma 2.6 and Lemma 2.5, we may choose $C_\nu > 0$ such that

$$\tilde{W}_n(U_\nu, C_\nu) \subset \Gamma^0_{n,m}\mathcal{F}^*_n(u), \tag{2.17}$$

and such that every $\gamma \in \Gamma_n$ satisfying

$$\gamma(\tilde{W}_n(U_\nu, C_\nu)) \cap \mathcal{F}^*_n(u) \neq \emptyset \tag{2.18}$$

is contained in $\Gamma_{n,m}$.

The union

$$S(F_m) = \bigcup_{\nu \geq 1} \tilde{W}_n(U_\nu, C_\nu) \subset \mathcal{D}^*_n$$

is an open neighborhood of $\mathcal{F}_m(u)$. By the choice of u, the set

$$V(F_m) := \Gamma_{n,m}(S(F_m)) \tag{2.19}$$

is an open neighborhood of the full boundary component F_m. Moreover, by construction, $V(F_m)$ is invariant under Γ_{n,F_m}.

We now show the injectivity of the natural map $\Gamma_{n,m}\backslash V(F_m) \to X^*_{\Gamma_n}$. To this end, let $z, w \in V(F_m)$ and $\gamma \in \Gamma_n$ such that

$$\gamma z = w.$$

We have to show that $\gamma \in \Gamma_{n,m}$.

Possibly shifting z and w by elements of $\Gamma_{n,m}$, we can assume that z and w lie in $S(F_m)$. By (2.17) we can further assume that they are also contained in $\mathcal{F}^*_n(u)$. Hence there exist $\mu, \nu \in \mathbb{Z}_{>0}$ such that

$$z \in \tilde{W}_n(U_\nu, C_\nu) \cap \mathcal{F}^*_n(u),$$
$$w \in \tilde{W}_n(U_\mu, C_\mu) \cap \mathcal{F}^*_n(u).$$

Using condition (2.18) we see that $\gamma \in \Gamma_{n,m}$. This concludes the proof of (i) and (ii) for the standard boundary component F_m.

If F is any rational boundary component of degree m, we choose $\delta \in \Gamma_n$ such that $F = \delta F_m$ and put

$$V(F) = \delta V(F_m).$$

By property (i) for $V(F_m)$ this is independent of the choice of δ. Employing the fact that $\Gamma_{n,F_m} = \delta^{-1}\Gamma_{n,F}\delta$, it is easily seen that $V(F)$ satisfies properties (i) and (ii) for F. Moreover, in this way (iii) also holds.

Finally, to prove (iv), let F and F' be rational boundary components of degree m, and let $z \in V(F) \cap V(F')$. Choose $\delta, \delta' \in \Gamma_n$ such that $F = \delta F_m$ and $F' = \delta' F_m$. Since $V(F) = \delta V(F_m)$ and $V(F') = \delta' V(F_m)$, there are $z_1, z_2 \in V(F_m)$ such that

$$z = \delta z_1 = \delta' z_2.$$

Then $\delta^{-1}\delta' z_2 = z_1$, and by (ii) we obtain $\delta^{-1}\delta' \in \Gamma_{n,m}$. Consequently,

$$F' = \delta' F_m = \delta(\delta^{-1}\delta')F_m = \delta F_m = F.$$

This concludes the proof of the proposition. □

Proposition 2.8. *Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{Q})$ be an arithmetic subgroup. For every rational boundary component F there exists an open neighborhood $W(F) \subset \mathcal{D}_n^*$ of F satisfying the following properties:*

(i) *$W(F)$ is invariant under the action of Γ_F.*
(ii) *The natural map*

$$\Gamma_F \backslash W(F) \to X_\Gamma^*$$

is injective.
(iii) *$W(\gamma F) = \gamma W(F)$ for all $\gamma \in \Gamma$.*
(iv) *If F and F' are rational boundary components of degree m, we have*

$$W(F) \cap W(F') \neq \emptyset \quad \Rightarrow \quad F = F'.$$

Proof. In the special case when $\Gamma \subset \Gamma_n$, it is easily seen that we may simply put $W(F) = V(F)$ with $V(F)$ as in Proposition 2.7.

Now let $\Gamma \subset \mathrm{Sp}_n(\mathbb{Q})$ be an arbitrary arithmetic subgroup. We choose a congruence subgroup

$$\Gamma' \subset \Gamma \cap \Gamma_n$$

which is normal in Γ. For any rational boundary component F, we put

$$W(F) = \bigcap_{\delta \in \Gamma} \delta^{-1} V(\delta F).$$

Since $\delta^{-1}V(\delta F) = V(F)$ for $\delta \in \Gamma'$, this is in fact an intersection of finitely many different open neighborhoods of F in $\mathcal{D}_n^*$. Hence it defines an open neighborhood of F. We leave it the the reader to verify that properties (i)–(iv) hold. □

Remark 2.9. We may in addition require that the open neighborhood $W(F)$ in Proposition 2.8 is connected. In fact, since the connected component of $W(F)$ containing F is stable under under Γ_F, we may replace $W(F)$ by this connected component if necessary.

2.4. **Siegel modular forms.** Let k be an integer. We denote the usual action of $G = \mathrm{Sp}_n(\mathbb{R})$ in weight k on functions $f : \mathbb{H}_n \to \mathbb{C}$ by

$$(f \mid_k \gamma)(\tau) = \det(c\tau + d)^{-k} f(\gamma\tau)$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. The composition of the quotient map and the inclusion defines a natural holomorphic map

$$p : \mathbb{H}_n \to X_\Gamma^*.$$

We denote by ω the sheaf of modular forms of weight 1 on X_Γ^*. Recall that if $V \subset X_\Gamma^*$ is open, the module of sections $\omega(V)$ is given by holomorphic functions $f : p^{-1}(V) \to \mathbb{C}$ satisfying

$$(f \mid_1 \gamma)(\tau) = f(\tau)$$

for all $\gamma \in \Gamma$ and all $\tau \in p^{-1}(V)$. Moreover, when $n = 1$, it is also required that f is holomorphic at the cusps contained in V. If $n > 1$, then regularity at the boundary is automatically satisfied by the local Koecher principle.

The sheaf ω is a coherent $\mathcal{O}_{X_\Gamma^*}$-module on X_Γ^*, which can be identified with the Hodge bundle. The global sections of $\omega^{\otimes k}$ are given by holomorphic modular forms of weight k for Γ, see, e.g., [BB, Section 10].

We now describe the sheaf $\omega^{\otimes k}$ near a rational boundary component F. By possibly conjugating by an element of Γ_n it suffices to do this near the standard boundary components F_m, where $0 \le m \le n$. Let $\tau_1 \in F_m$ be a boundary point. According to Proposition 2.8 and Definition 2.1 there exists a relatively compact open neighborhood $U \subset F_m$ of τ_1 and a $C > 0$ such that the open neighborhood $\tilde{W}_n(U, C) \subset \mathcal{D}_n^*$ defined in (2.11) satisfies

$$\gamma \tilde{W}_n(U, C) \cap \tilde{W}_n(U, C) \neq \emptyset \quad \Rightarrow \quad \gamma \in \Gamma_{F_m} \tag{2.20}$$

for $\gamma \in \Gamma$. If U is chosen sufficiently small, then the condition in (2.20) actually implies that γ is contained in the stabilizer $\Gamma_{\tau_1} \subset \Gamma$ of τ_1. This follows from the fact that $\Gamma_{F_m}/\Gamma_{F_m}^0$ acts properly discontinuously on the boundary component F_m. Possibly replacing U by a smaller open set we may further attain that $\tilde{W}_n(U, C)$ is invariant under Γ_{τ_1}. Then

$$V = \Gamma_{\tau_1} \backslash \tilde{W}_n(U, C) \subset X_\Gamma^* \tag{2.21}$$

is an open neighborhood of τ_1.

An element $f \in \omega^{\otimes k}(V)$ is given by a continuous function $f : \tilde{W}_n(U, C) \to \mathbb{C}$ which is holomorphic on $W_n(U, C)$ and satisfies $f \mid_k \gamma = f$ for all $\gamma \in \Gamma_{\tau_1}$. In particular, f is invariant under the action of translations of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma'_{F_m}, \tag{2.22}$$

where $s \in \mathrm{Sym}_{n-m}(\mathbb{Z})$. Therefore f has a partial Fourier expansion

$$f(\tau) = \sum_{T_2 \in \mathrm{Sym}_{n-m}(\mathbb{Q})} \phi_{T_2}(\tau_1, \tau_{12})\, e(\operatorname{tr} T_2 \tau_2), \tag{2.23}$$

which converges normally in a small neighborhood of U. Here the coefficients ϕ_{T_2} vanish unless T_2 is contained in a sublattice of $\mathrm{Sym}_{n-m}(\mathbb{Q})$ with bounded denominators (the character lattice of the torus G'_{F_m}/Γ'_{F_m}). Moreover, ϕ_{T_2} vanishes if T_2 is not positive semi-definite. We will refer to expansions as in (2.23) as Fourier-Jacobi expansions. The transformation behavior of f under matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & {}^tu^{-1} \end{pmatrix} \in \Gamma^0_{F_m} \tag{2.24}$$

implies that

$$\phi_{T_2[{}^tu^{-1}]}(\tau_1, \tau_{12}u) = \det(u)^k \cdot \phi_{T_2}(\tau_1, \tau_{12}) \tag{2.25}$$

for all $u \in \mathrm{GL}_{n-m}(\mathbb{Q})$ such that (2.24) belongs to $\Gamma^0_{F_m}$.

Now choose a small open neighborhood $W(F_m)$ of the *full* boundary component F_m as in Proposition 2.8. Then we easily obtain Lemma 2.10.

Lemma 2.10. *Let*

$$V = \Gamma_{F_m} \backslash W(F_m) \subset X^*_\Gamma$$

be the open neighborhood of the boundary stratum X_{Γ, F_m} induced by $W(F_m)$. The space of sections $\omega^{\otimes k}(V)$ is given by those continuous functions $f : W(F_m) \to \mathbb{C}$ which are holomorphic on $W(F_m) \cap \mathcal{D}_n$ and satisfy $f \mid_k \gamma = f$ for all $\gamma \in \Gamma_{F_m}$. The Fourier-Jacobi coefficients ϕ_{T_2} of f in the expansion (2.23) *have the transformation behavior*

$$\phi_{T_2}(\tau_1, \tau_{12}) e(\mathrm{tr}\, T_2\tau_2) \mid_k \gamma = \phi_{T_2}(\tau_1, \tau_{12}) e(\mathrm{tr}\, T_2\tau_2) \tag{2.26}$$

for all γ in the Jacobi group $\Gamma^J_{F_m} = \Gamma \cap G^J_{F_m}$.

Hence the coefficient ϕ_{T_2} is a weakly holomorphic Jacobi form of weight k and index T_2 for an arithmetic subgroup of $\mathrm{Sp}_m(\mathbb{Q}) \ltimes \mathrm{Mat}_{m,n-m}(\mathbb{Q})^2$ which only depends on Γ but not on T_2.

3. Formal Siegel modular forms

We define formal Siegel modular forms for arithmetic subgroups of $\mathrm{Sp}_n(\mathbb{Q})$. We interpret these as global sections of the formal completion of the sheaf Siegel modular forms along the Baily-Borel boundary.

3.1. **Formal Fourier-Jacobi series.**

Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{Q})$ be an arithmetic subgroup, and fix $k \in \mathbb{Z}$. Let F be a rational boundary component of degree m, where $0 \le m \le n$. The center G'_F of the unipotent radical of G_F is isomorphic to the additive group of $\mathrm{Sym}_{n-m}(\mathbb{R})$, and Γ'_F defines a lattice L_F in G'_F. Its dual lattice $L_F^\vee$ is the character lattice of the compact abelian group G'_F/Γ'_F.

Definition 3.1. A *formal Fourier-Jacobi series* of weight k for the boundary component F and the group Γ is a formal series

$$f(\tau) = \sum_{\substack{t \in L_F^\vee \\ t \ge 0}} f_t(\tau), \tag{3.1}$$

where the coefficients f_t are holomorphic functions on $\mathbb{H}_n$ satisfying the transformation laws

$$f_t \mid_k g = t(g) \cdot f_t \qquad \text{for all } g \in G'_F, \tag{3.2}$$

$$f_t \mid_k \gamma = f_{p_\ell(\gamma)(t)} \qquad \text{for all } \gamma \in \Gamma_F. \tag{3.3}$$

Here $p_\ell(\gamma)(t)$ denotes the action of γ on $t \in L_F^\vee$ induced by (2.4). In particular, for $\gamma \in \Gamma_F^J$ the condition in (3.3) reduces to $f_t \mid_k \gamma = f_t$.

Note that for $m = n$ the boundary component F is simply $\mathcal{D}_n$, the lattice L_F has rank 0, and hence the expansion in (3.1) is trivial.

Remark 3.2. Let F' be a second rational boundary component of degree m and let $\delta \in \Gamma_n$ such that $F' = \delta^{-1}F$. Then conjugation $g \mapsto \varphi_\delta(g) = \delta g \delta^{-1}$ with δ defines an automorphism of G, which restricts to an isomorphism $G_{F'} \to G_F$. In particular we have $G_{F'} = \delta^{-1} G_F \delta$ and $(\delta^{-1}\Gamma\delta)_{F'} = \delta^{-1}\Gamma_F\delta$. The map φ_δ induces an isomorphism $G'_{F'}/(\delta^{-1}\Gamma\delta)'_{F'} \to G'_F/\Gamma'_F$. Pull back via this isomorphism gives rise to an isomorphism $\varphi_\delta^* : L_F^\vee \to L_{F'}^\vee$, $t \mapsto \varphi_\delta^*(t) = t \circ \varphi_\delta$ of the corresponding character lattices. We define the pull back $f \mid_k \delta$ of the formal Fourier-Jacobi series f in (3.1) by defining the coefficients as

$$(f \mid_k \delta)_{\varphi_\delta^*(t)} := f_t \mid_k \delta \tag{3.4}$$

for $t \in L_F^\vee$. It is easily checked that

$$(f \mid_k \delta)(\tau) = \sum_{t \in L_F^\vee} (f_t \mid_k \delta)(\tau) \tag{3.5}$$

defines a formal Fourier-Jacobi series of weight k for the boundary component F' and the conjugate group $\delta^{-1}\Gamma\delta$. That is, we have

$$f_t \mid_k \delta \mid_k g = \varphi_\delta^*(t)(g) \cdot f_t \mid_k \delta \qquad \text{for all } g \in G'_{F'},$$
$$f_t \mid_k \delta \mid_k \gamma = f_{p_\ell(\gamma)(\varphi_\delta^*(t))} \mid_k \delta \qquad \text{for all } \gamma \in (\delta^{-1}\Gamma\delta)_{F'}.$$

This definition is compatible with the pull back of convergent Fourier-Jacobi series.

Remark 3.3. Assume that $\delta \in \Gamma_n$. Then f is a formal Fourier-Jacobi series of weight k for the boundary component F and the group Γ if and only if $f \mid_k \delta$ is a formal Fourier-Jacobi series of weight k for the boundary component $\delta^{-1}F$ and the group $\delta^{-1}\Gamma\delta$.

Let F be a rational boundary component of degree m and assume that $E < F$ is a rational boundary component of degree 0 which is adjacent to F. Then we have inclusions of groups

$$\begin{array}{ccccc} G'_F & \longrightarrow & G'_E & \longrightarrow & G_F \cap G_E \\ \uparrow & & \uparrow & & \uparrow \\ \Gamma'_F & \longrightarrow & \Gamma'_E & \longrightarrow & \Gamma_F \cap \Gamma_E \end{array},$$

which induce an inclusion $G'_F/\Gamma'_F \to G'_E/\Gamma'_E$ and a surjective homomorphism of the character lattices $L_E^\vee \to L_F^\vee$, $s \mapsto s \mid L_F$.

Let $f = \sum_{t \in L_F^\vee} f_t$ be a formal Fourier-Jacobi series of weight k for the boundary component F and the group Γ. The transformation law (3.3) implies that f_t has a normally convergent Fourier expansion

$$f_t = \sum_{\substack{s \in L_E^\vee \\ s \mid L_F = t}} f_{t,s}, \tag{3.6}$$

where the coefficients $f_s := f_{s|L_F,s}$ are holomorphic functions on $\mathbb{H}_n$ satisfying the transformation law

$$f_s \mid_k g = t(g) \cdot f_s \tag{3.7}$$

for all $g \in G'_E$ and $s \in L_E^\vee$. Putting these expansions together, we obtain the *formal Fourier expansion*

$$f = \sum_{s \in L_E^\vee} f_s \tag{3.8}$$

of f at E.

We call the formal Fourier-Jacobi series f *regular* at the boundary component E if its formal Fourier expansion (3.8) defines a formal Fourier-Jacobi series of weight k for the boundary component E in the sense of Definition 3.1. That is, $f_s = 0$ unless $s \geq 0$, and

$$f_s \mid_k \gamma = f_{p_\ell(\gamma)(s)} \qquad \text{for all } \gamma \in \Gamma_E. \tag{3.9}$$

Remark 3.4. Assume the above notation. If $E' < F$ is another rational boundary component of degree 0 which is adjacent to F, we may choose a $\delta \in \Gamma_n$ such that

$$F = \delta F; \qquad E = \delta E'.$$

Then the pull back $f \mid_k \delta$ is a formal Fourier-Jacobi series of weight k for F and the group $\delta^{-1}\Gamma\delta$. It is easily seen that f is regular at the boundary component E if and only if $f \mid_k \delta$ is regular at E'. In particular, f is regular at all adjacent degree 0 boundary components $E < F$ if and only if it regular for a set of representatives of the Γ_F-classes of such boundary components.

In the special case when $F = F_m$ is equal to the standard boundary component of degree m we identify $L_F^\vee$ with a sublattice of $\mathrm{Sym}_{n-m}(\mathbb{Q})$ via the symmetric bilinear form $(A, B) \mapsto \mathrm{tr}(AB)$. Then the action of $\gamma \in \Gamma_{F_m}$ as in (2.5) on $T_2 \in L_{F_m}^\vee$ is given by $p_\ell(\gamma)(T_2) = T_2[u]$. We may write the function f_{T_2} as

$$f_{T_2}(\tau) = \phi_{T_2}(\tau_1, \tau_{12}) e(\mathrm{tr}\, T_2\tau_2),$$

where ϕ_{T_2} is holomorphic on $\mathbb{H}_n$, independent of τ_2, and satisfies the transformation law

$$\phi_{T_2}(\tau_1, \tau_{12}) e(\mathrm{tr}\, T_2\tau_2) \mid_k \gamma = \phi_{T_2[u]}(\tau_1, \tau_{12}) e(\mathrm{tr}\, T_2[u]\tau_2) \tag{3.10}$$

for all $\gamma \in \Gamma_{F_m}$. The above formal series (3.1) can be rewritten as

$$f(\tau) = \sum_{\substack{T_2 \in L_{F_m}^\vee \\ T_2 \geq 0}} \phi_{T_2}(\tau_1, \tau_{12})\, e(\mathrm{tr}\, T_2\tau_2), \tag{3.11}$$

where the coefficients ϕ_{T_2} are weakly holomorphic Jacobi forms of index T_2 and weight k for the group $\Gamma_{F_m}^J$. Under this identification the Fourier expansion of f_{T_2} at the standard degree 0 boundary component $F_0 < F_m$ can be written as

(3.12)

$$\phi_{T_2}(\tau_1, \tau_{12}) = \sum_{T_1 \in \mathrm{Sym}_m(\mathbb{Q})} \sum_{T_{12} \in \mathrm{Mat}_{m,n-m}(\mathbb{Q})} a \begin{pmatrix} T_1 & T_{12} \\ {}^tT_{12} & T_2 \end{pmatrix} e(\mathrm{tr}\, T_1\tau_1 + 2\,\mathrm{tr}\, {}^tT_{12}\tau_{12}).$$

Putting these expansions together, we obtain the formal Fourier expansion

$$f(\tau) = \sum_{T \in \mathrm{Sym}_n(\mathbb{Q})} a(T)\, e(\mathrm{tr}\, T\tau) \tag{3.13}$$

at F_0. Moreover, f is regular at F_0 if $a(T) = 0$ unless $T \geq 0$, and

$$\det(u)^k a(T) = a(T[u]) \tag{3.14}$$

for all $T \in L_{F_0}^\vee \subset \mathrm{Sym}_n(\mathbb{Q})$ and all $\begin{pmatrix} u & s \\ 0 & {}^t u^{-1} \end{pmatrix} \in \Gamma_{F_0}$.

Now assume that F, F' are two rational boundary component of degree m and that there exist a rational boundary component E of degree 0 such that $F > E$ and $F' > E$. Let f, f' be formal Fourier-Jacobi series of weight k for the group Γ and the boundary components F, F', respectively. Then f and f' have formal Fourier expansions

$$f = \sum_{s \in L_E^\vee} f_s, \qquad f' = \sum_{s \in L_E^\vee} f'_s$$

at E as in (3.8). We say that f is *compatible* with f' at E if these formal expansions agree.

Definition 3.5. Let I_m be the set of all rational boundary components of degree m, and let $0 \leq l \leq n$. A *formal Siegel modular form* of weight k and cogenus l for the group Γ is a family $(f_F)_{F \in I_{n-l}}$, where f_F is a formal Fourier-Jacobi series of weight k for the boundary component F and the group Γ satisfying the following conditions:

(i) for all $F \in I_{n-l}$ and all $\gamma \in \Gamma$ we have $f_F \mid_k \gamma = f_{\gamma^{-1}F}$;

(ii) for all pairs $F, F' \in I_{n-l}$ and all degree 0 boundary components $E \in I_0$ with $F > E$ and $F' > E$, the formal Fourier-Jacobi series f_F and $f_{F'}$ are compatible at E.

We write $\mathrm{FM}_k^{(n,l)}(\Gamma)$ for the complex vector space of formal Siegel modular forms of weight k and cogenus l for Γ.

Remark 3.6.

(1) Because of condition (i), the family $(f_F)_{F \in I_{n-l}}$ is determined by the f_F for F in a system of representatives for I_{n-l}/Γ. According to Remark 3.4 it suffices to check condition (ii) for a set of representatives of Γ_F-classes of adjacent degree 0 rational boundary components $E < F$.

(2) Conditions (i) and (ii) imply that f_F is regular at all adjacent degree 0 rational boundary components $E < F$. In fact, if $\gamma \in \Gamma_E$, then $E < \gamma^{-1}F$. Applying condition (ii) for f_F and $f_{\gamma^{-1}F}$ we see that the transformation law (3.9) holds.

Example 3.7.

(1) Let $f \in M_k(\Gamma)$ be a holomorphic Siegel modular form of weight k for Γ. Then at each boundary component $F \in I_{n-l}$ the function f has a (convergent) Fourier-Jacobi expansion f_F as in (3.1), which we may as well view as a formal series. The family $(f_F)_{F \in I_{n-l}}$ defines a formal Siegel modular form of weight k and cogenus l for Γ.

(2) Let $V \subset X_\Gamma^*$ be an open neighborhood of the boundary ∂X_Γ^*. Then any section $f \in \omega^{\otimes k}(V)$ defines an element of $\mathrm{FM}_k^{(n,l)}(\Gamma)$ for every l.

(3) Assume that $\Gamma = \Gamma_n$. Then any symmetric formal Fourier–Jacobi series of weight k and cogenus l in the sense of [BR] defines an element of $\mathrm{FM}_k^{(n,l)}(\Gamma_n)$.

3.2. **Formal completion of the sheaf of modular forms.** Throughout this section we fix an arithmetic subgroup $\Gamma \subset \mathrm{Sp}_n(\mathbb{Q})$. We briefly write X for X_Γ and X^* for X_Γ^*. We denote by

$$Y = \partial X^* \subset X^*$$

the boundary of X as in (2.16). Then Y is a closed analytic subset of codimension n of the projective complex algebraic variety X^*. If $n > 1$ then Y is connected.

We let $\hat{X}^* = (\hat{X}^*, \mathcal{O}_{\hat{X}^*})$ be the formal complex space given by the completion of X^* along Y, and we write

$$i : \hat{X}^* \to X^* \tag{3.15}$$

for the natural morphism of formal complex spaces. We denote by $\hat{\omega}^{\otimes k}$ the completion of the sheaf $\omega^{\otimes k}$ of modular forms of weight k with respect Y. Since $\omega^{\otimes k}$ is coherent, the natural map $i^*(\omega^{\otimes k}) \to \hat{\omega}^{\otimes k}$ is an isomorphism. The adjunction map $\omega^{\otimes k} \to i_* i^* \omega^{\otimes k}$ defines an injective map on global sections

$$\omega^{\otimes k}(X^*) \to \hat{\omega}^{\otimes k}(\hat{X}^*). \tag{3.16}$$

Let $W(F_m)$ be a connected open neighborhood of F_m as in Proposition 2.8, and put

$$V = \Gamma_{F_m} \backslash W(F_m) \subset X^*.$$

As before, we view Γ'_{F_m} as a lattice L_{F_m} in $\mathrm{Sym}_{n-m}(\mathbb{Q})$, and identify its dual $L_{F_m}^\vee$ with a sublattice of the same space via the symmetric bilinear form $(A, B) \mapsto \mathrm{tr}(AB)$. Recall the description of $\omega^{\otimes k}$ on V given in Lemma 2.10. We now give a description of the completion $\hat{\omega}^{\otimes k}$ on this boundary stratum.

Proposition 3.8. *The space $\hat{\omega}^{\otimes k}(V)$ of sections is given by the space of all* formal *series*

$$f(\tau) = \sum_{\substack{T_2 \in L_{F_m}^\vee \\ T_2 \geq 0}} \phi_{T_2}(\tau_1, \tau_{12})\, e(\mathrm{tr}\, T_2 \tau_2) \tag{3.17}$$

satisfying the following conditions:

(i) *The coefficients $\phi_{T_2}(\tau_1, \tau_{12})$ are holomorphic on $W(F_m) \cap \mathcal{D}_n$ for all $T_2 \in L_{F_m}^\vee$.*

(ii) *The transformation law $f \mid_k \gamma = f$ holds for all $\gamma \in \Gamma_{F_m}$.*

(iii) *For every $S \in L_{F_m}$ that is primitive, positive semidefinite of rank 1, and for all $t \in \mathbb{Z}_{\geq 0}$, the sub-series*

$$\sum_{\substack{T_2 \in L_{F_m}^\vee \\ T_2 \geq 0 \\ \mathrm{tr}(T_2 S) = t}} \phi_{T_2}(\tau_1, \tau_{12})\, e(\mathrm{tr}\, T_2 \tau_2)$$

converges normally on $W(F_m) \cap \mathcal{D}$ and defines a holomorphic function there.

Remark 3.9.

(1) In condition (ii) the transformation law is to be understood coefficientwise as in (3.10).

(2) In condition (iii) of Proposition 3.8, the matrix S determines a 1-dimensional rational isotropic subspace of $U(F_m)$, and therefore a degree $n-1$ rational boundary component E with $E \geq F_m$. Now condition (iii) means that in the formal Fourier-Jacobi expansion of f with respect to E all formal Fourier-Jacobi coefficients (of cogenus 1) converge.

(3) In the case $m = n-1$, the lattice L_{F_m} has rank 1, and hence $L_{F_m} = h\mathbb{Z}$ for some positive rational number h. Then $L_{F_m}^\vee = h^{-1}\mathbb{Z}$ and the conditions of Proposition 3.8 mean that

$$f(\tau) = \sum_{T_2 \in \frac{1}{h}\mathbb{Z}_{\geq 0}} \phi_{T_2}(\tau_1, \tau_{12})\, e(T_2\tau_2) \tag{3.18}$$

is a formal series, whose coefficients $\phi_{T_2}(\tau_1, \tau_{12})$ are holomorphic on $\mathbb{H}_{n-1} \times \mathbb{C}^{n-1}$ and satisfy the transformation law (ii) of a Jacobi form for $\gamma \in \Gamma_{F_{n-1}}$.

We now turn to the proof of Proposition 3.8, which will occupy the rest of this subsection. We use the Grothendieck comparison theorem [EGA3, Theorem 4.1.5] in the category of formal complex analytic spaces [Ba, Theorem 2] to reduce the computation to a smooth toroidal compactification.

3.2.1. *Toroidal compactification.* We begin by recalling some facts about toroidal compactifications. Our main references are [AMRT] and [Na].

Let $\Sigma = (\Sigma_F)_F$ be a Γ-admissible collection of fans as in [AMRT], Definition 5.1 in Chapter 3. According to Theorem 5.2 of loc. cit. there exist a toroidal compactification $X^{\mathrm{tor}} = X^{\mathrm{tor}}_\Sigma$ of $X = X_\Gamma$ associated with Σ. Throughout we assume that Σ is smooth. Then X^{tor} is a compact Moishezon space (i.e. an algebraic space over $\mathbb{C}$) which is smooth in the orbifold sense. It contains X as a dense open subset and the complement is a divisor with normal crossings. Moreover, there is a proper morphism

$$\pi : X^{\mathrm{tor}} \to X^*$$

to the Baily-Borel compactification X^* which restricts to the identity on X. We define the sheaf of weight 1 modular forms on X^{tor} as the pull back $\omega^{\mathrm{tor}} = \pi^*(\omega)$. The local Koecher principle implies that $\pi_*(\omega^{\mathrm{tor}}) = \pi_*\pi^*(\omega) = \omega$.

Let F be a rational boundary component of degree m. Let $W(F)$ be a connected open neighborhood of F as in Proposition 2.8, and put

$$V = \Gamma_F \backslash W(F) \subset X^*.$$

We write $\hat{V} = i^{-1}(V)$ for the inverse image of V under the morphism $i : \hat{X}^* \to X^*$. Moreover, we denote by $\hat{X}^{\mathrm{tor}}$ the completion of X^{tor} with respect to the toroidal boundary divisor $\pi^{-1}(Y)$, and by $\hat{V}^{\mathrm{tor}}$ the completion of $V^{\mathrm{tor}} = \pi^{-1}(V)$ with respect to $\pi^{-1}(Y \cap V)$. Then we have the commutative diagram of formal complex spaces

$$\begin{array}{ccccc} \hat{V}^{\mathrm{tor}} & \longrightarrow & \hat{X}^{\mathrm{tor}} & \xrightarrow{i^{\mathrm{tor}}} & X^{\mathrm{tor}} \\ \downarrow & & \downarrow{\scriptstyle \hat{\pi}} & & \downarrow{\scriptstyle \pi} \\ \hat{V} & \longrightarrow & \hat{X}^* & \xrightarrow{i} & X^* \end{array},$$

where the vertical morphisms are proper. We denote by $\hat{\omega}^{\mathrm{tor}}$ the completion of ω^{tor} with respect to $\pi^{-1}(Y)$. Since ω^{tor} is coherent, the natural map $(i^{\mathrm{tor}})^*(\omega^{\mathrm{tor}}) \to \hat{\omega}^{\mathrm{tor}}$ is an isomorphism, see [Ba, Lemma 2.3].

We now recall the description of X^{tor} near a rational boundary component F of degree m. Let $\mathcal{D}(F) = G'_{F,\mathbb{C}} \cdot \mathcal{D}_n$ be the Siegel domain of the third kind associated with F, see [AMRT, Definition III.4.5], and put $\mathcal{D}(F)' = \mathcal{D}(F)/G'_{F,\mathbb{C}}$. Then there is a two step holomorphic fibration

$$\mathcal{D}(F) \to \mathcal{D}(F)' \to F,$$

which is equivariant for the action of $G_F \cdot G'_{F,\mathbb{C}}$. Here $\mathcal{D}(F) \to \mathcal{D}(F)'$ is a principal $G'_{F,\mathbb{C}}$-bundle, and $\mathcal{D}_n \subset \mathcal{D}(F)$ is an open subset. If $F = F_m$ is the standard boundary component of degree m we have

$$\mathcal{D}(F_m) = \left\{ \begin{pmatrix} \tau_1 & \tau_{12} \\ {}^t\tau_{12} & \tau_2 \end{pmatrix} \mid \tau_1 \in \mathbb{H}_m,\ \tau_{12} \in \mathbb{C}^{m\times(n-m)},\ \tau_2 \in \mathrm{Sym}_{n-m}(\mathbb{C}) \right\},$$

and the map $\mathcal{D}(F) \to \mathcal{D}(F)'$ is the natural projection to the $\mathbb{H}_m \times \mathbb{C}^{m\times(n-m)}$ part.

The quotient

$$T_F = G'_{F,\mathbb{C}}/\Gamma'_F$$

is a complex algebraic torus of rank $r = \frac{1}{2}(n-m)(n-m+1)$, and

$$\Gamma'_F\backslash\mathcal{D}(F) \to \mathcal{D}(F)' \tag{3.19}$$

is a principal T_F-bundle containing $\Gamma'_F\backslash\mathcal{D}_n$ as an open subset. The fan Σ_F determines a torus embedding

$$T_F \to T_{F,\Sigma_F}. \tag{3.20}$$

The toroidal variety on the right hand side has an open covering by affine toric varieties

$$T_F \to T_{F,\sigma} \tag{3.21}$$

for the cones $\sigma \in \Sigma_F$. Recall that if N denotes the co-character lattice of T_F and $N^\vee$ its dual (the character lattice), then $T_{F,\sigma} = \mathrm{Spec}\,\mathbb{C}[\sigma^\vee \cap N^\vee]$. By taking the contraction product of (3.19) and (3.20), we obtain the fiber bundle

$$(\Gamma'_F\backslash\mathcal{D}(F))_{\Sigma_F} := (\Gamma'_F\backslash\mathcal{D}(F)) \times^{T_F} T_{F,\Sigma_F} \tag{3.22}$$

over $\mathcal{D}(F)'$ associated to (3.19) with fiber T_{F,Σ_F}. Now define

$$(\Gamma'_F\backslash\mathcal{D}_n)_{\Sigma_F} \tag{3.23}$$

as the interior of the closure of $\Gamma'_F\backslash\mathcal{D}_n$ in $(\Gamma'_F\backslash\mathcal{D}(F)) \times^{T_F} T_{F,\Sigma_F}$. It can be viewed as a partial compactification of $\Gamma'_F\backslash\mathcal{D}_n$ in the direction F. The group Γ_F/Γ'_F acts properly discontinuously on $(\Gamma'_F\backslash\mathcal{D}_n)_{\Sigma_F}$, and there is a holomorphic map

$$(\Gamma_F/\Gamma'_F)\backslash(\Gamma'_F\backslash\mathcal{D}_n)_{\Sigma_F} \to X^{\mathrm{tor}},$$

which restricts to an isomorphism in a sufficiently small neighborhood of the F-stratum of the toroidal boundary, see [AMRT], p. 175. In particular, V^{tor} can be identified with an open subset of the left hand side as follows. Define

$$\big(\Gamma'_F\backslash(W(F)\cap\mathcal{D}_n)\big)_{\Sigma_F}$$

as the interior of the closure of $\Gamma'_F\backslash(W(F)\cap\mathcal{D}_n)$ in $(\Gamma'_F\backslash\mathcal{D}(F)) \times^{T_F} T_{F,\Sigma_F}$. The group Γ_F/Γ'_F acts properly discontinuously, and the holomorphic map

$$(\Gamma_F/\Gamma'_F)\backslash\big(\Gamma'_F\backslash(W(F)\cap\mathcal{D}_n)\big)_{\Sigma_F} \to X^{\mathrm{tor}}$$

is an open immersion with image V^{tor}.

For the rest of this subsection we assume that $F = F_m$ is the standard boundary component of degree m. The space of sections $\omega^{\otimes k}(V^{\mathrm{tor}})$ is given by all continuous functions $f : W(F_m) \to \mathbb{C}$ that are holomorphic on $W(F_m)\cap\mathcal{D}_n$ and satisfy $f\mid_k \gamma = f$ for all $\gamma\in\Gamma_{F_m}$. Then f has a Fourier-Jacobi expansion of the form

$$f(\tau) = \sum_{\substack{T_2\in L_{F_m}^\vee\\ T_2\geq 0}} \phi_{T_2}(\tau_1,\tau_{12})\, e(\mathrm{tr}\, T_2\tau_2). \tag{3.24}$$

Let $\rho\in\Sigma_{F_m}$ be a ray (i.e. a cone of dimension 1), and let $e_\rho\in L_{F_m}$ be the unique primitive ray generator. The ray determines a toroidal boundary divisor D_ρ, and we write $\mathcal{I}_\rho\subset\mathcal{O}_{X^{\mathrm{tor}}}$ for the corresponding ideal sheaf. Then $(\omega^{\otimes k}\otimes\mathcal{I}_\rho)(V^{\mathrm{tor}})$ is given by the subspace of those $f\in\omega^{\otimes k}(V^{\mathrm{tor}})$ whose Fourier-Jacobi coefficients ϕ_{T_2} vanish identically for all $T_2\in L_{F_m}^\vee$ with $(T_2,e_\rho)=0$. Proposition 3.10 gives a description of the formal completion $(\hat\omega^{\mathrm{tor}})^{\otimes k}$ over V^{tor}.

Proposition 3.10. *The space $(\hat\omega^{\mathrm{tor}})^{\otimes k}(V^{\mathrm{tor}})$ of sections is given by the space of all* formal *series as in* (3.24) *satisfying the following conditions:*

(i) *The coefficients $\phi_{T_2}(\tau_1,\tau_{12})$ are holomorphic on $W(F_m)\cap\mathcal{D}_n$ for all $T_2\in L_{F_m}^\vee$.*
(ii) *The transformation law $f\mid_k\gamma = f$ holds for all $\gamma\in\Gamma_{F_m}$ in the sense of* (3.10).
(iii) *For all rays $\rho\in\Sigma_{F_m}$ with primitive ray generator $e_\rho\in L_{F_m}$ and for all $t\in\mathbb{Z}_{\geq 0}$, the sub-series*

$$\sum_{\substack{T_2\in L_{F_m}^\vee\\ T_2\geq 0\\ (T_2,e_\rho)=t}} \phi_{T_2}(\tau_1,\tau_{12})\, e(\mathrm{tr}\, T_2\tau_2) \tag{3.25}$$

converges normally on $W(F_m)\cap\mathcal{D}_n$ and defines a holomorphic function there.

Proof. The result is a consequence of the fiber bundle structure (3.22) and Proposition 3.12 on toroidal varieties. ∎

Proof of Proposition 3.8. According to the Grothendieck comparison theorem [Ba, Theorem 2], the natural map

$$\hat{\omega} = (\pi_* \omega^{\mathrm{tor}})^{\wedge} \to \hat{\pi}_*(\hat{\omega}^{\mathrm{tor}}) \tag{3.26}$$

is an isomorphism of $\mathcal{O}_{\hat{X}^*}$-modules. In particular, there is a natural isomorphism

$$\hat{\omega}^{\otimes k}(V) \cong (\hat{\omega}^{\mathrm{tor}})^{\otimes k}(V^{\mathrm{tor}}),$$

and therefore we may compute the left hand side using Proposition 3.10. Since the first two conditions of Proposition 3.10 agree with the first two conditions of Proposition 3.8 it remains to compare the corresponding third conditions.

We first notice that any $S \in L_{F_m} \subset \mathrm{Sym}_{n-m}(\mathbb{Q})$ which is primitive and positive semidefinite of rank 1 determines a 1-dimensional rational subspace of the isotropic subspace $U(F_m) \cong \mathbb{R}^{n-m}$ as in (2.1). It is given as the orthogonal complement of $\ker(S)$ with respect to the standard scalar product on $\mathbb{R}^{n-m}$. Therefore S determines a rational boundary component E of degree $n-1$ with $E \geq F_m$, and $\mathbb{Z}S = L_E \subset L_{F_m}$. Moreover, it determines a ray $\mathbb{R}_{\geq 0} S \in \Sigma_E$, and hence, according to [AMRT] Definition 5.1 in Chapter 3, also a ray in Σ_{F_m}. Consequently, condition (iii) in Proposition 3.10 implies (iii) in Proposition 3.8.

It remains to show the implication in the other direction. Assume that condition (iii) in Proposition 3.8 holds. Let $\rho \in \Sigma_{F_m}$ be a ray whose primitive ray generator $e_\rho \in L_{F_m}$ has rank greater than 1, and let $t \in \mathbb{Z}_{\geq 0}$. We need to prove that the series (3.25) converges.

Let $\gamma \in \mathrm{GL}_{n-m}(\mathbb{Q})$ such that $D := \gamma^{-1} e_\rho {}^t\gamma^{-1}$ is diagonal. Without loss of generality, we may assume that $D = \mathrm{diag}(d_1, \ldots, d_{n-m})$ with $d_i \in \mathbb{Z}_{\geq 0}$ and $d_1 > 0$. Let $E_{ii} \in \mathrm{Sym}_{n-m}(\mathbb{Q})$ be the symmetric matrix with entries all 0 except for a 1 at the position (i,i). For every positive semidefinite $T_2 \in \mathrm{Sym}_{n-m}(\mathbb{Q})$ we have

$$0 \leq \sum_{i=1}^{n-m} d_i \cdot (T_2, E_{ii}) = (T_2, D).$$

Hence, $0 \leq d_1(T_2, E_{11}) \leq (T_2, D)$. In particular, we obtain

$$0 \leq d_1({}^t\gamma T_2 \gamma, E_{11}) \leq ({}^t\gamma T_2 \gamma, D),$$

and thereby

$$0 \leq d_1(T_2, \gamma E_{11} {}^t\gamma) \leq (T_2, \gamma D {}^t\gamma) = (T_2, e_\rho).$$

Let $c \in \mathbb{Q}_{>0}$ such that

$$S := c \cdot \gamma E_{11} {}^t\gamma \in L_{F_m}^\vee$$

is primitive. By construction, S is positive semidefinite of rank 1, and

$$0 \leq (T_2, S) \leq \frac{c}{d_1}(T_2, e_\rho).$$

Hence if $(T_2, e_\rho) = t$, then $0 \leq (T_2, S) \leq \frac{c}{d_1} t$. Consequently, the series in (3.25) is a sub-series of

$$\sum_{\substack{T_2 \in L_{F_m}^\vee \\ T_2 \geq 0 \\ (T_2, S) \leq \frac{c}{d_1} t}} \phi_{T_2}(\tau_1, \tau_{12})\, e(\mathrm{tr}\, T_2 \tau_2).$$

The latter series converges normally, since it is a finite sum of normally convergent series by condition (iii) of Proposition 3.8. Therefore, (3.25) is also normally convergent. This concludes the proof of the Proposition. □

3.2.2. *Completion of toroidal varieties.* Here we summarize some facts about the completion of toroidal varieties at their boundary. This can be reduced to considering affine toric varieties.

Let r and d be positive integers with $1 \leq r \leq d$. Let $\mathcal{O}_{\mathbb{C}^d}$ be the sheaf of holomorphic functions on $\mathbb{C}^d$ and consider the ideal sheaf in $I \subset \mathcal{O}_{\mathbb{C}^d}$ generated by $(z_1 \cdots z_r)$.

Proposition 3.11. *Let $a = (a_1, \dots, a_d) \in \mathbb{C}^d$ with $a_1 \cdots a_r = 0$. Let $U \subset \mathbb{C}^d$ be an open polydisc of radius $R = (R_1, \dots, R_d)$ around a. Assume that $R_i < |a_i|$ for all $i \in \{1, \dots, r\}$ with $a_i \neq 0$. The family of natural maps of sheaves*

$$\mathcal{O}_U/(z_1 \cdots z_r)^n \to \mathcal{O}_U/(z_1 - a_1, \dots, z_d - a_d)^n$$

for $n \in \mathbb{Z}_{>0}$ induces an injective map

$$\Big(\lim_n \mathcal{O}_U/(z_1 \cdots z_r)^n\Big)(U) \to \mathbb{C}[[z-a]].$$

Its image is given by those formal power series

$$f = \sum_{\nu=(\nu_1,\dots,\nu_d)\in\mathbb{N}_0^d} c_\nu \cdot (z-a)^\nu \tag{3.27}$$

for which the subseries

$$f_{i,t} = \sum_{\substack{\nu=(\nu_1,\dots,\nu_d)\in\mathbb{N}_0^d \\ \nu_i = t}} c_\nu \cdot (z-a)^\nu \tag{3.28}$$

converge normally on U for all $i \in J_a := \{i \in \mathbb{N} \mid 1 \leq i \leq r,\ a_i = 0\}$ and for all $t \in \mathbb{N}_0$, and hence define holomorphic functions there.

Proof. By the hypothesis on a the set J_a is non-empty. For $n \in \mathbb{N}$, consider the morphisms of $\mathcal{O}_U$-modules

(3.29)
$$\mathcal{O}_U/(z_1 \cdots z_r)^n \xrightarrow{\varphi} \prod_{i\in J_a} \mathcal{O}_U/(z_i)^n \xrightarrow{\pi_j} \mathcal{O}_U/((z_1-a_1)^n, \dots, (z_d-a_d)^n).$$

Here φ is induced by the quotient maps, and the π_j are obtained by composing the j-th projection with the quotient map for $j \in J_a$. It is easily checked on stalks that φ is injective. Notice that for $i \in \{1, \dots, r\}$ with $a_i \neq 0$ the class of z_i is invertible in $\mathcal{O}_{U,b}$ for all $b \in U$ by the assumption on U. By taking the limit over n and sections over U, we obtain

$$\big(\lim_n \mathcal{O}_U/(z_1 \cdots z_r)^n\big)(U) \xrightarrow{\varphi_U} \prod_{i\in J_a} \big(\lim_n \mathcal{O}_U/(z_i)^n\big)(U) \xrightarrow{\pi_{j,U}} \mathbb{C}[[z-a]]. \tag{3.30}$$

Since the limit and the global sections functors are left exact, the map φ_U is injective.

Let $i \in J_a$. Since U is a Stein domain, by taking power series expansions, the limit

$$\big(\lim_n \mathcal{O}_U/(z_i)^n\big)(U) \cong \lim_n \mathcal{O}_U(U)/(z_i)^n$$

can be identified with the subalgebra of $\mathbb{C}[[z-a]]$ consisting of those formal power series as in (3.27) for which the subseries

$$f_{i,t} = \sum_{\substack{\nu=(\nu_1,\dots,\nu_d)\in\mathbb{N}_0^d \\ \nu_i=t}} c_\nu \cdot (z-a)^\nu$$

converge on U for all $t \in \mathbb{N}_0$, and hence define holomorphic functions there. In particular, the maps $\pi_{j,U}$ are injective. Moreover, the composition $\pi_{j,U} \circ \varphi_U$ is independent of j. Therefore the image $\mathrm{Im}(\varphi_U)$ of the map φ_U is contained in the space B of formal power series as claimed in the statement of the proposition.

To prove the inclusion $B \subset \mathrm{Im}(\varphi_U)$, let $f \in B$ and denote its power series expansion as in (3.27). Let $n \in \mathbb{N}$. The convergence of the subseries in (3.28) for all $i \in J_a$ and all $t \in \mathbb{N}_0$ implies that

$$F_n := \sum_{\substack{\nu\in\mathbb{N}_0^d \\ \exists i\in J_a\colon \nu_i<n}} c_\nu \cdot (z-a)^\nu$$

converges and hence defines an element of $\mathcal{O}_U(U)$. Moreover, the image $F_n + (z_1\cdots z_r)^n \in \mathcal{O}_U(U)/(z_1\cdots z_r)^n$ is mapped to $F_{n-1} + (z_1\cdots z_r)^{n-1} \in \mathcal{O}_U(U)/(z_1\cdots z_r)^{n-1}$ under the natural projection. Consequently, the family $F = (F_n)_{n\in\mathbb{N}}$ defines an element of the projective limit $\lim_n \mathcal{O}_U(U)/(z_1\cdots z_r)^n$ with the property that $\varphi_U(F) = f$. This concludes the proof of the proposition. □

Let $N \cong \mathbb{Z}^d$ be a lattice with dual $M = N^\vee$. Write $\langle n, m\rangle$ for the natural $\mathbb{Z}$-bilinear pairing of $n \in N$ and $m \in N^\vee$. Recall that a *rational polyhedral cone* σ in $N_\mathbb{R}$ is a subset which is generated over $\mathbb{R}_{\geq 0}$ by finitely many vectors $n_1,\dots,n_l \in N$, that is,

$$\sigma = \{t_1 n_1 + \cdots + t_l n_l \mid \ t_1,\dots t_l \in \mathbb{R}_{\geq 0}\}.$$

The *dual cone* of σ is defined by

$$\sigma^\vee = \big\{m \in N_\mathbb{R}^\vee \mid \ \langle n, m\rangle \geq 0 \text{ for all } n \in \sigma\big\}.$$

It is a rational polyhedral cone in $N_\mathbb{R}^\vee$. The cone σ is called *strongly convex* if it contains no full lines. This is equivalent to $\sigma^\vee$ being of full dimension d in $N_\mathbb{R}^\vee$. For any rational polyhedral cone σ, the monoid $\sigma^\vee \cap N^\vee$ is finitely generated, and the monoid algebra $A_\sigma = \mathbb{C}[\sigma^\vee \cap N^\vee]$ is a finitely generated $\mathbb{C}$-algebra. The affine toric variety

$$T_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap N^\vee]$$

associated with σ is a normal irreducible affine algebraic variety over $\mathbb{C}$, containing the torus $T_N = \operatorname{Spec} \mathbb{C}[N^\vee]$ as an open dense subvariety.

Let Σ be a rational fan in $N_\mathbb{R}$, that is, a collection of strongly convex rational polyhedral cones in $N_\mathbb{R}$ satisfying the conditions:

- If $\sigma \in \Sigma$ and $\tau \leq \sigma$ is a face of σ, then $\tau \in \Sigma$.
- If $\sigma \in \Sigma$ and $\tau \in \Sigma$, then $\sigma \cap \tau \leq \sigma$ and $\sigma \cap \tau \leq \tau$.

We write T_Σ for the toric variety over $\mathbb{C}$ associated with Σ. It is obtained by gluing the T_σ for $\sigma \in \Sigma$ along common faces, see, e.g., [Fu].

The dimension of a cone σ in $N_\mathbb{R}$ is the dimension of the $\mathbb{R}$-vector space $\sigma + (-\sigma)$. A rational polyhedral cone is called *smooth* if it is generated by part of a basis of N. In this case it has exactly $l = \dim \sigma$ edges (i.e. 1-dimensional faces) and a canonical set of generators (which is part of a basis of N) is given by the ray generators of the edges of σ. A fan Σ is called smooth if all its cones are smooth. This is equivalent to T_Σ being smooth.

If X is a scheme of finite type over $\mathbb{C}$, we denote by X^{an} its analytification as in [SGA1, Exposé XII]. We now consider the toric variety T_Σ^{an} associated with a smooth rational fan Σ in $N_\mathbb{R}$ as a complex analytic space. Let $\hat{T}_\Sigma^{\mathrm{an}}$ be the completion of T_Σ^{an} at the toric boundary divisor

$$D_\Sigma = \sum_{\substack{\tau\in\Sigma \\ \dim\tau=1}} D_\tau,$$

where D_τ denotes the toric divisor associated with a ray τ. We write $e_\tau \in N\cap\tau$ for the ray generator of τ. Proposition 3.11 immediately implies the following result.

Proposition 3.12. *Let a be a point in the torus orbit corresponding to a cone $\sigma\in\Sigma$, and let $U\subset T_\sigma^{\mathrm{an}}$ be an open polydisc around a. The $\mathbb{C}$-algebra $\mathcal{O}_{\hat{T}_\Sigma^{\mathrm{an}}}(U)$ is given by those formal power series*

$$f = \sum_{\nu\in N^\vee} c_\nu\cdot z^\nu \in \mathbb{C}[[N^\vee]]$$

with the following properties:

(i) *We have $c_\nu = 0$ unless $\nu\in\sigma^\vee$.*
(ii) *For all edges $\tau\le\sigma$, and for all $t\in\mathbb{N}_0$ the subseries*

$$f_{\tau,t} = \sum_{\substack{\nu\in N^\vee \\ \langle e_\tau,\nu\rangle = t}} c_\nu\cdot z^\nu$$

converges normally and defines a holomorphic function on U.

Here we have put $z^\nu = e^{2\pi i\langle\zeta,\nu\rangle}$ for $\zeta\in N_\mathbb{C}/N$.

Proof. Denote by r the dimension of σ. Let $\tau_1,\dots,\tau_r$ be the edges of σ and let $e_i\in\tau_i\cap N$ be the corresponding ray generators. Then $e_1,\dots,e_r$ is the minimal set of generators of σ and there exist $e_{r+1},\dots,e_d\in N$ such that $e_1,\dots,e_d$ are a lattice basis of N. Let $e_1^\vee,\dots,e_d^\vee\in N^\vee$ be the corresponding dual basis of $N^\vee$. The assignment $e_i^\vee\mapsto X_i$ determines a $\mathbb{C}$-algebra isomorphism

$$A_\sigma = \mathbb{C}[\sigma^\vee\cap N^\vee] \to \mathbb{C}[X_{r+1}^{\pm1},\dots,X_d^{\pm1}][X_1,\dots,X_r], \tag{3.31}$$

and hence an isomorphism $(\mathbb{C}^\times)^{d-r}\times\mathbb{C}^r \cong T_\sigma^{\mathrm{an}}$. The vanishing ideal of the restriction of D_Σ to T_σ is given by

$$I_\sigma = (e_1^\vee+\cdots+e_r^\vee)\subset\mathbb{C}[\sigma^\vee\cap N^\vee].$$

It corresponds to the ideal $(X_1\cdots X_r) = (X_1\cdots X_d)\subset\mathbb{C}[X_{r+1}^{\pm1},\dots,X_d^{\pm1}][X_1,\dots,X_r]$ under (3.31). Hence we may use Proposition 3.11 to compute the completion

$$\hat{\mathcal{O}}_{T_\sigma^{\mathrm{an}},I_\sigma} = \lim_n \mathcal{O}_{T_\sigma^{\mathrm{an}}}/I_\sigma^n$$

of the structure sheaf $\mathcal{O}_{T_\sigma^{\mathrm{an}}}$ at the ideal I_σ. With the above identifications we obtain the assertion. ∎

3.3. **Formal Siegel modular forms of cogenus** 1. Let $f \in \hat{\omega}^{\otimes k}(\hat{X}^*)$ be a global section. If F is any rational boundary component of degree $n-1$, we may restrict f to $\Gamma_F\backslash W(F)$. By Proposition 3.8, we obtain a formal Fourier-Jacobi series f_F of weight k for F and the group Γ. In particular, for the standard boundary component F_{n-1} we obtain an expansion as in (3.18). The family $(f_F)_{F\in I_{n-1}}$ of formal Fourier-Jacobi series is compatible in the sense of Definition 3.5. Hence it determines a formal Siegel modular form of weight k and cogenus 1 for Γ. The assignment $f \mapsto (f_F)$ defines a homomorphism of complex vector spaces

$$\hat{\omega}^{\otimes k}(\hat{X}^*) \to \mathrm{FM}_k^{(n,1)}(\Gamma). \tag{3.32}$$

Since any global section $f \in \hat{\omega}^{\otimes k}(\hat{X}^*)$ is uniquely determined by its formal Fourier-Jacobi expansion of cogenus 1, the map is injective.

Theorem 3.13. *The above map in* (3.32) *is an isomorphism.*

Proof. We have to show that the map is surjective. Let $g = (g_F)_{F\in I_{n-1}} \in \mathrm{FM}_k^{(n,1)}(\Gamma)$. For every proper rational boundary component F we chose a sufficiently small open neighborhood of $V(F) = \Gamma_F\backslash W(F)$ as in Proposition 2.8. We show that g determines a section in $\hat{\omega}^{\otimes k}(V(F))$ for every F and that these sections agree on all pairwise intersections of these open neighborhoods. Hence they glue to a global section $f \in \hat{\omega}^{\otimes k}(\hat{X}^*)$ which maps to g under the map (3.32). First, we assume that $\Gamma = \Gamma_n(N)$ is the principal congruence subgroup of level N and genus n.

(1) We begin with the rank $n-1$ standard boundary component F_{n-1}. According to Proposition 3.8, the expansion

$$g_{F_{n-1}}(\tau) = \sum_{T_2\in\mathbb{Q}} \phi_{T_2}(\tau_1,\tau_{12})\, e(T_2\tau_2) \tag{3.33}$$

of $g_{F_{n-1}}$ as in (3.11), together with the transformation law (3.10), imply that $g_{F_{n-1}}$ defines an element of $\hat{\omega}^{\otimes k}(V(F_{n-1}))$.

(2) Now we consider the rank 0 standard boundary component F_0. The formal Fourier-Jacobi series $g_{F_{n-1}}$ has a formal Fourier expansion

$$g_{F_{n-1}} = \sum_{T\in\mathrm{Sym}_n(\mathbb{Q})} a(T)\, e(\mathrm{tr}\, T\tau) \tag{3.34}$$

as in (3.13). Since $g_{F_{n-1}}$ is regular at the boundary component F_0 (by Remark 3.6), the Fourier coefficients $a(T)$ satisfy the transformation law (3.14). Hence the series (3.34) satisfies conditions (i) and (ii) of Proposition 3.8 for $V(F_0)$. It remains to show that it also satisfies condition (iii).

To this end let $S \in L_{F_0} \subset \mathrm{Sym}_n(\mathbb{Q})$ be primitive and positive semi-definite of rank 1. Denote by $R \in L_{F_0}$ the positive semi-definite generator of the 1-dimensional lattice $L_{F_{n-1}}$. Then R is a matrix with entries all 0 except for the position (n,n). Write

$$S = h\cdot uR\,{}^tu$$

with $u \in \mathrm{GL}_n(\mathbb{Z})$ and $h \in \mathbb{Q}_{>0}$. Consider the boundary component $F = \delta F_{n-1}$, where $\delta = \left(\begin{smallmatrix} u & 0 \\ 0 & {}^tu^{-1}\end{smallmatrix}\right) \in \Gamma_{n,0}$, and the corresponding formal Fourier-Jacobi series g_F.

The formal Fourier expansions of $g_{F_{n-1}}$ and g_F at F_0 are given by

$$g_{F_{n-1}} = \sum_{t\geq 0} \sum_{\substack{T\geq 0\\ \operatorname{tr} TR=t}} a(T)q^T,$$

$$g_F = \sum_{t\geq 0} \sum_{\substack{T\geq 0\\ \operatorname{tr} TS=t}} b(T)q^T.$$

Since $F_{n-1} > F_0$ and $F > F_0$, the series $g_{F_{n-1}}$ and g_F must be compatible at F_0, and therefore $a(T) = b(T)$. Since g_F is a formal Fourier-Jacobi series for the boundary component F, the formal Fourier-Jacobi coefficients

$$\sum_{\substack{T\geq 0\\ \operatorname{tr} TS=t}} a(T)q^T$$

converge normally for all t. This gives the desired convergence condition (iii) of Proposition 3.8, and therefore shows that the formal Fourier expansion of $g_{F_{n-1}}$ defines an element of $\hat{\omega}^{\otimes k}(V(F_0))$, which agrees with $g_{F_{n-1}}$ on $V(F_0) \cap V(F_{n-1})$.

(3) Now let $0 \leq m \leq n-1$ and consider the boundary component F_m. We use the formal Fourier expansion (3.34) of $g_{F_{n-1}}$ to define Fourier-Jacobi coefficients

$$\psi_{T_2}(\tau_1, \tau_{12}) = \sum_{T=\left(\begin{smallmatrix} T_1 & T_{12}\\ {}^tT_{12} & T_2\end{smallmatrix}\right)\geq 0} a(T)e(\operatorname{tr} T_1\tau_1 + 2\operatorname{tr} T_{12}{}^t\tau_{12})$$

for $T_2 \in L_{F_m}^\vee \subset \operatorname{Sym}_{n-m}(\mathbb{Q})$. These series converge normally, since they are subseries of the Fourier expansions of the Fourier-Jacobi coefficients $\phi_{T_2^*}$ in (3.33), where T_2^* denotes the lower right entry of T_2. Then we have the identity of formal series

$$g_{F_{n-1}} = \sum_{T_2\in \operatorname{Sym}_{n-m}(\mathbb{Q})} \psi_{T_2}(\tau_1, \tau_{12})\, e(\operatorname{tr} T_2\tau_2). \tag{3.35}$$

By Lemma 3.14, the transformation law of (3.33) under $\Gamma_{F_{n-1}}$, and the transformation law of (3.34) under Γ_{F_0} imply that (3.35) has the transformation law of Proposition 3.8 (ii) for Γ_{F_m} (note that it is here where the choice of the group $\Gamma_n(N)$ plays a role). Moreover, condition (iii) of Proposition 3.8 follows from the corresponding condition for the formal Fourier expansion (3.34). Hence (3.35) defines a section in $\hat{\omega}^{\otimes k}(V(F_m))$, which is compatible with the sections in $\hat{\omega}^{\otimes k}(V(F_{n-1}))$ and $\hat{\omega}^{\otimes k}(V(F_0))$ that were constructed before.

(4) Let E be any proper rational boundary component. Let $0 \leq m \leq n-1$ and $\delta \in \Gamma_n$ such that $E = \delta F_m$. Consider the translated series $g_F \mid_k \delta$, which defines a formal Fourier Jacobi series for F_m and the group $\delta^{-1}\Gamma\delta$ by Remark 3.2. According to part (3) above it defines a section $\tilde{g} \in \hat{\omega}^{\otimes k}(\delta^{-1}\Gamma_E\delta\backslash\delta^{-1}W(E))$. By taking its pullback $\tilde{g} \mid_k \delta^{-1}$, we obtain a section in $\hat{\omega}^{\otimes k}(V(E))$.

(5) We now show that the local sections of (1)–(4) agree on all open intersections. Since the support of $\hat{\omega}^{\otimes k}$ is given by the boundary Y, it suffices to consider all pairs of proper rational boundary components F and F' for which

$$\bar{F} \cap \bar{F}' \neq \emptyset.$$

This implies that there exists a rational degree 0 boundary component E such that $F \geq E$ and $F' \geq E$. In view of part (3) we may assume without loss of generality that F and F' both have degree $n-1$. We have to show that the restrictions to

$V(E)$ of the sections of $\hat{\omega}^{\otimes k}(V(F))$ and $\hat{\omega}^{\otimes k}(V(F'))$ constructed in (4) agree, which is a consequence of the compatibility of g_F and $g_{F'}$ at E. This concludes the proof if $\Gamma = \Gamma_n(N)$.

(6) In the above argument we have only used the assumption that $\Gamma = \Gamma_n(N)$ in step (3) for $0 < m < n-1$ when we invoked Lemma 3.14. If $n \leq 2$ there are no such integers m and hence the argument also applies for any arithmetic subgroup $\Gamma \subset \mathrm{Sp}_n(\mathbb{Q})$. Finally, assume that $n \geq 2$ and that $\Gamma \subset \mathrm{Sp}_n(\mathbb{Q})$ is an arbitrary arithmetic subgroup. There exists a positive integer N such that $\Gamma' := \Gamma_n(N)$ is a subgroup of Γ of finite index. This corresponds to a finite covering $\pi : X^*_{\Gamma'} \to X^*_\Gamma$ of complex spaces and of their completions at the Satake boundaries. We may view the formal Siegel modular form g as an element $\mathrm{FM}_k^{(n,1)}(\Gamma')$. By the above argument, it determines a section $f \in \hat{\omega}^{\otimes k}(\hat{X}^*_{\Gamma'})$. Condition (i) of Definition 3.5 implies that f is actually invariant under the action of Γ, and hence descends to a section in $\omega^{\otimes k}(\hat{X}^*_\Gamma)$, which maps to g under the map (3.32). □

We conclude this subsection with the lemma that was used in the proof of Theorem 3.13.

Lemma 3.14. *Assume that $\Gamma = \Gamma_n(N)$ is the principal congruence subgroup of level N and genus n. Then for every triple of adjacent rational boundary components $F > F' > F''$, where F has degree $n-1$ and F'' has degree 0, the stabilizer $\Gamma_{F'}$ is generated by $\Gamma_F \cap \Gamma_{F'}$ and $\Gamma_{F''} \cap \Gamma_{F'}$.*

Proof. Since $\Gamma = \Gamma_n(N)$ is a normal subgroup of the full Siegel modular group Γ_n, and since Γ_n acts transitively on chains of adjacent rational boundary components of fixed degrees, it suffices to prove the assertion for triples of standard boundary components

$$F_{n-1} > F_m > F_0,$$

where $n-1 > m > 0$. Let

$$\gamma = \begin{pmatrix} a & 0 & * & * \\ * & u & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix}$$

be an element of Γ_{F_m}, where $u \in \mathrm{GL}_{n-m}(\mathbb{Z})$ is congruent to 1 modulo N. Then

$$\delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix}$$

belongs to $\Gamma_{F_0} \cap \Gamma_{F_m}$. The product $\delta^{-1}\gamma$ is contained in $\Gamma_{F_{n-1}} \cap \Gamma_{F_m}$, proving the assertion. □

4. Modularity of formal Siegel modular forms

4.1. **Affine covering numbers.** Let S be a scheme. Recall that the *affine covering number* $\mathrm{acn}(S)$ of S is defined as one less than the smallest number of open affine sets required to cover S, see, e.g., [At], [RV]. It gives an upper bound for the cohomological dimension of S, which is the largest integer j such that $H^j(S,\mathcal{F}) \neq 0$ for some quasicoherent sheaf $\mathcal{F}$ [RV, Proposition 4.12]. If S is quasi-projective, then according to [RV, Example 4.8], we have the trivial bound $\mathrm{acn}(S) \leq \dim(S)$.

We consider this notion for the Siegel modular variety $X_\Gamma = \Gamma\backslash\mathbb{H}_n$ associated with an arithmetic subgroup $\Gamma \subset \mathrm{Sp}_n(\mathbb{Q})$. Since the line bundle of modular forms on the Baily-Borel compactification X_Γ^* is ample, the complement of the divisor $\mathrm{div}(F)$ of any holomorphic modular form F of weight k defines an affine open subset of X_Γ^*. If F is a cusp form, then $X_\Gamma^* \setminus \mathrm{div}(F)$ is actually an affine open subset of X_Γ. Hence, $\mathrm{acn}(X_\Gamma)$ is the smallest non-negative integer j, for which there exist *cusp forms* $F_0, \dots, F_j$ for Γ having no common zero on X_Γ.

It is our goal to find upper bounds for $\mathrm{acn}(X_{\Gamma_n})$, where $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$. According to [At, Theorem 4], we have $\mathrm{acn}(X_{\Gamma_n}) \geq n(n-1)/2$. It is an interesting question whether this lower bound is actually an equality. If $n = 1$, then $\mathrm{acn}(X_{\Gamma_1}) = 0$, since X_{Γ_1} is affine. For $n = 2$ it is easy to see that

$$\mathrm{acn}(X_{\Gamma_2}) = 1,$$

since the Igusa cusp forms χ_{10} and χ_{12} for Γ_2 (see p. 848 in [Ig1]) have no common zeros on $\mathbb{H}_2$. However, for general n not much is known in this direction. It even seems to be difficult to find upper bounds that improve upon the trivial bound $n(n+1)/2$.

Here we use theta functions to obtain upper bounds for $\mathrm{acn}(X_{\Gamma_n})$ for further small values of n, see Propositions 4.3 and 4.5. Recall that for every theta characteristic $m = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{Z}^{2n}$ of genus n and for $\tau \in \mathbb{H}_n$ there is a theta constant defined by

$$\theta\left[m\right](\tau) = \theta\begin{bmatrix} a \\ b \end{bmatrix}(\tau) = \sum_{x \in \mathbb{Z}^n} \exp\left(\pi i \left({}^t(x+a/2)\tau(x+a/2) + {}^t(x+a/2)b\right)\right).$$

Since this function depends up to the sign only on m modulo $(2\mathbb{Z})^{2n}$, it is common to define $\theta\left[m\right](\tau) = \theta\left[\iota(m)\right](\tau)$ for $m \in \mathbb{F}_2^{2n}$. Here ι is defined by the embedding

$$\mathbb{F}_2 \to \mathbb{Z}, \quad 0 \mapsto 0,\ 1 \mapsto 1.$$

A theta characteristic $m = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{F}_2^{2n}$ is called even if ${}^t ab \equiv 0 \pmod 2$, and odd otherwise. There are $2^{n-1}(2^n+1)$ even and $2^{n-1}(2^n-1)$ odd theta characteristics in genus n. The function $\theta\left[m\right]$ vanishes identically if and only if m is odd. We denote by $m \mapsto \gamma.m$ the usual action of $\gamma \in \Gamma_n$ on the theta characteristcs, see, e.g., [Fr, Chapter I.3]. There are two orbits under this action, given by the even and the odd characteristics. We write $\mathcal{E}_n \subset \mathbb{F}_2^{2n}$ for the set of even theta characteristics. The theta transformation formula implies that

$$\theta^8[\gamma.m](\gamma\tau) = \det(c\tau+d)^4 \cdot \theta^8[m](\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_n$. In particular, the theta constants are modular forms for a congruence subgroup of Γ_n. Sightly more precisely, their eighth powers are modular forms of weight 4 for the principal congruence subgroup $\Gamma_n(2) \subset \Gamma_n$ of level two, see, e.g., [Ig2, Chapter V.1]. For a subset $\mathcal{S} \subset \mathcal{E}_n$ we write

$$\theta[\mathcal{S}](\tau) = \prod_{m \in \mathcal{S}} \theta[m](\tau)$$

for the corresponding product of theta constants. It is a modular form of weight $\#\mathcal{S}/2$.

If $n = n_1 + n_2$ and $m_i = \left(\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right) \in \mathbb{F}_2^{2n_i}$ are theta characteristics of genus n_1 and n_2, respectively, then

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{4.1}$$

determine a theta characteristic $m = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{F}_2^{2n}$. In this case we write $m = (m_1, m_2)$. If $\mathcal{S}_1 \subset \mathcal{E}_{n_1}$ and $\mathcal{S}_2 \subset \mathcal{E}_{n_2}$ are subsets, then $\mathcal{S}_1 \times \mathcal{S}_2 \subset \mathcal{E}_n$ and we may consider the corresponding theta products $\theta[\mathcal{S}_1 \times \mathcal{S}_2]$ of genus n. Lemma 4.1 is an easy consequence of the behavior of theta constants under the Siegel phi operator, see, e.g., Remark 3.10 in Chapter I.3 of [Fr].

Lemma 4.1. *Let $\mathcal{S} \subset \mathcal{E}_n$ and assume that $\#\mathcal{S} > 2^{n-1}(2^{n-1}+1)$. Then $\theta[\mathcal{S}]$ is a cusp form.*

As in [FP] and [Ig1] we consider the following modular forms of genus n:

$$F_{\mathrm{null}} = \theta[\mathcal{E}_n], \tag{4.2}$$

$$F_1 = \sum_{m \in \mathcal{E}_n} \theta[\mathcal{E}_n \setminus \{m\}]^8. \tag{4.3}$$

These forms have weight $2^{n-2}(2^n+1)$ and $2^{n+1}(2^n+1) - 4$, respectively. If $n \geq 2$, then according to Lemma 4.1, they are both cusp forms for Γ_n. If $n = 1$, then $F_{\mathrm{null}}^8 = \Delta$. For $n = 2$ we have have $F_{\mathrm{null}}^2 = \chi_{10}$.

We define the pushforward $M_k(\Gamma_n(2)) \to M_k(\Gamma_n)$ of modular forms of genus n by

$$P_n(f) = \sum_{\gamma \in \Gamma_n(2) \backslash \Gamma_n} f \mid_k \gamma \tag{4.4}$$

for $f \in M_k(\Gamma_n(2))$. It takes cusp forms to cusp forms. For example, let $\mathcal{E}_n^* = \mathcal{E}_n \setminus \{0\}$. Then the theta transformation formula and the fact that $\Gamma_n(2)\backslash\Gamma_n$ acts transitively on $\mathcal{E}_n$ imply that

$$P_n(\theta[\mathcal{E}_n^*]^8) = C \cdot F_1,$$

where C denotes the order of the stabilizer in $\Gamma_n(2)\backslash\Gamma_n$ of the zero characteristic.

For $n = 3$ the form F_{null} is a cusp form of weight 18, denoted χ_{18} in [Ig1], and F_1 is a cusp form of weight 140, denoted Σ_{140} in [Ig1]. By a result of Igusa [Ig1, Lemma 11], the common vanishing locus $\mathrm{div}(F_{\mathrm{null}}) \cap \mathrm{div}(F_1)$ is exactly the reducible locus of X_{Γ_3}, i.e., the image of the natural map $X_{\Gamma_1} \times X_{\Gamma_2} \to X_{\Gamma_3}$.

We now construct two cusp forms for Γ_3 whose simultaneous vanishing locus is disjoint from the reducible locus. To this end we consider the subsets

$$\mathcal{E}_{1,2} := \mathcal{E}_1 \times \mathcal{E}_2,$$
$$\mathcal{E}_{1,1,1} := \mathcal{E}_1 \times \mathcal{E}_1 \times \mathcal{E}_1$$

of $\mathcal{E}_3$. By Lemma 4.1, the corresponding theta products $\theta[\mathcal{E}_{1,2}]^8$ and $\theta[\mathcal{E}_{1,1,1}]^8$ are cusp forms for $\Gamma_3(2)$ of weight 120 and 108, respectively. We define cusp forms for Γ_3 by

$$F_{1,2} = P_3(\theta[\mathcal{E}_{1,2}]^8),$$
$$F_{1,1,1} = P_3(\theta[\mathcal{E}_{1,1,1}]^8).$$

Lemma 4.2.

(i) *The restriction of* $F_{1,2}$ *to* $X_{\Gamma_1} \times X_{\Gamma_2}$ *is given by*

$$F_{1,2} \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} = C \cdot \Delta(\tau_1)^{10} \cdot \chi_{10}(\tau_2)^{12},$$

where $\tau_1 \in \mathbb{H}_1$, $\tau_2 \in \mathbb{H}_2$, *and* C *is the order of the stabilizer in* $\Gamma_3(2)\backslash\Gamma_3$ *of the set* $\mathcal{E}_{1,2}$.

(ii) *The restriction of* $F_{1,1,1}$ *to* $X_{\Gamma_1}^3$ *is given by*

$$F_{1,1,1} \begin{pmatrix} \tau_1 & & \\ & \tau_2 & \\ & & \tau_3 \end{pmatrix} = C \cdot \Delta(\tau_1)^9 \Delta(\tau_2)^9 \Delta(\tau_3)^9,$$

where C *is the order of the stabilizer in* $\Gamma_3(2)\backslash\Gamma_3$ *of the set* $\mathcal{E}_{1,1,1}$.

Proof. We only carry out the proof of (i), since the proof of (ii) is analogous. We compute the restriction to $\mathbb{H}_1 \times \mathbb{H}_2$ of the summands

$$\theta[\mathcal{E}_{1,2}]^8 \mid_{120} \gamma = \theta[\gamma^{-1}\mathcal{E}_{1,2}]^8$$

in the definition of $P_3(\theta[\mathcal{E}_{1,2}]^8)$ for every $\gamma \in \Gamma_3(2)\backslash\Gamma_3$. There are two cases.

First, if γ takes $\mathcal{E}_{1,2}$ to itself, then $\theta[\gamma^{-1}\mathcal{E}_{1,2}]^8 = \theta[\mathcal{E}_{1,2}]^8$, and therefore the restriction is given by

$$\begin{aligned} \theta[\mathcal{E}_{1,2}]^8 \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} &= \prod_{\substack{m_1 \in \mathcal{E}_1 \\ m_2 \in \mathcal{E}_2}} \theta[m_1](\tau_1)^8 \theta[m_2](\tau_2)^8 \\ &= \theta[\mathcal{E}_1](\tau_1)^{80} \cdot \theta[\mathcal{E}_2](\tau_2)^{24} = \Delta(\tau_1)^{10} \cdot \chi_{10}(\tau_2)^{12}. \end{aligned}$$

Second, if γ does not take $\mathcal{E}_{1,2}$ to itself, then there exists an $m \in \mathcal{E}_{1,2}$ with $\tilde{m} := \gamma^{-1} m \notin \mathcal{E}_{1,2}$. Writing $\tilde{m} = (\tilde{m}_1, \tilde{m}_2)$ as in (4.1), we see that $\tilde{m}_1 \in \mathbb{F}_2^2$ must be odd. But this implies that

$$\theta[\tilde{m}]^8 \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} = \theta[\tilde{m}_1](\tau_1)^8 \theta[\tilde{m}_2](\tau_2)^8 = 0.$$

This proves the claim. □

Proposition 4.3. *The cusp forms* F_{null}, F_1, $F_{1,2}$, *and* $F_{1,1,1}$ *for* Γ_3 *have no common zero on* $\mathbb{H}_3$. *In particular,* $\operatorname{acn}(X_{\Gamma_3}) = 3$.

Proof. According to [Ig1, Lemma 11], the common vanishing locus of F_{null} and F_1 is exactly the reducible locus of X_{Γ_3}. But Lemma 4.2 and the fact that the vanishing divisor of χ_{10} is precisely the reducible locus of X_{Γ_2} imply that $F_{1,2}$ and $F_{1,1,1}$ never vanish simultaneously on the reducible locus of X_{Γ_3}. □

We now turn to the case $n = 4$. Following [FP] we consider the following modular forms, for Γ_n, which are defined for any n:

$$F_H = \sum_{A \subset \mathbb{F}_2^{2n}} \theta[\mathcal{E}_n \setminus A]^8, \tag{4.5}$$

$$F_T = 2^n \sum_{m \in \mathcal{E}_n} \theta[m]^{16} - \Big(\sum_{m \in \mathcal{E}_n} \theta[m]^8 \Big)^2. \tag{4.6}$$

In (4.5), the subsets A runs through all suitable sets of $v(n)$ characteristics corresponding to the irreducible components of the hyperelliptic locus of $X_{\Gamma_n(2)}$ as in [FP, Theorem 2]. These forms have weight $2\binom{2n+2}{n+1}$ and 8, respectively. For $n = 4$, we have $v(4) = 10$ and F_H is cuspidal by Lemma 4.1. Moreover, F_T is the Schottky cusp form.

Lemma 4.4. *For $n = 4$, the common vanishing locus of the cusp forms F_{null}, F_1, F_H, F_T is the reducible locus $X_{\Gamma_4}^{\mathrm{red}}$ of X_{Γ_4}.*

Proof. By a result of Igusa [Ig3, p. 544], the intersection $\mathrm{div}(F_{\mathrm{null}}) \cap \mathrm{div}(F_1) \cap \mathrm{div}(F_T)$ is equal to the union of the hyperelliptic locus and the reducible locus of X_{Γ_4}. According to [FP, Lemma 2], the form F_H never vanishes on the hyperelliptic locus, while it vanishes on the reducible locus by the argument in the proof of [FP, Theorem 1]. □

Proposition 4.5. *There are 5 cusp forms for Γ_4 that have no common zero on $X_{\Gamma_4}^{\mathrm{red}}$. In particular,* $\mathrm{acn}(X_{\Gamma_4}) \leq 8$.

Proof. We argue similarly as in the proof of Proposition 4.3. We use the pushforward of suitable theta products in genus 4 to construct the desired cusp forms. We let

$$\begin{aligned} F_{1,3} &= P_4(\theta[\mathcal{E}_1 \times \mathcal{E}_3]^8), \\ F_{1,3}^* &= P_4(\theta[\mathcal{E}_1 \times \mathcal{E}_3^*]^8), \\ F_{1,1,2} &= P_4(\theta[\mathcal{E}_1 \times \mathcal{E}_1 \times \mathcal{E}_2]^8), \\ F_{1,1,1,1} &= P_4(\theta[\mathcal{E}_1 \times \mathcal{E}_1 \times \mathcal{E}_1 \times \mathcal{E}_1]^8). \end{aligned}$$

By Lemma 4.1 these modular forms are cuspidal. Their weights are 432, 420, 360, 324, respectively. Their restriction to $\mathbb{H}_1 \times \mathbb{H}_3 \subset \mathbb{H}_4$ can be computed as in Lemma 4.2. For instance, we have

$$\begin{aligned} F_{1,3} \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} &= C_1 \cdot \Delta(\tau_1)^{36} F_{\mathrm{null}}(\tau_2)^{24}, \\ F_{1,3}^* \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} &= C_2 \cdot \Delta(\tau_1)^{35} F_1(\tau_2)^{3}, \end{aligned}$$

where C_1, C_2 are suitable positive integral constants and $\tau_1 \in \mathbb{H}_1$, $\tau_2 \in \mathbb{H}_3$. By means of Proposition 4.3 it can be concluded that these four cusp forms never vanish simultaneously on the image of the natural map $X_{\Gamma_1} \times X_{\Gamma_3} \to X_{\Gamma_4}$.

If we define in addition the cusp form

$$F_{2,2} = P_4(\theta[\mathcal{E}_2 \times \mathcal{E}_2]^8)$$

of weight 400, then a similar argument shows that the three cusp forms $F_{2,2}$, $F_{1,1,2}$, $F_{1,1,1,1}$ never vanish simultaneously on the image of the natural map $X_{\Gamma_2} \times X_{\Gamma_2} \to X_{\Gamma_4}$. Consequently, the five cusp forms $F_{1,3}$, $F_{1,3}^*$, $F_{1,1,2}$, $F_{1,1,1,1}$, $F_{2,2}$ have no common zero on $X_{\Gamma_4}^{\mathrm{red}}$.

The bound on $\mathrm{acn}(X_{\Gamma_4})$ now follows by means of Lemma 4.4. □

4.2. **A special case of Raynaud's Lefschetz Theorem.** Here we state a special case of an algebraization theorem of Raynaud. We will use it to show that the map (3.16) is surjective provided that a certain upper bound for the affine covering number holds.

Theorem 4.6. *Let k be a field and let X be a scheme over k which is of finite type, separated, and proper. Let $Y \subset X$ be a closed subscheme, and write $\hat{X}$ for the formal completion of X along Y. Assume that the complement $U = X \setminus Y$ is smooth and satisfies*

$$\mathrm{acn}(U) \leq \dim(U) - 2.$$

Then for every finite locally free sheaf $\mathcal{F}$ on X the canonical morphism

$$H^0(X, \mathcal{F}) \to H^0(\hat{X}, \hat{\mathcal{F}})$$

is an isomorphism.

Proof. This is a special case of [Ray, Corollaire 2.8]. To see this, we note that the assumptions imply that X has a dualizing complex. We put $c = \mathrm{acn}(U)$ and let d be an integer such that for all points $u \in U$ we have

$$\mathrm{depth}_u(\mathcal{F}) \geq \inf(d - c, d - \mathrm{trdeg}(\kappa(u)/k)), \tag{4.7}$$

where $\kappa(u)$ denotes the residue field of u. Then [Ray, Corollaire 2.8] states that for all $i < d - c - 1$ the canonical morphism

$$H^i(X, \mathcal{F}) \to H^i(\hat{X}, \hat{\mathcal{F}})$$

is an isomorphism.

To deduce the claimed result, we put $i = 0$ and $d = c + 2$. Since U is smooth and $\mathcal{F}$ is locally free, we have

$$\mathrm{depth}_u(\mathcal{F}) = \dim(U) - \mathrm{trdeg}(\kappa(u)/k).$$

Hence, condition (4.7) simplifies to

$$\dim(U) \geq \inf(2 + \mathrm{trdeg}(\kappa(u)/k), c + 2).$$

Our assumption $c = \mathrm{acn}(U) \leq \dim(U) - 2$ implies that this condition is satisfied. □

Remark 4.7. It is an interesting question whether the hypothesis of the theorem can be weakened. For instance, can the upper bound on $\mathrm{acn}(U)$ be replaced by an upper bound on the affine stratification number (see [RV]) or on the cohomological dimension of U?

We now specialize to the case that the ground field is $\mathbb{C}$. By means of GAGA we also have an analogous result for complex analytic spaces. Recall that if X is a scheme of finite type over $\mathbb{C}$, we denote by X^{an} its analytification, and we let $h : X^{\mathrm{an}} \to X$ be the analytification morphism, see [SGA1, Exposé XII]. If $\mathcal{F}$ is a coherent sheaf on X, we write $\mathcal{F}^{\mathrm{an}} = h^*(\mathcal{F})$ for its analytification.

Corollary 4.8. *Let X be a scheme over $\mathbb{C}$ which is of finite type, separated, and proper. Let $Y \subset X$ be a closed subscheme, and write $\hat{X}^{\mathrm{an}}$ for the formal completion of X^{an} along Y^{an} (in the category of formal complex analytic spaces as in* [Ba]*). Assume that the complement $U = X \setminus Y$ is smooth and satisfies*

$$\mathrm{acn}(U) \leq \dim(U) - 2.$$

Then for every finite locally free $\mathcal{O}_{X^{\mathrm{an}}}$-module $\mathcal{G}$ on X^{an} the canonical morphism

$$H^0(X^{\mathrm{an}}, \mathcal{G}) \to H^0(\hat{X}^{\mathrm{an}}, \hat{\mathcal{G}})$$

is an isomorphism. Here $\hat{\mathcal{G}}$ denotes the completion of $\mathcal{G}$ along Y^{an} in the category of formal complex analytic spaces.

Proof. By the GAGA theorem, there exists a unique finite locally free $\mathcal{O}_X$-module $\mathcal{F}$ on X such that $\mathcal{G} = \mathcal{F}^{\mathrm{an}}$. By Theorem 4.6 the canonical homomorphism

$$H^0(X, \mathcal{F}) \to H^0(\hat{X}, \hat{\mathcal{F}})$$

is an isomorphism. The left hand side is canonically isomorphic to $H^0(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf defining Y. The canonical homomorphism

$$H^0(\hat{X}, \hat{\mathcal{F}}) \to \lim_k H^0(X, \mathcal{F}/\mathcal{I}^k\mathcal{F}) \tag{4.8}$$

is an isomorphism. By GAGA for the coherent sheafs $\mathcal{F}/\mathcal{I}^k\mathcal{F}$, the right hand side of (4.8) is isomorphic to

$$\lim_k H^0(X^{\mathrm{an}}, (\mathcal{F}/\mathcal{I}^k\mathcal{F})^{\mathrm{an}}) \cong H^0(\hat{X}^{\mathrm{an}}, (\mathcal{F}^{\mathrm{an}})\hat{}\,).$$

Putting these maps together, we obtain a natural isomorphism

$$H^0(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}) \to H^0(\hat{X}^{\mathrm{an}}, (\mathcal{F}^{\mathrm{an}})\hat{}\,).$$

This completes the proof of the corollary. □

4.3. **Algebraization of formal Siegel modular forms.** Here we combine Corollary 4.8 and Theorem 3.13 to derive a modularity result for formal Siegel modular forms of cogenus 1. We use the notation of Section 3. In particular, $\Gamma \subset \mathrm{Sp}_n(\mathbb{Q})$ denotes an arithmetic subgroup.

Theorem 4.9. *Assume that* $\mathrm{acn}(X_\Gamma) \leq \frac{n(n+1)}{2} - 2$. *Then the natural map*

$$H^0(X_\Gamma^*, \omega^{\otimes k}) \to \mathrm{FM}_k^{(n,1)}(\Gamma) \tag{4.9}$$

taking a modular form to its cogenus 1 *formal Fourier-Jacobi expansions is an isomorphism.*

Proof. We first assume that $\Gamma = \Gamma_n(N)$ is the principal congruence subgroup of level $N \geq 3$ and that the condition $\mathrm{acn}(X_\Gamma) \leq \frac{n(n+1)}{2} - 2$ holds. Then Γ acts freely on $\mathbb{H}_n$ and ω^k is locally free of rank 1. The map (4.9) is part of the commutative diagram

$$\begin{array}{ccc} H^0(X_\Gamma^*, \omega^{\otimes k}) & \longrightarrow & \mathrm{FM}_k^{(n,1)}(\Gamma)\ , \\ \big\downarrow & \nearrow & \\ H^0(\hat{X}_\Gamma^*, \hat{\omega}^{\otimes k}) & & \end{array}$$

where the diagonal arrow is given by (3.32). According to Theorem 3.13 this diagonal map is an isomorphism. Moreover, by Corollary 4.8 the vertical map is an isomorphism. This implies the assertion.

Now we consider the case that Γ is an arbitrary arithmetic subgroup of $\mathrm{Sp}_n(\mathbb{Q})$. We choose a principal congruence subgroup Γ' of level $N \geq 3$ such that $\Gamma' \subset \Gamma$. Since finite morphisms are affine, we have $\mathrm{acn}(X_{\Gamma'}) \leq \mathrm{acn}(X_\Gamma)$. The covering map $\pi : X^*_{\Gamma'} \to X^*_\Gamma$ induces a pull back map

$$\pi^* : \mathrm{FM}_k^{(n,1)}(\Gamma) \to \mathrm{FM}_k^{(n,1)}(\Gamma'). \tag{4.10}$$

Hence, if $f = (f_F)_{F \in I_{n-1}} \in \mathrm{FM}_k^{(n,1)}(\Gamma)$, we may apply the above argument to deduce that $\pi^*(f) \in \mathrm{FM}_k^{(n,1)}(\Gamma')$ is the image of an element $g \in H^0(X^*_{\Gamma'}, \omega^{\otimes k})$. Then at each boundary component $F \in I_{n-1}$ the series f_F is the (convergent!) Fourier-Jacobi expansion as in (3.1) of the holomorphic modular form g for Γ'. But now condition (i) of Definition 3.5 implies that

$$g \mid_k \gamma = f_F \mid_k \gamma = f_{\gamma^{-1}F} = g$$

for every $\gamma \in \Gamma$. Hence $g \in H^0(X^*_\Gamma, \omega^{\otimes k})$, concluding the proof of the corollary. □

Corollary 4.10. *Assume that* $2 \leq n \leq 4$. *Then the natural map* (4.9) *is an isomorphism.*

Proof. If Γ is contained in the full Siegel modular group Γ_n, the assertion follows from Theorem 4.9 combined with the bounds on $\mathrm{acn}(X_{\Gamma_n})$ of Section 4.1.

Otherwise, we consider the auxiliary congruence subgroup $\Gamma' = \Gamma \cap \Gamma_n$. The covering map $\pi : X^*_{\Gamma'} \to X^*_\Gamma$ induces a pull back map as in (4.10). Pulling back $f \in \mathrm{FM}_k^{(n,1)}(\Gamma)$, we find that $\pi^*(f)$ is the image of an element $g \in H^0(X^*_{\Gamma'}, \omega^{\otimes k})$. As in the proof of Theorem 4.9 we conclude that g actually descends to an element of $H^0(X^*_\Gamma, \omega^{\otimes k})$. □

Remark 4.11.

(i) Theorem 4.9 and Corollary 4.10 have natural generalizations to vector valued modular forms transforming with a finite dimensional representation of Γ. Alternatively, one can deduce such results for vector valued forms from those for scalar forms by means of the argument of [Br].

(ii) Using induction on the cogenus as in [BR, Lemma 5.2] one can also deduce an analogue for formal Siegel modular forms of higher cogenus $l < n$.

(iii) The above results are also valid in the slighltly more general case when Γ is an arithmetic subgroup of $\mathrm{Sp}_n(\mathbb{R})$ (rather than of $\mathrm{Sp}_n(\mathbb{Q})$), that is, a subgroup which is commensurable with Γ_n. Note that by [Ch] the elements of such a subgroup are automatically projective rational.

4.4. **The case of the paramodular group.** Here we apply Theorem 4.9 to prove the modularity of certain formal Fourier-Jacobi series for the paramodular group of genus 2.

Throughout this subsection we let $n = 2$ and let N be a positive integer. Recall that the paramodular group of level N is the arithmetic subgroup $K(N) \subset \mathrm{Sp}_2(\mathbb{Q})$ consisting of matrices of the form

$$\begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix}, \tag{4.11}$$

where the stars stand for integral entries. For every exact divisor $d \| N$ we put $d' = N/d$ and choose $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ such that $\alpha\delta d - \beta\gamma d' = 1$. Then the matrix

$$V_d = \frac{1}{\sqrt{d}} \begin{pmatrix} d\delta & -N\gamma & 0 & 0 \\ -\beta & d\alpha & 0 & 0 \\ 0 & 0 & d\alpha & \beta \\ 0 & 0 & N\gamma & d\delta \end{pmatrix}$$

belongs to $\mathrm{Sp}_2(\mathbb{R})$ and is projective rational. The coset $V_d K(N)$ is independent of the choices of the parameters. Moreover, we have $V_d^2 \in K(N)$ and $V_d K(N) V_d = K(N)$. In the special case where $d = N$, we may choose $\beta = 1$, $\gamma = -1$, and $\alpha = \delta = 0$. Then we obtain for V_N the (projective) involution

$$\mu_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & N & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -N & 0 \end{pmatrix}.$$

We denote by $K(N)^*$ the arithmetic subgroup of $\mathrm{Sp}_2(\mathbb{R})$ which is generated by $K(N)$ and the V_d for $d \| N$. It contains $K(N)$ as a normal sugbroup, and

$$K(N)^*/K(N) \cong (\mathbb{Z}/2\mathbb{Z})^{\nu(N)},$$

where $\nu(N)$ denotes the number of prime divisors of N, see, e.g., [GH].

The structure of the rational boundary components of $K(N)$ and $K(N)^*$ is known, see, e.g., [PY]. If N is square-free, the situation is particularly simple. Then for $K(N)^*$ there is exactly one orbit of 0-dimensional rational boundary components and one orbit of 1-dimensional rational boundary components. For the group $K(N)$ there is one orbit of 0-dimensional rational boundary components, and there are $2^{\nu(N)}$ orbits on 1-dimensional rational boundary components. The orbits can be represented by the $V_d F_1$ for $d \| N$, and these representatives intersect exactly in the standard 0-dimensional boundary component F_0. Moreover, we have $K(N)^*_{F_1} = K(N)_{F_1}$, and $K(N)^*_{F_0}$ is obtained from $K(N)_{F_0}$ by extending with the involutions V_d.

Let f be a formal Fourier-Jacobi series of weight k for the standard boundary component F_1 and the group $K(N)$. Denote by

$$f(\tau) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Q})} a(T)\, e(\operatorname{tr} T\tau)$$

its formal Fourier expansion at the boundary component F_0 as in (3.13). We assume that f is *strongly symmetric* in the following strong sense: There exists a character $\chi_f : K(N)^*/K(N) \to \{\pm 1\}$ such that

$$f \mid_k \gamma = \chi_f(\gamma) f \tag{4.12}$$

for all $\gamma \in K(N)^*_{F_0}$. This condition can be rephrased as

$$a(T[{}^t u]) = \chi_f \begin{pmatrix} {}^t u^{-1} & 0 \\ 0 & u \end{pmatrix} \det(u)^k a(T) \tag{4.13}$$

for all $T \in \mathrm{Sym}_2(\mathbb{Z})^\vee$ positive semi-definite with $N \mid T_2$, and for all $u \in \Gamma_0(N)^* \subset \mathrm{SL}_2(\mathbb{R})$. Here the latter group denotes the extension of $\Gamma_0(N)$ by all Atkin-Lehner involutions W_d with $d \| N$. We obtain the following result.

Theorem 4.12. *Let N be a square-free positive integer. Let f be a strongly symmetric formal Fourier-Jacobi series of weight k for the boundary component F_1 and the group $K(N)$. Then f converges and defines the Fourier-Jacobi expansion of a modular form in $M_k(K(N))$.*

Proof. We define a formal Siegel modular form for $K(N)$ as follows. For any rational boundary component $F \in I_1$ we choose $\delta \in K(N)^*$ such that $F = \delta^{-1} F_1$ and put

$$f_F := \chi_f(\delta) f \mid_k \delta.$$

By Remark 3.3, f_F is a formal Fourier-Jacobi series for the boundary component F and the group $K(N)$. It is easily checked that the family $(f_F)_{F \in I_1}$ defines a formal Siegel modular form of weight k for $K(N)$. According to Corollary 4.10 it must come from a holomorphic Siegel modular form in $M_k(K(N))$. □

Remark 4.13. It suffices to check condition (4.13) for u running through a set of generators of $\Gamma_0(N)^*$. Therefore, Theorem 4.12 may be useful for computations with Siegel modular forms for the paramodular group. For instance, if $N = 1, 2, 3$, then $\Gamma_0(N)^*$ is generated by the translation matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and the Fricke involution. Hence, in this case the strong symmetry condition follows from the formal modularity of f for $K(N)_{F_1}$ and from (4.12) for the *single* matrix $\gamma = \mu_N$. This recovers part of the main result (Theorem 1.2) of [IPY].

In [IPY] the authors ask whether for general N it actually suffices to require (4.12) for the single element μ_N. More precisely, they consider a formal Fourier-Jacobi series f of weight k for the standard boundary component F_1 and the group $K(N)$ as above. They say that f satisfies the *involution condition*, if there is an $\varepsilon \in \pm 1$ such that the formal Fourier expansion of f satisfies $f \mid_k \mu_N = \varepsilon f$. In terms of the Fourier coefficients this means

$$a \begin{pmatrix} T_2/N & -T_{12} \\ -T_{12} & NT_1 \end{pmatrix} = \varepsilon \cdot a \begin{pmatrix} T_1 & T_{12} \\ T_{12} & T_2 \end{pmatrix} \tag{4.14}$$

for all $T = \left(\begin{smallmatrix} T_1 & T_{12} \\ T_{12} & T_2 \end{smallmatrix}\right) \in L^\vee_{F_1}$. It is asked in the introduction of loc. cit. whether any formal Fourier-Jacobi series f as before which satisfies the involution condition is the expansion of a classical holomorphic modular form of weight k for the group $K(N)$?

When N is prime, one could try to approach this question by using the fact that $K(N)^*$ is generated by $K(N)_{F_1}$ and μ_n to show that any formal Fourier-Jacobi series f which satisfies the involution condition (4.14) is actually strongly symmetric in the sense of (4.12). To this end one would have to show that for any $\gamma \in K(N)_{F_0}$ the formal Fourier-Jacobi series f_{F_1} and $f_{\gamma F_1}$ are compatible at F_0.

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