NON-STATIONARY GAUSSIAN RANDOM FIELDS ON HYPERSURFACES: SAMPLING AND STRONG ERROR ANALYSIS

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Abstract. A flexible model for non-stationary Gaussian random fields on hypersurfaces is introduced. The class of random fields on curves and surfaces is characterized by an amplitude spectral density of a second order elliptic differential operator. Sampling is done by a Galerkin–Chebyshev approximation based on the surface finite element method and Chebyshev polynomials. Strong error bounds are shown with convergence rates depending on the smoothness of the approximated random field. Numerical experiments that confirm the convergence rates are presented.

1. Introduction

Random fields are powerful tools for modeling spatially dependent data. They have found uses in a wide range of applications, for instance in geostatistics, cosmological data analysis, climate modeling, and biomedical imaging [22, 12]. One challenge in the modeling of spatial data is non-stationary behavior, i.e., different behaviors in different parts of the domain. Another challenge is that the domain may be a non-Euclidean space, for instance, a surface such as the sphere or on the cortical surface of the brain. In this paper, we present a surface finite element-based method to sample a flexible class of non-stationary random fields on curves and surfaces and show its strong convergence. The method, building on the foundational work for stationary fields introduced in [19], is an extension of the stochastic partial differential equation (SPDE) approach pioneered by [27] and popularized by [21]. The idea behind our method is to color white noise by applying a function of an elliptic differential operator $\mathcal{L}$. Formally, we study Gaussian random fields on curves and surfaces of the form

$$\mathcal{Z} = \gamma(\mathcal{L})\mathcal{W}, \tag{1}$$

where $\mathcal{L}$ is an elliptic differential operator, $\mathcal{W}$ denotes white noise, γ is a function, called *amplitude spectral density* in analogy to the spectral analysis of time signals [15]. By letting the coefficients of the differential operator vary over the domain, we can obtain local, non-stationary behaviors. If $1/\gamma$ is well-defined over $\mathbb{R}_+$, one may formally view $\mathcal{Z}$ as the solution

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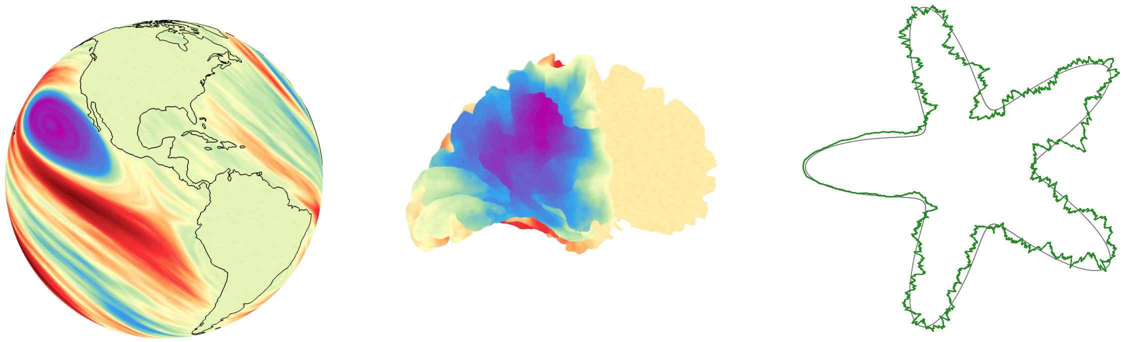


(a) A random field where the choice of coefficients are inspired the continents.

(b) A random field on the cortical surfaces only locally activated.

(c) A random field on a star-shaped domain.

FIGURE 1. Examples of random field samples generated with our method.

to the stochastic partial differential equation

$$(1/\gamma)(\mathcal{L})\mathcal{Z} = \mathcal{W}.$$

For instance, consider the three examples depicted in Figure 1. To generate random field samples, there are two components of $\mathcal{L}$ that we can vary: the diffusion matrix and the potential. In Figure 1(a), we use a diffusion matrix to give the field preferred directions, more specifically it elongates field in the northwest-southeast direction. The potential is large over the continents and small over the oceans, effectively "turning off" the random field over land. In Figure 1(b), the potential is small in the front of the brain and large elsewhere, so that the field is only large in the front of the brain. Finally, Figure 1(c) shows the method used to generate a non-stationary random field in the one-dimensional case. To illustrate the value of the field at a point, we move it in the normal direction for a distance proportional to the value of the field. In all three cases, we see that the field behaves locally varying over the domain. With the suggested model, we can achieve preferred directions, local activation, and local deactivation.

The computational method we use to solve Equation (1), i.e., sample the random fields, is based on the surface finite element method (SFEM), a computational method pioneered by [10, 11] and that has been used in the context of the generation of Whittle–Matérn random fields with Laplace–Beltrami operators in for instance [6, 18, 19]. Our main mathematical contribution is a strong convergence result for equations with more general elliptic operators and amplitude spectral densities than fractional powers. Using a functional calculus approach to the finite element discretization error, we obtain a strong rate of convergence of order $\mathcal{O}(C_\alpha(h)h^{2\min\{\alpha-d/4;1\}})$, where $d = 2$ for surfaces and $d = 1$ for curves, and $C_\alpha(h)$ is a dimension-dependent logarithmic factor.

The SPDE approach to random fields and their approximation have been studied previously for both surfaces and Euclidean geometries, examples include [6, 8, 3, 9, 18, 19, 21, 20]. However, to the best of our knowledge, we are the first to present a strong error analysis for the non-stationary case on hypersurfaces that are only given by mesh points. In contrast to the method presented in [16], we do not need to explicitly know the true hypersurface and perform computations on it. Instead, we only require a polyhedral approximation of the hypersurface.

Moreover, like in [9], we achieve this without requiring explicit approximation bounds on the eigenfunctions of the elliptic operator, and contrary to for instance [8], computing the eigenfunctions. In fact, the computation of eigenfunctions, with theoretical guarantees, is a notoriously difficult problem, see e.g., [2]. Our method circumvents this issue by introducing a Chebyshev approximation.

Our main contribution is a powerful, efficient, and flexible tool for the modeling and sampling of non-stationary random fields on curves and surfaces with proven accuracy. In particular, using tools from complex analysis and operator theory, we derive strong error bounds for approximations of arbitrary sufficiently smooth transformations of elliptic operators, where we do not require assumptions on the approximability of individual eigenfunctions.

The paper is structured as follows: In Section 2, we introduce the relevant deterministic framework. We provide the necessary background on geometry and functional analysis in Section 2.1. This is followed by a description of the main computational tool, surface finite elements in Section 2.2. Finally, Section 2.3 provides the relevant error estimates in the deterministic setting. In Section 3, we collect all material in the stochastic setting. We introduce first the class of considered random fields in Section 3.1 and their Galerkin–Chebyshev approximation in Section 3.2. The proof of its strong convergence is split into the SFEM error in Section 3.3 and the Chebyshev approximation error in Section 3.4. In Section 4, we present numerical experiments that confirm the strong error bounds. The source code used to generate the figures is available at this address: `https://github.com/mike-pereira/SFEMsim`.

2. Deterministic theory: geometry, functional analysis and finite elements

Before we are able to approximate random fields on hypersurfaces, we need to introduce and partially extend the existing literature on surface finite element approximations due to so far unconsidered error bounds required in our stochastic setting. We introduce the functional analytic setting in Section 2.1, discuss surface finite element methods in Section 2.2 and show error bounds in the deterministic setting in Section 2.3.

2.1. Geometric and functional analytic setting.

Let $\mathcal{M} \subset \mathbb{R}^{d+1}$ be a d-dimensional ($d \leq 2$) compact oriented smooth hypersurface ($k \geq 2$) without boundary, i.e., for any $x_0 \in \mathcal{M}$, there exists an open set $U_{x_0} \subset \mathbb{R}^{d+1}$ containing x_0 and a function $\phi_{x_0} \in C^\infty(U_{x_0})$ such that $\nabla \phi_{x_0} \neq 0$ on $\mathcal{M} \cap U_{x_0}$ and

$$\mathcal{M} \cap U_{x_0} = \{x \in U_{x_0}, \phi_{x_0}(x) = 0\}.$$

The tangent space of $\mathcal{M}$ at $x \in \mathcal{M}$ is the d-dimensional subspace of $\mathbb{R}^{d+1}$ given by $T_x\mathcal{M} = [\nabla \phi_x]^\perp$ (where ∇ denotes the usual gradient of functions of $C^1(\mathbb{R}^{d+1})$ and $\perp$ denotes the orthogonal complement in $\mathbb{R}^{d+1}$ with respect to the standard Euclidean inner product.). Since $\mathcal{M}$ is oriented, there exists a smooth map $\nu : \mathcal{M} \to \mathbb{R}^{d+1}$ assigning to each point $x \in \mathcal{M}$ a unit vector $\nu(x) = \pm \nabla \phi_x / \|\nabla \phi_x\|$ perpendicular to the tangent space $T_x\mathcal{M}$. Our hypersurface $\mathcal{M}$ is a Riemannian manifold equipped with the metric g that is the pullback of the Euclidean metric on $\mathbb{R}^{d+1}$. For instance, if $\mathcal{M} = \mathbb{S}^2$, this results in the standard round metric.

Let $\nabla_{\mathcal{M}}$ be the gradient operator acting on differentiable functions of $\mathcal{M}$, and let $\Delta_{\mathcal{M}}$ denote the Laplace–Beltrami operator on $(\mathcal{M}, g)$. We denote by $\mathrm{d}A$ the surface measure on $\mathcal{M}$, and by $L^2(\mathcal{M})$ the Hilbert space of $\mathrm{d}A$-measurable square integrable complex-valued functions, equipped with the inner product $(\cdot,\cdot)_{L^2(\mathcal{M})}$ defined by

$$(u, v)_{L^2(\mathcal{M})} = \int_{\mathcal{M}} u\overline{v}\,\mathrm{d}A, \quad u, v \in L^2(\mathcal{M}).$$

The *Sobolev spaces* with smoothness index $s \in \mathbb{R}^+$ are then defined via Bessel potentials by

$$H^s(\mathcal{M}) = (I - \Delta_{\mathcal{M}})^{-s/2} L^2(\mathcal{M}),$$

with corresponding norm $\| \cdot \|_{H^s(\mathcal{M})} = \| (I - \Delta_{\mathcal{M}})^{s/2} \cdot \|_{L^2(\mathcal{M})}$. For $s < 0$, $H^s(\mathcal{M})$ is defined as the space of distributions generated by

$$H^s(\mathcal{M}) = \left\{ u = (I - \Delta_{\mathcal{M}})^k v, \ v \in H^{2k+s}(\mathcal{M}) \right\}, \tag{2}$$

where $k \in \mathbb{N}$ is the smallest integer such that $2k + s > 0$. In this case, the corresponding norm is given by $\|u\|_{H^s(\mathcal{M})} = \|v\|_{H^{2k+s}(\mathcal{M})}$. We set $H^0(\mathcal{M}) = L^2(\mathcal{M})$. The reader is referred to [17], [25], [26], and references therein for more details on Sobolev spaces defined using Bessel potentials.

In this work, we consider elliptic differential operators associated to bilinear forms $\mathsf{A}_{\mathcal{M}}$ given by

$$\mathsf{A}_{\mathcal{M}}(u, v) = \int_{\mathcal{M}} (\mathcal{D} \nabla_{\mathcal{M}} u) \cdot (\nabla_{\mathcal{M}} \overline{v}) \, \mathrm{d}A + \int_{\mathcal{M}} (V u) \overline{v} \, \mathrm{d}A, \quad u, v \in H^1(\mathcal{M}), \tag{3}$$

where for any $x_0 \in \mathcal{M}$ the diffusion matrix $\mathcal{D}(x_0) = [D_{ij}(x_0)]_{i,j=1}^{d+1}$ is a real-valued, symmetric matrix such that for any $w \in T_{x_0}\mathcal{M}$, $\mathcal{D}(x_0)w \in T_{x_0}\mathcal{M}$ and $(\mathcal{D}(x_0)w) \cdot \overline{w} > 0$ if $w \neq 0$. In particular, since $T_{x_0}\mathcal{M} = [\nu(x_0)^{\perp}]$, $\mathcal{D}(x_0)$ is simply a matrix admitting $\nu(x_0)$ as an eigenvector with some eigenvalue $\mu_1(x_0) \in \mathbb{R}$, and such that the eigenvalues $\mu_i(x_0)$, $2 \leq i \leq d+1$ associated with its other eigenvectors are positive. Without loss of generality, we may assume that $\mu_1(x_0) = 0$, meaning that $\mathcal{D}\nu = 0$, that the eigenvalues $\mu_i(x_0)$, $2 \leq i \leq d+1$ are uniformly lower-bounded and upper-bounded on $\mathcal{M}$ by positive constants, and for simplicity that $D_{ij} \in C^{\infty}(\mathcal{M})$ for any $1 \leq i, j \leq d+1$. Finally, we assume that $V \in L^{\infty}(\mathcal{M})$ is a real-valued function that satisfies $V_- \leq V \leq V_+$ for some $0 < V_- \leq V_+ < +\infty$.

Throughout this paper, let $\mathsf{A}_{\mathcal{M}}$ be coercive and continuous, i.e., there exist positive constants δ and M such that for all $u, v \in H^1(\mathcal{M})$,

$$\mathsf{A}_{\mathcal{M}}(u, u) \geq \delta \|u\|^2_{H^1(\mathcal{M})}, \tag{4}$$

$$|\mathsf{A}_{\mathcal{M}}(u, v)| \leq M \|u\|_{H^1(\mathcal{M})} \|v\|_{H^1(\mathcal{M})}. \tag{5}$$

Following [28, Equation (1.33)], $\mathsf{A}_{\mathcal{M}}$ gives rise to an associated elliptic differential operator $\mathcal{L} : H^1(\mathcal{M}) \to H^{-1}(\mathcal{M})$ defined weakly by

$$\mathsf{A}_{\mathcal{M}}(u, v) = \int_{\mathcal{M}} (\mathcal{L}u) \overline{v} \, \mathrm{d}A, \quad u, v \in H^1(\mathcal{M}).$$

The spectral properties of this operator are detailed in the next proposition, which is proven in Appendix A.

Proposition 2.1. *Let $\delta > 0$ be the coercivity constant defined in Equation* (4)*. There exists a set of eigenpairs $\{(\lambda_i, e_i)\}_{i \in \mathbb{N}}$ of $\mathcal{L}$ consisting of a sequence of increasing real-valued eigenvalues $0 < \delta \leq \lambda_1 \leq \lambda_2 \leq \cdots$ with $\lambda_i \to +\infty$ as $i \to +\infty$, and $\{e_i\}_{i \in \mathbb{N}}$ forms an orthonormal basis of $L^2(\mathcal{M})$ where each e_i is real-valued.*

Since the operator $\mathcal{L}$ differs from the Laplace–Beltrami operator only by a zeroth-order potential term and a diffusion function in the second order term, switching between the two operators corresponds to a change of metric on $\mathcal{M}$. Therefore, the eigenvalue problem for $\mathcal{L}$ is equivalent to that for the Laplace–Beltrami operator on $\mathcal{M}$ equipped with a possibly rough metric if the coefficients of $\mathcal{D}$ are not smooth. The results in [1] imply growth rates on the eigenvalues in accordance with Weyl's law, and more specifically that there exist $c_{\mathcal{M}}, C_{\mathcal{M}} > 0$ such that for any $i \in \mathbb{N}$

$$c_{\mathcal{M}} i^{2/d} \leq \lambda_i \leq C_{\mathcal{M}} i^{2/d}. \tag{6}$$

As a last step in this subsection, we introduce nonlinear functions of $\mathcal{L}$ which allow later in Section 3 for the definition of a variety of Gaussian random fields. For that, we call a function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ an α-*amplitude spectral density* if

1) γ is extendable to a holomorphic function on $H_{\pi/2} := \{z \in \mathbb{C} : |\arg z| \leq \pi/2\}$.
2) There exist constants $C_\gamma > 0$ and $\alpha > 0$ such that for all $z \in H_{\pi/2}$,

$$|\gamma(z)| \leq C_\gamma |z|^{-\alpha}. \tag{7}$$

Applying a amplitude spectral density to $\mathcal{L}$ results in a linear operator $\gamma(\mathcal{L})$ whose action on functions $f \in L^2(\mathcal{M})$ is defined by

$$\gamma(\mathcal{L})f = \sum_{i=1}^{\infty} \gamma(\lambda_i)(f, e_i)_{L^2(\mathcal{M})} e_i, \tag{8}$$

where $\{(\lambda_i, e_i)\}_{i\in\mathbb{N}}$ are the eigenpairs of $\mathcal{L}$ defined in Proposition 2.1. A typical example is the function $\gamma(\lambda) = (\kappa^2 + \lambda)^{-\alpha}$ for $\alpha > d/4$ and $\kappa > 0$, which can be used to obtain Whittle–Matérn random fields [19].

Remark 2.2. For any α-amplitude spectral density γ and any $f \in L^2(\mathcal{M})$, $\gamma(\mathcal{L})f \in L^2(\mathcal{M})$. In fact, we have for any $s \in [0, \alpha]$, $\|\mathcal{L}^s(\gamma(\mathcal{L})f)\|_{L^2(\mathcal{M})} < \infty$. Indeed,

$$\|\mathcal{L}^s \gamma(\mathcal{L}) f\|_{L^2(\mathcal{M})} = \sum_{i=1}^{\infty} |\lambda_i^s \gamma(\lambda_i)|^2 |(f, e_i)_{L^2(\mathcal{M})}|^2,$$

where, since γ is an α-amplitude spectral density and $\lambda_i \in H_{\pi/2}$ for any $i \in \mathbb{N}$, $|\lambda_i^s \gamma(\lambda_i)| \lesssim |\lambda_i|^{-(\alpha - s)}$. Using then the fact that for any $i \in \mathbb{N}$, $\lambda_i \geq \lambda_1 > 0$, and that $\alpha - s \geq 0$, we conclude that $|\lambda_i^s \gamma(\lambda_i)| \lesssim |\lambda_1|^{-(\alpha-s)} \lesssim 1$ and therefore $\|\mathcal{L}^s \gamma(\mathcal{L}) f\|_{L^2(\mathcal{M})} \lesssim \sum_{i=1}^{\infty} |(f, e_i)_{L^2(\mathcal{M})}|^2 = \|f\|_{L^2(\mathcal{M})} < \infty$.

The goal of the remainder of this section is to study the approximation of functions of the form $u = \gamma(\mathcal{L})f$, where $f \in L^2(\mathcal{M})$. Formally, if $1/\gamma$ is well-defined on the spectrum of $\mathcal{L}$, then this is the solution to the partial differential equation $(1/\gamma)(\mathcal{L})u = f$.

2.2. **SFEM–Galerkin approximation.** The idea behind the surface finite element method, as introduced by [10], is to work on a polyhedral approximation of the surface that is in some sense close to the true surface $\mathcal{M}$. More precisely, fix $h > 0$ and let $\mathcal{M}_h$ be a piecewise polygonal surface consisting of non-degenerate simplices (for $d = 2$, triangles and for $d = 1$, line segments) with vertices on $\mathcal{M}$, and such that h is the size of the largest simplex defined as the in-ball radius. The set of simplices making up the discretized surface is denoted by $\mathcal{T}_h$, thus meaning that

$$\mathcal{M}_h = \bigcup_{T_j \in \mathcal{T}_h} T_j,$$

and we assume that for any two simplices in $\mathcal{T}_h$, it holds that their intersection is either empty, or a common edge or vertex.

Following [11, Section 1.4.1], we assume that the triangulation $\mathcal{T}_h$ is quasi-uniform, shape-regular, and that the number of simplices sharing the same vertex can be upper-bounded by

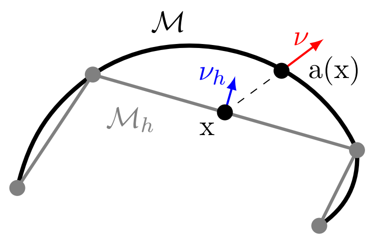


FIGURE 2. One-dimensional illustration of the lift. The lift is along the normal vector ν to the surface $\mathcal{M}$.

a constant independent of h. In turn, these two assumption imply that $N_h \propto h^{-d}$ where $N_h \in \mathbb{N}$ denotes the number of vertices of $\mathcal{M}_h$.

The discrete surface $\mathcal{M}_h$ is close to the true surface $\mathcal{M}$ in the sense that $\mathcal{M}_h$ is contained in a small neighborhood around $\mathcal{M}$ defined as follows. First, note that $\mathcal{M}$ can be seen as the boundary of some bounded open set $G \subset \mathbb{R}^{d+1}$ with exterior normal ν. Then, following [11, Section 2.3], we consider that there exists some (small) $\varpi > 0$ such that $\mathcal{M}_h$ is contained in a so-called *tubular neighborhood* U_ϖ of $\mathcal{M}$ defined by

$$U_\varpi = \{x \in \mathbb{R}^{d+1} : |d_s(x)| < \varpi\},$$

where $d_s : \mathbb{R}^{d+1} \to \mathbb{R}$ denotes the oriented distance function given by

$$d_s(x) = \begin{cases} \inf_{y \in \mathcal{M}} |x-y|, & x \in \mathbb{R}^{d+1} \setminus G, \\ \inf_{y \in \mathcal{M}} -|x-y|, & x \in G. \end{cases}$$

We denote by $\mathrm{d}A_h$ the surface measure on $\mathcal{M}_h$, and by $L^2(\mathcal{M}_h)$ the Hilbert space of $\mathrm{d}A_h$-measurable square integrable functions, equipped with the inner product $(\cdot,\cdot)_{L^2(\mathcal{M}_h)}$ defined by

$$(u_h, v_h)_{L^2(\mathcal{M}_h)} = \int_{\mathcal{M}_h} u_h \overline{v}_h \,\mathrm{d}A_h, \quad u_h, v_h \in L^2(\mathcal{M}_h).$$

Following [4, Section 1.2.1], we denote by $\sigma : \mathcal{M}_h \to \mathbb{R}^+$ the *area element* given by $\sigma = \mathrm{d}A/\mathrm{d}A_h$, such that for all $v \in L^2(\mathcal{M})$,

$$\int_{\mathcal{M}} v \,\mathrm{d}A = \int_{\mathcal{M}_h} \sigma v^{-\ell} \,\mathrm{d}A_h, \tag{9}$$

where we next introduce the lift and its inverse denoted by ℓ and $-\ell$, respectively.

A key element of SFEM is that we can move between $\mathcal{M}$ and $\mathcal{M}_h$ using the so-called *lift operator*. To construct the lift operator, we note that $d_s \in C^k(U_\varpi)$ for $k \geq 2$, and that for any $x \in U_\varpi$, there exists a unique $a(x) \in \mathcal{M}$ such that

$$x = a(x) + d_s(x)\nu(a(x)),$$

where ν denotes the normal at $a(x)$ to $\mathcal{M}$. In particular, this implies that any point $x \in U_\varpi$ can be uniquely described by the pair $(a(x), d_s(x)) \in \mathcal{M} \times \mathbb{R}$, and this procedure defines an isomorphism $p \colon \mathcal{M}_h \to \mathcal{M}$ given by

$$p(x) = x - d_s(x)\nu(a(x)), \quad x \in \mathcal{M}_h.$$

Therefore, any function $\eta \colon \mathcal{M}_h \to \mathbb{C}$ may be lifted to $\mathcal{M}$ by $\eta^\ell = \eta \circ p^{-1} : \mathcal{M} \to \mathbb{C}$. Likewise, the inverse lift of any function $\zeta : \mathcal{M} \to \mathbb{C}$ is given by $\zeta^{-\ell} = \zeta \circ p : \mathcal{M}_h \to \mathbb{C}$. The procedure is illustrated in Figure 2 in the one-dimensional setting. Note that the points on the discretized surface are *lifted* along the normal ν to the surface $\mathcal{M}$.

The mapping a is used to define the gradient of functions on $\mathcal{M}_h$ [11]. More specifically, for a differentiable $\eta : \mathcal{M}_h \to \mathbb{C}$, the gradient is given by

$$\nabla_{\mathcal{M}_h}\eta(x) = \nabla\check{\eta}(x) - (\nabla\check{\eta}(x)\cdot\nu_h(x))\nu_h(x) \in T_x\mathcal{M}_h, \tag{10}$$

where ν_h is the normal of $\mathcal{M}_h$ and $\check{\eta}$ is the continuous extension of η defined by $\check{\eta} : x \in U_\varpi \mapsto \check{\eta}(x) = \eta^\ell(a(x))$. With this definition, the Laplace–Beltrami operator on $\mathcal{M}_h$ can be defined by $\Delta_{\mathcal{M}_h} = \nabla_{\mathcal{M}_h}\cdot\nabla_{\mathcal{M}_h}$, and Sobolev spaces on $\mathcal{M}_h$ are defined in complete analogy to those on $\mathcal{M}$.

The analogue of the bilinear form $\mathsf{A}_\mathcal{M}$ on $\mathcal{M}_h$ is given by

$$\mathsf{A}_{\mathcal{M}_h}(u_h, v_h) = \int_{\mathcal{M}_h} (\mathcal{D}^{-\ell}\nabla_{\mathcal{M}_h}u_h)\cdot(\nabla_{\mathcal{M}_h}\overline{v}_h)\,\mathrm{d}A_h + \int_{\mathcal{M}_h}(V^{-\ell}u_h)\overline{v}_h\,\mathrm{d}A_h, \tag{11}$$

where $u_h, v_h \in H^1(\mathcal{M}_h)$ and $\mathcal{D}^{-\ell} = [D_{ij}^{-\ell}]_{i,j=1}^{d+1}$. As in [11], we assume that there exists $h_0 \in (0,1)$ small enough, such that $\mathsf{A}_{\mathcal{M}_h}$ is coercive (and continuous) whenever $h \le h_0$. Unless stated otherwise, we now assume that this last condition on h is fulfilled.

To conclude this subsection, we introduce the (linear) finite element space S_h on $\mathcal{M}_h$. The finite element space S_h is defined as the complex span of the standard real-valued *nodal basis*

$$\psi_1, \dots, \psi_{N_h} : \mathcal{M} \to \mathbb{R},$$

where for any $i \in \{1, \dots, N_h\}$, $\psi_i|_T$ is a polynomial of at most degree one taking the value 1 at the i-th vertex of $\mathcal{M}_h$ and 0 at all the other vertices, i.e.,

$$S_h = \operatorname{span}(\psi_1, \dots, \psi_{N_h}) \subset H^1(\mathcal{M}_h).$$

By construction, S_h is a vector space of dimension N_h. Its counterpart on $\mathcal{M}$ is the lifted finite element space S_h^ℓ given by

$$S_h^\ell = \left\{\phi_h^\ell, \phi_h \in S_h\right\} \subset H^1(\mathcal{M}).$$

On S_h, we can, as $\mathcal{L}$ on $\mathcal{M}$, associate to the bilinear form $\mathsf{A}_{\mathcal{M}_h}$ in Equation (11) a linear operator $\mathsf{L}_h : S_h \to S_h$ which maps any $u_h \in S_h$ to the unique $\mathsf{L}_h u_h \in S_h$ satisfying, for any $v_h \in S_h$, the equality

$$\mathsf{A}_{\mathcal{M}_h}(u_h, v_h) = (\mathsf{L}_h u_h, v_h)_{L^2(\mathcal{M}_h)}.$$

Similarly, if the bilinear form $\mathsf{A}_\mathcal{M}$ introduced in Equation (3) is restricted to S_h^ℓ, we can associate it to a linear operator $\mathcal{L}_h : S_h^\ell \to S_h^\ell$ that maps any $u_h^\ell \in S_h$ to the unique $\mathcal{L}_h u_h^\ell \in S_h^\ell$ that satisfies, for any $v_h^\ell \in S_h^\ell$, the equality

$$\mathsf{A}_\mathcal{M}(u_h^\ell, v_h^\ell) = (\mathcal{L}_h u_h^\ell, v_h^\ell)_{L^2(\mathcal{M})}.$$

Since the bilinear forms $\mathsf{A}_\mathcal{M}$ and $\mathsf{A}_{\mathcal{M}_h}$ are coercive, positive definite, Hermitian and have real coefficients, these two operators are diagonalizable in the sense that they each give rise to a set of N_h eigenpairs [14]. On the one hand, there exists a sequence $0 \le \Lambda_1^h \le \cdots \le \Lambda_{N_h}^h$ and an $L^2(\mathcal{M}_h)$-orthonormal basis $E_1^h, \dots, E_{N_h}^h$ of S_h such that

$$\mathsf{L}_h E_i^h = \Lambda_i^h E_i^h, \quad 1 \le i \le N_h,$$

and similarly there exists a sequence $0 \leq \lambda_1^h \leq \cdots \leq \lambda_{N_h}^h$ and an $L^2(\mathcal{M})$-orthonormal basis $e_1^h, \dots, e_{N_h}^h$ of S_h^ℓ such that

$$\mathcal{L}_h e_i^h = \lambda_i^h e_i^h, \quad 1 \leq i \leq N_h.$$

In particular, using the same approach as in Proposition 2.1, we can assume that the eigenfunctions $\{E_i^h\}_{1\leq i\leq N_h}$ and $\{e_i^h\}_{1\leq i\leq N_h}$ are all real-valued. The eigenvalues of the operators $\mathcal{L}$, $\mathcal{L}_h$ and L_h are linked to one another through the following lemma, due to [5, Lemma 3.1], [6, Lemma 4.1] and [24, Theorem 6.1].

In the following, $A \lesssim B$ is shorthand for that there is a constant $C > 0$ such that $A \leq CB$.

Lemma 2.3 (Eigenvalue error bounds). *Let $\{\lambda_i\}_{i\in\mathbb{N}}$, $\{\lambda_i^h\}_{1\leq i\leq N_h}$ and $\{\Lambda_i^h\}_{1\leq i\leq N_h}$ denote the eigenvalues of the operators $\mathcal{L}$, $\mathcal{L}_h$ and L_h, respectively. Then,*

$$\lambda_i \leq \lambda_i^h \lesssim (1 + h^2)\lambda_i, \quad 1 \leq i \leq N_h, \tag{12}$$

and

$$|\lambda_i^h - \Lambda_i^h| \lesssim h^2\lambda_i^h \lesssim h^2\lambda_i, \quad 1 \leq i \leq N_h. \tag{13}$$

Finally, we remark that the eigenvalues $\{\Lambda_i^h\}_{1\leq i\leq N_h}$ of L_h can be linked to the eigenvalues of some classical finite element matrices. Let $\boldsymbol{C}$ and $\boldsymbol{R}$ be the so-called *mass matrix* and *stiffness matrix*, respectively, and defined from the nodal basis by

$$\boldsymbol{C} = \left[(\psi_k, \psi_l)_{L^2(\mathcal{M}_h)}\right]_{1\leq k,l\leq N_h}, \quad \boldsymbol{R} \ = \left[\mathsf{A}_{\mathcal{M}_h}(\psi_k, \psi_l)\right]_{1\leq k,l\leq N_h}. \tag{14}$$

As defined, $\boldsymbol{C}$ is a symmetric positive definite matrix and $\boldsymbol{R}$ is a symmetric positive semi-definite matrix. Let then $\sqrt{\boldsymbol{C}} \in \mathbb{R}^{N_h\times N_h}$ be an invertible matrix satisfying $\sqrt{\boldsymbol{C}}(\sqrt{\boldsymbol{C}})^T = \boldsymbol{C}$. Then, by [19, Corollary 3.2], the eigenvalues $\{\Lambda_i^h\}_{1\leq i\leq N_h}$ are also the eigenvalues of the matrix $\boldsymbol{S} \in \mathbb{R}^{N_h\times N_h}$ defined by

$$\boldsymbol{S} = \big(\sqrt{\boldsymbol{C}}\big)^{-1}\boldsymbol{R}\big(\sqrt{\boldsymbol{C}}\big)^{-T}. \tag{15}$$

Besides, if $\boldsymbol{\psi}$ denotes the vector-valued function given by $\boldsymbol{\psi} = (\psi_1, \dots, \psi_{N_h})^T$, then the mapping $F : \mathbb{R}^{N_h} \to S_h$, defined by

$$F(\boldsymbol{v}) = F(\boldsymbol{v}) = \boldsymbol{\psi}^T\big(\sqrt{\boldsymbol{C}}\big)^{-T}\boldsymbol{v}, \quad \boldsymbol{v} \in \mathbb{R}^{N_h},$$

is an isomorphism whose inverse maps the eigenfunctions $\{E_i^h\}_{1\leq i\leq N_h}$ to (orthonormal) eigenvectors of $\boldsymbol{S}$. This means in particular that $\boldsymbol{S}$ can also be written as

$$\boldsymbol{S} = \boldsymbol{V}\,\mathrm{Diag}\big(\Lambda_1^h, \dots, \Lambda_1^{N_h}\big)\boldsymbol{V}^T, \tag{16}$$

where $\boldsymbol{V} = \big(F^{-1}(E_1^h)|\cdots|F^{-1}(E_{N_h}^h)\big) \in \mathbb{R}^{N_h\times N_h}$.

2.3. **Deterministic error analysis.** From now on, let us make the following assumption on the mesh size h. Let $\delta_0 \in (0, \delta/2)$ be fixed and arbitrary and let $h_0 \in (0, 1)$ such that $h_0^{-2} > \delta_0$ and $|\log h_0| > 1$. We now assume for the remainder of the paper n that mesh size satisfies $h \in (0, h_0)$.

Based on the introduced framework, we are now in place to quantify the error between functions of the operators $\mathcal{L}$, $\mathcal{L}_h$ and L_h. Let $P_h : L^2(\mathcal{M}) \to S_h^\ell$ be the L^2-projection onto S_h^ℓ

and let $\mathsf{P}_h : L^2(\mathcal{M}_h) \to S_h$ the L^2-projection onto S_h. We note that the operators $\mathcal{L}$, $\mathcal{L}_h$, and L_h define norms that are equivalent to the standard Sobolev norms, i.e.,

$$\|\mathcal{L}^{1/2}v\|_{L^2(\mathcal{M})} \sim \|v\|_{H^1(\mathcal{M})}, \ \|\mathsf{L}_h^{1/2}V_h\|_{L^2(\mathcal{M})} \sim \|V_h\|_{H^1(\mathcal{M}_h)}, \\ \|\mathcal{L}_h^{1/2}v_h\|_{L^2(\mathcal{M})} \sim \|v_h\|_{H^1(\mathcal{M})}, \tag{17}$$

for all $v \in H^1(\mathcal{M})$, and all $v_h \in S_h^\ell$ and $V_h \in S_h$.

With that at hand we are ready to state our main result in this section.

Proposition 2.4. *Let γ be an α-amplitude spectral density. There exists some constant such that for any $h \in (0, h_0)$, for all $f \in L^2(\mathcal{M})$ and for any $p \in [0, 1]$ such that $\|\mathcal{L}^p f\|_{L^2(\mathcal{M})} < \infty$, we have*

$$\|\gamma(\mathcal{L}_h)P_h f - \gamma(\mathcal{L})f\|_{L^2(\mathcal{M})} \lesssim C_{\alpha+p}(h)h^{2\min\{\alpha+p;1\}}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})},$$

where $C_{\alpha+p}(h) = |\log h|$ if $\alpha + p \le 1$, and $C_{\alpha+p}(h) = 1$ otherwise.

To prove this proposition we rely on a representation of functions of operators based on Cauchy–Stieltjes integrals, which are constructed as follows. Since the sesquilinear form defined by $\mathsf{A}_{\mathcal{M}}$ is continuous and coercive, the associated operators $\mathcal{L}$ and $\mathcal{L}_h$ are sectorial with some (common) angle $\theta \in (0, \pi/2)$ [28, Theorem 2.1]. Therefore, the spectra of $\mathcal{L}$ and $\mathcal{L}_h$ are contained in the complement of the set $G_\theta = \{z \in \mathbb{C}, \theta \le \arg(z) \le \pi\}$, as illustrated in Figure 3(a), and the following inequalities are satisfied for any $z \in G_\theta$ (cf. [28, Equation (2.2)]):

$$\|(z - \mathcal{L})^{-1}v\|_{L^2(\mathcal{M})} \le C_\theta|z|^{-1}\|v\|_{L^2(\mathcal{M})}, \quad v \in L^2(\mathcal{M}), \tag{18}$$

$$\|(z - \mathcal{L}_h)^{-1}v_h^\ell\|_{L^2(\mathcal{M})} \le C_\theta|z|^{-1}\|v_h^\ell\|_{L^2(\mathcal{M})}, \quad v_h^\ell \in S_h^\ell, \tag{19}$$

where $C_\theta > 0$ is a generic constant. Note in particular that by definition, G_θ is contained in the resolvent sets of $\mathcal{L}$ and $\mathcal{L}_h$, and that any amplitude spectral density γ is bounded, holomorphic and satisfies the inequality $|z|^\alpha|\gamma(z)| \lesssim 1$ for any $z \in G_\theta$. Hence, the operators $\gamma(\mathcal{L})$ and $\gamma(\mathcal{L}_h)$ can be defined as functional calculi of the operators $\mathcal{L}$ and $\mathcal{L}_h$ as [28, Chapter 16, Section 1.2] by

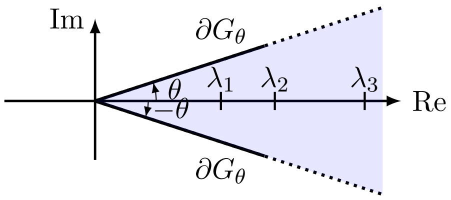
$$\gamma(\mathcal{L}) = \frac{1}{2\pi i}\int_\Gamma \gamma(z)(z - \mathcal{L})^{-1}\,\mathrm{d}z \quad \text{and} \quad \gamma(\mathcal{L}_h) = \frac{1}{2\pi i}\int_\Gamma \gamma(z)(z - \mathcal{L}_h)^{-1}\,\mathrm{d}z, \tag{20}$$

where $\Gamma \subset \mathbb{C}$ is any integral contour surrounding the spectra of $\mathcal{L}$ and $\mathcal{L}_h$ and contained in G_θ. In particular, these new definitions of functions of operators are independent of the choice of Γ, and coincide with the spectral definitions previously introduced in Equation (8) (cf. e.g. [28, Remark 2.7]).

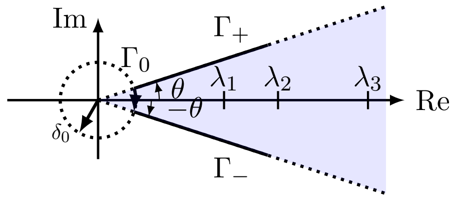
In the remainder, we split the contour Γ into

$$\Gamma = \Gamma_+ \cup \Gamma_0 \cup \Gamma_-, \tag{21}$$

where Γ_+ is parametrized by $g_+(t) = te^{i\theta}$ for $t \in (\infty, \delta_0]$, Γ_0 by $g_0(t) = \delta_0 e^{it}$ for $t \in [\theta, -\theta]$ and Γ_- by $g_-(t) = te^{i\theta}$ for $t \in [\delta_0, \infty)$ with $\delta_0 < \delta/2$ (see Figure 3(b) for an illustration). We use this contour to prove Proposition 2.4, while relying on the following bounds for the resolvent error along Γ. The proof is included in Appendix B.3 and is an adaption of results from [13, 6, 7].



(a) Illustration of G_θ being the complement of the shaded blue slice.



(b) Illustration of the contour $\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_0$ used in the proof of Proposition 2.4.

FIGURE 3. Contours used to define the Cauchy–Stieltjes integral representation of operators.

Lemma 2.5. *There exists some constant such that for any $h \in (0, h_0)$, for any $z \in \Gamma$, any $f \in L^2(\mathcal{M})$, for any $p \in [0,1]$ such that $\|\mathcal{L}^p f\|_{L^2(\mathcal{M})} < \infty$, for any $\beta \in [0,1]$ such that $p \in [0, (1+\beta)/2]$,*

$$\|(z-\mathcal{L}_h)^{-1}P_h f - P_h(z-\mathcal{L})^{-1}f\|_{L^2(\mathcal{M})} \lesssim h^{2\beta}|z|^{-(1+p-\beta)}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}. \tag{22}$$

We now provide a proof for Proposition 2.4.

Proof of Proposition 2.4. Let $h \in (0, h_0)$, $f \in L^2(\mathcal{M})$ and let $p \in [0,1]$ such that $\|\mathcal{L}^p f\|_{L^2(\mathcal{M})} < \infty$. We can therefore introduce $u = \mathcal{L}^p f \in L^2(\mathcal{M})$. First, note using the triangle inequality, we have

$$\begin{aligned}\|\gamma(\mathcal{L}_h)P_h f - \gamma(\mathcal{L})f\|_{L^2(\mathcal{M})} &\leq \|\gamma(\mathcal{L}_h)P_h f - P_h\gamma(\mathcal{L})f\|_{L^2(\mathcal{M})} \\ &\quad + \|P_h\gamma(\mathcal{L})f - \gamma(\mathcal{L})f\|_{L^2(\mathcal{M})} = S_1 + S_2\end{aligned}$$

where we take $S_1 = \|\gamma(\mathcal{L}_h)P_h f - P_h\gamma(\mathcal{L})f\|_{L^2(\mathcal{M})}$ and $S_2 = \|P_h\gamma(\mathcal{L})f - \gamma(\mathcal{L})f\|_{L^2(\mathcal{M})}$. We now bound these two terms.

For the term S_2, let us introduce the function $\widehat{\gamma}$ defined as $\widehat{\gamma}(\lambda) = \gamma(\lambda)\lambda^{-p}$. In particular, note that since γ is an α-amplitude spectral density, $\widehat{\gamma}$ is an $(\alpha+p)$-amplitude spectral density, and we have, by definition of u,

$$S_2 = \|(I-P_h)\gamma(\mathcal{L})f\|_{L^2(\mathcal{M})} = \|(I-P_h)\widehat{\gamma}(\mathcal{L})u\|_{L^2(\mathcal{M})}$$

We then use the Bramble–Hilbert lemma (cf. Lemma B.4) with $t = 2\min\{\alpha+p; 1\} \in (0,2]$, while noting that, following Remark 2.2, $\|\mathcal{L}^{t/2}(\widehat{\gamma}(\mathcal{L})u)\|_{L^2(\mathcal{M})} \lesssim \|u\|_{L^2(\mathcal{M})} < \infty$ (since $t/2 \leq \alpha + p$):

$$S_2 \lesssim h^t \|\mathcal{L}^{t/2}(\widehat{\gamma}(\mathcal{L})u)\|_{L^2(\mathcal{M})} \lesssim h^t\|u\|_{L^2(\mathcal{M})} = h^{2\min\{\alpha+p;1\}}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}.$$

Let us now bound the term S_1. For any $z \in \Gamma$, set $\mathcal{F}_h(z) = (z-\mathcal{L}_h)^{-1}P_h f - P_h(z-\mathcal{L})^{-1}f$, which yields, using the integral representations of $\gamma(\mathcal{L})$ and $\gamma(\mathcal{L}_h)$,

$$\gamma(\mathcal{L}_h)P_h f - P_h\gamma(\mathcal{L})f = \frac{1}{2\pi i}\int_\Gamma \gamma(z)\mathcal{F}_h(z)\,\mathrm{d}z.$$

The definition of Γ and its parametrization allow to decompose the integral as

$$\begin{aligned}\gamma(\mathcal{L}_h)P_hf - P_h\gamma(\mathcal{L})f = &\frac{-1}{2\pi i}\int_{\delta_0}^{\infty}\gamma(g_+(t))\mathcal{F}_h(g_+(t))g_+'(t)\,\mathrm{d}t\\ &- \frac{1}{2\pi i}\int_{-\theta}^{\theta}\gamma(g_0(t))\mathcal{F}_h(g_0(t))g_0'(t)\,\mathrm{d}t\\ &+ \frac{1}{2\pi i}\int_{\delta_0}^{\infty}\gamma(g_-(t))\mathcal{F}_h(g_-(t))g_-'(t)\,\mathrm{d}t,\end{aligned}$$

where we recall that $g_+'(t) = e^{i\theta},\ g_-'(t) = e^{-i\theta}$ and $g_0'(t) = i\delta_0 e^{it}$. Taking norms on both sides of this equality and using the triangle inequality then gives

$$\begin{aligned}S_1 &= \|\gamma(\mathcal{L}_h)P_hf - P_h\gamma(\mathcal{L})f\|_{L^2(\mathcal{M})}\\ &\le \frac{1}{2\pi}\int_{\delta_0}^{\infty}|\gamma(g_+(t))|\,\|\mathcal{F}_h(g_+(t))\|_{L^2(\mathcal{M})}\,\mathrm{d}t + \frac{\delta_0}{2\pi}\int_{-\theta}^{\theta}|\gamma(g_0(t))|\|\mathcal{F}_h(g_0(t))\|_{L^2(\mathcal{M})}\,\mathrm{d}t\\ &\quad+ \frac{1}{2\pi}\int_{\delta_0}^{\infty}|\gamma(g_-(t))|\,\|\mathcal{F}_h(g_-(t))\|_{L^2(\mathcal{M})}\,\mathrm{d}t\\ &= I_+ + I_0 + I_-.\end{aligned}$$

Let us start by bounding I_0. We apply Equation (7) and Equation (22) (with $\beta = 1$) to obtain

$$\begin{aligned}I_0 &\lesssim \int_{-\theta}^{\theta}|g_0(t)|^{-\alpha}\ h^2|g_0(t)|^{-p}\|\mathcal{L}^pf\|_{L^2(\mathcal{M})}\,\mathrm{d}t = h^2\|\mathcal{L}^pf\|_{L^2(\mathcal{M})}\int_{-\theta}^{\theta}\delta_0^{-(\alpha+p)}\,\mathrm{d}t\\ &\lesssim h^2\|\mathcal{L}^pf\|_{L^2(\mathcal{M})}.\end{aligned}$$

To bound I_+, we distinguish between the three cases $\alpha+p>1$, $\alpha+p<1$ and $\alpha+p=1$. When $\alpha+p>1$, we apply Equation (7) and Equation (22) (with $\beta=1$) to get

$$\begin{aligned}I_+ &\lesssim h^2\|\mathcal{L}^pf\|_{L^2(\mathcal{M})}\int_{\delta_0}^{\infty}|g_+(t)|^{-(\alpha+p)}\,\mathrm{d}t = h^2\|\mathcal{L}^pf\|_{L^2(\mathcal{M})}\int_{\delta_0}^{\infty}|t|^{-(\alpha+p)}\,\mathrm{d}t\\ &= h^2\|\mathcal{L}^pf\|_{L^2(\mathcal{M})}\frac{\delta_0^{-(\alpha+p-1)}}{\alpha+p-1}.\end{aligned}$$

Hence, we can conclude that if $\alpha+p>1$, we have $I_+ \lesssim h^2\|\mathcal{L}^pf\|_{L^2(\mathcal{M})}$.

For $\alpha+p<1$, recall that $0<h<h_0<1$ for some fixed h_0 such that $h_0^{-2}>\delta_0$. We then introduce $\tilde{\alpha} = (\alpha+p)/\alpha$ and split $I_+ = I_{+1} + I_{+2}$ with

$$\begin{aligned}I_{+1} &= \int_{\delta_0}^{h^{-2\tilde{\alpha}}}|\gamma(g_+(t))|\ \|\mathcal{F}_h(g_+(t))\|_{L^2(\mathcal{M})}\,\mathrm{d}t,\\ I_{+2} &= \int_{h^{-2\tilde{\alpha}}}^{\infty}|\gamma(g_+(t))|\ \|\mathcal{F}_h(g_+(t))\|_{L^2(\mathcal{M})}\,\mathrm{d}t.\end{aligned}$$

Let then $e_h = (1-(\alpha+p))/|\log h| \in (0, 1-(\alpha+p))$ and $\beta = \alpha+p+e_h \in (0,1)$. Note in particular that $p\in[0,(1+\beta)/2]$ since $(1+\beta)/2-p = (1-p+\alpha+e_h)/2>0$. We can therefore use Equation (7) and Equation (22) to bound I_{+1} by

$$\begin{aligned}
I_{+1} &\lesssim h^{2\beta}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}\int_{\delta_0}^{h^{-2\tilde{\alpha}}} |g_+(t)|^{-(\alpha+1+p-\beta)}\,\mathrm{d}t = h^{2\beta}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}\int_{\delta_0}^{h^{-2\tilde{\alpha}}} |t|^{-(1-e_h)}\,\mathrm{d}t \\
&\lesssim h^{2\beta}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}\int_{0}^{h^{-2\tilde{\alpha}}} |t|^{-(1-e_h)}\,\mathrm{d}t = h^{2\beta}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}\frac{h^{-2\tilde{\alpha}e_h}}{e_h} \\
&= h^{2(\alpha+p)}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}\frac{h^{-2(\tilde{\alpha}-1)e_h}}{e_h} \lesssim h^{2(\alpha+p)}|\log h|\|\mathcal{L}^p f\|_{L^2(\mathcal{M})},
\end{aligned}$$

since $h^{-2\tilde{\alpha}e_h} = \exp(2\tilde{\alpha}(1-(\alpha+p))) \lesssim 1$. We proceed in the same manner to bound I_{+2}, using this time Equation (22) (with $\beta = p$):

$$\begin{aligned}
I_{+2} &\lesssim h^{2p}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}\int_{h^{-2\tilde{\alpha}}}^{\infty} |g_+(t)|^{-(\alpha+1)}\,\mathrm{d}t = h^{2p}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}\int_{h^{-2\tilde{\alpha}}}^{\infty} |t|^{-(\alpha+1)}\,\mathrm{d}t \\
&\lesssim h^p h^{2\tilde{\alpha}\alpha}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})} \lesssim h^{2(\alpha+p)}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}.
\end{aligned}$$

Hence, we conclude that if $\alpha+p<1$, we have $I_+ \lesssim |\log h|\; h^{2(\alpha+p)}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}$.

Finally, for $\alpha+p=1$ we repeat the same approach as for the case $\alpha+p<1$. On the one hand we use Equation (22) (with $\beta=1$) to get for the term I_{+1} the bound

$$\begin{aligned}
I_{+1} &\lesssim h^{2}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}\int_{\delta_0}^{h^{-2\tilde{\alpha}}} |t|^{-1}\,\mathrm{d}t = h^{2}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}(\log(h^{-2\tilde{\alpha}}) - \log\delta_0) \\
&\lesssim h^{2}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}|\log h|,
\end{aligned}$$

where we used the fact that h is upper-bounded by a fixed constant h_0 satisfying $h_0^{-2} > \delta_0$ to derive the last inequality. On the other hand, we once again use Equation (22) with $\beta = p$ to bound the term I_{+2}

$$\begin{aligned}
I_{+2} &\lesssim h^{2p}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}\int_{h^{-2\tilde{\alpha}}}^{\infty} |t|^{-(\alpha+1)}\,\mathrm{d}t \lesssim h^p h^{2\tilde{\alpha}\alpha}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})} \\
&\lesssim h^{2(\alpha+p)}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})} = h^{2}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}.
\end{aligned}$$

Hence, we have $I_+ \lesssim h^2|\log h|\;\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}$ when $\alpha+p=1$.

Putting together the three cases, we can therefore conclude that

$$I_+ \lesssim C_{\alpha+p}(h)h^{2\min\{\alpha+p;1\}}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})},$$

where $C_{\alpha+p}(h) = |\log h|$ if $\alpha+p \leq 1$, and $C_{\alpha+p}(h) > 1$ if $\alpha+p \neq 1$. Finally, note that by symmetry, I_- satisfies the same bounds as I_+, meaning that we also get $I_- \lesssim C_{\alpha+p}(h)h^{2\min\{\alpha+p;1\}}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}$.

In conclusion, putting together the estimates for I_0, I_+ and I_- we get

$$S_1 \lesssim C_{\alpha+p}(h)h^{2\min\{\alpha+p;1\}}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})},$$

and putting together the estimates pf S_1 and S_2 yields

$$\|\gamma(\mathcal{L}_h)P_h f - \gamma(\mathcal{L})f\|_{L^2(\mathcal{M})} \lesssim C_{\alpha+p}(h)h^{2\min\{\alpha+p;1\}}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})},$$

which concludes the proof. $\square$

A result similar to Proposition 2.4 can be derived to quantify the error between functions of the operators $\mathcal{L}_h$ and L_h. To do so, we note that since $\mathsf{A}_{\mathcal{M}_h}$ is continuous and coercive, the associated operator L_h is also sectorial with some angle in $(0, \pi/2)$. Hence, without loss of generality, the angle $\theta \in (0, \pi/2)$ can be assumed to be large enough to ensure that Equation (19) also holds for L_h, i.e. that for any $z \in G_\theta$,

$$\|(z - \mathsf{L}_h)^{-1} v_h\|_{L^2(\mathcal{M}_h)} \leq C_\theta |z|^{-1} \|v_h\|_{L^2(\mathcal{M}_h)}, \quad v_h \in S_h, \tag{23}$$

and that an integral representation similar to Equation (20) also holds for $\gamma(\mathsf{L}_h)$, namely:

$$\gamma(\mathsf{L}_h) = \frac{1}{2\pi i} \int_\Gamma \gamma(z)(z - \mathsf{L}_h)^{-1} \, \mathrm{d}z.$$

Proposition 2.6. *Let γ be an α-amplitude spectral density with $\alpha > d/4$. Then, there exists some constant such that for any $h \in (0, h_0)$, for any $\tilde{f} \in S_h^\ell$,*

$$\left\| \left(\gamma(\mathcal{L}_h)\tilde{f}\right)^{-\ell} - \gamma(\mathsf{L}_h)\mathsf{P}_h(\sigma \tilde{f}^{-\ell}) \right\|_{L^2(\mathcal{M}_h)} \lesssim h^2 \, \|\mathcal{L}_h^{-\min\{\alpha+d/4;1\}/2} \tilde{f}\|_{L^2(\mathcal{M})}. \tag{24}$$

This proposition can be seen as an extension of [6, Lemma 4.4] relying on extensions of the estimates proven in [6, Lemma A.1]. Its proof is similar and therefore postponed to Appendix B.

3. Stochastic theory: random fields on surfaces and convergence of SFEM approximation

In this section, we introduce random fields and white noise on surfaces, thus allowing us to make sense of Equation (1) in Section 1. Further, we give approximation methods based on SFEM and prove strong error bounds.

3.1. **Random fields on surfaces.** Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a complete probability space. We are interested in Gaussian random fields on $\mathcal{M}$ defined as $L^2(\mathcal{M})$-valued random variables through expansions of the form

$$\mathcal{Z} = \sum_{i \in \mathbb{N}} Z_i e_i, \tag{25}$$

where $\{e_i\}_{i\in\mathbb{N}}$ denotes a real-valued orthonormal basis of $L^2(\mathcal{M})$ composed of eigenfunctions of the operator $\mathcal{L}$ (cf. Proposition 2.1), and $\{Z_i\}_{i\in\mathbb{N}}$ is a sequence of real Gaussian random variables such that $\mathbb{E}[Z_i] = 0$ for any $i \in \mathbb{N}$ and $\sum_{i\in\mathbb{N}} \mathbb{E}[|Z_i|^2] < \infty$. As such, $\mathcal{Z}$ can be seen as en element of the Hilbert space $L^2(\Omega; L^2(\mathcal{M}))$ of $L^2(\mathcal{M})$-valued random variables, to which we associate the inner product $(\cdot,\cdot)_{L^2(\Omega;L^2(\mathcal{M}))}$ (and norm $\|\cdot\|_{L^2(\Omega;L^2(\mathcal{M}))}$) defined by

$$(\mathcal{Z}, \mathcal{Z}')_{L^2(\Omega;L^2(\mathcal{M}))} = \mathbb{E}\left[(\mathcal{Z}, \mathcal{Z}')_{L^2(\mathcal{M})}\right], \quad \mathcal{Z}, \mathcal{Z}' \in L^2(\Omega; L^2(\mathcal{M})).$$

Finally, in analogy to Equation (25), we formally define the Gaussian white noise on $\mathcal{M}$ by the expansion

$$\mathcal{W} = \sum_{i \in \mathbb{N}} W_i e_i, \tag{26}$$

where $\{W_i\}_{i\in\mathbb{N}}$ is a sequence of independent real standard Gaussian random variables. We observe that even though this expansion does not converge in $L^2(\Omega; L^2(\mathcal{M}))$, it does however converge in $L^2(\mathcal{M}; H^s(\mathcal{M}))$ for $s < -d/2$. Moreover, we have that for any $\phi \in$

$L^2(\mathcal{M})$, the expansion $(\phi, \mathcal{W})_{L^2(\mathcal{M})} = \sum_{i\in\mathbb{N}} W_i(\phi, e_i)_{L^2(\mathcal{M})}$ converges in $L^2(\Omega)$. Further, $(\mathcal{W}, \phi)_{L^2(\mathcal{M})}$ defines a complex Gaussian variable with mean 0, and for any $\phi' \in L^2(\mathcal{M})$, $\mathsf{Cov}((\phi, \mathcal{W})_{L^2(\mathcal{M})}, (\phi', \mathcal{W})_{L^2(\mathcal{M})}) = \mathbb{E}[(\phi, \mathcal{W})_{L^2(\mathcal{M})}\overline{(\phi', \mathcal{W})_{L^2(\mathcal{M})}}] = (\phi, \phi')_{L^2(\mathcal{M})}$. As such, the Gaussian white noise (26) can be interpreted as a *generalized Gaussian random field* over $L^2(\mathcal{M})$.

Circling back to the class of random fields defined in Section 1, we can now make sense of Equation (1) through Equation (8) and Equation (26), thus yielding the definition

$$\mathcal{Z} = \gamma(\mathcal{L})\mathcal{W} = \sum_{i\in\mathbb{N}} \gamma(\lambda_i) W_i e_i, \tag{27}$$

which results in $\mathcal{Z} \in L^2(\Omega; H^s(\mathcal{M}))$ for any $s \geq 0$ such that $4\alpha - d > 2s$. Note in particular that all summands in Equation (27) are real-valued functions, and that therefore $\mathcal{Z}$ is real-valued. In Figure 4 we illustrate the influence of the parameter choices on the resulting field on $\mathbb{S}^2$ for generalized *non-stationary Whittle–Matérn* fields on $\mathbb{S}^2$

$$\mathcal{Z} = (\mathcal{L})^{-\alpha}\mathcal{W},$$

where $\alpha > d/2$. By selecting $D_{ij}(x) = \delta_{ij}$ and $V(x) = \kappa^2$ with $\kappa > 0$ one recovers the classical, stationary Whittle–Matérn fields studied in various settings in for instance [9, 3, 6, 21, 18, 27]. In Figures 4(a) and 4(b) we show this case with $\kappa^2 = 10$ for a rougher field with $\alpha = 0.55$ and a smoother one with $\alpha = 1.5$, respectively.

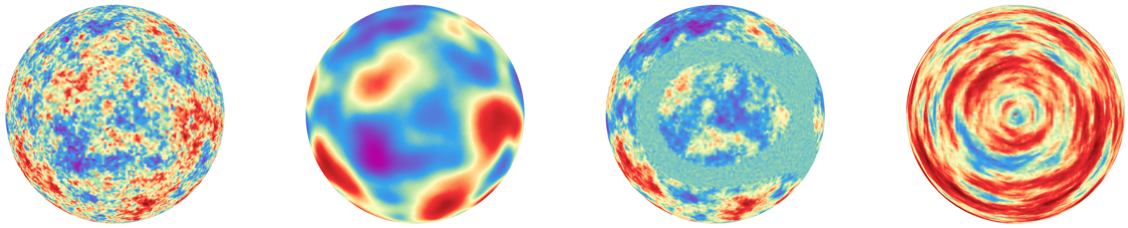
Two non-stationary fields are shown in Figure 4(c) and Figure 4(d), obtained by varying the coefficient functions D_{ij} and V and setting $\alpha = 0.75$. In Figure 4(c), we keep $D_{ij} = \delta_{ij}$ but use

$$V(x) = \begin{cases} 10^5 & \text{for } x_2^6 + x_1^3 - x_3^2 \in (0.1, 0.5), \\ 10 & \text{else} \end{cases}$$

resulting in the observed localized behavior, where the field is essentially turned off in the region with large V. More specifically, V describes the local correlation length, where a large $V(x)$ corresponds to a small correlation length around x.

Finally, setting $V = 10$ constant again, we show the influence of varying parameters $\mathcal{D}$ in Figure 4(d). To derive suitable coefficients, we select a smooth function f and compute its gradient $\nabla_{\mathbb{S}^2} f$ as well as its skew-gradient X_f given by $x \times \nabla f(x)$ at each point $x \in \mathbb{S}^2$. We set for fixed $\rho_1, \rho_2 > 0$, $\mathcal{D}(x)u = \rho_1(\nabla f(x) \cdot u)\nabla f(x) + \rho_2(X_f(x) \cdot u)X_f(x)$ for any $u \in T_x\mathbb{S}^2$. Here, the inner product refers to the Riemannian inner product associated with the standard round metric on $\mathbb{S}^2$. Since $\nabla_{\mathbb{S}^2} f(x)$ and $X_f(x)$ both are in $T_x\mathbb{S}^2$, $\mathcal{D}(x)$ is a linear mapping from $T_x\mathbb{S}^2$ into itself. Further, as $\nabla_{\mathbb{S}^2} f$ is perpendicular to $X_f(x)$, by selecting ρ_1 and ρ_2, we obtain a field that is elongated either orthogonally to the level sets of f (large ρ_1, small ρ_2) or tangentially to the level sets (small ρ_1, and large ρ_2). To generate Figure 4(d), we selected $f(x) = x_2$, i.e., the function returning the second coordinate in Cartesian coordinates, $\rho_1 = 1$ and $\rho_2 = 25$. Finally, we remark that other types of amplitude spectral densities can be considered. For instance, to generate Figure 1(c), we used an amplitude spectral density of the form $\gamma(\lambda) = \sin(\lambda)\lambda^{-0.6}$.

3.2. **Approximation of random fields with surface finite elements.** Let γ be an α-amplitude spectral density with $\alpha > d/4$. Following the approach presented by [19], the field $\mathcal{Z}$ is approximated by an expansion similar to that of Equation (27), but involving



(a) Stationary Whittle–Matérn field with low decay parameter.

(b) Stationary Whittle–Matérn field with high decay parameter.

(c) Non-stationary Whittle–Matérn field by varying potential.

(d) Non-stationary Whittle–Matérn field by varying diffusion matrix.

FIGURE 4. Some examples to highlight the influence of the different model parameters.

only quantities defined on the polyhedral surface $\mathcal{M}_h$. More precisely, we define the SFEM–Galerkin approximation Z_h of the field $\mathscr{Z}$ by the relation

$$\mathsf{Z}_h = \sum_{i=1}^{N_h} \gamma(\Lambda_i^h) W_i^h E_i^h, \tag{28}$$

where $\{W_i^h\}_{1\leq i\leq N_h}$ is a sequence of independent standard Gaussian random variables, whose precise definition is clarified later in this section, and $\{(\Lambda_i^h, E_i^h)\}_{1\leq i\leq N_h}$ are the eigenpairs of L_h introduced in Section 2.2. Then, as proven in [19, Theorem 3.4], Z_h can be decomposed in the nodal basis $\{\psi_i\}_{1\leq i\leq N_h}$ of S_h as

$$\mathsf{Z}_h = \sum_{i=1}^{N_h} Z_i \psi_i,$$

where the weights $(Z_1, \dots, Z_{N_h})$ form a centered Gaussian vector which covariance matrix $\boldsymbol{\Sigma}_Z$ can be expressed using the matrices $\boldsymbol{C}$ and $\boldsymbol{S}$ introduced in Equations (14) and (15) as follows:

$$\boldsymbol{\Sigma}_Z = \left(\sqrt{\boldsymbol{C}}\right)^{-T} \gamma(\boldsymbol{S})^2 \left(\sqrt{\boldsymbol{C}}\right)^{-1}, \tag{29}$$

and, following Equation (16), the function of matrix $\gamma(\boldsymbol{S})$ is defined as

$$\gamma(\boldsymbol{S})^2 = \boldsymbol{V}\,\mathrm{Diag}\left(\gamma\big(\Lambda_1^h\big)^2, \dots, \gamma\big(\Lambda_1^{N_h}\big)^2\right) \boldsymbol{V}^T. \tag{30}$$

Note that sampling the weights $(Z_1, \dots, Z_{N_h})$, and therefore the field Z_h, using directly the expression of their covariance matrix requires in practice to fully diagonalize $\boldsymbol{S}$ (since Equation (29) involves a function of a matrix). Such an operation would result in a prohibitive computational cost (of order $\mathcal{O}(N_h^3)$ operations). To avoid this cost, we use the Chebyshev trick proposed by [19, Section 4], and approximate Z_h by the field $\mathsf{Z}_{h,M}$ defined by

$$\mathsf{Z}_{h,M} = \sum_{i=1}^{N_h} P_{\gamma,M}(\Lambda_i^h) W_i^h E_i^h, \tag{31}$$

where $P_{\gamma,M}$ is a Chebyshev polynomial approximation of degree $M \in \mathbb{N}$ of γ over an interval $[\lambda_{\min}, \lambda_{\max}]$ containing all the eigenvalues $\{\Lambda_i^h\}_{1\le i\le N_h}$ of L_h (which we recall, coincide with the eigenvalues of $\boldsymbol{S}$). Such an interval can be obtained as follows. On the one hand, one can take $\lambda_{\min} = V_-$. On the other hand, following [19], a candidate for $\lambda_{\max}$ is obtained by applying the Gershgorin circle theorem to $\boldsymbol{S}$.

The field $\mathsf{Z}_{h,M}$, called Galerkin–Chebyshev approximation of $\mathcal{Z}$, is in essence defined by just replacing the amplitude spectral density γ by the polynomial $P_{\gamma,M}$ in the definition of SFEM–Galerkin approximation Z_h. The expansion of $\mathsf{Z}_{h,M}$ into the nodal basis,

$$\mathsf{Z}_{h,M} = \sum_{i=1}^{N_h} Z_i^{(M)}\psi_i, \tag{32}$$

is such that the weights $(Z_1^{(M)},\dots,Z_{N_h}^{(M)})$ now form a centered Gaussian vector with covariance matrix $\boldsymbol{\Sigma}_{Z^{(M)}}$ given by

$$\boldsymbol{\Sigma}_{Z^{(M)}} = \big(\sqrt{\boldsymbol{C}}\big)^{-T} P_{\gamma,M}^2(\boldsymbol{S})\,\big(\sqrt{\boldsymbol{C}}\big)^{-1}. \tag{33}$$

Hence, the matrix function in Equation (29) is now replaced by a matrix polynomial $P_{\gamma,M}^2(\boldsymbol{S})$. This eliminates the eigendecomposition need associated with matrix functions and therefore speeds up the computations. Indeed, the weights $(Z_1^{(M)},\dots,Z_{N_h}^{(M)})$ can be sampled through

$$\begin{pmatrix} Z_1^{(M)} \\ \vdots \\ Z_{N_h}^{(M)} \end{pmatrix} = \big(\sqrt{\boldsymbol{C}}\big)^{-T} P_{\gamma,M}(\boldsymbol{S})\boldsymbol{W} \tag{34}$$

where $\boldsymbol{W} = (w_1,\dots,w_{N_h})^T$ is a vector of independent standard Gaussian random variables, and the matrix-vector product by $P_{\gamma,M}(\boldsymbol{S})$ can be computed iteratively while just requiring products between $\boldsymbol{S}$ and vectors. In the next two subsections, we provide error estimates quantifying the error between our target random field $\mathcal{Z}$ and its successive SFEM and polynomial approximations by Z_h and $\mathsf{Z}_{h,M}$.

3.3. **Error analysis of the SFEM discretization.** We start with analyzing the error between the random field $\mathcal{Z}$ defined on $\mathcal{M}$ and its SFEM approximation Z_h, as stated in the next theorem.

Theorem 3.1. *Let γ be an α-amplitude spectral density with $\alpha > d/4$. Then, for any $h \in (0,h_0)$, the strong approximation error of the random field $\mathcal{Z}$ by its discretization Z_h^ℓ satisfies the bound*

$$\|\mathcal{Z} - \mathsf{Z}_h^\ell\|_{L^2(\Omega;L^2(\mathcal{M}))} \le C_\alpha(h) h^{2\min\{\alpha-d/4;1\}}, \tag{35}$$

where $C_\alpha(h) = |\log h|$ if $d/4 < \alpha \le 1$, $C_\alpha(h) = |\log h|^{3/2}$ if $1 < \alpha < 1 + d/2$, and $C_\alpha(h) = |\log h|^{1/2}$ if $\alpha \ge 1 + d/2$.

To prove the strong error estimate, we rely on the deterministic error bounds proven in the previous section, and on several intermediate approximations of $\mathcal{Z}$ defined on the spaces S_h^ℓ (i.e., on $\mathcal{M}$) and S_h (i.e., on $\mathcal{M}_h$). These intermediate approximations require in turn to define approximations of the Gaussian white noise $\mathcal{W}$ on the spaces S_h^ℓ and S_h.

We first define on S_h^ℓ (i.e., on $\mathcal{M}$), the projected white noise $\widetilde{\mathcal{W}}$ as

$$\widetilde{\mathcal{W}} = \sum_{j=1}^{N_h} \xi_j e_j^h, \tag{36}$$

where we recall that $\{e_j^h\}_{1\le i\le N_h}$ denotes an orthonormal basis of eigenfunctions of $\mathcal{L}_h$, and for any $j \in \{1,\dots,N_h\}$, we take $\xi_j = \left(e_j^h, \mathcal{W}\right)_{L^2(\mathcal{M})}$. In particular, this last relation implies (by definition of the white noise $\mathcal{W}$) that $\xi_1,\dots,\xi_{N_h}$ are independent standard Gaussian random variables.

Remark 3.2. By injecting the representation (26) of $\mathcal{W}$ in the definition of ξ_j, we get that $\widetilde{\mathcal{W}}$ can itself be formally represented by $\widetilde{\mathcal{W}} = \sum_{k=1}^\infty W_k P_h e_k = P_h\mathcal{W}$. This explains why we refer to it as a projected white noise. In particular, note that $\tilde{\mathcal{W}} \in L^2(\Omega; L^2(\mathcal{M}))$ and $\|\tilde{\mathcal{W}}\|^2_{L^2(\Omega;L^2(\mathcal{M}))} = \|\sum_{j=1}^{N_h} \xi_j e_j^h\|^2_{L^2(\Omega;L^2(\mathcal{M}))} = N_h < \infty$.

Based on $\widetilde{\mathcal{W}}$, we can then introduce a first approximation, on the space S_h^ℓ, of the field $\mathcal{Z}$. We denote this approximation by $\tilde{\mathcal{Z}}_h$ and define it in analogy to Equation (27) as

$$\tilde{\mathcal{Z}}_h = \gamma(\mathcal{L}_h)\widetilde{\mathcal{W}} = \sum_{j=1}^{N_h} \gamma(\lambda_j^h)\xi_j e_j^h. \tag{37}$$

Now, on the space S_h, we define *two* white noise approximations $\widehat{\mathsf{W}}$ and W which are based on the projected white noise $\widetilde{\mathcal{W}}$:

$$\widehat{\mathsf{W}} = \mathsf{P}_h(\sigma\widetilde{\mathcal{W}}^{-\ell}) \quad \text{and} \quad \mathsf{W} = \mathsf{P}_h(\sigma^{1/2}\widetilde{\mathcal{W}}^{-\ell}),$$

where σ is the ratio of area measures introduced in Section 2.2. On the one hand, we associate to $\widehat{\mathsf{W}}$ an approximation $\widehat{\mathsf{Z}}_h$ of the field $\mathcal{Z}$ on S_h, which we define in analogy to Equation (38) as

$$\widehat{\mathsf{Z}}_h = \gamma(\mathsf{L}_h)\widehat{\mathsf{W}}.$$

On the other hand, by expanding $\widehat{\mathsf{W}}$ and W in the orthonormal basis $\{E_j^h\}_{1\le i\le N_h}$ of eigenfunctions of L_h, we obtain alternative representations of these fields, and we can draw a link between W and the SFEM–Galerkin approximation Z_h, as stated in the next lemma.

Lemma 3.3. *The noises* $\widehat{\mathsf{W}}$ *and* W *can be written as*

$$\widehat{\mathsf{W}} = \sum_{j=1}^{N_h} \alpha_j E_j^h \quad \textit{and} \quad \mathsf{W} = \sum_{j=1}^{N_h} \beta_j E_j^h,$$

where $(\alpha_1,\dots,\alpha_{N_h})$ *and* $(\beta_1,\dots,\beta_{N_h})$ *are multivariate normal with mean* 0 *and respective covariance matrices* $\boldsymbol{A} = [(\sigma E_i^h, E_j^h)_{L^2(\mathcal{M}_h)}]_{1\le i,j\le N_h}$ *and* $\boldsymbol{B} = \boldsymbol{I}$. *In particular, it holds that the SFEM–Galerkin approximation* Z_h *defined in Equation* (28) *satisfies*

$$\mathsf{Z}_h = \gamma(\mathsf{L}_h)\mathsf{W}, \tag{38}$$

where we take for any $j \in \{1,\dots,N_h\}$, $W_j^h = \beta_j$.

Proof. As $\widehat{\mathsf{W}} \in S_h$, we can expand it in the orthonormal basis $\{E_j^h\}_{1\leq i\leq N_h}$ to get

$$\widehat{\mathsf{W}} = \sum_{j=1}^{N_h} \alpha_j E_j^h,$$

where $\alpha_j = (\widehat{\mathsf{W}}, E_j^h)_{L^2(\mathcal{M}_h)} = (\sigma\widetilde{\mathcal{W}}^{-\ell}, E_j^h)_{L^2(\mathcal{M}_h)}$. Then, by definition of σ, of $\widetilde{\mathcal{W}}$ and since $(E_j^h)^\ell \in S_h^\ell$, we further get for any $1 \leq j \leq N_h$,

$$\begin{aligned}
\alpha_j = (\widetilde{\mathcal{W}}, (E_j^h)^\ell)_{L^2(\mathcal{M})} &= \sum_{k=1}^{N_h} (\mathcal{W}, e_k^h)_{L^2(\mathcal{M})} (e_k^h, (E_j^h)^\ell)_{L^2(\mathcal{M})} \\
&= \left(\mathcal{W}, \sum_{k=1}^{N_h} (e_k^h, (E_j^h)^\ell)_{L^2(\mathcal{M})} e_k^h\right)_{L^2(\mathcal{M})} = (\mathcal{W}, (E_j^h)^\ell)_{L^2(\mathcal{M})}.
\end{aligned}$$

Therefore, by definition of the white noise $\mathcal{W}$, we can conclude that for any $1 \leq i, j \leq N_h$, α_j is normally distributed with mean 0, and that

$$\mathbb{E}[\alpha_i\alpha_j] = \mathsf{Cov}(\alpha_i, \alpha_j) = \left((E_i^h)^\ell, (E_j^h)^\ell\right)_{L^2(\mathcal{M})} = \left(\sigma E_i^h, E_j^h\right)_{L^2(\mathcal{M}_h)} = A_{ij}.$$

Hence, $(\alpha_1, \ldots, \alpha_{N_h})$ is indeed multivariate normal (any linear combination of the α_j being Gaussian by definition of $\mathcal{W}$) with mean 0 and covariance matrix $\boldsymbol{A}$.

Similarly, since $\mathsf{W} \in S_h$, we can write again

$$\mathsf{W} = \sum_{j=1}^{N_h} \beta_j E_j^h,$$

where $\beta_j = (\mathsf{W}, E_j^h)_{L^2(\mathcal{M}_h)} = (\sigma^{1/2}\widetilde{\mathcal{W}}^{-\ell}, E_j^h)_{L^2(\mathcal{M}_h)} = (\sigma\widetilde{\mathcal{W}}^{-\ell}, \sigma^{-1/2}E_j^h)_{L^2(\mathcal{M}_h)}$, and the same computations as before yield that $\beta_j = \big(\mathcal{W}, (\sigma^{-1/2}E_j^h)^\ell\big)_{L^2(\mathcal{M})}$. Hence, we conclude this time that for any $1 \leq i, j \leq N_h$, β_j is also normally distributed with mean 0, and that

$$\mathbb{E}[\beta_i\beta_j] = \left(\sigma(\sigma^{-1/2}E_i^h), (\sigma^{-1/2}E_j^h)\right)_{L^2(\mathcal{M}_h)} = \left(\sigma(\sigma^{-1/2}E_i^h), (\sigma^{-1/2}E_j^h)\right)_{L^2(\mathcal{M}_h)} = B_{ij},$$

by orthonormality of $\{E_j^h\}_{1\leq i\leq N_h}$. In conclusion, $(\beta_1, \ldots, \beta_{N_h})$ is indeed multivariate normal with mean 0 and covariance matrix $\boldsymbol{B} = \boldsymbol{I}$. In particular, this means that $\beta_1, \ldots, \beta_{N_h}$ are independent standard Gaussian variables.

Finally, note that we can write

$$\begin{aligned}
\gamma(\mathsf{L}_h)\mathsf{P}_h\mathsf{W} &= \sum_{j=1}^{N_h} \gamma(\Lambda_j^h)\big(\mathsf{P}_h\mathsf{W}, E_j^h\big)_{L^2(\mathcal{M}_h)} E_j^h = \sum_{j=1}^{N_h} \gamma(\Lambda_j^h)\big(\mathsf{W}, E_j^h\big)_{L^2(\mathcal{M}_h)} E_j^h \\
&= \sum_{j=1}^{N_h} \gamma(\Lambda_j^h)\beta_j E_j^h = \mathsf{Z}_h
\end{aligned}$$

where we take $W_j^h = \beta_j$ in Equation (28). $\square$

We now circle back to proving Theorem 3.1. Using the intermediate approximations $\tilde{\mathcal{Z}}_h$ and $\widehat{\mathsf{Z}}_h$ and the equivalence of the L^2 norms on $\mathcal{M}$ and $\mathcal{M}_h$, we can upper-bound the error between the field $\mathcal{Z}$ and its SFEM–Galerkin approximation Z_h by

$$\begin{aligned}
&\|\mathcal{Z}-\mathsf{Z}_h^\ell\|_{L^2(\Omega;L^2(\mathcal{M}))}\\
&\quad\lesssim \|\mathcal{Z}-\tilde{\mathcal{Z}}_h\|_{L^2(\Omega;L^2(\mathcal{M}))}+\|\tilde{\mathcal{Z}}_h-\widehat{\mathsf{Z}}_h^\ell\|_{L^2(\Omega;L^2(\mathcal{M}))}+\|\widehat{\mathsf{Z}}_h^\ell-\mathsf{Z}_h^\ell\|_{L^2(\Omega;L^2(\mathcal{M}))}\\
&\quad\lesssim \|\mathcal{Z}-\tilde{\mathcal{Z}}_h\|_{L^2(\Omega;L^2(\mathcal{M}))}+\big\|\big(\tilde{\mathcal{Z}}_h\big)^{-\ell}-\widehat{\mathsf{Z}}_h\big\|_{L^2(\Omega;L^2(\mathcal{M}_h))}+\big\|\widehat{\mathsf{Z}}_h-\mathsf{Z}_h\big\|_{L^2(\Omega;L^2(\mathcal{M}_h))}.
\end{aligned}$$

We derive error estimates for each one of the three terms obtained in the last inequality. We start with the term $\|\mathcal{Z}-\tilde{\mathcal{Z}}_h\|_{L^2(\Omega;L^2(\mathcal{M}))}$.

Lemma 3.4. *Let $h\in(0,h_0)$. It holds that*

$$\|\mathcal{Z}-\tilde{\mathcal{Z}}_h\|_{L^2(\Omega;L^2(\mathcal{M}))}\lesssim C_\alpha(h)h^{2\min\{\alpha-d/4;1\}},$$

where $C_\alpha(h)=|\log h|$ if $d/4<\alpha\le 1$, $C_\alpha(h)=|\log h|^{3/2}$ if $1<\alpha<1+d/2$, and $C_\alpha(h)=|\log h|^{1/2}$ if $\alpha\ge 1+d/2$.

Proof. Our aim is to bound the error between $\tilde{\mathcal{Z}}_h$ and $\mathcal{Z}$, where we remark in particular that $\mathcal{Z}=\gamma(\mathcal{L})\mathcal{W}$ and $\mathcal{Z}_h=\gamma(\mathcal{L}_h)P_h\mathcal{W}$ (cf. Remark 3.2). Recall that we consider $h\in(0,h_0)$ (meaning in particular that $|\log h|>1$).

Similarly to the proof of [6, Theorem 5.2], we introduce $\varepsilon_h=\min\{d/2;\alpha-d/2\}/(2|\log h|)$, $\eta=2(d/4+\varepsilon_h)$, and let $\zeta=2(\alpha-d/4-\varepsilon_h)>0$, so that $\alpha=(\eta+\zeta)/2$. Note in particular that $\varepsilon_h>0$ since $d\in\{1,2\}$ and $\alpha>1>d/2$, and $\varepsilon_h<(\alpha-d/2)/2<\alpha-d/2<\alpha-d/4$. Besides, $\eta>d/2$ and $\zeta>0$. We distinguish two cases: if $\alpha\le 1$, and if $\alpha>1$.

<u>Case $\alpha\le 1$</u>. The proof follows the same approach as [6, Theorem 5.2]. Indeed, note that the triangle inequality yields

$$\|\mathcal{Z}-\tilde{\mathcal{Z}}_h\|_{L^2(\Omega;L^2(\mathcal{M}))}=\|\gamma(\mathcal{L})\mathcal{W}-\gamma(\mathcal{L}_h)P_h\mathcal{W}\|_{L^2(\Omega;L^2(\mathcal{M}))}\le S_1+S_2,$$

where we take

$$S_1=\|\gamma(\mathcal{L})\mathcal{W}-\gamma(\mathcal{L})P_h\mathcal{W}\|_{L^2(\Omega;L^2(\mathcal{M}))},\quad S_2=\|\gamma(\mathcal{L})P_h\mathcal{W}-\gamma(\mathcal{L}_h)P_h\mathcal{W}\|_{L^2(\Omega;L^2(\mathcal{M}))}.$$

To bound the term S_1, we use

$$\begin{aligned}
S_1^2&=\|\gamma(\mathcal{L})(I-P_h)\mathcal{W}\|^2_{L^2(\Omega;L^2(\mathcal{M}))}=\|\sum_{i\in\mathbb{N}}\gamma(\lambda_i)W_i(I-P_h)e_i\|^2_{L^2(\Omega;L^2(\mathcal{M}))}\\
&=\sum_{i\in\mathbb{N}}\gamma(\lambda_i)^2\|(I-P_h)e_i\|^2_{L^2(\mathcal{M})},
\end{aligned}$$

and since γ is an α-amplitude spectral density, we get

$$S_1^2\lesssim\sum_{i\in\mathbb{N}}\lambda_i^{-2\alpha}\|(I-P_h)e_i\|^2_{L^2(\mathcal{M})},\tag{39}$$

Applying the Bramble–Hilbert lemma (67) with $t=2((\alpha-d/4)-\varepsilon_h)\in(0,2)$ together with Equation (39) gives

$$S_1^2\lesssim\sum_{i\in\mathbb{N}}h^{2t}\lambda_i^{-2\alpha+t}=h^{4(\alpha-d/4-\varepsilon_h)}\sum_{i\in\mathbb{N}}\lambda_i^{-2(d/4+\varepsilon_h)}\lesssim h^{4(\alpha-d/4)}\sum_{i\in\mathbb{N}}\lambda_i^{-2(d/4+\varepsilon_h)},$$

since $h^{-4\varepsilon_h} = e^{4(\alpha-d/4)} \lesssim 1$. We can then write, using Equation (6),

$$S_1^2 \lesssim h^{4(\alpha-d/4)} \sum_{i\in\mathbb{N}} i^{-(1+4\varepsilon_h/d)} \lesssim h^{4(\alpha-d/4)} \int_1^\infty t^{-(1+4\varepsilon_h/d)}\,\mathrm{d}t$$
$$\lesssim \frac{1}{4\varepsilon_h/d} h^{4(\alpha-d/4)} \lesssim |\log h| h^{4(\alpha-d/4)},$$

where we used the definition of ε_h for the last inequality. So, we conclude $S_1 \lesssim |\log h|^{1/2} h^{2(\alpha-d/4)}$.

For the term S_2, we note that

$$S_2 = \|\gamma(\mathcal{L})P_h\mathcal{W} - \gamma(\mathcal{L}_h)P_h(P_h\mathcal{W})\|_{L^2(\Omega;L^2(\mathcal{M}))},$$

where, following Equation (36), $\|P_h\mathcal{W}\|^2_{L^2(\Omega;L^2(\mathcal{M}))} = \|\sum_{j=1}^{N_h} \xi_j e_j^h\|^2_{L^2(\Omega;L^2(\mathcal{M}))} = N_h < \infty$ (cf. Remark 3.2). Hence, we can use Proposition 2.4 to deduce that since $\alpha \le 1$,

$$S_2 \lesssim |\log h| h^{2\alpha} \|P_h\mathcal{W}\|_{L^2(\Omega;L^2(\mathcal{M}))}.$$

Note then that since $N_h \propto h^{-d}$ (uniform mesh assumption), we can write $\|P_h\mathcal{W}\|_{L^2(\Omega;L^2(\mathcal{M}))} = N_h^{1/2} \lesssim h^{-d/2}$. Therefore, we can conclude that

$$S_2 \lesssim |\log h| h^{2\alpha-d/2}.$$

Putting together the bounds obtained for S_1 and S_2, we can then conclude that if $\alpha \le 1$,

$$\|\mathcal{Z} - \tilde{\mathcal{Z}}_h\|_{L^2(\Omega;L^2(\mathcal{M}))} \le S_1 + S_2 \lesssim |\log h| h^{2\alpha-d/2} = |\log h| h^{2\min\{(\alpha-d/4);1\}},$$

where we use the fact that $\alpha - d/4 < \alpha < 1$ to derive the last inequality.

Case $\alpha > 1$. We now assume that $\alpha > 1$. We define the function $\tilde{\gamma}(x) = \gamma(x)x^{\zeta/2}$. Note that γ is an α-amplitude spectral density and $\tilde{\gamma}$ decays as $|\tilde{\gamma}(x)| \lesssim |x|^{-(\alpha-\zeta/2)} = |x|^{-\eta/2}$. Hence, $\tilde{\gamma}$ is a $(\eta/2)$-amplitude spectral density. Now, by definition of $\tilde{\gamma}$,

$$\|\mathcal{Z} - \tilde{\mathcal{Z}}_h\|_{L^2(\Omega;L^2(\mathcal{M}))} = \|\tilde{\gamma}(\mathcal{L})\mathcal{L}^{-\zeta/2}\mathcal{W} - \tilde{\gamma}(\mathcal{L}_h)\mathcal{L}_h^{-\zeta/2}P_h\mathcal{W}\|_{L^2(\Omega;L^2(\mathcal{M}))},$$

so that the triangle inequality yields $\|\mathcal{Z} - \tilde{\mathcal{Z}}_h\|_{L^2(\Omega;L^2(\mathcal{M}))} \le E_1 + E_2$ where we take

$$E_1 = \|\big(\tilde{\gamma}(\mathcal{L}) - \tilde{\gamma}(\mathcal{L}_h)P_h\big)\mathcal{L}^{-\zeta/2}\mathcal{W}\|_{L^2(\Omega;L^2(\mathcal{M}))},$$
$$E_2 = \|\tilde{\gamma}(\mathcal{L}_h)\big(P_h\mathcal{L}^{-\zeta/2}\mathcal{W} - \mathcal{L}_h^{-\zeta/2}P_h\mathcal{W}\big)\|_{L^2(\Omega;L^2(\mathcal{M}))}.$$

For the term E_1, we first note that $\zeta = d/2 + 2(\alpha - d/2 - \varepsilon_h) > d/2$ since $\varepsilon_h < \alpha - d/2$. Therefore, we can use [6, Lemma 4.2] and the fact that h (and therefore ε_h) is upper-bounded to deduce that

$$\|\mathcal{L}^{-\zeta/2}\mathcal{W}\|^2_{L^2(\Omega;L^2(\mathcal{M}))} \le \frac{\zeta}{\zeta - d/2} \lesssim 1.$$

In particular, $\mathcal{L}^{-\zeta/2}\mathcal{W}$ is in $L^2(\Omega; L^2(\mathcal{M}))$. Let then $p = \min\{\alpha - d/2; 1\} - 2\varepsilon_h$. In particular, we have $p \in (0,1)$ since $p \le 1 - 2\varepsilon_h < 1$ and $p > \min\{\alpha - d/2; 1\} - \min\{\alpha - d/2; d/2\} \ge 0$. Moreover, we have $\zeta/2 - p = d/4 + (\alpha - d/2) - \min\{\alpha - d/2; 1\} + \varepsilon_h \ge d/4 + \varepsilon_h > d/4$. Hence, using once again [6, Lemma 4.2], we can conclude that

$$\begin{aligned}\|\mathcal{L}^p\mathcal{L}^{-\zeta/2}\mathcal{W}\|^2_{L^2(\Omega;L^2(\mathcal{M}))} &= \|\mathcal{L}^{-(\zeta/2-p)}\mathcal{W}\|^2_{L^2(\Omega;L^2(\mathcal{M}))}\\ &\lesssim \frac{\zeta/2-p}{\zeta/2-p-d/4} \lesssim \begin{cases} |\log h|, & \text{if } \alpha-d/2\le 1\\ 1, & \text{if } \alpha - d/2 > 1\end{cases}\end{aligned}$$

where we used the definition of ε_h to derive the last inequality. Hence, we can apply Proposition 2.4 to get

$$\begin{aligned}E_1^2 &= \mathbb{E}[\|\big(\tilde\gamma(\mathcal{L})-\tilde\gamma(\mathcal{L}_h)P_h\big)\mathcal{L}^{-\zeta/2}\mathcal{W}\|^2_{L^2(\mathcal{M})}]\\ &\lesssim C_{\eta/2+p}(h)^2h^{4\min\{\eta/2+p;1\}}\,\mathbb{E}[\|\mathcal{L}^p\mathcal{L}^{-\zeta/2}\mathcal{W}\|^2_{L^2(\mathcal{M})}]\\ &\lesssim C_{\eta/2+p}(h)^2h^{4\min\{\eta/2+p;1\}}K'_\alpha(h)^2,\end{aligned}$$

where we take on the one hand, $C_{\eta/2+p}(h) = |\log h|$ if $\eta/2+p\le 1$, and $C_{\eta/2+p}(h)=1$ otherwise, and on the other hand $K'_\alpha(h) = |\log h|^{1/2}$ if $\alpha \le 1+d/2$ and $K'_\alpha(h)=1$ otherwise.

Finally, note that according to Lemma A.1, $\eta/2+p = \min\{\alpha-d/2;1\}+d/4-\varepsilon_h \ge \min\{\alpha-d/4;1\}-\varepsilon_h$ which implies that $h^{4\min\{\eta/2+p;1\}} \le h^{4(\min\{\alpha-d/4;1\}-\varepsilon_h)} = h^{4\min\{\alpha-d/4;1\}}\exp(2\min\{d/2;\alpha-d/2\})$. Besides, we note that if $\alpha > 1+d/2$, then $\eta/2+p = 1+d/4-\varepsilon_h > 1$ (since by construction $\varepsilon_h < d/4$), and therefore $C_{\eta/2+p}(h)=1$. We can therefore conclude that

$$E_1 \lesssim K_1(h)h^{2\min\{\alpha-d/4;1\}}$$

where $K_1(h) = |\log h|^{3/2}$ if $1<\alpha\le 1+d/4+\varepsilon_h$, and $K_1(h)=|\log h|^{1/2}$ if $1+d/4+\varepsilon_h < \alpha \le 1+d/2$ and $K_1(h)=1$ if $\alpha > 1+d/2$.

For E_2, we first use the definition of the projection operator P_h and the self-adjointness of $\mathcal{L}$ and $\mathcal{L}_h$ (which is a consequence of the definition of the associated bilinear form $\mathsf{A}_{\mathcal{M}}$)

$$\begin{aligned}E_2^2 &= \left\|\sum_{i=1}^{N_h}\tilde\gamma(\lambda_i^h)\Big(\big(P_h\mathcal{L}^{-\zeta/2}\mathcal{W},e_i^h\big)_{L^2(\mathcal{M})} - \big(\mathcal{L}_h^{-\zeta/2}P_h\mathcal{W},e_i^h\big)_{L^2(\mathcal{M})}\Big)e_i^h\right\|^2_{L^2(\Omega;L^2(\mathcal{M}))}\\ &= \left\|\sum_{i=1}^{N_h}\tilde\gamma(\lambda_i^h)\Big(\big(\mathcal{L}^{-\zeta/2}\mathcal{W},e_i^h\big)_{L^2(\mathcal{M})} - \big(P_h\mathcal{W},\mathcal{L}_h^{-\zeta/2}e_i^h\big)_{L^2(\mathcal{M})}\Big)e_i^h\right\|^2_{L^2(\Omega;L^2(\mathcal{M}))}\\ &= \left\|\sum_{i=1}^{N_h}\tilde\gamma(\lambda_i^h)\Big(\big(\mathcal{W},\mathcal{L}^{-\zeta/2}e_i^h\big)_{L^2(\mathcal{M})} - \big(\mathcal{W},\mathcal{L}_h^{-\zeta/2}e_i^h\big)_{L^2(\mathcal{M})}\Big)e_i^h\right\|^2_{L^2(\Omega;L^2(\mathcal{M}))}\\ &= \left\|\sum_{i=1}^{N_h}\tilde\gamma(\lambda_i^h)\big(\mathcal{W},\mathcal{L}^{-\zeta/2}e_i^h - \mathcal{L}_h^{-\zeta/2}e_i^h\big)_{L^2(\mathcal{M})}e_i^h\right\|^2_{L^2(\Omega;L^2(\mathcal{M}))}.\end{aligned}$$

Using the orthonormality of the basis $\{e_i^h\}_{1\le i\le N_h}$ and the definition of the white noise $\mathcal{W}$, we conclude that

$$\begin{aligned}E_2^2 &= \mathbb{E}\left[\sum_{i=1}^{N_h}\tilde\gamma(\lambda_i^h)^2|\big(\mathcal{W},\mathcal{L}^{-\zeta/2}e_i^h-\mathcal{L}_h^{-\zeta/2}e_i^h\big)_{L^2(\mathcal{M})}|^2\right]\\ &= \sum_{i=1}^{N_h}\tilde\gamma(\lambda_i^h)^2\|\mathcal{L}^{-\zeta/2}e_i^h-\mathcal{L}_h^{-\zeta/2}e_i^h\|^2_{L^2(\mathcal{M})}.\end{aligned}$$

On the one hand, using Proposition 2.4 with $f = e_i^h \in S_h^\ell$ $(1 \leq i \leq N_h)$, which has norm $\|e_i^h\|_{L^2(\mathcal{M})} = 1$, we have

$$\|\mathcal{L}^{-\zeta/2} e_i^h - \mathcal{L}_h^{-\zeta/2} e_i^h\|_{L^2(\mathcal{M})} \lesssim C_\zeta(h) h^{2\min\{\zeta/2;1\}}, \quad \text{where} \quad C_\zeta(h) = \begin{cases} 1, & \zeta > 2, \\ |\log h|, & \zeta \leq 2. \end{cases}$$

Note that by definition of ζ and using Lemma A.1, we have $\min\{\zeta/2;1\} \geq \min\{\alpha - d/4;1\} - \varepsilon_h > 0$ (since $\varepsilon_h \in (0, \alpha - d/4)$), which in turn gives $h^{2\min\{\zeta/2;1\}} \leq h^{2(\min\{\alpha-d/4;1\}-\varepsilon_h)} \lesssim h^{2\min\{\alpha-d/4;1\}}$. Hence, we can conclude that

$$\|\mathcal{L}^{-\zeta/2} e_i^h - \mathcal{L}_h^{-\zeta/2} e_i^h\|_{L^2(\mathcal{M})} \lesssim C_\zeta(h) h^{2\min\{\alpha-d/4;1\}}$$

and therefore, $E_2^2 \lesssim C_\zeta(h)^2 h^{4\min\{\alpha-d/4;1\}} \sum_{i=1}^{N_h} \tilde\gamma(\lambda_i^h)^2$. On the other hand, the term $\sum_{i=1}^{N_h} \tilde\gamma(\lambda_i^h)^2$can be bounded using Equation (12) and Equation (6) which yield that

$$\sum_{i=1}^{N_h} \tilde\gamma(\lambda_i^h)^2 \lesssim \sum_{i=1}^{N_h} (\lambda_i^h)^{-\eta} \lesssim \sum_{i=1}^{N_h} i^{-2\eta/d} \lesssim \int_1^\infty t^{-2\eta/d} = \frac{1}{2\eta/d - 1} \lesssim |\log h|,$$

since $2\eta/d = 1 + 4\varepsilon_h/d > 1$ and by definition of ε_h. Hence, we can conclude that

$$E_2 \lesssim C_\zeta(h) |\log h|^{1/2} h^{2\min\{\alpha-d/4;1\}} \lesssim K_2(h) h^{2\min\{\alpha-d/4;1\}},$$

where $K_2(h) = C_\zeta(h)|\log h|^{1/2}$, i.e., $K_2(h) = |\log h|^{3/2}$ if $1 < \alpha \leq 1 + d/4 + \varepsilon_h$, and $K_2(h) = |\log h|^{1/2}$ if $\alpha > 1 + d/4 + \varepsilon_h$.

Hence, by putting together the estimates for E_1 and E_2, and noting that $K_1(h) \leq K_2(h)$, we can conclude that if $\alpha > 1$, then

$$\|\mathcal{Z} - \tilde{\mathcal{Z}}_h\|_{L^2(\Omega;L^2(\mathcal{M}))} \leq E_1 + E_2 \lesssim K_2(h) h^{2\min\{\alpha-d/4;1\}}.$$

In particular note that for any $\alpha > 1$, we can clearly upper-bound $K_2(h)$ by $K_2(h) \lesssim |\log h|^{3/2}$. But if moreover $\alpha \geq 1 + d/2$, we have y definition of ε_h, $1 + d/4 + \varepsilon_h < 1 + d/4 + \min\{d/2; \alpha - d/2\}/2 \leq 1 + d/2 \leq \alpha$, and therefore $K_2(h) = |\log h|^{1/2}$. Hence, we can conclude that if $\alpha > 1$, $\|\mathcal{Z} - \tilde{\mathcal{Z}}_h\|_{L^2(\Omega;L^2(\mathcal{M}))} \lesssim K_2'(h) h^{2\min\{\alpha-d/4;1\}}$ where $K_2'(h) = |\log h|^{3/2}$ if $1 < \alpha < 1 + d/2$ and $K_2'(h) = |\log h|^{1/2}$ if $\alpha \geq 1 + d/2$. And finally gathering the estimates obtained in the two cases $\alpha \leq 1$ and $\alpha > 1$, we obtain the result stated in the lemma. ☐

For the error between $\big(\tilde{\mathcal{Z}}_h\big)^{-\ell}$ and $\widehat{\mathsf{Z}}_h$ we get the following estimate, inspired by [6, Lemma 4.4].

Lemma 3.5. *It holds that*

$$\big\|\big(\tilde{\mathcal{Z}}_h\big)^{-\ell} - \widehat{\mathsf{Z}}_h\big\|_{L^2(\Omega;L^2(\mathcal{M}_h))} \lesssim |\log(h)|^{(d-1)/2}\ h^2.$$

Proof. Let

$$\mathcal{E}_h = \big\| \left(\tilde{\mathcal{Z}}_h\right)^{-\ell} - \widehat{\mathsf{Z}}_h\big\|_{L^2(\Omega;L^2(\mathcal{M}_h))} = \big\| \left(\gamma(\mathcal{L}_h)\widetilde{\mathcal{W}}\right)^{-\ell} - \gamma(\mathsf{L}_h)\mathsf{P}_h(\sigma\widetilde{\mathcal{W}}^{-\ell})\big\|_{L^2(\Omega;L^2(\mathcal{M}_h))}.$$

We note that $\widetilde{\mathcal{W}}$ is an S_h^ℓ-valued random variable. Therefore, we can apply Proposition 2.6 to realizations of $\widetilde{\mathcal{W}}$, and take the expectation on both sides to get

$$\mathcal{E}_h \lesssim h^2 \|\mathcal{L}_h^{-\min\{\alpha+d/4;1\}/2}\widetilde{\mathcal{W}}\|_{L^2(\Omega;L^2(\mathcal{M}))}.$$

We then distinguish two cases. First, if $d = 1$, then we note that $\min\{\alpha + d/4; 1\} \geq \alpha + d/4 > d/2$ since $\alpha > d/4$. Besides, [6, Lemma 4.2] yields that for all $r \in (d/2, 2)$, there is a constant $C > 0$ such that

$$\|\mathcal{L}_h^{-r/2}\widetilde{\mathcal{W}}\|^2_{L^2(\Omega;L^2(\mathcal{M}))} \leq C\frac{r}{r - d/2}. \tag{40}$$

Hence, using estimate (40), we conclude that

$$\mathcal{E}_h \lesssim h^2 \frac{\min\{\alpha + d/4; 1\}}{\min\{\alpha + d/4; 1\} - d/2} \lesssim h^2.$$

If now $d = 2$, since $\alpha + d/4 > d/2 = 1$, we have $\min\{\alpha + d/4; 1\} = 1$. We then note that by the proof of [6, Lemma 4.2], for any $\varepsilon > 0$

$$\|\mathcal{L}_h^{-1/2}\widetilde{\mathcal{W}}\|^2_{L^2(\Omega;L^2(\mathcal{M}))} \lesssim \sum_{j=1}^{N_h} \lambda_j^{-1} \lesssim \lambda_{N_h}^{\varepsilon} \sum_{j=1}^{N_h} \lambda_j^{-1-\varepsilon}.$$

Apply now Equation (6) to see that

$$\|\mathcal{L}_h^{-1/2}\widetilde{\mathcal{W}}\|^2_{L^2(\Omega;L^2(\mathcal{M}))} \lesssim N_h^{\varepsilon} \sum_{j=1}^{N_h} j^{-(1+\varepsilon)} \lesssim N_h^{\varepsilon}\frac{1+\varepsilon}{\varepsilon} \lesssim h^{-2\varepsilon}\frac{1+\varepsilon}{\varepsilon},$$

meaning that for any $\varepsilon > 0$,

$$\mathcal{E}_h \lesssim \left(\frac{1+\varepsilon}{\varepsilon}\right)^{1/2} h^{2-\varepsilon}.$$

In particular, taking $\varepsilon = |\log(h)|^{-1}$ yields

$$\mathcal{E}_h \lesssim |\log(h)|^{1/2} h^2,$$

which concludes the proof. □

Finally, for the error between $\widehat{\mathsf{Z}}_h$ and Z_h we get the following estimate.

Lemma 3.6. *Let $h \in (0, h_0)$. It holds that there is a constant such that*

$$\|\mathsf{Z}_h - \widehat{\mathsf{Z}}_h\|_{L^2(\Omega;L^2(\mathcal{M}_h))} \lesssim h^2.$$

Proof. To prove this statement, we note that, using the same notations as the ones in the proof of Lemma 3.3,

$$\begin{aligned}
\|\mathsf{Z}_h - \widehat{\mathsf{Z}}_h\|^2_{L^2(\Omega;L^2(\mathcal{M}_h))} &= \left\|\gamma(\mathsf{L}_h)\left(\widehat{\mathsf{W}} - \mathsf{W}\right)\right\|^2_{L^2(\Omega;L^2(\mathcal{M}_h))} \\
&= \mathbb{E}\left[\left\|\sum_{j=1}^{N_h} \gamma(\Lambda_j^h)(\alpha_j - \beta_j)E_j^h\right\|^2_{L^2(\mathcal{M}_h)}\right] = \sum_{j=1}^{N_h} \gamma(\Lambda_j^h)^2\, \mathbb{E}[|\alpha_j - \beta_j|^2],
\end{aligned}$$

where we applied the orthogonality of the eigenfunctions in the last step. Now, following the definition of α_j and β_j given in the proof of Lemma 3.3,

$$\begin{aligned}
\mathbb{E}[|\alpha_j - \beta_j|^2] &= \mathbb{E}\left[\left|\left(\mathcal{W}, ((1-\sigma^{-1/2})E_j^h)^{\ell}\right)_{L^2(\mathcal{M})}\right|^2\right] = \left\|\left((1-\sigma^{-1/2})E_j^h\right)^{\ell}\right\|^2_{L^2(\mathcal{M})} \\
&= \left\|(\sigma^{1/2} - 1)E_j^h\right\|^2_{L^2(\mathcal{M}_h)} \leq \left\|(\sigma^{1/2} - 1)\right\|^2_{L^\infty(\mathcal{M}_h)}.
\end{aligned}$$

Besides, for all $x \in \mathcal{M}_h$, it holds that

$$\sigma^{1/2}(x) - 1 = \sqrt{1 + \sigma(x) - 1} - 1 \leq \sqrt{1 + |\sigma(x) - 1|} - 1 \leq 1 + \frac{1}{2}|\sigma(x) - 1| - 1 = \frac{1}{2}|\sigma(x) - 1|.$$

Therefore,

$$\left\| (\sigma^{1/2} - 1) \right\|^2_{L^\infty(\mathcal{M}_h)} \leq \|(\sigma - 1)\|^2_{L^\infty(\mathcal{M}_h)} \leq Ch^2,$$

where [11, Lemma 4.1] was applied in the final step. We conclude that

$$\|\mathsf{Z}_h - \widehat{\mathsf{Z}}_h\|^2_{L^2(\Omega;L^2(\mathcal{M}_h))} \leq Ch^4 \sum_{j=1}^{N_h} \gamma(\Lambda_j^h)^2,$$

and it remains to show that $\sum_{j=1}^{N_h} \gamma(\Lambda_j^h)^2$ is bounded by a constant. To this end, we use Lemma 2.3, which implies that there exists some constant $C_\lambda > 0$ such that

$$|\Lambda_j^h / \lambda_j^h - 1| \leq C_\lambda h^2.$$

Recall then that the mesh size h satisfies $h < h_0$, where $h_0 \in (0, 1)$. Without loss of generality, let us further assume that $C_\lambda h_0 < 1$. Now, by the growth assumption on γ,

$$\begin{aligned}\sum_{j=1}^{N_h} \gamma(\Lambda_j^h)^2 &\lesssim \sum_{j=1}^{N_h} |\Lambda_j^h|^{-2\alpha} = \sum_{j=1}^{N_h} |\lambda_j^h|^{-2\alpha} \left| 1 + \left(\frac{\Lambda_j^h}{\lambda_j^h} - 1 \right) \right|^{-2\alpha} \leq (1 - Ch_0^2)^{-2\alpha} \sum_{j=1}^{N_h} |\lambda_j^h|^{-2\alpha} \\ &\lesssim \sum_{j=1}^{N_h} |\lambda_j^h|^{-2\alpha}.\end{aligned}$$

Using Equation (12) and Equation (6), we bound

$$\sum_{j=1}^{N_h} |\lambda_j^h|^{-2\alpha} \leq \sum_{j=1}^{N_h} |\lambda_j|^{-2\alpha} \lesssim \sum_{j=1}^{\infty} j^{-4\alpha/d} = \zeta(4\alpha/d),$$

where ζ denotes the Riemann zeta function. Thus, $\sum_{j=1}^{N_h} \gamma(\Lambda_j^h)^2$ is bounded by a constant and

$$\|\mathsf{Z}_h - \widehat{\mathsf{Z}}_h\|^2_{L^2(\Omega;L^2(\mathcal{M}_h))} \leq Ch^4,$$

which proves the lemma. □

Equipped with the estimates derived in Lemmas 3.4 to 3.6 we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. By summing the three estimates derived in Lemmas 3.4 to 3.6, we get

$$\|\mathcal{Z} - \mathsf{Z}_h^\ell\|_{L^2(\Omega;L^2(\mathcal{M}))} \lesssim C_\alpha(h) h^{2\min\{\alpha - d/4; 1\}} + |\log(h)|^{(d-1)/2}\, h^2 + h^2.$$

Noe then that $2\min\{\alpha - d/4; 1\} \leq 2$ and that $|\log(h)|^{(d-1)/2} \lesssim C_\alpha(h)$. Hence, we have the bound $|\log(h)|^{(d-1)/2}\, h^2 \lesssim C_\alpha(h) h^{2\min\{\alpha - d/4;1\}}$ which allows us to conclude that

$$\|\mathcal{Z} - \mathsf{Z}_h^\ell\|_{L^2(\Omega;L^2(\mathcal{M}))} \lesssim C_\alpha(h) h^{2\min\{\alpha - d/4;1\}},$$

and the proof is complete. □

Having bounded the SFEM error, we are now ready to derive the additional error associated with the Chebyshev approximation which we use to compute SFEM–Galerkin approximations in practice.

3.4. **Error analysis of the Galerkin–Chebyshev approximation.** The error between the field $\mathcal{Z}$ and its Galerkin–Chebyshev approximation $\mathsf{Z}_{h,M}$ described in Section 3.2 is derived by combining Theorem 3.1 with an error bound between the SFEM–Galerkin approximation Z_h and the Galerkin–Chebyshev approximation $\mathsf{Z}_{h,M}$ obtained using [19, Theorem 5.8]. The latter bound is expressed in the next result.

Lemma 3.7. *Let $[\lambda_{\min}, \lambda_{\max}] \subset (0, \infty)$ be the interval on which the Chebyshev polynomial approximation $P_{\gamma,M}$ is computed, and let $\xi = \lambda_{\min}/(\lambda_{\max} - \lambda_{\min})$. Then, there exists a constant $C > 0$ such that the error between the discretized field Z_h and its approximation $\mathsf{Z}_{h,M}$ is upper-bounded by*

$$\|\mathsf{Z}_{h,M} - \mathsf{Z}_h\|_{L^2(\Omega;L^2(\mathcal{M}_h))} \leq Ch^{-d/2}\epsilon_\xi^{-1}(1+\epsilon_\xi)^{-M} \tag{41}$$

with $\epsilon_\xi = \xi + \sqrt{\xi(2+\xi)}$. In particular, setting $\lambda_{\min} = V_-$ and $\lambda_{\max} = \Lambda^h_{N_h}$, the error is bounded by

$$\|\mathsf{Z}_{h,M} - \mathsf{Z}_h\|_{L^2(\Omega;L^2(\mathcal{M}_h))} \leq C_V^{-1} h^{-(d/2+1)} \exp(-C_V hM) \tag{42}$$

for some constant $C_V > 0$ proportional to $\sqrt{V_-}$.

Proof. Let $E_\xi \subset \mathbb{C}$ be the ellipse centered at $z = (\lambda_{\min} + \lambda_{\max})/2$, with foci $z_1 = \lambda_{\min}$ and $z_2 = \lambda_{\max}$, and semi-major axis $a_\xi = \lambda_{\max}/2$. In particular, note that $E_\xi \subset H_{\pi/2}$ and for any $z \in E_\xi$, $\mathrm{Re}(z) \geq \lambda_{\min}/2 > 0$. Hence, since γ is an amplitude spectral density, by definition γ is holomorphic and bounded inside E_ξ. We can then adapt the same proof as in [19, Theorem 5.8] to obtain the stated proposition. □

Hence, for a fixed mesh size h, the approximation error $\|\mathsf{Z}_{h,M} - \mathsf{Z}_h\|_{L^2(\Omega;L^2(\mathcal{M}_h))}$ converges to 0 as the order of the polynomial approximation M goes to infinity. Choosing M as a function of h that grows fast enough then allows us to ensure the convergence of the approximation error as n goes to infinity [19, Section 5.2]. In particular, by taking $M = M_h = (d/2 + 3)C_V^{-1}h^{-1}|\log h| = (d/2+3)C_V^{-1}h^{-1}\log(h^{-1})$ the bound (42) becomes

$$\begin{aligned} \|\mathsf{Z}_{h,M} - \mathsf{Z}_h\|_{L^2(\Omega;L^2(\mathcal{M}_h))} &\leq C_V^{-1} \exp((d/2+1)\log(h^{-1}) - (d/2+3)\log(h^{-1})) \\ &= C_V^{-1} h^2. \end{aligned} \tag{43}$$

Putting together Theorem 3.1 and Equation (43), we can now state the following result which provides a bound between the field $\mathcal{Z}$ and its Galerkin–Chebyshev approximation Z_{h,M_h}.

Theorem 3.8. *Let γ be an α-amplitude spectral density with $\alpha > d/4$, let $h \in (0, h_0)$ and take $M_h = (d/2+3)C_V^{-1}h^{-1}|\log h|$. Then, The strong error between the random field $\mathcal{Z}$ by its Galerkin–Chebyshev approximation Z^ℓ_{h,M_h} satisfies the bound*

$$\|\mathcal{Z} - \mathsf{Z}^\ell_{h,M_h}\|_{L^2(\Omega;L^2(\mathcal{M}))} \lesssim C_\alpha(h) h^{2\min\{\alpha - d/4;1\}}, \tag{44}$$

where $C_\alpha(h) = |\log h|$ if $d/4 < \alpha \leq 1$, $C_\alpha(h) = |\log h|^{3/2}$ if $1 < \alpha < 1 + d/2$, and $C_\alpha(h) = |\log h|^{1/2}$ if $\alpha \geq 1 + d/2$.

Remark 3.9. In practice, it is common to "chop" a Chebyshev polynomial approximation, i.e. to truncate the polynomial approximation at an early order. Indeed, let us assume that we have computed the coefficients $\{c_k\}_{0\le k\le M_h}$ of the Chebyshev series up to the order $M = M_h$ and let $c_{\max} = \max_{0\le k\le M_h} |c_k|$. Let $\varepsilon > 0$ be fixed but arbitrary and assume that there exists $m_\varepsilon \in \{0, \dots, M_h - 1\}$ such that for any $k > m_\varepsilon$, $|c_k|/c_{\max} < \varepsilon$. The error between the field Z_{h,M_h} and its "chopped" counterpart $\mathsf{Z}_{h,m_\varepsilon}$ is given by

$$\begin{aligned}\|\mathsf{Z}_{h,M_h} - \mathsf{Z}_{h,m_\varepsilon}\|^2_{L^2(\Omega;L^2(\mathcal{M}_h))} &= \|\sum_{i=1}^{N_h}(P_{\gamma,M_h}(\Lambda_i^h) - P_{\gamma,m_\varepsilon}(\Lambda_i^h))W_i^h E_i^h\|^2_{L^2(\Omega;L^2(\mathcal{M}_h))}\\ &= \sum_{i=1}^{N_h}(P_{\gamma,M_h}(\Lambda_i^h) - P_{\gamma,m_\varepsilon}(\Lambda_i^h))^2\end{aligned}$$

where $P_{\gamma,M_h}(\lambda) = \sum_{k=0}^{M_h} c_k T_k(\lambda)$ is the M_h-th order Chebyshev approximation of γ and T_k denotes the k-th (shifted) Chebyshev polynomial. In particular, we have $|P_{\gamma,M_h}(\Lambda_i^h) - P_{\gamma,m_\varepsilon}(\Lambda_i^h)| = |\sum_{k=m_\varepsilon+1}^{M_h} c_k T_k(\lambda)| \le \sum_{k=m_\varepsilon+1}^{M_h} |c_k|\,|T_k(\lambda)| \le \sum_{k=m_\varepsilon+1}^{M_h} |c_k| \le (M_h - m_\varepsilon)c_{\max}\varepsilon \le M_h c_{\max}\varepsilon$. Hence, the error between Z_{h,M_h} and its "chopped" counterpart $\mathsf{Z}_{h,m_\varepsilon}$ satisfies

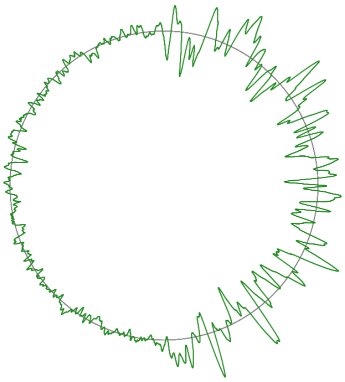
$$\|\mathsf{Z}_{h,M_h} - \mathsf{Z}_{h,m_\varepsilon}\|_{L^2(\Omega;L^2(\mathcal{M}_h))} \le N_h^{1/2} M_h c_{\max}\varepsilon \lesssim h^{-(1+d/2)}|\log h|\ \varepsilon$$

Hence, if we have $\varepsilon \lesssim |\log h|^{-1}h^{(3+d/2)}$ then we can conclude that $\|\mathsf{Z}_{h,M_h} - \mathsf{Z}_{h,m_\varepsilon}\|_{L^2(\Omega;L^2(\mathcal{M}_h))} \lesssim h^2$, thus implying that once again we have $\|\mathcal{Z} - \mathsf{Z}^\ell_{h,m_\varepsilon}\|_{L^2(\Omega;L^2(\mathcal{M}))} \lesssim C_\alpha(h)h^{2\min\{\alpha-d/4;1\}}$. This means that in practice, if we notice that the coefficients of the Chebyshev polynomial approximation $\{c_k\}_{0\le k\le M_h}$ decay fast enough that we can find some $m_\varepsilon \in \{0, \dots, M_h - 1\}$ such that for any $k > m_\varepsilon$, $|c_k|/c_{\max} < \varepsilon \lesssim |\log h|^{-1}h^{(3+d/2)}$, we can replace the field Z_{h,M_h} by its chopped counterpart while still maintaining the same strong error bound with respect to the field $\mathcal{Z}$.

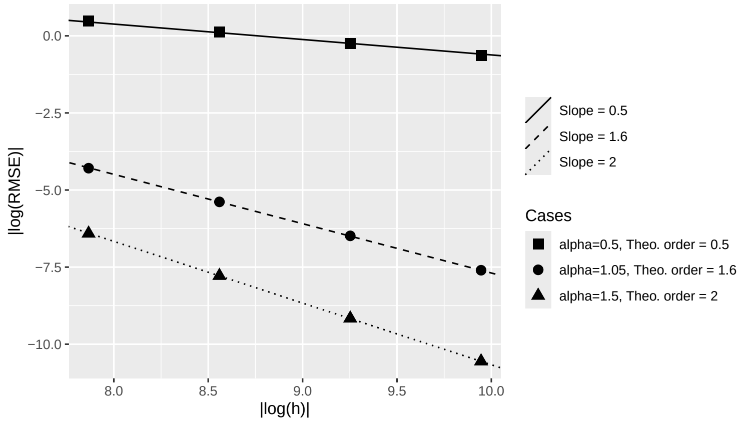
4. Numerical experiments

In this section, we present numerical experiments confirming the strong error bound in Equation (44). We consider two cases: when the manifold $\mathcal{M}$ is a circle (hence $d = 1$), and when the manifold $\mathcal{M}$ is a sphere (hence $d = 2$). Indeed, in both cases we are able to easily define nested meshes of various sizes h, and define appropriate noise projections at the different levels.

In each case, the field $\mathcal{Z}$ is approximated by a Galerkin–Chebyshev field $\mathsf{Z}_{h_{\text{fine}},M_{h_{\text{fine}}}}$ computed at a very fine mesh size h_{fine}. More precisely, 25 samples of $\{\mathsf{Z}^{(i)}_{h_{\text{fine}},M_{h_{\text{fine}}}}\}_{1\le i\le 25}$ of the field $\mathsf{Z}_{h_{\text{fine}},M_{h_{\text{fine}}}}$ are computed. In particular, let us denote by $\mathsf{W}^{(i)}_{h_{\text{fine}}}$ the projected white noise used to compute the sample $\mathcal{Z}^{(i)} = P_{\gamma,M_{h_{\text{fine}}}}(\mathsf{L}_{h_{\text{fine}}})\mathsf{W}^{(i)}_{h_{\text{fine}}}$. For a coarse mesh size $h > h_{\text{fine}}$, each sample $\mathcal{Z}^{(i)}$ is compared to the sample of $\mathsf{Z}^{(i)}_{h,M_h}$ computed from the white noise $\mathsf{W}^{(i)}_h$ obtained by projecting the noise $\mathsf{W}^{(i)}_{h_{\text{fine}}}$ (defined in the fine mesh) down to the coarse mesh. Note that, following Remark 3.9, the Galerkin–Chebyshev fields are systematically computed by chopping the Chebyshev polynomial approximations at a small level $\epsilon = 10^{-12}$. Besides, when computing the representation of a Galerkin–Chebyshev field in the nodal basis through (34), the matrix $\sqrt{\boldsymbol{C}}$ is computed as a Cholesky factorization of the mass matrix $\boldsymbol{C}$ in the first experiment, and is approximate using a (diagonal) mass lumping approach in the second



(a) Sample (in green) of the random field with $\alpha = 1.75$ on the fine circle mesh (in black). The distance to the circle indicates the value of taken by the field.



(b) Strong error as a function the mesh size.

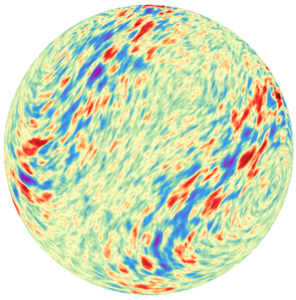
FIGURE 5. First numerical experiment (on the circle).

experiment. Finally, the strong error at a mesh size h is then estimated from the samples using

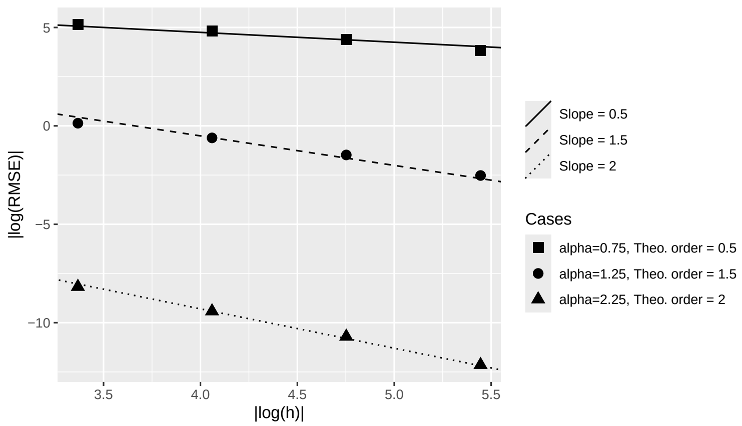
$$\|\mathcal{Z} - \mathsf{Z}^{\ell}_{h,M_h}\|_{L^2(\Omega;L^2(\mathcal{M}))} \approx \left(\frac{1}{25}\sum_{i=1}^{25}\|\mathsf{Z}^{(i)}_{h_{\text{fine}},M_{h_{\text{fine}}}} - \mathsf{Z}^{(i)}_{h,M_h}\|^2_{L^2(\mathcal{M}_h)}\right)^{1/2}. \tag{45}$$

In the first experiment, we consider a random field $\mathcal{Z}$ defined on the unit circle $\mathcal{M}$ with amplitude spectral density given by the formula $\gamma(\lambda) = (V_0)^{\alpha}(\lambda + \cos(0.9\pi)\sqrt{\lambda})^{-\alpha}$ where $V_0 = 10^4$ and for $\alpha \in \{0.5, 1.05, 1.5\}$. The bilinear form $\mathcal{M}$ is taken as follows. The diffusion matrix $\mathcal{D}$ is defined by $\mathcal{D}(x) = I - \nu(x)\nu(x)^T$, $x \in \mathcal{M}$, where $\nu(x) = x/\|x\|$ is the normal vector to a point $x \in \mathcal{M}$ of the unit circle. The function V is defined as $V(x) = 3V_0$, $x \in \mathcal{M}$, if $\theta(x) \in]\pi/2, 3\pi/2[$ and $V(x) = V_0$ otherwise, and $\theta(x) \in [0, 2\pi[$ denotes the circular coordinate of the point $x \in \mathcal{M}$. The fine mesh size is taken to be $h_{\text{fine}} = 2^{-19}\pi$ and the coarse mesh sizes are $h = \{2^{-13}\pi, 2^{-14}\pi, 2^{-15}\pi, 2^{-16}\pi\}$. An example of sample (on the fine mesh) obtained with this choice of parameters is displayed in Figure 5(a). The results of this first experiment are presented in Figure 5(b), where the log of the strong error (45) is plotted against the log of the mesh size h, for the three amplitude spectral density obtained by taking $\alpha \in \{0.5, 1.05, 1.5\}$. As shown in the figure, we retrieve the rates (respectively 0.5,1.6 and 2) predicted by Theorem 3.8.

In the second experiment, we consider a random field $\mathcal{Z}$ defined on the unit sphere $\mathcal{M}$ with amplitude spectral density given by the formula $\gamma(\lambda) = C_0\lambda^{-\alpha}$ where $C_0 = 500$ and for $\alpha \in \{0.75, 1.25, 2.25\}$. The bilinear form $\mathcal{M}$ is taken as follows. The diffusion matrix $\mathcal{D}$ is defined $\mathcal{D}(x) = \nabla_{\mathcal{M}} f(x)(\nabla_{\mathcal{M}} f(x))^T + \rho(x)\nabla^{\perp}_{\mathcal{M}} f(x)(\nabla^{\perp}_{\mathcal{M}} f(x))^T$, where $f(x) = 2\cos(\theta(x))\cos(\phi(x))\sin^2(\theta(x))$ and $\rho(x) = 0.1 + 0.6/(1 + e^{-4\cos(\theta(x))})$. Here $x \in \mathcal{M}$ and $(\theta(x), \phi(x))$ denotes spherical coordinates. The function V is defined as $V(x) = 500(1 + 5\cos^2(\pi\theta(x)))$. The fine mesh size is taken to be $h_{\text{fine}} = 2.16 \times 10^{-3}$ and the coarse mesh



(a) Sample of the random field with $\alpha = 1.75$ on the fine mesh. The colors indicate the value of taken by the field.



(b) Strong error as a function the mesh size.

FIGURE 6. Second numerical experiment (on the sphere).

sizes are $h = \{3.45 \times 10^{-2}, 1.73 \times 10^{-2}, 8.63 \times 10^{-3}, 4.32 \times 10^{-3}\}$. An example of sample (on the fine mesh) obtained with this choice of parameters is displayed in Figure 6(a). The results of this second experiment are presented in Figure 6(b), where once again the log of the strong error (45) is plotted against the log of the mesh size h, for the three amplitude spectral density obtained by taking $\alpha \in \{0.75, 1.25, 2.25\}$. As seen in Figure 6(b), we retrieve the rates (respectively 0.5,1.5 and 2) predicted by Theorem 3.8.

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Appendix A. Deterministic proofs

A.1. **Proof of Proposition 2.1.**

Proof. A standard result in the spectral theory of elliptic operators on compact Riemannian manifolds (see e.g. [23, Section 8]) ensures that there exists a set of eigenpairs $\{(\lambda_i, f_i)\}_{i\in\mathbb{N}}$ of $\mathcal{L}$ such that $0 \le \lambda_1 \le \lambda_2 \le \cdots$ with $\lambda_i \to +\infty$ as $i \to +\infty$, and $\{f_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathcal{M})$ composed of possibly complex-valued functions.

Hence, let us prove that $\lambda_1 \ge \delta$ and that we can build an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of $L^2(\mathcal{M})$ such that each e_i is real-valued and an eigenfunction of $\mathcal{L}$ with eigenvalue λ_i. On the one hand, by definition of $\mathcal{L}$, λ_1 and f_1, we obtain

$$\lambda_1 = \lambda_1 (f_1, f_1)_{L^2(\mathcal{M})} = \mathsf{A}_{\mathcal{M}}(f_1, f_1) \ge \delta \|f_1\|^2_{H^1(\mathcal{M})} \ge \delta \|f_1\|^2_{L^2(\mathcal{M})} = \delta,$$

where for the last two inequalities we used the coercivity of $\mathsf{A}_{\mathcal{M}}$ and the definition of the H^1-norm.

On the other hand, let $\lambda > 0$ be one of the eigenvalues of $\mathcal{L}$, and $E_\lambda \subset L^2(\mathcal{M})$ the associated eigenspace. Following again the results from [23, Section 8], we get that $E_\lambda \subset C^\infty(\mathcal{M})$, $\dim E_\lambda < \infty$ and that if $\lambda' \ne \lambda$ is another eigenvalue of $\mathcal{L}$, then E_λ and $E_{\lambda'}$ are orthogonal. Besides, E_λ is in fact generated by the set $\{f_j\}_{j\in J_\lambda}$, where $J_\lambda = \{i \in \mathbb{N} : \lambda_i = \lambda\}$ is finite (since $\dim E_\lambda < \infty$).

Take then $u \in E_\lambda$. Hence, for any $v \in H^1(\mathcal{M})$, $\mathsf{A}_{\mathcal{M}}(u, v) = \lambda (u, v)_{L^2(\mathcal{M})}$. But also, by definition of $\mathsf{A}_{\mathcal{M}}$,

$$\mathsf{A}_{\mathcal{M}}(\overline{u}, v) = \mathsf{A}_{\mathcal{M}}(\overline{v}, u) = \overline{\mathsf{A}_{\mathcal{M}}(u, \overline{v})} = \overline{\lambda (u, \overline{v})_{L^2(\mathcal{M})}} = \lambda (\overline{u}, v)_{L^2(\mathcal{M})},$$

where used the fact that $\mathcal{D}$ is a real symmetric matrix and V is real-valued for the first two equalities. Consequently, we also have $\overline{u} \in E_\lambda$, and so, the real-valued functions $\mathrm{Re}(u) = (u + \overline{u}))/2$ and $\mathrm{Im}(u) = (u - \overline{u}))/2i$ (corresponding to real and imaginary parts of u) are also in E_λ.

Circling back to the orthonormal basis $\{f_j\}_{j\in J_\lambda}$ of E_λ, we consider the set of real-valued functions $F_\lambda = \{\mathrm{Re}(f_j)\}_{j\in J_\lambda} \cup \{\mathrm{Im}(f_j)\}_{j\in J_\lambda} \subset E_\lambda$, and the subspace $V_\lambda \subset E_\lambda$ generated by F_λ. In particular $\dim V_\lambda \le \dim E_\lambda$. By applying the Gram–Schmidt orthogonalization process to F_λ, we get an orthonormal basis $\{e_k\}_{1\le k\le \dim V_\lambda}$ of V_λ which by construction is composed of real-valued functions (since F_λ is composed of real-valued functions). Let us show that $\{e_k\}_{1\le k\le \dim V_\lambda}$ is in fact a basis of E_λ, or equivalently that $\dim V_\lambda = \dim E_\lambda$.

We proceed by contradiction. Assume that $\dim V_\lambda < \dim E_\lambda$. This means in particular that the orthogonal complement of V_λ in E_λ, denoted by $V_\lambda^\perp$, is not reduced to 0. Let then $0 \ne w \in V_\lambda^\perp$. By linearity, we have, for any $j \in J_\lambda$, $(f_j, w)_{L^2(\mathcal{M})} = (\mathrm{Re}(f_j), w)_{L^2(\mathcal{M})} + i(\mathrm{Im}(f_j), w)_{L^2(\mathcal{M})} = 0$, since $\mathrm{Re}(f_j), \mathrm{Im}(f_j) \in F_\lambda \subset V_\lambda$. Hence, since $w \in E_\lambda$ and $\{f_j\}_{j\in J_\lambda}$ is an orthonormal basis of E_λ, it must hold that $w = 0$, which contradicts our initial claim. Consequently, $\dim V_\lambda = \dim E_\lambda$, and therefore $\{e_k\}_{1\le k\le \dim V_\lambda}$ is an orthonormal basis of E_λ.

Finally, by repeating the construction above to each eigenspace E_λ associated with distinct eigenvalues, and concatenating the obtained bases, we obtain an orthonormal basis $L^2(\mathcal{M})$ (due to the fact that these eigenspaces are orthogonal to one another and span $L^2(\mathcal{M})$). Each element in this basis is an eigenfunction of $\mathcal{L}$ since it is built from a given eigenspace, and is a real-valued function. This concludes our proof. □

A.2. **A useful inequality.** We end this section by introducing a lemma which will be used to obtain practical error bounds.

Lemma A.1. *For any $a, b \in \mathbb{R}$ and any $\varepsilon > 0$,* $\min\{a-\varepsilon; b\} + \varepsilon \geq \min\{a; b\}$.

Proof. Indeed, if $a < b$, we have $\min\{a-\varepsilon; b\}+\varepsilon = a-\varepsilon+\varepsilon = a = \min\{a; b\}$. If $b \leq a < b+\varepsilon$, $\min\{a-\varepsilon; b\}+\varepsilon = a-\varepsilon+\varepsilon = a \geq b = \min\{a; b\}$. And finally if $a \geq b+\varepsilon$, $\min\{a-\varepsilon; b\}+\varepsilon = b+\varepsilon > b = \min\{a; b\}$. □

Appendix B. Error estimates

B.1. **Geometric consistency estimate.** The following geometric consistency estimate quantifies the error between the bilinear forms $\mathsf{A}_{\mathcal{M}}$ and $\mathsf{A}_{\mathcal{M}_h}$. Its proof is a straightforward adaptation of the proof of [11, Lemma 4.7] to account for the diffusion matrix $\mathcal{D}$.

Lemma B.1. *There is a constant $C > 0$ such that for all $h < h_0$ and $u_h, v_h \in S_h^\ell$,*

$$\left|\mathsf{A}_{\mathcal{M}}(u_h^\ell, v_h^\ell) - \mathsf{A}_{\mathcal{M}_h}(u_h, v_h)\right| \leq Ch^2 \|u_h^\ell\|_{H^1(\mathcal{M})} \|v_h^\ell\|_{H^1(\mathcal{M})}. \tag{46}$$

Proof. We first note that, by definition of $\mathcal{D}$, for any $x_0 \in \mathcal{M}$, and any $w, w' \in T_{x_0}\mathcal{M}$, $(\mathcal{D}(x_0)w)\cdot\overline{w'}$ defines an inner product on $T_{x_0}\mathcal{M}$. We denote by $\|\cdot\|$ the usual Euclidean norm of vectors of $T_{x_0}\mathcal{M} \subset \mathbb{C}^{d+1}$ and by $\|\cdot\|_{\mathcal{D}(x_0)}$ the norm defined by $\|w\|^2_{\mathcal{D}(x_0)} = (\mathcal{D}(x_0)w)\cdot\overline{w}$, $w \in T_{x_0}\mathcal{M}$.

Let $\Pi = I - \nu\nu^T$ (resp. $\Pi_h = I - \nu_h\nu_h^T$) be the orthogonal projection onto the tangent planes of $\mathcal{M}$ (resp. $\mathcal{M}_h$), and let $\mathcal{H} : \mathcal{M} \to \mathbb{R}^{(d+1)\times(d+1)}$ be the extended Weingarten map of $\mathcal{M}$ (cf. [11, Definition 2.5]). Recall in particular that $\mathcal{H}(x)\nu(x) = 0$ for any $x \in \mathcal{M}$, meaning in particular that $\mathcal{H}\Pi = \Pi\mathcal{H} = \mathcal{H}$. Finally, we introduce the map $\mathcal{Q}_h : \mathcal{M} \to \mathbb{R}^{(d+1)\times(d+1)}$ defined as

$$\mathcal{Q}_h = \frac{1}{\sigma^\ell}\Pi(I - d_s^\ell\mathcal{H})\Pi_h^\ell\mathcal{D}\Pi_h^\ell(I - d_s^\ell\mathcal{H})\Pi,$$

where d_s is the oriented distance function restricted to $\mathcal{M}_h$ and introduced in Section 2.2. On the one hand, note that for any $u_h, v_h \in S_h$,

$$\begin{aligned}
&(\mathcal{D}^{-\ell}\nabla_{\mathcal{M}_h}u_h)\cdot\nabla_{\mathcal{M}_h}\overline{v}_h \\
&\quad= \big(\mathcal{D}^{-\ell}\Pi_h(I - d_s\mathcal{H}^{-\ell})\Pi^{-\ell}(\nabla_{\mathcal{M}}u_h^\ell)^{-\ell}\big)\cdot\big(\Pi_h(I - d_s\mathcal{H}^{-\ell})\Pi^{-\ell}(\nabla_{\mathcal{M}}\overline{v}_h^\ell)^{-\ell}\big) \\
&\quad= \sigma\big(\mathcal{Q}_h^{-\ell}(\nabla_{\mathcal{M}}u_h^\ell)^{-\ell}\big)\cdot(\nabla_{\mathcal{M}}\overline{v}_h^\ell)^{-\ell},
\end{aligned}$$

which gives, after integrating both sides over $\mathcal{M}_h$ and using Equation (9),

$$\int_{\mathcal{M}_h}(\mathcal{D}^{-\ell}\nabla_{\mathcal{M}_h}u_h)\cdot(\nabla_{\mathcal{M}_h}\overline{v}_h)\,\mathrm{d}A_h = \int_{\mathcal{M}}\big(\mathcal{Q}_h\nabla_{\mathcal{M}}u_h^\ell\big)\cdot(\nabla_{\mathcal{M}}\overline{v}_h^\ell)\,\mathrm{d}A. \tag{47}$$

Let then $\mathsf{A}_{\mathcal{M}_h}^\ell : S_h^\ell \times S_h^\ell \to \mathbb{R}$ be the Hermitian form defined for any $u_h^\ell, v_h^\ell \in S_h^\ell$ by

$$\mathsf{A}_{\mathcal{M}_h}^\ell(u_h^\ell, v_h^\ell) = \int_{\mathcal{M}}\big(\mathcal{Q}_h\nabla_{\mathcal{M}}u_h^\ell\big)\cdot(\nabla_{\mathcal{M}}\overline{v}_h^\ell)\,\mathrm{d}A + \int_{\mathcal{M}}\big(\sigma^\ell\big)^{-1}Vu_h^\ell\overline{v}_h^\ell\,\mathrm{d}A. \tag{48}$$

Note that following Equations (9) and (47), $\mathsf{A}^{\ell}_{\mathcal{M}_h}$ satisfies for any $u_h, v_h \in S_h$ the equality $\mathsf{A}^{\ell}_{\mathcal{M}_h}(u_h^{\ell}, v_h^{\ell}) = \mathsf{A}_{\mathcal{M}_h}(u_h, v_h)$. Therefore, for any $u_h, v_h \in S_h$, we bound

$$\begin{aligned}\big|\mathsf{A}_{\mathcal{M}}(u_h^{\ell}, v_h^{\ell}) - \mathsf{A}_{\mathcal{M}_h}(u_h, v_h)\big| &= \big|\mathsf{A}_{\mathcal{M}}(u_h^{\ell}, v_h^{\ell}) - \mathsf{A}^{\ell}_{\mathcal{M}_h}(u_h^{\ell}, v_h^{\ell})\big| \\ &\leq \left|\int_{\mathcal{M}} ((\mathcal{Q}_h - \mathcal{D})\nabla_{\mathcal{M}} u_h^{\ell}) \cdot (\nabla_{\mathcal{M}} \overline{v}_h^{\ell}) \, \mathrm{d}A\right| + \left|\int_{\mathcal{M}} (1 - (\sigma^{\ell})^{-1}) V u_h^{\ell} \overline{v}_h^{\ell} \, \mathrm{d}A\right|. \end{aligned} \tag{49}$$

We now bound these two terms. Recall that [11, Lemma 4.1] shows

$$\|\sigma\|_{L^\infty(\mathcal{M}_h)} \lesssim 1, \quad \|\sigma^{-1}\|_{L^\infty(\mathcal{M}_h)} \lesssim 1, \quad \|\sigma - 1\|_{L^\infty(\mathcal{M}_h)} \lesssim h^2, \quad \|\sigma^{-1} - 1\|_{L^\infty(\mathcal{M}_h)} \lesssim h^2. \tag{50}$$

Hence, since V takes positive values,

$$\begin{aligned}\Big|\int_{\mathcal{M}} (1 - (\sigma^{\ell})^{-1}) V u_h^{\ell} \overline{v}_h^{\ell} \, \mathrm{d}A\Big| &\leq \int_{\mathcal{M}} |1 - (\sigma^{\ell})^{-1}| \; V \; |u_h^{\ell}| \; |{v_h}^{\ell}| \, \mathrm{d}A \\ &\leq \|1 - \sigma^{-1}\|_{L^\infty(\mathcal{M}_h)} \int_{\mathcal{M}} V \; |u_h^{\ell}| \; |{v_h}^{\ell}| \, \mathrm{d}A,\end{aligned}$$

which in turn gives (using the Cauchy–Schwartz inequality and Equation (50)),

$$\Big|\int_{\mathcal{M}} (1 - (\sigma^{\ell})^{-1}) V u_h^{\ell} \overline{v}_h^{\ell} \, \mathrm{d}A\Big| \lesssim h^2 \bigg(\int_{\mathcal{M}} V |u_h^{\ell}|^2 \, \mathrm{d}A\bigg)^{1/2} \bigg(\int_{\mathcal{M}} V \; |{v_h}^{\ell}|^2 \, \mathrm{d}A\bigg)^{1/2}. \tag{51}$$

To bound the other term, we first introduce for any $B \in \mathbb{R}^{(d+1)\times(d+1)}$ the notation $\|B\| = \sup_{\|x\|=1} \|Bx\|$. Then we have

$$\begin{aligned}\|\mathcal{Q}_h - \mathcal{D}\| &= \big\|(\sigma^{\ell})^{-1}\big(\Pi(I - d_s^{\ell}\mathcal{H})\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}(I - d_s^{\ell}\mathcal{H})\Pi - \mathcal{D}\big) + \big((\sigma^{\ell})^{-1} - 1\big)\mathcal{D}\big\| \\ &\leq \big\|(\sigma^{\ell})^{-1}\big\|_{L^\infty(\mathcal{M})} \big\|\Pi(I - d_s^{\ell}\mathcal{H})\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}(I - d_s^{\ell}\mathcal{H})\Pi - \mathcal{D}\big\| \\ &\quad + \big\|(\sigma^{\ell})^{-1} - 1\big\|_{L^\infty(\mathcal{M})} \big\|\mathcal{D}\big\|.\end{aligned}$$

By Equation (50) and since $\mathcal{D}$ has bounded eigenvalues over $\mathcal{M}$, we obtain

$$\|\mathcal{Q}_h - \mathcal{D}\| \lesssim \big\|\Pi(I - d_s^{\ell}\mathcal{H})\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}(I - d_s^{\ell}\mathcal{H})\Pi - \mathcal{D}\big\| + h^2, \tag{52}$$

where the constant in the inequality is independent of the location on $\mathcal{M}$. We split the first term on the right into

$$\begin{aligned}&\big\|\Pi(I - d_s^{\ell}\mathcal{H})\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}(I - d_s^{\ell}\mathcal{H})\Pi - \mathcal{D}\big\| \\ &\qquad = \big\|\Pi\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}\Pi - \mathcal{D} - \Pi\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}d_s^{\ell}\mathcal{H}\Pi - d_s^{\ell}\mathcal{H}\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}(I - d_s^{\ell}\mathcal{H})\Pi\big\| \\ &\qquad \leq \big\|\Pi\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}\Pi - \mathcal{D}\big\| + \big\|\Pi\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}d_s^{\ell}\mathcal{H}\Pi\big\| + \big\|d_s^{\ell}\mathcal{H}\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}(I - d_s^{\ell}\mathcal{H})\Pi\big\|.\end{aligned}$$

Since $\|d_s\|_{L^\infty(\mathcal{M}_h)} \lesssim h^2$ by [11, Lemma 4.1] and $\mathcal{H}$ is defined independently of h, we conclude that

$$\big\|\Pi(I - d_s^{\ell}\mathcal{H})\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}(I - d_s^{\ell}\mathcal{H})\Pi - \mathcal{D}\big\| \lesssim \big\|\Pi\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}\Pi - \mathcal{D}\big\| + h^2. \tag{53}$$

We notice that $\mathcal{D} = \Pi\mathcal{D}\Pi$, since by definition of $\mathcal{D}$, $\mathcal{D}\nu = 0$, which implies

$$\begin{aligned}\big\|\Pi\Pi_h^{\ell}\mathcal{D}\Pi_h^{\ell}\Pi - \mathcal{D}\big\| &= \big\|\Pi\Pi_h^{\ell}\Pi\mathcal{D}\Pi\Pi_h^{\ell}\Pi - \Pi\mathcal{D}\Pi\big\| = \big\|(\Pi\Pi_h^{\ell}\Pi - \Pi)\mathcal{D}\Pi\Pi_h^{\ell}\Pi + \Pi\mathcal{D}(\Pi\Pi_h^{\ell}\Pi - \Pi)\big\| \\ &\leq \big\|\Pi\Pi_h^{\ell}\Pi - \Pi\big\|\big\|\mathcal{D}\Pi\Pi_h^{\ell}\Pi\big\| + \big\|\Pi\mathcal{D}\big\|\big\|\Pi\Pi_h^{\ell}\Pi - \Pi\big\|.\end{aligned}$$

Using that $\left\|\Pi\Pi_h^\ell\Pi-\Pi\right\|\lesssim h^2$ by the proof of [11, Lemma 4.1], we deduce that $\left\|\Pi\Pi_h^\ell\mathcal{D}\Pi_h^\ell\Pi-\mathcal{D}\right\|\lesssim h^2$. Injecting this inequality into Equation (53), and the resulting inequality into Equation (52), we conclude that

$$\|\mathcal{Q}_h-\mathcal{D}\|\lesssim h^2.$$

This allows us to write

$$\begin{aligned}\left|\int_{\mathcal{M}}((\mathcal{Q}_h-\mathcal{D})\nabla_{\mathcal{M}}u_h^\ell)\cdot(\nabla_{\mathcal{M}}\overline{v}_h^\ell)\,\mathrm{d}A\right| &\leq \int_{\mathcal{M}}\|(\mathcal{Q}_h-\mathcal{D})\nabla_{\mathcal{M}}u_h^\ell\|\;\|\nabla_{\mathcal{M}}{v_h}^\ell\|\,\mathrm{d}A\\ \lesssim \int_{\mathcal{M}}h^2\|\nabla_{\mathcal{M}}u_h^\ell\|\;\|\nabla_{\mathcal{M}}{v_h}^\ell\|\,\mathrm{d}A &\leq h^2\int_{\mathcal{M}}(\mu_{\min})^{-1}\|\nabla_{\mathcal{M}}u_h^\ell\|_{\mathcal{D}}\;\|\nabla_{\mathcal{M}}{v_h}^\ell\|_{\mathcal{D}}\,\mathrm{d}A,\end{aligned}$$

where $\mu_{\min}:\mathcal{M}\to\mathbb{R}_+$ maps any $x\in\mathcal{M}$ to the smallest eigenvalue of $\mathcal{D}(x)$ associated with an eigenvector in $\nu^\perp$. This last inequality is a consequence of the fact that by construction $\nabla_{\mathcal{M}}u_h^\ell,\nabla_{\mathcal{M}}v_h^\ell\in\nu^\perp$ and using the characterization of eigenvalues through Rayleigh quotients. Since the non-zero eigenvalues of $\mathcal{D}$ are uniformly bounded above and below by positive constants, we conclude that

$$\left|\int_{\mathcal{M}}((\mathcal{Q}_h-\mathcal{D})\nabla_{\mathcal{M}}u_h^\ell)\cdot(\nabla_{\mathcal{M}}\overline{v}_h^\ell)\;\mathrm{d}A\right|\lesssim h^2\int_{\mathcal{M}}\|\nabla_{\mathcal{M}}u_h^\ell\|_{\mathcal{D}}\;\|\nabla_{\mathcal{M}}{v_h}^\ell\|_{\mathcal{D}}\,\mathrm{d}A.$$

Then, using the Cauchy–Schwartz inequality yields

$$\begin{aligned}&\left|\int_{\mathcal{M}}((\mathcal{Q}_h-\mathcal{D})\nabla_{\mathcal{M}}u_h^\ell)\cdot(\nabla_{\mathcal{M}}\overline{v}_h^\ell)\;\mathrm{d}A\right|\\ &\qquad\lesssim h^2\left(\int_{\mathcal{M}}\|\nabla_{\mathcal{M}}u_h^\ell\|_{\mathcal{D}}^2\;\mathrm{d}A\right)^{1/2}\left(\int_{\mathcal{M}}\|\nabla_{\mathcal{M}}v_h^\ell\|_{\mathcal{D}}^2\;\mathrm{d}A\right)^{1/2}.\end{aligned}\tag{54}$$

Inserting the derived bounds Equation (51) and Equation (54) into Equation (49), we derive

$$\begin{aligned}\left|\mathsf{A}_{\mathcal{M}}(u_h^\ell,v_h^\ell)-\mathsf{A}_{\mathcal{M}_h}(u_h,v_h)\right| &\lesssim h^2\left(\int_{\mathcal{M}}\|\nabla_{\mathcal{M}}u_h^\ell\|_{\mathcal{D}}^2\;\mathrm{d}A\right)^{1/2}\left(\int_{\mathcal{M}}\|\nabla_{\mathcal{M}}v_h^\ell\|_{\mathcal{D}}^2\;\mathrm{d}A\right)^{1/2}\\ &\quad+h^2\left(\int_{\mathcal{M}}V|u_h^\ell|^2\;\mathrm{d}A\right)^{1/2}\left(\int_{\mathcal{M}}V\;|{v_h}^\ell|^2\;\mathrm{d}A\right)^{1/2}.\end{aligned}$$

Note that for any $u\in H^1(\mathcal{M})$,

$$\mathsf{A}_{\mathcal{M}}(u,u)\geq\int_{\mathcal{M}}V|u|^2\,\mathrm{d}A,\quad\text{and}\quad\mathsf{A}_{\mathcal{M}}(u,u)\geq\int_{\mathcal{M}}\mathcal{D}\nabla_{\mathcal{M}}u\cdot\nabla_{\mathcal{M}}\overline{u}\,\mathrm{d}A=\int_{\mathcal{M}}\|\nabla_{\mathcal{M}}u\|_{\mathcal{D}}^2\,\mathrm{d}A,$$

so we obtain

$$\left|\mathsf{A}_{\mathcal{M}}(u_h^\ell,v_h^\ell)-\mathsf{A}_{\mathcal{M}_h}^\ell(u_h^\ell,v_h^\ell)\right|\lesssim h^2\sqrt{\mathsf{A}_{\mathcal{M}}(u_h^\ell,u_h^\ell)}\sqrt{\mathsf{A}_{\mathcal{M}}(v_h^\ell,v_h^\ell)}.$$

Finally, due to Equation (5),

$$\sqrt{\mathsf{A}_{\mathcal{M}}(u_h^\ell,u_h^\ell)}\sqrt{\mathsf{A}_{\mathcal{M}}(v_h^\ell,v_h^\ell)}\lesssim\|u_h^\ell\|_{H^1(\mathcal{M})}\|v_h^\ell\|_{H^1(\mathcal{M})},$$

and the result follows. □

B.2. **Norm estimates.** We start by introducing a few norm estimates involving the resolvents of $\mathcal{L}$ and $\mathcal{L}_h$. These results are straightforward extensions of the results introduced in [7, Lemmma 6.3], but adapted to the contour Γ used to define functions of operators in this paper. We also recall a finite element error estimate which derives from a result in [7, Lemma 6.1].

Lemma B.2. *For any $z \in \Gamma$, $s \in [0,1]$, $v \in L^2(\mathcal{M})$, and $v_h \in S_h^\ell$, it holds*

$$\|\mathcal{L}^s(z-\mathcal{L})^{-1}v\|_{L^2(\mathcal{M})} \lesssim |z|^{-(1-s)}\|v\|_{L^2(\mathcal{M})}, \tag{55}$$

$$\|\mathcal{L}_h^s(z-\mathcal{L}_h)^{-1}v_h\|_{L^2(\mathcal{M})} \lesssim |z|^{-(1-s)}\|v_h\|_{L^2(\mathcal{M})}. \tag{56}$$

Besides, for any $\beta \in [0,1]$ and for any $\varphi \in L^2(\mathcal{M})$,

$$\|\mathcal{L}_h^{(1-\beta)/2}P_h(\mathcal{L}^{-1}-\mathcal{L}_h^{-1}P_h)\varphi\|_{L^2(\mathcal{M})} \lesssim h^{2\beta}\|\mathcal{L}^{-(1-\beta)/2}\varphi\|_{L^2(\mathcal{M})} \tag{57}$$

Proof. We start with the proof of Equation (55). To this end, let $z \in \Gamma$ and $s \in [0,1]$. Let then $v \in L^2(\mathcal{M})$ and $w = (z-\mathcal{L})^{-1}v \in \mathcal{D}(\mathcal{L})$, where $\mathcal{D}(\cdot)$ denotes the domain of an operator. Since $\mathcal{L}$ is self-adjoint, we have $\mathcal{D}(\mathcal{L}^s) = [L^2(\mathcal{M}), \mathcal{D}(\mathcal{L})]_s$ with isometry, where $[L^2(\mathcal{M}), \mathcal{D}(\mathcal{L})]_s$ the intermediate space between $L^2(\mathcal{M})$ and $\mathcal{D}(\mathcal{L})$ obtained by the complex interpolation method [28, Theorem 16.1]. In particular, we have

$$\|\mathcal{L}^s w\|_{L^2(\mathcal{M})} = \|w\|_{[L^2(\mathcal{M}),\mathcal{D}(\mathcal{L})]_s} \leq \|\mathcal{L}w\|_{L^2(\mathcal{M})}^s \|w\|_{L^2(\mathcal{M})}^{1-s} \tag{58}$$

where the inequality derives from a classical result on interpolation spaces (see e.g., [28, Section 5.1]). On the one hand, we obtain by Equation (18) that

$$\|w\|_{L^2(\mathcal{M})} = \|(z-\mathcal{L})^{-1}v\|_{L^2(\mathcal{M})} \lesssim |z|^{-1}\|v\|_{L^2(\mathcal{M})}.$$

On the other hand, adding and subtracting $z(z-\mathcal{L})^{-1}v$ yields

$$\|\mathcal{L}w\|_{L^2(\mathcal{M})} = \|z(z-\mathcal{L})^{-1}v - v\|_{L^2(\mathcal{M})} \leq |z|\|(z-\mathcal{L})^{-1}v\|_{L^2(\mathcal{M})} + \|v\|_{L^2(\mathcal{M})} \lesssim 2\|v\|_{L^2(\mathcal{M})},$$

where we once again use Equation (18) to derive the last inequality. Combining these two estimates with Equation (58), we get

$$\|\mathcal{L}^s(z-\mathcal{L})^{-1}v\|_{L^2(\mathcal{M})} = \|\mathcal{L}^s w\|_{L^2(\mathcal{M})} \lesssim \|v\|_{L^2(\mathcal{M})}^s |z|^{-(1-s)}\|v\|_{L^2(\mathcal{M})}^{1-s} = |z|^{-(1-s)}\|v\|_{L^2(\mathcal{M})},$$

hence proving Equation (55). As for Equation (56), it is proven using the same steps as Equation (55) but substituting $\mathcal{L}$ by $\mathcal{L}_h$, Equation (18) by Equation (19), and $\mathcal{D}(\mathcal{L})$ by S_h^ℓ.

Finally, let us prove Equation (57). Let $\beta \in (0,1]$ and $\varphi \in L^2(\mathcal{M})$. Then, we have

$$\begin{aligned}\|\mathcal{L}_h^{(1-\beta)/2}P_h(\mathcal{L}^{-1}-\mathcal{L}_h^{-1}P_h)\varphi\|_{L^2(\mathcal{M})} &\lesssim \|\mathcal{L}^{(1-\beta)/2}P_h(\mathcal{L}^{-1}-\mathcal{L}_h^{-1}P_h)\varphi\|_{L^2(\mathcal{M})}\\ &\lesssim \|\mathcal{L}^{(1-\beta)/2}(\mathcal{L}^{-1}-\mathcal{L}_h^{-1}P_h)\varphi\|_{L^2(\mathcal{M})}\end{aligned}$$

where we use [7, Lemma 5.2] to derive the first inequality, and [7, Lemma 5.1] to derive the second inequality. Finally, noting that $\mathcal{L}$ satisfies elliptic regularity for indices $\alpha \in (0,1]$ (cf. [7, Assumption 1]), we can apply [7, Lemma 6.1] with $\alpha = \beta$ and $s = (1-\beta)/2$ to conclude that $\|\mathcal{L}^{(1-\beta)/2}(\mathcal{L}^{-1}-\mathcal{L}_h^{-1}P_h)\varphi\|_{L^2(\mathcal{M})} \lesssim h^{2\beta}\|\mathcal{L}^{-(1-\beta)/2}\varphi\|_{L^2(\mathcal{M})}$ and therefore

$$\|\mathcal{L}_h^{(1-\beta)/2}P_h(\mathcal{L}^{-1}-\mathcal{L}_h^{-1}P_h)\varphi\|_{L^2(\mathcal{M})} \lesssim h^{2\beta}\|\mathcal{L}^{-(1-\beta)/2}\varphi\|_{L^2(\mathcal{M})}.$$

This concludes the proof of Lemma B.2. □

We now prove some estimates for the norm of shifted inverses of the operators $\mathcal{L}_h$ and L_h, and for the error between inverses of these two operators. These results can be seen as extensions of the ones stated in [6, Lemma A.1].

Lemma B.3. *Let $v \in L^2(\mathcal{M})$, $v_h \in S_h^\ell$ and $V_h \in S_h$ be arbitrary. Then, for all $z \in \Gamma$, for any $q \in [-1,1]$, and any $r \in (0,2)$, and any $s \in [0,1]$,*

$$\|\mathsf{L}_h(z-\mathsf{L}_h)^{-1}V_h\|_{L^2(\mathcal{M}_h)} \lesssim |z|^{-1/2}\|V_h\|_{H^1(\mathcal{M}_h)}, \tag{59}$$

$$\|(z-\mathcal{L}_h)^{-1}v_h\|_{L^2(\mathcal{M})} \lesssim |z|^{r/2-1}\|\mathcal{L}_h^{-r/2}v_h\|_{L^2(\mathcal{M})}, \tag{60}$$

$$\left\|(\mathcal{L}_h^{-1}v_h)^{-\ell} - \mathsf{L}_h^{-1}\mathsf{P}_h(\sigma v_h^{-\ell})\right\|_{H^1(\mathcal{M}_h)} \lesssim h^2\|\mathcal{L}_h^{-1/2}v_h\|_{L^2(\mathcal{M})}. \tag{61}$$

Proof. To prove Equation (59), we first observe that the estimate in Equation (56) carry over to the case when L_h is used instead of $\mathcal{L}_h$, and $L^2(\mathcal{M})$ (resp. S_h^ℓ) is replaced by its counterpart $L^2(\mathcal{M}_h)$ (resp. S_h) on the polyhedral surface. Hence, for any $z \in \Gamma$, $V_h \in L^2(\mathcal{M}_h)$, we have

$$\|\mathsf{L}_h^s(z-\mathsf{L}_h)^{-1}V_h\|_{L^2(\mathcal{M}_h)} \lesssim |z|^{-(1-s)}\|V_h\|_{L^2(\mathcal{M}_h)}$$

In particular, we retrieve (by taking $s = 1/2$)

$$\begin{aligned}\|\mathsf{L}_h(z-\mathsf{L}_h)^{-1}V_h\|_{L^2(\mathcal{M}_h)} &= \|\mathsf{L}_h^{1/2}(z-\mathsf{L}_h)^{-1}\mathsf{L}_h^{1/2}V_h\|_{L^2(\mathcal{M}_h)} \\ &\lesssim |z|^{-1/2}\|\mathsf{L}_h^{1/2}V_h\|_{L^2(\mathcal{M}_h)} \lesssim |z|^{-1/2}\|V_h\|_{H^1(\mathcal{M}_h)},\end{aligned} \tag{62}$$

where we used the equivalence of norms (17) in the last inequality.

To bound Equation (60), we apply Equation (56) with $s = r/2$ to obtain

$$\|(z-\mathcal{L}_h)^{-1}v_h\|_{L^2(\mathcal{M})} = \|\mathcal{L}_h^{r/2}(z-\mathcal{L}_h)^{-1}\mathcal{L}_h^{-r/2}v_h\|_{L^2(\mathcal{M})} \lesssim |z|^{-(1-r/2)}\|\mathcal{L}_h^{-r/2}v_h\|_{L^2(\mathcal{M})}.$$

Finally, to prove the bound in Equation (61), we rely on the geometric consistency estimate of Lemma B.1. Let $u_h = \mathcal{L}_h^{-1}v_h$ and let $U_h = \mathsf{L}_h^{-1}\mathsf{P}_h(\sigma v_h^{-\ell})$. Note that by definition of $\mathcal{L}_h^{-1}$,

$$(\mathcal{L}_h u_h, w_h)_{L^2(\mathcal{M})} = \mathsf{A}_{\mathcal{M}}(u_h, w_h) = (v_h, w_h)_{L^2(\mathcal{M})},$$

for all $w_h \in S_h^\ell$. Likewise, for L_h^{-1} we obtain

$$\begin{aligned}(\mathsf{L}_h U_h, W_h)_{L^2(\mathcal{M}_h)} &= \mathsf{A}_{\mathcal{M}_h}(U_h, W_h) = (\mathsf{P}_h(\sigma v_h^{-\ell}), W_h)_{L^2(\mathcal{M}_h)} \\ &= (\sigma v_h^{-\ell}, W_h)_{L^2(\mathcal{M}_h)} = (v_h, W_h^\ell)_{L^2(\mathcal{M})},\end{aligned}$$

for all $W_h \in S_h$, where we used the definition of σ in the last step. Let us now select a fixed, but arbitrary, $\Xi_h \in S_h$. Then, by combining the last two equations,

$$\begin{aligned}|\mathsf{A}_{\mathcal{M}_h}(u_h^{-\ell} - U_h, \Xi_h)| &= |\mathsf{A}_{\mathcal{M}_h}(u_h^{-\ell}, \Xi_h) - \mathsf{A}_{\mathcal{M}_h}(U_h, \Xi_h)| \\ &= |\mathsf{A}_{\mathcal{M}_h}(u_h^{-\ell}, \Xi_h) - (v_h, \Xi_h^\ell)_{L^2(\mathcal{M})}| = |\mathsf{A}_{\mathcal{M}_h}(u_h^{-\ell}, \Xi_h) - \mathsf{A}_{\mathcal{M}}(u_h, \Xi_h^\ell)|,\end{aligned}$$

meaning that an application of Lemma B.1 results in the bound

$$|\mathsf{A}_{\mathcal{M}_h}(u_h^{-\ell} - U_h, \Xi_h)| \lesssim h^2\|u_h\|_{H^1(\mathcal{M})}\|\Xi_h^\ell\|_{H^1(\mathcal{M})}. \tag{63}$$

Further, note that for any $\Xi_h^\ell \in S_h^\ell$, the equivalence of norms (17) gives

$$\|\Xi_h^\ell\|_{H^1(\mathcal{M})}^2 \sim \|\mathcal{L}_h^{1/2}\Xi_h^\ell\|_{L^2(\mathcal{M})}^2 = (\mathcal{L}_h\Xi_h^\ell, \Xi_h^\ell)_{L^2(\mathcal{M})} = \mathsf{A}_{\mathcal{M}}(\Xi_h^\ell, \Xi_h^\ell), \tag{64}$$

where the last equality comes from the definition of $\mathcal{L}_h$. And similarly, for any $\Xi_h \in S_h$, we have

$$\|\Xi_h\|^2_{H^1(\mathcal{M}_h)} \sim \mathsf{A}_{\mathcal{M}_h}(\Xi_h, \Xi_h). \tag{65}$$

Then, applying the triangle inequality to the (last) right-hand side of Equation (64) gives

$$\begin{aligned}\|\Xi_h^\ell\|^2_{H^1(\mathcal{M})} &\lesssim \mathsf{A}_{\mathcal{M}_h}(\Xi_h, \Xi_h) + \left|\mathsf{A}_{\mathcal{M}}(\Xi_h^\ell, \Xi_h^\ell) - \mathsf{A}_{\mathcal{M}_h}(\Xi_h, \Xi_h)\right| \\ &\lesssim \|\Xi_h\|^2_{H^1(\mathcal{M}_h)} + h^2\|\Xi_h^\ell\|^2_{H^1(\mathcal{M})},\end{aligned}$$

where we used Equation (65) and Lemma B.1 to derive the second inequality. This means in particular that there exists $C > 0$ independent of h such that $\|\Xi_h^\ell\|^2_{H^1(\mathcal{M})} \le C(\|\Xi_h\|^2_{H^1(\mathcal{M}_h)} + h^2\|\Xi_h^\ell\|^2_{H^1(\mathcal{M})})$. Recall that $h \in (0, h_0)$ for some $h_0 \in (0, 1)$ small enough. Assuming that especially $1 - Ch_0^2 > 0$ yields $\|\Xi_h^\ell\|^2_{H^1(\mathcal{M})} \le C(1-Ch^2)^{-1}\|\Xi_h\|^2_{H^1(\mathcal{M}_h)} \le C(1-Ch_0^2)^{-1}\|\Xi_h\|^2_{H^1(\mathcal{M}_h)}$, which allows us to conclude that

$$\|\Xi_h^\ell\|^2_{H^1(\mathcal{M})} \lesssim \|\Xi_h\|^2_{H^1(\mathcal{M}_h)}. \tag{66}$$

Now, applying successively Equation (65) and Equation (63) with $\Xi_h = u_h^{-\ell} - U_h$, we obtain

$$\|u_h^{-\ell} - U_h\|^2_{H^1(\mathcal{M}_h)} \lesssim h^2\|u_h\|_{H^1(\mathcal{M})}\|u_h - U_h^\ell\|_{H^1(\mathcal{M})} \lesssim h^2\|u_h\|_{H^1(\mathcal{M})}\|u_h^{-\ell} - U_h\|_{H^1(\mathcal{M}_h)},$$

where the last inequality is derived from applying Equation (66). Therefore, we end up with

$$\|u_h - U_h^\ell\|_{H^1(\mathcal{M})} \lesssim h^2\|u_h\|_{H^1(\mathcal{M})} \lesssim h^2\|\mathcal{L}_h^{-1/2}v_h\|_{L^2(\mathcal{M})},$$

where the equivalence of norms (17) together with the definition of u_h are used in the final step. This concludes the proof of Equation (61). □

Finally, we recall the Bramble-–Hilbert lemma (cf. [6, Equation (4.8)]), which is used in several proofs in this paper.

Lemma B.4 (Bramble-–Hilbert lemma)**.** *For any $t \in [0, 2]$ and any $\varphi \in H^t(\mathcal{M})$,*

$$\|(I - P_h)\varphi\|_{L^2(\mathcal{M})} \lesssim h^t\|\varphi\|_{H^t(\mathcal{M})} \lesssim h^t\|\mathcal{L}^{t/2}\varphi\|_{L^2(\mathcal{M})}, \tag{67}$$

where we used the equivalence of Sobolev norms dot-spaces norms in the last inequality.

B.3. **Proof of Lemma 2.5.**

Proof. Let $h \in (0, h_0)$, $z \in \Gamma$. Let $f \in L^2(\mathcal{M})$ and let $p \in [0, 1]$ such that $\|\mathcal{L}^p f\|_{L^2(\mathcal{M})} < \infty$. Finally, let $\beta \in [0, 1]$ such that $p \in [0, (1+\beta)/2]$. We then define $\mathcal{F}_h(z)$ by $\mathcal{F}_h(z) = (z - \mathcal{L}_h)^{-1}P_h - P_h(z - \mathcal{L})^{-1}$. Note then

$$\begin{aligned}\mathcal{F}_h(z) &= (z - \mathcal{L}_h)^{-1}\mathcal{L}_h(\mathcal{L}_h^{-1}P_h(z - \mathcal{L})\mathcal{L}^{-1} - \mathcal{L}_h^{-1}(z - \mathcal{L}_h)P_h\mathcal{L}^{-1})\mathcal{L}(z - \mathcal{L})^{-1} \\ &= (z - \mathcal{L}_h)^{-1}\mathcal{L}_h(P_h\mathcal{L}^{-1} - \mathcal{L}_h^{-1}P_h)\mathcal{L}(z - \mathcal{L})^{-1}\end{aligned}$$

Hence,

$$\begin{aligned}\|\mathcal{F}_h(z)f\|_{L^2(\mathcal{M})} &= \|(z - \mathcal{L}_h)^{-1}\mathcal{L}_h(P_h\mathcal{L}^{-1} - \mathcal{L}_h^{-1}P_h)\mathcal{L}(z - \mathcal{L})^{-1}f\|_{L^2(\mathcal{M})} \\ &= \|\mathcal{L}_h^{1-(1-\beta)/2}(z - \mathcal{L}_h)^{-1}\mathcal{L}_h^{(1-\beta)/2}(P_h\mathcal{L}^{-1} - \mathcal{L}_h^{-1}P_h)\mathcal{L}(z - \mathcal{L})^{-1}f\|_{L^2(\mathcal{M})} \\ &\lesssim |z|^{-(1-\beta)/2}\|\mathcal{L}_h^{(1-\beta)/2}(P_h\mathcal{L}^{-1} - \mathcal{L}_h^{-1}P_h)\mathcal{L}(z - \mathcal{L})^{-1}f\|_{L^2(\mathcal{M})}\end{aligned}$$

where we used Equation (56) with $s = (1-\beta)/2 \in [0, 1/2]$ to derive the inequality. Note then that, using Equation (57), we have

$$\begin{aligned}\|\mathcal{F}_h(z)f\|_{L^2(\mathcal{M})} &\lesssim |z|^{-(1-\beta)/2} h^{2\beta} \|\mathcal{L}^{-(1-\beta)/2}\mathcal{L}(z-\mathcal{L})^{-1}f\|_{L^2(\mathcal{M})} \\ &= h^{2\beta}|z|^{-(1-\beta)/2}\|\mathcal{L}^{(1+\beta)/2}(z-\mathcal{L})^{-1}f\|_{L^2(\mathcal{M})} \\ &= h^{2\beta}|z|^{-(1-\beta)/2}\|\mathcal{L}^{(1+\beta)/2-p}(z-\mathcal{L})^{-1}\mathcal{L}^p f\|_{L^2(\mathcal{M})}\end{aligned}$$

Using then Equation (55) with $s = (1+\beta)/2 - p \in [0, 1]$, we retrieve

$$\begin{aligned}\|\mathcal{F}_h(z)f\|_{L^2(\mathcal{M})} &\lesssim h^{2\beta}|z|^{-(1-\beta)/2}|z|^{-(1-(1+\beta)/2+p)}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})} \\ &= h^{2\beta}|z|^{-(1-\beta+p)}\|\mathcal{L}^p f\|_{L^2(\mathcal{M})}.\end{aligned}$$

This concludes the proof. □

B.4. **Proof of Proposition 2.6.** Based on the results in the previous subsections, we can now move on to the proof of Proposition 2.6.

Proof. Let $\tilde{f} \in S_h^\ell$. We introduce the inverse lift operator $\mathcal{P}_\ell : L^2(\mathcal{M}) \to L^2(\mathcal{M}_h)$ which maps any $F \in L^2(\mathcal{M})$ to $\mathcal{P}_\ell F = F^{-\ell}$. Let then $\mathcal{E}_h = \left\|\left(\gamma(\mathcal{L}_h)\tilde{f}\right)^{-\ell} - \gamma(\mathsf{L}_h)\mathsf{P}_h(\sigma\tilde{f}^{-\ell})\right\|_{L^2(\mathcal{M}_h)} = \left\|\mathcal{P}_\ell\gamma(\mathcal{L}_h)\tilde{f} - \gamma(\mathsf{L}_h)\mathsf{P}_h(\sigma\mathcal{P}_\ell\tilde{f})\right\|_{L^2(\mathcal{M}_h)}$. Note that by the integral representations of the operators (20)

$$\mathcal{E}_h = \Big\|\frac{1}{2\pi i}\int_\Gamma \gamma(z)\mathcal{F}(z)\tilde{f}\,\mathrm{d}z\Big\|_{L^2(\mathcal{M}_h)},$$

where we take for any $z \in \Gamma$, $\mathcal{F}(z) = \mathcal{P}_\ell(z-\mathcal{L}_h)^{-1} - (z-\mathsf{L}_h)^{-1}\mathsf{P}_h\sigma\mathcal{P}_\ell$. Similarly, as in the proof of Proposition 2.4, we use the splitting (21) of Γ and the triangle inequality to deduce that

$$\begin{aligned}\mathcal{E}_h \leq{}& \frac{1}{2\pi}\int_{\Omega_+}|\gamma(g_+(t))|\left\|\mathcal{F}(g_+(t))\tilde{f}\right\|_{L^2(\mathcal{M}_h)}\mathrm{d}t + \frac{\delta_0}{2\pi}\int_{\Omega_0}|\gamma(g_0(t))|\|\mathcal{F}(g_0(t))\tilde{f}\|_{L^2(\mathcal{M}_h)}\,\mathrm{d}t \\ &+ \frac{1}{2\pi}\int_{\Omega_-}|\gamma(g_-(t))|\left\|\mathcal{F}(g_-(t))\tilde{f}\right\|_{L^2(\mathcal{M}_h)}\mathrm{d}t,\end{aligned}\tag{68}$$

where we take $\Omega_+ = \Omega_- = [\delta_0, \infty)$ and $\Omega_0 = [-\theta, \theta]$. For $* \in \{+, 0, -\}$, let us then introduce the quantity

$$\mathcal{E}_h^* = \int_{\Omega_*}|\gamma(g_*(t))|\,\|\mathcal{F}(g_*(t))\|_{L^2(\mathcal{M}_h)}\,\mathrm{d}t,$$

so that Equation (68) may be rewritten as $\mathcal{E}_h \lesssim \mathcal{E}_h^+ + \mathcal{E}_h^0 + \mathcal{E}_h^-$ and in particular, $g_*(t) \in \Gamma$ for any $t \in \Omega_*$.

We now fix $* \in \{+, 0, -\}$ and bound the term $\mathcal{E}_h^*$. First, for any $z \in \Gamma$, we rewrite $\mathcal{F}(z)$ and split

$$\begin{aligned}\mathcal{F}(z) &= (z-\mathsf{L}_h)^{-1}\mathsf{L}_h\left((z\mathsf{L}_h^{-1} - I)\mathcal{P}_\ell\mathcal{L}_h^{-1} - \mathsf{L}_h^{-1}\mathsf{P}_h\sigma\mathcal{P}_\ell(z\mathcal{L}_h^{-1} - I)\right)\mathcal{L}_h(z-\mathcal{L}_h)^{-1} \\ &= (z-\mathsf{L}_h)^{-1}\mathsf{L}_h\left(z\mathsf{L}_h^{-1}(I-\mathsf{P}_h\sigma)\mathcal{P}_\ell\mathcal{L}_h^{-1} + \mathsf{L}_h^{-1}\mathsf{P}_h\sigma\mathcal{P}_\ell - \mathcal{P}_\ell\mathcal{L}_h^{-1}\right)\mathcal{L}_h(z-\mathcal{L}_h)^{-1} \\ &= \mathcal{F}_1(z) + \mathcal{F}_2(z),\end{aligned}$$

where we take $\mathcal{F}_1(z) = (z - \mathsf{L}_h)^{-1}\mathsf{L}_h \left(z\mathsf{L}_h^{-1}(I - \mathsf{P}_h\sigma)\mathcal{P}_\ell\mathcal{L}_h^{-1}\right)\mathcal{L}_h(z-\mathcal{L}_h)^{-1} = z(z-\mathsf{L}_h)^{-1}(I - \mathsf{P}_h\sigma)\mathcal{P}_\ell(z-\mathcal{L}_h)^{-1}$ and $\mathcal{F}_2(z) = (z - \mathsf{L}_h)^{-1}\mathsf{L}_h\left(\mathsf{L}_h^{-1}\mathsf{P}_h\sigma\mathcal{P}_\ell - \mathcal{P}_\ell\mathcal{L}_h^{-1}\right)\mathcal{L}_h(z-\mathcal{L}_h)^{-1}$. Hence, by the triangle inequality,

$$\mathcal{E}_h^* \lesssim \int_{\Gamma_t} |\gamma(g_*(t))| \Big(\left\|\mathcal{F}_1(g_*(t))\tilde{f}\right\|_{L^2(\mathcal{M}_h)} + \left\|\mathcal{F}_2(g_*(t))\tilde{f}\right\|_{L^2(\mathcal{M}_h)}\Big)\,\mathrm{d}t. \tag{69}$$

We first bound $\left\|\mathcal{F}_1(z)\tilde{f}\right\|_{L^2(\mathcal{M}_h)}$. Using successively Equation (23) and the geometric estimates in [5, Corollary 2.2] results in

$$\begin{aligned}\left\|\mathcal{F}_1(z)\tilde{f}\right\|_{L^2(\mathcal{M}_h)} &= |z|\left\|(z-\mathsf{L}_h)^{-1}(I-\mathsf{P}_h\sigma)\mathcal{P}_\ell(z-\mathcal{L}_h)^{-1}\tilde{f}\right\|_{L^2(\mathcal{M}_h)} \\ &\lesssim \left\|(I-\mathsf{P}_h\sigma)\mathcal{P}_\ell(z-\mathcal{L}_h)^{-1}\tilde{f}\right\|_{L^2(\mathcal{M}_h)} \lesssim h^2\left\|(z-\mathcal{L}_h)^{-1}\tilde{f}\right\|_{L^2(\mathcal{M})}.\end{aligned}$$

Using then Equation (60), we conclude that, for any $p \in (0,2)$,

$$\left\|\mathcal{F}_1(z)\tilde{f}\right\|_{L^2(\mathcal{M}_h)} \lesssim h^2|z|^{-(1-p/2)}\left\|\mathcal{L}_h^{-p/2}\tilde{f}\right\|^2_{L^2(\mathcal{M})}. \tag{70}$$

To bound $\left\|\mathcal{F}_2(z)\tilde{f}\right\|_{L^2(\mathcal{M}_h)}$, we apply Equations (59) and (61) to obtain

$$\begin{aligned}\left\|\mathcal{F}_2(z)\tilde{f}\right\|_{L^2(\mathcal{M}_h)} &\lesssim |z|^{-1/2}\left\|\left(\mathsf{L}_h^{-1}\mathsf{P}_h\sigma\mathcal{P}_\ell - \mathcal{P}_\ell\mathcal{L}_h^{-1}\right)\mathcal{L}_h(z-\mathcal{L}_h)^{-1}\tilde{f}\right\|_{H^1(\mathcal{M}_h)} \\ &\lesssim |z|^{-1/2}h^2\left\|\mathcal{L}_h^{1/2}(z-\mathcal{L}_h)^{-1}\tilde{f}\right\|_{L^2(\mathcal{M})} \\ &= |z|^{-1/2}h^2\left\|\mathcal{L}_h^{(1+p)/2}(z-\mathcal{L}_h)^{-1}\mathcal{L}_h^{-p/2}\tilde{f}\right\|_{L^2(\mathcal{M})}\end{aligned}$$

and with Equation (56) (applied with $s = (1+p)/2$)

$$\left\|\mathcal{F}_2(z)\tilde{f}\right\|_{L^2(\mathcal{M}_h)} \lesssim h^2|z|^{-(1-p/2)}\left\|\mathcal{L}_h^{-p/2}\tilde{f}\right\|_{L^2(\mathcal{M})}. \tag{71}$$

Using Equations (70) and (71) with $p = \min\{\alpha + d/4; 1\}$ together with Equation (69) gives

$$\mathcal{E}_h^* \lesssim \int_{\Omega_*} |\gamma(g_*(t))| h^2 |g_*(t)|^{-(1-p/2)} \|\mathcal{L}_h^{-p/2}\tilde{f}\|_{L^2(\mathcal{M})}\,\mathrm{d}t,$$

which yields in turn (since γ is an α-amplitude spectral density)

$$\mathcal{E}_h^* \lesssim h^2\|\mathcal{L}_h^{-p/2}\tilde{f}\|_{L^2(\mathcal{M})}\int_{\Omega_*}|g_*(t)|^{-(1+\alpha-p/2)}\,\mathrm{d}t \lesssim h^2\|\mathcal{L}_h^{-\min\{\alpha+d/4;1\}/2}\tilde{f}\|_{L^2(\mathcal{M})},$$

since $\alpha - p/2 = \max\{\alpha - (\alpha+d/4)/2; \alpha - 1/2\} = \max\{(\alpha - d/4)/2; \alpha - 1/2\} \geq (\alpha - d/4)/2 > 0$. Finally, since this inequality holds for any $* \in \{+, 0, -\}$, we retrieve the claim (24) using Equation (68). $\square$

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