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## RANKS OF SOFT OPERATORS IN NOWHERE SCATTERED C\*-ALGEBRAS

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*Abstract* We show that for C\*-algebras with the global Glimm property, the rank of every operator can be realized as the rank of a soft operator, that is, an element whose hereditary sub-C\*-algebra has no nonzero, unital quotients. This implies that the radius of comparison of such a C\*-algebra is determined by the soft part of its Cuntz semigroup.

Under a mild additional assumption, we show that every Cuntz class dominates a (unique) largest soft Cuntz class. This defines a retract from the Cuntz semigroup onto its soft part, and it follows that the covering dimensions of these semigroups differ by at most 1.

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## 1. Introduction

Realizing every strictly positive, lower-semicontinuous, affine function on the tracial state space of a separable, simple, nuclear, nonelementary  $C^*$ -algebra as the rank of an operator in its stabilization is a deep and open problem, first studied in [19]. A positive solution to this problem would imply that every separable, simple, nonelementary  $C^*$ -algebra of locally finite nuclear dimension and strict comparison of positive elements is  $\mathcal{Z}$ -stable, thus proving the remaining implication of the prominent Toms–Winter conjecture ([50, Section 5]) in this case; see, for example, [37, Section 9] and the discussion in [15, Section 5].

When the  $C^*$ -algebra  $A$  is not simple, the problem is still of much interest, but one needs to replace the tracial state space by the cone  $\text{QT}(A)$  of lower-semicontinuous, extended-valued 2-quasitraces on  $A$ . Each such quasitrace extends canonically to the stabilization  $A \otimes \mathbb{K}$ , and the *rank* of an operator  $a \in (A \otimes \mathbb{K})_+$  is defined as the map  $\widehat{[a]}: \text{QT}(A) \rightarrow [0, \infty]$  given by

$$\widehat{[a]}(\tau) := d_\tau(a) := \lim_{n \rightarrow \infty} \tau(a^{1/n})$$

for  $\tau \in \text{QT}(A)$ . The *rank problem* is then to determine which functions on  $\text{QT}(A)$  arise as the rank of a positive operator in  $A$  or  $A \otimes \mathbb{K}$ .

A natural obstruction arises if  $A$  has a nonzero elementary ideal quotient, that is, if there are closed ideals  $I \subseteq J \subseteq A$  such that  $J/I$  is  $*$ -isomorphic to  $\mathbb{K}(H)$  for some Hilbert space  $H$ . In this case, the natural trace on  $\mathbb{K}(H)$  induces a quasitrace  $\tau \in \text{QT}(A)$  that is discrete in the sense that  $d_\tau(a) \in \{0, 1, 2, \dots, \infty\}$  for every  $a \in (A \otimes \mathbb{K})_+$ . A similar obstruction arises in the representation of interpolation groups by continuous, affine functions on their state space; see [26, Chapter 8].

To avoid this obstruction, it is therefore natural to assume that  $A$  has no nonzero elementary ideal quotients, a condition termed *nowhere scatteredness* in [39]. Building on the results from [37], the rank problem was solved in [2] for nowhere scattered  $C^*$ -algebras that have stable rank one: Every function on  $\text{QT}(A)$  that satisfies the ‘obvious’ conditions arises as the rank of an operator in  $A \otimes \mathbb{K}$ ; see [2, Theorem 7.13] for the precise statement. Moreover, one can arrange for the operator to be *soft*, which means that it generates a hereditary sub- $C^*$ -algebra that has no nonzero unital quotients; see [40, Definition 3.1].

As a consequence, in a nowhere scattered, stable rank one  $C^*$ -algebra, the rank of every operator can be realized as the rank of a *soft* operator.

The aim of this paper is to study this phenomenon in greater generality and, more concretely, to investigate when the rank of every operator in a  $C^*$ -algebra  $A$  can be realized as the rank of a soft element. We show that this holds whenever  $A$  satisfies the *Global Glimm Property* – a notion conjectured to be equivalent to nowhere scatteredness; see Paragraph 2.3. Namely, we prove:

**Theorem A (5.11).** *Let  $A$  be a stable  $C^*$ -algebra with the global Glimm property. Then, for any  $a \in A_+$  there exists a soft element  $b \in A_+$  with  $b \precsim a$  and such that*

$$d_\tau(a) = d_\tau(b)$$

for every  $\tau \in \text{QT}(A)$ .

In Theorem A above, we use  $\simeq$  to denote the *Cuntz subequivalence*, a relation between positive elements introduced by Cuntz in [18]. This relation allows one to define the *Cuntz semigroup*, an object that has played an important role in the structure and classification theory of  $C^*$ -algebras; see Paragraph 2.1 and [2, 17, 37, 45, 49]. As explained in Paragraph 2.2, the study of the Cuntz semigroup has often come in hand with the development of abstract Cuntz semigroups, also known as *Cu-semigroups*; see [5, 6, 7, 14, 32, 47] among many others.

If an operator  $a$  is soft, then its Cuntz class  $[a]$  is strongly soft (we recall the definition at the beginning of Section 3). If  $A$  has the global Glimm property, then every strongly soft Cuntz class arises this way, and it follows that the submonoid  $\text{Cu}(A)_{\text{soft}}$  of strongly soft Cuntz classes agrees with the subset of Cuntz classes with a soft representative; see Corollary 3.4.

The cone  $\text{QT}(A)$  is naturally isomorphic to the cone  $F(\text{Cu}(A))$  of functionals on the Cuntz semigroup  $\text{Cu}(A)$ ; see [20, Theorem 4.4]. As an application of Theorem A, we show that the same is true for the cone of functionals on  $\text{Cu}(A)_{\text{soft}}$ .

**Theorem B (5.15).** *Let  $A$  be a  $C^*$ -algebra with the global Glimm property. Then,  $\text{QT}(A)$  is naturally isomorphic to  $F(\text{Cu}(A)_{\text{soft}})$ .*

We introduce in Section 4 a weak notion of cancellation for Cuntz semigroups, which we term *left-soft separativity*; see Definition 4.2. Whenever a  $C^*$ -algebra with the global Glimm property has a left-soft separative Cuntz semigroup, the relation between arbitrary and soft elements from Theorem A can be made more precise:

**Theorem C (6.3, 6.6).** *Let  $A$  be a  $C^*$ -algebra with the global Glimm property. Assume that  $\text{Cu}(A)$  is left-soft separative. Then;*

- (i) *For every element  $x \in \text{Cu}(A)$ , there exists a greatest element in  $\text{Cu}(A)_{\text{soft}}$  below  $x$ , denoted by  $\sigma(x)$ .*
- (ii) *We have  $\lambda(\sigma(x)) = \lambda(x)$  for every  $x \in \text{Cu}(A)$  and  $\lambda \in F(\text{Cu}(A))$ .*
- (iii) *The map  $\sigma: \text{Cu}(A) \rightarrow \text{Cu}(A)_{\text{soft}}$ , defined by  $x \mapsto \sigma(x)$ , preserves order, suprema of increasing sequences and is superadditive.*

We show in section 4 that the Cuntz semigroup is left-soft separative whenever the  $C^*$ -algebra has stable rank one or strict comparison of positive elements. Under these assumptions, we also show that  $\sigma$  is subadditive and, consequently, a generalized Cu-morphism; see Theorem 6.9. Then  $\text{Cu}(A)_{\text{soft}}$  is a retract of  $\text{Cu}(A)$ , as defined in [42].

Using structure results of retracts and soft elements, we study the covering dimension ([42]) and the radius of comparison ([13]) of  $C^*$ -algebras with the global Glimm property in terms of their soft elements.

**Theorem D (7.3).** *Let  $A$  be a  $C^*$ -algebra with the global Glimm property. Assume one of the following holds:*

- (i)  *$A$  has strict comparison of positive elements;*
- (ii)  *$A$  has stable rank one;*
- (iii)  *$A$  has topological dimension zero, and  $\text{Cu}(A)$  is left-soft separative.*

*Then  $\dim(\text{Cu}(A)_{\text{soft}}) \leq \dim(\text{Cu}(A)) \leq \dim(\text{Cu}(A)_{\text{soft}}) + 1$ .*

**Theorem E (8.6).** *Let  $A$  be a unital, separable  $C^*$ -algebra with the global Glimm property. Assume that  $A$  has stable rank one. Then*

$$\text{rc}(\text{Cu}(A), [1]) = \text{rc}(\text{Cu}(A)_{\text{soft}}, \sigma([1])).$$

We finish the paper with some applications of Theorems D and E to crossed products; see Theorem 7.9 and Example 8.9.

## 2. Preliminaries

In this section, we recall definitions and results that will be used in the sections that follow. The reader is referred to [8], [4] and [24] for an extensive introduction to the theory of Cu-semigroups and their interplay with Cuntz semigroups.

Given a  $C^*$ -algebra  $A$ , we use  $A_+$  to denote the set of its positive elements.

**2.1** (The Cuntz semigroup). Let  $A$  be a  $C^*$ -algebra. Given  $a, b \in A_+$ , one says that  $a$  is *Cuntz subequivalent* to  $b$ , written  $a \preceq b$  if there exists a sequence  $(v_n)_n$  in  $A$  such that  $a = \lim_n v_n b v_n^*$ . Further, one says that  $a$  is *Cuntz equivalent* to  $b$ , written  $a \sim b$ , if  $a \preceq b$  and  $b \preceq a$ .

The *Cuntz semigroup* of  $A$ , denoted by  $\text{Cu}(A)$ , is the positively ordered monoid defined as the quotient  $(A \otimes \mathcal{K})_+ / \sim$  equipped with the order induced by  $\preceq$  and the addition induced by addition of orthogonal elements. For further details, we refer to [4, 8, 24].

**2.2** (Cu-semigroups). Let  $(P, \leq)$  be a partially ordered set. Suppose that every increasing sequence in  $P$  has a supremum. Given two elements  $x, y$  in  $P$ , one says that  $x$  is *way below*  $y$ , denoted  $x \ll y$ , if for every increasing sequence  $(z_n)_n$  in  $P$  satisfying  $y \leq \sup_n z_n$ , there exists some  $m \in \mathbb{N}$  such that  $x \leq z_m$ .

As defined in [17], a *Cu-semigroup* is a positively ordered monoid  $S$  satisfying two domain-type conditions and two compatibility conditions:

- (O1) Every increasing sequence in  $S$  has a supremum.
- (O2) For every element  $x$  in  $S$ , there exists a sequence  $(x_n)_n$  in  $S$  such that  $x_0 \ll x_1 \ll x_2 \ll \dots$  and  $x = \sup_n x_n$ .
- (O3) The addition is compatible with the way-below relation, that is, for every  $x', x, y', y \in S$  satisfying  $x' \ll x$  and  $y' \ll y$ , we have  $x' + y' \ll x + y$ .
- (O4) The addition is compatible with suprema of increasing sequences, that is, for every increasing sequences  $(x_n)_n$  and  $(y_n)_n$  in  $S$ , we have

$$\sup_n (x_n + y_n) = \sup_n x_n + \sup_n y_n.$$

It follows from [17] that the Cuntz semigroup of any  $C^*$ -algebra always satisfies (O1)–(O4). Specifically, the Cuntz semigroup of any  $C^*$ -algebra is a Cu-semigroup.

Given a monoid morphism  $\varphi$  between two Cu-semigroups, we say that  $\varphi$  is a *Cu-morphism* if it preserves the order, suprema of increasing sequences and the way-below relation. A *generalized Cu-morphism* is a monoid map that preserves order and suprema of increasing sequences (but not necessarily the way-below relation).

The following properties, which will often be considered throughout the paper, are also satisfied in the Cuntz semigroup of any  $C^*$ -algebra; see [4, Proposition 4.6] and its

precursor [35, Lemma 7.1] for (O5), [32, Proposition 5.1.1] for (O6) and [1, Proposition 2.2] for (O7).

(O5) For every  $x, y, x', y', z \in S$  satisfying  $x + y \leq z$  and  $x' \ll x$  and  $y' \ll y$ , there exists  $c \in S$  such that  $y' \ll c$  and  $x' + c \leq z \leq x + c$ .

This property is often applied with  $y' = y = 0$ .

(O6) For every  $x, x', y, z \in S$  satisfying  $x' \ll x \ll y + z$ , there exist  $v, w \in S$  such that

$$v \leq x, y, \quad w \leq x, z, \quad \text{and} \quad x' \leq v + w.$$

(O7) For every  $x, x', y, y', w \in S$  satisfying  $x' \ll x \leq w$  and  $y' \ll y \leq w$ , there exists  $z \in S$  such that  $x', y' \ll z \leq w, x + y$ .

Given an element  $x$  in a Cu-semigroup, we denote by  $\infty x$  the supremum of the increasing sequence  $(nx)_n$ .

**2.3** (The Global Glimm Property and nowhere scatteredness). A  $C^*$ -algebra  $A$  is said to be *nowhere scattered* if no hereditary sub- $C^*$ -algebra of  $A$  has a nonzero one-dimensional representation. Equivalently,  $A$  is nowhere scattered if and only if  $A$  has no nonzero elementary ideal quotients; see [39, Definition A] and [39, Theorem 3.1].

We say that  $A$  has the *global Glimm property* (in the sense of [29, Definition 4.12]) if, for every  $a \in A_+$  and  $\varepsilon > 0$ , there exists a square-zero element  $r \in \overline{aAa}$  such that  $(a - \varepsilon)_+ \in \overline{\text{span}ArA}$ ; see [43, Section 3].

A  $C^*$ -algebra satisfying the global Glimm property is always nowhere scattered. The converse remains open and is known as the *global Glimm problem*. The problem has been answered affirmatively under the additional assumption of real rank zero ([21]) or stable rank one ([2]).

A Cu-semigroup is said to be  $(2, \omega)$ -divisible if, for every pair  $x', x \in S$  with  $x' \ll x$ , there exists  $y \in S$  such that  $2y \leq x$  and  $x' \leq \infty y$ ; see [33, Definition 5.1].

For a detailed study of the global Glimm problem and its relation with the Cuntz semigroup, we refer to [43]; see also [48]. Among other results, it follows from [43, Theorem 3.6] that a  $C^*$ -algebra  $A$  has the global Glimm property if and only if  $\text{Cu}(A)$  is  $(2, \omega)$ -divisible.

### 3. Soft operators and strongly soft Cuntz classes

In this section, we first recall the definitions of (completely) soft operators in  $C^*$ -algebras and of strongly soft elements in Cu-semigroups. We then connect these notions and show that, for a  $C^*$ -algebra  $A$  with the global Glimm property, an element in the Cuntz semigroup  $\text{Cu}(A)$  is strongly soft if and only if it has a soft representative; see Theorem 3.3 and Corollary 3.4.

As defined in [40, Definition 4.2], an element  $x$  in a Cu-semigroup  $S$  is *strongly soft* if for all  $x' \in S$  with  $x' \ll x$  there exists  $t \in S$  such that

$$x' + t \ll x, \quad \text{and} \quad x' \ll \infty t.$$

This notion of softness is stronger than the one considered in [4, Definition 5.3.1]. However, if  $S$  is residually stably finite, both notions agree; see [40, Proposition 4.6]. In particular, this applies to weakly cancellative Cu-semigroups (see Paragraph 4.1 below).

As mentioned in the introduction, a positive element  $a$  in a  $C^*$ -algebra  $A$  is said to be *soft* if its hereditary sub- $C^*$ -algebra has no nonzero unital quotients. This definition can be seen as a generalization of *pure positivity*, a notion introduced in [30, Definition 2.1] for simple  $C^*$ -algebras. An element  $a \in A_+$  is said to be *completely soft* if  $(a - \varepsilon)_+$  is soft for every  $\varepsilon > 0$ , where  $(a - \varepsilon)_+$  denotes the ‘cut down’ of  $a$  given by applying functional calculus to  $a$  with the function  $f(t) = \max\{t - \varepsilon, 0\}$ .

As in [40, Definition 5.2], we say that a  $C^*$ -algebra  $A$  has an *abundance of soft elements* if, for every  $a \in A_+$  and  $\varepsilon > 0$ , there exists a positive, soft element  $b \in \overline{aAa}$  such that  $(a - \varepsilon)_+ \in \overline{\text{span}AbA}$ . By [40, Proposition 7.7], any  $C^*$ -algebra with the global Glimm property has an abundance of soft elements.

If  $a \in A_+$  is soft, then its Cuntz class  $[a]$  is strongly soft; see [40, Proposition 4.16]. Conversely, we prove in Theorem 3.3 below that if  $A$  has an abundance of soft elements (in particular, if  $A$  has the global Glimm property), then every strongly soft Cuntz class arises this way, that is, a Cuntz class  $[b] \in \text{Cu}(A)$  is strongly soft if and only if there exists a soft element  $a \in (A \otimes \mathbb{K})_+$  with  $b \sim a$ . It remains unclear if this also holds for general  $C^*$ -algebras; see [40, Question 4.17].

Given  $a, b \in A_+$ , we will write  $a \triangleleft b$  whenever  $a \in \overline{\text{span}AbA}$ . We say that two positive elements  $a$  and  $b$  in a  $C^*$ -algebra are *orthogonal* if  $ab = 0$ .

The next result is the  $C^*$ -algebraic analog of [40, Theorem 4.14(2)].

**Proposition 3.1.** *Let  $a$  and  $b$  be orthogonal positive elements in a  $C^*$ -algebra such that  $a \triangleleft b$  and such that  $b$  is soft. Then  $a + b$  is soft.*

**Proof.** By [40, Proposition 3.6], a positive element  $c$  in a  $C^*$ -algebra is soft if and only if for every  $\varepsilon > 0$  there exists  $r \in \overline{(cAc)_+}$  such that  $r$  is orthogonal to  $(c - \varepsilon)_+$  and such that  $c \triangleleft r$ . Using this characterization for  $b$ , we show that it is satisfied for  $a + b$ .

To verify that  $a + b$  is soft, let  $\varepsilon > 0$ . Using that  $b$  is soft, we obtain  $r \in \overline{(bAb)_+}$  such that  $r$  is orthogonal to  $(b - \varepsilon)_+$  and such that  $b \triangleleft r$ . Since  $a$  and  $b$  are orthogonal, we have

$$((a + b) - \varepsilon)_+ = (a - \varepsilon)_+ + (b - \varepsilon)_+.$$

Since  $r$  belongs to  $\overline{bAb}$ , it is also orthogonal to  $a$ , and thus also orthogonal to  $((a + b) - \varepsilon)_+$ . Further, we have  $a + b \triangleleft b \triangleleft r$ , as desired. □

**Lemma 3.2.** *Let  $A$  be a  $C^*$ -algebra with an abundance of soft elements, let  $a \in A_+$  be such that  $x := [a] \in \text{Cu}(A)$  is strongly soft, and let  $x' \in \text{Cu}(A)$  satisfy  $x' \ll x$ . Then there exists a positive, completely soft element  $b \in \overline{aAa}$  such that*

$$x' \ll [b] \ll x.$$

**Proof.** Choose  $x'' \in \text{Cu}(A)$  such that  $x' \ll x'' \ll x$ . Using that  $x$  is strongly soft, we know that there exists  $t \in \text{Cu}(A)$  such that  $x'' \ll \infty t$  and  $x'' + t \ll x$ . Choose orthogonal positive elements  $c, d \in A \otimes \mathbb{K}$  and  $\varepsilon > 0$  such that

$$x'' = [c], \quad t = [d], \quad x' \ll [(c - \varepsilon)_+], \quad \text{and} \quad x'' \ll \infty[(d - \varepsilon)_+].$$

Using that  $c + d \lesssim a$ , we can apply Rørdam’s lemma (see, for example, [36, Theorem 2.30]) to obtain  $x \in A \otimes \mathbb{K}$  such that

$$((c + d) - \varepsilon)_+ = xx^*, \text{ and } x^*x \in \overline{aAa}.$$

Set

$$c' := x^*(c - \varepsilon)_+x, \text{ and } d' := x^*(d - \varepsilon)_+x.$$

Then  $c', d' \in \overline{aAa}$ . Since  $c$  and  $d$  are orthogonal, we have

$$((c + d) - \varepsilon)_+ = (c - \varepsilon)_+ + (d - \varepsilon)_+.$$

It follows that  $c'$  and  $d'$  are orthogonal and that  $c' \sim (c - \varepsilon)_+$  and  $d' \sim (d - \varepsilon)_+$ .

In particular, we have  $x'' \ll \infty[(d - \varepsilon)_+] = \infty[d']$ , and we obtain  $\delta > 0$  such that  $x'' \ll \infty[(d' - \delta)_+]$ . Applying that  $A$  has an abundance of soft elements for  $d'$  and  $\delta$ , we obtain a soft element  $e \in \overline{d'Ad'}$  such that  $(d' - \delta)_+ \triangleleft e$ . Since  $c'$  and  $d'$  are orthogonal, and  $e$  belongs to  $\overline{d'Ad'}$ , it follows that  $c'$  and  $e$  are orthogonal.

Using that positive elements  $g, h$  in a C\*-algebra satisfy  $g \triangleleft h$  if and only if  $[g] \leq \infty[h]$ , we have

$$[c'] = [(c - \varepsilon)_+] \leq [c] = x'' \leq \infty[(d' - \delta)_+] \leq \infty[e]$$

and thus  $c' \triangleleft e$ . By Proposition 3.1,  $c' + e$  is soft.

Note that  $c'$  and  $e$  belong to  $\overline{aAa}$ . In particular,  $c' + e$  belongs to  $A_+$ , and we can apply [40, Theorem 6.9] to obtain a completely soft element  $f \in A_+$  such that  $fAf = \overline{(c' + e)A(c' + e)} \subseteq \overline{aAa}$ . Then  $f \in \overline{aAa}$ , and therefore  $[f] \leq [a] = x$ . Further, we have

$$x' \ll [(c - \varepsilon)_+] = [c'] \leq [c' + e] = [f].$$

Choose  $\delta > 0$  such that

$$x' \ll [(f - \delta)_+],$$

and set  $b := (f - \delta)_+$ . Since cut downs of  $(f - \delta)_+$  are also cut downs of  $f$ , we see that  $b$  is completely soft. Further, we have

$$x' \ll [b] = [(f - \delta)_+] \ll [f] \leq x,$$

which shows that  $b$  has the desired properties. □

A unital C\*-algebra is said to have *stable rank one* if its invertible elements are norm dense, and a general C\*-algebra is said to have stable rank one if its minimal unitization does; see [12, Section V.3.1].

A C\*-algebra is said to have *weak stable rank one* if  $A \subseteq \overline{\text{Gl}(\tilde{A})}$ . Any stable C\*-algebra has weak stable rank one; see [13, Lemma 4.3.2].

**Theorem 3.3.** *Let  $A$  be a C\*-algebra with an abundance of soft elements, and let  $a \in A_+$  be such that  $[a] \in \text{Cu}(A)$  is strongly soft. Then there exists a sequence  $(a_n)_n$  of completely soft elements in  $(\overline{aAa})_+$  such that  $([a_n])_n$  in  $\text{Cu}(A)$  is  $\ll$ -increasing with  $[a] = \sup_n [a_n]$ .*



If, moreover,  $A$  has weak stable rank one, then  $[a]$  is strongly soft if and only if there exists a completely soft element  $b \in A_+$  such that  $[a] = [b]$ .

**Proof.** Choose a  $\ll$ -increasing sequence  $(x_n)_n$  in  $\text{Cu}(A)$  with supremum  $[a]$ . We will inductively choose completely soft elements  $a_n \in (\overline{aAa})_+$  such that

$$x_n \ll [a_n] \ll [a], \text{ and } [a_n] \ll [a_{n+1}]$$

for  $n \in \mathbb{N}$ . To start, apply Lemma 3.2 for  $x_0 \ll [a]$  to obtain a completely soft element  $a_0 \in (\overline{aAa})_+$  such that  $x_0 \ll [a_0] \ll [a]$ . Assuming we have chosen  $a_0, \dots, a_n$ , find  $x'_n \in \text{Cu}(A)$  such that  $[a_n], x_n \ll x'_n \ll [a]$ . Applying Lemma 3.2 for  $x'_n \ll [a]$ , we obtain a completely soft element  $a_{n+1} \in (\overline{aAa})_+$  such that  $x'_n \ll [a_{n+1}] \ll [a]$ . Proceeding inductively, we obtain the desired sequence  $(a_n)_n$ .

Next, assume that  $A$  has weak stable rank one. By [40, Proposition 4.16], soft operators have strongly soft Cuntz classes. Conversely, assuming that  $[a]$  is strongly soft, we will show that  $[a] = [b]$  for some completely soft element  $b \in A_+$ .

Let  $(a_n)_n$  be as above. We will show that  $\sup_n [a_n]$  (which is  $[a]$ ), has a soft representative. Given  $c, d \in A_+$ , we will write  $c \sim_u d$  if there exists a unitary  $u \in \tilde{A}$  such that  $c = udu^*$ , and we write  $c \subseteq d$  if  $\overline{cAc} \subseteq \overline{dAd}$ .

Using [36, §2.5], one can find a sequence  $(\delta_n)_n$  in  $(0, \infty)$  and a sequence of contractive elements  $(b_n)_n$  in  $A_+$  such that

$$\begin{array}{ccccccc} a_1 & \rightsquigarrow & a_2 & \rightsquigarrow & a_3 & \rightsquigarrow & \dots \\ \leq & & \leq & & \leq & & \\ (a_1 - \delta_1)_+ & & (a_2 - \delta_2)_+ & & (a_3 - \delta_3)_+ & & \dots \\ \sim_u & & \sim_u & & \sim_u & & \\ b_1 & \subseteq & b_2 & \subseteq & b_3 & \subseteq & \dots \end{array}$$

and, setting  $b_\infty := \sum_n \frac{1}{2^n \|b_n\|} b_n$ , such that  $[b_\infty] = \sup_n [a_n]$ .

For each  $n \in \mathbb{N}$ , since  $a_n$  is completely soft, so is the element  $(a_n - \delta_n)_+$ . Since  $(a_n - \delta_n)_+$  and  $b_n$  are unitarily equivalent, they generate  $*$ -isomorphic hereditary sub- $C^*$ -algebras of  $A$ , and it follows that  $b_n$  is completely soft as well.

Further, since  $b_0 \subseteq b_1 \subseteq \dots$  and  $b_\infty = \sum_n \frac{1}{2^n \|b_n\|} b_n$ , the sequence of hereditary sub- $C^*$ -algebras  $\overline{b_n A b_n}$  is increasing with  $\overline{b_\infty A b_\infty} = \bigcup_n \overline{b_n A b_n}$ . Since each  $\overline{b_n A b_n}$  has no nonzero unital quotients, it follows from [40, Proposition 2.17] that neither does  $\overline{b_\infty A b_\infty}$ . This proves that  $b_\infty$  is soft.

Note that  $b_\infty$  belongs to  $A_+$ . Applying [40, Theorem 6.9], we obtain a completely soft element  $b \in A_+$  such that  $\overline{bAb} = \overline{b_\infty A b_\infty}$ . Then  $[b] = [b_\infty] = [a]$ , as desired.  $\square$

**Corollary 3.4.** *Let  $A$  be a  $C^*$ -algebra with the global Glimm property, and let  $x \in \text{Cu}(A)$ . Then  $x$  is strongly soft if and only if there exists a soft element  $a \in (A \otimes \mathbb{K})_+$  with  $x = [a]$ .*

**Proof.** It follows from [43, Theorem 3.6] that  $A \otimes \mathbb{K}$  has the global Glimm property. Hence,  $A \otimes \mathbb{K}$  has an abundance of soft elements by [43, Proposition 7.7]. Further,  $A \otimes \mathbb{K}$  has weak stable rank one by [13, Lemma 4.3.2]. Now, the result follows from Theorem 3.3.  $\square$

**3.5** (The strongly soft subsemigroup). Given a Cu-semigroup  $S$ , we let  $S_{\text{soft}}$  denote the set of strongly soft elements in  $S$ . By Corollary 3.4, given a  $C^*$ -algebra  $A$  with the global Glimm property, we have

$$\text{Cu}(A)_{\text{soft}} = \{[a] : a \in (A \otimes \mathbb{K})_+ \text{ soft}\}.$$

In particular, if  $A$  is stably finite, simple and unital, it follows from [40, Proposition 4.16] that the subset  $\text{Cu}(A)_{\text{soft}} \setminus \{0\}$  coincides with  $\text{Cu}_+(A)$ , the set of Cuntz classes of purely positive elements as introduced in [30, Definition 2.1]; see also [11, Definition 3.8].

Given a Cu-semigroup  $S$ , a *sub-Cu-semigroup* in the sense of [38, Definition 4.1] is a submonoid  $T \subseteq S$  that is a Cu-semigroup for the inherited order and such that the inclusion map  $T \rightarrow S$  is a Cu-morphism.

**Proposition 3.6.** *Let  $S$  be a  $(2, \omega)$ -divisible Cu-semigroup that satisfies (O5). Then,  $S_{\text{soft}}$  is a sub-Cu-semigroup that also satisfies (O5).*

*If  $S$  also satisfies (O6) (respectively (O7)), then so does  $S_{\text{soft}}$ .*

**Proof.** By [40, Proposition 7.7], if a Cu-semigroup is  $(2, \omega)$ -divisible and satisfies (O5), then it has an abundance of soft elements, which then by [40, Proposition 5.6] implies that its strongly soft elements form a sub-Cu-semigroup. Thus,  $S_{\text{soft}}$  is a sub-Cu-semigroup.

Let us verify that  $S_{\text{soft}}$  satisfies (O5). By [4, Theorem 4.4(1)] it suffices to show that for all  $x', x, y', y, z', z \in S_{\text{soft}}$  satisfying

$$x' \ll x, \quad y' \ll y, \quad \text{and} \quad x + y \ll z' \ll z, \tag{1}$$

there exist  $c', c \in S_{\text{soft}}$  such that

$$x' + c \ll z, \quad z' \ll x + c', \quad \text{and} \quad y' \ll c' \ll c. \tag{2}$$

So let  $x', x, y', y, z', z \in S_{\text{soft}}$  satisfy Equation (1). Choose  $v', v \in S_{\text{soft}}$  such that

$$z' \ll v' \ll v \ll z.$$

Applying (O5), we obtain  $b \in S$  such that

$$x' + b \leq v' \leq x + b, \quad \text{and} \quad y' \ll b.$$

Using that  $v' \ll v$  and that  $v$  is strongly soft, we apply [40, Proposition 4.13] to find  $t \in S_{\text{soft}}$  such that  $v' + t \leq v \leq \infty t$ . Set  $c := b + t$ . Since  $b \leq v' \leq v \leq \infty t$  and  $t$  is strongly soft, we have  $c \in S_{\text{soft}}$  by [40, Theorem 4.14(2)]. Thus, one gets

$$x' + c = x' + b + t \leq v' + t \leq v \ll z,$$

and

$$z' \ll v' \leq x + b \leq x + c, \quad \text{and} \quad y' \ll b \leq c.$$

Using also that  $S_{\text{soft}}$  is a Cu-semigroup and  $c \in S_{\text{soft}}$ , we can find  $c' \in S_{\text{soft}}$  such that

$$c' \ll c, \quad z' \ll x + c', \quad \text{and} \quad y' \ll c'.$$

This shows that  $c'$  and  $c$  satisfy Equation (2), as desired.

That  $S_{\text{soft}}$  satisfies (O6) (respectively (O7)) whenever  $S$  does is proven analogously.  $\square$

#### 4. Separative Cu-semigroups

We introduce in Definition 4.2 the notion of left-soft separativity, a weakening of weak cancellation (Paragraph 4.1) that is satisfied in the Cuntz semigroup of every  $C^*$ -algebra with stable rank one or strict comparison of positive elements; see Proposition 4.3 and Proposition 4.8, respectively. We also prove in Proposition 4.6 that, among strongly soft elements, the notions of unperforation and almost unperforation coincide.

**4.1** (Cuntz semigroups of stable rank one  $C^*$ -algebras). Let  $A$  be a stable rank one  $C^*$ -algebra. As shown in [35, Theorem 4.3], the Cuntz semigroup  $\text{Cu}(A)$  satisfies a cancellation property termed *weak cancellation*: If  $x, y, z \in \text{Cu}(A)$  satisfy  $x + z \ll y + z$ , then  $x \ll y$ .

If  $A$  is also separable, then  $\text{Cu}(A)$  is *inf-semilattice ordered*, that is, for every pair of elements  $x, y \in \text{Cu}(A)$  their infimum  $x \wedge y$  exists, and for every  $x, y, z \in \text{Cu}(A)$  one has  $(x + z) \wedge (y + z) = (x \wedge y) + z$ ; see [2, Theorem 3.8].

As defined in [41], a Cu-semigroup is *separative* if  $x \ll y$  whenever  $x + t \ll y + t$  with  $t \ll \infty x, \infty y$ . This and other cancellation properties will be studied in more detail in [41].

For the results in this paper, we will need the following tailored definition:

**Definition 4.2.** We say that a Cu-semigroup  $S$  is *left-soft separative* if, for any triple of elements  $y, t \in S$  and  $x \in S_{\text{soft}}$  satisfying

$$x + t \ll y + t, \quad t \ll \infty x, \quad \text{and} \quad t \ll \infty y,$$

we have  $x \ll y$ .

**Proposition 4.3.** *Every weakly cancellative Cu-semigroup is separative, and every separative Cu-semigroup is left-soft separative.*

*In particular, the Cuntz semigroup of every stable rank one  $C^*$ -algebra is left-soft separative.*

**Proof.** It follows directly from the definitions that weak cancellation is stronger than left-soft separativity. By [35, Theorem 4.3], the Cuntz semigroup of a stable rank one  $C^*$ -algebra is weakly cancellative.  $\square$

**Lemma 4.4.** *Let  $S$  be a  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5). Then,  $S$  is left-soft separative if and only if for all  $y, t', t \in S$  and  $x \in S_{\text{soft}}$  satisfying*

$$x + t \leq y + t', \quad t' \ll t, \quad t' \ll \infty y, \quad \text{and} \quad t' \ll \infty x,$$

*we have  $x \leq y$ .*

**Proof.** The backwards implication is straightforward to verify and even holds for general Cu-semigroups. To show the forward implication, assume that  $S$  is left-soft separative, and let  $x, y, t', t \in S$  as in the statement. By Proposition 3.6, we know that  $S_{\text{soft}}$  is a sub-Cu-semigroup. In particular,  $x$  can be written as the supremum of a  $\ll$ -increasing sequence of strongly soft elements.

Take  $x' \in S_{\text{soft}}$  such that  $x' \ll x$ . We have

$$x' + t' \ll x + t \leq y + t', \quad t' \ll \infty x, \quad \text{and} \quad t' \ll \infty y.$$

By left-soft separativity, we deduce  $x' \ll y$ . Since  $x$  is the supremum of such  $x'$ , one gets  $x \leq y$ , as required. □

**Lemma 4.5.** *Let  $S$  be a left-soft separative,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5), and let  $x, t \in S_{\text{soft}}$  and  $y, t' \in S$  satisfy*

$$x + t \leq y + t', \quad t' \ll t, \quad t' \ll \infty y.$$

*Then  $x \leq y$ .*

**Proof.** Take  $t'' \in S$  such that  $t' \ll t'' \ll t$ . Using that  $t$  is strongly soft, one finds  $s \in S_{\text{soft}}$  such that  $t'' + s \leq t \leq \infty s$ ; see [40, Proposition 4.13]. Note that, since  $x$  and  $s$  are strongly soft, so is  $x + s$  by [40, Theorem 4.14]. We get

$$(x + s) + t'' = x + (s + t'') \leq x + t \leq y + t'.$$

Further, we have  $t' \ll \infty y$  and  $t' \ll t'' \leq \infty s \leq \infty(x + s)$ .

An application of Lemma 4.4 shows that  $x + s \leq y$  and, therefore, that  $x \leq y$ . □

The following result shows that three different versions of unperforation coincide for the semigroup of strongly soft elements in a Cu-semigroup. Given elements  $x$  and  $y$  in a partially ordered monoid, one writes  $x <_s y$  if there exists  $n \geq 1$  such that  $(n + 1)x \leq ny$ , and one writes  $x \leq_p y$  if there exists  $n_0 \in \mathbb{N}$  such that  $nx \leq ny$  for all  $n \geq n_0$ . We refer to [4, Chapter 5] for details regarding these definitions.

**Proposition 4.6.** *Let  $S$  be a Cu-semigroup. The following are equivalent:*

- (1)  $S_{\text{soft}}$  is unperforated: If  $x, y \in S_{\text{soft}}$  and  $n \geq 1$  satisfy  $nx \leq ny$ , then  $x \leq y$ .
- (2)  $S_{\text{soft}}$  is nearly unperforated: If  $x, y \in S_{\text{soft}}$  satisfy  $x \leq_p y$ , then  $x \leq y$ .
- (3)  $S_{\text{soft}}$  is almost unperforated: If  $x, y \in S_{\text{soft}}$  satisfy  $x <_s y$ , then  $x \leq y$ .

**Proof.** In general, (1) implies (2), which implies (3); see [4, Proposition 5.6.3]. To verify that (3) implies (1), let  $x, y \in S_{\text{soft}}$  and  $n \geq 1$  satisfy  $nx \leq ny$ . Then  $\widehat{x} \leq \widehat{y}$ ; see Paragraph 5.1. By [40, Proposition 4.5],  $x$  is functionally soft. Thus, we deduce from [4, Theorem 5.3.12] that  $x \leq y$ , as desired. □

**Lemma 4.7.** *Every almost unperforated Cu-semigroup satisfying (O5) is left-soft separative.*

**Proof.** Let  $S$  be an almost unperforated Cu-semigroup satisfying (O5). To verify that  $S$  is left-soft separative, let  $y, t \in S$  and  $x \in S_{\text{soft}}$  satisfy  $x + t \ll y + t$  and  $t \ll \infty x, \infty y$ . Choose  $y' \in S$  such that

$$x + t \ll y' + t, \quad t \ll \infty y', \quad \text{and} \quad y' \ll y.$$

Then  $x \leq_p y'$  by [4, Proposition 5.6.8(ii)]. In particular, there exists  $k \in \mathbb{N}$  such that  $kx \leq ky'$ , and thus  $\widehat{x} \leq \widehat{y}'$ ; see Paragraph 5.1. By [40, Proposition 4.5],  $x$  is

functionally soft. Using that  $S$  is almost unperforated, we obtain that  $x \leq y' \ll y$ , by [4, Theorem 5.3.12].  $\square$

A  $C^*$ -algebra  $A$  is said to have *strict comparison of positive elements* if, for all  $a, b \in (A \otimes \mathbb{K})_+$  and some  $\varepsilon > 0$ , one has that  $d_\tau(a) \leq (1 - \varepsilon)d_\tau(b)$  for all  $\tau \in \text{QT}(A)$  implies  $a \precsim b$ .

**Proposition 4.8.** *Let  $A$  be a  $C^*$ -algebra with strict comparison of positive elements. Then  $\text{Cu}(A)$  is left-soft separative.*

**Proof.** A  $C^*$ -algebra has strict comparison of positive elements if and only if its Cuntz semigroup is almost unperforated; see [20, Proposition 6.2]. Since every Cuntz semigroup satisfies (O5), the result follows from Lemma 4.7.  $\square$

Since every  $\mathcal{Z}$ -stable  $C^*$ -algebra has strict comparison of positive elements (see [34, Theorem 4.5]), one gets the following:

**Corollary 4.9.** *The Cuntz semigroup of every  $\mathcal{Z}$ -stable  $C^*$ -algebra is left-soft separative.*

### 5. Ranks and soft elements

Given a  $(2, \omega)$ -divisible Cu-semigroup  $S$  satisfying (O5)–(O7) (for example, the Cuntz semigroup of a  $C^*$ -algebra with the global Glimm property) and an element  $x \in S$ , we show in Theorem 5.10 that there exists a strongly soft element  $w$  below  $x$  which agrees with  $x$  at the level of functionals, that is, the rank of  $x$  coincides with the rank of  $w$ ; see Paragraph 5.1. Paired with Theorem 3.3, this implies that the rank of any positive element in a  $C^*$ -algebra satisfying the global Glimm property is the rank of a soft element (Theorem 5.11).

Using Theorem 5.10, we also prove that  $F(S)$ , the set of functionals on  $S$ , is homeomorphic to  $F(S_{\text{soft}})$ ; see Theorem 5.14.

**5.1 (Functionals and ranks).** Given a Cu-semigroup  $S$ , we will denote by  $F(S)$  the set of its *functionals*, that is to say, the set of monoid morphisms  $S \rightarrow [0, \infty]$  that preserve the order and suprema of increasing sequences. If  $S$  satisfies (O5), then  $F(S)$  becomes a compact, Hausdorff space – and even an algebraically ordered compact cone [1, Section 3] – when equipped with a natural topology [20, 28, 32].

Given a  $C^*$ -algebra, the cone  $\text{QT}(A)$  of lower-semicontinuous 2-quasitraces on  $A$  is naturally isomorphic to  $F(\text{Cu}(A))$ , as shown in [20, Theorem 4.4].

We let  $\text{LAff}(F(S))$  denote the monoid of lower-semicontinuous, affine functions  $F(S) \rightarrow (-\infty, \infty]$ , equipped with pointwise order and addition. For  $x \in S$ , the *rank* of  $x$  is defined as the map  $\hat{x}: F(S) \rightarrow [0, \infty]$  given by

$$\hat{x}(\lambda) := \lambda(x)$$

for  $\lambda \in F(S)$ . The function  $\hat{x}$  belongs to  $\text{LAff}(F(S))$  and the *rank problem* of determining which functions in  $\text{LAff}(F(S))$  arise this way has been studied extensively in [37] and [2].

Sending an element  $x \in S$  to its rank  $\hat{x}$  defines a monoid morphism from  $S$  to  $\text{LAff}(F(S))$  which preserves both the order and suprema of increasing sequences.

**Lemma 5.2.** *Let  $S$  be a  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5), and let  $u \in S_{\text{soft}}$  and  $u', x \in S$  be such that*

$$u' \ll u \ll x.$$

*Then, there exists  $c \in S_{\text{soft}}$  satisfying*

$$u' + 2c \leq x \leq \infty c.$$

**Proof.** Let  $u'' \in S$  be such that  $u' \ll u'' \ll u$ . By [40, Proposition 4.13], there exists  $s \in S$  satisfying

$$u'' + s \leq u \leq \infty s.$$

Since  $u'' \ll u \leq \infty s$ , there exists  $s' \in S$  such that

$$s' \ll s, \text{ and } u'' \ll \infty s'.$$

We have

$$u'' + s \leq x, \quad u' \ll u'', \text{ and } s' \ll s.$$

Applying (O5), we obtain  $d \in S$  such that  $u' + d \leq x \leq u'' + d$  with  $s' \leq d$ . Since  $u'' \leq \infty s'$ , it follows that  $x \leq \infty d$ . Finally, apply [40, Proposition 7.7] to  $d$  in order to obtain  $c \in S_{\text{soft}}$  such that  $2c \leq d \leq \infty c$ . This element satisfies the required conditions.  $\square$

A Cu-semigroup  $S$  is said to be *countably based* if it contains a countable subset  $D \subseteq S$  such that every element in  $S$  can be written as the supremum of an increasing sequence of elements in  $D$ . Separable  $C^*$ -algebras have countably based Cuntz semigroups; see, for example, [3].

**Lemma 5.3.** *Let  $S$  be a countably based,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7), and let  $x \in S$ . Consider the set*

$$L_x := \{u' \in S : u' \ll u \ll x \text{ for some } u \in S_{\text{soft}}\}.$$

*Then, for every  $k \in \mathbb{N}$ ,  $x' \in S$  such that  $x' \ll x$ , and  $u', v' \in L_x$ , there exists a strongly soft element  $w' \in L_x$  such that*

$$u' \ll w', \quad x' \ll \infty w', \text{ and } \frac{k}{k+1} \widehat{v'} \leq \widehat{w'} \text{ in } \text{LAff}(F(S)).$$

*If, additionally,  $S$  is left-soft separative,  $w'$  may be chosen such that  $v' \ll w'$ .*

**Proof.** Let  $u', v' \in L_x$ , let  $x' \in S$  satisfy  $x' \ll x$ , and let  $k \in \mathbb{N}$ . By definition, there exist  $u, v \in S_{\text{soft}}$  such that

$$u' \ll u \ll x, \text{ and } v' \ll v \ll x.$$

Choose  $y', y \in S$  such that

$$x' \ll y' \ll y \ll x, \quad v \ll y', \text{ and } u \ll y'.$$

Using that  $S_{\text{soft}}$  is a sub-Cu-semigroup by Proposition 3.6, we can choose elements  $u'', u''', v'' \in S_{\text{soft}}$  such that

$$u' \ll u'' \ll u''' \ll u, \text{ and } v' \ll v'' \ll v.$$

Applying Lemma 5.2 for  $u''' \ll u \ll y$  and  $v'' \ll v \ll y$ , we obtain  $c, d \in S_{\text{soft}}$  such that

$$u''' + c \leq y \leq \infty c, \text{ and } v'' + 2d \leq y \leq \infty d.$$

Then, applying [43, Proposition 4.10] for  $y' \ll y \leq \infty c, \infty d$ , we get  $e \in S$  such that

$$y' \ll \infty e, \text{ and } e \ll c, d.$$

By [40, Proposition 7.7], there exists a strongly soft element  $e_0$  such that  $e_0 \leq e \leq \infty e_0$ . Replacing  $e$  by  $e_0$ , we may assume that  $e \in S_{\text{soft}}$ . Using again that  $S_{\text{soft}}$  is a sub-Cu-semigroup, we can find  $e', e'' \in S_{\text{soft}}$  satisfying

$$y' \ll \infty e', \text{ and } e' \ll e'' \ll e.$$

By [40, Proposition 4.13], there exists  $r \in S$  such that

$$e'' + r \leq e \leq \infty r.$$

Since  $e'' \ll e$ , we can find  $r' \in S$  such that

$$r' \ll r, \text{ and } e'' \leq \infty r'.$$

Thus, one has

$$e'' + (r + u''') \leq e + u''' \leq c + u''' \leq y, \quad e' \ll e'', \text{ and } r' + u'' \ll r + u''.$$

Applying (O5), we obtain  $z \in S$  such that

$$e' + z \leq y \leq e'' + z, \text{ and } r' + u'' \leq z.$$

Using again that  $S_{\text{soft}}$  is a sub-Cu-semigroup, choose  $d' \in S_{\text{soft}}$  such that

$$e \ll d' \ll d.$$

We have

$$(v'' + d) + d = v'' + 2d \leq y \leq z + e'' \leq z + d', \tag{3}$$

with  $v'' + d \in S_{\text{soft}}$ . Note that

$$d' \ll d \leq \infty(v'' + d), \text{ and } d' \ll d \leq y \leq z + e'' \leq z + \infty r' \leq \infty z.$$

In particular, since  $d' \ll \infty z$ , there exists  $M \in \mathbb{N}$  such that  $d' \leq Mz$ . Set

$$l := \infty(u'' + v''), \text{ and } w := e' + (z \wedge l),$$

where  $z \wedge l$  exists because  $l$  is idempotent, and  $S$  is countably based and satisfies (O7); see [1, Theorem 2.4].

Note that, since  $l \leq \infty y' \leq \infty e'$  and  $e' \in S_{\text{soft}}$ , it follows from [40, Theorem 4.14] that  $w \in S_{\text{soft}}$ . We get

$$w \leq e' + z \leq y \ll x, \quad x' \ll y' \leq \infty e' \leq \infty w, \text{ and } u' \ll u'' \leq z \wedge l \leq w.$$

By [1, Theorem 2.5], the map  $S \rightarrow S, s \mapsto s \wedge l$ , is additive. Using this at the second and fourth step, we get

$$\begin{aligned} v'' + 2(d' \wedge l) &= (v'' \wedge l) + 2(d' \wedge l) = (v'' + 2d') \wedge l \\ &\leq (z + d') \wedge l = (z \wedge l) + (d' \wedge l) \leq w + (d' \wedge l). \end{aligned}$$

We also have  $d' \wedge l \leq (Mz) \wedge l = M(z \wedge l) \leq Mw$ , and this implies that

$$\widehat{v''} \leq \widehat{w}.$$

Now, since  $v' \ll v''$  and  $\frac{k}{k+1} < 1$ , we can apply [32, Lemma 2.2.5] to obtain

$$\frac{k}{k+1} \widehat{v'} \ll \widehat{v''} \leq \widehat{w}.$$

Since  $w$  is strongly soft and  $S_{\text{soft}}$  is a sub-Cu-semigroup, there exists a  $\ll$ -increasing sequence of soft elements with supremum  $w$ . Using that the rank map  $x \mapsto \widehat{x}$  preserves suprema of increasing sequences, we can find  $w' \in S_{\text{soft}}$  such that

$$w' \ll w, \quad \frac{k}{k+1} \widehat{v'} \leq \widehat{w'}, \quad x' \ll \infty w', \quad \text{and} \quad u' \ll w'.$$

Further, we have  $w' \ll w \ll x$ . This shows that  $w'$  is a strongly soft element in  $L_x$ , as desired.

If, additionally,  $S$  is left-soft separative, we can apply Lemma 4.4 on (3) to obtain that  $v'' + d \leq z$ , and so  $v'' \leq z$ . We also have  $v'' \leq l$  and thus

$$v' \ll v'' \leq z \wedge l \leq w.$$

We also have  $u' \ll u'' \leq w$  and  $x' \ll \infty w$ . Using that  $w$  is strongly soft and that  $S_{\text{soft}}$  is a sub-Cu-semigroup, we can find  $w' \in S_{\text{soft}}$  such that  $u', v' \ll w' \ll w$  and  $x' \ll \infty w'$ . Then  $w'$  has the desired properties.  $\square$

**Remark 5.4.** The assumption of  $S$  being countably based in Lemma 5.3 is only used to prove the existence of the infimum  $z \wedge l$ . If  $S$  is the Cuntz semigroup of a C\*-algebra, this infimum always exists; see [16]. Thus, the first part of Lemma 5.3 holds for every C\*-algebra with the global Glimm property.

**Proposition 5.5.** *Let  $S$  be a countably based,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7), let  $x', x \in S$  with  $x' \ll x$ , let  $k \in \mathbb{N}$  and let  $u' \in L_x$ . Then, for every finite subset  $C \subseteq L_x$ , there exists a strongly soft element  $w' \in L_x$  such that*

$$u' \ll w', \quad x' \ll \infty w', \quad \text{and} \quad \frac{k}{k+1} \widehat{v'} \leq \widehat{w'} \text{ in } \text{LAff}(F(S))$$

for every  $v' \in C$ .

**Proof.** We will prove the result by induction on  $|C|$ , the size of  $C$ . If  $|C| = 1$ , the result follows from Lemma 5.3.



Thus, fix  $n \in \mathbb{N}$  with  $n \geq 2$ , and assume that the result holds for any finite subset of  $n - 1$  elements. Given  $C \subseteq L_x$  with  $|C| = n$ , pick some  $v_0 \in C$ . Applying the induction hypothesis, we get an element  $w'' \in L_x$  such that

$$u' \ll w'', \quad x' \ll \infty w'', \quad \text{and} \quad \frac{k}{k+1} \widehat{v}' \leq \widehat{w}''$$

for every  $v' \in C \setminus \{v_0\}$ .

Now, applying Lemma 5.3 to  $x'$ ,  $w''$  and  $v_0$ , we get a strongly soft element  $w' \in L_x$  such that

$$w'' \ll w', \quad x' \ll \infty w', \quad \text{and} \quad \frac{k}{k+1} \widehat{v}_0 \leq \widehat{w}'.$$

Then  $\widehat{w}'' \leq \widehat{w}'$ , which shows that  $w'$  satisfies the required conditions. □

**Proposition 5.6.** *Let  $S$  be a countably based,  $(2, \omega)$ -divisible Cu-semigroup satisfying  $(O5)$ – $(O7)$ , let  $x \in S$  and let  $u' \in L_x$ . Then there exists  $w \in S_{\text{soft}}$  such that*

$$u' \ll w \leq x \leq \infty w, \quad \text{and} \quad \lambda(w) = \sup_{v' \in L_x} \lambda(v'),$$

for every  $\lambda \in F(S)$ .

**Proof.** By definition of  $L_x$ , we obtain  $u \in S_{\text{soft}}$  such that  $u' \ll u \ll x$ . Let  $(x_n)_n$  be a  $\ll$ -increasing sequence with supremum  $x$  and such that  $u \ll x_0$ . Note that the sets  $L_{x_n}$  form an increasing sequence of subsets of  $S$  with  $L_x = \bigcup_n L_{x_n}$ .

Let  $B$  be a countable basis for  $S$ . Then

$$B \cap L_x = \bigcup_n (B \cap L_{x_n}),$$

and we can choose a  $\subseteq$ -increasing sequence  $(C_n)_n$  of finite subsets of  $B \cap L_x$  such that

$$B \cap L_x = \bigcup_n C_n, \quad \text{and} \quad C_n \subseteq B \cap L_{x_n} \text{ for each } n.$$

We have  $u' \in L_{x_0} \subseteq L_{x_1}$ . Apply Proposition 5.5 to  $k = 1, (0 \ll x_1), u'$ , and  $C_1$  to obtain a strongly soft element  $w'_1 \in L_{x_1}$  such that

$$u' \ll w'_1, \quad 0 \ll \infty w'_1, \quad \text{and} \quad \frac{1}{2} \widehat{v}' \leq \widehat{w}'_1$$

for every  $v' \in C_1$ .

We have  $w'_1 \in L_{x_2}$ . Applying Proposition 5.5 again to  $k = 2, (x_1 \ll x_2), w'_1$  and  $C_2$ , we obtain a strongly soft element  $w'_2 \in L_{x_2}$  such that

$$w'_1 \ll w'_2, \quad x_1 \ll \infty w'_2, \quad \text{and} \quad \frac{2}{3} \widehat{v}' \leq \widehat{w}'_2$$

for every  $v' \in C_2$ .

Proceeding inductively, we get a  $\ll$ -increasing sequence of strongly soft elements  $(w'_n)_n$  such that

$$w'_n \in L_{x_n}, \quad x_{n-1} \ll \infty w'_n, \quad \text{and} \quad \frac{n}{n+1} \widehat{v}' \leq \widehat{w}'_n$$

for every  $v' \in C_n$  and  $n \geq 2$ .

Set  $w := \sup_n w'_n$ , which is strongly soft by [40, Theorem 4.14]. Note that we get  $u' \ll w'_1 \leq w \leq x$  by construction. Further, since  $x_n \leq \infty w'_{n+1} \leq \infty w$  for each  $n \geq 2$ , we deduce that  $x \leq \infty w$ .

Now, take  $\lambda \in F(S)$ . Given  $v' \in B \cap L_x$ , choose  $n_0 \geq 2$  such that  $v' \in C_{n_0}$ . We have

$$\frac{n}{n+1} \lambda(v') \leq \lambda(w'_n) \leq \lambda(w)$$

for every  $n \geq n_0$ . Thus, it follows that  $\lambda(v') \leq \lambda(w)$  for every  $v' \in B \cap L_x$ .

Since  $L_x$  is downward hereditary, every element in  $L_x$  is the supremum of an increasing sequence from  $B \cap L_x$ . Using also that functionals preserve suprema of increasing sequences, we obtain

$$\sup_{v' \in L_x} \lambda(v') \leq \sup_{v' \in B \cap L_x} \lambda(v') \leq \lambda(w) = \sup_n \lambda(w'_n) \leq \sup_{v' \in L_x} \lambda(v'),$$

which shows that  $w$  has the desired properties. □

**Lemma 5.7.** *Let  $S$  be a  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7), and let  $x', x, t \in S$  be such that  $x' \ll x \leq \infty t$ . Then there exists a strongly soft element  $u' \in L_x$  such that*

$$x' \ll u' + t.$$

**Proof.** Choose  $x'' \in S$  such that  $x' \ll x'' \ll x$ . Applying [43, Proposition 4.10] to

$$x'' \ll x \leq \infty x, \infty t,$$

we get  $s \in S$  such that

$$x'' \ll \infty s, \text{ and } s \ll x, t.$$

By [40, Proposition 7.7], we can choose  $s' \in S_{\text{soft}}$  such that

$$x'' \leq \infty s', \text{ and } s' \ll s.$$

Then  $x'' \ll \infty s'$ . Applying (O5) to  $s' \ll s \leq x$ , we obtain  $v \in S$  satisfying

$$v + s' \leq x \leq v + s.$$

In particular, one has  $x'' \ll v + s$ . Applying (O6) to  $x' \ll x'' \leq v + s$ , we find  $u \in S$  such that

$$x' \ll u + s, \text{ and } u \ll x'', v.$$

Since  $u \ll x'' \leq \infty s'$ , it follows from [40, Theorem 4.14] that  $u + s'$  is soft. Further, we get

$$x' \ll u + s \leq u + t \leq (u + s') + t, \text{ and } u + s' \leq v + s' \leq x.$$

Using that  $S_{\text{soft}}$  is a sub-Cu-semigroup by Proposition 3.6, we can find  $u' \in S_{\text{soft}}$  such that

$$x' \ll u' + t, \text{ and } u' \ll u + s' \leq x.$$

Then  $u' \in L_x$ , which shows that  $u'$  has the desired properties. □

**Lemma 5.8.** *Let  $S$  be a  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7), and let  $t \in S_{\text{soft}}$  and  $t', x', x \in S$  be such that*

$$x' \ll x \leq \infty t, \text{ and } t' \ll t.$$

*Then, there exists a strongly soft element  $v' \in L_x$  such that*

$$x' + t' \leq v' + t.$$

**Proof.** By [40, Proposition 4.13], there exists  $s \in S_{\text{soft}}$  such that

$$t' + s \leq t \leq \infty s.$$

Applying Lemma 5.7 to  $x' \ll x \leq \infty s$ , we obtain a strongly soft element  $v' \in L_x$  satisfying  $x' \leq v' + s$ . Consequently, we obtain

$$x' + t' \leq v' + s + t' \leq v' + t. \quad \square$$

We refer to [38, Section 5] for an introduction to the basic technique to reduce certain proofs about Cu-semigroups to the countably based setting. In particular, a property  $\mathcal{P}$  for Cu-semigroups is said to satisfy the Löwenheim–Skolem condition if, for every Cu-semigroup  $S$  satisfying  $\mathcal{P}$ , there exists a  $\sigma$ -complete and cofinal subcollection of countably based sub-Cu-semigroups of  $S$  satisfying  $\mathcal{P}$ .

**Lemma 5.9.** *Let  $S$  be a Cu-semigroup, let  $u \in S_{\text{soft}}$  and let  $\mathcal{R}$  be the family of countably based sub-Cu-semigroups  $T \subseteq S$  containing  $u$  and such that  $u$  is strongly soft in  $T$ . Then  $\mathcal{R}$  is  $\sigma$ -complete and cofinal.*

**Proof.** Strong softness is preserved under Cu-morphisms, and the inclusion map of a sub-Cu-semigroup is a Cu-morphism. Hence, given sub-Cu-semigroups  $T_1 \subseteq T_2 \subseteq S$  containing  $u$ , if  $u$  is strongly soft in  $T_1$ , then it is also strongly soft in  $T_2$ . This implies in particular that  $\mathcal{R}$  is  $\sigma$ -complete.

To show that  $\mathcal{R}$  is cofinal, let  $T_0 \subseteq S$  be a countably based sub-Cu-semigroup, and let  $B_0 \subseteq T_0$  be a countable basis, that is, a countable subset such that every element in  $T_0$  is the supremum of an increasing sequence from  $B_0$ .

Let  $(u_n)_n$  be a  $\ll$ -increasing sequence in  $S$  with supremum  $u$ . Since  $u$  is strongly soft in  $S$ , for each  $n$  we obtain  $t_n \in S$  such that

$$u_n + t_n \ll u, \text{ and } u_n \ll \infty t_n.$$

By [38, Lemma 5.1], there exists a countably based sub-Cu-semigroup  $T \subseteq S$  containing

$$B_0 \cup \{u_0, u_1, \dots\} \cup \{t_0, t_1, \dots\}.$$

One checks that  $T_0 \subseteq T$ , and that  $u$  is strongly soft in  $T$ . □

**Theorem 5.10.** *Let  $S$  be a  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7), let  $x \in S$  and let  $u' \in L_x$ . Then there exists  $w \in S_{\text{soft}}$  such that*

$$u' \ll w \leq x \leq \infty w, \text{ and } \widehat{w} = \widehat{x}.$$

**Proof.** We first prove the result under the additional assumption that  $S$  is countably based. Use Proposition 5.6 to obtain  $w \in S_{\text{soft}}$  such that

$$u' \ll w \leq x \leq \infty w, \text{ and } \lambda(w) = \sup_{v' \in L_x} \lambda(v'),$$

for every  $\lambda \in F(S)$ . Since  $w \leq x$ , we have  $\widehat{w} \leq \widehat{x}$ . To show the reverse inequality, let  $\lambda \in F(S)$ . We need to prove that  $\lambda(x) \leq \lambda(w)$ .

Take  $x', w' \in S$  such that  $x' \ll x$  and  $w' \ll w$ . Applying Lemma 5.8, we obtain an element  $v' \in L_x$  such that

$$x' + w' \leq v' + w.$$

Since  $v'$  belongs to  $L_x$ , we have  $\lambda(v') \leq \lambda(w)$ . This implies

$$\lambda(x') + \lambda(w') \leq \lambda(v') + \lambda(w) \leq 2\lambda(w).$$

Passing to the supremum over all  $x'$  way below  $x$ , and all  $w'$  way below  $w$ , we get

$$\lambda(x) + \lambda(w) \leq 2\lambda(w).$$

This proves  $\lambda(x) \leq \lambda(w)$ . Indeed, if  $\lambda(w) = \infty$ , then there is nothing to prove. If  $\lambda(w) \neq \infty$ , we can cancel  $\lambda(w)$  from the previous inequality.

We now consider the case that  $S$  is not countably based. Choose  $u \in S_{\text{soft}}$  such that  $u' \ll u \ll x$ . Since  $(2, \omega)$ -divisibility and (O5)–(O7) each satisfy the Löwenheim-Skolem condition, and using also Lemma 5.9, we can use the technique from [38, Section 5] to deduce that there exists a countably based,  $(2, \omega)$ -divisible sub-Cu-semigroup  $H \subseteq S$  satisfying (O5)–(O7), containing  $x, u$  and  $u'$ , and such that  $u$  is strongly soft in  $H$ .

Applying the first part of the proof to  $H$ , we find  $w \in H_{\text{soft}}$  such that

$$u' \ll w \leq x \leq \infty w, \text{ and } \lambda(x) = \lambda(w)$$

for every  $\lambda \in F(H)$ .

Since the inclusion  $\iota: H \rightarrow S$  is a Cu-morphism, it follows that  $w$  is strongly soft in  $S$ . Further, any functional  $\lambda$  on  $S$  induces the functional  $\lambda \iota$  on  $H$ . This shows that  $w$  satisfies the required conditions. □

**Theorem 5.11.** *Let  $A$  be a stable  $C^*$ -algebra with the global Glimm property. Then, for any  $a \in A_+$  there exists a soft element  $b \in A_+$  with  $b \lesssim a$  and such that*

$$d_\tau(a) = d_\tau(b)$$

for every  $\tau \in \text{QT}(A)$ .

**Proof.** Let  $a \in A_+$ . Since  $A$  has the global Glimm property, it follows from [43, Theorem 3.6] that  $\text{Cu}(A)$  is  $(2, \omega)$ -divisible. Using Theorem 5.10, find  $w \in \text{Cu}(A)_{\text{soft}}$  such that  $w \leq [a]$  and  $\lambda(w) = \lambda([a])$  for every  $\lambda \in F(\text{Cu}(A))$ .

By Theorem 3.3, there exists a soft element  $b \in A_+$  such that  $w = [b]$ . The result now follows from the fact that the map

$$\tau \mapsto ([a] \mapsto d_\tau(a))$$

is a natural bijection from  $\text{QT}(A)$  to  $F(\text{Cu}(A))$ ; see [20, Theorem 4.4]. □

**Lemma 5.12.** *Let  $S$  be a  $(2, \omega)$ -divisible Cu-semigroup  $S$  satisfying (O5), let  $x \in S$  and let  $\lambda \in F(S)$ . Then*

$$\sup_{\{v \in S_{\text{soft}} : v \leq x\}} \lambda(v) = \sup_{v' \in L_x} \lambda(v').$$

**Proof.** Given  $v' \in L_x$ , there exists  $v \in S_{\text{soft}}$  with  $v' \leq v \leq x$ , which shows the inequality ‘ $\geq$ ’.

Conversely, let  $v \in S_{\text{soft}}$  with  $v \leq x$ . Since  $S_{\text{soft}}$  is a sub-Cu-semigroup by Proposition 3.6, there exists a  $\ll$ -increasing sequence  $(v'_n)_n$  in  $S_{\text{soft}}$  with supremum  $v$ . Each  $v'_n$  belongs to  $L_x$ , and one gets

$$\lambda(v) = \sup_n \lambda(v'_n) \leq \sup_{v' \in L_x} \lambda(v').$$

This shows the the inequality ‘ $\leq$ ’. □

We will prove in Theorem 5.14 that the inclusion  $\iota : S_{\text{soft}} \rightarrow S$  induces a homeomorphism  $\iota^* : F(S) \rightarrow F(S_{\text{soft}})$ . The inverse of  $\iota^*$  is constructed in the next result.

**Proposition 5.13.** *Let  $S$  be a  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7), and let  $\lambda \in F(S_{\text{soft}})$ . Then  $\lambda_{\text{soft}} : S \rightarrow [0, \infty]$  given by*

$$\lambda_{\text{soft}}(x) := \sup_{\{v \in S_{\text{soft}} : v \leq x\}} \lambda(v)$$

for  $x \in S$ , is a functional on  $S$ .

**Proof.** It is easy to see that  $\lambda_{\text{soft}}$  preserves order. Further, given an increasing sequence  $(x_n)_n$  with supremum  $x$  in  $S$ , we have that for every  $v' \in L_x$  there exists  $n \in \mathbb{N}$  with  $v' \in L_{x_n}$ . Thus, using Lemma 5.12, we get

$$\lambda_{\text{soft}}(x) = \sup_{v' \in L_x} \lambda(v') \leq \sup_n \left( \sup_{v' \in L_{x_n}} \lambda(v') \right) = \sup_n \lambda_{\text{soft}}(x_n).$$

Since  $\lambda_{\text{soft}}$  is order preserving, we also have  $\sup_n \lambda_{\text{soft}}(x_n) \leq \lambda_{\text{soft}}(x)$ , which shows that  $\lambda_{\text{soft}}$  preserves suprema of increasing sequences.

Given  $x, y \in S$  and  $u, v \in S_{\text{soft}}$  such that  $u \leq x$  and  $v \leq y$ , we have  $u + v \in S_{\text{soft}}$  and  $u + v \leq x + y$ . This implies that

$$\lambda_{\text{soft}}(x) + \lambda_{\text{soft}}(y) \leq \lambda_{\text{soft}}(x + y).$$

Thus,  $\lambda_{\text{soft}}$  is subadditive.

Finally, we show that  $\lambda_{\text{soft}}$  is superadditive. Given  $x, y \in S$  and  $w' \in L_{x+y}$ , take  $x', x'', y', y'' \in S$  such that

$$x' \ll x'' \ll x, \quad y' \ll y'' \ll y, \quad \text{and} \quad w' \ll x' + y'.$$

By [40, Proposition 7.7], there exist  $s, t \in S_{\text{soft}}$  such that

$$s \leq x'' \leq \infty s, \quad \text{and} \quad t \leq y'' \leq \infty t.$$

Take  $s', t' \in S$  such that  $s' \ll s$  and  $t' \ll t$ . Using Lemma 5.8, we find  $u' \in L_x$  and  $v' \in L_y$  such that

$$x' + s' \leq u' + s, \text{ and } y' + t' \leq v' + t.$$

Consequently, one has

$$w' + s' + t' \leq x' + y' + s' + t' \leq u' + s + v' + t.$$

Applying Theorem 5.10, find  $u, v \in S_{\text{soft}}$  such that

$$u' \ll u \leq x \leq \infty u, \text{ and } v' \ll v \leq y \leq \infty v.$$

This implies

$$w' + s' + t' \leq u + s + v + t$$

and, therefore,

$$\lambda(w') + \lambda(s' + t') \leq \lambda(u) + \lambda(v) + \lambda(s + t).$$

Passing to the suprema over all  $s'$  way below  $s$ , and all  $t'$  way below  $t$ , we deduce that

$$\lambda(w') + \lambda(s + t) \leq \lambda(u) + \lambda(v) + \lambda(s + t).$$

Note that  $s + t \leq x'' + y'' \ll x + y \leq \infty(u + v)$ . This allows us to cancel  $\lambda(s + t)$ , and we obtain

$$\lambda(w') \leq \lambda(u) + \lambda(v) \leq \lambda_{\text{soft}}(x) + \lambda_{\text{soft}}(y).$$

Since this holds for every  $w' \in L_{x+y}$ , we can apply Lemma 5.12 to get

$$\lambda_{\text{soft}}(x + y) = \sup_{\{w \in S_{\text{soft}} : w \leq x + y\}} \lambda(w) = \sup_{w' \in L_{x+y}} \lambda(w') \leq \lambda_{\text{soft}}(x) + \lambda_{\text{soft}}(y).$$

This show that  $\lambda_{\text{soft}}$  is superadditive and thus a functional. □

**Theorem 5.14.** *Let  $S$  be a  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7). Let  $\iota : S_{\text{soft}} \rightarrow S$  be the canonical inclusion. Then the map  $\iota^* : F(S) \rightarrow F(S_{\text{soft}})$  given by  $\iota^*(\lambda) := \lambda \circ \iota$  is a natural homeomorphism.*

**Proof.** Given  $\lambda \in F(S_{\text{soft}})$ , let  $\lambda_{\text{soft}} \in F(S)$  be defined as in Proposition 5.13. This defines a map  $\phi : F(S_{\text{soft}}) \rightarrow F(S)$  by  $\phi(\lambda) := \lambda_{\text{soft}}$ . We verify that  $\iota^* \phi = \text{id}_{F(S_{\text{soft}})}$  and  $\phi \iota^* = \text{id}_{F(S)}$ .

Given  $\lambda \in F(S_{\text{soft}})$  and  $w \in S_{\text{soft}}$ , we have

$$\iota^* \phi(\lambda)(w) = \iota^* \lambda_{\text{soft}}(w) = \lambda_{\text{soft}}(\iota(w)) = \sup_{\{v \in S_{\text{soft}} : v \leq w\}} \lambda(v) = \lambda(w),$$

which shows  $\iota^* \phi = \text{id}_{F(S_{\text{soft}})}$ .

Conversely, if  $\lambda \in F(S)$  and  $x \in S$ , we can use Theorem 5.10 at the last step to obtain

$$\phi \iota^*(\lambda)(x) = \phi(\lambda \iota)(x) = \sup_{\{v \in S_{\text{soft}} : v \leq x\}} \lambda(v) = \lambda(x).$$

This shows that  $\iota^*$  is a bijective, continuous map. Since  $F(S)$  and  $F(S)_{\text{soft}}$  are both compact, Hausdorff spaces, it follows that  $\iota^*$  is a homeomorphism.  $\square$

Since simple, nonelementary  $C^*$ -algebras automatically have the global Glimm property, the next result can be considered as a generalization of [31, Lemma 3.8] to the nonsimple setting.

**Theorem 5.15.** *Let  $A$  be a  $C^*$ -algebra with the global Glimm property. Then  $QT(A)$  is naturally homeomorphic to  $F(\text{Cu}(A)_{\text{soft}})$ .*

**Proof.** The result follows from Theorem 5.14 and the fact that  $QT(A)$  is naturally homeomorphic to  $F(\text{Cu}(A))$ ; see [20, Theorem 4.4].  $\square$

## 6. Retraction onto the soft part of a Cuntz semigroup

Let  $S$  be a countably based, left-soft separative,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7). Given any  $x \in S$ , we have seen in Lemma 5.3 that  $L_x$  is upward directed. It then follows from [4, Remarks 3.1.3] that  $L_x$  has a supremum, which justifies the following:

**Definition 6.1.** Let  $S$  be a countably based, left-soft separative,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7). We define  $\sigma : S \rightarrow S$  by

$$\sigma(x) := \sup L_x = \sup \{u' \in S : u' \ll u \ll x \text{ for some } u \in S_{\text{soft}}\}$$

for  $x \in S$ .

We will see in Proposition 6.3 that  $\sigma(x)$  is the largest strongly soft element dominated by  $x$ . Therefore, we often view  $\sigma$  as a map  $S \rightarrow S_{\text{soft}}$ . In Theorem 6.6, we show that  $\sigma$  is close to being a generalized Cu-morphism, and in Proposition 6.8 we give sufficient conditions ensuring that it is.

If  $A$  is a separable  $C^*$ -algebra satisfying the global Glimm property and with left-soft separative Cuntz semigroup, then  $\text{Cu}(A)$  satisfies the assumptions of Definition 6.1. If  $A$  also has stable rank one or strict comparison of positive elements, then  $\sigma : \text{Cu}(A) \rightarrow \text{Cu}(A)_{\text{soft}}$  is a generalized Cu-morphism; see Theorem 6.9. Then  $\text{Cu}(A)_{\text{soft}}$  is a retract of  $S$ ; see Definition 6.7. This generalizes the construction of predecessors in the context of simple  $C^*$ -algebras from [22], as well as the constructions from [4, Section 5.4] and [37, Proposition 2.9].

**Remark 6.2.** Let  $S$  be a weakly cancellative Cu-semigroup satisfying (O5)–(O7) (for instance, the Cuntz semigroup of a stable rank one  $C^*$ -algebra). Take  $x \in S$ , and consider the set

$$L'_x := \{u' : u' \ll u \leq \infty s, \text{ and } u + s \ll x \text{ for some } u, s \in S\}.$$

A slight modification of Proposition 5.5 shows that  $L'_x$  is upward directed.

If  $S$  is countably based and  $(2, \omega)$ -divisible, it is readily checked that

$$\sigma(x) = \sup L_x = \sup L'_x.$$

However, if  $S$  is not  $(2, \omega)$ -divisible,  $\sup L'_x$  may not be strongly soft. For example, the Cuntz semigroup of  $\mathbb{C}$  is  $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ , which is weakly cancellative. One can check that

$$\sup L'_x = \begin{cases} 0, & \text{if } x = 0 \\ x - 1, & \text{if } x \neq 0, \infty \\ \infty, & \text{if } x = \infty \end{cases}$$

In particular, if  $x \neq 0, \infty$ , we get  $\sup L'_x = x - 1$ , which is not strongly soft.

As another example, there are Cu-semigroups whose order structure is deeply related to its soft elements but where  $\sup L'_x$  is rarely strongly soft: Let  $S$  be a Cu-semigroup of the form  $\text{Lsc}(X, \overline{\mathbb{N}})$  for some  $T_1$ -space  $X$  (these were called *Lsc-like* in [46]). An element  $f \in \text{Lsc}(X, \overline{\mathbb{N}})$  is strongly soft if and only if  $f = \infty \chi_U$  for the indicator function  $\chi_U$  of some open subset  $U \subseteq X$ . Thus, if  $x \in S$  satisfies  $x \ll \infty$ , we have  $\sup L'_x \ll \infty$ , which implies that  $\sup L'_x$  is not strongly soft, unless it is zero.

**Proposition 6.3.** *Let  $S$  be a countably based, left-soft separative,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7), and let  $x \in S$ . Then:*

- (1) *The element  $\sigma(x)$  is the largest strongly soft element dominated by  $x$ .*
- (2) *We have  $\infty x = \infty \sigma(x)$ .*
- (3) *We have  $x = \sigma(x)$  if and only if  $x$  is strongly soft.*
- (4) *We have  $x \leq \sigma(x) + t$  for all  $t \in S$  with  $x \leq \infty t$ .*

**Proof.** To verify (1), note that the members of  $L_x$  are bounded by  $x$ , and consequently  $\sigma(x) \leq x$ . To see that  $\sigma(x)$  is strongly soft, let  $s \in S$  be such that  $s \ll \sigma(x)$ . We will find  $t \in S$  such that  $s + t \ll \sigma(x)$  and  $s \ll \infty t$ .

Since  $\sigma(x) = \sup L_x$ , there exists  $u' \in L_x$  such that  $s \ll u' \leq \sigma(x)$ . Using that  $u' \in L_x$ , we find  $u \in S_{\text{soft}}$  with  $u' \ll u \ll x$ . By Proposition 3.6,  $S_{\text{soft}}$  is a sub-Cu-semigroup, and we obtain  $u'' \in S_{\text{soft}}$  such that

$$s \ll u' \ll u'' \ll u \ll x.$$

Then  $s \ll u'' \in S_{\text{soft}}$  and by the definition of strong softness we obtain  $t \in S$  such that  $s + t \ll u''$  and  $s \ll \infty t$ . We have  $u'' \in L_x$  and therefore  $u'' \leq \sigma(x)$ , which shows that  $t$  has the desired properties.

Thus,  $\sigma(x)$  is a strongly soft element dominated by  $x$ . To show that it is the largest element with these properties, let  $w \in S_{\text{soft}}$  satisfy  $w \leq x$ . We can use once again that  $S_{\text{soft}}$  is a sub-Cu-semigroup to find a  $\ll$ -increasing sequence  $(w_n)_n$  of strongly soft elements with supremum  $w$ . Then  $w_n \in L_x$  for each  $n$ , and consequently

$$w = \sup_n w_n \leq \sup L_x = \sigma(x).$$

This also shows that  $x = \sigma(x)$  if and only if  $x$  is strongly soft. We have proved (1) and (3).



To verify (2), we first note that  $\infty\sigma(x) \leq \infty x$  since  $\sigma(x) \leq x$ . For the converse inequality, use Proposition 5.6 to obtain  $w \in S_{\text{soft}}$  with  $w \leq x \leq \infty w$ . By (1), we have  $w \leq \sigma(x)$ , and we get

$$\infty x = \infty w \leq \infty\sigma(x).$$

Finally, to prove (4), let  $t \in S$  satisfy  $x \leq \infty t$ . Let  $x' \in S$  satisfy  $x' \ll x$ . Applying Lemma 5.7, we obtain  $u' \in L_x$  such that  $x' \ll u' + t$ . Then

$$x' \ll u' + t \leq \sigma(x) + t.$$

Passing to the supremum over all  $x'$  way below  $x$ , we get  $x \leq \sigma(x) + t$ , as desired. □

**Example 6.4.** Let  $A$  be a separable,  $\mathcal{W}$ -stable  $C^*$ -algebra, that is,  $A \cong A \otimes \mathcal{W}$  where  $\mathcal{W}$  denotes the Jacelon–Razak algebra. Then, every element in  $\text{Cu}(A)$  is strongly soft. Thus, Proposition 6.3 implies that  $\sigma(x) = x$  for every  $x \in \text{Cu}(A)$ . We refer to [4, Section 7.5] for details.

Similarly, given a separable  $\mathcal{Z}$ -stable  $C^*$ -algebra  $A$ , where  $\mathcal{Z}$  denotes the Jiang–Su algebra, then it follows from [4, Theorem 7.3.11] that  $\text{Cu}(A)$  has  $\mathcal{Z}$ -multiplication. Here,  $Z = (0, \infty] \sqcup \mathbb{N}$  is the Cuntz semigroup of  $\mathcal{Z}$ , and  $(0, \infty]$  is the subsemigroup of nonzero, strongly soft elements. Let  $1' \in Z$  be the strongly soft element corresponding to  $1 \in [0, \infty]$ . As noted in [4, Proposition 7.3.16], one has

$$1' \text{Cu}(A) = \text{Cu}(A)_{\text{soft}} \cong \text{Cu}(A) \otimes [0, \infty].$$

This implies that  $\sigma(x) = 1'x$  for each  $x \in \text{Cu}(A)$ .

**Lemma 6.5.** *Let  $S$  be a countably based, left-soft separative,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7), and let  $x \in S$ . Then*

$$2\sigma(x) = x + \sigma(x).$$

**Proof.** Using that  $\sigma(x) \leq x$ , we have  $2\sigma(x) \leq x + \sigma(x)$ . To show the reverse inequality, let  $w \in S$  satisfy  $w \ll \sigma(x)$ . Since  $\sigma(x)$  is strongly soft, it follows from [40, Proposition 4.13] that there exists  $t \in S$  with  $w + t \leq \sigma(x) \leq \infty t$ .

We have  $x \leq \infty\sigma(x)$  by Proposition 6.3 (2), and thus  $x \leq \infty t$ . Therefore,  $x \leq \sigma(x) + t$  by Proposition 6.3 (4). Thus, we have

$$x + w \leq \sigma(x) + t + w \leq 2\sigma(x).$$

Passing to the supremum over all  $w$  way below  $\sigma(x)$ , we get  $x + \sigma(x) \leq 2\sigma(x)$ . □

**Theorem 6.6.** *Let  $S$  be a countably based, left-soft separative,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7). Then, the map  $\sigma : S \rightarrow S_{\text{soft}}$  preserves order, suprema of increasing sequences and is superadditive. Further, we have*

$$2\sigma(x + y) = \sigma(x + y) + (\sigma(x) + \sigma(y)) = 2(\sigma(x) + \sigma(y))$$

for every  $x, y \in S$ .

**Proof.** To show that  $\sigma$  is order preserving, let  $x, y \in S$  satisfy  $x \leq y$ . Then  $L_x \subseteq L_y$ , and thus

$$\sigma(x) = \sup L_x \leq \sup L_y = \sigma(y).$$

To show that  $\sigma$  preserves suprema of increasing sequences, let  $(x_n)_n$  be an increasing sequence in  $S$  with supremum  $x$ . Since  $\sigma$  is order-preserving, one gets  $\sup_n \sigma(x_n) \leq \sigma(x)$ . Conversely, given  $u' \in L_x$ , choose  $u \in S_{\text{soft}}$  with  $u' \ll u \ll x$ . Then there exists  $n \in \mathbb{N}$  such that  $u \ll x_n$ , and thus  $u' \in L_{x_n}$ . We deduce that

$$u' \leq \sup L_{x_n} = \sigma(x_n) \leq \sup_n \sigma(x_n).$$

Hence,  $\sigma(x) = \sup L_x \leq \sup_n \sigma(x_n)$ , as desired.

To see that  $\sigma$  is superadditive, let  $x, y \in S$ . Note that  $\sigma(x) + \sigma(y)$  is a strongly soft element bounded by  $x + y$ . Using Proposition 6.3 (1), we get  $\sigma(x) + \sigma(y) \leq \sigma(x + y)$ .

Next, given  $x, y \in S$ , let us show that  $2\sigma(x + y) \leq 2\sigma(x) + 2\sigma(y)$ . To prove this, let  $w \in S$  satisfy  $w \ll \sigma(x + y)$ . By [40, Proposition 4.13], there exists  $s \in S$  satisfying

$$w + s \leq \sigma(x + y) \leq \infty s.$$

Applying [40, Proposition 7.7], we find  $t \in S$  such that  $2t \leq s \leq \infty t$ . Using also Proposition 6.3 (2), we deduce that

$$w + 2t \leq w + s \leq \sigma(x + y), \quad \text{and} \quad x, y \leq \infty(x + y) = \infty\sigma(x + y) \leq \infty s \leq \infty t.$$

Using Proposition 6.3 (4) at the second step and Lemma 6.5 at last step, we get

$$\begin{aligned} \sigma(x + y) + w &\leq x + y + w \leq \sigma(x) + \sigma(y) + w + 2t \leq \sigma(x) + \sigma(y) + \sigma(x + y) \\ &\leq \sigma(x) + \sigma(y) + x + y = 2\sigma(x) + 2\sigma(y). \end{aligned}$$

Passing to the supremum over all elements  $w$  way below  $\sigma(x + y)$ , we obtain

$$2\sigma(x + y) \leq 2\sigma(x) + 2\sigma(y).$$

Next, given  $x, y \in S$ , using the above inequality together with the established superadditivity of  $\sigma$ , we get

$$2\sigma(x + y) \leq 2\sigma(x) + 2\sigma(y) \leq \sigma(x + y) + (\sigma(x) + \sigma(y)) \leq 2\sigma(x + y),$$

as desired.  $\square$

Recall that a *generalized Cu-morphism* is a monoid morphism between Cu-semigroups that preserves order and suprema of increasing sequences. We recall the definition of *retract* from [42, Definition 3.14].

**Definition 6.7.** Let  $S, T$  be Cu-semigroups. We say that  $S$  is a *retract* of  $T$  if there exist a Cu-morphism  $\iota: S \rightarrow T$  and a generalized Cu-morphism  $\sigma: T \rightarrow S$  such that  $\sigma \circ \iota = \text{id}_S$ .

**Proposition 6.8.** *Let  $S$  be a countably based, left-soft separative,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7). Additionally, assume one of the following:*

- (i)  $S$  is almost unperforated;
- (ii)  $S$  is inf-semilattice ordered;
- (iii)  $S \otimes \{0, \infty\}$  is algebraic.

*Then,  $\sigma$  is a generalized Cu-morphism and  $S_{\text{soft}}$  is a retract of  $S$ .*

**Proof.** By Theorem 6.6, we only need to check that  $\sigma$  is subadditive.

(i): If  $S$  is almost unperforated, then it follows from Proposition 4.6 that  $S_{\text{soft}}$  is unperforated. Given any pair  $x, y \in S$ , we know from Theorem 6.6 that

$$2\sigma(x + y) = 2(\sigma(x) + \sigma(y)).$$

Since this equality is in  $S_{\text{soft}}$ , it follows that  $\sigma(x + y) = \sigma(x) + \sigma(y)$ .

For (ii) and (iii), note that it is enough to prove that  $\sigma(x + y) \leq x + \sigma(y)$  for all  $x, y \in S$ . Indeed, if this inequality holds, one can use it at the second and last steps to get

$$\sigma(x + y) = \sigma(\sigma(x + y)) \leq \sigma(x + \sigma(y)) = \sigma(\sigma(y) + x) \leq \sigma(y) + \sigma(x),$$

as required.

Given  $x, y \in S$ , we proceed to verify that  $\sigma(x + y) \leq x + \sigma(y)$ . Let  $w \in S$  satisfy  $w \ll \sigma(x + y)$ . Choose  $y' \in S$  such that

$$y' \ll y, \text{ and } w \ll x + y'.$$

Since  $\sigma(x + y)$  is strongly soft, it follows from [40, Proposition 4.13] that there exists  $r \in S_{\text{soft}}$  such that

$$w + r \leq \sigma(x + y) \leq \infty r.$$

Applying Proposition 6.3 (2), one gets

$$y' \ll y \leq \infty \sigma(x + y) \leq \infty r.$$

Applying [43, Proposition 4.7], we obtain  $t', t \in S$  such that

$$y' \leq \infty t', \text{ and } t' \ll t \ll r, y.$$

Using that  $S$  is  $(2, \omega)$ -divisible, it follows from [40, Proposition 5.6] that we may assume both  $t'$  and  $t$  to be strongly soft. Thus, as in the proof of Lemma 5.2, we can apply (O5) to obtain an element  $b$  satisfying

$$t' + b \leq y \leq t + b, \text{ and } y \leq \infty b,$$

which implies

$$w + r \leq \sigma(x + y) \leq x + y \leq x + t + b$$

with  $t \ll r \leq \infty(x + y) = \infty(x + b)$ .

Thus, since both  $w$  and  $r$  are strongly soft, left-soft separativity (in the form of Lemma 4.5) implies that  $w \leq x + b$ . Since  $S$  is countably based and satisfies (O7), the

infimum  $(b \wedge \infty t')$  exists. Note that  $(b \wedge \infty t') + t'$  is soft because  $(b \wedge \infty t') \leq \infty t'$ ; see [40, Theorem 4.14]. Then

$$(b \wedge \infty t') + t' \leq b + t' \leq y,$$

and thus  $b \wedge \infty t' \leq (b \wedge \infty t') + t' \leq \sigma(y)$  by Proposition 6.3 (1).

(ii): Assuming that  $S$  is inf-semilattice ordered, it now follows that

$$w \leq (x + b) \wedge (x + \infty t') = x + (b \wedge \infty t') \leq x + \sigma(y).$$

Passing to the supremum over all  $w$  way below  $\sigma(x + y)$ , we get  $\sigma(x + y) \leq x + \sigma(y)$ , as desired. This proves the case (ii).

(iii): Let us additionally assume that  $y \ll \infty y$ . Then, given  $w$  and  $r$  as before, we have that  $y \ll \infty y \leq \infty r$ . This implies that there exists  $r' \in S$  such that  $r' \ll r$  and  $y \leq \infty r'$ . Using Proposition 6.3 at the last step, one gets

$$w + r \leq \sigma(x + y) \leq x + y \leq x + \sigma(y) + r'$$

with  $r' \ll r \leq \infty(x + y) = \infty(x + \sigma(y))$ .

Therefore, we can use Lemma 4.4 to deduce that  $w \leq x + \sigma(y)$ . Since this holds for every  $w$  way below  $\sigma(x + y)$ , it follows that  $\sigma(x + y) \leq x + \sigma(y)$  whenever  $y \ll \infty y$ .

If  $S \otimes \{0, \infty\}$  is algebraic, then by [43, Lemma 4.16] every  $y \in S$  is the supremum of an increasing sequence  $(y_n)_n$  of elements  $y_n \in S$  such that  $y_n \ll \infty y_n$ . Using the above for each  $y_n$  and using that  $\sigma$  preserves suprema of increasing sequences, we get

$$\sigma(x + y) = \sup_n \sigma(x + y_n) \leq \sup_n (x + \sigma(y_n)) = x + \sigma(y),$$

as desired. □

**Theorem 6.9.** *Let  $A$  be a separable  $C^*$ -algebra with the global Glimm property. Additionally, assume one of the following holds:*

- (i)  $A$  has strict comparison of positive elements;
- (ii)  $A$  has stable rank one;
- (iii)  $A$  has topological dimension zero, and  $\text{Cu}(A)$  is left-soft separative.

*Then,  $\text{Cu}(A)_{\text{soft}}$  is a retract of  $\text{Cu}(A)$ .*

**Proof.** The Cuntz semigroup  $\text{Cu}(A)$  is countably based and satisfies (O5)–(O7). Since  $A$  has the global Glimm property, it follows from [43, Theorem 3.6] that  $\text{Cu}(A)$  is  $(2, \omega)$ -divisible. We check that the additional conditions of Proposition 6.8 are satisfied:

(i): Assume that  $A$  has strict comparison of positive elements. Then  $\text{Cu}(A)$  is almost unperforated by [20, Proposition 6.2] and left-soft separative by Proposition 4.8. This verifies Proposition 6.8 (i).

(ii): Assume that  $A$  has stable rank one. Then  $\text{Cu}(A)$  is inf-semilattice ordered by [2, Theorem 3.8], and left-soft separative by Proposition 4.3. This verifies Proposition 6.8 (ii).

(iii): Assume that  $A$  has topological dimension zero, and  $\text{Cu}(A)$  is left-soft separative. Then  $\text{Cu}(A) \otimes \{0, \infty\}$  is algebraic by [43, Proposition 4.18]. This verifies Proposition 6.8 (iii). □

**Question 6.10.** Let  $S$  be a countably based, weakly cancellative,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7). Is the map  $\sigma: S \rightarrow S_{\text{soft}}$  subadditive?

With view towards the proof of subadditivity in Theorem 6.6, we ask the following question.

**Question 6.11.** Let  $S$  be the Cuntz semigroup of a  $C^*$ -algebra. Let  $x, y, z, w \in S$  satisfy

$$w = 2w, \quad x \leq y + z, \quad \text{and} \quad x \leq y + w.$$

We know that  $z \wedge w$  exists. Does it follow that  $x \leq y + (z \wedge w)$ ?

Question 6.11 above has a positive answer if  $S$  satisfies the *interval axiom*, as defined in [39, Definition 9.3].

### 7. Dimension of a Cuntz semigroup and its soft part

Let  $S$  be a countably based, left-soft separative,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7), and assume that  $\sigma: S \rightarrow S_{\text{soft}}$  is a generalized Cu-morphism. We show that the (covering) dimension of  $S$  and  $S_{\text{soft}}$ , as defined in [42, Definition 3.1], are closely related: We have  $\dim(S_{\text{soft}}) \leq \dim(S) \leq \dim(S_{\text{soft}}) + 1$ ; see Proposition 7.2.

Using the technique developed in [38, Section 5], we remove the assumption that the Cu-semigroup is countably based; see Theorem 7.3. The result applies, in particular, to the Cuntz semigroup of every  $C^*$ -algebra with the global Glimm property that has either strict comparison of positive elements, stable rank one or topological dimension zero; see Corollary 7.4.

We also study the dimension of the fixed-point algebra  $A^\alpha$  for a finite group action  $\alpha$ ; see Theorem 7.9.

**7.1** (Dimension of Cu-semigroups). Recall from [42, Definition 3.1] that, given a Cu-semigroup  $S$  and  $n \in \mathbb{N}$ , we say that  $S$  has *dimension  $n$* , in symbols  $\dim(S) = n$ , if  $n$  is the least integer such that, for any  $r \in \mathbb{N}$ , any pair  $x', x \in S$ , and any tuple  $y_1, \dots, y_r \in S$  with  $x' \ll x \ll y_1 + \dots + y_r$ , there exist elements  $z_{j,k} \in S$  with  $j = 1, \dots, r$  and  $k = 0, \dots, n$  such that:

- (i)  $z_{j,k} \ll y_j$  for every  $j$  and  $k$ ;
- (ii)  $x' \ll \sum_{j,k} z_{j,k}$ ;
- (iii)  $\sum_j z_{j,k} \ll x$  for each  $k$ .

If no such  $n$  exists, we say that  $S$  has dimension  $\infty$ , in symbols  $\dim(S) = \infty$ .

The next result generalizes [42, Proposition 3.17] to the nonsimple setting.

**Proposition 7.2.** *Let  $S$  be a countably based, left-soft separative,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7), and assume that  $\sigma: S \rightarrow S_{\text{soft}}$  is a generalized Cu-morphism. Then,*

$$\dim(S_{\text{soft}}) \leq \dim(S) \leq \dim(S_{\text{soft}}) + 1.$$

**Proof.** Since  $\sigma$  is a generalized Cu-morphism, the first inequality follows from [42, Proposition 3.15]. To show the second inequality, set  $n := \dim(S_{\text{soft}})$ , which we may assume to be finite. To verify that  $\dim(S) \leq n + 1$ , let  $x' \ll x \ll y_1 + \dots + y_r$  in  $S$ . We need to find  $z_{j,k} \in S$  for  $j = 1, \dots, r$  and  $k = 0, \dots, n + 1$  such that

- (i)  $z_{j,k} \ll y_j$  for each  $j$  and  $k$ ;
- (ii)  $x' \ll \sum_{j,k} z_{j,k}$ ;
- (iii)  $\sum_j z_{j,k} \ll x$  for each  $k$ .

First, choose  $x'', x''' \in S$  such that  $x' \ll x'' \ll x''' \ll x$ . Applying that  $S$  satisfies (O6) for  $x'' \ll x''' \leq y_1 + \dots + y_r$ , we obtain  $s_1, \dots, s_r \in S$  such that

$$x'' \ll s_1 + \dots + s_r, \text{ and } s_j \ll x''', y_j \text{ for each } j = 1, \dots, r.$$

Choose  $s'_1, \dots, s'_r \in S$  such that

$$x'' \ll s'_1 + \dots + s'_r, \text{ and } s'_j \ll s_j \text{ for each } j = 1, \dots, r.$$

Using that  $S$  is  $(2, \omega)$ -divisible (and consequently also  $(r, \omega)$ -divisible by [43, Paragraph 2.4]), we obtain  $v \in S$  such that

$$rv \leq x, \text{ and } x''' \leq \infty v.$$

For each  $j$ , we have  $s_j \ll x''' \leq \infty v$ . Applying [43, Proposition 4.10] to

$$s'_j \ll s_j \ll \infty v, \infty y_j,$$

we obtain  $v_j \in S$  such that

$$s'_j \ll \infty v_j, \text{ and } v_j \ll v, y_j.$$

Note that

$$x'' \ll s'_1 + \dots + s'_r \leq \infty(v_1 + \dots + v_r), \text{ and } v_1 + \dots + v_r \ll rv \leq x.$$

Now, applying Proposition 6.3 at the second step, we have

$$x' \ll x'' \leq \sigma(x'') + (v_1 + \dots + v_r).$$

Using that  $S_{\text{soft}}$  is a sub-Cu-semigroup by Proposition 3.6, we can choose an element  $w \in S_{\text{soft}}$  such that

$$x' \ll w + (v_1 + \dots + v_r), \text{ and } w \ll \sigma(x'').$$

Applying that  $\dim(S_{\text{soft}}) \leq n$  for  $w \ll \sigma(x'') \leq \sigma(y_1) + \dots + \sigma(y_r)$ , we obtain  $z_{j,k} \in S_{\text{soft}}$  for  $j = 1, \dots, r$  and  $k = 0, \dots, n$  such that

- (i')  $z_{j,k} \ll \sigma(y_j)$  for each  $j$  and  $k = 0, \dots, n$ ;

- (ii')  $w \ll \sum_j \sum_{k=0}^n z_{j,k}$ ;
- (iii')  $\sum_j z_{j,k} \ll \sigma(x'')$  for each  $k = 0, \dots, n$ .

Set  $z_{j,n+1} := v_j$  for each  $j$ . These elements satisfy conditions (i) and (iii). To verify (ii), we note that

$$x' \ll w + (v_1 + \dots + v_r) \ll \left( \sum_j \sum_{k=0}^n z_{j,k} \right) + (v_1 + \dots + v_r) = \sum_j \sum_{k=0}^{n+1} z_{j,k},$$

as desired. □

**Theorem 7.3.** *Let  $S$  be a left-soft separative,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7). Additionally, assume one of the following:*

- (i)  $S$  is almost unperforated;
- (ii)  $S$  satisfies the Riesz interpolation property and the interval axiom;
- (iii)  $S \otimes \{0, \infty\}$  is algebraic.

Then,  $\dim(S_{\text{soft}}) \leq \dim(S) \leq \dim(S_{\text{soft}}) + 1$ .

**Proof.** By [38, Proposition 5.3], properties (O5), (O6) and (O7) each satisfy the Löwenheim–Skolem condition. Similarly, one can see that left-soft separativity,  $(2, \omega)$ -divisibility, and the properties listed in (i)–(iii) each satisfy the Löwenheim–Skolem condition. (For (iii), one can use [43, Lemma 4.16].) The proof is now analogous to [38, Proposition 5.9] using Proposition 7.2. □

**Corollary 7.4.** *Let  $A$  be a  $C^*$ -algebra with the global Glimm property. Additionally, assume one of the following:*

- (i)  $A$  has strict comparison of positive elements;
- (ii)  $A$  has stable rank one;
- (iii)  $A$  has topological dimension zero, and  $\text{Cu}(A)$  is left-soft separative.

Then,  $\dim(\text{Cu}(A)_{\text{soft}}) \leq \dim(\text{Cu}(A)) \leq \dim(\text{Cu}(A)_{\text{soft}}) + 1$ .

**Proof.** As in the proof of Theorem 6.9, we see that  $\text{Cu}(A)$  satisfies the corresponding assumptions of Theorem 7.3, from which the result follows. □

**Notation 7.5.** Let  $A$  be a  $C^*$ -algebra, and let  $\alpha: G \rightarrow \text{Aut}(A)$  be an action of a finite group  $G$  on  $A$ . We will denote by  $C^*(G, A, \alpha)$  the induced crossed product.

The *fixed-point algebra*  $A^\alpha$  is defined as

$$A^\alpha := \{a \in A : \alpha_g(a) = a \text{ for all } g \in G\}.$$

**7.6 (Fixed-point semigroups).** For a group action  $\alpha$  on a  $C^*$ -algebra  $A$ , there are three natural objects that may be seen as the fixed-point semigroup of  $\text{Cu}(A)$ : The Cuntz semigroup  $\text{Cu}(A^\alpha)$ , the fixed-point semigroup  $\text{Cu}(A)^\alpha$ , and the fixed-point Cu-semigroup  $\text{Cu}(A)^{\text{Cu}(\alpha)}$ . We give some details.

The *fixed-point semigroup*  $\text{Cu}(A)^\alpha$  is defined as

$$\text{Cu}(A)^\alpha := \{x \in \text{Cu}(A) : \text{Cu}(\alpha_g)(x) = x \text{ for all } g \in G\}.$$

This is a submonoid of  $\text{Cu}(A)$  that is closed under passing to suprema of increasing sequences. In general, it is not known if or when  $\text{Cu}(A)^\alpha$  is a sub-Cu-semigroup of  $\text{Cu}(A)$ .

An indexed collection  $(x_t)_{t \in (0,1]}$  of elements in  $S$  is a *path* if  $x_t \ll x_r$  whenever  $r < t$  and  $x_t = \sup_{r < t} x_r$  for every  $t \in (0,1]$ . The *fixed-point Cu-semigroup*, as defined in [25, Definition 2.8], is

$$\text{Cu}(A)^{\text{Cu}(\alpha)} = \left\{ x \in \text{Cu}(A) : \exists (x_t)_{t \in (0,1]} \text{ path in } \text{Cu}(A) : \begin{array}{l} x_1 = x, \text{ and} \\ \text{Cu}(\alpha_g)(x_t) = x_t \forall t, g \end{array} \right\}.$$

Using [25, Lemma 2.9], one can show that  $\text{Cu}(A)^{\text{Cu}(\alpha)}$  is always a sub-Cu-semigroup of  $\text{Cu}(A)$ . Note that  $\text{Cu}(A)^{\text{Cu}(\alpha)}$  is contained in  $\text{Cu}(A)^\alpha$ . In Proposition 7.8, we will see a situation in which  $\text{Cu}(A)^\alpha$  and  $\text{Cu}(A)^{\text{Cu}(\alpha)}$  agree.

**Lemma 7.7.** *Let  $S$  be an inf-semilattice ordered Cu-semigroup, and let  $\alpha$  be an action of a finite group  $G$  on  $S$  by Cu-isomorphisms on  $S$ . Then the fixed-point semigroup  $S^\alpha := \{x \in S : \alpha_g(x) = x \text{ for all } g \in G\}$  is a sub-Cu-semigroup of  $S$ .*

*Moreover, if  $S$  satisfies weak cancellation (resp. (O5), (O6), (O7)), then so does  $S^\alpha$ .*

**Proof.** Define  $\Phi : S \rightarrow S^\alpha$  by

$$\Phi(x) := \bigwedge_{g \in G} \alpha_g(x)$$

for  $x \in S$ . For each  $x \in S$ , we have  $\Phi(\Phi(x)) = \Phi(x) \leq x$ ; and we have  $\Phi(x) = x$  if and only if  $x \in S^\alpha$ .

It is straightforward to verify that  $S^\alpha$  is a submonoid that is closed under suprema of increasing sequences. To show that  $S^\alpha$  is a sub-Cu-semigroup, it remains to verify that for given  $x \in S^\alpha$  and  $y \in S$  with  $y \ll x$ , there exists  $x' \in S^\alpha$  with  $y \leq x' \ll x$ .

Let  $(x_n)_n$  be a  $\ll$ -increasing sequence in  $S$  with supremum  $x$ . For each  $g \in G$ , we have  $x = \alpha_g(x) = \sup_n \alpha_g(x_n)$ , and it follows that

$$x = \Phi(x) = \sup_n \Phi(x_n).$$

Hence, there exists  $n_0$  such that  $y \leq \Phi(x_{n_0})$ . Set  $x' := \Phi(x_{n_0})$ . Then  $x' \in S^\alpha$  and

$$y \leq x' \leq x_{n_0} \ll x,$$

which shows that  $x'$  has the desired properties. Thus,  $S^\alpha$  is a sub-Cu-semigroup.

Since  $S^\alpha$  is a sub-Cu-semigroup of  $S$ , it follows that  $S^\alpha$  is weakly cancellative whenever  $S$  is. Assuming that  $S$  satisfies (O5), let us verify that so does  $S^\alpha$ . Let  $x', x, y', y, z \in S^\alpha$  satisfy

$$x' \ll x, \quad y' \ll y, \quad \text{and} \quad x + y \leq z.$$



Choose  $y'' \in S^\alpha$  satisfying  $y' \ll y'' \ll y$ . Applying (O5) in  $S$ , we obtain  $c \in S$  such that

$$x' + c \leq z \leq x + c, \text{ and } y'' \ll c.$$

We claim that  $\Phi(c)$  has the desired properties. Indeed, for each  $g \in G$ , we have

$$z = \alpha_g(z) \leq \alpha_g(x + c) = x + \alpha_g(c).$$

Using that  $S$  is semilattice ordered, we get

$$z \leq \bigwedge_{g \in G} (x + \alpha_g(c)) = x + \bigwedge_{g \in G} \alpha_g(c) = x + \Phi(c).$$

We also have

$$x' + \Phi(c) \leq x' + c \leq z, \text{ and } y' \ll y'' = \Phi(y'') \leq \Phi(c).$$

Assuming that  $S$  satisfies (O6), let us verify that so does  $S^\alpha$ . Let  $x', x, y, z \in S^\alpha$  satisfy

$$x' \ll x \leq y + z.$$

It suffices to find  $\tilde{e} \in S^\alpha$  such that

$$x' \leq \tilde{e} + z, \text{ and } \tilde{e} \leq x, y.$$

(One can then apply this argument with the roles of  $y$  and  $z$  reversed to verify (O6).)

Applying (O6) in  $S$ , we obtain  $e \in S$  such that

$$x' \leq e + z, \text{ and } e \leq x, y.$$

For each  $g \in G$ , we have

$$x' = \alpha_g(x') \leq \alpha_g(e + z) = \alpha_g(e) + z.$$

Using that  $S$  is semilattice-ordered, we get

$$x' \leq \bigwedge_{g \in G} (\alpha_g(e) + z) = \left( \bigwedge_{g \in G} \alpha_g(e) \right) + z = \Phi(e) + z.$$

Further, we have

$$\Phi(e) \leq e \leq x, y,$$

which shows that  $\tilde{e} := \Phi(e) \in S^\alpha$  has the desired properties.

Similarly, one shows that (O7) passes from  $S$  to  $S^\alpha$ . □

We refer to [23, Definition 2.2] for the definition of the weak tracial Rokhlin property. The first isomorphism in the statement below is well known, but we add it here for the convenience of the reader.

**Proposition 7.8.** *Let  $A$  be a nonelementary, stably finite, simple, unital  $C^*$ -algebra, and let  $\alpha$  be a finite group action on  $A$  that has the weak tracial Rokhlin property. Then we have*

$$\text{Cu}(C^*(G, A, \alpha)) \cong \text{Cu}(A^\alpha), \text{ and } \text{Cu}(A)^{\text{Cu}(\alpha)} = \text{Cu}(A)^\alpha.$$

Restricting to the soft parts, we obtain:

$$\text{Cu}(C^*(G, A, \alpha))_{\text{soft}} \cong \text{Cu}(A^\alpha)_{\text{soft}} \cong \text{Cu}(A)_{\text{soft}}^{\text{Cu}(\alpha)} = \text{Cu}(A)^\alpha \cap \text{Cu}(A)_{\text{soft}}.$$

If, moreover,  $A$  is separable and has stable rank one, then  $\text{Cu}(A)^\alpha$  is a simple, countably based, weakly cancellative,  $(2, \omega)$ -divisible sub-Cu-semigroup of  $\text{Cu}(A)$  satisfying (O5)–(O7).

**Proof.** For any action of a finite group on a unital  $C^*$ -algebra, the fixed-point algebra is  $*$ -isomorphic to a corner of the crossed product; see [11, Lemma 4.3(4)]. By [27, Corollary 5.4],  $C^*(G, A, \alpha)$  is simple, which implies that  $C^*(G, A, \alpha)$  and  $A^\alpha$  are Morita equivalent and therefore have isomorphic Cuntz semigroups.

As noted in Paragraph 7.6,  $\text{Cu}(A)^{\text{Cu}(\alpha)}$  is contained in  $\text{Cu}(A)^\alpha$  in general, and  $\text{Cu}(A)^{\text{Cu}(\alpha)}$  is always a sub-Cu-semigroup of  $\text{Cu}(A)$ . Let  $\iota: A^\alpha \rightarrow A$  denote the inclusion map, and note that  $\text{Cu}(\iota)$  takes image in  $\text{Cu}(A)^{\text{Cu}(\alpha)}$ .

To show that  $\text{Cu}(A)^\alpha$  is contained in  $\text{Cu}(A)^{\text{Cu}(\alpha)}$ , let  $x \in \text{Cu}(A)^\alpha$ . If  $x$  is compact in  $\text{Cu}(A)$ , then we can use the constant path  $x_t = x$  to see that  $x \in \text{Cu}(A)^{\text{Cu}(\alpha)}$ . On the other hand, if  $x$  is soft, then we can apply [11, Lemma 5.4] to obtain  $y \in \text{Cu}(A^\alpha)_{\text{soft}}$  such that  $x = \text{Cu}(\iota)(y)$ . Since  $\text{Cu}(\iota)$  takes image in  $\text{Cu}(A)^{\text{Cu}(\alpha)}$ , we have  $x \in \text{Cu}(A)^{\text{Cu}(\alpha)}$ . Since  $A$  is simple and stably finite, every Cuntz class is either compact or soft, and we have  $\text{Cu}(A)^{\text{Cu}(\alpha)} = \text{Cu}(A)^\alpha$ .

We have shown

$$\text{Cu}(C^*(G, A, \alpha)) \cong \text{Cu}(A^\alpha), \text{ and } \text{Cu}(A)^{\text{Cu}(\alpha)} = \text{Cu}(A)^\alpha.$$

We know from [11, Theorem 5.5] that  $\text{Cu}(\iota)$  induces an order-isomorphism between the soft part of  $\text{Cu}(A^\alpha)$  and  $\text{Cu}(A)^\alpha \cap \text{Cu}(A)_{\text{soft}}$ , the  $\alpha$ -invariant elements in  $\text{Cu}(A)_{\text{soft}}$ . It is easy to see that  $\text{Cu}(\iota)$  maps  $\text{Cu}(A^\alpha)_{\text{soft}}$  into  $\text{Cu}(A)^{\text{Cu}(\alpha)}$  and that  $\text{Cu}(A)^{\text{Cu}(\alpha)}$  is contained in  $\text{Cu}(A)^\alpha \cap \text{Cu}(A)_{\text{soft}}$ . Together, we get

$$\text{Cu}(A^\alpha)_{\text{soft}} \xrightarrow[\text{Cu}(\iota)]{\cong} \text{Cu}(A)^{\text{Cu}(\alpha)}_{\text{soft}} = \text{Cu}(A)^\alpha \cap \text{Cu}(A)_{\text{soft}}.$$

Since  $A^\alpha$  is a simple, nonelementary  $C^*$ -algebra,  $\text{Cu}(A^\alpha)$  is a simple,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7). It follows from Proposition 3.6 that  $\text{Cu}(A^\alpha)_{\text{soft}}$  is a Cu-semigroup that also satisfies (O5)–(O7).

Finally, assume that  $A$  is also separable and has stable rank one. Then  $\text{Cu}(A)$  is a Cu-semigroup satisfying (O5)–(O7). Further,  $\text{Cu}(A)$  is weakly cancellative and inf-semilattice ordered by [35, Theorem 4.3] and [2, Theorem 3.8]. Hence,  $\text{Cu}(A)^\alpha$  satisfies (O5)–(O7) by Lemma 7.7.

We have seen that  $\text{Cu}(A)^\alpha$  is a sub-Cu-semigroup of  $\text{Cu}(A)$ . Thus, since  $\text{Cu}(A)$  is simple and weakly cancellative, so is  $\text{Cu}(A)^\alpha$ . To verify  $(2, \omega)$ -divisibility, let  $x \in \text{Cu}(A)^\alpha$ . Since  $A$  is simple and nonelementary, we know from Paragraph 2.3 that  $\text{Cu}(A)$  is  $(2, \omega)$ -divisible. Hence, there exists  $y \in \text{Cu}(A)$  such that  $2y \leq x \leq \infty y$ . Using [11, Lemma 5.2], we find a nonzero element  $z \in \text{Cu}(A)^\alpha$  satisfying  $z \leq y$ . Then  $2z \leq x \leq \infty z$ , a priori in  $\text{Cu}(A)$  but then also in  $\text{Cu}(A)^\alpha$  since the inclusion  $\text{Cu}(A)^\alpha \rightarrow \text{Cu}(A)$  is an order-embedding.  $\square$

**Theorem 7.9.** *Let  $A$  be a nonelementary, separable, simple, unital  $C^*$ -algebra of stable rank one, and let  $\alpha$  be a finite group action on  $A$  that has the weak tracial Rokhlin property. Then*

$$\dim(\text{Cu}(C^*(G, A, \alpha))) = \dim(\text{Cu}(A^\alpha)), \tag{4}$$

and

$$\dim(\text{Cu}(A)^{\text{Cu}(\alpha)}) - 1 \leq \dim(\text{Cu}(A^\alpha)) \leq \dim(\text{Cu}(A)^{\text{Cu}(\alpha)}) + 1.$$

**Proof.** By Proposition 7.8, we have

$$\text{Cu}(C^*(G, A, \alpha)) \cong \text{Cu}(A^\alpha),$$

which immediately proves (4).

It also follows from Proposition 7.8 that  $\text{Cu}(A)^{\text{Cu}(\alpha)}$  is a simple, weakly cancellative (hence left-soft separative),  $(2, \omega)$ -divisible sub-Cu-semigroup of  $\text{Cu}(A)$  satisfying (O5)–(O7). Since  $S$  is simple,  $S \otimes \{0, \infty\}$  is algebraic. (In fact,  $S \otimes \{0, \infty\} \cong \{0, \infty\}$ .) Therefore, we can apply Theorem 7.3 (iii) to obtain

$$\dim(\text{Cu}(A)_{\text{soft}}^{\text{Cu}(\alpha)}) \leq \dim(\text{Cu}(A)^{\text{Cu}(\alpha)}) \leq \dim(\text{Cu}(A)_{\text{soft}}^{\text{Cu}(\alpha)}) + 1.$$

Further, since  $A^\alpha$  is simple and stably finite, we know from [42, Remark 3.18] that

$$\dim(\text{Cu}(A^\alpha)_{\text{soft}}) \leq \dim(\text{Cu}(A^\alpha)) \leq \dim(\text{Cu}(A^\alpha)_{\text{soft}}) + 1.$$

The result now follows since  $\text{Cu}(A^\alpha)_{\text{soft}} \cong \text{Cu}(A)_{\text{soft}}^{\text{Cu}(\alpha)}$ ; see Proposition 7.8. □

**Example 7.10.** Let  $n \geq 2$ , and let  $G$  be  $S_n$ , the symmetric group on the set  $\{1, \dots, n\}$ . Let  $A = \mathcal{Z}^{\otimes n} \cong \mathcal{Z}$ , and let  $\alpha: G \rightarrow \text{Aut}(A)$  be the permutation action given by

$$\alpha_\theta(a_1 \otimes a_2 \otimes \dots \otimes a_n) = a_{\theta^{-1}(1)} \otimes a_{\theta^{-1}(2)} \otimes \dots \otimes a_{\theta^{-1}(n)}.$$

It follows from [27, Example 5.10] that  $\alpha$  has the weak tracial Rokhlin property. Thus, using Theorem 7.9, one has

$$\dim(\text{Cu}(A^\alpha)) = \dim(\text{Cu}(C^*(G, A, \alpha))).$$

The crossed product  $\text{Cu}(C^*(G, A, \alpha))$  is simple and  $\mathcal{Z}$ -stable; see Corollaries 5.4 and 5.7 from [27]. Therefore, it follows from [42, Proposition 3.22] that

$$\dim(\text{Cu}(A^\alpha)) = \dim(\text{Cu}(C^*(G, A, \alpha))) \leq 1,$$

and, moreover, we have  $\dim(\text{Cu}(A)^{\text{Cu}(\alpha)}) \leq 2$  by Theorem 7.9.

### 8. Radius of comparison of a Cuntz semigroup and its soft part

In this section, we show that, under the assumptions of Section 5, the radius of comparison of a Cu-semigroup is equal to that of its soft part; see Theorem 8.5. We deduce that the radius of comparison of a  $C^*$ -algebra  $A$  is equal to that of the soft part of its Cuntz semigroup whenever  $A$  is unital and separable, satisfies the global Glimm property, and has either stable rank one or strict comparison of positive elements; see Theorem 8.6.

This can be seen as a generalization of [31, Theorem 6.14] to the setting of nonsimple C\*-algebras; see Remark 8.8.

We also study in Example 8.9 the radius of comparison of certain crossed products.

**Proposition 8.1.** *Let  $S$  be a countably based, left-soft separative,  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7), and let  $x \in S$ . Then  $\widehat{x} = \widehat{\sigma(x)}$ .*

**Proof.** By Theorem 5.10, there exists  $w \in S_{\text{soft}}$  such that  $w \leq x$  and  $\widehat{x} = \widehat{w}$ . Since  $\sigma(x)$  is the largest strongly soft element dominated by  $x$  (Proposition 6.3), we get  $w \leq \sigma(x)$ , and so

$$\widehat{x} = \widehat{w} \leq \widehat{\sigma(x)} \leq \widehat{x},$$

as required.  $\square$

With the homeomorphism from Theorem 5.14 at hand, we can now relate the radius of comparison of  $S$  and  $S_{\text{soft}}$ . Let us first recall the definition of the radius of the comparison of Cu-semigroups from Section 3.3 of [13].

**Definition 8.2.** Given a Cu-semigroup  $S$ , a full element  $e \in S$  and  $r > 0$ , one says that the pair  $(S, e)$  satisfies condition (R1) for  $r$  if  $x, y \in S$  satisfy  $x \leq y$  whenever

$$\lambda(x) + r\lambda(e) \leq \lambda(y)$$

for all  $\lambda \in F(S)$ .

The *radius of comparison* of  $(S, e)$ , denoted by  $\text{rc}(S, e)$ , is the infimum of the positive elements  $r$  such that  $(S, e)$  satisfies (R1) for  $r$ .

**Remark 8.3.** In [13, Definition 3.3.2], for a C\*-algebra  $A$  and a full element  $a \in (A \otimes \mathcal{K})_+$ , the notation  $r_{A, a}$  is used for  $\text{rc}(\text{Cu}(A), [a])$ . Also, it was shown in [13, Proposition 3.2.3] that for unital C\*-algebras all of whose quotients are stably finite, the radius of comparison  $\text{rc}(\text{Cu}(A), [1_A])$  coincides with the original notion of radius of comparison  $\text{rc}(A)$  as introduced in [44, Definition 6.1].

**Proposition 8.4.** *Let  $\varphi: S \rightarrow T$  be a generalized Cu-morphism between Cu-semigroups that is also an order embedding, and let  $e \in S$  be a full element such that  $\varphi(e)$  is full in  $T$ . Then,  $\text{rc}(S, e) \leq \text{rc}(T, \varphi(e))$ .*

**Proof.** Take  $r > 0$ . We show that  $(S, e)$  satisfies condition (R1) for  $r$  whenever  $(T, \varphi(e))$  does, which readily implies the claimed inequality.

Thus, assume that  $(T, \varphi(e))$  satisfies condition (R1) for  $r$ . In order to verify that  $(S, e)$  satisfies (R1) for  $r$  as well, let  $x, y \in S$  satisfy

$$\lambda(x) + r\lambda(e) \leq \lambda(y)$$

for all  $\lambda \in F(S)$ .

Note that, for every  $\rho \in F(T)$ , we have that  $\rho \circ \varphi \in F(S)$ . Thus, we get

$$\rho(\varphi(x)) + r\rho(\varphi(e)) \leq \rho(\varphi(y))$$

for every  $\rho \in F(T)$ . It follows from our assumption that  $\varphi(x) \leq \varphi(y)$ , and, since  $\varphi$  is an order-embedding, we deduce that  $x \leq y$ , as desired.  $\square$

**Theorem 8.5.** *Let  $S$  be a  $(2, \omega)$ -divisible Cu-semigroup satisfying (O5)–(O7), and let  $e \in S$  be a full element. Then, there exists  $w \in S_{\text{soft}}$  such that*

$$\text{rc}(S, e) = \text{rc}(S_{\text{soft}}, w), \quad w \leq e \leq \infty w, \quad \text{and} \quad \widehat{e} = \widehat{w}.$$

*If  $S$  is also countably based and left-soft separative, we have*

$$\text{rc}(S, e) = \text{rc}(S_{\text{soft}}, \sigma(e)).$$

**Proof.** By Theorem 5.10, we can pick  $w \in S_{\text{soft}}$  such that

$$w \leq e \leq \infty w, \quad \text{and} \quad \widehat{e} = \widehat{w}.$$

Using at the first step that the inclusion map  $\iota: S_{\text{soft}} \rightarrow S$  is a Cu-morphism and an order-embedding and applying Proposition 8.4 and using at the last step that  $\widehat{e} = \widehat{w}$ , we get

$$\text{rc}(S_{\text{soft}}, w) \leq \text{rc}(S, \iota(w)) = \text{rc}(S, w) = \text{rc}(S, e).$$

To prove the converse inequality, let  $r > 0$  and assume that  $(S_{\text{soft}}, w)$  satisfies condition (R1) for  $r$ . Take  $\varepsilon > 0$ . We will show that  $(S, e)$  satisfies (R1) for  $r + \varepsilon$ .

Now, let  $x, y \in S$  be such that  $\lambda(x) + (r + \varepsilon)\lambda(e) \leq \lambda(y)$  for every  $\lambda \in F(S)$  or, equivalently, such that

$$\widehat{x} + (r + \varepsilon)\widehat{e} \leq \widehat{y}$$

in  $\text{LAff}(F(S))$ .

Applying [40, Proposition 7.7], we find  $k \in \mathbb{N}$  and then  $t \in S_{\text{soft}}$  such that

$$kt \leq e \leq \infty t, \quad \text{and} \quad 1 \leq k\varepsilon.$$

Thus, we get

$$\widehat{x + t + r\widehat{e}} \leq \widehat{x} + k\varepsilon\widehat{t} + r\widehat{e} \leq \widehat{x} + \varepsilon\widehat{e} + r\widehat{e} = \widehat{x} + (\varepsilon + r)\widehat{e} \leq \widehat{y}.$$

Note that, since  $e$  is full in  $S$ , so is  $t$ . By [40, Theorem 4.14(2)], this implies that  $x + t$  is strongly soft.

By Theorem 5.10, there exists  $v \in S_{\text{soft}}$  such that  $v \leq y$  and  $\widehat{v} = \widehat{y}$ . One gets

$$\widehat{x + t + r\widehat{w}} = \widehat{x + t + r\widehat{e}} \leq \widehat{y} = \widehat{v}$$

or, equivalently, that

$$\lambda(x + t) + r\lambda(w) \leq \lambda(v)$$

for every  $\lambda \in F(S)$ .

Using that  $F(S) \cong F(S_{\text{soft}})$  (Theorem 5.14) and that  $(S_{\text{soft}}, w)$  satisfies condition (R1) for  $r$ , it follows that

$$x \leq x + t \leq v \leq y.$$

This shows that, given any  $\varepsilon > 0$ ,  $(S, e)$  satisfies condition (R1) for  $r + \varepsilon$  whenever  $(S_{\text{soft}}, w)$  satisfies (R1) for  $r$ . Consequently, we have  $\text{rc}(S, e) \leq \text{rc}(S_{\text{soft}}, w)$ , as required.

Finally, if  $S$  is also countably based and left-soft separative, then we can use  $w := \sigma(e)$  by Proposition 8.1.  $\square$

**Theorem 8.6.** *Let  $A$  be a unital, separable  $C^*$ -algebra with the global Glimm property. Assume that  $A$  has stable rank one. Then*

$$\text{rc}(\text{Cu}(A), [1]) = \text{rc}(\text{Cu}(A)_{\text{soft}}, \sigma([1])).$$

**Proof.** Proceeding as in the proof of Theorem 6.9, we see that the assumptions on  $A$  imply that  $\text{Cu}(A)$  is a countably based, left-soft separative,  $(2, \omega)$ -divisible  $\text{Cu}$ -semigroup satisfying (O5)–(O7) and that  $[1]$  is full. Hence, the result follows from Theorem 8.5.  $\square$

**Corollary 8.7.** *Let  $A$  be a unital, separable, nowhere scattered  $C^*$ -algebra of stable rank one. Then*

$$\text{rc}(A) = \text{rc}(\text{Cu}(A)_{\text{soft}}, \sigma([1])).$$

**Proof.** By [43, Proposition 7.3],  $A$  has the global Glimm property; see also [2, Section 5]. Further, by [13, Proposition 3.2.3], we have  $\text{rc}(A) = \text{rc}(\text{Cu}(A), [1])$ , and so the result follows from Theorem 8.6.  $\square$

**Remark 8.8.** For a large subalgebra  $B$  of a simple, unital, stably finite, nonelementary  $C^*$ -algebra  $A$ , it is shown in [31, Theorem 6.8] that  $\text{Cu}(A)_{\text{soft}} \cong \text{Cu}(B)_{\text{soft}}$ ; see also Paragraph 3.5. Thus, using Theorem 8.5 at the first and last steps, one gets

$$\text{rc}(A) = \text{rc}(\text{Cu}(A)_{\text{soft}}, \sigma_A([1])) = \text{rc}(\text{Cu}(B)_{\text{soft}}, \sigma_B([1])) = \text{rc}(B),$$

which recovers [31, Theorem 6.14].

Note that in this case the existence of  $\sigma$  is provided by [22].

**Example 8.9.** Let  $A$  be a nonelementary, separable, simple, unital  $C^*$ -algebra of stable rank one, real rank zero and such that the order of projections over  $A$  is determined by traces, and let  $\alpha$  be a finite group action on  $A$  that has the tracial Rokhlin property. Then

$$\text{rc}(\text{Cu}(A^\alpha), [1]) = \text{rc}(\text{Cu}(A)^{\text{Cu}(\alpha)}, [1]).$$

Indeed, by [9], the crossed product  $C^*(G, A, \alpha)$  has stable rank one and then so does the fixed point algebra  $A^\alpha$  by [11, Lemma 4.3]. The question of when stable rank one passes to crossed products by a finite group action with the (weak) tracial Rokhlin property is discussed after Corollary 5.6 in [11]. One can also see that  $A^\alpha$  is nonelementary, separable, simple and unital. Therefore,  $\text{Cu}(A^\alpha)$  is a countably based, weakly cancellative (hence, left-soft separative),  $(2, \omega)$ -divisible  $\text{Cu}$ -semigroups satisfying (O5)–(O7). By Proposition 7.8, the  $\text{Cu}$ -semigroup  $\text{Cu}(A)^{\text{Cu}(\alpha)}$  has the same properties. Further, the soft parts of  $\text{Cu}(A^\alpha)$  and  $\text{Cu}(A)^{\text{Cu}(\alpha)}$  are isomorphic by Proposition 7.8.

This allows us to apply Theorem 8.5 at the first and last steps, and we get

$$\begin{aligned} \text{rc}(\text{Cu}(A^\alpha), [1]) &= \text{rc}(\text{Cu}(A^\alpha)_{\text{soft}}, \sigma([1])) \\ &= \text{rc}(\text{Cu}(A)_{\text{soft}}^{\text{Cu}(\alpha)}, \sigma([1])) = \text{rc}(\text{Cu}(A)^{\text{Cu}(\alpha)}, [1]). \end{aligned}$$

Other examples where our results might be applicable are those obtained in [10].

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