On proper intersections on a singular analytic space

Citation for the original published paper (version of record):
Andersson, M., Samuelsson Kalm, H. (2025). On proper intersections on a singular analytic space.
Mathematische Zeitschrift, 309(2). http://dx.doi.org/10.1007/s00209-024-03642-1

Mathematische Zeitschrift (2025) 309:23
https://doi.org/10.1007/s00209-024-03642-1

Mathematische Zeitschrift

On proper intersections on a singular analytic space

Mats Andersson[1] · Håkan Samuelsson Kalm[1]

Received: 9 October 2023 / Accepted: 27 October 2024

Abstract
Given a reduced analytic space Y we introduce a class of *nice* cycles, including all effective $\mathbb{Q}$-Cartier divisors. Equidimensional nice cycles that intersect properly allow for a natural intersection product. Using $\bar\partial$-potentials and residue calculus we provide an intrinsic way of defining this product. The intrinsic definition makes it possible to prove global formulas. In case Y is smooth all cycles are differences of nice cycles, and so we get a new way to define classical proper intersections.

1 Introduction

Let Y be a complex manifold of dimension n. In this paper a cycle μ in Y is a locally finite sum $\sum_j \nu_j Z_j$, where $\nu_j \in \mathbb{Q}\setminus\{0\}$ and Z_j are distinct irreducible analytic subsets of Y. The cycle is effective if $\nu_j > 0$. The support, $|\mu|$, of μ is the union of the Z_j and the (co)dimension of μ is the (co)dimension of $\cup_j Z_j$. Assume that $\mu_1, \dots, \mu_r$ are cycles in Y of pure codimensions. It is well-known that then

$$\operatorname{codim}\big(|\mu_1| \cap \cdots \cap |\mu_r|\big) \le \operatorname{codim}|\mu_1| + \cdots + \operatorname{codim}|\mu_r|. \tag{1.1}$$

If equality holds in (1.1), then $\mu_1, \dots, \mu_r$ are said to intersect properly. In that case there is a well-defined cycle

$$\mu_r \cdots \mu_1 = \sum_j m_j V_j,$$

the proper intersection product, where V_j are the irreducible components of the set-theoretical intersection $|\mu_1| \cap \cdots \cap |\mu_r|$ and $m_j \in \mathbb{Q}$. In general some m_j may be 0. However, if μ_j are effective, then $\mu_r \cdots \mu_1$ is effective and all $m_j > 0$.

Classically, this intersection product was defined algebraically, see, e.g., [15]. On the analytic side, given a cycle $\mu = \sum_j \nu_j Z_j$, recall that there is an associated closed current $[\mu] = \sum_j \nu_j [Z_j]$, the Lelong current of μ, where $[Z_j]$ is integration over the regular part of

The first author was partially supported by the Swedish Research Council.

✉ Håkan Samuelsson Kalm
hasam@chalmers.se

Mats Andersson
matsa@chalmers.se

1 Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, 412 96 Göteborg, Sweden

Published online: 23 December 2024

Z_j. It is quite remarkable that one can define the Lelong current of the proper intersection as

$$[\mu_r \cdots \mu_1] = [\mu_r] \wedge \cdots \wedge [\mu_1], \tag{1.2}$$

where the current product on the right-hand side is given a meaning via suitable regularizations of the various factors, see, e.g., [12] and [14].

Let us now assume that Y is a reduced analytic space of pure dimension n. Then there is no known general analogous intersection theory; not even (1.1) holds in general. We say that a cycle μ in Y is *nice* if locally there is an embedding $i\colon Y \to Y'$, where Y' is smooth, and an effective pure-dimensional cycle μ' in Y' such that $i_*\mu$, i.e., μ considered as a cycle in Y', is the proper intersection $\mu' \cdot i_*Y$ in Y'. By definition thus a nice cycle is effective and locally equidimensional. However, the irreducible components of a nice cycle are in general not nice. In a neighborhood of a given point one can always take the embedding i to be the minimal embedding, see Lemma 5.5 below. If $\mu_1, \ldots, \mu_r$ are nice and of pure codimensions, then (1.1) holds, and as in the smooth case we say that they intersect properly if equality holds. In that case we have an intrinsic nice cycle $\mu = \mu_r \cdots \mu_1$. As in the smooth case, $\mu_r \cdots \mu_1$ is commutative. If $i\colon Y \to Y'$ is a local embedding and μ_j' are effective cycles in Y' such that $i_*\mu_j = \mu_j' \cdot i_*Y$, then $\mu_1', \ldots, \mu_r', i_*Y$ intersect properly and $i_*\mu = \mu_r' \cdots \mu_1' \cdot i_*Y$. See Sect. 5 for details.

It is well-known that the proper intersection in a complex manifold of cycles with integer coefficients has integer coefficients. However, in a reduced analytic space we need to consider cycles with rational coefficients. In fact, even if μ has integer coefficients, μ' may need to have rational coefficients. Moreover, the intersection product of nice cycles with integer coefficients in general has rational coefficients.

The intersection theory for nice cycles in Y is in a way a quite simple consequence of the proper intersection theory in ambient space. Our first main result is an intrinsic way to define proper intersections in Y, i.e., with no explicit reference to any ambient space. The approach in [12] to use regularizations of $[\mu_j]$ seems to be difficult to extend when Y is singular. Instead we introduce *good* $\bar\partial$-potentials. If μ is a cycle in Y of pure codimension κ we say that a current u in Y of bidegree $(\kappa, \kappa - 1)$ is a good $\bar\partial$-potential of μ if $\bar\partial u = [\mu]$, u is smooth outside $|\mu|$, and u is pseudomeromorphic in Y. This last requirement is an intrinsic regularity property that will be explained in Sect. 2.1.

Theorem 1.1 *Let Y be a reduced analytic space of pure dimension.*

(i) Each nice cycle in Y locally has a good $\bar\partial$-potential.
(ii) Assume that μ_1 and μ_2 are nice cycles that intersect properly and u_2 is a good $\bar\partial$-potential of μ_2. Then $u_2 \wedge [\mu_1]$, a priori defined outside $|\mu_2|$, has a unique pseudomeromorphic extension to Y of pure bidegree. Moreover,

$$\bar\partial(u_2 \wedge [\mu_1]) = [\mu_2 \cdot \mu_1]. \tag{1.3}$$

As in the smooth case, cf. (1.2), we write $[\mu_2] \wedge [\mu_1]$ for the current in (1.3). Since $\mu_2 \cdot \mu_1 = \mu_1 \cdot \mu_2$, clearly $[\mu_2] \wedge [\mu_1]$ is commutative and independent of the choice of u_2. This follows also quite easily from the intrinsic definition and residue calculus, see Proposition 3.4.

If $\mu_1, \ldots, \mu_r$ are nice and intersect properly, then in a neighborhood of any $x \in |\mu_1| \cap \cdots \cap |\mu_r|$ this result can be iterated to give the Lelong current of $\mu_r \cdots \mu_1$ there. Since all effective cycles in a manifold are nice this gives in particular a new definition of proper intersection when Y is smooth.

Remark 1.2 In view of the influential paper [16] by Gillet–Soulé it might look more natural to use dd^c-potentials rather than $\bar\partial$-potentials. However, the latter choice gives access to residue theory, without which we cannot show existence of local potentials, let alone define the product $u_2 \wedge [\mu_1]$; cf. Remark 5.13.

Our intrinsic definition of the proper intersection of nice cycles makes it possible to prove global results for a compact singular Y.

Theorem 1.3 *Let Y be compact, ω a Kähler form, and let $\mu_1, \mu_2, \dots, \mu_r$ be nice cycles in Y of pure codimensions $\kappa_1, \dots, \kappa_r$, respectively. Assume that $\mu_1, \dots, \mu_k$ intersect properly for $k = 2, \dots, r$. Assume also that for each $j = 1, \dots, r$ there is a smooth (closed) form α_j, and a pseudomeromorphic current a_j of bidegree $(\kappa_j, \kappa_j - 1)$, smooth in $Y \setminus |\mu_j|$, such that*

$$\bar\partial a_j = [\mu_j] - \alpha_j, \ \ j = 1, \dots, r. \tag{1.4}$$

In addition, suppose that all of the α_j, except possibly α_1, locally have smooth $\bar\partial$-potentials. If $\kappa = \kappa_1 + \cdots + \kappa_r$, then

$$\int_Y [\mu_r] \wedge \cdots \wedge [\mu_1] \wedge \omega^{n-\kappa} = \int_Y \alpha_r \wedge \cdots \wedge \alpha_1 \wedge \omega^{n-\kappa}. \tag{1.5}$$

Formula (1.5) suggests that the intersection product is “cohomologically sound”.

Example 1.4 Assume that μ_j in Theorem 1.3 are the fundamental cycles of ideals defined by global sections of Hermitian vector bundles E_j of ranks $\kappa_j = \operatorname{codim} \mu_j$. We will see in Sect. 6 that then μ_j are nice, and in Sect. 7 we prove that there are global pseudomeromorphic currents a_j, smooth in $Y \setminus |\mu_j|$, such that $\bar\partial a_j = da_j = [\mu_j] - c_{\kappa_j}(E_j)$. Since the Chern forms $c_{\kappa_j}(E_j)$ locally have smooth potentials we can thus take $\alpha_j = c_{\kappa_j}(E_j)$ in (1.4) and (1.5).

Cycles as in Example 1.4 will be called RE-cycles, and are discussed in Sect. 6.

There is a proper intersection theory on normal surfaces due to Mumford [17]. Recently Barlet and Magnússon, [8], defined proper intersections on a so-called nearly smooth Y by analytic methods. In Sect. 9 we show that for RE-cycles our intersection product coincides with the intersection in [8].

Our approach relies on residue theory, and in Sect. 2 we have collected some material that we need. In Sect. 3 we present our $\bar\partial$-potential approach to proper intersection. In Sect. 4 we use this approach in case Y is smooth and show that it gives the usual intersection product. Proper intersection of nice cycles is discussed in Sect. 5 and Theorem 1.1 is proved. The special case of RE-cycles is considered in Sect. 6. We prove the global Theorem 1.3 in Sect. 7 and provide various examples in Sect. 8. In the last section we show that our product, at least for RE-cycles, coincides with the product in [8] when Y is nearly smooth.

2 Some notions and results in residue theory

Throughout this section Y is a (reduced) analytic space of pure dimension n. A smooth form α on Y_{reg} is smooth on Y, $\alpha \in \mathcal{E}(Y)$, if for a local embedding $i\colon Y \to Y'$ into a manifold Y' there is a smooth form $\tilde\alpha$ in Y' such that $\alpha = i^*\tilde\alpha$ on Y_{reg}. It follows that $\bar\partial$, d, and ∂ are well-defined on $\mathcal{E}(Y)$. If X is a reduced analytic space and $g\colon X \to Y$ is a holomorphic mapping, then there is a functorial pullback mapping $g^*\colon \mathcal{E}(Y) \to \mathcal{E}(X)$, see [9, Corollary 3.2.21].

Currents on Y are dual to the compactly supported smooth forms on Y. More concretely, if $i: Y \to Y'$ is an embedding, then the currents on Y can be identified with the currents on Y' that vanish on test forms ξ such that $i^*\xi = 0$. Equivalently, the currents on Y can be identified with the currents τ on Y' such that $\xi \wedge \tau = 0$ for all test forms ξ with $i^*\xi = 0$.

If $g: X \to Y$ is a proper holomorphic mapping, then there is a pushforward mapping $g_*: \mathcal{C}(X) \to \mathcal{C}(Y)$ from currents on X to currents on Y defined as $\langle g_*\tau, \xi\rangle = \langle \tau, g^*\xi\rangle$. If τ is a current on X and α is smooth on Y, then

$$\alpha \wedge g_*\tau = g_*(g^*\alpha \wedge \tau). \tag{2.1}$$

If $Z \subset Y$ is an analytic subset of pure codimension κ and $j: Z \to Y$ is the inclusion, then the Lelong current $[Z]$ has bidegree (κ, κ) and $j_*1 = [Z]$.

If $g: X \to Y$ is proper, then there is a mapping $g_*: \mathcal{Z}(X) \to \mathcal{Z}(Y)$, where $\mathcal{Z}(Y)$ are the cycles in Y, defined as follows. Let $\mu \in \mathcal{Z}(X)$ be an irreducible analytic subset of X of dimension k. If $\dim g(\mu) < k$, then $g_*\mu = 0$, and if $\dim g(\mu) = k$, then $g_*\mu = \deg(g|_\mu)g(\mu)$, where $\deg(g|_\mu)$ is the number of points in $g^{-1}(x)$ for generic $x \in g(\mu)$. By linearity, g_* extends to $\mathcal{Z}(X)$. We have, cf. [3, Section 2],

$$g_*[\mu] = [g_*\mu]. \tag{2.2}$$

In what follows we will usually identify a cycle with its Lelong current. In view of (2.2), this is consistent with pushforward. With this convention, Y (considered as a cycle in Y) is identified with the constant function 1, and if $j: Z \to Y$ is an embedding of a reduced analytic space, then $j_*1 = [j(Z)] = j(Z) = j_*Z$. If Z is an analytic subset of Y and j is the inclusion, then we often identify Z and $j(Z)$.

2.1 Pseudomeromorphic currents

The function $1/z^\ell$ in $\mathbb{C} \setminus \{0\}$ extends to $\mathbb{C}$ as a principal value current. The current $\bar\partial(1/z^\ell)$ is the associated residue current. If $\mathcal{U} \subset \mathbb{C}^r$ is open, $(z_1, \dots, z_r)$ are coordinates in $\mathcal{U}$, and α is a smooth compactly supported form in $\mathcal{U}$, thus

$$\alpha \wedge \frac{1}{z_1^{\ell_1}} \cdots \frac{1}{z_s^{\ell_s}} \bar\partial \frac{1}{z_{s+1}^{\ell_{s+1}}} \wedge \cdots \wedge \bar\partial \frac{1}{z_r^{\ell_r}}$$

exists as a tensor product in $\mathcal{U}$. Such a current is an *elementary pseudomeromorphic current*.

The notion of pseudomeromorphic currents was introduced in [6] and further developed in [4]. We take the characterization in [7, Theorem 2.15] of pseudomeromorphic currents as the definition in this paper. If Y is smooth, then a germ of a current τ at $x \in Y$ is *pseudomeromorphic* if it is a finite sum

$$\sum_\ell (f_\ell)_* \nu_\ell,$$

where $f_\ell: \mathcal{U}_\ell \to Y$ are holomorphic mappings, $\mathcal{U}_\ell \subset \mathbb{C}^{N_\ell}$ are open, and ν_ℓ are elementary in $\mathcal{U}_\ell$. If Y is a reduced analytic space, then a germ of a current τ at $x \in Y$ is pseudomeromorphic if there is a smooth modification $\pi: Y' \to Y_x$ of a neighborhood Y_x of x and a pseudomeromorphic current τ' in Y' such that $\pi_*\tau' = \tau$.

The set of germs of pseudomeromorphic currents in Y is an open subset of the sheaf of currents on Y and thus is a sheaf, the sheaf $\mathcal{PM}_Y$ of pseudomeromorphic current on Y. This sheaf is closed under $\bar\partial$ and multiplication by smooth forms. We refer to, e.g., [7] for proofs of the statements below about pseudomeromorphic currents.

If τ is pseudomeromorphic and the holomorphic function h vanishes on the support of τ, then $\bar{h}\tau = 0 = d\bar{h} \wedge \tau$. Furthermore, we have the
Dimension principle: If τ is pseudomeromorphic, has bidegree $(*, q)$ and support contained in a subvariety of codimension $> q$, then $\tau = 0$.

Example 2.1 If f is a holomorphic function on Y, not identically 0 on any irreducible component, then $1/f$, a priori defined outside $Z = f^{-1}(0)$, has a pseudomeromorphic extension to Y; cf. Example 2.2. By the dimension principle such an extension must be unique. The residue current, $\bar{\partial}(1/f)$, clearly has support in Z.

If τ is pseudomeromorphic in $\mathcal{U}$ and Z is a subvariety, then the natural restriction of τ to the open subset $\mathcal{U} \setminus Z$ has a pseudomeromorphic extension $\mathbf{1}_{\mathcal{U}\setminus Z}\tau$ to $\mathcal{U}$ such that

$$\mathbf{1}_{\mathcal{U}\setminus Z}\tau = \lim_{\epsilon\to 0} \chi(|f|^2 v/\epsilon)\tau, \tag{2.3}$$

if f is any tuple of holomorphic functions with $\{f = 0\} = Z$, χ is a smooth function on $[0, \infty)$ that is 0 in a neighborhood of 0 and 1 in a neighborhood of ∞, and v is a smooth strictly positive function. The right-hand side of (2.3) is indeed independent of the choice of f, χ, and v. It follows that

$$\mathbf{1}_Z\tau := \tau - \mathbf{1}_{\mathcal{U}\setminus Z}\tau \tag{2.4}$$

is pseudomeromorphic and has support on Z. If Z' is another subvariety, then

$$\mathbf{1}_{Z'}\mathbf{1}_Z\tau = \mathbf{1}_{Z\cap Z'}\tau. \tag{2.5}$$

If τ is pseudomeromorphic and α is a smooth form, then[1]

$$\mathbf{1}_Z\alpha \wedge \tau = \alpha \wedge \mathbf{1}_Z\tau. \tag{2.6}$$

If $g\colon X \to Y$ is proper, ν and $g_*\nu$ are pseudomeromorphic, and Z is a subvariety of Y, then

$$\mathbf{1}_Z g_*\nu = g_*(\mathbf{1}_{g^{-1}(Z)}\nu). \tag{2.7}$$

Example 2.2 With the setting in Example 2.1 we have $\mathbf{1}_{Y\setminus Z}(1/f) = 1/f$ by the dimension principle. In particular, cf. (2.3), $\lim_{\epsilon\to 0} \chi(|f|^2 v/\epsilon)/f = 1/f$. Thus, $1/f$ can be defined as a principal value current, which is the original definition of Herrera–Lieberman.

A current a on Y is *almost semi-meromorphic*, $a \in ASM(Y)$, if $a = \pi_*(\alpha/\sigma)$, where $\pi : X \to Y$ is a modification, σ is a section of a line bundle $L \to X$, and α is a smooth form with values in L. Notice that if L is Hermitian and β is a smooth form, then $\pi_*(\partial \log |\sigma|^2 \wedge \beta) \in ASM(Y)$. The smallest Zariski closed set outside which $a \in ASM(Y)$ is smooth is called the Zariski singular support of a. If V is the Zariski singular support of a, then in view of (2.7) and Example 2.2 we have $\mathbf{1}_V a = 0$. The following lemma is [7, Theorem 4.8].

Lemma 2.3 *If $a \in ASM(Y)$ has Zariski singular support V and $\tau \in \mathcal{PM}(Y)$, then $a \wedge \tau$, a priori defined on $Y \setminus V$, has a unique pseudomeromorphic extension T to Y such that* $\mathbf{1}_V T = 0$.

We write $a \wedge \tau$ for the extension as well. By (2.3) it follows that if f is a tuple of holomorphic functions with $\{f = 0\} = V$, then $\chi(|f|^2 v/\epsilon)a \wedge \tau \to a \wedge \tau$.

Example 2.4 Let f be a holomorphic function on Y and $\tau \in \mathcal{PM}(Y)$. Then $1/f \in ASM(Y)$ and $(1/f)\tau \in \mathcal{PM}(Y)$ by Lemma 2.3. Moreover, $f(1/f)\tau = \mathbf{1}_{Y\setminus\{f=0\}}\tau$.

[1] If noting else is suggested by brackets, $\mathbf{1}_Z$ is always assumed to act on the whole expression on its right.

2.2 Two operations on $\mathcal{PM}$

Let σ be a section of a Hermitian holomorphic vector bundle $E \to Y$ with zero set[2] Z.

Lemma 2.5 *For each $k = 1, 2, \ldots$, there is a (necessarily unique) almost semi-meromorphic current m_k^σ in Y that coincides with $(2\pi i)^{-1}\partial \log|\sigma|^2 \wedge (dd^c \log|\sigma|^2)^{k-1}$ outside Z and such that $\mathbf{1}_Z m_k^\sigma = 0$.*

Proof Let $\pi : X \to Y$ be a normal modification such that for each connected component of X, either $\pi^*\sigma = 0$ or $\pi^*\sigma = \sigma^0\sigma'$, where σ^0 is a generically non-vanishing section of a line bundle L and σ' is a non-vanishing section of $L^* \otimes \pi^* E$. Equip L with a Hermitian metric by setting $|s|_L^2 := |s\sigma'|^2$ for any section s of L. In particular, $|\pi^*\sigma|^2 = |\sigma^0|_L^2$ on the union X' of the components of X where $\pi^*\sigma$ is generically non-vanishing. By the Poincaré–Lelong formula thus

$$dd^c \log\left|\pi^*\sigma|_{X'}\right|^2 = dd^c \log|\sigma^0|_L^2 = \mathrm{div}\sigma^0 - c_1(L).$$

Since π is a biholomorphism generically we have $1 = \pi_* 1$ as currents. By (2.1) thus,

$$(2\pi i)^{-1}\partial \log|\sigma|^2 \wedge (dd^c \log|\sigma|^2)^{k-1} = \pi_*\big((2\pi i)^{-1}\partial \log|\sigma^0|_L^2 \wedge (-c_1(L))^{k-1}\big) \quad (2.8)$$

outside Z. Defining $\partial \log|\sigma^0|_L^2 \wedge (-c_1(L))^{k-1} = 0$ on $X \backslash X'$ the lemma follows since the right-hand side then is in $ASM(Y)$ and vanishes on irreducible components of Y contained in Z. □

By Lemma 2.3 we can now define our first operation:

$$m_k^\sigma : \mathcal{PM}_Y \to \mathcal{PM}_Y, \qquad \tau \mapsto m_k^\sigma \wedge \tau, \quad k = 1, 2, \ldots. \quad (2.9)$$

In view of Lemma 2.3, $\mathbf{1}_Z m_k^\sigma \wedge \tau = 0$, and if τ has support in Z, then $m_k^\sigma \wedge \tau = 0$. For any pseudomeromorphic τ, by the comment after Lemma 2.3, we have

$$m_k^\sigma \wedge \tau = \lim_{\epsilon\to 0} \chi(|\sigma|^2/\epsilon)(2\pi i)^{-1}\partial \log|\sigma|^2 \wedge (dd^c \log|\sigma|^2)^{k-1} \wedge \tau. \quad (2.10)$$

Our second operation is the following:

$$M_k^\sigma : \mathcal{PM}_Y \to \mathcal{PM}_Y, \quad M_0^\sigma \wedge \tau = \mathbf{1}_Z \tau, \quad M_k^\sigma \wedge \tau = \mathbf{1}_Z \bar\partial(m_k^\sigma \wedge \tau), \quad k = 1, 2, \ldots.$$

If τ has support in Z, then $M_0^\sigma \wedge \tau = \tau$ and $M_k^\sigma \wedge \tau = 0$, $k = 1, 2, \ldots$. One can check that for any pseudomeromorphic τ,

$$M_0^\sigma \wedge \tau = \lim_{\epsilon\to 0}(1 - \chi(|\sigma|^2/\epsilon))\tau, \quad M_k^\sigma \wedge \tau = \lim_{\epsilon\to 0} \bar\partial\chi(|\sigma|^2/\epsilon) \wedge m_k^\sigma \wedge \tau, \quad k = 1, 2, \ldots. \quad (2.11)$$

If α is a smooth form, then by (2.11),

$$\alpha \wedge M_k^\sigma \wedge \tau = M_k^\sigma \wedge (\alpha \wedge \tau). \quad (2.12)$$

Moreover, if $\tau = g_*\nu$ is pseudomeromorphic, where $g : X \to Y$ is a proper holomorphic mapping and ν is pseudomeromorphic on X, then in view of (2.1), (2.11), and (2.10)

$$M_k^\sigma \wedge \tau = g_*(M_k^{g^*\sigma} \wedge \nu), \qquad m_k^\sigma \wedge \tau = g_*(m_k^{g^*\sigma} \wedge \nu). \quad (2.13)$$

We write M_k^σ instead of $M_k^\sigma \wedge 1$. By the dimension principle, $M_k^\sigma = 0$ for $k < \operatorname{codim} Z$.

[2] Z may contain irreducible components of Y.

Example 2.6 Let Z be a germ of an irreducible analytic subset of codimension κ in Y and let σ be a tuple of holomorphic functions defining the ideal $\mathcal{J}_Z$ of holomorphic functions vanishing on Z. Then by [3, Corollary 1.3], $M_\kappa^\sigma = \nu[Z]$, where ν is a positive integer. It follows that (the Lelong current of) any cycle in Y is pseudomeromorphic. If Z is not contained in Y_{sing}, then $\nu = 1$ in view of [5, Corollary 1.3]. However, if Z is contained in Y_{sing}, then it may happen that $\nu \geq 2$.

Remark 2.7 Assume that $\tau = g_*\beta$ is pseudomeromorphic, where $g\colon X \to Y$ is proper and β is a product of components of Chern forms of various Hermitian vector bundles over X. Then τ is a *generalized cycle*, a notion that was introduced in [3]. It follows from [3, Section 5] that

$$M_k^\sigma \wedge \tau = \mathbf{1}_Z[dd^c \log|\sigma|^2]^k \wedge \tau,$$

where

$$[dd^c \log|\sigma|^2]^k \wedge \tau = \lim_{\epsilon\to 0}(dd^c \log(|\sigma|^2 + \epsilon))^k \wedge \tau.$$

2.3 Regular sequences

Assume that $f = (f_1, \ldots, f_\kappa)$ is a regular sequence at $x \in Y$ and let $Z = \{f = 0\}$; in particular then codim $Z_x = \kappa$. We can consider f as a section of the trivial rank-κ vector bundle equipped with the trivial metric. Then $\bar\partial m_\kappa^f = 0$ outside Z. By Lemma 2.3 we get that if τ is pseudomeromorphic, then $\mathbf{1}_{Y\setminus Z}\bar\partial(m_\kappa^f \wedge \tau) = -m_\kappa^f \wedge \bar\partial\tau$. It follows that

$$\bar\partial(m_\kappa^f \wedge \tau) = \mathbf{1}_Z\bar\partial(m_\kappa^f \wedge \tau) + \mathbf{1}_{Y\setminus Z}\bar\partial(m_\kappa^f \wedge \tau) = M_\kappa^f \wedge \tau - m_\kappa^f \wedge \bar\partial\tau \tag{2.14}$$

and, by applying $\bar\partial$ to (2.14), that

$$\bar\partial(M_\kappa^f \wedge \tau) = M_\kappa^f \wedge \bar\partial\tau. \tag{2.15}$$

Example 2.8 Let (ζ, z) be coordinates in $\mathbb{C}^n \times \mathbb{C}^n$ and let $\eta = \zeta - z$. Let $p_1, p_2\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ be the projections on the first and second factor, respectively, and let $i\colon \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$ be the diagonal embedding. We claim that if μ is pseudomeromorphic in $\mathcal{U} \subset \mathbb{C}^n$ then

$$M_n^\eta \wedge (1 \otimes \mu) = i_*\mu \tag{2.16}$$

in $\mathcal{U} \times \mathcal{U}$. If additionally μ has compact support, then

$$\mu = \bar\partial\big((p_1)_*(m_n^\eta \wedge (1 \otimes \mu))\big) + (p_1)_*(m_n^\eta \wedge (1 \otimes \bar\partial\mu)). \tag{2.17}$$

To see (2.16), let $[\Delta] := i_*1$ be the diagonal. Then $[\Delta] \wedge (1 \otimes \mu)$ is well-defined as a tensor product; in fact, in the coordinates $(\eta, z) = (\zeta - z, z)$ it is just $\delta_0(\eta) \otimes \mu(z)$. It follows that

$$[\Delta] \wedge (1 \otimes \mu) = i_*\mu$$

in $\mathcal{U} \times \mathcal{U}$. Since by Example 2.6 we have $M_n^\eta = [\Delta]$, (2.16) follows. To see (2.17), notice that by (2.14) we have

$$\bar\partial(m_n^\eta \wedge (1 \otimes \mu)) = M_n^\eta \wedge (1 \otimes \mu) - m_n^\eta \wedge (1 \otimes \bar\partial\mu).$$

If μ has compact support then we can apply $(p_1)_*$. By (2.16), and since $p_1 \circ i = \mathrm{id}$, thus (2.17) follows. Notice that $m_n^\eta = \partial|\eta|^2 \wedge (\bar\partial\partial|\eta|^2)^{n-1}/(2\pi i|\eta|^2)^n$ is the Bochner-Martinelli kernel and that (2.17) is the Bochner-Martinelli formula.

2.4 The inequality (1.1)

Let us recall why (1.1) holds in a complex manifold Y. By induction it is enough to verify it for two analytic sets A_1 and A_2 in Y. Let $n = \dim Y$. Notice that if $i\colon Y \to Y \times Y$ is the diagonal embedding and $\Delta = i_* Y$, then

$$i_*(A_1 \cap A_2) = (A_1 \times A_2) \cap \Delta.$$

Locally in $Y \times Y$, Δ is defined by n functions so that the right-hand side is obtained by successive intersection by n divisors (this is not true when Y is singular so then the argument breaks down). It is well-known that each such intersection can decrease the dimension by at most one unit, see, e.g., [13, Theorem II 6.2]. Hence $\dim i_*(A_1 \cap A_2) \geq \dim A_1 + \dim A_2 - n$ and so (1.1) follows.

From (1.1) we get the following useful observation.

Lemma 2.9 *Assume that $A_1, \ldots, A_r$ are germs of analytic sets of pure codimensions at a point in a smooth manifold Y. If the intersection $A_1 \cap \cdots \cap A_r$ is proper, then each intersection $A_1 \cap \cdots \cap A_\nu$, $\nu \leq r$, is proper.*

Proof Assume that $A_1, \ldots, A_r$ have codimensions $\kappa_1, \ldots \kappa_r$, respectively. By assumption, $\operatorname{codim}(A_1 \cap \cdots \cap A_r) = \kappa_1 + \cdots + \kappa_r$. In view of (1.1) thus

$$\begin{aligned}\kappa_1 + \cdots + \kappa_r &= \operatorname{codim}\big((A_1 \cap \cdots \cap A_\nu) \cap A_{\nu+1} \cap \cdots \cap A_r\big)\\ &\leq \operatorname{codim}(A_1 \cap \cdots \cap A_\nu) + \kappa_{\nu+1} + \cdots + \kappa_r \leq \kappa_1 + \cdots \kappa_r.\end{aligned}$$

Both inequalities thus are equalities and we conclude that $\operatorname{codim}(A_1 \cap \cdots \cap A_\nu) = \kappa_1 + \cdots + \kappa_\nu$ as desired. □

3 A $\bar\partial$-potential approach to proper intersections

Let Y be a reduced analytic space of pure dimension n. We now describe our intrinsic approach to a proper intersection product in Y of cycles which locally admit good potentials.

Definition 3.1 Assume that μ is a cycle in Y of pure codimension $\kappa \geq 1$ with support Z. We say that a pseudomeromorphic current u of bidegree $(\kappa, \kappa - 1)$ in an open subset $\mathcal{U} \subset Y$ is a *good $\bar\partial$-potential* (or simply a *good potential*) of μ if u is smooth in $\mathcal{U} \setminus Z$ and $\bar\partial u = \mu$.

It follows from the dimension principle (see Sect. 2.1) that $\mathbf{1}_Z u = 0$. If $\chi_\epsilon = \chi(|f|^2/\epsilon)$, where $Z(f) = Z$, therefore $\chi_\epsilon u$ are smooth and $\chi_\epsilon u \to u$; cf. (2.3) and (2.4). Since $\bar\partial u = 0$ outside Z thus $\mu_\epsilon = \bar\partial \chi_\epsilon \wedge u$ are smooth, $\bar\partial$-closed, and $\mu_\epsilon \to \mu$.

Let μ_1 and μ_2 be (germs of) cycles of pure codimensions $\kappa_j \geq 1$ at a point $x \in Y$, let $Z_j = |\mu_j|$, and assume that u_j is a good potential of μ_j for $j = 1, 2$.

Proposition 3.2 *The smooth form $u_2 \wedge u_1$, a priori defined in the Zariski open set $Z_2^c \cap Z_1^c$, has a unique pseudomeromorphic extension T to Y such that $\mathbf{1}_{Z_2 \cup Z_1} T = 0$.*

Proof If T and T' are pseudomeromorphic currents with the stated properties, then

$$T - T' = \mathbf{1}_{Z_2^c \cap Z_1^c}(T - T') + \mathbf{1}_{Z_2 \cup Z_1}(T - T') = 0$$

since they coincide in $Z_2^c \cap Z_1^c$ and both vanish on the complement.

For the existence of the extension, let $i\colon Y \to Y \times Y$ be the diagonal embedding and let $\Delta = i_* Y$. Since Δ is not contained in $(Y \times Y)_{sing}$ it follows from Example 2.6 that $M_n^\eta = [\Delta]$, where η is a tuple of holomorphic functions defining Δ. By (2.12) we have

$$M_n^\eta \wedge (u_2 \otimes u_1) = (u_2 \otimes u_1) \wedge M_n^\eta = (u_2 \otimes u_1) \wedge [\Delta] = i_* i^*(u_2 \otimes u_1)$$

in $Z_2^c \times Z_1^c$. If $p\colon Y \times Y \to Y$ is the projection, e.g., on the first factor, then $p \circ i = \mathrm{id}_Y$ and it follows that $p_*\big(M_n^\eta \wedge (u_2 \otimes u_1)\big)$ is equal to $u_2 \wedge u_1$ in $Z_2^c \cap Z_1^c$. Since p is a simple projection, p_* preserves pseudomeromorphicity and thus

$$T := \mathbf{1}_{Z_2^c \cap Z_1^c}\, p_*\big(M_n^\eta \wedge (u_2 \otimes u_1)\big)$$

is the desired pseudomeromorphic extension. □

We denote the extension T by $u_2 \wedge u_1$ as well. It is immediate from the proposition that this product is anti-commutative since its restriction to $Z_2^c \cap Z_1^c$ is.

We will now define the product of u_2 and μ_1. Notice that $u_2 \wedge \mu_1$ is well-defined in Z_2^c. We claim that $\mathcal{T} = -\mathbf{1}_{Z_2^c} \bar\partial (u_2 \wedge u_1)$ is the unique pseudomeromorphic extension of $u_2 \wedge \mu_1$ to Y such that $\mathbf{1}_{Z_2} \mathcal{T} = 0$. The uniqueness of such an extension is clear, and since u_2 is $\bar\partial$-closed in Z_2^c, $\mathcal{T}$ is indeed an extension, so the claim follows. We denote this extension by $u_2 \wedge \mu_1$. That is,

$$u_2 \wedge \mu_1 = -\mathbf{1}_{Z_2^c} \bar\partial (u_2 \wedge u_1). \tag{3.1}$$

It follows from the dimension principle that if (3.4) below holds and T is any pseudomeromorphic current of pure bidegree such that $T = u_2 \wedge \mu_1$ in Z_2^c and $\operatorname{supp} T \subset Z_1$, then $T = u_2 \wedge \mu_1$ in Y.

We now define

$$\mu_2 \wedge \mu_1 := \bar\partial (u_2 \wedge \mu_1). \tag{3.2}$$

It is clear that $\mu_2 \wedge \mu_1$ has support on $Z_2 \cap Z_1$. However, without further assumptions it may depend on the choice of potential u_2. If f is a holomorphic tuple with zero set Z_2 we have

$$u_2 \wedge \mu_1 = \lim_{\epsilon \to 0} \chi(|f|^2/\epsilon) u_2 \wedge \mu_1 \tag{3.3}$$

since $\mathbf{1}_{Z_2} u_2 \wedge \mu_1 = 0$, cf. (2.3) and (2.4). Since $\bar\partial(u_2 \wedge \mu_1) = 0$ in Z_2^c it follows that

$$\mu_2 \wedge \mu_1 = \lim_{\epsilon \to 0} \bar\partial \chi(|f|^2/\epsilon) \wedge u_2 \wedge \mu_1.$$

Definition 3.3 We say that the pure-dimensional cycles μ_2 and μ_1 intersect properly if

$$\operatorname{codim} (Z_2 \cap Z_1) \geq \operatorname{codim} Z_2 + \operatorname{codim} Z_1. \tag{3.4}$$

Since $\mu_2 \wedge \mu_1$ has support in $Z_2 \cap Z_1$ it follows from the dimension principle that $\mu_2 \wedge \mu_1 = 0$ if $\operatorname{codim} (Z_2 \cap Z_1) > \operatorname{codim} Z_2 + \operatorname{codim} Z_1$.

Proposition 3.4 *Assume that Z_2 and Z_1 intersect properly. With the notation above,*

$$\mu_2 \wedge \mu_1 = \mu_1 \wedge \mu_2, \tag{3.5}$$

and the product is independent of the choice of good potentials in the definition.

We have already noticed that $\mu_2 \wedge \mu_1$ has support on $Z_2 \cap Z_1$.

Proof We claim that

$$\bar{\partial}(u_2 \wedge u_1) = -u_2 \wedge \mu_1 + u_1 \wedge \mu_2. \tag{3.6}$$

In fact, $\bar{\partial}(u_2 \wedge u_1) = 0$ in the Zariski open set $Z_1^c \cap Z_2^c$, and so

$$\mathbf{1}_{Z_1^c \cap Z_2^c} \bar{\partial}(u_2 \wedge u_1) = 0.$$

Recall that $\kappa_j = \operatorname{codim} Z_j$. The current $T := \mathbf{1}_{Z_1 \cap Z_2} \bar{\partial}(u_2 \wedge u_1)$ has bidegree $(*, \kappa_2 + \kappa_1 - 1)$, and by the assumption of proper intersection it has support on a set of codimension $\geq \kappa_2 + \kappa_1$. Thus T vanishes by the dimension principle. In view of (3.1) and the equality

$$\mathbf{1} = \mathbf{1}_{Z_1^c} + \mathbf{1}_{Z_2^c} - \mathbf{1}_{Z_1^c \cap Z_2^c} + \mathbf{1}_{Z_1 \cap Z_2} \tag{3.7}$$

now (3.6) follows. If we apply $\bar{\partial}$ to (3.6) we get (3.5), cf. (3.2).

From the very definition of $\mu_2 \wedge \mu_1$ it is clear that it does not depend on the potential u_1. In view of (3.5) it does not depend on u_2. □

The product $\mu_2 \wedge \mu_1$ is $\mathbb{Q}$-bilinear in the following sense. If $\mu_2 = a\mu_2' + b\mu_2''$, where μ_2' and μ_2'' have good potentials and intersect μ_1 properly and $a, b \in \mathbb{Q}$, then

$$a\mu_2' \wedge \mu_1 + b\mu_2'' \wedge \mu_1 = \mu_2 \wedge \mu_1. \tag{3.8}$$

Since $\mu_2 \wedge \mu_1 = \mu_1 \wedge \mu_2$ the roles of μ_2 and μ_1 can be interchanged. To see (3.8), let u_2' and u_2'' be good potentials of μ_2' and μ_2'', respectively, and let $\tilde{u}_2 = au_2' + bu_2''$. Then $\bar{\partial}\tilde{u}_2 = \mu_2$ but $\tilde{u}_2$ is not necessarily a good potential of μ_2. However, the proof of Proposition 3.4 goes through with u_2 replaced by $\tilde{u}_2$ and Z_2 replaced by $|\mu_2'| \cup |\mu_2''|$. It follows that

$$\bar{\partial}(\tilde{u}_2 \wedge \mu_1) = \mu_1 \wedge \mu_2,$$

from which we see that (3.8) holds.

Remark 3.5 With similar techniques as in the proof of Proposition 3.4 one can prove that $\mu_2 \wedge \mu_1$ is d-closed. However, unfortunately we cannot show, in general, that $\mu_2 \wedge \mu_1$ is (the Lelong current of) a cycle.

Example 3.6 Assume that $f = (f_1, \ldots, f_{\kappa_2})$ is a regular sequence at $x \in Y$ and let $\mu_2 = M_{\kappa_2}^f$, cf. Example 2.6. In view of Sect. 2.2, $m_{\kappa_2}^f$ is a good potential of μ_2. If μ_1 is a cycle which has a local good potential, then

$$\mu_2 \wedge \mu_1 = M_{\kappa_2}^f \wedge \mu_1, \tag{3.9}$$

where the right-hand side is defined in Sect. 2.2. To see this, recall that $m_{\kappa_2}^f \wedge \mu_1$, defined as in Sect. 2.2, is the unique pseudomeromorphic extension to Y of the natural product of $m_{\kappa_2}^f$ and μ_1 in Z_2^c such that $\mathbf{1}_{Z_2} m_{\kappa_2}^f \wedge \mu_1 = 0$. Therefore $m_{\kappa_2}^f \wedge \mu_1$ is the product of the good potential $m_{\kappa_2}^f$ and μ_1. Now (3.9) follows by (3.2) and (2.14).

For degree reasons, cycles in Y of codimension 0 cannot have good potentials. Such a cycle is a linear combination of the irreducible components of Y. If $\mu_1 = aY$, $a \in \mathbb{Q}$, and μ_2 is a cycle with a good potential u_2, then we define $u_2 \wedge \mu_1 := au_2$ and $\mu_2 \wedge \mu_1 := a\mu_2$, which is consistent with (3.2). If μ_1 is any other cycle of codimension 0, then we do not give any meaning to $\mu_2 \wedge \mu_1$.

Remark 3.7 One can extend the approach in this section to more than two factors. If $u_1, \ldots, u_r$ are good potentials of cycles $\mu_1, \ldots, \mu_r$, respectively, with $|\mu_j| = Z_j$, then $u_r \wedge \cdots \wedge u_1$, a priori defined in $\cap_j Z_j^c$, has a unique pseudomeromorphic extension to Y whose restriction to $\cup_j Z_j$ vanishes. One can then recursively define

$$u_r \wedge \cdots \wedge u_{\ell+1} \wedge \mu_\ell \wedge \cdots \wedge \mu_1 := \pm \mathbf{1}_{\cap_{\ell+1}^r Z_j^c} \bar\partial (u_r \wedge \cdots \wedge u_\ell \wedge \mu_{\ell-1} \wedge \cdots \wedge \mu_1)$$

for $\ell = 1, 2, \ldots$. We say that $\mu_1, \ldots, \mu_r$ intersect properly if, for any permutation $j_1, \ldots, j_r$ of $1, \ldots, r$, all successive intersections

$$Z_{j_2} \cap Z_{j_1}, \quad Z_{j_3} \cap (Z_{j_2} \cap Z_{j_1}), \quad Z_{j_4} \cap (Z_{j_3} \cap Z_{j_2} \cap Z_{j_1}), \ldots$$

are proper in the sense of (3.4). If this is the case, then $\mu_r \wedge \cdots \wedge \mu_1$ is independent of the ordering of the factors and choice of good potentials, has support in $\cap_1^r Z_j$, and is d-closed. Since this extension is not needed in this paper we omit the details.

4 Proper intersections when *Y* is smooth

We shall now use the approach in the previous section in the case when Y is smooth, and see that we get back the classical proper intersection product.

Lemma 4.1 *If Y is smooth, then any cycle μ in Y admits, locally, a good $\bar\partial$-potential.*

Proof It is sufficient to show the lemma when Y is an open subset of $\mathbb{C}^n$. Let $\mathcal{U} \Subset Y$ be pseudoconvex and let ρ be a smooth cutoff function in Y that is 1 in $\mathcal{U}$. Let m_n^η be the Bochner–Martinelli kernel in $\mathbb{C}^n \times \mathbb{C}^n$, see Example 2.8. By (2.17) applied to $\rho\mu$ we get that

$$\mu = \bar\partial u' + (p_1)_*\big(m_n^\eta \wedge (1 \otimes \bar\partial\rho \wedge \mu)\big) \tag{4.1}$$

in $\mathcal{U}$, where $u' = (p_1)_*(m_n^\eta \wedge (1 \otimes \rho\mu))$. Since p_1 is a simple projection, u' is pseudomeromorphic. Notice that u' and the last term in (4.1) are the convolutions of the Bochner-Martinelli form in $\mathbb{C}^n$ by $\rho\mu$ and $\bar\partial\rho \wedge \mu$, respectively. Hence they are smooth where $\rho\mu$ and $\bar\partial\rho \wedge \mu$, respectively, are smooth. In $\mathcal{U}$ thus u' is smooth outside $|\mu|$ and the last term in (4.1) is smooth. Moreover, the last term in (4.1) is clearly $\bar\partial$-closed in $\mathcal{U}$ and so it is $\bar\partial u''$ for some smooth form u'' there. We conclude that $u' + u''$ is a good potential of μ in $\mathcal{U}$. □

We can thus use the approach in Sect. 3 and obtain a commutative current product $\mu_2 \wedge \mu_1$ for any cycles μ_1, μ_2 that intersect properly, and it has support on $|\mu_2| \cap |\mu_1|$.

Proposition 4.2 *If μ_1 and μ_2 are any cycles in Y that intersect properly, with supports Z_1 and Z_2, respectively, then $\mu_2 \wedge \mu_1$ is (the Lelong current of) a cycle $\mu_2 \cdot \mu_1$ with support on $Z_1 \cap Z_2$. Moreover, if $i\colon Y \to Y \times Y$ is the diagonal embedding and $\Delta = i_* Y$, then*

$$i_*(\mu_2 \wedge \mu_1) = i_*(\mu_2 \cdot \mu_1) = [\Delta] \cdot (\mu_2 \otimes \mu_1). \tag{4.2}$$

Proof The proposition is local so we may assume that Y is an open subset of $\mathbb{C}^n$. In view of (3.8) and Lemma 4.1 we can also assume that $\mu_j = |\mu_j| = Z_j$. Let u_2 be a good potential of μ_2. We first claim that

$$u_2 \wedge \mu_1 = p_*\big(M_n^\eta \wedge (u_2 \otimes \mu_1)\big), \tag{4.3}$$

where $p\colon Y \times Y \to Y$ is the projection on the first factor.

Notice that $u_2 \otimes 1$ is smooth in $Z_2^c \times Y$. Therefore, by (2.12), (2.16), and (2.1), in $Z_2^c \times Y$ we have

$$M_n^\eta \wedge (u_2 \otimes \mu_1) = (u_2 \otimes 1) \wedge M_n^\eta \wedge (1 \otimes \mu_1) = (u_2 \otimes 1) \wedge i_* \mu_1 = i_*(u_2 \wedge \mu_1).$$

Since $p \circ i = \mathrm{id}_Y$, (4.3) holds in Z_2^c. Now $M_n^\eta \wedge (u_2 \otimes \mu_1)$ has support in $\Delta \cap (Y \times Z_1)$. Thus, its restriction to $Z_2 \times Y$ has support in $\Delta \cap (Z_2 \times Z_1)$ and hence it vanishes in view of the dimension principle. By (2.7) thus,

$$\mathbf{1}_{Z_2} p_* \big(M_n^\eta \wedge (u_2 \otimes \mu_1)\big) = p_* \big(\mathbf{1}_{Z_2 \times Y} M_n^\eta \wedge (u_2 \otimes \mu_1\big) = 0.$$

Since also $\mathbf{1}_{Z_2} u_2 \wedge \mu_1 = 0$, cf. (3.1), it follows that (4.3) holds in Y.

Applying $\bar\partial$ to (4.3) and using (2.15) we get

$$\mu_2 \wedge \mu_1 = p_* \big(M_n^\eta \wedge (\mu_2 \otimes \mu_1)\big). \tag{4.4}$$

To see that this is a cycle, notice that if $\iota \colon \mu_2 \times \mu_1 \to Y \times Y$ is the inclusion, then by (2.13), $M_n^\eta \wedge (\mu_2 \otimes \mu_1) = \iota_* M_n^{\iota^* \eta}$. By King's formula, cf. [5], it follows that $M_n^{\iota^*\eta}$, and hence $M_n^\eta \wedge (\mu_2 \otimes \mu_1)$, are cycles. Since $M_n^\eta \wedge (\mu_2 \otimes \mu_1)$ has support in $\Delta \simeq Y$ there is a cycle μ in Y such that

$$i_* \mu = M_n^\eta \wedge (\mu_2 \otimes \mu_1). \tag{4.5}$$

Thus, by (4.4), $\mu_2 \wedge \mu_1 = p_* i_* \mu = \mu$ is a cycle.

For the last statement, notice that m_n^η is a good potential of $[\Delta]$ in view of Example 3.6. Thus, cf. Example 3.6,

$$M_n^\eta \wedge (\mu_2 \otimes \mu_1) = [\Delta] \wedge (\mu_2 \otimes \mu_1). \tag{4.6}$$

Since this is a cycle we write the right-hand side as $[\Delta] \cdot (\mu_2 \otimes \mu_1)$. Now (4.2) follows from (4.4), (4.5), and (4.6). □

Remark 4.3 If Y is singular, then the diagonal $\Delta \subset Y \times Y$ is not a regular embedding, i.e., defined by a locally complete intersection. The proof of Proposition 4.2 then breaks down because if η (locally) defines Δ, then m_n^η is not $\bar\partial$-closed outside Δ and thus not a good potential of Δ; cf. Example 8.10 below.

4.1 Comparison to Chirka's approach

In [12] the product of two properly intersecting cycles μ_1 and μ_2 can be obtained as follows, see the theorem in [12, Ch. 3, §16.2, p. 212]. We denote it here by $\mu_2 \curlywedge \mu_1$ to distinguish it from our product. If μ_2^ϵ is a regularization of μ_2 obtained by any standard approximate identity, then

$$\lim_{\epsilon \to 0} \mu_2^\epsilon \wedge \mu_1 = \mu_2 \curlywedge \mu_1. \tag{4.7}$$

Proposition 4.4 *If μ_1 and μ_2 intersect properly in the manifold Y, then $\mu_2 \curlywedge \mu_1 = \mu_2 \wedge \mu_1$.*

Proof The statement can be checked locally so we can assume that Y is an open subset of $\mathbb{C}^n$. Let $\phi(\zeta) = \bar\partial \chi(|\zeta|^2) \wedge m_n^\zeta$, where $\chi' \geq 0$, and let $\chi^\epsilon(\zeta) = \chi(|\zeta|^2/\epsilon)$. Then $\phi(\zeta)$ is a positive (n,n)-form and

$$\phi_\epsilon(\zeta) := \phi(\zeta/\sqrt{\epsilon}) = \bar\partial \chi^\epsilon(\zeta) \wedge m_n^\zeta \tag{4.8}$$

is an approximate identity (considered as an (n,n)-form) in $\mathbb{C}^n$. If $p\colon Y\times Y\to Y$ is the projection on the second factor and $\eta=\zeta-z$, then

$$\mu_2^\epsilon := p_*(\phi_\epsilon(\eta)\wedge(\mu_2\otimes 1)) \tag{4.9}$$

is the convolution of μ_2 and ϕ_ϵ. By (4.7) thus $\lim_{\epsilon\to 0}\mu_2^\epsilon\wedge\mu_1=\mu_2\curlywedge\mu_1$. The proposition now follows from the next lemma. $\square$

Lemma 4.5 *We have* $\lim_{\epsilon\to 0}\mu_2^\epsilon\wedge\mu_1=\mu_2\wedge\mu_1$.

Proof This can be checked locally so let $\mathcal{U}\Subset Y$ be pseudoconvex and let ρ be a smooth cutoff function in Y that is 1 in a neighborhood of $\overline{\mathcal{U}}$. In view of (4.9) and (4.8), if $\epsilon>0$ is sufficiently small, then in $\mathcal{U}$ we have

$$\begin{aligned}\mu_2^\epsilon &= p_*\big(\phi_\epsilon(\eta)\wedge(\rho\mu_2\otimes 1)\big)\\ &= \bar\partial p_*\big(\chi^\epsilon(\eta)m_n^\eta\wedge(\rho\mu_2\otimes 1)\big)+p_*\big(\chi^\epsilon(\eta)m_n^\eta\wedge(\bar\partial\rho\wedge\mu_2\otimes 1)\big).\end{aligned}$$

The last term on the right-hand side is independent of $\epsilon>0$ in $\mathcal{U}$ if ϵ is sufficiently small. As in the proof of Lemma 4.1 it follows that

$$\mu_2^\epsilon=\bar\partial(u'_\epsilon+u''),$$

where $u'_\epsilon=p_*\big(\chi^\epsilon(\eta)m_n^\eta\wedge(\rho\mu_2\otimes 1)\big)$ and u'' is smooth in $\mathcal{U}$. By (2.10),

$$\lim_{\epsilon\to 0}\chi^\epsilon(\eta)m_n^\eta\wedge(\rho\mu_2\otimes 1)=m_n^\eta\wedge(\rho\mu_2\otimes 1)$$

is pseudomeromorphic, and since p is a simple projection,

$$u':=\lim_{\epsilon\to 0}u'_\epsilon=p_*\big(m_n^\eta\wedge(\rho\mu_2\otimes 1)\big)$$

is pseudomeromorphic. Moreover, u' is smooth outside $|\mu_2|$, and we notice that the convergence $u'_\epsilon\to u'$ is locally uniform in $\mathcal{U}\setminus|\mu_2|$. Thus $u:=u'+u''$ is a good potential of μ_2 in $\mathcal{U}$.

We claim that $u\wedge\mu_1=\lim_{\epsilon\to 0}(u'_\epsilon+u'')\wedge\mu_1$, where the left-hand side is the product in Sect. 3. Taking the claim for granted the lemma follows since

$$\mu_2^\epsilon\wedge\mu_1=\bar\partial(u'_\epsilon+u'')\wedge\mu_1=\bar\partial\big((u'_\epsilon+u'')\wedge\mu_1\big)\to\bar\partial(u\wedge\mu_1)=\mu_2\wedge\mu_1.$$

To show the claim, notice first that by standard distribution theory,

$$u'_\epsilon\wedge\mu_1=p_*\big(\chi^\epsilon(\eta)m_n^\eta\wedge(\rho\mu_2\otimes\mu_1)\big).$$

Then $\lim_{\epsilon\to 0}u'_\epsilon\wedge\mu_1=p_*\big(m_n^\eta\wedge(\rho\mu_2\otimes\mu_1)\big)$ is pseudomeromorphic, has support in $|\mu_1|$, and is equal to $u'\wedge\mu_1$ in $\mathcal{U}\setminus|\mu_2|$ since $u'_\epsilon\to u'$ locally uniformly there. Hence $T:=\lim_{\epsilon\to 0}(u'_\epsilon+u'')\wedge\mu_1$ is pseudomeromorphic, has support in $|\mu_1|$, and $T=u\wedge\mu_1$ in $\mathcal{U}\setminus|\mu_2|$. Since μ_2 and μ_1 intersect properly thus the claim follows by the dimension principle; cf. the comment after (3.1). $\square$

4.2 Comparison to the algebraic definition in [15]

The intersection product in [15] of two cycles μ_1 and μ_2, that we here denote by $\mu_2\bullet\mu_1$, is defined by the intersection of the diagonal $\Delta\subset Y\times Y$ and the product cycle $\mu_2\times\mu_1$. In view of Proposition 4.2, to see that the intersection product in [15] in the case of proper intersections coincides with our product it thus suffices to see that if $\mu \subset Y$ is an irreducible subvariety and $A \subset Y$ is a submanifold intersecting μ properly, then $A \bullet \mu = [A] \wedge \mu$.

Assume that $\operatorname{codim} A = \kappa$ and $\dim \mu = d$. In general, if A and μ do not necessarily intersect properly, then $A \bullet \mu$ is a Chow class of dimension $d - \kappa$ in the set-theoretic intersection $A \cap \mu$. If A and μ intersect properly, then $\dim A \cap \mu = d - \kappa$ and this Chow class is the zeroth Segre class of the subspace $A \cap \mu$ of μ, $s_0(A \cap \mu, \mu)$, which is a cycle with support $A \cap \mu$. If $\iota\colon \mu \to Y$ and $j\colon A \cap \mu \to \mu$ are the inclusions, then as cycles in Y we have

$$A \bullet \mu = \iota_* j_* s_0(A \cap \mu, \mu).$$

Let $f = (f_1, \ldots, f_\kappa)$ be a tuple defining A (locally) in Y. By Example 2.6 and (2.14) then m_κ^f is a good potential of $[A]$. In view of (2.14) and (2.13) thus

$$[A] \wedge \mu = \bar\partial(m_\kappa^f \wedge \mu) = M_\kappa^f \wedge \mu = \iota_* M_\kappa^{\iota^* f}.$$

It follows from [3, Proposition 1.5] that $M_\kappa^{\iota^* f} = j_* s_0(A \cap \mu, \mu)$ and hence $[A] \wedge \mu = A \bullet \mu$.

5 Proper intersection of nice cycles

To begin with, assume that Y is a complex manifold and let $i\colon Y \to Y'$ be a local embedding into a complex manifold Y'. Let $\mu_1, \ldots, \mu_r$ be germs of effective cycles in Y at a point x and let μ_j' be effective cycles in Y' intersecting i_*Y properly and such that $i_*\mu_j = \mu_j' \cdot_{Y'} i_*Y$. For instance one can take μ_j' as follows. Let (x, y) be coordinates in Y' such that $i_*Y = \{y = 0\}$. Possibly after shrinking Y' and Y, we have $Y' = i_*Y \times U$ for some neighborhood U of 0 in some $\mathbb{C}^d$ and we can take $\mu_j' = i_*\mu_j \times U$. For any choice of such μ_j' we have that $\mu_1', \ldots, \mu_r', i_*Y$ intersect properly in Y' if and only if $\mu_1, \ldots, \mu_r$ intersect properly in Y, cf. Lemma 5.4 below. In this case,

$$i_*(\mu_1 \cdot_Y \cdots \cdot_Y \mu_r) = \mu_1' \cdot_{Y'} \cdots \cdot_{Y'} \mu_r' \cdot_{Y'} i_*Y. \tag{5.1}$$

We can thus relate proper intersections in Y to proper intersections in a larger smooth ambient space. This is the idea for defining proper intersections when Y is singular and motivates the definition of nice cycles.

Definition 5.1 Let Y be a reduced analytic space of pure dimension n. A germ of a cycle μ at $x \in Y$ is nice if there is a local embedding $i\colon Y \to Y'$, where Y' is smooth, and a germ of an effective pure-dimensional cycle μ' at $i(x) \in Y'$ intersecting i_*Y properly, such that

$$i_*\mu = \mu' \cdot_{Y'} i_*Y. \tag{5.2}$$

We say that μ' is a *representative* of μ in Y' at $i(x)$. Since the intersection $\mu' \cdot_{Y'} i_*Y$ is proper it follows that

$$\operatorname{codim}_Y \mu = \operatorname{codim}_{Y'} \mu'. \tag{5.3}$$

Since μ and μ' are effective we have that

$$|i_*\mu| = |\mu'| \cap i_*Y. \tag{5.4}$$

Moreover, if $\mu = \mu_1 + \mu_2$, where μ_j are effective, then $|\mu| = |\mu_1| \cup |\mu_2|$. Neither this last statement nor (5.4) holds if the assumptions on effectivity are dropped, not even if Y is smooth.

A cycle μ in Y is nice if the germ of μ at x is nice for all $x \in Y$. From the definition we see that a $\mathbb{Q}_+$-linear combination of nice cycles is nice.

Example 5.2 The germ of Y at any point $x \in Y$ is nice since there is a local embedding $i\colon Y \to Y'$ of a neighborhood of x into some open $Y' \subset \mathbb{C}^N$ and $i_*Y = Y' \cdot_{Y'} i_*Y$. Since aY', $a \in \mathbb{Q}_+$, are the only effective cycles in Y' of codimension 0, the only nice cycles in Y at x of codimension 0 are aY. In particular, if Y has several irreducible components at x, then neither of these are nice.

Example 5.3 Let $Y = \{z^2 = w^3\} \subset \mathbb{C}^2$ and let $p = 0$. Then $Y \cdot_{\mathbb{C}^2} \{w = 0\} = 2p$ and $Y \cdot_{\mathbb{C}^2} \{z = 0\} = 3p$. Considering p as a cycle in Y thus $(1/2)\{w = 0\}$ and $(1/3)\{z = 0\}$ are representatives of p in $\mathbb{C}^2$.

Lemma 5.4 *If $\mu_1, \dots, \mu_r$ are germs of nice cycles at $x \in Y$ of pure codimensions $\kappa_1, \dots, \kappa_r$, respectively, then*

$$\operatorname{codim}\big(|\mu_1| \cap \cdots \cap |\mu_r|\big) \le \kappa_1 + \cdots + \kappa_r. \tag{5.5}$$

Let μ'_j be representatives of μ_j in the same Y'. Then equality holds in (5.5) if and only if $\mu'_1, \dots, \mu'_r, i_*Y$ intersect properly.

Proof Notice first that if $A \subset Y$, then

$$\operatorname{codim} A + \operatorname{codim}_{Y'} i_*Y = \operatorname{codim}_{Y'} i_*A. \tag{5.6}$$

In view of (5.4), $i_*\big(|\mu_1| \cap \cdots \cap |\mu_r|\big) = |\mu'_1| \cap \cdots \cap |\mu'_r| \cap i_*Y$. By (1.1) in Y' and (5.6) thus

$$\begin{aligned}
&\operatorname{codim}\big(|\mu_1| \cap \cdots \cap |\mu_r|\big) + \operatorname{codim}_{Y'} i_*Y = \operatorname{codim}_{Y'} i_*\big(|\mu_1| \cap \cdots \cap |\mu_r|\big) \\
&= \operatorname{codim}_{Y'}(|\mu'_1| \cap \cdots \cap |\mu'_r| \cap i_*Y) \le \operatorname{codim}_{Y'}|\mu'_1| + \cdots + \operatorname{codim}_{Y'}|\mu'_1| + \operatorname{codim}_{Y'} i_*Y.
\end{aligned}$$

In view of (5.6) and (5.3) both (5.5) and the last statement of the lemma follow. □

Lemma 5.5 *Let μ be a germ of a nice cycle in Y at x and let $j\colon Y \to Y'_m$ be a minimal embedding of a neighborhood of x. Then there is a representative of μ in Y'_m.*

Proof By definition there is an embedding $i\colon Y \to Y'$ and a representative μ' of μ in Y'. Since j is a minimal embedding there is an embedding $\iota\colon Y'_m \to Y'$ such that the composition

$$Y \xrightarrow{j} Y'_m \xrightarrow{\iota} Y' \tag{5.7}$$

is the embedding i. By choosing suitable local coordinates in Y', and possibly shrinking Y' and Y'_m, we can assume that $Y' = \iota_*Y'_m \times U$ for some open U in some $\mathbb{C}^d$.

We claim that μ', $i_*Y \times U$, and $\iota_*Y'_m$ intersect properly in Y'. This follows since by definition μ' and i_*Y intersect properly in Y', and since we clearly have the proper intersection

$$(i_*Y \times U) \cdot_{Y'} \iota_*Y'_m = i_*Y. \tag{5.8}$$

By Lemma 2.9 thus all pairs of μ', $i_*Y \times U$, $\iota_*Y'_m$ intersect properly. Let

$$\mu'_m := \mu' \cdot_{Y'} \iota_*Y'_m.$$

Then μ' is a representative of μ'_m, and in view of (5.8), $i_*Y \times U$ is a representative of j_*Y in Y'. By Lemma 5.4 thus μ'_m and j_*Y intersect properly in Y'_m. From (5.1) and (5.8) we now get

$$\iota_*(\mu'_m \cdot_{Y'_m} j_*Y) = \mu' \cdot_{Y'} (i_*Y \times U) \cdot_{Y'} \iota_*Y'_m = \mu' \cdot_{Y'} i_*Y = i_*\mu = \iota_* j_*\mu.$$

Since ι_* is injective thus $j_*\mu = \mu'_m \cdot j_*Y$, i.e., μ'_m is a representative of μ in Y'_m. □

We can thus assume that $i\colon Y \to Y'$ is a minimal embedding in Definition 5.1. By uniqueness of minimal embeddings it follows that if $\mu_1, \ldots, \mu_r$ are nice cycles in Y at x, then all of them have representatives μ'_j in the same (minimal) Y'. Moreover, if μ is a nice cycle at x and $i\colon Y \to Y'$ is any local embedding, then there is a representative μ' of μ in Y'. Indeed, using the notation of the above proof, if ν' is a representative of μ in Y'_m we can take $\mu' = \iota_* \nu' \times U$.

We say that germs $\mu_1, \ldots, \mu_r$ of pure-dimensional nice cycles at $x \in Y$ intersect properly if equality holds in (5.5). It follows that if $\mu_1, \ldots, \mu_r$ intersect properly, then each subset of them do as well, cf. (the proof of) Lemma 2.9. Notice that if $i\colon Y \to Y'$ is a local embedding and μ'_j are representatives of μ_j in Y', then by Lemma 5.4, μ_j intersect properly if and only if $\mu'_1, \ldots, \mu'_r, i_*Y$ intersect properly.

Proposition 5.6 *Assume that $\mu_1, \ldots, \mu_r$ are germs of nice cycles at $x \in Y$ that intersect properly. Then there is a unique germ of a nice cycle $\mu_1 \cdots \mu_r$ at x such that if $i\colon Y \to Y'$ is a local embedding and μ'_j are representatives of μ_j in Y', then*

$$i_*(\mu_1 \cdots \mu_r) = \mu'_1 \cdot_{Y'} \cdots \mu'_r \cdot_{Y'} i_*Y. \tag{5.9}$$

Proof The uniqueness is clear since i_* is injective. To show existence we take (5.9) for one fixed local embedding i as the definition of $\mu_1 \cdots \mu_r$ and show that it is independent of representatives μ'_j and i.

If we have other representatives μ''_j in Y' for μ_j, then $\mu''_j \cdot_{Y'} i_*Y = \mu'_j \cdot_{Y'} i_*Y$ and hence, by commutativity of proper intersections in a smooth space, $\mu''_1 \cdot_{Y'} \cdots \mu''_r \cdot_{Y'} i_*Y = \mu'_1 \cdot_{Y'} \cdots \mu'_r \cdot_{Y'} i_*Y$.

Let $j\colon Y \to Y'_m$ be a minimal embedding and factorize i as in (5.7). As in the proof of Lemma 5.5 we get representatives $\mu'_{m,j}$ of μ_j in Y'_m. With the notation in that proof we have $\iota_*(\mu'_{m,1} \cdots \mu'_{m,r} \cdot j_*Y) = \mu'_1 \cdots \mu'_r \cdot i_*Y = i_*(\mu_1 \cdots \mu_r)$. Since $i_* = \iota_* j_*$ thus $\mu'_{m,1} \cdots \mu'_{m,r} \cdot j_*Y = j_*(\mu_1 \cdots \mu_r)$. By uniqueness of minimal embeddings and the independence of representatives it follows that $\mu_1 \cdots \mu_r$ is independent of the embedding.

Clearly, $\mu'_1 \cdot_{Y'} \cdots \mu'_r$ is a representative of $\mu_1 \cdots \mu_r$ in Y' and so $\mu_1 \cdots \mu_r$ is nice. ◻

In view of Lemma 5.5 and Proposition 5.6 the following definition makes sense.

Definition 5.7 The proper intersection product of properly intersecting germs of nice cycles $\mu_1, \ldots, \mu_r$ in Y at x is the nice cycle $\mu_1 \cdots \mu_r$ such that (5.9) holds.

The product $\mu_1 \cdots \mu_r$ is commutative since the proper intersection product in Y' is. Moreover, $\mu_1 \cdots \mu_r$ is $\mathbb{Q}_+$-linear in each factor in the sense that if, for some j, $\mu_j = a\nu_1 + b\nu_2$, where $a, b \in \mathbb{Q}_+$ and ν_1 and ν_2 are nice, then

$$\mu_1 \cdots \mu_r = a\mu_1 \cdots \nu_1 \cdots \mu_r + b\mu_1 \cdots \nu_2 \cdots \mu_r. \tag{5.10}$$

Notice that both ν_1 and ν_2 intersect $\mu_1, \ldots, \mu_{j-1}, \mu_{j+1}, \ldots, \mu_r$ properly since $|\mu_j| = |\nu_1| \cup |\nu_2|$.

Example 5.8 Assume that p is a point in Y. By the local parametrization theorem, Y is locally embedded as a branched cover in a neighborhood of 0 in $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^{N-n}$. It follows that $p \in Y$ is the proper intersection of Y and $z_1 = \cdots = z_n = 0$ in $\mathbb{C}^N$. Hence $\{p\}$ is a nice cycle in Y.

Example 5.9 Let $Y = \{x_1x_2 + x_3x_4 = 0\}$ in $\mathbb{C}^4$. Then the analytic sets $\mu_1 = \{x_1 = x_3 = 0\}$ and $\mu_2 = \{x_2 = x_4 = 0\}$ both have codimension 1 in Y but their set-theoretic intersection is just the point 0, which has codimension 3. In view of Lemma 5.5 not both of them, and by symmetry thus none of them, is nice.

Here is an example of an irreducible Y and a nice cycle of positive codimension whose irreducible components are not nice, cf. Example 5.2.

Example 5.10 Let Y be as in Example 5.9; it is irreducible. The subvariety $\{x \in Y;\ x_1 = 0\}$ is certainly nice and it has the two irreducible components $\mu_1 = \{x_1 = 0,\ x_3 = 0\}$ and $\{x_1 = 0,\ x_4 = 0\}$. It follows from Example 5.9 that none of them is nice.

Proposition 5.11 *Assume that $i\colon Y \to Y'$ is a local embedding and Y' is smooth. If μ' is a representative of the nice cycle μ in Y' and u' is a good potential of μ', then there is a unique good potential u of μ such that $u = i^*u'$ in $Y\setminus|\mu|$.*

Proof The uniqueness is clear in view of the dimension principle. By Lemma 4.1, i_*Y has a good potential in Y' and from Sect. 3 it follows that $u' \wedge i_*Y$, a priori defined in $Y' \setminus |\mu'|$, has a unique pseudomeromorphic extension to Y' such that $\mathbf{1}_{|\mu'|}(u' \wedge i_*Y) = 0$. If $\chi_\epsilon = \chi(|f|^2/\epsilon)$ and $\{f = 0\} = |\mu'|$, then by (3.3) we have

$$\chi_\epsilon u' \wedge i_*Y \to u' \wedge i_*Y. \tag{5.11}$$

Let ξ be any smooth form in Y' such that $i^*\xi = 0$. Then clearly $\xi \wedge \chi_\epsilon u' \wedge i_*Y = 0$ for $\epsilon > 0$. By (5.11) therefore $\xi \wedge u' \wedge i_*Y = 0$. This means that there is a (unique) current u in Y such that $i_*u = u' \wedge i_*Y$; cf. Sect. 2. We claim that

$$\mathbf{1}_{i_*Y_{sing}}(u' \wedge i_*Y) = 0. \tag{5.12}$$

Taking this claim for granted for the moment we can complete the proof. Since $u' \wedge i_*Y$ is pseudomeromorphic and in addition (5.12) holds, it follows from [2, Theorem 1.1] that u is pseudomeromorphic in Y.

Moreover, in view of (3.1) and Sect. 4.1 or 4.2,

$$i_*\bar\partial u = \bar\partial i_*u = \bar\partial(u' \wedge i_*Y) = \mu' \wedge i_*Y = \mu' \cdot i_*Y = i_*\mu$$

so that $\bar\partial u = \mu$. In $Y'\setminus|\mu'|$, where u' is smooth, we have $i_*i^*u' = u' \wedge i_*Y$. Thus $u = i^*u'$ in $Y \setminus |\mu|$ and is smooth there.

To show the claim, notice first that $\mathbf{1}_{|\mu'|}\mathbf{1}_{i_*Y_{sing}}(u' \wedge i_*Y) = 0$ in view of (2.5) since $\mathbf{1}_{|\mu'|}(u' \wedge i_*Y) = 0$. Moreover, u is smooth in $Y' \setminus |\mu'|$ and so, by (2.6), $\mathbf{1}_{i_*Y_{sing}}(u' \wedge i_*Y) = u' \wedge \mathbf{1}_{i_*Y_{sing}} i_*Y = 0$ there. It follows that $\mathbf{1}_{Y'\setminus|\mu'|}\mathbf{1}_{i_*Y_{sing}}(u' \wedge i_*Y) = 0$. Hence,

$$\mathbf{1}_{i_*Y_{sing}}(u' \wedge i_*Y) = \mathbf{1}_{|\mu'|}\mathbf{1}_{i_*Y_{sing}}(u' \wedge i_*Y) + \mathbf{1}_{Y'\setminus|\mu'|}\mathbf{1}_{i_*Y_{sing}}(u' \wedge i_*Y) = 0$$

concluding the proof. □

From Lemma 4.1 we know that μ' has a good potential in Y' and hence we have

Corollary 5.12 *If μ is a nice cycle in Y, then μ locally has a good potential.*

We will now prove Theorem 1.1, which says that the extrinsic Definition 5.7 of the proper intersection product coincides with intrinsic $\bar\partial$-potential-theoretic definition (3.2).

Proof of Theorem 1.1 Statement (i) follows from Corollary 5.12. For the first part of statement (ii), notice that $u_2 \wedge \mu_1$ is canonically defined outside $|\mu_2| \cap |\mu_1|$ since u_2 is smooth outside $|\mu_2|$ and $\mu_1 = 0$ outside $|\mu_1|$. Let T be the right-hand side of (3.1). Notice that T is a pseudomeromorphic current equal to $u_2 \wedge \mu_1$ outside $|\mu_2| \cap |\mu_1|$ and of the same bidegree as $u_2 \wedge \mu_1$. Since μ_1 and μ_2 intersect properly it follows from the dimension principle that $\mathbf{1}_{|\mu_2| \cap |\mu_1|} T = 0$. The first part of (ii) thus follows.

To prove (1.3), let $i : Y \to Y'$ be a local embedding, μ'_j representatives of μ_j in Y', u'_2 a good potential of μ'_2, and u_2 the good potential of μ_2 such that $i^* u'_2 = u_2$ outside $|\mu_2|$; cf. Proposition 5.11. By Lemma 5.4, μ'_1, μ'_2, and $i_* Y$ intersect properly, and by Lemma 2.9 any subset intersect properly too. We claim that

$$i_*(u_2 \wedge \mu_1) = u'_2 \wedge (\mu'_1 \cdot i_* Y). \tag{5.13}$$

Indeed, it holds outside $|\mu'_2|$ since there

$$u'_2 \wedge (\mu'_1 \cdot i_* Y) = u'_2 \wedge i_* \mu_1 = i_*(i^* u'_2 \wedge \mu_1) = i_*(u_2 \wedge \mu_1).$$

Moreover, both sides of (5.13) have support in $|\mu'_1| \cap i_* Y$. Thus (5.13) holds outside $|\mu'_2| \cap |\mu'_1| \cap i_* Y$. Since μ'_2, μ'_1, and $i_* Y$ intersect properly it follows from the dimension principle that (5.13) holds in Y'.

In view of (3.2) and Sect. 4.1 or 4.2 we now get

$$\begin{aligned} i_*(\mu_2 \wedge \mu_1) &= i_* \bar\partial (u_2 \wedge \mu_1) = \bar\partial i_*(u_2 \wedge \mu_1) = \bar\partial (u'_2 \wedge (\mu'_1 \cdot i_* Y)) = \mu'_2 \wedge (\mu'_1 \cdot i_* Y) \\ &= \mu'_2 \cdot \mu'_1 \cdot i_* Y = i_*(\mu_2 \cdot \mu_1), \end{aligned}$$

and so the proof is finished. □

Remark 5.13 Notice that goodness of $\bar\partial$-potentials is a regularity assumption that is crucial in the intrinsic definition of the proper intersection product. In order to use dd^c-potentials for such a construction, again one would need some regularity assumption. One idea is to assume the dd^c-potential to have singularities of logarithmic type along the cycle, as is done by Gillet and Soulé in [16] to define their $*$-product of properly intersecting cycles. As long as Y is smooth any cycle has such a dd^c-potential, but we do not know if this is true for, say, nice cycles on a singular space.

Remark 5.14 From Sect. 3, the product $\mu_2 \wedge \mu_1$ is defined assuming only that μ_j have good $\bar\partial$-potentials, and it is commutative and d-closed if μ_1 and μ_2 intersect properly. But unless μ_j are nice we do not know if $\mu_2 \wedge \mu_1$ is (the Lelong current of) a cycle. However, we have no example of a cycle in Y that is not nice but has a good $\bar\partial$-potential.

6 RE-cycles

Let Y be a reduced analytic space of pure dimension n, and $\mathcal{J} \subset \mathcal{O}_Y$ a locally complete intersection ideal sheaf with zero set Z and codimension κ. Let $\pi : \widetilde{Y} \to Y$ be the normalization of the blowup along $\mathcal{J}$, let D be the exceptional divisor, and L the corresponding line bundle. It is well-known, see, e.g., [15, Ch. 1 and 4], that if Y is smooth, then the (Lelong current of the) fundamental cycle $\mu_{\mathcal{J}}$ of $\mathcal{J}$ satisfies

$$\mu_{\mathcal{J}} = \pi_*\big([D] \wedge \hat{c}_1(L^*)^{\kappa-1}\big), \tag{6.1}$$

where $\hat{c}_1(L^*)$ is the first Chern form of L^* with respect to some arbitrary Hermitian metric. If Y is not smooth we take (6.1) as the definition of the fundamental cycle of $\mathcal{J}$. It is well-known that (6.1) is independent of the choice of metric on L; it follows for instance from the dimension principle since Chern forms of different metrics in particular are $\bar{\partial}$-cohomologous. It is also well-known that (6.1) is an effective integral cycle, cf., e.g., Example 6.1 below.

The sheaf $\mathcal{J}$ being a locally complete intersection precisely means that the analytic subspace of Y with structure sheaf $\mathcal{O}_Y/\mathcal{J}$ is a regular embedding. We say that a cycle μ in Y is a *regular embedding cycle*, an RE-cycle, if μ is a locally finite sum $\sum_k \nu_k \mu_{\mathcal{J}_k}$, where $\nu_k \in \mathbb{Q}_+$ and $\mathcal{J}_k$ are locally complete intersection ideals.

Example 6.1 Assume that $f = (f_1, \ldots, f_\kappa)$ is a tuple of holomorphic functions generating $\mathcal{J}$ so that f is a regular sequence at each $x \in Z$, or more generally, f is a section of a Hermitian vector bundle of rank κ such that f generates $\mathcal{J}$. By [3, Proposition 1.5] then $M^f_\kappa = \mu_{\mathcal{J}}$. It follows, see [5, Theorem 1.1], that $\mu_{\mathcal{J}}$ is an effective integral cycle. Notice also that $\bar{\partial} m^f_\kappa = 0$ outside $Z(f)$ so that, cf. (2.14), m^f_κ is a good potential of $\mu_{\mathcal{J}}$.

In case $\kappa = 1$ this can be made more explicit. In this case, if Y is normal, then the fundamental cycle $\mu_{\mathcal{J}}$ is the divisor, $\operatorname{div} f$, of f. If Y is not normal, then $\mu_{\mathcal{J}} = \pi_*(\operatorname{div} \pi^* f)$, where $\pi : \widetilde{Y} \to Y$ is the normalization, and we take $\mu_{\mathcal{J}}$ as the definition of $\operatorname{div} f$. Notice that $m^f_1 = \partial \log |f|^2 / 2\pi i$ is a good potential of $\operatorname{div} f$.

If the tuple f generates $\mathcal{J}$ it is sometimes convenient to write μ_f rather than $\mu_{\mathcal{J}}$.

Example 6.2 Any point in Y is an RE-cycle in view of Example 5.8.

Proposition 6.3 *If $\mathcal{J}$ is a complete intersection ideal at x, then $\mu_{\mathcal{J}}$ is a nice cycle.*

Proof Assume that $f = (f_1, \ldots, f_\kappa)$ is a minimal generating tuple for $\mathcal{J}$ at x so that $M^f_\kappa = \mu_{\mathcal{J}}$. Let $i \colon Y \to Y'$ be an embedding and let $F = (F_1, \ldots, F_\kappa)$ be a tuple of holomorphic functions at $i(x) \in Y'$ such that $f = i^* F$. Since $\{F = 0\} \cap i_* Y = i_*\{f = 0\}$ has codimension $\kappa + \operatorname{codim}_{Y'} Y$ it follows from (1.1) that $\operatorname{codim}\{F = 0\} = \kappa$. Hence, F defines a regular embedding at $i(x)$ and $\mu_F = M^F_\kappa$ intersects $i_* Y$ properly in Y'. Since $f = i^* F$ it follows from (2.13) that $i_* M^f_\kappa = M^F_\kappa \wedge i_* Y$. By Example 3.6 and Lemma 4.1 thus

$$i_* \mu_{\mathcal{J}} = i_* M^f_\kappa = M^F_\kappa \wedge i_* Y = M^F_\kappa \cdot_{Y'} i_* Y = \mu_F \cdot_{Y'} i_* Y. \tag{6.2}$$

We conclude that $\mu_{\mathcal{J}}$ is a nice cycle in Y. $\square$

Let us illustrate the connection between the intrinsic fundamental cycle $\mu_{\mathcal{J}}$ and the representing fundamental cycle μ_F in Y' given by (6.2) with the following example.

Example 6.4 Let $Y = \{z^2 = w^3\} \subset \mathbb{C}^2_{z,w}$, $i \colon Y \to \mathbb{C}^2$ the inclusion, and $p = 0 \in Y$; cf. Example 5.3. Then $\pi : \mathbb{C} \to Y$, $\pi(t) = (t^3, t^2)$, is the normalization and, cf. Example 6.1,

$$\operatorname{div}(z|_Y) = \pi_* \operatorname{div} t^3 = 3p, \qquad \operatorname{div}(w|_Y) = \pi_* \operatorname{div} t^2 = 2p.$$

Thus, $3p$ and $2p$ are the fundamental cycles of the regular embeddings defined by $\langle z|_Y \rangle$ and $\langle w|_Y \rangle$, respectively. From the extrinsic viewpoint, the fundamental cycles $\operatorname{div} z = \{z = 0\}$ and $\operatorname{div} w = \{w = 0\}$ are representatives in $\mathbb{C}^2$ of $3p$ and $2p$, respectively, since $\{z = 0\} \cdot i_* Y = 3p$ and $\{w = 0\} \cdot i_* Y = 2p$.

Proposition 6.5 *If $\mathcal{J}$ and $\widetilde{\mathcal{J}}$ define properly intersecting regular embeddings, then $\mathcal{J} + \widetilde{\mathcal{J}}$ defines a regular embedding and $\mu_{\mathcal{J}+\widetilde{\mathcal{J}}} = \mu_{\mathcal{J}} \cdot \mu_{\widetilde{\mathcal{J}}}$.*

Proof If Y is smooth this is well-known and follows from, e.g., [15]; cf. also Remark 6.7 below. The general case can be reduced to that case as follows. As in the proof of Proposition 6.3, choose minimal tuples f and $\tilde{f}$ generating $\mathcal{J}$ and $\widetilde{\mathcal{J}}$, respectively, at x, a local embedding $i\colon Y \to Y'$, and F and $\widetilde{F}$ in Y' such that $f = i^*F$ and $\tilde{f} = i^*\widetilde{F}$. Then F and $\widetilde{F}$ define regular embeddings and by (6.2), $i_*\mu_{\mathcal{J}} = \mu_F \cdot i_*Y$ and $i_*\mu_{\widetilde{\mathcal{J}}} = \mu_{\widetilde{F}} \cdot i_*Y$.

Since $\mu_{\mathcal{J}}$ and $\mu_{\widetilde{\mathcal{J}}}$ intersect properly it follows that $(f, \tilde{f})$, which generates $\mathcal{J} + \widetilde{\mathcal{J}}$, is a regular sequence and that μ_F, $\mu_{\widetilde{F}}$, i_*Y intersect properly; cf. Lemma 5.4. In particular, μ_F and $\mu_{\widetilde{F}}$ intersect properly in the smooth space Y' and so, in view of (5.9) and (2.13),

$$i_*(\mu_{\mathcal{J}} \cdot \mu_{\widetilde{\mathcal{J}}}) = \mu_F \cdot \mu_{\widetilde{F}} \cdot i_*Y = \mu_{F,\widetilde{F}} \cdot i_*Y = M^{F,\widetilde{F}}_{\kappa+\tilde{\kappa}} \wedge i_*Y = i_*(M^{f,\tilde{f}}_{\kappa+\tilde{\kappa}}) = i_*\mu_{\mathcal{J}+\widetilde{\mathcal{J}}},$$

where κ and κ' are the codimensions of $\mathcal{J}$ and $\mathcal{J}'$, respectively. This finishes the proof. □

By $\mathbb{Q}_+$-linearity, cf. (5.10), we have

Corollary 6.6 *Any RE-cycle in Y is nice, and if μ_1 and μ_2 are RE-cycles that intersect properly, then $\mu_1 \cdot \mu_2$ is an RE-cycle.*

Remark 6.7 If f and $\tilde{f}$ are minimal tuples that defines $\mathcal{J}$ and $\widetilde{\mathcal{J}}$, respectively, then Proposition 6.5 says that $(f, \tilde{f})$ is a regular sequence and

$$M^{f,\tilde{f}}_{\kappa+\tilde{\kappa}} = M^f_\kappa \wedge M^{\tilde{f}}_{\tilde{\kappa}}, \tag{6.3}$$

where κ and κ' are the codimensions of $\mathcal{J}$ and $\mathcal{J}'$, respectively. This may have independent interest. If Y is smooth, then (6.3) is known and follows from, e.g., [5, Eq. (7.6)].

Example 6.8 ($\mathbb{Q}$-Cartier divisors) Assume that Y is normal. A cycle μ in Y is a $\mathbb{Q}$-Cartier divisor if locally there is a meromorphic function $f = g/h$ and a positive integer q such that $q\mu = \operatorname{div} f := \operatorname{div} g - \operatorname{div} h$. If μ is effective it follows that f is holomorphic on Y_{reg}, and therefore holomorphic on Y by normality. Thus effective $\mathbb{Q}$-Cartier divisors are RE-cycles. Assume that μ_1 and μ_2 are effective $\mathbb{Q}$-Cartier divisors that intersect properly and suppose that $\operatorname{div} f_j = q_j\mu_j$, $j = 1, 2$, for some $q_j \in \mathbb{Z}_+$. Then

$$\mu_2 \cdot \mu_1 = \mu_2 \wedge \mu_1 = \frac{1}{q_1 q_2} \operatorname{div} f_2 \wedge \operatorname{div} f_1. \tag{6.4}$$

7 Global intersection formulas

We begin with the proof of Theorem 1.3.

Proof of Theorem 1.3 By assumption, $\mu_1, \ldots, \mu_k$ intersect properly and we let $\nu_k = \mu_k \cdots \mu_1$, $k = 1, \ldots, r$. We inductively define currents A_k of bidegree $(\kappa_1 + \cdots + \kappa_k, \kappa_1 + \cdots + \kappa_k - 1)$ such that

$$\bar{\partial} A_k = \nu_k - \alpha_k \wedge \cdots \wedge \alpha_1. \tag{7.1}$$

Let $A_1 = a_1$. Assume now that A_k is found. We then define

$$A_{k+1} := a_{k+1} \wedge \nu_k + A_k \wedge \alpha_{k+1}.$$

The second product is not problematic since α_{k+1} is smooth. We claim that the first product, a priori defined outside $|\mu_{k+1}|$, has a unique pseudomeromorphic extension to Y, denoted by $a_{k+1} \wedge \nu_k$ as well, and that

$$\bar{\partial}(a_{k+1} \wedge \nu_k) = \mu_{k+1} \wedge \nu_k - \alpha_{k+1} \wedge \nu_k. \tag{7.2}$$

Taking the claim for granted for the moment, using that $\bar\partial\alpha_{k+1} = 0$ and (7.1) we get

$$\begin{aligned}\bar\partial A_{k+1} &= (\mu_{k+1} - \alpha_{k+1}) \wedge \nu_k + (\nu_k - \alpha_k \wedge \cdots \wedge \alpha_1) \wedge \alpha_{k+1}\\ &= \mu_{k+1} \wedge \nu_k - \alpha_{k+1} \wedge \cdots \wedge \alpha_1.\end{aligned}$$

Hence, A_{k+1} has the desired properties. For degree reasons,

$$d(A_r \wedge \omega^{n-\kappa}) = \bar\partial(A_r \wedge \omega^{n-\kappa}) = \nu_r \wedge \omega^{n-\kappa} - \alpha_r \wedge \cdots \wedge \alpha_1 \wedge \omega^{n-\kappa}$$

and thus (1.5) follows by Stokes' theorem.

It remains to show the claim. The uniqueness is clear in view of the dimension principle since $\mu_1, \dots, \mu_{k+1}$ intersect properly. For the existence it is thus sufficient to check that $a_{k+1} \wedge \nu_k$ can be extended across $|\mu_{k+1}|$ locally in Y. Let γ be a local smooth $\bar\partial$-potential of α_{k+1}. Then

$$a_{k+1} = a_{k+1} + \gamma - \gamma =: u_{k+1} - \gamma,$$

and in view of (1.4) thus u_{k+1} is a local good potential of μ_{k+1}. Therefore,

$$u_{k+1} \wedge \nu_k - \gamma \wedge \nu_k \tag{7.3}$$

is defined in view of Sect. 3 and is a local pseudomeromorphic extension of $a_{k+1} \wedge \nu_k$ across $|\mu_{k+1}|$. Checking (7.2) is also a local problem. We can thus replace $a_{k+1} \wedge \nu_k$ by (7.3) and then (7.2) follows by (3.2). This finishes the proof of the claim and Theorem 1.3. □

In the rest of this section, given a nice cycle μ in Y, we consider two cases where there are a and α as in Theorem 1.3; cf. (1.4).

Proposition 7.1 *Suppose that Y is compact and let μ be a nice cycle in Y of pure codimension κ. Assume that there is a global embedding $i: Y \to Y'$ into a compact Kähler manifold Y' and an effective cycle μ' in Y' intersecting i_*Y properly such that $i_*\mu = \mu' \cdot_{Y'} i_*Y$. Then there is a pseudomeromorphic current a in Y, smooth in $Y \setminus |\mu|$, and a smooth closed (κ, κ)-form α such that $\bar\partial a = \mu - \alpha$. Moreover, α locally has smooth $\bar\partial$-potentials.*

Proof By [16, Theorem 1.3.5], there is a smooth form α' in Y' and a dd^c-potential v' such that

$$dd^c v' = \mu' - \alpha',$$

v' is smooth in $Y' \setminus |\mu'|$ and has singularities of logarithmic type along $|\mu'|$. This latter property means the following: There is a proper surjective mapping $p: Y'' \to Y'$ and a current v'' in Y'' such that $v' = p_* v''$, v'' is smooth in $Y'' \setminus p^{-1}|\mu'|$, and in suitable local coordinates $s = (s_1, \dots, s_{N''})$ in Y'',

$$v'' = c_1 \log|s_1|^2 + \cdots + c_k \log|s_k|^2 + b,$$

where c_j are smooth closed forms and b is a smooth form.

Notice that $a' := \partial v'/2\pi i$ is smooth in $Y' \setminus |\mu'|$ and $\bar\partial a' = \mu' - \alpha'$. We claim that, in addition, a' is pseudomeromorphic in Y'. In fact, $a' = p_* a''$ where $a'' := \partial v''/2\pi i$ locally has the form

$$c_1 \wedge \frac{ds_1}{2\pi i s_1} + \cdots + c_k \wedge \frac{ds_k}{2\pi i s_k} + \partial b/2\pi i.$$

Thus a'' is pseudomeromorphic in Y'' and it follows from [7, Corollary 2.16] that a' is pseudomeromorphic in Y'.

Let $\alpha := i^*\alpha'$. As in the proof of Proposition 5.11 it follows that there is a unique pseudomeromorphic current a in Y such that $a = i^* a'$ in $Y \setminus |\mu|$. To see that α and a have the desired properties, let γ' be a local smooth $\bar\partial$-potential of α'. Then $i^*\gamma'$ is a local smooth $\bar\partial$-potential of α. Moreover, $u' := a' + \gamma'$ is a local good potential of μ'. By Proposition 5.11 there is a (unique) local good potential u of μ such that $u = i^*u'$ outside $|\mu|$. Thus $u - i^*\gamma'$ is pseudomeromorphic and equal to i^*a' outside $|\mu|$. Hence, $a = u - i^*\gamma'$ locally, and it follows that $\bar\partial a = \mu - \alpha$. □

The assumptions in Proposition 7.1 are rather restrictive since in general one cannot expect that μ has a global representative in a smooth ambient space. We will now give an intrinsic condition on μ ensuring that there are a and α as in Theorem 1.3; see Proposition 7.5 below. The proof of Proposition 7.5 is based on the following generalization of the Poincaré–Lelong formula. It is precisely Theorem 1.1 in [1], but since the formula in [1] is formulated only when Y is smooth, we give a proof here.

Proposition 7.2 *Let Y be a reduced pure-dimensional analytic space and let σ be a holomorphic section of a Hermitian vector bundle $E \to Y$ (of any rank) such that the zero set $Z = Z(\sigma)$ has codimension $\kappa > 0$. Let S denote the trivial line bundle over $Y \setminus Z$ so that*

$$0 \to S \xrightarrow{\sigma} E \to Q \to 0$$

is exact in $Y \setminus Z$. If S and Q are equipped with the induced Hermitian metrics, then the associated Chern forms $c(S)$ and $c(Q)$ have (unique) locally integrable closed extensions $C(S)$ and $C(Q)$ to Y. Moreover, $\log|\sigma|^2 \cdot C(Q)$ is locally integrable, $dd^c(\log|\sigma|^2 \cdot C(Q))$ has order 0, and there is a real locally integrable form W such that

$$dd^c W = M^{Q,\sigma} - c(E) + C(Q), \tag{7.4}$$

where

$$M^{Q,\sigma} = \mathbf{1}_Z dd^c(\log|\sigma|^2 \cdot C(Q)). \tag{7.5}$$

Moreover,

$$M_\kappa^{Q,\sigma} = M_\kappa^\sigma. \tag{7.6}$$

Proof Let $\pi : X \to Y$ be a modification such that X is smooth and the ideal generated by $\pi^*\sigma$ is principal, cf. the proof of Lemma 2.5. Then $\pi^*\sigma = \sigma^0\sigma'$, where σ^0 is a section of a line bundle $L \to X$ defining a divisor D and σ' is a non-vanishing section of $\pi^*E \otimes L^*$. We then have the exact sequence

$$0 \to L \xrightarrow{\sigma'} \pi^*E \longrightarrow Q' \to 0.$$

Equip L and Q' with the induced metrics and notice that $|\sigma^0|_L = |\sigma^0\sigma'| = |\pi^*\sigma|$. Outside $\pi^{-1}(Z)$ we thus have:

(a) $\pi^*S \xrightarrow{\sigma^0} L$ is an isomorphism of Hermitian line bundles,

(b) the mapping $\pi^*S \xrightarrow{\pi^*\sigma} \pi^*E$ factorizes as $\pi^*S \xrightarrow{\sigma^0} L \xrightarrow{\sigma'} \pi^*E$.

By (a) we get that $c(S) = \pi_*c(\pi^*S) = \pi_*c(L)$ outside Z, and thus $C(S) := \pi_*c(L)$ is a locally integrable closed extension of $c(S)$. From (b) it follows that, outside Z, $\pi^*Q = Q'$ as Hermitian bundles. As with $c(S)$ we see that $C(Q) := \pi_*c(Q')$ is a locally integrable closed extension of $c(Q)$. Since $\log|\sigma|^2 \cdot c(Q) = \pi_*(\log|\pi^*\sigma|^2 \cdot c(Q'))$ outside Z and $\log|\pi^*\sigma|^2 \cdot c(Q')$ is locally integrable in X it follows that $\log|\sigma|^2 \cdot c(Q)$ is locally integrable in Y. Moreover, since $dd^c \log|\pi^*\sigma|^2$ has order 0 and $c(Q')$ is smooth and closed it follows that $dd^c(\log|\sigma|^2 \cdot C(Q)) = \pi_* dd^c(\log|\pi^*\sigma|^2 \cdot c(Q'))$ has order 0.

By [11] there is a smooth form v in X such that

$$dd^c v = c(\pi^* E) - c(L) \wedge c(Q'). \tag{7.7}$$

Moreover, by the Poincaré–Lelong formula on X,

$$dd^c \log |\pi^* \sigma|^2 = dd^c \log |\sigma^0|_L^2 = [D] - c_1(L). \tag{7.8}$$

If

$$w := \log |\pi^* \sigma|^2 c(Q') - v, \tag{7.9}$$

then a simple calculation gives

$$dd^c w = [D] \wedge c(Q') - c(\pi^* E) + c(Q'). \tag{7.10}$$

Since $\log |\sigma|^2 \cdot c(S)$ is locally integrable, in view of (2.7) and (7.8) we have

$$\begin{aligned} M^{Q,\sigma} &= \pi_*\big(\mathbf{1}_{\pi^{-1}Z} dd^c (\log |\pi^* \sigma|^2 \cdot c(Q'))\big) = \pi_*\big(\mathbf{1}_{|D|}([D] - c_1(L)) \wedge c(Q')\big) \\ &= \pi_*([D] \wedge c(Q')). \end{aligned} \tag{7.11}$$

Thus we get (7.4) with $W := \pi_* w$ after applying π_* to (7.10).

To see (7.6) notice that (7.7) gives

$$c(Q') = s(L) \wedge c(\pi^* E) - dd^c (s(L) \wedge v),$$

where[3] $s(L) := 1/c(L) = \sum_{k=1}^{\infty} (-c_1(L))^{k-1}$. Hence,

$$M^{Q,\sigma} = \pi_*\big([D] \wedge c(Q')\big) = \pi_*\big([D] \wedge s(L) \wedge c(\pi^* E)\big) - dd^c \pi_*\big([D] \wedge s(L) \wedge v\big). \tag{7.12}$$

On the other hand, by Lemma 2.5 and (2.8),

$$\begin{aligned} M^{\sigma} &= \sum_{k \geq 1} M_k^{\sigma} = \sum_{k \geq 1} \mathbf{1}_Z \bar{\partial} m_k^{\sigma} = \sum_{k \geq 1} \pi_* \left(\mathbf{1}_{|D|} \bar{\partial} \left(\frac{\partial \log |\sigma^0|_L^2}{2\pi i} \wedge (-c_1(L))^{k-1} \right) \right) \\ &= \pi_*\big(\mathbf{1}_{|D|}([D] - c_1(L)) \wedge s(L)\big) = \pi_*\big([D] \wedge s(L)\big). \end{aligned}$$

In view of (7.12) it thus follows that

$$M^{Q,\sigma} = c(E) \wedge M^{\sigma} + dd^c \gamma, \tag{7.13}$$

where $\gamma = -\pi_*([D] \wedge s(L) \wedge v)$. Clearly, γ has support in Z, and since π is a modification γ is pseudomeromorphic. Taking the component of (7.13) of bidegree (κ, κ) we obtain (7.6) since $M_k^{\sigma} = 0$ for $k < \kappa$ and the component of bidegree $(\kappa - 1, \kappa - 1)$ of γ vanishes by the dimension principle. □

Remark 7.3 If $\kappa = 1 = \operatorname{rank} E$, then (7.4) is the Poincaré–Lelong formula, albeit the underlying space is possibly non-normal; cf. Example 6.1 and [3, Proposition 2.1].

Corollary 7.4 *If $A = \partial W / 2\pi i$, then A is pseudomeromorphic, smooth outside Z, and*

$$\bar{\partial} A = dA = M^{Q,\sigma} - c(E) + C(Q). \tag{7.14}$$

[3] $s(L)$ is the total Segre form of L.

Proof Notice that $W = \pi_* w$, cf. (7.9). Thus, $2\pi i A$ is π_* of

$$\frac{\partial|\pi^*\sigma|^2}{|\pi^*\sigma|^2} \wedge c(Q') - \partial v = \frac{\partial|\sigma^0|^2}{|\sigma^0|^2} \wedge c(Q') - \partial v$$

and hence pseudomeromorphic since π is a modification and v is smooth. Now (7.14) follows from (7.4). □

Proposition 7.5 *Assume that $\mu_{\mathcal{J}}$ is the fundamental cycle of a locally complete intersection ideal $\mathcal{J}$ of codimension κ generated by a global holomorphic section σ of a Hermitian vector bundle $E \to Y$ of rank κ. Then there is a pseudomeromorphic current a in Y, smooth in $Y \setminus |\mu_{\mathcal{J}}|$, such that*

$$\bar{\partial} a = da = \mu_{\mathcal{J}} - c_\kappa(E). \tag{7.15}$$

Proof With the notation above, let $a = A_{\kappa,\kappa-1}$. Then a is pseudomeromorphic and smooth outside $|\mu_{\mathcal{J}}|$. By [3, Proposition 1.5], $\mu_{\mathcal{J}} = M^\sigma_\kappa$, and since rank $Q = \kappa - 1$ we have $C_\kappa(Q) = 0$. Taking the component of bidegree (κ, κ) of (7.14) thus (7.15) follows from (7.6). □

If $\mu = \sum_\ell q_\ell \mu_{\mathcal{J}_\ell}$, $q_\ell \in \mathbb{Q}_+$, is an RE-cycle where each $\mu_{\mathcal{J}_\ell}$ is as in the proposition, then by $\mathbb{Q}_+$-linearity, cf. (5.10), we get a pseudomeromorphic a and a smooth α such that $\bar{\partial} a = \mu - \alpha$ and α locally has smooth $\bar{\partial}$-potentials.

8 Examples

Example 8.1 If Y is smooth and μ is any cycle, then the proper intersection of μ and Y is μ, since μ is the intersection of Δ and $\mu \times Y$ in $Y \times Y$.

In case Y is singular this holds if μ is a nice cycle; recall from Example 5.2 that Y is nice. To see this, let $i : Y \to Y'$ be a local embedding into a smooth Y' and let μ' be a representative of μ in Y'. Since Y' is a representative of Y in Y' we have $i_*(\mu \cdot Y) = \mu' \cdot_{Y'} Y' \cdot_{Y'} i_* Y = \mu' \cdot_{Y'} i_* Y = i_*\mu$ by (5.9), and thus $\mu \cdot Y = \mu$. One can also choose a good potential u of μ and notice that $\mu \cdot Y = \bar{\partial}(u \wedge 1) = \bar{\partial} u = \mu$; cf. (3.2).

Example 8.2 Assume that μ_1 and μ_2 are nice in Y and intersect properly. Moreover, assume that $\mu_1 = |\mu_1| =: Z$, let $\iota : Z \to Y$ be the inclusion, and let τ be the cycle in Z such that $\iota_*\tau = \mu_2 \cdot Z$. We first claim that τ is nice.

Let $i : Y \to Y'$ be a local embedding and let μ_2' and Z' be representatives of μ_2 and Z, respectively, in Y'. Since μ_2 and Z intersect properly, μ_2', Z', i_*Y intersect properly. We have

$$(i \circ \iota)_*\tau = i_*(\mu_2 \cdot Z) = \mu_2' \cdot_{Y'} Z' \cdot_{Y'} i_*Y = \mu_2' \cdot_{Y'} i_* Z = \mu_2' \cdot_{Y'} (i \circ \iota)_* Z,$$

where we in the second last equality consider Z as a cycle in Y, and in the last equality as a cycle in Z. Hence, μ_2' is a representative of τ in Y' and so τ is nice.

Let us also notice that if u_2' is a local good potential of μ_2', then by Proposition 5.11 there is a unique local good potential u of τ such that $u = \iota^* i^* u_2'$ outside $|\tau|$. In particular, if $\mu_2 = \mu_f$ is the fundamental cycle of a locally complete intersection ideal of codimension κ generated by a holomorphic κ-tuple f, then $m_\kappa^{\iota^* f}$ is a good potential of τ; cf. Example 6.1 and the proof of Proposition 6.3. In this case thus τ is the fundamental cycle $\mu_{\iota^* f}$ and

$$\mu_f \cdot Z = \iota_* \mu_{\iota^* f}. \tag{8.1}$$

Example 8.3 Let $Y = \{xy - z^2 = 0\} \subset \mathbb{P}^3_{[x_0,x,y,z]}$ and let $i\colon Y \to \mathbb{P}^3$ be the inclusion. Then Y has an isolated singular point $p = [1, 0, 0, 0]$; the so-called A_1-singularity. Consider the lines

$$L_1 = \{[x_0, x, y, z];\ x = z = 0\}, \quad L_2 = \{[x_0, x, y, z];\ y = z = 0\}.$$

It is straightforward to check that the sections $\sigma_1 = i^*x$ and $\sigma_2 = i^*y$ of $i^*\mathcal{O}(1)$ vanish to order 2 on $L_1\backslash\{0\}$ and $L_2\backslash\{0\}$, respectively. Thus, $\mathrm{div}\sigma_1 = 2L_1$ and $\mathrm{div}\sigma_2 = 2L_2$, so L_1 and L_2 are 1/2-Cartier divisors. Let $\iota\colon L_1 \to Y$ be the inclusion. Noticing that $\iota^*\sigma_2 = y|_{L_1}$, by (8.1) we have that

$$2L_2 \cdot L_1 = \iota_*\mathrm{div}\iota^*\sigma_2 = [p], \tag{8.2}$$

and so $L_2 \cdot L_1 = (1/2)[p]$.

To see that $L_2 \cdot L_1 = (1/2)[p]$ one can also use Theorem 1.3 as follows.

Example 8.4 We keep the notation from Example 8.3 and let ω be the Fubini–Study metric form on $\mathbb{P}^3$. By the Poincaré–Lelong formula we have $dd^c \log|\sigma_j|^2 = \mathrm{div}\sigma_j - i^*\omega$, and by Theorem 1.3 thus

$$\int_Y 2L_2 \cdot 2L_1 = \int_Y \mathrm{div}\sigma_2 \wedge \mathrm{div}\sigma_1 = \int_Y \omega^2.$$

Now, $\int_Y \omega^2 = 2$ since Y has degree 2, and hence $\int_Y L_2 \cdot L_1 = 1/2$. Since $|L_2 \cdot L_1| = \{p\}$ it follows that $L_2 \cdot L_2 = (1/2)[p]$.

We remark that there is nothing special about the lines L_1 and L_2. In fact, the set of pairs of lines in Y through p (including double lines) is in one-to-one correspondence with the set of divisors, containing p, of sections of $i^*\mathcal{O}(1)$.

The next example shows that (5.4) does not hold in general if the representative μ' of μ in Definition 5.1 is not effective.

Example 8.5 We continue to keep the notation of Example 8.3 and let $\sigma_3 = i^*z$. Then $\{\sigma_3 = 0\} = L_1 \cup L_2$ and σ_3 vanishes to order 1 along $L_1 \cup L_2 \setminus \{0\}$. Hence, $\mathrm{div}\,\sigma_3 = L_1 + L_2$. Since $\mathrm{div}\,\sigma_2 = 2L_2$ thus $\mathrm{div}\,\sigma_3 - (1/2)\mathrm{div}\,\sigma_2 = L_1$. It follows that $\mu' := \mathrm{div}\,z - (1/2)\mathrm{div}\,y$ is a "representative" of L_1 in $\mathbb{C}^3$ such that $|\mu'| \cap Y = L_1 \cup L_2$ strictly contains L_1.

Example 8.6 Let Z_1 and Z_2 be two 2-dimensional planes in $\mathbb{C}^4$. Clearly, Z_j are fundamental cycles of complete intersection ideals so the cycle $Z_1 + Z_2$ is an RE-cycle. However, if Z_1 and Z_2 intersect properly, so that $Z_1 \cap Z_2$ is just a point p, then no multiple of $Z_1 + Z_2$ is the fundamental cycle of a complete intersection. This follows from Hartshorne's connectedness theorem, which says that a set-theoretic complete intersection is connected in codimension 1. Since $Z_1 \cup Z_2$ is 2-dimensional and becomes disconnected by removing p thus $Z_1 \cup Z_2$ is not a complete intersection.

If Z_1 and Z_2 do not intersect properly, then $Z_1 + Z_2$ is the fundamental cycle of a complete intersection ideal. This is clear since then either $Z_1 = Z_2$ or $Z_1 \cap Z_2$ is a line.

Here is an example of a singular Y where similar phenomena occur.

Example 8.7 Let $f\colon \mathbb{C}^4_z \to \mathbb{C}^{10}_w$ be the mapping

$$f(z) = (z^{\alpha_1}, \ldots, z^{\alpha_{10}}) = (w_1, \ldots, w_{10}),$$

where z^{α_j} are the monomials in $\mathbb{C}^4$ of degree 2. Let $Y = f(\mathbb{C}^4)$ and let $i\colon Y \to \mathbb{C}^{10}$ be the inclusion. The differential of f is injective outside 0 so Y is smooth outside $0 = f(0)$. One can check that f is 2 : 1 outside 0, and using, e.g., [3, (6.1)] that the multiplicity of Y at 0 is 8. In particular, $0 \in Y$ is a singular point.

Let $\widetilde{Z}_1 = \{z_1 = z_2 = 0\} \subset \mathbb{C}^4$ and $Z_1 = f(\widetilde{Z}_1)$. Choose the monomials z^{α_j} so that $z^{\alpha_1} = z_1^2$ and $z^{\alpha_2} = z_2^2$. Then, since $f_* 1 = 2$, by (2.13) and Remark 6.7 we have

$$\begin{aligned} 2M_2^{i^*w_1, i^*w_2} &= f_*(M_2^{z_1^2, z_2^2}) = f_*(M_1^{z_1^2} \wedge M_1^{z_2^2}) = f_*(2[z_1 = 0] \wedge 2[z_2 = 0]) \\ &= 4 f_* \widetilde{Z}_1 = 8 Z_1. \end{aligned} \tag{8.3}$$

Hence, $4Z_1$ is the fundamental cycle of $\langle i^*w_1, i^*w_2 \rangle$, so Z_1 is in particular an RE-cycle.

If $\widetilde{Z}_2$ is another 2-dimensional linear subspace of $\mathbb{C}^4$, then in the same way $Z_2 := f(\widetilde{Z}_2)$ is an RE-cycle, and hence $Z_1 + Z_2$ is an RE-cycle. If $\widetilde{Z}_2$ intersects $\widetilde{Z}_1$ properly, then $\widetilde{Z}_1 \cup \widetilde{Z}_2$ is not a complete intersection; cf. Example 8.6. In this case, since $f^{-1}(Z_1 \cup Z_2) = \widetilde{Z}_1 \cup \widetilde{Z}_2$, it follows that no multiple of $Z_1 + Z_2$ is the fundamental cycle of a complete intersection ideal. If, on the other hand, $\widetilde{Z}_2$ and $\widetilde{Z}_1$ do not intersect properly, then a multiple of $Z_1 + Z_2$ is the fundamental cycle of a complete intersection ideal. For instance, if $\widetilde{Z}_2 = \{z_2 = z_3 = 0\}$ and $w_3 = z^{\alpha_3} = z_1 z_3$, then in a similar way as in (8.3),

$$M_2^{i^*w_2, i^*w_3} = 2(Z_1 + Z_2).$$

In the next example we will see that one can give a meaning to the intersection $\mathcal{J} \cdot \mu$, where $\mathcal{J}$ generates a regular embedding and μ is any cycle such that $\operatorname{codim} Z(\mathcal{J}) \cap |\mu| = \operatorname{codim} Z(\mathcal{J}) + \operatorname{codim} |\mu|$.

Example 8.8 Let $\mathcal{J}$ be a locally complete intersection ideal sheaf on Y of codimension κ and let μ be a cycle in Y such that $\operatorname{codim} Z(\mathcal{J}) \cap |\mu| = \kappa + \operatorname{codim} |\mu|$. Let $f = (f_1, \dots, f_\kappa)$ be a holomorphic tuple locally generating $\mathcal{J}$. We claim that

$$\mathcal{J} \cdot \mu := M_\kappa^f \wedge \mu,$$

cf. Sect. 2.2, is the Lelong current of a cycle with support $Z(\mathcal{J}) \cap |\mu|$ that only depends on the integral closure class of $\mathcal{J}$.

To see this, assume first that μ is an irreducible subvariety and let $\iota \colon \mu \to Y$ be the inclusion. Then, by (2.13) we have $M_\kappa^f \wedge \mu = \iota_* M_\kappa^{\iota^* f}$. Since $Z(\mathcal{J})$ and μ intersect properly it follows that $\iota^* f$ is a regular sequence at all points where $\iota^* f = 0$. In view of Example 6.1 thus $M_\kappa^{\iota^* f}$ is a cycle in μ and we conclude that $M_\kappa^f \wedge \mu$ is a cycle in Y. Moreover, in view of [5, Remark 4.1], $M_\kappa^{\iota^* f}$ only depends on the integral closure class of $\langle \iota^* f \rangle$. The claim now follows for an arbitrary μ by linearity.

If μ in Example 8.8 has a good potential, e.g., is a nice cycle, then in view of Sect. 3, $\mathcal{J} \cdot \mu = \mu_{\mathcal{J}} \wedge \mu = \mu \wedge \mu_{\mathcal{J}}$. Thus, in this case, $\mathcal{J} \cdot \mu$ only depends on the fundamental cycle $\mu_{\mathcal{J}}$ of $\mathcal{J}$. We will see in the following example that in general, even for a principal ideal $\mathcal{J}$, $\mathcal{J} \cdot \mu$ depends on (the integral closure class of) $\mathcal{J}$ and not only on its fundamental cycle.

Example 8.9 Let $Y = \{(x, y, z) \in \mathbb{C}^3;\ xy = 0\}$ and let, for positive integers p and q, $f_{p,q}$ be the restriction to Y of $x^p + y^q$. Let $\pi : Y' \to Y$ be the normalization and notice that Y' is the disjoint union of $Y_1' \simeq \{y = 0\}$ and $Y_2' \simeq \{x = 0\}$. We have that, cf. Example 6.1,

$$\operatorname{div} f_{p,q} = \pi_* \operatorname{div} \pi^* f_{p,q} = \pi_*(\operatorname{div} x^p|_{Y_1'} + \operatorname{div} y^q|_{Y_2'}) = (p+q)[Z],$$

where $Z = \{x = y = 0\}$. Thus, $(p+q)[Z]$ is the fundamental cycle of the ideal $\mathcal{J}_{p,q} \subset \mathcal{O}_Y$ generated by $f_{p,q}$, and so $[Z]$ is an RE-cycle. Notice that $\mathcal{J}_{p,q}$ and $\mathcal{J}_{p',q'}$ have the same fundamental cycle if $p + q = p' + q'$.

Let $[W] = \{y = z = 0\}$ and let $\iota\colon W \to Y$ be the inclusion. Then, cf. Example 8.8,

$$\mathcal{J}_{p,q} \cdot [W] = \mathrm{div} f_{p,q} \wedge [W] = \iota_* \mathrm{div} \iota^* f_{p,q} = \iota_* \mathrm{div}(x^p|_W) = p[0].$$

Since $\mathcal{J}_{1,2}$ and $\mathcal{J}_{2,1}$ have the same fundamental cycle it follows that the product defined in Example 8.8 does not only depend on the fundamental cycle of $\mathcal{J}$. We notice also that $[W]$ cannot have a good potential; in particular it cannot be an RE-cycle, or even a nice cycle. Indeed, if it had, then $\mathcal{J}_{p,q} \cdot [W]$ would only depend on the fundamental cycle of $\mathcal{J}_{p,q}$ in view of the discussion just before this example.

Assume that Y has pure dimension n, let $\iota\colon Y \to Y \times Y$ be the diagonal embedding and η a tuple of holomorphic functions defining $\Delta = \iota_* Y$ in $Y \times Y$. In view of Example 2.6 we have $M_n^\eta = [\Delta]$. By (2.12) thus

$$M_n^\eta \wedge (\alpha \otimes \beta) = \iota_*(\alpha \wedge \beta) \tag{8.4}$$

if α and β are smooth forms. If α and β are good $\bar\partial$-potentials u_j of cycles μ_j, then we used that (8.4) holds outside $|\mu_1| \times |\mu_2|$ to define the product $u_1 \wedge u_2$; see the proof of Proposition 3.2. Moreover, if α and β are generically smooth and have certain mild singularities, then the left-hand side of (8.4) can be used to give a reasonable meaning to the product $\alpha \wedge \beta$, see, e.g., [18]. However, contrary to the case when Y is smooth, if α and β are cycles, then the left-hand side of (8.4) cannot in general be used to give a reasonable definition of $\alpha \wedge \beta$ for singular Y as the following example shows.

Example 8.10 Let Y and p be as in Example 8.3, but here considered in $\mathbb{C}^3_{x,y,z}$ so that $p = (0, 0, 0)$. By Example 6.2, $[p]$ is an RE-cycle. Let $\iota\colon Y \to Y \times Y$ be the diagonal embedding and η the restriction to $Y \times Y$ of $(x - x', y - y', z - z')$ so that η defines the diagonal $\iota_* Y$. We claim that

$$M_2^\eta \wedge (Y \otimes [p]) = 2\iota_*[p]. \tag{8.5}$$

Using the left-hand side as a definition of a proper intersection product of Y and $[p]$ would thus not be in agreement with Example 8.1.

To see (8.5), consider the generically $2:1$ mapping

$$\pi : \mathbb{C}^2 \to Y; \quad (u, v) \mapsto (u^2, v^2, uv).$$

and let g be the mapping $Y \to Y \times Y$, $\xi \mapsto (\xi, p)$. Since $g_* 1 = Y \otimes [p]$ and $\pi_* 1 = 2$, by (2.13) we have that

$$M_2^\eta \wedge (Y \otimes [p]) = g_* M_2^{g^*\eta} = g_* M_2^{x,y,z} = \frac{1}{2} g_* \pi_* M_2^{u^2,v^2,uv}. \tag{8.6}$$

The ideal (u^2, v^2, uv) has the same integral closure as the regular sequence (u^2, v^2) and hence, cf. [5, Remark 4.1],

$$M_2^{u^2,v^2,uv} = M_2^{u^2,v^2} = 4[0]. \tag{8.7}$$

Since $\pi_*[0] = [p]$ thus (8.5) follows from (8.6) and (8.7).

9 The case when *Y* is nearly smooth

Recently, in [8], Barlet and Magnússon introduced a class of analytic spaces called *nearly smooth*. An analytic space Y is nearly smooth if it is normal and if for each $y \in Y$ there is a neighborhood $\mathcal{U}$ of y, a connected complex manifold $\widetilde{Y}$, and a proper holomorphic surjective finite mapping $q\colon \widetilde{Y} \to \mathcal{U}$. Such a mapping q is called a local model. The number of points of $q^{-1}(y)$ for generic $y \in Y$ is constant and denoted by $\deg q$.

Recall that if $\mu_{\mathcal{J}}$ is the fundamental cycle of a locally complete intersection ideal $\mathcal{J}$ of codimension κ generated by a tuple $f = (f_1, \ldots, f_\kappa)$, then $\mu_{\mathcal{J}} = M^f_\kappa$; cf. Example 6.1.

Lemma 9.1 *Suppose that $f_j = (f_{j,1}, \ldots, f_{j,\kappa_j})$, $j = 1, 2$, are holomorphic tuples such that codim $f_j^{-1}(0) = \kappa_j$ and that the corresponding fundamental cycles $\mu_j = M^{f_j}_{\kappa_j}$ intersect properly. If $q\colon \widetilde{Y} \to Y$ is a local model, then $M^{q^* f_j}_{\kappa_j}$ are properly intersecting fundamental cycles and*

$$\mu_2 \cdot \mu_1 = \frac{1}{deg\, q} q_*(M^{q^* f_2}_{\kappa_2} \wedge M^{q^* f_1}_{\kappa_1}).$$

Proof Since $q^{-1}(y)$ is 0-dimensional for all $y \in Y$ it follows that $M^{q^* f_j}_{\kappa_j}$ are properly intersecting fundamental cycles of complete intersection ideals.

Since $q_* 1 = \deg q$, by (2.13) we have that $\deg q \cdot M^{f_1}_{\kappa_1} = q_* M^{q^* f_1}_{\kappa_1}$. By (2.13) again thus

$$\deg q \cdot M^{f_2}_{\kappa_2} \wedge M^{f_1}_{\kappa_1} = q_* \left(M^{q^* f_2}_{\kappa_2} \wedge M^{q^* f_1}_{\kappa_1} \right).$$

The lemma now follows in view of Example 3.6. □

By [8, Proposition 1.1.5], the inequality (1.1) holds for nearly smooth spaces. For such spaces proper intersection thus has a clear meaning, and an intersection product $\mu_2 \cap_Y \mu_1$ for any two properly intersecting cycles μ_1 and μ_2 in Y is introduced in [8]. The main result of this section is the following proposition.

Proposition 9.2 *Let Y be nearly smooth and let μ_1 and μ_2 be properly intersecting RE-cycles in Y. Then $\mu_2 \cdot \mu_1 = \mu_2 \cap_Y \mu_1$.*

Lemma 9.3 *Let $f = (f_1, \ldots, f_\kappa)$ be a holomorphic tuple in Y such that codim $f^{-1}(0) = \kappa$. Then there is a neighborhood S of $0 \in \mathbb{C}^\kappa$ such that $\left(M^{f-s}_\kappa\right)_{s \in S}$ is an analytic family of cycles in Y parametrized by S. In particular, $\lim_{s\to 0} M^{f-s}_\kappa = M^f_\kappa$ as currents.*

The definition of an analytic family of cycles can be found, e.g., in [9, Section 4.3.1]. The precise definition is not needed in this paper; instead we will recall and use various natural properties of such families when needed.

Proof Notice first that there is a neighborhood $S \subset \mathbb{C}^\kappa$ of 0 such that codim $f^{-1}(s) = \kappa$ for all $s \in S$; see, e.g., [9, Proposition 2.4.60].

It is a local problem in Y to show that $\left(M^{f-s}_\kappa\right)_{s \in S}$ is an analytic family so we can assume that we have a local model $q\colon \widetilde{Y} \to Y$. As in the proof of Lemma 9.1 we have $q_* M^{q^* f - s}_\kappa = \deg q \cdot M^{f-s}_\kappa$ and codim $(q^* f)^{-1}(s) = \kappa$, $s \in S$. If $\left(M^{q^* f - s}_\kappa\right)_{s \in S}$ is an analytic family in $\widetilde{Y}$, then $q_* M^{q^* f - s}_\kappa$ is an analytic family in Y by [9, Theorem 4.3.22]. Hence, to show the lemma it suffices to show that $\left(M^{q^* f - s}_\kappa\right)_{s \in S}$ is analytic. We may thus assume that Y is smooth. Possibly replacing Y by $f^{-1}(S)$ we may also assume that $f\colon Y \to S$ has fibers of constant codimension κ.

Let $G \subset Y \times S$ be the graph of f and let $H_s := Y \times \{s\} \subset Y \times S$. Then G and H_s intersect properly and $X_s := G \cdot H_s$ is after the identification $H_s \simeq Y$ a cycle in Y. In view of [10, Ch. VII, Proposition 1.5.1], $(X_s)_{s \in S}$ is an analytic family of cycles in Y.

Let $F(x,s)=f(x)-s$. By Example 2.6 we have $M_\kappa^F=[G]$. Fix an arbitrary $s_0\in S$ and let $i\colon Y\to Y\times S$ be the embedding $x\mapsto(x,s_0)$ so that $i_*1=H_{s_0}$. In view of Sect. 4.1 and Sect. 4.2, and (2.13),

$$i_*X_{s_0}=G\cdot H_{s_0}=M_\kappa^F\wedge i_*1=i_*M_\kappa^{i^*F}=i_*M_\kappa^{f-s_0}.$$

Hence, $X_{s_0}=M_\kappa^{f-s_0}$ and it follows that $(M_\kappa^{f-s})_{s\in S}$ is an analytic family. The last statement of the lemma follows from [9, Proposition 4.2.17]. □

If $q\colon\widetilde{Y}\to Y$ is a local model and μ is a cycle in Y, then there is a natural pullback cycle $q^*\mu$ in $\widetilde{Y}$, see [8, Section 2.1]. We have the following corollary of Lemma 9.3.

Corollary 9.4 *Let $f=(f_1,\dots,f_\kappa)$ be as in Lemma 9.3, let $q\colon\widetilde{Y}\to Y$ be a local model, and let $\mu=M_\kappa^f$. Then $q^*\mu=M_\kappa^{q^*f}$.*

Proof Let $\widetilde{V}=\{x\in\widetilde{Y};\ \mathrm{rank}_x\, q<\dim Y\}$ and $V=q(\widetilde{V})$. Then, since q is proper and surjective, $\widetilde{V}$ and V are nowhere dense analytic subsets of $\widetilde{Y}$ and Y, respectively.

By [8, Section 2.1], q^* has the following two properties. First, if Z is a cycle with no component contained in $V\cup Y_{sing}$ and $Z=|Z|$, then

$$q^*Z=q^{-1}|Z|. \tag{9.1}$$

Second, if $(\mu_s)_{s\in S}$ is an analytic family of cycles in Y, then $(q^*\mu_s)_{s\in S}$ is an analytic family of cycles in $\widetilde{Y}$.

Let now $\mu_s:=M_\kappa^{f-s}$ and $\widetilde{\mu}_s:=(M_\kappa^{q^*f-s})_{s\in S}$, where S is as in Lemma 9.3. By that lemma, $(\mu_s)_{s\in S}$ and $(\widetilde{\mu}_s)_{s\in S}$ are analytic families of cycles in Y and $\widetilde{Y}$, respectively. We claim that

$$q^*\mu_s=\widetilde{\mu}_s \tag{9.2}$$

for all $s\in S$, from which the corollary follows. To show the claim it suffices to check that (9.2) holds for generic s in S since both $q^*\mu_s$ and $\widetilde{\mu}_s$ are analytic, in particular continuous, in s. Let $\widetilde{A}=\{x\in\widetilde{Y};\ \mathrm{rank}_x\, f\circ q<\kappa\}$ and $A=q(\widetilde{A})$. Then $\widetilde{A}$ and A are nowhere dense analytic subsets of $\widetilde{Y}$ and Y, respectively.

For generic s, by, e.g., [9, Corollary 2.4.61], μ_s has no component contained in $A\cup V\cup Y_{sing}$ and $\widetilde{\mu}_s$ has no component contained in $\widetilde{A}$. Fix such an s. Outside $A\cup V\cup Y_{sing}$, f has constant rank κ and so $\{f=s\}$ is a submanifold and $f-s$ generates its radical ideal there. By Example 2.6 thus

$$\mu_s=|\mu_s| \tag{9.3}$$

outside $A\cup V\cup Y_{sing}$. Since μ_s has no component contained in $A\cup V\cup Y_{sing}$ it follows that (9.3) holds in Y. In the same way it follows that $\widetilde{\mu}_s=|\widetilde{\mu}_s|$. Since μ_s in particular has no component contained in $V\cup Y_{sing}$ it now follows from (9.1) that

$$q^*\mu_s=q^{-1}|\mu_s|=q^{-1}\{f=s\}=\{f\circ q=s\}=|\widetilde{\mu}_s|=\widetilde{\mu}_s$$

and the claim and the corollary are proved. □

Proof of Proposition 9.2 This is a local statement so after shrinking Y we may assume that there is a local model $q\colon\widetilde{Y}\to Y$. Moreover, by linearity we can assume that $\mu_j=M_{\kappa_j}^{f_j}$ for holomorphic tuples $f_j=(f_{j1},\dots,f_{j\kappa_j})$ such that $\mathrm{codim}\, f_j^{-1}(0)=\kappa_j$.

The pullback cycles $q^*\mu_j$ intersect properly in $\widetilde{Y}$ and, by [8, Theorem 3.1.5],

$$\mu_2 \cap_Y \mu_1 = \frac{1}{\deg q} q_*(q^*\mu_2 \cdot q^*\mu_1), \tag{9.4}$$

where $q^*\mu_2 \cdot q^*\mu_1$ is the proper intersection product in $\widetilde{Y}$. By Lemma 9.1 thus the proposition follows from Corollary 9.4. □

Remark 9.5 By [8, Proposition 1.1.5], any Weil divisor in a nearly smooth Y is a $\mathbb{Q}$-Cartier divisor. Effective Weil divisors in Y thus are RE-cycles, see Example 6.8. In view of Example 8.1 and since any 0-dimensional cycle is RE by Example 6.2 we thus see that for 2-dimensional nearly smooth spaces the proper intersection product in [8] coincides with our product for effective cycles.

Acknowledgements We are grateful to the referee for careful reading and many important comments and suggestions that have improved the paper.

Funding Open access funding provided by Chalmers University of Technology.

References

1. Andersson, M.: A generalized Poincaré–Lelong formula. Math. Scand. **101**(2), 195–218 (2007)
2. Andersson, M.: Pseudomeromorphic currents on subvarieties. Complex Var. Elliptic Equ. **61**(11), 1533–1540 (2016)
3. Andersson, M., Eriksson, D., Samuelsson Kalm, H., Wulcan, E., Yger, A.: Global representation of Segre numbers by Monge–Ampère products. Math. Ann. **380**, 349–391 (2021)
4. Andersson, M., Samuelsson, H.: A Dolbeault–Grothendieck lemma on complex spaces via Koppelman formulas. Invent. Math. **190**(2), 261–297 (2012)
5. Andersson, M., Samuelsson Kalm, H., Wulcan, E., Yger, A.: Segre numbers, a generalized King formula, and local intersections. J. Reine Angew. Math. **728**, 105–136 (2017)
6. Andersson, M., Wulcan, E.: Decomposition of residue currents. J. Reine Angew. Math. **638**, 103–118 (2010)
7. Andersson, M., Wulcan, E.: Direct images of semi-meromorphic currents. Ann. Inst. Fourier Grenoble **68**(2), 875–900 (2018)
8. Barlet, D., Magnússon, J.: On nearly smooth complex spaces. Math. Z. **292**, 317–341 (2019)
9. Barlet, D., Magnússon, J.: Complex analytic cycles. I. Basic results on complex geometry and foundations for the study of cycles. In: Grundlehren der Mathematischen Wissenschaften, vol. 356. Springer, Cham [2019] (2019)
10. Barlet, D., Magnússon, J.: Cycles analytiques complexes. II. l'espace des cycles. In: Cours Spécialisés, vol. 27. Société Mathématique de France, Paris, [2020] (2020)
11. Bott, R., Chern, S.S.: Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections. Acta Math. **114**, 71–112 (1965)
12. Chirka, E.M.: Complex Analytic Sets. Kluwer Academic Publishers Group, Dordrecht (1989). Accessed 6 Feb 2018
13. Demailly, J.-P.: Complex analytic and differential geometry. http://www-fourier.ujf-grenoble.fr/~demailly. Accessed 6 Feb 2018
14. Demailly, J.-P.: Courants positifs et théorie de l'intersection. Gaz. Math. **53**, 131–159 (1992)
15. Fulton, W.: Intersection Theory, 2nd edn. Springer, Berlin (1998)

16. Gillet, H., Soulé, C.: Arithmetic intersection theory. Inst. Hautes Études Sci. Publ. Math. **72**, 93–174 (1990)
17. Mumford, D.: The topology of normal singularities of an algebraic surface and a criterion for simplicity. Inst. Hautes Études Sci. Publ. Math. No. **9**, 5–22 (1961)
18. Ruppenthal, J., Samuelsson Kalm, H., Wulcan, E.: Explicit Serre duality on complex spaces. Adv. Math. **305**, 1320–1355 (2017)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.