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Citation for the original published paper (version of record):

Austad, A., Thiel, H. (2025). The ideal separation property for reduced group C^* -algebras. *Journal of Functional Analysis*, 289(1).
<http://dx.doi.org/10.1016/j.jfa.2025.110904>

N.B. When citing this work, cite the original published paper.



Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa



Regular Article

The ideal separation property for reduced group C^* -algebras [☆]



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ARTICLE INFO

Article history:

Received 11 September 2024

Accepted 13 February 2025

Available online 3 March 2025

Communicated by Stefaan Vaes

MSC:

primary 22D15, 43A20, 46H10

secondary 22D20, 22D25, 46L05

Keywords:

Ideal separation property

Ideal intersection property

Locally compact groups

Group C^* -algebra

C^* -uniqueness

*-regularity

ABSTRACT

We say that an inclusion of an algebra A into a C^* -algebra B has the ideal separation property if closed ideals in B can be recovered by their intersection with A . Such inclusions have attractive properties from the point of view of harmonic analysis and noncommutative geometry. We establish several permanence properties of locally compact groups for which $L^1(G) \subseteq C^*_{\text{red}}(G)$ has the ideal separation property.

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[☆] The first named author was partially supported by the Independent Research Fund Denmark through grant number 1026-00371B and The Research Council of Norway project 324944. The second named author was partially supported by the Knut and Alice Wallenberg Foundation (KAW 2021.0140).

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1. Introduction

A locally compact group G is called **-regular* if for any closed, two-sided ideal I in the full group C^* -algebra $C^*(G)$, we have $I = \overline{I \cap L^1(G)}$, where the closure is in $C^*(G)$ -norm. An in-depth study of groups with this property was first initiated by Boidol, Leptin, Schurman and Vahle in [12]. They identified that groups of polynomial growth are *-regular. Boidol then continued to add to the list of *-regular groups, showing that semidirect products of abelian groups belong to this class [11]. He also characterized the connected groups which are *-regular [10]. Later, Hauenschild, Kaniuth and Voigt [21] showed that the class of *-regular groups is closed under Cartesian products. Boidol conjectured that every symmetric (=hermitian) locally compact group is *-regular, and that conversely every almost connected, *-regular group is symmetric [11]. To honor his contributions to the theory, *-regular groups are sometimes referred to as Boidol groups [38, Definition 10.5.8].

A closely related notion is that of C^* -uniqueness, where a locally compact group G is said to be C^* -unique if the Banach *-algebra $L^1(G)$ has a unique C^* -norm. It is not difficult to see that $L^1(G)$ has a unique C^* -norm if and only if, for any non-zero, closed, two-sided ideal I in $C^*(G)$, we have $I \cap L^1(G) \neq \{0\}$. From this it follows that every *-regular group is C^* -unique, and that every C^* -unique group has the weak containment property (the full and reduced group C^* -algebra coincide), and therefore is amenable; see also [13, Proposition 2]. In recent work, the first named author and Raum [5] showed that locally virtually polycyclic groups are C^* -unique.

Boidol gave examples of amenable groups which are not C^* -unique, and of C^* -unique groups which are not *-regular [13, p.230], thereby showing strict containment between these classes of groups. However, it remains unknown if such examples exist among discrete groups, prompting Leung and Ng to pose the question if all discrete, amenable group are C^* -unique [30]. An even more ambitious, yet also unresolved, question is:

Question A. *Are all discrete, amenable groups *-regular?*

While Boidol focused on *-regularity and uniqueness of C^* -norm for L^1 -algebras of locally compact groups, the notions were studied for more general *-algebras by Barnes in [6]. We also mention that *-regularity of more general convolution algebras has been studied by Leung and Ng [31,32], Leung [29], and more recently by Flores [15]. On the other hand, C^* -uniqueness was studied for twisted group algebras by the first named author [3], and for twisted groupoid algebras in collaboration with Ortega [4].

A closely related topic is *algebraic C^* -uniqueness* for a discrete group Γ , defined by uniqueness of the C^* -norm on the group algebra $\mathbb{C}[\Gamma]$. It is straightforward to see that this implies uniqueness of the C^* -norm on $\ell^1(\Gamma)$, and consequently, it also implies amenability. However, examples like the group of integers show that C^* -uniqueness is a strictly weaker property than algebraic C^* -uniqueness. The question of which discrete

groups are algebraically C^* -unique has attracted much attention in recent years [18,1,43]. Interestingly, $*$ -regularity of $\mathbb{C}[\Gamma]$ appears to not have been investigated so far.

One can view $*$ -regularity and C^* -uniqueness of a locally compact group G as the ideal separation and ideal intersection properties for the inclusion $L^1(G) \subseteq C^*(G)$. Here, we say that a subalgebra A of a C^* -algebra B has the *ideal separation property* (ISP for short) if for any closed, two-sided ideals I and J in B we have $I \subseteq J$ whenever $I \cap A \subseteq J \cap A$. Likewise, we say $A \subseteq B$ has the *ideal intersection property* (IIP for short) if for any non-zero, closed, two-sided ideal I in B we have $I \cap A \neq \{0\}$; see Definition 2.2.

Barnes’ notions of $*$ -regularity and C^* -uniqueness for a $*$ -algebra A then correspond precisely to the ISP and the IIP for the inclusion $A \subseteq C^*(A)$, where $C^*(A)$ is the enveloping C^* -algebra of A . (One needs to assume that A admits some C^* -norm so that A embeds into $C^*(A)$.)

The ISP and the IIP have also been extensively studied for C^* -algebras arising from dynamical systems or groupoids. For example, Kawamura and Tomiyama [23] showed that the action of an amenable, discrete group Γ on a compact, Hausdorff space X is topologically free if and only if $C(X) \subseteq C(X) \rtimes \Gamma$ has the ISP. For actions on possibly noncommutative C^* -algebras A , the ISP and the IIP for the inclusion $A \subseteq A \rtimes_{\text{red}} \Gamma$ have been studied for example by Sierakowski [44], Kennedy and Schafhauser [25] and Geffen and Ursu [17]. The IIP and the ISP for the inclusion $C(\mathcal{G}^{(0)}) \subseteq C^*_{\text{red}}(\mathcal{G})$ for a groupoid \mathcal{G} have for example been studied by Kennedy, Kim, Li, Raum and Ursu [24] and Brix, Carlsen and Sims [14].

We note that the ISP and the IIP are always studied with respect to the *reduced* crossed product and the *reduced* groupoid algebra. The reason is that the kernels of the quotient maps $A \rtimes G \rightarrow A \rtimes_{\text{red}} G$ and $C^*(\mathcal{G}) \rightarrow C^*_{\text{red}}(\mathcal{G})$ have trivial intersection with A and $C(\mathcal{G}^{(0)})$, respectively.

Similarly, for the canonical quotient map $\lambda: C^*(G) \rightarrow C^*_{\text{red}}(G)$, we always have $L^1(G) \cap \ker \lambda = \{0\}$. Therefore, if G is nonamenable, then $L^1(G) \subseteq C^*(G)$ does not have the IIP, let alone the ISP. We therefore believe that the natural setting to study ideal separation and ideal intersection for a locally compact group G is with respect to the inclusion of $L^1(G)$ in the reduced C^* -algebra $C^*_{\text{red}}(G)$, and we define:

Definition B. We say that a locally compact group G has

- (1) the *ideal separation property*, ISP for short, if for any distinct closed, two-sided ideals $I, J \subseteq C^*_{\text{red}}(G)$ we have $I \cap L^1(G) \neq J \cap L^1(G)$.
- (2) the *ideal intersection property*, IIP for short, if for every non-zero, closed, two-sided ideal $I \subseteq C^*_{\text{red}}(G)$ we have $I \cap L^1(G) \neq \{0\}$.

We make the following easy, yet useful, observation, whose proof follows from the above discussion:

Proposition C. Let G be a locally compact group. The following statements hold:

- (1) The group G is $*$ -regular if and only if G is amenable and has the ISP.
- (2) The group G is C^* -unique if and only if G is amenable and has the IIP.

Thus, the known classes of $*$ -regular or C^* -unique groups provide examples of (automatically amenable) groups with the ISP or the IIP. Moreover, there are many non-amenable groups with the ISP or the IIP. For example, it is immediate that the class of groups with the ISP includes all C^* -simple groups, that is, all groups G for which $C_{\text{red}}^*(G)$ is simple.

We note that Definition B is inspired by the analogous property for C^* -dynamical systems from [5], where it is termed the L^1 -IIP.

In this paper we establish several permanence properties of the ISP and the IIP for locally compact groups, as summarized in the following theorem.

Theorem D. Let G and H be locally compact groups.

- (1) If G has the ISP (the IIP) and $N \subseteq G$ is a closed (compact), amenable, normal subgroup, then G/N has the ISP (the IIP); see Proposition 3.1.
- (2) If every open, relatively compact subset of G is contained in a closed subgroup with the ISP (the IIP), then G has the ISP (the IIP); see Theorem 3.3.
- (3) If $G = \varprojlim_{\lambda} G_{\lambda}$ is an inverse limit such that the maps $G \rightarrow G_{\lambda}$ are surjective with compact kernel, then G has the ISP (the IIP) if and only if each G_{λ} does; see Theorem 3.9.
- (4) The product group $G \times H$ has the IIP (the ISP) if and only if both G and H have the IIP (the ISP, and the pair of group C^* -algebras $C_{\text{red}}^*(G)$ and $C_{\text{red}}^*(H)$ has Tomiyama's property (F)); see Theorem 4.13.

Statement (2) shows in particular that the ISP and the IIP are local properties; see Corollary 3.4. Further, statement (3) implies that the ISP and the IIP for an almost connected, locally compact group are determined by its Lie group quotients; see Corollary 3.11.

We note that our results for Cartesian products allow us to deduce that groups such as $\mathbb{Z} \times \mathbb{F}_2$ have the ISP, which was not known with the current technology. Further, Theorem 4.13 offers a strategy to exhibit a counterexample to the following natural extension of Question A to the setting of nonamenable groups:

Question E. *Do all discrete groups have the ideal separation property?*

Indeed, if Γ and Δ are discrete groups for which the pair of group C^* -algebras $C_{\text{red}}^*(\Gamma)$ and $C_{\text{red}}^*(\Delta)$ does not have Tomiyama's property (F), then $\Gamma \times \Delta$ does not have the ISP. It remains unclear if such a pair of groups exists; see Question 4.16 and the discussion preceding it.

The paper is structured as follows. In Section 2 we introduce the ISP and the IIP for an inclusion of a subalgebra in a C^* -algebra. We give characterizations in terms of primitive ideals and $*$ -structure spaces.

Specializing to locally compact groups, we treat the inclusion $L^1(G) \subseteq C_{\text{red}}^*(G)$ in detail in Section 3, and we investigate preservation of the ISP and the IIP with respect to quotients, approximation by closed subgroups, inductive limits, and inverse limits. Lastly, Section 4 deals with the ISP and the IIP for the inclusion of the algebraic and projective tensor product in the minimal tensor product of C^* -algebras. This is then applied to characterize the ISP and the IIP for Cartesian products of locally compact groups.

Acknowledgments The authors thank Leonel Robert for valuable comments on the minimal tensor product of C^* -algebras.

2. Definition and first properties

In this section, we introduce the ideal separation property (ISP) and the ideal intersection property (IIP) for the inclusion of a subalgebra into a C^* -algebra, and we study basic properties of these notions. We neither assume that the subalgebra is dense nor closed, to allow for the study of inclusions such as $A \subseteq A \rtimes_{\text{red}} G$ as well as $L^1(G) \subseteq C_{\text{red}}^*(G)$.

Later we will specialize to the inclusion $A \subseteq B$ of a dense $*$ -subalgebra A in a C^* -algebra B , and we give characterizations of the ISP (Proposition 2.6) and the IIP (Proposition 2.8) in terms of the connection between the $*$ -structure space of A and the primitive ideal space of B . We also characterize the ISP and the IIP using estimates on C^* -seminorms; see Proposition 2.9.

Convention 2.1. All ideals in this paper are assumed to be two-sided.

Definition 2.2. Let $A \subseteq B$ be an inclusion of a subalgebra into a C^* -algebra B .

- (1) We say that $A \subseteq B$ has the *ideal separation property* (ISP for short), or that A *separates ideals in B* , if for all distinct closed ideals I and J in B we have $I \cap A \neq J \cap A$.
- (2) We say that $A \subseteq B$ has the *ideal intersection property* (IIP for short), or that A *detects ideals in B* , if for every non-zero, closed ideal I in B we have $I \cap A \neq \{0\}$.

The next result provides basic characterizations of the ISP. Most useful is condition (3), which states that every closed ideal in B can be *reconstructed* from its intersection with A .

Proposition 2.3. *Let $A \subseteq B$ be an inclusion of a subalgebra into a C^* -algebra B . The following statements are equivalent:*

- (1) The inclusion $A \subseteq B$ has the ISP.
- (2) For every closed ideal $I \subseteq B$, the inclusion $(A + I)/I \subseteq B/I$ has the IIP.
- (3) Every closed ideal $I \subseteq B$ is equal to the closed ideal of B generated by $I \cap A$.
- (4) For any closed ideals $I, J \subseteq B$ we have $I \subseteq J$ whenever $I \cap A \subseteq J \cap A$.

Proof. We show the implications ‘(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)’. Assuming (1), let us verify (2). Let $I \subseteq B$ be a closed ideal, and let $J \subseteq B/I$ be a non-zero, closed ideal. Let $\pi_I: B \rightarrow B/I$ denote the quotient map, and set $J' := \pi_I^{-1}(J)$, which is a closed ideal of B with $I \subseteq J'$ and $I \neq J'$. Since A separates ideals in B , we obtain $a \in A$ with $a \in J'$ but $a \notin I$. Then $0 \neq \pi_I(a) \in J \cap \pi_I(A)$.

Assuming (2), let us verify (3). Let $I \subseteq B$ be a closed ideal, and let H denote the closed ideal of B generated by $I \cap A$. We clearly have $H \subseteq I$, and we need to show that $I \subseteq H$. Let $\pi_H: B \rightarrow B/H$ denote the quotient map. We have $I \cap A \subseteq H$, and using $H \subseteq I$ a straightforward calculation will show that $\pi_H(I) \cap \pi_H(A) = \pi_H(I \cap A) = \{0\}$. Using that $\pi_H(A) \subseteq B/H$ has the IIP, we get $\pi_H(I) = \{0\}$, and so $I \subseteq H$.

Assuming (3), let us verify (4). Let $I, J \subseteq B$ be closed ideals with $I \cap A \subseteq J \cap A$. Then the closed ideal of B generated by $I \cap A$ is contained in the closed ideal generated by $J \cap A$, and by assumption this implies that $I \subseteq J$.

Assuming (4), let us verify (1). To show that A separates ideals in B , let $I, J \subseteq B$ be closed ideals with $I \cap A = J \cap A$. We need to show that $I = J$. But the assumption immediately implies that $I \subseteq J$ and $J \subseteq I$. \square

It follows easily from Definition 2.2 that both the ISP and the IIP pass to superalgebras in the following sense.

Proposition 2.4. *Let B be a C^* -algebra, and let $A_1 \subseteq A_2 \subseteq B$ be subalgebras. If $A_1 \subseteq B$ has the ISP (the IIP), so does $A_2 \subseteq B$.*

We now turn to characterizations of the ISP and the IIP in terms of primitive ideal spaces and $*$ -structure spaces. We recall the basic definitions and refer the reader to [38, Chapter 10.5] for further details.

For a $*$ -algebra A we denote by Π_A^* the set of all kernels of topologically irreducible $*$ -representations of A , and we equip it with the hull-kernel topology. This topological space is known as the *$*$ -structure space* of A . For any subset S of A the *hull* of S is defined as

$$h(S) = \{P \in \Pi_A^* \mid S \subseteq P\}.$$

For any subset L of Π_A^* the *kernel* of L is defined as

$$k(L) = \bigcap_{Q \in L} Q \subseteq A.$$

We also write $h_A(S)$ for $h(S)$, and $k_A(L)$ for $k(L)$. As the space Π_A^* is equipped with the hull-kernel topology, a subset $K \subseteq \Pi_A^*$ is defined to be closed if $K = h(k(K))$. For every subset $S \subseteq A$, the set $h(S)$ is closed.

Under mild assumptions on A (that are always satisfied for Banach $*$ -algebras, in particular for $L^1(G)$ for a locally compact group G), whenever $I \subseteq A$ is a $*$ -ideal, then there is a natural homeomorphism of $\Pi_A^* \setminus h(I)$ with Π_I^* given by the map $P \mapsto P \cap I$; see [38, Theorem 10.5.5].

If A is a C^* -algebra, then the $*$ -structure space coincides with the primitive ideal space, which we denote by Π_A . Moreover, in a C^* -algebra any closed ideal is the intersection of the primitive ideals containing it, so in this case we have $I = k(h(I))$ for every closed ideal $I \subseteq A$.

Let $A \subseteq B$ be an inclusion of a dense $*$ -subalgebra A into a C^* -algebra B . We will characterize the ISP and the IIP by conditions that capture to what extent A separates the primitive ideal space of B . Since A is dense in B , the restriction of a topologically irreducible $*$ -representation of B to A is again topologically irreducible. We therefore have a well-defined map between the $*$ -structure spaces

$$\Psi: \Pi_B \rightarrow \Pi_A^*, \quad P \mapsto P \cap A. \tag{2.1}$$

Given a subset $S \subseteq A$, we have

$$\Psi^{-1}(h_A(S)) = h_B(S),$$

which shows that Ψ is continuous. We show in Proposition 2.6 that Ψ is open if and only if $A \subseteq B$ has the ISP.

Lemma 2.5. *Let $A \subseteq B$ be an inclusion of a dense $*$ -subalgebra into a C^* -algebra. Let $Y \subseteq \Pi_B$, and let $P \in \Pi_B$. Then $\Psi(P)$ belongs to the closure of $\Psi(Y)$ in Π_A^* if and only if $k_B(Y) \cap A \subseteq P \cap A$.*

Proof. We first note that $\Psi(P)$ belongs to the closure of $\Psi(Y)$ in Π_A^* if and only if $\Psi(P) \in h_A(k_A(\Psi(Y)))$, that is, if and only if

$$k_A(\Psi(Y)) \subseteq \Psi(P) = P \cap A.$$

We have

$$k_B(Y) \cap A = \left(\bigcap \{Q : Q \in Y\} \right) \cap A = \bigcap \{Q \cap A : Q \in Y\}, = k_A(\Psi(Y))$$

which completes the proof. \square

Proposition 2.6. *Let $A \subseteq B$ be an inclusion of a dense $*$ -subalgebra A into a C^* -algebra B . The following statements are equivalent:*

- (1) The inclusion $A \subseteq B$ has the ISP.
- (2) We have $I = \overline{I \cap A}^{\|\cdot\|_B}$ for every closed ideal $I \subseteq B$.
- (3) The map Ψ of (2.1) is a homeomorphism onto its image.
- (4) The map Ψ of (2.1) is open onto its image.
- (5) For every closed and proper subset $K \subseteq \Pi_B$ and every $P \in \Pi_B \setminus K$ there exists $f \in A$ such that $f \in k_B(K)$ and $f \notin P$.

Proof. Assuming (1), let us verify (2). Let $I \subseteq B$ be a closed ideal. Since $A \subseteq B$ is dense, the closed ideal generated by $I \cap A$ is the closure of $I \cap A$. Therefore $I = \overline{I \cap A}$ by Proposition 2.3(3).

Assuming (2), let us verify (3). Observing that $P = \overline{P \cap A}$ for all $P \in \Pi_B$ we deduce that Ψ is injective. Since Ψ is continuous, it remains to show that Ψ maps closed subsets of Π_B to relatively closed subsets of $\Psi(\Pi_B)$ in Π_A^* . Let $K \subseteq \Pi_B$ be a closed subset. To show that $\Psi(K)$ is closed inside the image of Ψ , let $P \in \Pi_B$ such that $\Psi(P)$ belongs to the closure of $\Psi(K)$ in Π_A^* . Using that assumption at the first and last step, and using Lemma 2.5 at the second step, we get

$$k_B(K) = \overline{k_B(K) \cap A} \subseteq \overline{P \cap A} = P$$

Thus, $P \in h_B(k_B(K))$, which implies $P \in K$ since K is closed. This shows that $\Psi(K)$ is closed relative to the image of Ψ .

It is clear that (3) implies (4). Assuming (4), let us verify (5). Suppose that $K \subseteq \Pi_B$ is a proper, closed subset, and let $P \in \Pi_B \setminus K$. By assumption, $\Psi(K)$ is closed relative to the image of Ψ , and thus $\Psi(P) \notin h_A(k_A(\Psi(K)))$. This means that $k_A(\Psi(K))$ is not a subset of $\Psi(P)$. Choose $f \in k_A(\Psi(K))$ with $f \notin \Psi(P)$. We have seen in the proof of Lemma 2.5 that $k_A(\Psi(K)) = k_B(K) \cap A$. Thus, we have $f \in A$ and $f \in k_B(K)$, but $f \notin \Psi(B) = P \cap A$, which implies that $f \notin P$.

Assuming (5), let us verify (1). Let I and J be distinct closed ideals in B . We need to show that $I \cap A \neq J \cap A$. Without loss of generality, we may assume that $I \not\subseteq J$. Then $h_B(I) \not\subseteq h_B(J)$, which allows us to pick $P \in h_B(I) \setminus h_B(J)$. Since $h_B(J)$ is a closed subset of Π_B , we can apply the assumption to find $f \in A$ with $f \in k_B(h_B(J)) = J$ but $f \notin P$. Since $I \subseteq P$, we have $f \notin I$, and thus $I \cap A \neq J \cap A$. \square

Lemma 2.7. *Let $A \subseteq B$ be an inclusion of a dense $*$ -subalgebra into a C^* -algebra, and let $Y \subseteq \Pi_B$. The following statements hold:*

- (1) The subset $Y \subseteq \Pi_B$ is dense if and only if $k_B(Y) = \{0\}$.
- (2) The subset $\Psi(Y) \subseteq \text{Im}(\Psi)$ is dense if and only if $k_B(Y) \cap A = \{0\}$.

Proof. (1) The closure of Y in Π_B is $h_B(k_B(Y))$. Thus, $Y \subseteq \Pi_B$ is dense if and only if $h_B(k_B(Y)) = \Pi_B$. Since B is a C^* -algebra, we have $k_B(\Pi_B) = \{0\}$. It follows that $h_B(k_B(Y)) = \Pi_B$ if and only if $k_B(Y) = \{0\}$.

(2) Given $P \in \Pi_B$, the element $\Psi(P)$ belongs to the closure of $\Psi(Y)$ if and only if $k_B(Y) \cap A \subseteq P \cap A$; see Lemma 2.5. Thus, $\Psi(Y) \subseteq \text{Im}(\Psi)$ is dense if and only if $k_B(Y) \cap A \subseteq P \cap A$ for every $P \in \Pi_B$, which in turn is equivalent to $k_B(Y) \cap A = \{0\}$ since $\bigcap \{P : P \in \Pi_B\} = \{0\}$. \square

Proposition 2.8. *Let $A \subseteq B$ be an inclusion of a dense $*$ -subalgebra A into a C^* -algebra B . The following statements are equivalent:*

- (1) *The inclusion $A \subseteq B$ has the IIP.*
- (2) *A subset $Y \subseteq \Pi_B$ is dense if (and only if) $\Psi(Y) \subseteq \text{Im}(\Psi)$ is dense.*
- (3) *For every closed and proper subset $K \subseteq \Pi_B$ there exists a non-zero $f \in A$ such that $f \in k_B(K)$.*

Proof. Assuming (1), let us verify (2). Let $Y \subseteq \Pi_B$. In general, if Y is dense in Π_B , then $\Psi(Y)$ is dense in the image of Ψ . To show the converse, assume that $\Psi(Y)$ is dense in $\text{Im}(\Psi)$. Applying Lemma 2.7, we get $k_B(Y) \cap A = \{0\}$. Using that $A \subseteq B$ has the IIP, we deduce that $k_B(Y) = \{0\}$. Applying Lemma 2.7 again, we get that Y is dense in Π_B .

Assuming (2), let us verify (3). Let $K \subseteq \Pi_B$ be a closed and proper subset. The assumption implies that $\Psi(K)$ is not dense in $\text{Im}(\Psi)$. By Lemma 2.7 we get $k_B(K) \cap A \neq \{0\}$, which shows that there exists a non-zero $f \in A$ with $f \in k_B(K)$.

Assuming (3), let us verify (1). Suppose that $I \subseteq B$ is a non-zero, closed ideal. Then $h_B(I) \subseteq \Pi_B$ is a closed and proper subset. By assumption, we find a non-zero element $f \in A$ with $f \in k_B(h_B(I)) = I$. Thus, $I \cap A \neq \{0\}$. \square

We will have use for relating the ISP and the IIP for an inclusion $A \subseteq B$ of a dense $*$ -subalgebra into a C^* -algebra to statements about C^* -seminorms on the $*$ -algebra A . In order to state and prove our next result, let us introduce some notation. Denote by R_A^B the collection of $*$ -representations of A which extend to $*$ -representations of the C^* -algebra B . Assuming that A admits a C^* -envelope $C^*(A)$ (then A is called a G^* -algebra [38, Definition 10.1.1]), we may identify R_A^B with

$$\{\pi \in R_A^{C^*(A)} \mid \ker(\pi_*) \supseteq \ker(C^*(A) \rightarrow B)\}$$

where π_* is the natural extension of π to $C^*(A)$, and $C^*(A) \rightarrow B$ is the canonical surjection. Indeed, this identification is justified by observing that by the universal property of $C^*(A)$, $\pi \in R_A^B$ if and only if π_* factors through the surjection $C^*(A) \rightarrow B$.

Proposition 2.9. *Let $A \subseteq B$ be an inclusion of a dense $*$ -subalgebra A into a C^* -algebra B . For $\pi \in R_A^B$, denote by $\bar{\pi}$ the extension to B . The following statements hold:*

- (1) *The inclusion $A \subseteq B$ has the ISP if and only if for every $\pi, \varrho \in R_A^B$ the containment $\ker(\pi) \subseteq \ker(\varrho)$ in A implies $\|\bar{\varrho}(b)\| \leq \|\bar{\pi}(b)\|$ for all $b \in B$.*

(2) The inclusion $A \subseteq B$ has the IIP if and only if for every $\pi \in R_A^B$, the statement $\ker(\pi) = \{0\}$ in A implies $\|\bar{\pi}(b)\| = \|b\|_B$ for all $b \in B$.

Proof. We begin by proving (1). We first note that $\pi, \varrho \in R_A^B$ satisfy $\ker(\bar{\pi}) \subseteq \ker(\bar{\varrho})$ if and only if $\|\bar{\varrho}(b)\| \leq \|\bar{\pi}(b)\|$ for all $b \in B$.

Suppose now that $A \subseteq B$ has the ISP, and let $\pi, \varrho \in R_A^B$ satisfy $\ker(\pi) \subseteq \ker(\varrho)$ in A . Then

$$\ker(\bar{\pi}) \cap A = \ker(\pi) \subseteq \ker(\varrho) = \ker(\bar{\varrho}) \cap A.$$

By Proposition 2.3, we get $\ker(\bar{\pi}) \subseteq \ker(\bar{\varrho})$, as desired.

Conversely, suppose that $\ker(\pi) \subseteq \ker(\varrho)$ in A implies $\ker(\bar{\pi}) \subseteq \ker(\bar{\varrho})$ for all $\pi, \varrho \in R_A^B$. To show that $A \subseteq B$ has the ISP, we verify statements (4) in Proposition 2.3. Let $I, J \subseteq B$ be closed ideals satisfying $I \cap A \subseteq J \cap A$. Choose $*$ -representations α and β of B with $I = \ker(\alpha)$ and $J = \ker(\beta)$. Set $\pi = \alpha|_A$ and $\varrho = \beta|_A$, which belong to R_A^B and satisfy $\alpha = \bar{\pi}$ and $\beta = \bar{\varrho}$. We have

$$\ker(\pi) = \ker(\bar{\pi}) \cap A = I \cap A \subseteq J \cap A = \ker(\bar{\varrho}) \cap A = \ker(\varrho).$$

By assumption, we get $\ker(\bar{\pi}) \subseteq \ker(\bar{\varrho})$, and thus

$$I = \ker(\bar{\pi}) \subseteq \ker(\bar{\varrho}) = J.$$

Statement (2) is shown similarly. \square

3. Permanence properties of the ISP and IIP for groups

In this section we investigate permanence of the ideal separation property (ISP) and the ideal intersection property (IIP) for locally compact groups when passing to quotients (Proposition 3.1), inductive limits (Theorem 3.5) and inverse limits (Theorem 3.9). We also show that a locally compact group has the ISP (the IIP) if sufficiently many of its closed subgroups do (Theorem 3.3), and in particular that the ISP and the IIP are local properties (Corollary 3.4).

Every locally compact group will be assumed to come equipped with a fixed left Haar measure. Recall from Definition B that a locally compact group G is said to have the ISP or the IIP if this property holds for the inclusion $L^1(G) \subseteq C_{\text{red}}^*(G)$.

The next result generalizes results about the passage of $*$ -regularity and C^* -uniqueness to quotient groups; see Corollary 3.2.

Proposition 3.1. *Let G be a locally compact group, and suppose $N \subseteq G$ is a closed, normal subgroup. The following statements hold:*

- (1) *If G has the ISP and N is amenable, then G/N has the ISP.*
- (2) *If G has the IIP and N is compact, then G/N has the IIP.*

Proof. Using that N is a closed, normal subgroup, we obtain a well-defined map $\varrho: L^1(G) \rightarrow L^1(G/N)$ given by

$$\varrho(f)(xN) = \int_N f(xn)dn$$

for $f \in L^1(G)$ and $n \in N$. By [41, Proposition 3.4.5 and Theorem 3.5.4], ϱ is a surjective $*$ -homomorphism.

Assuming that N is also amenable (which holds under the assumption of (1), as well as of (2)), the map ϱ extends to a surjective $*$ -homomorphism $\pi: C^*_{\text{red}}(G) \rightarrow C^*_{\text{red}}(G/N)$. (This is folklore but the details can be found in e.g. [34, Corollary 2.4]). We obtain the following commutative diagram:

$$\begin{CD} L^1(G) @<\varrho<< C^*_{\text{red}}(G) \\ @VV\varrho V @VV\pi V \\ L^1(G/N) @<\varrho<< C^*_{\text{red}}(G/N). \end{CD}$$

We have

$$\pi(\pi^{-1}(I) \cap L^1(G)) = I \cap L^1(G/N), \quad \text{and} \quad \pi^{-1}(I) \cap L^1(G) = \pi^{-1}(I \cap L^1(G/N))$$

for every closed ideal $I \subseteq C^*_{\text{red}}(G/N)$.

(1) Assuming that G has the ISP and that N is amenable, we verify that G/N has the ISP. Let $I, J \subseteq C^*_{\text{red}}(G/N)$ be closed ideals with $I \cap L^1(G/N) \subseteq J \cap L^1(G/N)$. Then

$$\pi^{-1}(I) \cap L^1(G) = \pi^{-1}(I \cap L^1(G/N)) \subseteq \pi^{-1}(J \cap L^1(G/N)) = \pi^{-1}(J) \cap L^1(G)$$

Applying that G has the ISP, we get $\pi^{-1}(I) \subseteq \pi^{-1}(J)$, and thus $I \subseteq J$.

(2) Assuming that G has the IIP and that N is compact, we verify that G/N has the IIP. We first identify $C^*_{\text{red}}(G/N)$ with a direct summand of $C^*_{\text{red}}(G)$. For full group C^* -algebras a similar result is contained in [33, Lemma 5.2].

The normalized Haar measure on N defines a central idempotent z in the measure algebra $M(G)$, which we identify with the multiplier algebra $M(L^1(G))$ of $L^1(G)$. We have $\ker(\varrho) = (1 - z)L^1(G)$. Using that $L^1(G)$ contains a contractive approximate identity for $C^*_{\text{red}}(G)$, we obtain a natural contractive $*$ -homomorphism $\varphi: M(L^1(G)) \rightarrow M(C^*_{\text{red}}(G))$. Then $\varphi(z)$ is a central projection in $M(C^*_{\text{red}}(G))$, and we obtain that $\ker(\pi) = (1 - \varphi(z))C^*_{\text{red}}(G)$. This means that ϱ and π induce bijections

$$zL^1(G) \rightarrow L^1(G/N), \quad \text{and} \quad \varphi(z)C^*_{\text{red}}(G) \rightarrow C^*_{\text{red}}(G/N).$$

To verify that G/N has the IIP, let $I \subseteq C_{\text{red}}^*(G/N)$ be a non-zero, closed ideal, and set

$$J := \varphi(z)\pi^{-1}(I).$$

Then J is a closed ideal in $C_{\text{red}}^*(G)$ with $\pi(J) = I$. Since G has the IIP, it follows that $J \cap L^1(G)$ is non-zero. Since $J \cap L^1(G)$ is contained in $zL^1(G)$, and since ϱ is injective on $zL^1(G)$, it follows that

$$\{0\} \neq \varrho(J \cap L^1(G)) \subseteq I \cap L^1(G/N),$$

as desired. \square

The following result is (implicitly) contained in [10] and [13]. We note that it is not clear if C^* -uniqueness passes to quotients by closed, normal subgroups, even in the case of discrete groups; see [30, Remark 3.6].

Corollary 3.2. *Let G be a locally compact group, and suppose $N \subseteq G$ is a closed, normal subgroup. The following statements hold:*

- (1) *If G is $*$ -regular, then G/N is $*$ -regular.*
- (2) *If G is C^* -unique and N is compact, then G/N is C^* -unique.*

Proof. As noted in Proposition C, a locally compact group is $*$ -regular (C^* -unique) if and only if it is amenable and has the ISP (the IIP). Using that amenability passes to quotients by closed, normal subgroups ([39, Proposition 1.13]), the result follows from Proposition 3.1. \square

The next result shows that a locally compact group has the ISP (the IIP) if ‘sufficiently’ many closed subgroups do. Analogous results were shown for $*$ -regularity in [11, Theorem 1], and for C^* -uniqueness (implicitly) in [30, Proposition 2.1].

Theorem 3.3. *Let G be a locally compact group, and let $(H_i)_{i \in I}$ be a family of closed subgroups of G such that every open, relatively compact subset of G is contained in some H_i . The following statements hold:*

- (1) *If every H_i has the ISP, then G has the ISP.*
- (2) *If every H_i has the IIP, then G has the IIP.*

Proof. Our proof is modeled after that of [11, Theorem 1].

We first prove (1). To verify that G has the ISP, we use Proposition 2.9 for the inclusion $L^1(G) \subseteq C_{\text{red}}^*(G)$. Let $\pi, \varrho \in R_{L^1(G)}^{C^*(G)}$ with $\ker(\pi) \subseteq \ker(\varrho)$ in $L^1(G)$. Let $f \in C_c(G)$, and set $K = \{g \in G \mid f(g) \neq 0\}$. Then K is open and relatively compact, so by assumption

we find $i_0 \in I$ for which $\overline{K} \subseteq H_{i_0}$. Then H_{i_0} is an open subgroup of G , and thus we may assume that the Haar measure on H_{i_0} is given by the restriction of the Haar measure on G . Then $L^1(H_{i_0})$ embeds isometrically into $L^1(G)$, and we can restrict π and ϱ to $*$ -representations of $L^1(H_{i_0})$, which then satisfy $\ker(\pi|_{L^1(H_{i_0})}) \subseteq \ker(\varrho|_{L^1(H_{i_0})})$. Using that H_{i_0} has the ISP, we get

$$\|\varrho|_{L^1(H_{i_0})}(f|_{H_{i_0}})\| \leq \|\pi|_{L^1(H_{i_0})}(f|_{H_{i_0}})\|.$$

Noting that $\pi|_{L^1(H_{i_0})}(f|_{H_{i_0}}) = \pi(f)$, and $\varrho|_{L^1(H_{i_0})}(f|_{H_{i_0}}) = \rho(f)$, we get $\|\rho(f)\| \leq \|\pi(f)\|$. Since f was arbitrary, this holds for all $f \in C_c(G)$, and by density we deduce it holds for all $f \in C_r^*(G)$. By Proposition 2.9, we conclude that $L^1(G) \subseteq C_r^*(G)$ has the ISP.

The proof of (2) is very similar. Let $\pi \in R_{L^1(G)}^{C_r^*(G)}$ be such that $\ker(\pi) = \{0\}$ in $L^1(G)$. Let $f \in C_c(G)$ be arbitrary, and set $K = \{g \in G \mid f(g) \neq 0\}$. As above, there is $i_0 \in I$ for which $\overline{K} \subseteq H_{i_0}$, and using that H_{i_0} has the IIP, we get

$$\|\pi|_{L^1(H_{i_0})}(f|_{H_{i_0}})\| = \|f\|_{C_r^*(H_{i_0})}.$$

Noting that $C_r^*(H_{i_0}) \hookrightarrow C_r^*(G)$ isometrically, we deduce that $\|\pi(f)\| = \|f\|_{C_r^*(G)}$. Since f was arbitrary, this holds for all $f \in C_c(G)$, and by density we deduce it holds for all $f \in C_r^*(G)$. By Proposition 2.9 we conclude that $L^1(G) \subseteq C_r^*(G)$ has the IIP. \square

Using Theorem 3.3 we can easily establish that both the ISP and the IIP are local properties for groups. Analogous results for $*$ -regularity and C^* -uniqueness were shown in [11, Corollary 1] and [30, Proposition 2.1], respectively.

Corollary 3.4. *Let G be a locally compact group. The following statements hold:*

- (1) *If every open, compactly generated subgroup of G has the ISP, then G has the ISP.*
- (2) *If every open, compactly generated subgroup of G has the IIP, then G has the IIP.*

Proof. Every open, relatively compact subset of G is contained in some open, compactly generated subgroup of G . Therefore, the result follows from Theorem 3.3. \square

Next, we consider preservation of the ISP and the IIP when passing to inductive limits. We restrict to the case of discrete groups, which ensures that inductive limits exist. (For a discussion of the subtleties in defining even a group topology on the inductive limit of non-discrete, locally compact groups, we refer to [46].)

If $\Gamma_1 \rightarrow \Gamma_2 \rightarrow \dots$ is a sequential inductive system of discrete groups with injective connecting maps, then the natural maps $\Gamma_n \rightarrow \varinjlim_k \Gamma_k$ are automatically injective, which shows that the conditions on the kernels in the next result are automatically satisfied in this case.

Theorem 3.5. *Let $\Gamma = \varinjlim_{\lambda} \Gamma_{\lambda}$ be an inductive limit of discrete groups. The following statements hold.*

- (1) *If each Γ_{λ} has the ISP, and if the kernel of each $\Gamma_{\lambda} \rightarrow \Gamma$ is amenable, then Γ has the ISP.*
- (2) *If each Γ_{λ} has the IIP, and if the kernel of each $\Gamma_{\lambda} \rightarrow \Gamma$ is finite, then Γ has the IIP.*

Proof. For each λ , let Δ_{λ} denote the image of the natural map $\Gamma_{\lambda} \rightarrow \Gamma$. Then $(\Delta_{\lambda})_{\lambda}$ is a directed family of (automatically closed) subgroups of Γ . Hence, every open and relatively compact (=finite) subset of Γ is contained in Δ_{λ} for some λ , which shows that $(\Delta_{\lambda})_{\lambda}$ satisfies the general assumptions of Theorem 3.3.

(1) For each λ , using that Δ_{λ} is the quotient of Γ_{λ} by an amenable subgroup, and that Γ_{λ} has the ISP, it follows from Proposition 3.1 that Δ_{λ} has the ISP. Consequently, Γ has the ISP by Theorem 3.3.

(2) Analogously to (1), we see that each Δ_{λ} has the IIP by Proposition 3.1, and thus Γ has the IIP by Theorem 3.3. \square

We deduce that $*$ -regularity passes to inductive limits of discrete groups, which seems to not have been noticed so far.

Corollary 3.6. *Let $\Gamma = \varinjlim_{\lambda} \Gamma_{\lambda}$ be an inductive limit of discrete groups. The following statements hold:*

- (1) *If each Γ_{λ} is $*$ -regular, then Γ is $*$ -regular.*
- (2) *If each Γ_{λ} is C^* -unique and if all connecting maps $\Gamma_{\lambda} \rightarrow \Gamma_{\kappa}$ are injective, then Γ is C^* -unique.*

Proof. (1) Since $*$ -regular groups are amenable, and since amenability passes to (closed) subgroups, we see that every subgroup of a discrete, $*$ -regular group is amenable. Therefore, the kernel of the map $\Gamma_{\lambda} \rightarrow \Gamma$ is automatically amenable for every λ , and the result follows from Theorem 3.5.

(2) The assumption implies that $\Gamma_{\lambda} \rightarrow \Gamma$ is injective for every λ , and the result follows from Theorem 3.5. \square

Remark 3.7. In light of $*$ -regularity passing to inductive limits of discrete groups, it would be of great interest to determine if $*$ -regularity passes to extensions of discrete groups. A positive answer (even just for extensions by finite groups and \mathbb{Z}) would immediately provide a good partial answer to Question A by showing that elementary amenable groups are $*$ -regular. Indeed, a result of Osin [36, Corollary 2.1] shows that the class of elementary amenable groups is the smallest class \mathcal{C} of groups which contains the trivial group and is closed under inductive limits, and such that $\Gamma \in \mathcal{C}$ for every extension

$$1 \rightarrow N \rightarrow \Gamma \rightarrow \Gamma/N \rightarrow 1$$

with $N \in \mathcal{C}$ and Γ/N a finite group or $\Gamma/N = \mathbb{Z}$.

Since in the result of Osin it suffices to assume that the class is closed under inductive limits with injective connecting maps, one could similarly deduce that elementary amenable groups are C^* -unique if one can show that C^* -uniqueness passes to extensions by finite groups and \mathbb{Z} .

We deduce from Theorem 3.3 the existence of certain maximal subgroups that have the ISP or the IIP. An analogous result for maximal $*$ -regular subgroups of discrete groups was shown in [30, Proposition 3.2].

Theorem 3.8. *Let G be a locally compact group. The following statements hold:*

- (1) *Every (normal) open subgroup $H \leq G$ having the ISP is contained in a maximal (normal) open subgroup of G having the ISP.*
- (2) *Every (normal) open subgroup $H \leq G$ having the IIP is contained in a maximal (normal) open subgroup of G having the IIP.*

Proof. To verify (1), let $H \leq G$ be an open (normal) subgroup that has the ISP. Let \mathcal{L} denote the set of open (normal) subgroups of G that contain H and that have the ISP. We order \mathcal{L} by inclusion. The existence of maximal elements in \mathcal{L} follows once we have verified the assumptions of Zorn’s lemma for \mathcal{L} .

Given a chain \mathcal{C} in \mathcal{L} , we need to find an upper bound of \mathcal{C} in \mathcal{L} . Set $H' := \bigcup_{C \in \mathcal{C}} C$, which is an open (normal) subgroup of G containing H . Since \mathcal{C} is a chain, every relatively compact subset of H' is contained in some $C \in \mathcal{C}$. This shows that the family \mathcal{C} of open (hence also closed) subgroups of H' satisfies the assumptions of Theorem 3.3, and since each group in \mathcal{C} has the ISP, we deduce that H' has the ISP. Hence H' belongs to \mathcal{L} , and it is clearly an upper bound for \mathcal{C} .

An analogous argument proves (2). \square

We now turn to inverse limits. As for inductive limits, inverse limits of locally compact groups only exist under additional assumptions. We restrict our attention to the important case of an inverse limit $G = \varprojlim_{\lambda} G_{\lambda}$ where each map $G \rightarrow G_{\lambda}$ is surjective and has compact kernel K_{λ} . Then $(K_{\lambda})_{\lambda}$ is a downward directed family of compact, normal subgroups with $\bigcap_{\lambda} K_{\lambda} = \{1\}$, and given such a family of compact, normal subgroups we can realize G as an inverse limit of the quotients G/K_{λ} . In this setting, we show that G has the ISP (the IIP) if and only if each G_{λ} has the ISP (the IIP). We discuss in Remark 3.10 how our result generalizes similar statements about $*$ -regularity and C^* -uniqueness from [10, Lemma 9] and [30, Corollary 2.3].

Theorem 3.9. *Let G be a locally compact group which can be realized as an inverse limit $G = \varprojlim_{\lambda} G_{\lambda}$ such that each canonical map $G \rightarrow G_{\lambda}$ is surjective and has compact kernel. The following statements hold:*

- (1) *The group G has the ISP if and only if every G_{λ} has the ISP.*
- (2) *The group G has the IIP if and only if every G_{λ} has the IIP.*

Proof. For each λ , let K_{λ} denote the kernel of the natural map $G \rightarrow G_{\lambda}$, so that $G_{\lambda} \cong G/K_{\lambda}$. By Proposition 3.1, the ISP and the IIP pass to quotients by compact, normal subgroups, which shows the forward implications in (1) and (2).

Given λ , using that K_{λ} is a compact, normal subgroup of G , we have seen in the proof of Proposition 3.1 that there exist natural surjective $*$ -homomorphisms $\pi_{\lambda}: C_{\text{red}}^*(G) \rightarrow C_{\text{red}}^*(G_{\lambda})$ and $\varrho_{\lambda}: L^1(G) \rightarrow L^1(G_{\lambda})$ such that the following diagram commutes:

$$\begin{array}{ccc}
 L^1(G) & \hookrightarrow & C_{\text{red}}^*(G) \\
 \varrho_{\lambda} \downarrow & & \downarrow \pi_{\lambda} \\
 L^1(G_{\lambda}) & \hookrightarrow & C_{\text{red}}^*(G_{\lambda}).
 \end{array}$$

Set $I_{\lambda} := \ker(\pi_{\lambda})$, which is a closed ideal in $C_{\text{red}}^*(G)$.

Claim 1: *We have $\bigcap_{\lambda} I_{\lambda} = \{0\}$.* We have seen in the proof of Proposition 3.1 that for each λ there exists a central, contractive idempotent $z_{\lambda} \in M(G)$ such that

$$\ker(\varrho_{\lambda}) = (1 - z_{\lambda})L^1(G), \quad \text{and} \quad I_{\lambda} = \ker(\pi_{\lambda}) = (1 - \varphi(z_{\lambda}))C_{\text{red}}^*(G), \tag{3.1}$$

where $\varphi: M(G) \rightarrow M(C_{\text{red}}^*(G))$ denotes the natural, contractive $*$ -homomorphism. Using that $\bigcap_{\lambda} K_{\lambda} = \{1\}$ and [41, Proposition 3.5.6], it follows that $\lim_{\lambda} \|z_{\lambda}f - f\|_1 = 0$ for every $f \in L^1(G)$. Given $a \in C_{\text{red}}^*(G)$, let us verify that $\lim_{\lambda} \|\varphi(z_{\lambda})a - a\| = 0$. Given $\varepsilon > 0$, choose $f \in L^1(G)$ with $\|a - f\| < \varepsilon$. Then

$$\begin{aligned}
 \|\varphi(z_{\lambda})a - a\| &\leq \|\varphi(z_{\lambda})a - \varphi(z_{\lambda})f\| + \|\varphi(z_{\lambda})f - f\| + \|f - a\| \\
 &\leq \|\varphi(z_{\lambda})\| \|a - f\| + \|z_{\lambda}f - f\|_1 + \|f - a\| < 3\varepsilon.
 \end{aligned}$$

Now, given $a \in \bigcap_{\lambda} I_{\lambda} \subseteq C_{\text{red}}^*(G)$, we get $(1 - z_{\lambda})a = 0$ and thus $z_{\lambda}a = a$ for each λ , which then gives $a = 0$. Thus, $\bigcap_{\lambda} I_{\lambda} = \{0\}$. This proves the claim.

Claim 2: *Given a closed ideal $I \subseteq C_{\text{red}}^*(G)$, we have*

$$\pi_{\lambda}(I) \cap L^1(G_{\lambda}) = \pi_{\lambda}(I \cap L^1(G))$$

for every λ . Indeed, the inclusion ‘ \supseteq ’ is clear. To show the converse inclusion, let $a \in \pi_{\lambda}(I) \cap L^1(G_{\lambda})$, and choose $b \in I$ and $f \in L^1(G)$ such that

$$\pi_\lambda(b) = a, \quad \text{and} \quad \varrho_\lambda(f) = a.$$

Then $z_\lambda f \in L^1(G)$. Since I is a C^* -algebra, every element factors and we can choose $b_1, b_2 \in I$ such that $b = b_1 b_2$. Then

$$\varphi(z_\lambda)b = \varphi(z_\lambda)b_1 b_2 \in C_{\text{red}}^*(G)I \subseteq I.$$

Note that using Equation (3.1) we obtain

$$\pi_\lambda(z_\lambda f) = \pi_\lambda(f) = a = \pi_\lambda(b) = \pi_\lambda(\varphi(z_\lambda)b)$$

and since π_λ is injective on $\varphi(z_\lambda)C_{\text{red}}^*(G)$ by Equation (3.1), we get $z_\lambda f = \varphi(z_\lambda)b \in I \cap L^1(G)$. This proves the claim.

(1) Assuming that each G_λ has the ISP, let us verify that G has the ISP. Let $I, J \subseteq C_{\text{red}}^*(G)$ be closed ideals with $I \cap L^1(G) \subseteq J \cap L^1(G)$. For each λ , using Claim 2, we deduce that

$$\pi_\lambda(I) \cap L^1(G_\lambda) = \pi_\lambda(I \cap L^1(G)) \subseteq \pi_\lambda(J \cap L^1(G)) = \pi_\lambda(J) \cap L^1(G_\lambda).$$

Using that G_λ has the ISP, we get $\pi_\lambda(I) \subseteq \pi_\lambda(J)$, and thus $I \subseteq J + I_\lambda$. Applying Claim 1 at the last step, we obtain

$$I \subseteq \bigcap_\lambda (J + I_\lambda) = J + \bigcap_\lambda I_\lambda = J,$$

as desired.

(2) Assuming that each G_λ has the IIP, let us verify that G has the IIP. Let $I \subseteq C_{\text{red}}^*(G)$ be a non-zero, closed ideal. By Claim 1, we have $\bigcap_\lambda I_\lambda = \{0\}$, and we find λ such that $I \not\subseteq I_\lambda$. Then $\pi_\lambda(I) \subseteq C_{\text{red}}^*(G_\lambda)$ is a non-zero, closed ideal. Using that G_λ has the IIP at the first step, and using Claim 2 at the second step, we get

$$\{0\} \neq \pi_\lambda(I) \cap L^1(G_\lambda) = \pi_\lambda(I \cap L^1(G)),$$

which shows that $I \cap L^1(G)$ is non-zero, as desired. \square

Remark 3.10. Let G be a locally compact group, and let $(K_\lambda)_\lambda$ be a downward directed family of compact, normal subgroups with $\bigcap_\lambda K_\lambda = \{1\}$. By [10, Lemma 9], the group G is $*$ -regular if and only if all G/K_λ are $*$ -regular. Similarly, by [30, Corollary 2.3], the group G is C^* -unique if and only if G/K_λ is C^* -unique for large enough λ .

Theorem 3.9 recovers and generalizes these results (and even strengthens the result about C^* -uniqueness). Indeed, by [39, Proposition 1.13], if N is a closed, normal subgroup in G , then G is amenable if and only if N and G/N are amenable. Since compact groups are amenable, it follows that G is amenable if and only if G/K_λ is amenable for some

λ (equivalently, for every λ). Therefore, [10, Lemma 9] follows from Theorem 3.9 using that by Proposition C a locally compact group is $*$ -regular if and only if it is amenable and satisfies the ISP. Similarly, we recover [30, Corollary 2.3], and in fact we obtain that G is C^* -unique if and only if all G/K_λ are C^* -unique.

Given a locally compact group G , let G_0 denote the connected component of the identity, which is always a closed, normal subgroup. One says that G is *connected* if $G = G_0$, that G is *almost connected* if G/G_0 is compact, and that G is *totally disconnected* if $G_0 = \{1\}$. The quotient group G/G_0 is totally disconnected, and it follows from van Dantzig's theorem (see, for example [45, Theorem 1.6.7]) that there exists an open, compact subgroup $U \subseteq G/G_0$. Then the pre-image of U in G is an open subgroup $V \subseteq G$ such that $V/V_0 (= V/G_0)$ is compact. This shows that every locally compact group contains an almost connected, open subgroup.

By the Gleason-Yamabe theorem ([35, Theorem 4.6]), every almost connected, locally compact group is an inverse limit of (finite-dimensional, real) Lie groups. (See also [45, Theorem 1.6.1].) As a consequence of Theorem 3.9, we see that the ISP and the IIP for an almost connected, locally compact group is determined by its Lie group quotients. This generalizes a similar statement for C^* -uniqueness from [30, Proposition 2.2(b)]. One also obtains that an almost connected, locally compact group is $*$ -regular if and only if all of its Lie group quotients are $*$ -regular.

Corollary 3.11. *Let G be an almost connected, locally compact group, and set*

$$\mathcal{L} := \{K \subseteq G : K \text{ is a compact, normal subgroup such that } G/K \text{ is a Lie group}\}.$$

Then G has the ISP (the IIP) if and only if G/K has the ISP (the IIP) for every $K \in \mathcal{L}$.

Proof. This follows from Theorem 3.9 using that the family \mathcal{L} is downward directed (possibly with a minimal element $\{1\}$ if G is already a Lie group). \square

4. Ideal separation in tensor products of C^* -algebras

Given C^* -algebras A and B , we observe that the inclusion of the algebraic tensor product $A \odot B$ in the minimal tensor product $A \otimes B$ always has the ideal intersection property (IIP); see Theorem 4.8. For the ideal separation property (ISP) this is no longer true: The inclusion $A \odot B \subseteq A \otimes B$ has the ISP if and only if the pair of C^* -algebras A and B has Tomiyama's property (F); see Theorem 4.5. Analogous results holds for the inclusion of the projective tensor product $A \widehat{\otimes} B$ in $A \otimes B$.

Using this, we show that for two locally compact groups G and H , the Cartesian product $G \times H$ has the IIP if and only if G and H have the IIP. Further, $G \times H$ has the ISP if and only if G and H have the ISP and the pair of reduced group C^* -algebras $C_{\text{red}}^*(G_1)$ and $C_{\text{red}}^*(G_2)$ has Tomiyama's property (F); see Theorem 4.13. Since

Tomiyama’s property (F) is automatic whenever one of the C^* -algebras is exact, we deduce that for two locally compact groups G and H , at least one of which is exact, $G \times H$ has the ISP if and only if G and H do; see Corollary 4.14.

We begin by recalling some basic facts about algebraic, injective, projective, minimal and maximal tensor products of C^* -algebras. Let A and B be two C^* -algebras. We use $A \odot B$ to denote the algebraic tensor product as \mathbb{C} -vector spaces, which is a complex $*$ -algebra with involution induced by $(a \otimes b)^* := a^* \otimes b^*$.

Viewing A and B as Banach spaces, we let $\|\cdot\|_\varepsilon$ and $\|\cdot\|_\pi$ denote the injective and projective norms on $A \odot B$, with the respective completions denoted by $A \check{\otimes} B$ and $A \widehat{\otimes} B$; see [42, Sections 2 and 3] for details. We further let $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ denote the minimal and maximal C^* -norms, with completions denoted by $A \otimes_{\max} B$ and $A \otimes B$; see [7, Section II.9] for details.

While $A \otimes B$ and $A \otimes_{\max} B$ are C^* -algebras, $A \widehat{\otimes} B$ is ‘only’ a Banach $*$ -algebra. Even worse, $A \check{\otimes} B$ is in general not even a Banach algebra, since the multiplication on $A \odot B$ may not extend to this completion; see [9, Corollary 4]. We have

$$\|\cdot\|_\varepsilon \leq \|\cdot\|_{\min} \leq \|\cdot\|_{\max} \leq \|\cdot\|_\pi,$$

and in general all norms can be different. We obtain contractive maps with dense ranges as shown in the following commutative diagram:

$$\begin{array}{ccc} A \otimes_{\max} B & \twoheadrightarrow & A \otimes B \\ \uparrow & & \downarrow \\ A \widehat{\otimes} B & \hookrightarrow & A \check{\otimes} B. \end{array}$$

The map from the maximal to the minimal tensor product is a quotient map, and it is an isomorphism whenever A or B are nuclear (and also sometimes when neither A nor B are nuclear [40]). Haagerup showed in [20, Proposition 2.2] that the canonical map $A \widehat{\otimes} B \rightarrow A \check{\otimes} B$ is injective. It follows that the map $A \widehat{\otimes} B \rightarrow A \otimes B$ is injective as well since the first map in a composition of maps making up an injection must be injective. We will therefore usually identify $A \odot B$ and $A \widehat{\otimes} B$ with subalgebras of $A \otimes B$:

$$A \odot B \subseteq A \widehat{\otimes} B \subseteq A \otimes B.$$

Given $*$ -homomorphisms $\pi_1: A_1 \rightarrow A$ and $\pi_2: B_1 \rightarrow B$, the naturally induced $*$ -homomorphism $A_1 \odot B_1 \rightarrow A \odot B$ extends to the respective completions, when both tensor products are equipped with the minimal (maximal, projective) norm. If π_1 and π_2 are injective, then so are the maps $A_1 \odot B_1 \rightarrow A \odot B$ and $A_1 \otimes B_1 \rightarrow A \otimes B$, but $A_1 \otimes_{\max} B_1 \rightarrow A \otimes_{\max} B$ may not be injective. If π_1 and π_2 are surjective, then so are $A_1 \odot B_1 \rightarrow A \odot B$, as well as $A_1 \otimes B_1 \rightarrow A \otimes B$ and $A_1 \otimes_{\max} B_1 \rightarrow A \otimes_{\max} B$.

If J is a closed ideal in B , then the kernel of the map $A \odot B \rightarrow A \odot (B/J)$ is equal to $A \odot J$. Thus, the sequence

$$0 \rightarrow A \odot J \rightarrow A \odot B \rightarrow A \odot (B/J) \rightarrow 0$$

is exact. The maximal tensor product is also exact in this sense: The sequence

$$0 \rightarrow A \otimes_{\max} J \rightarrow A \otimes_{\max} B \rightarrow A \otimes_{\max} (B/J) \rightarrow 0$$

is exact. This is no longer true for the minimal tensor product, and one says that A is *exact* if taking minimal tensor products with A preserves short exact sequences.

The projective tensor product of C^* -algebras has the remarkable property of preserving exact sequences and isometric embeddings induced by sub- C^* -algebras. First, analogous to the minimal tensor product, if $A_1 \subseteq A$ and $B_1 \subseteq B$ are sub- C^* -algebras, then the map $A_1 \odot B_1 \rightarrow A \odot B$ is isometric for the projective tensor products and induces a natural isometric embedding $A_1 \widehat{\otimes} B_1 \rightarrow A \widehat{\otimes} B$; see [19, Theorem 1]. Second, analogous to the maximal tensor product, if $J \subseteq B$ is a closed ideal, then the sequence

$$0 \rightarrow A \widehat{\otimes} J \rightarrow A \widehat{\otimes} B \rightarrow A \widehat{\otimes} (B/J) \rightarrow 0$$

is exact, that is, the map $A \widehat{\otimes} J \rightarrow A \widehat{\otimes} B$ is an isometric embedding and its image agrees with the kernel of the quotient map $A \widehat{\otimes} B \rightarrow A \widehat{\otimes} (B/J)$; see [42, Proposition 2.5], see also [19, Lemma 1].

Next, we study closed, prime ideals in projective tensor products of C^* -algebras. By an *ideal* we mean a ring-theoretic ideal, that is, an additive subgroup that is invariant under taking products on the left or right with elements from the containing algebra. We note that ideals in C^* -algebras are not necessarily subspaces; see [7, Examples II.5.2.1].

We begin by generalizing the well-known result that the minimal tensor product of prime C^* -algebras is prime to the setting of projective tensor products.

Lemma 4.1. *Let A and B be prime C^* -algebras. Then $A \widehat{\otimes} B$ is prime.*

Proof. The proof is based on ideas in the proof of [8, Lemma 2.13]. To reach a contradiction, let $I_1, I_2 \subseteq A \widehat{\otimes} B$ be nonzero ideals such that $I_1 I_2 = \{0\}$. Let $\overline{I_1}$ and $\overline{I_2}$ denote the closures in $A \widehat{\otimes} B$. By [19, Corollary 2], there exist nonzero elementary tensors $a_1 \otimes b_1 \in \overline{I_1}$ and $a_2 \otimes b_2 \in \overline{I_2}$. Since A is prime and $a_1, a_2 \neq 0$, we obtain $x \in A$ such that $a_1 x a_2 \neq 0$. Similarly, we get $y \in B$ such that $b_1 y b_2 \neq 0$. Then

$$0 \neq (a_1 x a_2) \otimes (b_1 y b_2) = (a_1 \otimes b_1)(x \otimes y)(a_2 \otimes b_2) \in \overline{I_1} \cdot \overline{I_2}$$

But $I_1 I_2 = \{0\}$ implies $\overline{I_1} \cdot \overline{I_2} = \{0\}$, a contradiction. \square

The next result establishes a natural bijection between the set of closed, prime ideals in $A \widehat{\otimes} B$, and the Cartesian product of the spaces of closed, prime ideals in A and in B . The result is probably well-known to experts, but we could not locate it in the literature. The analogous statement identifying closed, maximal ideals (and maximal

modular ideals) in $A \widehat{\otimes} B$ with Cartesian products of respective ideals in A and B is [19, Theorem 9]. An analogous statement for closed, prime ideals in the operator space projective tensor product is [22, Theorem 6].

Proposition 4.2. *Let A and B be C^* -algebras, and let $I \subseteq A \widehat{\otimes} B$ be a closed, prime ideal. Set*

$$J := \{a \in A : a \odot B \subseteq I\}, \quad \text{and} \quad K := \{b \in B : A \odot b \subseteq I\}.$$

Then $J \subseteq A$ and $K \subseteq B$ are closed, prime ideals, and we have

$$I = (J \widehat{\otimes} B) + (A \widehat{\otimes} K) \subseteq A \widehat{\otimes} B.$$

Conversely, if $J \subseteq A$ and $K \subseteq B$ are closed, prime ideals, then $(J \widehat{\otimes} B) + (A \widehat{\otimes} K)$ is a closed, prime ideal in $A \widehat{\otimes} B$.

Proof. The proof is based on ideas in the proof of [8, Lemma 2.13]. First, it is straightforward to check that J is a closed, additive subgroup. To show that J is a left ideal, let $a \in J$ and $a_1 \in A$. Given $b \in B$, we need to verify $(a_1 a) \otimes b \in I$. Since every element in a C^* -algebra can be written as a product of two elements ([7, Proposition II.3.2.1]), we find $b_1, b_2 \in B$ such that $b = b_1 b_2$. Then

$$(a_1 a) \otimes b = (a_1 a) \otimes (b_1 b_2) = (a_1 \otimes b_1)(a \otimes b_2) \in I.$$

Similarly, one shows that J is a right ideal. To show that J is a prime ideal, let $J_1, J_2 \subseteq A$ be ideals such that $J_1 J_2 \subseteq J$. Let $\overline{J_1}$ and $\overline{J_2}$ denote the closures in A . Since J is closed, we have $\overline{J_1 J_2} \subseteq J$. Then $\overline{J_1} \widehat{\otimes} B$ and $\overline{J_2} \widehat{\otimes} B$ are ideals in $A \widehat{\otimes} B$ such that

$$(\overline{J_1} \widehat{\otimes} B)(\overline{J_2} \widehat{\otimes} B) \subseteq (\overline{J_1 J_2}) \widehat{\otimes} B \subseteq J \widehat{\otimes} B \subseteq I.$$

Since I is prime, we get $\overline{J_1} \widehat{\otimes} B \subseteq I$ or $\overline{J_2} \widehat{\otimes} B \subseteq I$, which entails $J_1 \subseteq \overline{J_1} \subseteq J$ or $J_2 \subseteq \overline{J_2} \subseteq J$. Analogously, it follows that K is a closed, prime ideal in B .

It is clear that I contains $(J \widehat{\otimes} B) + (A \widehat{\otimes} K)$. To show the other inclusion, we consider the natural map $\pi: A \widehat{\otimes} B \rightarrow (A/J) \widehat{\otimes} (B/K)$. Since the projective tensor product is exact (see [42, Proposition 2.5]), the kernel of π is $(J \widehat{\otimes} B) + (A \widehat{\otimes} K)$. We will show that $\pi(I) = \{0\}$.

Since π is a quotient map, $\pi(I)$ is a closed ideal in $(A/J) \widehat{\otimes} (B/K)$. To reach a contradiction, assume that $\pi(I) \neq \{0\}$. Then $\pi(I)$ contains a nonzero, elementary tensor, say $x \otimes y \in \pi(I)$ for $x \in A/J$ and $y \in B/K$; see [19, Corollary 2]. Choosing lifts $a \in A$ and $b \in B$, and using that the kernel of π is contained in I , we deduce that $a \otimes b \in I$. Let $J_1 \subseteq A$ and $K_1 \subseteq B$ denote the closed ideals generated by a and b , respectively. Then

$$(J_1 \widehat{\otimes} B)(A \widehat{\otimes} K_1) \subseteq J_1 \widehat{\otimes} K_1 \subseteq I.$$

Using that I is prime, it follows that $J_1 \widehat{\otimes} B \subseteq I$ or $A \widehat{\otimes} K_2 \subseteq I$, and therefore $a \in J$ or $b \in K$, which implies that $\pi(a \otimes b) = 0$, a contradiction. By the above identification of the kernel of π with $(\widehat{J \otimes B}) + (A \widehat{\otimes} K)$ we therefore have

$$I \subseteq \ker(\pi) = (\widehat{J \otimes B}) + (A \widehat{\otimes} K).$$

Next, assume that $J \subseteq A$ and $K \subseteq B$ are closed, prime ideals. As above, we consider the map $\pi: A \widehat{\otimes} B \rightarrow (A/J) \widehat{\otimes} (B/K)$, whose kernel is $(\widehat{J \otimes B}) + (A \widehat{\otimes} K)$. By Lemma 4.1, the algebra $(A/J) \widehat{\otimes} (B/K)$ is prime, and therefore the kernel of π is a closed, prime ideal. \square

An ideal I in a ring R is said to be semiprime if for every ideal J in R one has $J \subseteq I$ whenever $J^2 \subseteq I$. It is known that an ideal is semiprime if and only if it is the intersection of some family of prime ideals; see [27, Section 10]. In [16, Corollary 5.5] it is shown that semiprime (not necessarily closed) ideals in C^* -algebras are self-adjoint subspaces. With view to the next result, it is therefore natural to ask if semiprime (not necessarily closed) ideals in projective tensor products of C^* -algebras are self-adjoint subspaces.

Corollary 4.3. *Let A and B be C^* -algebras. Then every closed, semiprime ideal in $A \widehat{\otimes} B$ is a self-adjoint subspace.*

Proof. It suffices to show the result for closed, prime ideals, since being a self-adjoint subspace passes to intersections. So let $I \subseteq A \widehat{\otimes} B$ be a closed, prime ideal. By Proposition 4.2, there are closed, prime ideals $J \subseteq A$ and $K \subseteq B$ such that

$$I = (\widehat{J \otimes B}) + (A \widehat{\otimes} K).$$

Since closed ideals in C^* -algebras are self-adjoint subspaces, it follows that I has the claimed properties. \square

The next result is mostly contained in [8, Lemma 2.13].

Proposition 4.4. *Let A and B be C^* -algebras, and let $I \subseteq A \otimes B$ be a closed, prime ideal. Set*

$$J := \{a \in A : a \odot B \subseteq I\}, \quad \text{and} \quad K := \{b \in B : A \odot b \subseteq I\}.$$

Then $J \subseteq A$ and $K \subseteq B$ are closed, prime ideals, and we have

$$(J \otimes B) + (A \otimes K) \subseteq I \subseteq I_{J,K} := \ker(A \otimes B \rightarrow (A/J) \otimes (B/K)).$$

Moreover,

$$(J \otimes B + A \otimes K) \cap (A \odot B) = I \cap A \odot B = I_{J,K} \cap A \odot B = J \odot B + A \odot K$$

and

$$(J \otimes B + A \otimes K) \cap (A \widehat{\otimes} B) = I \cap A \widehat{\otimes} B = I_{J,K} \cap A \widehat{\otimes} B = J \widehat{\otimes} B + A \widehat{\otimes} K.$$

If $I = \overline{I \cap A \widehat{\otimes} B}$ (in particular, if $I = \overline{I \cap A \odot B}$), then

$$J \otimes B + A \otimes K = I.$$

Proof. As in the proof of Proposition 4.2, we see that J and K are closed, prime ideals, and that $(J \otimes B) + (A \otimes K) \subseteq I \subseteq I_{J,K}$. (This is also contained in [8, Lemma 2.13].)

Using that the algebraic tensor product is exact, we see that the kernel of the map $A \odot B \rightarrow (A/J) \odot (B/K)$ is $(J \odot B) + (A \odot K)$, which implies that

$$I_{J,K} \cap (A \odot B) = (J \odot B) + (A \odot K).$$

The inclusions

$$(J \odot B) + (A \odot K) \subseteq ((J \otimes B) + (A \otimes K)) \cap (A \odot B) \subseteq I \cap (A \odot B) \subseteq I_{J,K} \cap (A \odot B)$$

are clear, and therefore are all equalities. The equalities for the intersection with $A \widehat{\otimes} B$ follow analogously, using that the projective tensor product is exact.

Lastly, let us assume that $I = I \cap A \widehat{\otimes} B$. Using at the last step that the sum of closed ideals in a C^* -algebra is closed, we get

$$I = \overline{I \cap A \widehat{\otimes} B} = \overline{J \widehat{\otimes} B + A \widehat{\otimes} K} = \overline{J \otimes B + A \otimes K} = J \otimes B + A \otimes K,$$

as desired. \square

A pair of C^* -algebras A and B is said to satisfy *Tomiyama’s property (F)* if closed ideals in $A \otimes B$ are separated by states of the form $\varphi \otimes \psi$, where φ is a pure state on A and ψ is a pure state on B , [47]. We say that an ideal in the minimal tensor product $A \otimes B$ of two C^* -algebras is a product ideal if it is of the form $I \otimes J$ for an ideal $I \subseteq A$ and an ideal $J \subseteq B$.

The next result characterizes Tomiyama’s property (F) in terms of the ISP for the inclusion of the algebraic and the projective tensor product in the minimal tensor product of C^* -algebras. For the algebraic tensor product this is essentially contained in [8, Proposition 2.16], see also [28, Proposition 5.1], and our contribution is the characterization in terms of the projective tensor product. This is crucial for showing that if the Cartesian product $G \times H$ of two locally compact groups has the ISP, then the pair of C^* -algebras $C^*_{\text{red}}(G)$ and $C^*_{\text{red}}(H)$ has Tomiyama’s property (F); see Theorem 4.13.

Theorem 4.5. *Let A and B be C^* -algebras. Then the following are equivalent:*

- (1) The pair (A, B) satisfies Tomiyama’s property (F).
- (2) Every closed ideal in $A \otimes B$ is the closure of the sum of all contained product ideals.
- (3) The inclusion $A \odot B \subseteq A \otimes B$ has the ideal separation property, that is, closed ideals $I_1, I_2 \subseteq A \otimes B$ satisfy $I_1 \subseteq I_2$ whenever $I_1 \cap (A \odot B) \subseteq I_2 \cap (A \odot B)$.
- (4) We have $I = \overline{I \cap (A \odot B)}$ for every closed ideal $I \subseteq A \otimes B$.
- (5) The inclusion $\widehat{A \otimes B} \subseteq A \otimes B$ has the ideal separation property.
- (6) We have $I = \overline{I \cap (\widehat{A \otimes B})}$ for every closed ideal $I \subseteq A \otimes B$.
- (7) We have $P = P \cap (\widehat{A \otimes B})$ for every closed, prime ideal $P \subseteq A \otimes B$.

Proof. The equivalence of (1) and (2) is shown in [8, Proposition 2.16]. The equivalence of (3) and (4) follows from Proposition 2.6 using that $A \odot B$ is dense in $A \otimes B$. Similarly we see that (5) and (6) are equivalent. It is also easy to see that (2) implies (4), that (4) implies (6), and that (6) implies (7).

Let us show that (7) implies (1). Given a closed, prime ideal $P \subseteq A \otimes B$, using the assumption $P = P \cap (\widehat{A \otimes B})$, it follows from Proposition 4.4 that $P = (J \otimes B) + (A \otimes K)$ for closed, prime ideals $I \subseteq A$ and $K \subseteq B$. This verifies condition (iii) in [8, Proposition 2.16], which then shows that the pair (A, B) satisfies Tomiyama’s property (F). \square

Remark 4.6. Consider the following statements for a closed ideal $I \subseteq A \otimes B$:

- (1) I is the closure of the sum of all product ideals contained in I .
- (2) $I = \overline{I \cap (A \odot B)}$.
- (3) $I = I \cap (\widehat{A \otimes B})$.

If a closed ideal satisfies (1), then it satisfies (2), which in turn implies that it satisfies (3). Further, Theorem 4.5 shows that if all closed ideals in $A \otimes B$ satisfy (3), then the pair (A, B) satisfies Tomiyama’s property (F) and consequently all closed ideals in $A \otimes B$ satisfy (1) and (2).

If the pair (A, B) does not satisfy Tomiyama’s property (F), then there necessarily exists a closed ideal in $A \otimes B$ that does not satisfy (3) (hence neither (1), nor (2)). In this case, it remains unclear if there also exist closed ideals in $A \otimes B$ that satisfy (3) but not (2), or that satisfy (2) but not (1).

Note that statement (2) in Theorem 4.8 below shows that, unlike the ISP, the IIP always holds for the inclusion of the algebraic into the minimal tensor product of C^* -algebras.

Paragraph 4.7. Let A and B be C^* -algebras. Given $\psi \in B^*$, that is, a bounded, linear functional $\psi : B \rightarrow \mathbb{C}$, we consider the left slice map $L_\psi^{(0)} : A \odot B \rightarrow A$ given by

$$L_\psi^{(0)} \left(\sum_k a_k \otimes b_k \right) = \sum_k \psi(b_k) a_k.$$

If ψ is a positive, linear functional, then it is well-known that $L_\psi^{(0)}$ extends to the completion $A \otimes B$ and induces a positive, linear map $L_\psi: A \otimes B \rightarrow A$; see [7, Paragraph II.9.7.1]. Using that every bounded, linear functional on a C^* -algebra is a linear combination of positive, linear functionals ([7, Theorem II.6.3.4]), we see that $L_\psi^{(0)}$ extends to a bounded, linear map $L_\psi: A \otimes B \rightarrow A$ for every $\psi \in B^*$.

Theorem 4.8. *Let A and B be C^* -algebras containing subalgebras $A_0 \subseteq A$ and $B_0 \subseteq B$. The following statements hold:*

- (1) *The inclusion $A_0 \odot B_0 \subseteq A \otimes B$ has the ISP if and only if $A_0 \subseteq A$ and $B_0 \subseteq B$ have the ISP and the pair (A, B) has Tomiyama’s property (F).*
- (2) *The inclusion $A_0 \odot B_0 \subseteq A \otimes B$ has the IIP if and only if $A_0 \subseteq A$ and $B_0 \subseteq B$ have the IIP.*

Proof. (1) To show the forward implication, assume that $A_0 \odot B_0 \subseteq A \otimes B$ has the ISP. Then $A \odot B \subseteq A \otimes B$ has the ISP (see Proposition 2.4), and thus (A, B) has Tomiyama’s property (F) by Theorem 4.5. To verify that $A_0 \subseteq A$ has the ISP, let $I, J \subseteq A$ be distinct closed ideals. Then $I \otimes B$ and $J \otimes B$ are distinct closed ideals in $A \otimes B$. By assumption, their intersections with $A_0 \odot B_0$ are also distinct. Without loss of generality, we assume that there is $t \in A \otimes B$ with

$$t \in (I \otimes B) \cap (A_0 \odot B_0), \quad \text{and} \quad t \notin J \otimes B.$$

Since (A, B) has Tomiyama’s property (F), the slice map problem has a positive solution with respect to closed ideals; see [48, Remark 24]. Thus, if an element $x \in A \otimes B$ satisfies $L_\psi(x) \in J$ for every $\psi \in B^*$, then $x \in J \otimes B$. Since our element t does not belong to $J \otimes B$, we can pick $\psi \in B^*$ such that

$$L_\psi(t) \notin J.$$

We have $L_\psi(I \otimes B) \subseteq I$, and therefore $L_\psi(t) \in I$. Finally, since $t \in A_0 \odot B_0$, say $t = \sum_k a_k \otimes b_k$ for a finite sum of elementary tensors with $a_k \in A_0$ and $b_k \in B_0$, we have

$$L_\psi(t) = \sum_k \psi(b_k) a_k \in A_0.$$

Thus, the element $L_\psi(t)$ belongs to $I \cap A_0$ but not to J , which shows that A_0 separates ideals in A . Analogously one shows that $B_0 \subseteq B$ has the ISP.

To show the backwards implication, assume that $A_0 \subseteq A$ and $B_0 \subseteq B$ have the ISP and that the pair (A, B) has Tomiyama’s property (F). Given a subset S in a C^* -algebra, let us use $\langle S \rangle$ to denote the closed ideal generated by S . To show the ISP for $A_0 \odot B_0 \subseteq A \otimes B$, we verify condition (3) in Proposition 2.3, that is, we prove that

$I = \langle I \cap (A_0 \odot B_0) \rangle$ for every closed ideal $I \subseteq A \otimes B$. We first consider the case that $I = J \otimes K$ for closed ideals $J \subseteq A$ and $K \subseteq B$. Note that

$$(J \otimes K) \cap (A_0 \odot B_0) \supseteq (J \cap A_0) \odot (K \cap B_0).$$

Since $A_0 \subseteq A$ has the ISP, we have $J = \langle J \cap A_0 \rangle$ by Proposition 2.3. Similarly, we get $K = \langle K \cap B_0 \rangle$, and thus

$$\langle (J \otimes K) \cap (A_0 \odot B_0) \rangle \supseteq \langle (J \cap A_0) \odot (K \cap B_0) \rangle \supseteq \langle J \odot K \rangle = J \otimes K.$$

Now, let $I \subseteq A \otimes B$ be an arbitrary closed ideal. Since (A, B) has Tomiyama’s property (F), we obtain a family of product ideals I_λ such that I is the closure of $\sum_\lambda I_\lambda$. For each λ , we have

$$\langle I \cap (A_0 \odot B_0) \rangle \supseteq \langle I_\lambda \cap (A_0 \odot B_0) \rangle = I_\lambda,$$

and thus

$$\langle I \cap (A_0 \odot B_0) \rangle \supseteq \langle \sum_\lambda I_\lambda \rangle = I,$$

as desired.

(2) To show the forward implication, assume that $A_0 \odot B_0 \subseteq A \otimes B$ has the IIP. To verify that $A_0 \subseteq A$ has the IIP, let $I \subseteq A$ be a non-zero, closed ideal. Then $I \otimes B$ is a non-zero, closed ideal in $A \otimes B$. By assumption, we obtain a non-zero element $t \in (I \otimes B) \cap (A_0 \odot B_0)$. Since slice maps detect if an element in $A \otimes B$ is zero (see S3 on p.541 in [48]), we obtain $\psi \in B^*$ such that

$$L_\psi(t) \neq 0.$$

As in (1), we get $L_\psi(t) \in I \cap A_0$. Analogously one shows that $B_0 \subseteq B$ has the IIP.

To show the backwards implication, assume that $A_0 \subseteq A$ and $B_0 \subseteq B$ have the IIP. Let $I \subseteq A \otimes B$ be a non-zero, closed ideal. Then I contains a nonzero elementary tensor $a \otimes b$; see, for example, [8, Lemma 2.12(ii)]. Let J be the closed ideal of A generated by a , and analogously for K . Then $J \otimes K \subseteq I$. Since $A_0 \subseteq A$ has the IIP, we obtain a nonzero element $x \in J \cap A_0$. Similarly, we obtain a nonzero element $y \in K \cap B_0$. Then

$$0 \neq x \otimes y \in (J \cap A_0) \odot (K \cap B_0) \subseteq (J \odot K) \cap (A_0 \odot B_0) \subseteq I \cap (A_0 \odot B_0)$$

as desired. \square

The following corollary is immediate by Theorem 4.8.

Corollary 4.9. *Let A be a C^* -algebra, and suppose $A_0 \subseteq A$ is a subalgebra with the ISP (the IIP). Then $M_n(A_0) \subseteq M_n(A)$ has the ISP (the IIP).*

The next result recovers [1, Corollary 2.3] and also proves the converse direction.

Corollary 4.10. *Let Γ and Δ be discrete groups. Then $\Gamma \times \Delta$ is algebraically C^* -unique if and only if Γ and Δ are.*

Proof. Similarly as in Proposition C, we see that a discrete group H is algebraically C^* -unique if and only if H is amenable and the inclusion $\mathbb{C}[H] \subseteq C_{\text{red}}^*(H)$ has the IIP. The natural isomorphism between $C_{\text{red}}^*(\Gamma \times \Delta)$ and $C_{\text{red}}^*(\Gamma) \otimes C_{\text{red}}^*(\Delta)$ identifies $\mathbb{C}[\Gamma \times \Delta]$ with $\mathbb{C}[\Gamma] \odot \mathbb{C}[\Delta]$, as shown in the following commutative diagram:

$$\begin{array}{ccc}
 \mathbb{C}[\Gamma \times \Delta] & \hookrightarrow & C_{\text{red}}^*(\Gamma \times \Delta) \\
 \cong \downarrow & & \downarrow \cong \\
 \mathbb{C}[\Gamma] \odot \mathbb{C}[\Delta] & \hookrightarrow & C_{\text{red}}^*(\Gamma) \otimes C_{\text{red}}^*(\Delta).
 \end{array}$$

Now the result follows from Theorem 4.8 together with the result that $\Gamma \times \Delta$ is amenable if and only if Γ and Δ are amenable ([39, Proposition 1.13]). \square

Example 4.11. Let $\Gamma = \mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2$ be one of the torsion-free, algebraically C^* -unique groups constructed by Scarparo in [43], and let Δ be any locally finite group. Then $\Gamma \times \Delta$ is algebraically C^* -unique.

Theorem 4.12. *Let $\iota_A: A_0 \rightarrow A$ and $\iota_B: B_0 \rightarrow B$ be bounded homomorphisms from Banach algebras A_0 and B_0 to C^* -algebras A and B , and let $\iota: A_0 \widehat{\otimes} B_0 \rightarrow A \otimes B$ be the naturally induced map. The following statements hold:*

- (1) *The inclusion $\iota(A_0 \widehat{\otimes} B_0) \subseteq A \otimes B$ has the ISP if and only if $\iota_A(A_0) \subseteq A$ and $\iota_B(B_0) \subseteq B$ have the ISP and the pair (A, B) has Tomiyama’s property (F).*
- (2) *The inclusion $\iota(A_0 \widehat{\otimes} B_0) \subseteq A \otimes B$ has the IIP if and only if $\iota_A(A_0) \subseteq A$ and $\iota_B(B_0) \subseteq B$ have the IIP.*

Proof. Given $\psi \in B^*$, we consider the left slice map $L_\psi: A \otimes B \rightarrow A$ as in Paragraph 4.7. It follows easily from the universal property of the projective tensor product that there exists a bounded, linear map $\widehat{L}_\psi: A_0 \widehat{\otimes} B_0 \rightarrow A_0$ determined on finite sums of elementary tensors by

$$\widehat{L}_\psi \left(\sum_k a_k \otimes b_k \right) = \sum_k \psi(\iota_B(b_k)) a_k.$$

Using that $L_\psi \circ \iota$ and $\iota_A \circ \widehat{L}_\psi$ are bounded and linear and agree on elementary tensors, we see that both maps agree. Thus, the following diagram commutes:

$$\begin{array}{ccc}
 A_0 \widehat{\otimes} B_0 & \xrightarrow{\iota} & A \otimes B \\
 \widehat{L}_\psi \downarrow & & \downarrow L_\psi \\
 A_0 & \xrightarrow{\iota_A} & A.
 \end{array}$$

We will use that if $I \subseteq A$ is a closed ideal, then $L_\psi(I \otimes B) \subseteq I$.

(1) To show the forward implication, assume that $\iota(A_0 \widehat{\otimes} B_0) \subseteq A \otimes B$ has the ISP. Since $\iota(A_0 \widehat{\otimes} B_0)$ is contained in $A \widehat{\otimes} B$, it follows that $A \widehat{\otimes} B \subseteq A \otimes B$ has the ISP (see Proposition 2.4), and thus (A, B) has Tomiyama’s property (F) by Theorem 4.5. To verify that $\iota_A(A_0) \subseteq A$ has the ISP, let $I, J \subseteq A$ be distinct closed ideals. Then $I \otimes B$ and $J \otimes B$ are distinct closed ideals in $A \otimes B$. By assumption, their intersections with $\iota(A_0 \widehat{\otimes} B_0)$ are also distinct. Without loss of generality, we assume that there is $t \in A_0 \widehat{\otimes} B_0$ with

$$\iota(t) \in I \otimes B, \quad \text{and} \quad \iota(t) \notin J \otimes B.$$

As in the proof of Theorem 4.8, we pick $\psi \in B^*$ such that

$$L_\psi(\iota(t)) \notin J.$$

Set $x := \widehat{L}_\psi(t) \in A_0$. Then

$$\iota_A(x) = \iota_A(\widehat{L}_\psi(t)) = L_\psi(\iota(t)) \in L_\psi(I \otimes B) \subseteq I, \quad \text{and} \quad \iota_A(x) = L_\psi(\iota(t)) \notin J.$$

Thus, $\iota_A(x)$ separates I and J . Analogously, one shows that $\iota_B(B_0) \subseteq B$ has the ISP.

To show the backwards implication, assume that $\iota_A(A_0) \subseteq A$ and $\iota_B(B_0) \subseteq B$ have the ISP and that the pair (A, B) has Tomiyama’s property (F). By Theorem 4.8, it follows that

$$\iota_A(A_0) \odot \iota_B(B_0) \subseteq A \otimes B$$

has the ISP. Since $\iota_A(A_0) \odot \iota_B(B_0)$ is contained in $\iota(A_0 \widehat{\otimes} B_0)$, it follows from Proposition 2.4 that $\iota(A_0 \widehat{\otimes} B_0) \subseteq A \otimes B$ has the ISP.

(2) To show the forward implication, assume that $\iota(A_0 \widehat{\otimes} B_0) \subseteq A \otimes B$ has the IIP. To show that $\iota_A(A_0) \subseteq A$ has the IIP, let $I \subseteq A$ be a non-zero, closed ideal. Then $I \otimes B \subseteq A \otimes B$ is a non-zero, closed ideal, and by assumption we obtain $t \in A_0 \widehat{\otimes} B_0$ such that $0 \neq \iota(t) \in I \otimes B$. As in the proof of Theorem 4.8, we can pick $\psi \in B^*$ such that

$$L_\psi(\iota(t)) \neq 0.$$

Set $x := \widehat{L}_\psi(t) \in A_0$. As in (1), we get $\iota_A(x) \in I$. Further, we have

$$\iota_A(x) = \iota_A(\widehat{L}_\psi(t)) = L_\psi(\iota(t)) \neq 0,$$

and thus $I \cap \iota_A(A_0) \neq \{0\}$. Analogously, one shows that $\iota_B(B_0) \subseteq B$ has the IIP.

The backwards implication follows from Theorem 4.8, similarly as in (1). \square

We are now ready to prove the main result of this section, characterizing when the Cartesian product of two locally compact groups has the ISP or the IIP. We discuss in Remark 4.15 how our result generalizes the characterization of *-regularity and C^* -uniqueness of product groups from [21, Corollary 3.5].

Theorem 4.13. *Let G and H be locally compact groups. The following statements hold:*

- (1) *The product $G \times H$ has the ISP if and only if G and H have the ISP and the pair of C^* -algebras $C_{\text{red}}^*(G)$ and $C_{\text{red}}^*(H)$ has Tomiyama’s property (F).*
- (2) *The product $G \times H$ has the IIP if and only if G and H have the IIP.*

Proof. Using the natural isomorphisms of $L^1(G \times H)$ with $L^1(G) \widehat{\otimes} L^1(H)$ (see, for example, [37, Corollary 1.10.14]), and of $C_{\text{red}}^*(G \times H)$ with $C_{\text{red}}^*(G) \otimes C_{\text{red}}^*(H)$, we obtain the following commutative diagram:

$$\begin{array}{ccc} L^1(G \times H) & \hookrightarrow & C_{\text{red}}^*(G \times H) \\ \cong \downarrow & & \downarrow \cong \\ L^1(G) \widehat{\otimes} L^1(H) & \longrightarrow & C_{\text{red}}^*(G) \otimes C_{\text{red}}^*(H). \end{array}$$

It follows that the natural map $L^1(G) \widehat{\otimes} L^1(H) \rightarrow C_{\text{red}}^*(G) \otimes C_{\text{red}}^*(H)$ is an inclusion, and we use it to identify $L^1(G) \widehat{\otimes} L^1(H)$ with a dense subalgebra of $C_{\text{red}}^*(G) \otimes C_{\text{red}}^*(H)$.

Using this, we see that $G \times H$ has the ISP (the IIP) if and only if the inclusion $L^1(G) \widehat{\otimes} L^1(H) \subseteq C_{\text{red}}^*(G) \otimes C_{\text{red}}^*(H)$ has the ISP (the IIP). Now the result follows from Theorem 4.12. \square

Following Kirchberg and Wassermann [26], a locally compact group G is said to be *exact* if taking reduced crossed products by G preserves short exact sequences of C^* -algebras with continuous G -actions. By considering trivial actions on C^* -algebras in a short exact sequence, we see that $C_{\text{red}}^*(G)$ is an exact C^* -algebra whenever G is an exact, locally compact group. Conversely, a discrete group G is exact if (and only if) $C_{\text{red}}^*(G)$ is exact ([26, Theorem 5.2]), and it remains a major open problem if this also holds for nondiscrete groups; see [2, Problem 9.3] and [34].

Corollary 4.14. *Let G and H be locally compact groups and assume that at least one of them is exact. Then $G \times H$ has the ISP if and only if G and H have the ISP.*

Proof. A pair of C^* -algebras automatically satisfies Tomiyama's property (F) if one of the C^* -algebras is exact. To see this, note that [8, Proposition 2.17(2)] implies that [8, Proposition 2.16(ii)] is satisfied, and the statement is now a consequence of Theorem 4.5. Thus, it follows from the assumption that the pair of C^* -algebras $C_{\text{red}}^*(G)$ and $C_{\text{red}}^*(H)$ satisfies Tomiyama's property (F). Now the result follows from Theorem 4.13. \square

Remark 4.15. Let G and H be locally compact groups. By [21, Corollary 3.5], the product $G \times H$ is $*$ -regular (C^* -unique) if and only if G and H are $*$ -regular (C^* -unique).

Theorem 4.13 recovers and generalizes these results. Indeed, it follows from [39, Proposition 1.13] that $G \times H$ is amenable if and only if G and H are amenable. Therefore, [21, Corollary 3.5] follows from Theorem 4.13 (and Corollary 4.14) using that by Proposition C a locally compact group is $*$ -regular (C^* -unique) if and only if it is amenable and satisfies the ISP (the IIP).

With view towards Theorem 4.13, our Question E if every discrete group has the ISP is closely related to Question 4.16 below. Indeed, if Γ_1 and Γ_2 are discrete groups such that the pair of C^* -algebras $C_{\text{red}}^*(\Gamma_1)$ and $C_{\text{red}}^*(\Gamma_2)$ does not have Tomiyama's property (F), then $\Gamma_1 \times \Gamma_2$ does not have the ISP. Note that such groups are necessarily non-exact. They also cannot both be C^* -simple.

Question 4.16. Are there discrete groups Γ_1 and Γ_2 such that the pair of C^* -algebras $C_{\text{red}}^*(\Gamma_1)$ and $C_{\text{red}}^*(\Gamma_2)$ does not have Tomiyama's property (F)?

Data availability

No data was used for the research described in the article.

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