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Equitable tradable parking permit scheme for shared nonpublic parking management☆

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ABSTRACT

Parking issues have been one of the significant transport problems in central metropolitan areas due to intensive land use and lacking parking spaces. This study presents an approach to evaluating and optimizing a new Tradable Parking Permit scheme for Sharing Parking (TPPSP) in nonpublic parking areas (NPAs). Special emphasis is put on improving the utilization of parking places under multimodal transport systems and equity among different users, which has been overlooked in previous relevant studies. The nested-logit-based user equilibrium condition under multimodal transport systems for the TPPSP scheme is modeled and solved by a variational inequality model, incorporating a quantal response equilibrium dedicated to depicting non-cooperative cruise-for-parking competition. Especially, the model takes into account the heterogeneous users' choice behavior regarding travel modes and routes. The proposed model not only investigates the impacts of the TPPSP scheme on the performance of the road networks and parking facilities but also the equity of various users. Afterward, a mathematical program with equilibrium constraints is utilized to optimize equity measured by the Gini coefficient and solved by the Kriging metamodel algorithm. Numerical experiments are utilized to validate the proposed approaches. The results show that the proposed TPPSP scheme is an effective approach to promoting social welfare by improving the performance of road networks and parking facilities outside the NPAs and balancing the equity of different users simultaneously. The outcomes provide a potential and useful pricing and management strategy for improving parking issues in urban contexts.

1. Introduction

The escalation of car ownership and the intensive utilization of land in the core districts of megacities have rendered parking difficulties as a paramount transportation challenge (Chen et al., 2021; Wang et al., 2021). Particularly, a significant portion of parking resources in the central zones of numerous Chinese metropolises, such as Shenzhen, Hangzhou, Xi'an, Shanghai, and Beijing, are situated within "non-public" parking areas (NPAs) including university campuses and sizable corporate entities(Bridgelall, 2014;

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Chen, 2021; Zhao et al., 2021; Wang and Liu, 2014; Chen et al., 2015; Shen et al., 2020). Within these NPAs, parking access is typically granted to individuals associated with these institutions (hereinafter "internal beneficiaries") and certain visitors. However, the users who are not working or visiting the workplaces of NPAs (hereinafter referred to as "external users") are not allowed to park in the NPAs, even though there are available parking spaces in these NPAs. This exclusionary practice leads to suboptimal utilization of NPA parking facilities, compelling external users to extensively search for parking in urban and other congested areas. Conversely, making NPA parking facilities accessible to external users could significantly enhance the utilization of these spaces, thereby improving overall transportation system efficiency (Shao et al., 2016; Xiao et al., 2018; Xiao et al., 2020; Peng et al., 2022) and contributing to societal welfare (Jian et al., 2020; Xiao et al., 2020; Takayama and Kuwahara, 2020; Hu et al., 2022).

In the realm of parking management, the concept of tradable parking permits emerges as a notable strategy within the framework of shared parking resources (Bao and Ng, 2022). This approach posits that internal beneficiaries, once allocated tradable parking permits, possess the flexibility to either utilize these permits for their personal parking needs or to sell permits to others, thereby garnering financial benefits. The transaction of parking permits occurs directly among users, bypassing the operator, and thus, the financial incentives derived from sharing parking spaces and trading permits accrue directly to the users involved. The strategy that facilitates the sharing of unoccupied parking facilities within NPAs with external users holds the promise of enhancing the utilization rates of these NPAs, mitigating the pressing issue of parking scarcity in the central districts of large urban areas.

However, Tradable Parking Permit for Sharing Parking (TPPSP), while innovative, may have potential equity concerns stemming from the diversity of Origin-Destination (OD) pairs and levels of car ownership among internal beneficiaries. Specifically, the commuting journey (originating from home and culminating at the workplace) experiences variance in travel time due to the geographical diversity of homes and work locations within central urban areas. Consequently, travelers embarking on longer OD journeys incur longer travel times compared to their counterparts with shorter commutes. Meanwhile, commutes typically involve one of three modes: public transit, driving with a parking spot reserved via permit in a NPA, or driving while competing for public parking spaces. Public transit, in many large Chinese cities, often results in longer travel times due to less direct routes and lower accessibility, thereby contributing to travel time disparities. Similarly, searching for public parking spaces extends travel time, creating a contrast in convenience and time expenditure among commuters (Jiang and Fan, 2020).

In TPPSP, initial permit allocation schemes that do not consider the varied OD pairs and car ownership profiles risk exacerbating disparities in generalized travel costs (comprising both travel time and monetary expenses) among internal beneficiaries, potentially leading to equity issues. In this context, equity pertains to equalizing generalized travel costs across internal beneficiaries, regardless of their distinct OD patterns and car ownership.

The equity concern is particularly pronounced if permits are exclusively allocated to car-owning internal beneficiaries who can gain either time savings by securing parking through their permit or financial benefits by selling their permit. Those without cars are already inferior in travel time to work due to longer travel time of using public transit than driving, and they despite not utilizing NPA parking facilities directly, are entitled to financial compensation through the sale of their permits, implicitly contributing to the pool of shared parking spaces by not driving. Therefore, the equitable distribution of parking permits, taking into account various OD pairs and car ownership statuses, necessitates careful consideration. To our knowledge, the potential equity implications associated with TPPSP have not been adequately examined or addressed in the system's design, indicating a critical oversight that merits further exploration.

To improve the research gaps, this study aims to propose an equitable TPPSP scheme of NPA. The main contributions of this paper are as follows. First, the shared parking scheme considers equity among various users in terms of different OD pairs and car ownership as one of the objectives via the appropriate initial assignment of tradable parking permits for internal beneficiaries. Second, a multimodal network equilibrium model characterizing different users' travel modes and route choices is developed to explicitly measure the impacts of TPPSP on the operational condition of the multimodal transport network. Third, the nested-logit-based multimodal network equilibrium model incorporating heterogeneous quantal response equilibrium (Rogers et al., 2009) is utilized to model the non-cooperative cruise-for-parking behavior in the competitive parking around the road network outside the NPA. The formulated mathematical program with equilibrium constraints is solved by the Kriging metamodel algorithm. The proposed method is validated through numerical experiments to compare with existing schemes in terms of improving parking efficiency and equity.

The remainder of this paper is organized as follows. The relevant studies are reviewed in Section 2. Sections 3 and 4 present models with the nested-logit-based user equilibrium condition for various users, travel modes, and routes. In Section 5, a solver is leveraged to obtain solutions. Section 6 uses the proposed method for a case study based on a real urban multimodal network, followed by conclusions and avenues for future research in Section 7.

2. Literature review

To design efficient parking management strategies, many studies investigated mathematical models and optimization of parking pricing and quantity control strategies. Models for pricing and tradable parking permit schemes are originally derived from road pricing (Yang and Huang, 2005) and road capacity allocation schemes (Yang and Wang, 2011; Miralinaghi and Peeta, 2016; Zhu et al., 2017; Xiao et al., 2019; Krabbenborg et al., 2021; Lessan and Fu, 2022), respectively. Multimodal transportation network equilibrium models considering both public transit and private cars are the basis for parking pricing (Qian et al., 2012; Fosgerau and de Palma, 2013; Liu et al., 2014a; Ji et al., 2022; Takayama and Kuwahara, 2020) and tradable parking permits (Zhang et al., 2011; Yang et al., 2013; Liu et al., 2014b). The multimodal equilibrium model generally consists of network modeling, traffic assignment and mathematical approaches to solve equilibrium conditions. To model the transport network for parking strategy design, the homes and central areas are generally regarded as nodes and are connected via paralleled transit lines and highways, forming a simple many-to-one network, which cannot reflect the realistic topology of multimodal transportation networks in reality. The highway is constrained by a bottleneck at the entrance of central areas and the number of parking spaces in central areas. Parking-related travel time when travelers drive to commute is considered with the queuing delay at the bottleneck due to the competition for parking in central areas, quantified by the bottleneck models (Vickrey, 1969; Arnott et al., 1991; Anderson and de Palma, 2004; Tang et al., 2021). The underlying assumption for such models is that all parking facilities in central areas are homogenous and cruising time for parking is negligible compared with travel time from home to central areas, which is good for model formulation but not realistic. Nevertheless, these models are insufficient when we evaluate the shared parking policy. The central area of a megacity is a large area, so the locations of workplaces and parking facilities in central areas are significantly heterogeneous, which cannot be neglected. Moreover, the parking facilities are heterogeneous in terms of parking fee, search time (Qian and Rajagopal, 2014b; Inci and Lindsey, 2015; Mackowski et al., 2015; Lei and Ouyang, 2017) and cruising probability (Du and Gong, 2016).

To improve traffic assignment in multimodal equilibrium models, some studies utilized various travel behavior models (such as multinomial logit, nested logit, and cross-nested logit) to capture the travelers' mode choice and route choice (Florian et al., 1999; Zarrinmehr et al., 2019; Zhang et al., 2023). Then, logit form-based variational inequality models were proposed to derive the equilibrium condition of the multimodal transportation network (Liu et al. 2021; Fan et al., 2022). Generally, the nested logit is suitable for modeling mode and route choices when there are overlaps among different modes and routes, e.g., some links are overlapped among different routes and travel times of different routes are related (De Cea et al., 2005; Ben-Akiva, 1973; Anderson et al., 1992; Wu et al., 2012). Such multimodal equilibrium models have been utilized by a number of studies regarding multimodal transport management strategies, such as park-and-ride (Liu et al., 2018; Yuan and Yu, 2018; Pi et al., 2019; Ye et al., 2021; Fan et al., 2022), exclusive bus lane (Zhang et al., 2018a; Wang et al., 2020), tradable mobility credit (Wu et al., 2012) and so forth. However, to the best of our knowledge, the multimodal transportation network equilibrium model that is able to describe the parking-related travel cost is hardly investigated. Parking-related behaviors include searching a parking place in a parking node and switching to another parking node in the network if failing to find a parking spot. In each node, parking fees and search time (Chen et al., 2024; Cheng et al., 2023; Huang et al., 2005; Lam et al., 2006; Qian and Rajagopal, 2014a) are counted in travelers' parking-related travel costs. However, the failure of a parking search should also be considered in the parking-related travel cost in the TPPSP scheme. Searching for parking includes both finding parking spots in a parking cluster and travel time to another cluster if failing to search for an available parking spot (Boyles et al., 2015; Du et al., 2019). This could result in extra nonnegligible travel time in the parking-intensive central areas of a megacity. For driving while competing for public parking facilities, travelers probably search in the same public parking cluster repeatedly due to cyclic cruising routes. In the morning commute, the main reason for travelers cycling around public parking facilities is that travelers are in a non-cooperative parking game with the bounded rational choice of cruising for parking (Du et al., 2019; Karaliopoulos et al., 2017). Hence, travelers may cruise in the same parking facility repeatedly to find an available parking spot. To our best knowledge, users' non-cooperative cruise-for-parking behavior is hardly integrated into the nested logit-based multimodal transportation network equilibrium model to precisely describe various travelers' choices of travel modes and routes and equilibrium conditions in the TPPSP scheme.

In short summary, existing research about parking pricing or tradable parking permit management optimization mainly takes system efficiency and final equilibrium as the objectives, but ignores the potential equity due to tradable parking permit assignment. Meanwhile, from the technical perspective, existing models often simplify the complexity of urban environments and assume homogeneity in parking facilities and overlook the significant variability in location, parking fees, search times, and the probability of finding parking within central urban areas. This simplification limits the models' applicability in evaluating shared parking policies, where the heterogeneous nature of workplaces and parking facilities plays a crucial role. Moreover, neglecting non-cooperative adequately integrate non-cooperative cruise-for-parking behavior overlooks the strategic decisions made by drivers competing for limited public parking resources, particularly in high-density urban areas centers where searching for parking can significantly impact overall travel time and route choice. Addressing this oversight could enhance the precision of models in predicting travel behavior under parking management strategies like TPPSP scheme.

3. Problem description and formulation

3.1. Abbreviation and notation

The following abbreviation and notation will be adopted throughout this paper.

TPPSP	Tradable parking permit scheme for sharing parking
SPBP	Shared parking policy of paying for booked parking
AP	Assigned-permit
NP	Non-assigned-permit
NPA	Nonpublic parking area
VOT	Value of time
VI	Variational inequality
NPP	Non-cooperative public parking
User group related notations	
N^{TPP}	Number of parking permits
$\mathscr{U}$	Set of user groups, $\mathscr{U} = \{u \vert u = 1, 2\}$

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TPPSP	Tradable parking permit scheme for sharing parking
u	User group, Group (1) ($u=1$) represents the internal beneficiaries without cars, and Group (2) ($u=2$) represents the internal beneficiaries with cars
β^u	VOTs for the group $u\in\mathscr{U}$
$\mathscr{W}$	Set of OD pairs
$\mathscr{W}_N$	Set of OD pairs between homes and workplaces with NPA
$\mathscr{W}_P$	Set of OD pairs between homes and workplaces without NPA
$\mathscr{O}$	Set of origins
$\mathscr{D}$	Set of destinations
o	Origin $o\in\mathscr{O}$
d	Destination $d\in\mathscr{D}$
w	OD pair, $w=(o,d)\in\mathscr{W}$ where $o\in\mathscr{O}$, $d\in\mathscr{D}$
$\mathbf{Q}^U$	Vector of number of users in groups, representing the initial assignment of the permits to OD pairs in user groups,$\mathbf{Q}^U=\left(\cdots,\widehat{Q}^{w,u},\cdots,\check{Q}^{w,u},\cdots\right)$
$Q^{w,u}$	Number of users in group u for OD pair w
$\widehat{Q}^{w,u}$	Number of AP users in group u for OD pair w
$\check{Q}^{w,u}$	Number of NP users in group u for OD pair w
Travel mode related notations	
$\mathscr{M}$	Set of travel modes,$\mathscr{M}=\{m\mid m=1,2,3\}$
m	Travel mode, $m=1,2,3$ represent the travel modes of public transit, driving with a parking permit in NPA, and driving while competing for public parking facilities, respectively.
$\mathbf{q}^U$	Vector of demands of modes for user groups,$\mathbf{q}^U=(\cdots,\widehat{q}^{w,m,u},\cdots,\check{q}^{w,m,u},\cdots)$
$\widehat{q}^{w,m,u}$	Flow of AP users choosing mode $m\in\mathscr{M}$ in group u for OD pair w
$\check{q}^{w,m,u}$	Flow of NP users choosing mode $m\in\mathscr{M}$ in group u for OD pair w
$\check{\mathbf{q}}^{3,U}$	Vector of demands of NP external users cruising for public parking facilities,$\check{\mathbf{q}}^{3,U}=\left(\cdots,\check{q}^{w,3,u},\cdots\right)$
Network related notations	
$\mathscr{G}$	Multimodal network,$\mathscr{G}=(\mathscr{N},\mathscr{A})$
$\mathscr{N}$	Set of nodes
n	Node,$n\in\mathscr{N}$
$\mathscr{N}_t(n)$	Set of ending points of links that head to node n
$\mathscr{N}_h(n)$	Set of starting points of links that emanate from node n
$\delta_{o,n}$	Binary indicator where $\delta_{o,n}=1$ if node n is the origin $o\in\mathscr{O}$, $\delta_{o,n}=0$ otherwise
$\delta_{n,d}$	Binary indicator where $\delta_{n,d}=1$ if node n is the destination $d\in\mathscr{D}$, $\delta_{n,d}=0$ otherwise
$\mathscr{A}$	Set of links
(i,j)	Link, $(i,j)\in\mathscr{A}$ where $i\in\mathscr{N}$ and $j\in\mathscr{N}$
$\mathscr{A}_S$	Set of subway links
$\mathscr{N}_S$	Set of subway stations
$\mathscr{A}_W$	Set of walking links
$\mathscr{A}_R$	Set of links for driving
$\mathscr{N}_B$	Set of nodes for booked parking facilities in NPA
$\mathscr{N}_P$	Set of nodes for public parking facilities around NPA
$\delta_n^{\mathscr{N}_P}$	Indicator variables
N_n^B	Supply of booked parking facility $n\in\mathscr{N}_B$
N_n^P	Supply of public parking facility $n\in\mathscr{N}_P$
Route-link flow related notations	
$\mathscr{R}^{w,m}$	Set of routes between OD pair w by mode m
$\mathscr{R}_l^{w,3}$	Set where all routes of external users cruising for public parking facilities connect OD pair w by traversing l links.
$\mathbf{f}^U$	Vector of route flows of modes for user groups,$\mathbf{f}^U=\left(\cdots,\widehat{f}_r^{w,m,u},\cdots,\check{f}_r^{w,m,u},\cdots\right)$
$\widehat{f}_r^{w,m,u}$	Flow of u th group in AP users on the r th route in $\mathscr{R}^{w,m}$
$\check{f}_r^{w,m,u}$	Flow of u th group in NP users on the r th route in $\mathscr{R}^{w,m}$
$\widehat{P}_r^{w,m,u}$	Choice probability of AP users in group u between OD pair w on the r th route in $\mathscr{R}^{w,m}$
$\check{P}_r^{w,m,u}$	Choice probability of NP users in group u between OD pair w on the r th route in $\mathscr{R}^{w,m}$
$\breve{f}_{r_l}^{w,3,u}$	the flow and of the r th route in $\mathscr{R}_l^{w,3}$
$\check{P}_{r_l}$	the choice probability of the r th route in $\mathscr{R}_l^{w,3}$
$\mathbf{x}$	Vector of link flows,$\mathbf{x}=(\cdots,x_{ij},\cdots)$
x_{ij}	Aggregated flow for all users on the link $(i,j)\in\mathscr{A}$
$\delta_{ij,r}^{w,m}$	Route-link indicator where $\delta_{ij,r}^{w,m}=1$ if link (i,j) is on the route $r\in\mathscr{R}^{w,m}$, and $\delta_{ij,r}^{w,m}=0$ otherwise
x_n	Flow traversing the node $n\in\mathscr{N}$
$\mathbf{x}^U$	Vector of link flows of modes for user groups,$\mathbf{x}^U=\left(\cdots,\widehat{x}_{ij}^{w,m,u},\cdots,\check{x}_{ij}^{w,m,u},\cdots\right)$
$\widehat{x}_{ij}^{w,m,u}$	Flow of u th group in AP users between OD pair w by mode m on the link (i,j)
$\check{x}_{ij}^{w,m,u}$	Flow of u th group in NP users between OD pair w by mode m on the link (i,j)
$\check{\mathbf{x}}^{3,U}$	Vector of link flows of NP external users cruising for public parking facilities,$\check{\mathbf{x}}^{3,U}=\left(\cdots,\check{x}_{ij}^{w,3,u},\cdots\right)$
Travel cost related notations	

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(*continued*)

TPPSP	Tradable parking permit scheme for sharing parking
$t_{ij,S}$	Travel time on the link $(i,j) \in \mathscr{A}_S$
p_{ij}^S	Subway fee on the link $(i,j) \in \mathscr{A}_S$
$t_{ij,R}(x_{ij})$	Travel time on links $(i,j) \in \mathscr{A}_R$
$t_{n,B}$	Average time for external users searching the parking spaces
$t_{n,P}(x_n)$	Search time in the public parking facility which is a polynomial-type function with respect to parking occupancy
$p_{k_{l-1}}$	the parking fee for the public parking facility $k_{l-1} \in \mathscr{N}_P$
$t_{oj,W}$ or $t_{jd,W}$	Walking time in the walking link $(o,j) \in \mathscr{A}_W$, $o \in \mathscr{O}$, $j \in \mathscr{N}_S$, or $(i,d) \in \mathscr{A}_W$, $d \in \mathscr{D}$, $i \in \mathscr{N}_S, \mathscr{N}_B or \mathscr{N}_P$.
p	Average trade price of parking permits
E	Set of scale parameter representing rationality level
ε^w	Scale parameter representing rationality level varying by OD pairs
$C_r^{w,m,u}(\bullet)$	Perceived generalized travel cost of group u between OD pair w on the r th route in $\mathscr{R}^{w,m}$.
$c_r^{w,m,u}(\bullet)$	Measured generalized travel cost of group u between OD pair w on the r th route in $\mathscr{R}^{w,m}$.
$\zeta_r^{w,m,u}$	Perception error of users in group u between OD pair w on the r th route in $\mathscr{R}^{w,m}$.
$\theta^{w,m,u}$	Parameter measuring the correlations of different routes of mode m for OD pair w and group u
$\theta^{w,u}$	Parameter measuring of the degree of independence for different modes for OD pair w and group u
$\overline{c}^{w,m,u}(\mathbf{x},p)$	the perceived cost for selecting mode m
$\widehat{E}^{w,u}(\bullet;\mathbf{Q}^U)$	Benefits for AP internal beneficiaries in group u between OD pair w under the assignment scheme for TPPSP $\mathbf{Q}^U$
$\check{E}^{w,u}(\bullet;\mathbf{Q}^U)$	Benefits for NP internal beneficiaries in group u between OD pair w under the assignment scheme for TPPSP $\mathbf{Q}^U$
$\widehat{c}^{w,u}(\bullet;\mathbf{Q}^U)$	Travel cost for AP internal beneficiaries in group u between OD pair w after the introduction of TPPSP $\mathbf{Q}^U$
$\check{c}^{w,u}(\bullet;\mathbf{Q}^U)$	Travel cost for NP internal beneficiaries in group u between OD pair w after the introduction of TPPSP $\mathbf{Q}^U$
$\widehat{c}^{w,u}(\bullet;0)$	Travel cost for AP internal beneficiaries in group u between OD pair w before the introduction of TPPSP $\mathbf{Q}^U$
$\check{c}^{w,u}(\bullet;0)$	Travel cost for NP internal beneficiaries in group u between OD pair w before the introduction of TPPSP $\mathbf{Q}^U$
SW	Social welfare
GN	Gini coefficient
Others	
Superscript *	Variable's optimal value or the value in the equilibrium condition

3.2. Shared tradable parking permit scheme

In the TPPSP scheme, it is assumed that the operators of NPA leave internal beneficiaries free to make their own decisions in the permit trading market and the transaction cost in the market is ignorable. Therefore, the supply and demand of the market will determine the trade price of the parking permit (Yang and Wang, 2011). Operators (i.e., the operators of NPA) solely makes the initial assignment of permits to the different internal beneficiaries.

The number of parking permits is denoted by N^{TPP}. The internal beneficiary who is initially assigned with a permit denoted as an assigned-permit (AP) internal beneficiary, can book a parking spot via permit or sell the permits to others and take transit to commute. Non-assigned-permit (NP) users include the internal beneficiary who is not assigned a parking permit initially and external users who chose to drive to work but have no parking permits in NPAs. NP users can purchase the permit from others for a parking spot. The external users herein refer to the demand of external users who drive cars for commuting and do not include the demand of external users who use public transport in the initial scenario. The reason is that our study focus is parking management and users who originally use public transit would not compete for parking resources. An external user (also counted as an NP user) can purchase the permit from AP internal beneficiaries and book a parking space in NPA or compete for a parking spot around the NPA if they fail to (or are reluctant to) obtain a permit. Once a parking spot is booked by a permit, it cannot be cancelled, and the used parking permit cannot be re-traded.

For users, we set different groups with different values of times (VOTs). We assume that the VOTs for the users in a certain user group are identical and use β^u to denote the VOTs for the group $u \in \mathscr{U}$ where $\mathscr{U}$ denotes the set for user groups. Group (1) ($u = 1$) represents the internal beneficiaries without cars, and Group (2) ($u = 2$) represents the internal beneficiaries with cars. Internal beneficiaries have three possible options: public transit, driving with a parking permit in NPA and driving while competing for public parking facilities. For external users driving to work, there are two options: driving with a parking permit in NPA and driving with competing for public parking facilities. We denote the set of travel modes as $\mathscr{M} = \{m|m = 1,2,3\}$, where $m = 1,2,3$ represent the travel modes of public transit, driving with a parking permit in NPA, and driving while competing for public parking facilities, respectively.

The users are grouped by car ownership for each OD pair. The number of users in a group $u \in \mathscr{U}$ for OD pair $w \in \mathscr{W}$ is denoted by $Q^{w,u}$ where $\mathscr{W}$ denotes the set of OD pairs. $\mathscr{W}$ consists of two disjoint subsets $\mathscr{W}_N$ and $\mathscr{W}_P$, representing the OD pairs between homes and workplaces with and without NPA, respectively. In group u for OD pair w, the number of AP users is denoted as $\widehat{Q}^{w,u}$, while the number of NP users is denoted as $\check{Q}^{w,u}$. Let $\widehat{q}^{w,m,u}$ and $\check{q}^{w,m,u}$ denote the flow of AP and NP users choosing mode $m \in \mathscr{M}$, respectively. Therefore, we have

$$Q^{w,u} = \widehat{Q}^{w,u} + \check{Q}^{w,u}, \forall w \in \mathscr{W}, \forall u \in \mathscr{U} \tag{1}$$

$$\widehat{Q}^{w,u} \geq 0, \forall w \in \mathscr{W}, \forall u \in \mathscr{U} \tag{2}$$

$$\check{Q}^{w,u} \geq 0, \forall w \in \mathscr{W}, \forall u \in \mathscr{U} \tag{3}$$

$$\sum_{w \in W} \sum_{u \in U} \widehat{Q}^{w,u} = N^{TPP} \tag{4}$$

$$\sum_{w \in W} \sum_{u=2} \left(\widehat{q}^{w,2,u} + \check{q}^{w,2,u} \right) = N^{TPP} \tag{5}$$

$$\sum_{m \in M} \widehat{q}^{w,m,u} = \widehat{Q}^{w,u}, \forall w \in \mathscr{W}, \forall u \in \mathscr{U} \tag{6}$$

$$\sum_{m \in M} \check{q}^{w,m,u} = \check{Q}^{w,u}, \forall w \in \mathscr{W}, \forall u \in \mathscr{U} \tag{7}$$

$$\widehat{q}^{w,m,u} \geq 0, \forall w \in \mathscr{W}, m \in \mathscr{M}, u \in \mathscr{U} \tag{8}$$

$$\check{q}^{w,m,u} \geq 0, \forall w \in \mathscr{W}, m \in \mathscr{M}, u \in \mathscr{U} \tag{9}$$

3.3. Multimodal network

To evaluate the generalized travel cost of users using different transport modes, we construct a multimodal network $\mathscr{G} = (\mathscr{N}, \mathscr{A})$ where $\mathscr{N}$ and $\mathscr{A}$ are the sets of nodes and links, respectively.

3.3.1. Network structure

A conceptual multimodal network is illustrated in Fig. 1. The origin and destination represent home and workplace, respectively. The network includes facilities for public transit (e.g., routes and stations) and driving (roadway and parking facilities). It should be noted that we merely consider the subway as the representative of public transit in this study for demonstration. However, the proposed method and solutions are scalable and can be easily extended to large networks with more public transit routes. The subway service is represented as links for taking the subway, as the red dash line in Fig. 1. The sets of subway links and stations are denoted by $\mathscr{A}_S$ and $\mathscr{N}_S$, respectively. In the morning commuting, travelers can walk to the starting subway station near their homes, take the

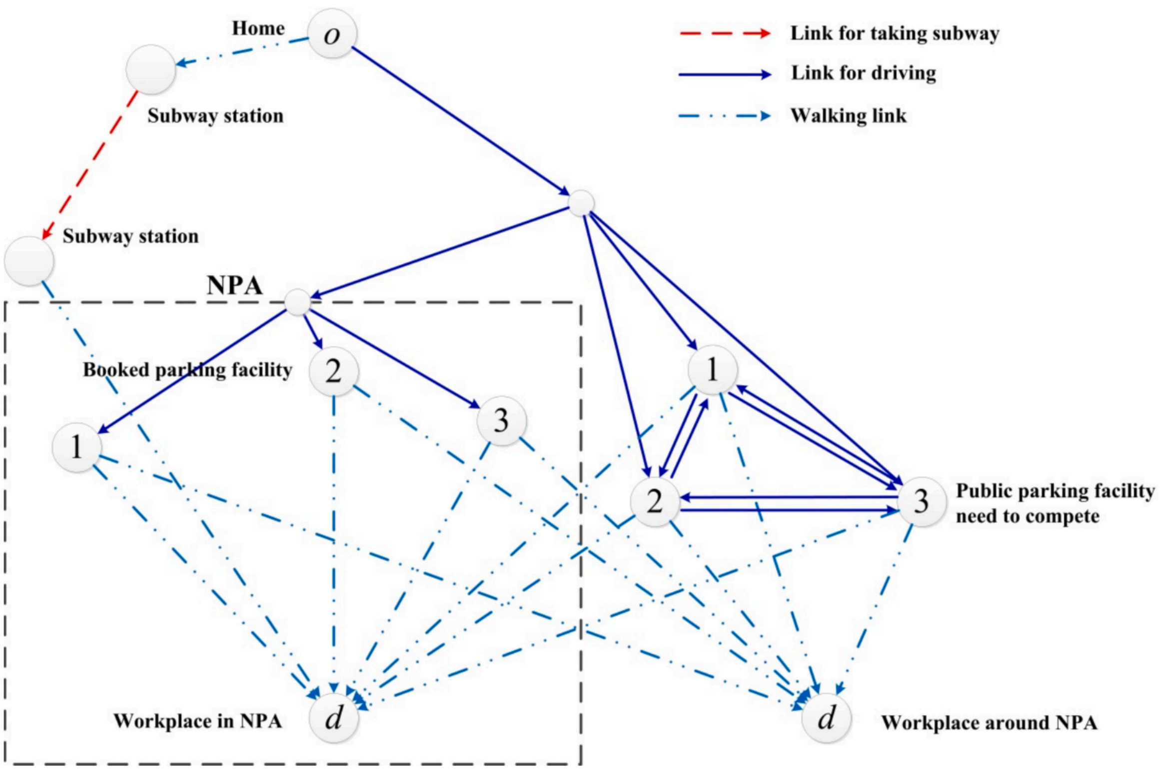

Fig. 1. Multimodal network structure.

subway and walk from the ending station to their destinations. Hence, walking links are included in the network as well. It is assumed that there is only one walking access between a destination (or an origin) and a subway station. The set of walking links is denoted as $\mathscr{A}_W$. For the roadway with parking facilities, the parking facilities are represented as the nodes in the network and connected with links for driving. The set of links for driving is denoted by $\mathscr{A}_R$. The public parking facilities surrounding the NPA are connected, so the cruise for parking among public parking facilities can be modelled. The sets of nodes for booked parking facilities in NPA and public parking facilities around NPA are denoted by $\mathscr{N}_B$ and $\mathscr{N}_P$, respectively. It is assumed that there is only one walking access between a parking facility and a destination.

In $\mathscr{G}$, the directional link $(i,j) \in \mathscr{A}$ emanates from the node $i \in \mathscr{N}$ to node $j \in \mathscr{N}$. Users of various groups are presented as the traffic flows in the network. Let $\mathscr{R}^{w,m}$ denote the route set between OD pair w by mode m. The flows of u th group in AP and NP users on the r th route in $\mathscr{R}^{w,m}$ are denoted by $\widehat{f}_r^{w,m,u}$ and $\check{f}_r^{w,m,u}$. The flows of u th group in AP and NP users between OD pair w by mode m on the link (i,j) are denoted by $\widehat{x}_{ij}^{w,m,u}$ and $\check{x}_{ij}^{w,m,u}$. Although there is only one walking access between a subway station and a destination, the urban area may have more than one subway station near the NPAs and users can choose which stations to get off. The route flows for AP and NP internal beneficiaries by subway are $\widehat{f}_r^{w,1,u}$ and $\check{f}_r^{w,1,u}$, respectively, where $r \in \mathscr{R}^{w,1}$, $w \in \mathscr{W}_N$. For users of an OD pair who drive with parking permits in NPA, they do not need to search for parking. The route flows for AP and NP users driving with parking permits are $\widehat{f}_r^{w,2,u}$ and $\check{f}_r^{w,2,u}$, respectively, where $r \in \mathscr{R}^{w,2}$, $w \in \mathscr{W}$. For external users and NP internal beneficiaries of an OD pair who drive with competing for public parking facilities, they have to cruise around the network looking for parking spaces outside the NPA. Cyclic routes are common in the course of cruising for parking. To avoid enumerating all possible routes in the cruising, we use link flow to describe the external users and NP internal beneficiaries competing for public parking, denoted by $\check{x}_{ij}^{w,3,u}$ where $(i,j) \in \mathscr{A}$, $w \in \mathscr{W}$. To guarantee flow conservation, we propose the following assumption.

Assumption. *All the users who drive to commute will find parking spots at the end. Once a car is parked in a spot, it will not leave during the morning commute.*

This assumption is a widely-used implicit assumption in the models for parking in the morning commute, such as bottleneck models (Liu et al., 2014b; Tang et al., 2021) and multimodal transportation network equilibrium models (Huang et al., 2005; Ye et al., 2021). It can give the validity of bounded rationality assumption in cruising-for-parking competition. The existence of cyclic routes for users has two reasons: stochastic availability of parking spots and boundedly rational choice behavior (Du et al., 2019). The assumption leads the parking of arriving commuters to be a birth process, rather than a birth-and-death process (Boyles et al., 2015). The number of occupied parking spaces accumulates over time and the stochasticity of parking availability is not involved. Hence, the existence of cyclic routes relies on users' perceptions about the utility of the cruising strategy for parking with bounded rationality. Therefore, we have the following flow-demand conservation and non-negativity conditions.

$$\sum_{r\in R^{w,m}} \widehat{f}_r^{w,m,u} = \widehat{q}^{w,m,u},\ \forall w \in \mathscr{W},\ m = 1,2,\ \forall u \in \mathscr{U} \tag{10}$$

$$\sum_{r\in R^{w,m}} \check{f}_r^{w,m,u} = \check{q}^{w,m,u},\ \forall w \in \mathscr{W},\ m = 1,2,\ \forall u \in \mathscr{U} \tag{11}$$

$$\sum_{i\in \mathscr{N}_t(n)} \check{x}_{in}^{w,3,u} - \sum_{j\in \mathscr{N}_h(n)} \check{x}_{nj}^{w,3,u} - \check{q}^{w,3,u}\delta_{n,d} + \check{q}^{w,3,u}\delta_{o,n} = 0,\ \forall w \in \mathscr{W}_P,\ u = 2,\ \forall n \in \mathscr{N} \tag{12}$$

$$\widehat{f}_r^{w,m,u} \geq 0,\ \forall r \in \mathscr{R}^{w,m},\ \forall w \in \mathscr{W},\ m = 1,2,\ \forall u \in \mathscr{U} \tag{13}$$

$$\check{f}_r^{w,m,u} \geq 0,\ \forall r \in \mathscr{R}^{w,m},\ \forall w \in \mathscr{W},\ m = 1,2,\ \forall u \in \mathscr{U} \tag{14}$$

$$\check{x}_{ij}^{w,3,u} \geq 0,\ \forall w \in \mathscr{W},\ u = 2,\ \forall (i,j) \in \mathscr{A} \tag{15}$$

where $\mathscr{N}_t(n)$ is the set of ending points of links that head to node n, and $\mathscr{N}_h(n)$ refers to the set of starting points of links that emanate from node n. $\delta_{o,n} = 1$ if node n is the origin o, $\delta_{o,n} = 0$ otherwise; $\delta_{n,d} = 1$ if node n is the destination d, $\delta_{n,d} = 0$ otherwise. Let x_{ij} denote the aggregated flow for all users on the link $(i,j) \in \mathscr{A}$ and we have

$$x_{ij} = \sum_{w\in \mathscr{W}} \sum_{u\in \mathscr{U}} \left(\sum_{m=1,2} \sum_{r\in R^{w,m}} \left(\widehat{f}_r^{w,m,u} + \check{f}_r^{w,m,u} \right) \delta_{ij,r}^{w,m} + \check{x}_{ij}^{w,3,u} \right),\ \forall (i,j) \in \mathscr{A} \tag{16}$$

where $\delta_{ij,r}^{w,m} = 1$ if link (i,j) is on the route $r \in \mathscr{R}^{w,m}$, and $\delta_{ij,r}^{w,m} = 0$ otherwise. Let x_n denote the flow traversing the node $n \in \mathscr{N}$ and we defined that

$$x_n = \sum_{i\in \mathscr{N}_t(n)} x_{in},\ \forall n \in \mathscr{N} \tag{17}$$

Let $\widehat{x}_n^{w,m,u}$ and $\check{x}_n^{w,m,u}$ denote the flows of AP and NP users traversing the node $n \in \mathscr{N}$ and we have

$$\widehat{x}_n^{w,m,u} = \sum_{i \in \mathscr{N}_t(n)} \widehat{x}_{in}^{w,m,u}, \forall n \in \mathscr{N} \tag{18}$$

$$\check{x}_n^{w,m,u} = \sum_{i \in \mathscr{N}_t(n)} \check{x}_{in}^{w,m,u}, \forall n \in \mathscr{N} \tag{19}$$

3.3.2. Link and node costs

In this part, we present the generalized costs of links and nodes in the network in both NPAs and areas around the NPAs. For the subway links, the travel time is assumed to be constant. The travel cost for user group $u \in \mathscr{U}$ on the link $(i,j) \in \mathscr{A}_S$, $c_{ij,S}^u$ is given by

$$c_{ij,S}^u = \beta^u t_{ij,S} + p_{ij}^S, \forall u \in \mathscr{U}, \forall (i,j) \in \mathscr{A}_S \tag{20}$$

where $t_{ij,S}$ and p_{ij}^S denote the travel time and subway fee on the link $(i,j) \in \mathscr{A}_S$. For the links of driving, the travel times on links $(i,j) \in \mathscr{A}_R$ are denoted by $t_{ij,R}(x_{ij})$. This travel time is measured by the widely-used Bureau of Public Roadside (BPR) function. The travel cost for Group (2) on the link $(i,j) \in \mathscr{A}_R$, $c_{ij,R}^u(\bullet)$ is given by

$$c_{ij,R}^u(x_{ij}) = \beta^u t_{ij,R}(x_{ij}), u = 2, \forall (i,j) \in \mathscr{A}_R \tag{21}$$

Admittedly, drivers who travel through the studied area are neglected which underestimates the congestion level of roads. This is because of lacking data about traffic flow driving through the study area. The multimodal transportation network will be too complex and large if we involve these drivers in the model. The road congestion in the areas near the NPAs is evaluated via BPR function. In practice, the capacity in BPR function needs to be adjusted by using the original road capacity of road links subtracting the average traffic flows travelling through distributed on the road links based on long-term historical observation data which can infer the road-segment traffic flow travelling through (Rizvi and Friedrich, 2024). This could be challenging but feasible using emerging traffic monitoring techniques such as car plate recognition systems and GPS data of vehicles (Xu et al., 2023). We acknowledge the limitation of this aspect in our study, which is also a dilemma of many parking network-related studies. However, the travel times along the roads far from the NPAs are assumed as constant according to the road length since the users are not considered as the main cause of congestion in these roads. For the nodes of booked parking facilities in NPA, the average time of finding the booked parking spaces is denoted by $t_{n,B}$, $n \in \mathscr{N}_B$ and is assumed to be constant. The travel cost for Group (2) on the booked parking facility $n \in \mathscr{N}_B$, $c_{n,B}^u(\bullet)$ is

$$c_{n,B}^u = \beta^u t_{n,B}, u = 2, \forall n \in \mathscr{N}_B \tag{22}$$

For the nodes of public parking facilities around the NPA, the average time for external users searching the parking spaces is denoted by $t_{n,P}(x_n)$, $n \in \mathscr{N}_P$. The travel cost for Group (2) on the public parking facility $n \in \mathscr{N}_P$, $c_{n,P}^u(\bullet)$ is

$$c_{n,P}^u(x_n) = \beta^u t_{n,P}(x_n), u = 2, \forall n \in \mathscr{N}_P \tag{23}$$

where $t_{n,P}(x_n)$ denotes the search time in the public parking facility which is a polynomial-type function with respect to parking occupancy (Balijepalli et al., 2015). In the morning commuting, the occupancy increases temporally. The search time increases with the occupancy. A stable situation will be reached when all parking spaces are taken. Hence, $t_{n,P}(x_n)$ is defined as

$$t_{n,P}(x_n) = t_{n,P}^0\left(1 + \eta_S \bullet OCC_n\left(x_n; N_n^P\right)^{\alpha_S}\right), u = 2, \forall i \in \mathscr{N}_P \tag{24}$$

where $t_{n,P}^0$ is the search time in the public parking facility $n \in \mathscr{N}_P$ when occupancy is zero. $OCC_n(x_n; N_n^P)$ is the average occupancy for the public parking facility $n \in \mathscr{N}_P$ when x_n external users are searching with N_n^P parking spaces. η_S and α_S are both positive coefficients. According to **Assumption**, the average occupancy (Du et al. 2019) is

$$OCC_n\left(x_n; N_n^P\right) = 1 + \frac{1}{N_n^P} \sum_{j=0}^{N_n^P - 1} \frac{x_n^{j+1}}{j!} e^{-x_n} - \sum_{j=0}^{N_n^P} \frac{x_n^j}{j!} e^{-x_n} \tag{25}$$

According to the definition of walking links, a walking link is denoted by $(o,j) \in \mathscr{A}_W$ where $o \in \mathscr{O}, j \in \mathscr{N}_S$, or $(i,d) \in \mathscr{A}_W$ where $d \in \mathscr{D}$, $i \in \mathscr{N}_S, \mathscr{N}_B or \mathscr{N}_P$. $\mathscr{O}$ and $\mathscr{D}$ denote the sets of origins and destinations, respectively where $\mathscr{O} \subset \mathscr{N}$ and $\mathscr{D} \subset \mathscr{N}$. The walking time in each link is assumed to be a constant and denoted as $t_{oj,W}$ or $t_{jd,W}$. The travel cost for user group $u \in \mathscr{U}$ on the walking link is

$$c_{oj,W}^u = \beta^u t_{oj,W}, \forall u \in \mathscr{U}, \forall (o,j) \in \mathscr{A}_W \text{ or } c_{id,W}^u = \beta^u t_{id,W}, \forall u \in \mathscr{U}, \forall (i,d) \in \mathscr{A}_W \tag{26}$$

Let N_n^B denote the supplies of booked parking facility $n \in \mathscr{N}_B$. The supply constraints for booked and public parking facilities are

$$x_n \le N_n^B, \forall n \in \mathscr{N}_B \tag{27}$$

$$\sum_{d \in \mathscr{D}} x_{nd} \le N_n^P, \forall n \in \mathscr{N}_P \tag{28}$$

Besides, we also have $\sum_{n\in\mathscr{N}_B} N_n^B = N^{TPP}$. In summary, the travel costs without the parking fee for user groups on the links are presented as

$$c_{ij}^u(\mathbf{x}) = \begin{cases} c_{ij,S}^u & \text{if}(i,j) \in \mathscr{A}_S \\ c_{ij,R}^u(x_{ij}) & \text{if}(i,j) \in \mathscr{A}_R \\ c_{ij,W}^u & \text{if}(i,j) \in \mathscr{A}_W \end{cases}, \forall u \in \mathscr{U} \tag{29}$$

4. Cyclic cruising for public parking facilities

Cyclic flows exist during searching for available parking lots. If occupied parking spaces are not available again, users' bounded rationality in making decisions could lead to repeatedly searching, namely cyclic cruising for parking (Du et al, 2019). Since the parking game is non-cooperative, boundedly rational users perceive the travel time of their routes and speculate all other users' route choice behavior in the parking competition. When the parking game reaches equilibrium, the heterogeneous quantal response equilibrium is utilized to depict all users' route choices probabilistically. For external users who cruise for public parking facilities, the flow on each route is

$$\check{f}_{r_l}^{w,3,u} = \check{P}_{r_l}\check{q}^{w,3,u}, \forall r \in \mathscr{R}_l^{w,3},\ l = 1,2,\cdots, \forall w \in \mathscr{W}_P,\ u = 2 \tag{30}$$

where $\overset{\smile}{f}{}_{r_l}^{w,3,u}$ and $\check{P}_{r_l}$ are the flow and choice probability of the r th route in $\mathscr{R}_l^{w,3}$, respectively. $\mathscr{R}_l^{w,3}$ represents the set where all routes of external users cruising for public parking facilities connect OD pair w by traversing l links. The model where the parking fee is taken into account is the extension of the model in Du et al. (2019). The external users and NP internal beneficiaries competing for public parking lots are assumed to present the flows of cruising for parking with infinite cycles. The route set for these external users $\mathscr{R}^{w,3}$ consists of $\{\mathscr{R}_1^{w,3}, \mathscr{R}_2^{w,3}, \cdots, \mathscr{R}_l^{w,3}, \cdots\}$. We assume that the rationality level varies by OD pairs and is quantified via scale parameter ε^w. The choice probability on the r th route in $\mathscr{R}_l^{w,3}$ of external users cruising for public parking facilities in the heterogeneous quantal response equilibrium condition is

$$\check{P}_{r_l} = \frac{\exp\left(-\varepsilon^w c_{r_l}^{w,3,u}(\boldsymbol{x})\right)}{\sum_{r\in\mathscr{R}^{w,3}}\exp\left(-\varepsilon^w c_r^{w,3,u}(\boldsymbol{x})\right)}, \forall r \in \mathscr{R}_l^{w,3},\ l = 1,2,\cdots, \forall w \in \mathscr{W},\ u = 2 \tag{31}$$

where $c_{r_l}^{w,3,u}(\bullet)$ and $c_r^{w,3,u}(\bullet)$ are the travel time on the r th route in $\mathscr{R}_l^{w,3}$ and $\mathscr{R}^{w,3}$, respectively. $c_{r_l}^{w,3,u}(\bullet)$ is defined as

$$\begin{aligned} c_{r_l}^{w,3,u}(\mathbf{x}) = \sum_{k_1\in N_h(o)}\sum_{k_2\in N_h(k_1)}\cdots\sum_{d\in N_h(k_{l-1})}\Big(c_{ok_1}^u(\mathbf{x})\delta_{ok_1,r}^{w,3} + \left(c_{k_1k_2}^u(\mathbf{x}) + c_{k_2,P}^u(x_{k_2})\delta_{k_2}^{\mathscr{N}_P}\right)\delta_{k_1k_2,r}^{w,3} + \cdots + \left(c_{k_{l-1}d}^u(\mathbf{x}) + p_{k_{l-1}}\right)\delta_{k_{l-1}d,r}^{w,3}\Big), \forall r \in \mathscr{R}_l^{w,3}, l \\ = 1,2,\cdots, \forall w \in \mathscr{W}, u = 2 \end{aligned} \tag{32}$$

where $\mathbf{x}$ is the vector of link flows, $\mathbf{x} = (\cdots, x_{ij}, \cdots)$, $(i,j) \in \mathscr{A}_R$. $\delta_{ij,r}^{w,3}$ and $\delta_n^{\mathscr{N}_P}$ denote two indicator variables. $\delta_{ij,r}^{w,3} = 1$ if link (i,j) is on r th route in $\mathscr{R}_l^{w,3}$; $\delta_{ij,r}^{w,3} = 0$ otherwise. $\delta_n^{\mathscr{N}_P} = 1$ if the node $n \in \mathscr{N}_P$; $\delta_n^{\mathscr{N}_P} = 0$ otherwise. $p_{k_{l-1}}$ denotes the parking fee for the public parking facility $k_{l-1} \in \mathscr{N}_P$. $p_{k_{l-1}}$ is associated with the destination in Eq. (32), since the node for the parking facility is connected with the destination by a walking link, and if a user is on the link $(k_{l-1}, d) \in \mathscr{A}_W$, he or she must have parked the car in the parking facility $k_{l-1} \in \mathscr{N}_P$. As illustrated in Eq. (32), one public parking facility can be revisited in a cyclic route, and the parking spaces in it will be searched repeatedly. However, the parking fee is charged only if the user is successfully parked. The flows of external users cruising for parking in the heterogeneous quantal response equilibrium condition can be presented as Lemma 1.

Lemma 1. *The pattern of flows and demands for external users and NP internal beneficiaries cruising for public parking facilities, $\check{\mathbf{x}}^{3,U^*} \in \Omega_X\left(\check{\mathbf{q}}^{3,U}\right)$ is in Eq. (31) under the given demands of user groups $\check{\mathbf{q}}^{3,U}$, if it satisfies the below variational inequality (VI) conditions for non-cooperative public parking (NPP).*

[VI-NPP].

$$\sum_{u=2}\sum_{w\in\mathscr{W}}\sum_{(i,j)\in\mathscr{A}}\left(c_{ij}^u(\mathbf{x}) + c_{j,P}^u\left(x_j^*\right)\delta_j^{\mathscr{N}_P} + p_i\delta_j^{\mathscr{D}} + \frac{1}{\varepsilon^w}\left(\ln\check{x}_{ij}^{w,3,u^*} - \ln\sum_{k\in\mathscr{N}_t(j)}\check{x}_{kj}^{w,3,u^*}\right)\right)\left(\check{x}_{ij}^{w,3,u} - \check{x}_{ij}^{w,3,u^*}\right) \geq 0,\ \forall\check{\mathbf{x}}^{3,U} \in \Omega_X\left(\check{\mathbf{q}}^{3,U}\right) \tag{33}$$

and the following equality holds:

$$\exp\left[\varepsilon^w\left(\check{\xi}_d^{w,u^*} - \check{\xi}_o^{w,u^*}\right)\right] = \sum_{r\in\mathscr{R}^{w,3}}\exp\left[-\varepsilon^w c_r^{w,3,u}(\mathbf{x}^*)\right],\ u = 2, \forall w \in \mathscr{W} \tag{34}$$

where $\check{\mathbf{x}}^{3,U} = \left(\cdots, \check{x}_{ij}^{w,3,u}, \cdots\right)$ and $\check{\mathbf{q}}^{3,U} = \left(\cdots, \check{q}^{w,3,u}, \cdots\right)$ denote the link flow and demand vectors of NP external users cruising for public parking facilities, respectively. $\check{\mathbf{x}}^{3,U^*} = \left(\cdots, \check{x}_{ij}^{w,3,u^*}, \cdots\right)$ denotes the vectors of equilibrium link flows for external users and NP internal beneficiaries cruising for public parking facilities. x_{ij}^* is the equilibrium flow on the link in Eq. (18). $\delta_j^{\mathscr{D}} = 1$ if the node $j \in \mathscr{D}$;

$\delta_j^{\mathscr{D}} = 0$ otherwise. The feasible domain $\Omega_X\left(\check{\mathbf{q}}^{3,U}\right)$ is constituted with constraints (11) and (15). $\check{\xi}_d^{w,u*}$ and $\check{\xi}_o^{w,u*}$ are the optimal Lagrangian multipliers associated with Eqs.(35) and (36) in constraint (11), respectively. The proof is presented in the Appendix for readability.

$\sum_{i\in\mathscr{N}_t(n)}\check{x}_{id}^{w,3,u} - \check{q}^{w,3,u} = 0, \forall w \in \mathscr{W}, u = 2$(35)

$-\sum_{j\in\mathscr{N}_h(n)}\check{x}_{oj}^{w,3,u} + \check{q}^{w,3,u} = 0, \forall w \in \mathscr{W}, u = 2$(36)

Lemma 2. *The VI-NPP has at least one solution.*

Proof. (.) *The VI-NPP can be reformulated as another equivalent VI problem:*

$$\sum_{u=2}\sum_{w\in\mathscr{W}}\sum_{(i,j)\in\mathscr{A}}\left(s_{ij}\left(\check{\mathbf{x}}^{3,U*}\right)+e_{ij}^{w,3,u}\left(\check{\mathbf{x}}^{3,U*}\right)\right)\left(\check{x}_{ij}^{w,3,u}-\check{x}_{ij}^{w,3,u*}\right)\geq 0,\ \forall\check{\mathbf{x}}^{3,U}\in\Omega_X\left(\check{\mathbf{q}}^{3,U}\right) \tag{37}$$

where

$$s_{ij}\left(\check{\mathbf{x}}^{3,U*}\right)=c_{ij}^{u}(\mathbf{x})+c_{j,P}^{u}\left(x_j^{*}\right)\delta_j^{\mathscr{N}_P}+p_i\delta_j^{\mathscr{D}},\ \forall(i,j)\in\mathscr{A} \tag{38}$$

$$e_{ij}^{w,3,u}\left(\check{\mathbf{x}}^{3,U*}\right)=\frac{1}{\varepsilon^{w}}\left(\ln\check{x}_{ij}^{w,3,u*}-\ln\sum_{k\in\mathscr{N}_t(j)}\check{x}_{kj}^{w,3,u*}\right),\ u=2,\ \forall w\in\mathscr{W},\ \forall(i,j)\in\mathscr{A} \tag{39}$$

Since the feasible domain $\Omega_X\left(\check{\mathbf{q}}^{3,U}\right)$ is compact and convex, and $s_{ij}\left(\check{\mathbf{x}}^{3,U*}\right)+e_{ij}^{w,3,u}\left(\check{\mathbf{x}}^{3,U*}\right)$ is continuous on $\Omega_X\left(\check{\mathbf{q}}^{3,U}\right)$, the VI problem admits at least one solution (Facchinei and Pang, 2003). In addition, since the average time searching the public parking spaces, $t_{n,P}(x_n)$, $n\in\mathscr{N}_P$ is not monotone, the VI-NPP does not have unique solutions.

5. Model descriptions

5.1. Multimodal equilibrium

We use a nested logit model to model the users' choices of travel modes and routes. In the model, users first make their decisions about travel modes and then the travel routes available for a specific transport mode. The perceived generalized travel cost of group u between OD pair w on the r th route in $\mathscr{R}^{w,m}$, $C_r^{w,m,u}(\bullet)$ is.

$C_r^{w,m,u}(\bullet) = c_r^{w,m,u}(\bullet) + \zeta_r^{w,m,u}, \forall u\in\mathscr{U}, \forall m\in\mathscr{M}, \forall w\in\mathscr{W}, \forall r\in\mathscr{R}^{w,m}$(40).

where $c_r^{w,m,u}(\bullet)$ denotes the measured generalized travel cost of group u between OD pair w on the r th route in $\mathscr{R}^{w,m}$. $\zeta_r^{w,m,u}$ reflects the users' perception error. The flow on each route is given by

$\widehat{f}_r^{w,m,u} = \widehat{Q}^{w,u}\widehat{P}_r^{w,m,u}, \forall u\in\mathscr{U}, \forall m\in\mathscr{M}, \forall w\in\mathscr{W}, \forall r\in\mathscr{R}^{w,m}$(41).

$\check{f}_r^{w,m,u} = \check{Q}^{w,u}\check{P}_r^{w,m,u}, \forall u\in\mathscr{U}, \forall m\in\mathscr{M}, \forall w\in\mathscr{W}, \forall r\in\mathscr{R}^{w,m}$(42).

where $\widehat{P}_r^{w,m,u}$ and $\check{P}_r^{w,m,u}$ denote the choice probability of AP and NP users in group u between OD pair w on the r th route in $\mathscr{R}^{w,m}$, respectively. Here, we consider the travel cost without the trade price of the parking permit. When internal beneficiaries take the subway, we have.

$c_r^{w,1,u} = \sum_{(i,j)\in\mathscr{A}}c_{ij}^{u}\delta_{ij,r}^{w,1}, \forall u\in\mathscr{U}, \forall w\in\mathscr{W}_N, \forall r\in\mathscr{R}^{w,1}$(43).

When users drive with parking permits in NPA, they do not need to search for parking and hence we have.

$c_r^{w,2,u}(\mathbf{x}) = \sum_{(i,j)\in\mathscr{A}}\left(c_{ij}^{u}(\mathbf{x})+c_{j,B}^{u}\delta_j^{\mathscr{N}_B}\right)\delta_{ij,r}^{w,2}, u = 2, \forall w\in\mathscr{W}, \forall r\in\mathscr{R}^{w,2}$(44).

where $\delta_n^{\mathscr{N}_B} = 1$ if the node $n\in\mathscr{N}_B$; $\delta_n^{\mathscr{N}_B} = 0$ otherwise. When users drive without parking permits in NPA, they cruise around the network for competitive parking outside the NPA. The travel cost is derived from Eq. (32). Incorporating the heterogeneous quantal response equilibrium model into the nested logit model, the probability for the group u between OD pair w to choose the r th route in $\mathscr{R}^{w,m}$ is given by

$\widehat{P}_r^{w,1,u} = \frac{\exp\left(-\theta^{w,1,u}\left(c_r^{w,1,u}-p\right)\right)}{\sum_{r\in\mathscr{R}^{w,1}}\exp\left(-\theta^{w,1,u}\left(c_r^{w,1,u}-p\right)\right)}\bullet\exp\left(-\theta^{w,u}\frac{\overline{c}^{w,1,u}(p)}{\sum_{m\in\mathscr{M}}\exp\left(-\theta^{w,u}\overline{c}^{w,m,u}(\mathbf{x},p)\right)}\right), \forall u\in\mathscr{U}, \forall w\in\mathscr{W}_N, \forall r\in\mathscr{R}^{w,1}$(45a).

$\check{P}_r^{w,1,u} = \frac{\exp\left(-\theta^{w,1,u}c_r^{w,1,u}\right)}{\sum_{r\in\mathscr{R}^{w,1}}\exp\left(-\theta^{w,1,u}c_r^{w,1,u}\right)}\bullet\exp\left(-\theta^{w,u}\frac{\overline{c}^{w,1,u}(p)}{\sum_{m\in\mathscr{M}}\exp\left(-\theta^{w,u}\overline{c}^{w,m,u}(\mathbf{x},p)\right)}\right), \forall u\in\mathscr{U}, \forall w\in\mathscr{W}_N, \forall r\in\mathscr{R}^{w,1}$(45b).

$\widehat{P}_r^{w,2,u} = \frac{\exp\left(-\theta^{w,2,u}c_r^{w,2,u}(\mathbf{x})\right)}{\sum_{r\in\mathscr{R}^{w,2}}\exp\left(-\theta^{w,2,u}c_r^{w,2,u}(\mathbf{x})\right)}\bullet\exp\left(-\theta^{w,u}\frac{\overline{c}^{w,2,u}(p)}{\sum_{m\in\mathscr{M}}\exp\left(-\theta^{w,u}\overline{c}^{w,2,u}(\mathbf{x},p)\right)}\right), u = 2, \forall w\in\mathscr{W}, \forall r\in\mathscr{R}^{w,2}$(46a).

$\check{P}_r^{w,2,u} = \frac{\exp\left(-\theta^{w,2,u}\left(c_r^{w,2,u}+p\right)\right)}{\sum_{r\in\mathscr{R}^{w,2}}\exp\left(-\theta^{w,2,u}\left(c_r^{w,2,u}+p\right)\right)}\bullet\exp\left(-\theta^{w,u}\frac{\overline{c}^{w,2,u}(p)}{\sum_{m\in\mathscr{M}}\exp\left(-\theta^{w,u}\overline{c}^{w,2,u}(\mathbf{x},p)\right)}\right), u = 2, \forall w\in\mathscr{W}, \forall r\in\mathscr{R}^{w,2}$(46b).

$\check{P}_{r_l}^{w,3,u} = \frac{\exp\left(-\varepsilon^{w}c_{r_l}^{w,3,u}(\mathbf{x})\right)}{\sum_{r\in\mathscr{R}^{w,3}}\exp\left(-\varepsilon^{w}c_r^{w,3,u}(\mathbf{x})\right)}\bullet\exp\left(-\theta^{w,u}\frac{\overline{c}^{w,3,u}(\mathbf{x})}{\sum_{m\in\mathscr{M}}\exp\left(-\theta^{w,u}\overline{c}^{w,3,u}(\mathbf{x})\right)}\right), u = 2, \forall w\in\mathscr{W}, \forall r\in\mathscr{R}_l^{w,3}, l = 1,2,\cdots$(47).

where p denotes the average trade price of parking permits. $\theta^{w,m,u}$ measures the correlations of different routes of mode m for OD pair w and group u. $\theta^{w,u}$ is a measure of the degree of independence for different modes for OD pair w and group u. $\overline{c}^{w,m,u}(x,p)$ is the perceived cost for selecting mode m:

$\bar{c}^{w,1,u}(p) = -\frac{1}{\theta^{w,1,u}}\ln\left(\sum_{r\in\mathscr{R}^{w,1}}\exp\left(-\theta^{w,1,u}\left(c_r^{w,1,u}-p\delta_{AP}\right)\right)\right), \forall u\in\mathscr{U}, \forall w\in\mathscr{W}_N$(48).

$\bar{c}^{w,2,u}(\mathbf{x},p) = -\frac{1}{\theta^{w,2,u}}\ln\left(\sum_{r\in\mathscr{R}^{w,2}}\exp\left(-\theta^{w,2,u}\left(c_r^{w,2,u}(\mathbf{x})+p\delta_{NP}\right)\right)\right), u=2, \forall w\in\mathscr{W}$(49).

$\bar{c}^{w,3,u}(\mathbf{x}) = -\frac{1}{\varepsilon^{w}}\ln\left(\sum_{r\in\mathscr{R}^{w,3}}\exp\left(-\varepsilon^{w}c_r^{w,3,u}(\mathbf{x})\right)\right), u=2, \forall w\in\mathscr{W}$(50).

where $\delta_{AP}=1$ if users are AP ones; $\delta_{AP}=0$ if users are NP ones. $\delta_{NP}=1-\delta_{AP}$. The above travel choices result in the multimodal equilibrium condition. The flows of users under TPPSP can be obtained by solving the VI problem in the below Proposition.

Proposition. *Users' flows and demands, $\left(\mathbf{f}^{U^*},\mathbf{x}^{U^*},\mathbf{q}^{U^*}\right)\in\Theta\left(\mathbf{Q}^U\right)$ are in the equilibrium conditions (45a), (45b), (46a), (46b), (47) and the average trade prices of parking permits, $p^*\in\mathbb{R}^+$ is in the market equilibrium under the given demands of user groups $\mathbf{Q}^U$, if it is the solution to the VI problem:*

[VI-TPPSP].

$$\sum_{u\in\mathscr{U}}\sum_{w\in\mathscr{W}_N}\sum_{r\in\mathscr{R}^{w,1}}\left(c_r^{w,1,u}-p^*+\frac{1}{\theta^{w,1,u}}\ln\widehat{f}_r^{w,1,u^*}\right)\left(\widehat{f}_r^{w,1,u}-\widehat{f}_r^{w,1,u^*}\right)+\left(c_r^{w,1,u}+\frac{1}{\theta^{w,1,u}}\ln\check{f}_r^{w,1,u^*}\right)\left(\check{f}_r^{w,1,u}-\check{f}_r^{w,1,u^*}\right)+\sum_{u=2}\sum_{w\in\mathscr{W}}\sum_{r\in\mathscr{R}^{w,2}}\left(c_r^{w,2,u}(\mathbf{x})+\frac{1}{\theta^{w,2,u}}\ln\widehat{f}_r^{w,2,u^*}\right)\left(\widehat{f}_r^{w,2,u}-\widehat{f}_r^{w,2,u^*}\right)+\left(c_r^{w,2,u}(\mathbf{x})+p^*+\frac{1}{\theta^{w,2,u}}\ln\check{f}_r^{w,2,u^*}\right)\left(\check{f}_r^{w,2,u}-\check{f}_r^{w,2,u^*}\right)+\sum_{u=2}\sum_{w\in\mathscr{W}}\sum_{(i,j)\in\mathscr{A}}\left(c_{ij}^u(\mathbf{x})+c_j^u\left(x_j^*\right)\delta_j^{\mathscr{N}_P}+p_i\delta_j^{\mathscr{D}}+\frac{1}{\varepsilon^w}\left(\ln\check{x}_{ij}^{w,3,u^*}-\ln\sum_{k\in\mathscr{N}_t(j)}\check{x}_{kj}^{w,3,u^*}\right)\right)\left(\check{x}_{ij}^{w,3,u}-\check{x}_{ij}^{w,3,u^*}\right)+\sum_{u\in\mathscr{U}}\sum_{m=1}^{2}\sum_{w\in\mathscr{W}}\left(\frac{1}{\theta^{w,u}}-\frac{1}{\theta^{w,m,u}}\right)\left(\ln\widehat{q}^{w,m,u^*}\left(\widehat{q}^{w,m,u}-\widehat{q}^{w,m,u^*}\right)+\ln\check{q}^{w,m,u^*}\left(\check{q}^{w,m,u}-\check{q}^{w,m,u^*}\right)\right)+\sum_{u=2}\sum_{w\in\mathscr{W}}\frac{1}{\theta^{w,u}}\ln\check{q}^{w,3,u^*}\left(\check{q}^{w,3,u}-\check{q}^{w,3,u^*}\right)+\sum_{n\in\mathscr{N}_B}\left(N_n^B-x_n^*\right)(p-p^*)\geq 0,$$

$$\forall\left(\mathbf{f}^U,\mathbf{x}^U,\mathbf{q}^U\right)\in\Theta\left(\mathbf{Q}^U\right), \forall p\in\mathbb{R}^+ \tag{51}$$

where $\mathbf{f}^U=\left(\cdots,\widehat{f}_r^{w,m,u},\cdots,\check{f}_r^{w,m,u},\cdots\right)$, $\mathbf{x}^U=\left(\cdots,\check{x}_{ij}^{w,3,u},\cdots\right)$ and $\mathbf{q}^U=\left(\cdots,\widehat{q}^{w,m,u},\cdots,\check{q}^{w,m,u},\cdots\right)$ denote the vectors of route flows, link flows, and demands of modes for user groups, respectively. $\mathbf{f}^{U^*}=\left(\cdots,\widehat{f}_r^{w,m,u^*},\cdots,\check{f}_r^{w,m,u^*},\cdots\right)$, $\mathbf{x}^{U^*}=\left(\cdots,\check{x}_{ij}^{w,3,u^*},\cdots\right)$, $\mathbf{q}^{U^*}=\left(\cdots,\widehat{q}^{w,m,u^*},\cdots,\check{q}^{w,m,u^*},\cdots\right)$ and p^* denote the vectors of route flows, link flows, demands of modes for user groups and average trade price of permits in the equilibrium condition, respectively. $\mathbf{Q}^U=\left(\cdots,\widehat{Q}^{w,u},\cdots,\check{Q}^{w,u},\cdots\right)$ denotes the vector of number of users in groups, representing the assignment of the permits to OD pairs in user groups. The feasible domain $\Theta\left(\mathbf{Q}^U\right)$ is constituted with constraints (5)-(15), (27) and (28). It should be noted that $\mathbf{Q}^U$ is associated with N^{TPP} and thus, the number of parking permits will influence the choice of travel modes and routes.

Proposition 1 can be proved based on Lemma 1 whose detail is presented in the Appendix. Since the feasible domain $\Theta\left(\mathbf{Q}^U\right)$ is compact and convex, and all functions are continuous on $\Theta\left(\mathbf{Q}^U\right)$, the VI problem VI-TPPSP admits at least one solution. The solution to the VI problem is denoted as $\mathrm{SOLVI}\left(\mathbf{F}\left(\mathbf{f}^U,\mathbf{x}^U,\mathbf{q}^U,p\right),\Theta\left(\mathbf{Q}^U\left(N^{TPP}\right)\right)\right)$. Since the search time in public parking facilities around NPA, $c_j^u(x_j)$ is not strictly monotone, the solution of VI-TPPSP is not unique. There are different solvers for the above VI problem. For instance, the solution to the VI problem can be obtained by solving the nonlinear program (Aghassi et al., 2006).

5.2. Equity measure

One of our focuses is to consider equity as one of the objective in the TPPSP policy. Various methods of measuring equity are proposed in the literature such as logarithmic variance, Theil's entropy and Kolm measure (Litman 2002, Levinson 2010). The Gini coefficient is one of the most widely used measures to quantify the equity in resource (i.e., income, natural resource and pollution allowance, etc.) allocation problem among different groups (Alvaredo, 2011; Bowles and Carlin, 2020; Dai et al., 2018; Paleti et al., 2016; Zhang et al., 2018b; Chen et al., 2020; Ma et al., 2020; He et al., 2022; Sarkar, 2023). It is an appealing and prevalent equity measure since it provides a single measure based on the entire distribution and allows easy comparisons across groups (Sarkar, 2023), which makes it stand out among other metrics. Hence, the Gini coefficient is utilized as the equity measure among different users in the TPPSP scheme. **It should be noted that the equity of both AP and NP internal beneficiaries (but not external users) is considered in the formulation of the Gini coefficient**, as the stakeholders of shared parking permits are internal beneficiaries who have the rights to use the NPAs, even though it may lead to benefits for external users as well. Due to market behavior in reality, the operators of NPAs mainly consider the benefits of internal beneficiaries rather than external users.

Let $\widehat{E}^{w,u}\left(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U\right)$ and $\check{E}^{w,u}\left(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U\right)$ denote the benefits for AP and NP internal beneficiaries in group u between OD pair w under the assignment scheme for TPPSP $\mathbf{Q}^U=\left(\cdots,\widehat{Q}^{w,u},\cdots,\check{Q}^{w,u},\cdots\right)$, respectively, represented as the difference in the generalized costs (including travel time and monetary costs) before and after the TPPSP introduction for all users of OD pair w for group u. $\widehat{E}^{w,u}\left(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U\right)$ and $\check{E}^{w,u}\left(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U\right)$ are given by Eq. (52a) and Eq. (52b), respectively.

$$\widehat{E}^{w,u}\left(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U\right)=\widehat{e}^{w,u}\left(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U\right)-\widehat{e}^{w,u}\left(\mathbf{f}^U,\mathbf{x}^U,p;0\right), \forall u\in\mathscr{U}, \forall w\in\mathscr{W} \tag{52a}$$

$$\check{E}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)=\check{e}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)-\check{e}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;0),\ \forall u\in\mathscr{U},\ \forall w\in\mathscr{W} \tag{52b}$$

where $\widehat{e}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;0)$ and $\check{e}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;0)$ denote the travel costs (including travel time and monetary costs) for AP and NP internal beneficiaries in group u between OD pair w before the introduction of TPPSP policy, respectively. $\widehat{e}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)$ and $\check{e}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)$ denote the travel costs after the introduction of the TPPSP policy with the assignment scheme $\mathbf{Q}^U$, respectively. $\widehat{e}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)$ and $\check{e}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)$ are given by Eq.(53a) and Eq.(53b), respectively.

$$\widehat{e}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)=\sum_{m=1,2}\sum_{r}\left(c_r^{w,m,u}-(1-\delta_{m,2})p(\mathbf{Q}^U)\right)\widehat{f}_r^{w,m,u}(\mathbf{Q}^U),\ \forall u\in\mathscr{U},\ \forall w\in\mathscr{W} \tag{53a}$$

$$\check{e}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)=\sum_{m=1,2}\sum_{r}\left(c_r^{w,m,u}+\delta_{m,2}p(\mathbf{Q}^U)\right)\check{f}_r^{w,m,u}(\mathbf{Q}^U)+\sum_{(i,j)\in\mathscr{A}}\left(c_{ij}^u(\mathbf{x})+c_j^u\left(x_j(\mathbf{Q}^U)\right)\delta_j^{\mathscr{N}_P}+p_i\delta_j^{\mathscr{D}}\right)\check{x}_{ij}^{w,3,u}(\mathbf{Q}^U),\ \forall u\in\mathscr{U},\ \forall w\in\mathscr{W} \tag{53b}$$

where the flows $\widehat{f}_r^{w,m,u}(\mathbf{Q}^U)$, $\check{f}_r^{w,m,u}(\mathbf{Q}^U)$, $\check{x}_{ij}^{w,3,u}(\mathbf{Q}^U)$, $x_j(\mathbf{Q}^U)$ and the trade price $p(\mathbf{Q}^U)$ are derived from $\mathrm{SOLVI}\left(\boldsymbol{F}\left(\mathbf{f}^U,\mathbf{x}^U,\mathbf{q}^U,p\right),\Theta(\mathbf{Q}^U)\right)$. The Gini coefficient $G\left(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U\right)$ is represented by Eq. (54).

$$\begin{aligned}G(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)=&\frac{1}{2\left(\sum_u\sum_w Q^{w,u}\right)^2\bullet\overline{E(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)}}\sum_{u_1,u_2}\sum_{w_1,w_2}\widehat{Q}^{w_1,u_1}\bullet\widehat{Q}^{w_2,u_2}\bullet\left|\widehat{E}^{w_1,u_1}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)-\widehat{E}^{w_2,u_2}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)\right|+\check{Q}^{w_1,u_1}\\&\bullet\check{Q}^{w_2,u_2}\bullet\left|\check{E}^{w_1,u_1}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)-\check{E}^{w_2,u_2}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)\right|+\widehat{Q}^{w_1,u_1}\bullet\check{Q}^{w_2,u_2}\bullet\left|\widehat{E}^{w_1,u_1}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)-\check{E}^{w_2,u_2}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)\right|\end{aligned} \tag{54}$$

where $\overline{E(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)}$ denotes the expected benefits for all users and we have

$$\overline{E(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)}=\frac{\sum_u\sum_w\widehat{E}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)+\check{E}^{w,u}(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)}{\sum_u\sum_w Q^{w,u}} \tag{55}$$

For the Gini coefficient, the smaller value of $G(\mathbf{f}^U,\mathbf{x}^U,p;\mathbf{Q}^U)$ represents higher equity of the TPPSP policy.

5.3. Mathematical programming with equilibrium constraints for permit assignment

To promote equity by optimizing the assignment of parking permits, we formulate a multimodal equilibrium condition of travel modes and routes as a constraint. The optimization problem is formulated as Mathematical programming with equilibrium constraints (MPEC) with integer variables, denoted as MPEC-GN:

[MPEC-GN]

$$\underset{\widehat{Q}^{w,u}}{\mathrm{Min}}\,GN(\mathbf{f}^{U^*},\mathbf{x}^{U^*},p^*;\cdots,\widehat{Q}^{w,u},\cdots,N^{TPP}) \tag{56}$$

subject to

$$\widehat{Q}^{w,u}\in\mathbb{Z}^+ \tag{57}$$

$$0\le\widehat{Q}^{w,u}\le Q^{w,u},\ \forall w\in\mathscr{W},\ \forall u\in\mathscr{U} \tag{58}$$

$$(\mathbf{f}^{U^*},\mathbf{x}^{U^*},p^*)\in\mathrm{SOLVI}\left(\mathbf{F}(\mathbf{f}^U,\mathbf{x}^U,\mathbf{q}^U,p),\Theta(\mathbf{Q}^U(N^{TPP}))\right) \tag{59}$$

In Eq.(1), the decision variables $\mathbf{Q}^U=\left(\cdots,\widehat{Q}^{w,u},\cdots,\check{Q}^{w,u},\cdots\right)$ can be reformulated as $\mathbf{Q}^U=(\cdots,\widehat{Q}^{w,u},\cdots,Q^{w,u}-\widehat{Q}^{w,u},\cdots)$ to reduce the number of variables by half.

6. Solution algorithm

Due to the non-uniqueness of the solution of VI-TPPSP, it is hard to obtain the exact solution for the integer (global) optimization problem MPEC-GN. The original model is computationally expensive. There exists a variety of algorithms that can make an approximation to the solution. We utilize the metamodel (or surrogate) method to solve the MPEC-GN on account of the determinacy of the metamodel response and its computation efficiency. Different metamodel methods such as the radial basis function, regression spline method, the spatial correlation (i.e., the Kriging) method and the neural network method, could be utilized (Yu et al., 2022; Yu et al., 2023a; Yu et al., 2023b). We select Kriging metamodel for this study as it has higher flexibility in fitting arbitrary smooth response functions and is more robust to small changes (Cheng et al., 2019). The integer (global) optimization problem is reformulated as:

$$\mathrm{Min}G\left(\mathbf{Q}^{U}\right) \tag{60}$$

Subject to (58)

where $G\left(\mathbf{Q}^{U}\right)$ is the objective function in Eq. (56). The approximation method of Kriging metamodel to $G\left(\mathbf{Q}^{U}\right)$ and the framework of Kriging metamodel algorithm is explicated in previous studies (Xia et al., 2018; Yu et al., 2021). Kriging metamodel algorithm (KMA) for MPEC-GN is described as follows.

[KMA-MPEC-GN].

Step 0: Initialization. Given n_s feasible solution set $\left\{\cdots,\mathbf{Q}^{U[i]},\cdots\right\}$ generated by a symmetric Latin hypercube design (Ye et al. 2000). n_s is the original sample size of Ω_x.

Step 1: Obtain the best points in the feasible domain $\Theta\left(\mathbf{Q}^{U}\right)$. Compute the values of $G\left(\mathbf{Q}^{U[i]}\right)$ for every point $\mathbf{Q}^{U[i]}$ in $\Theta\left(\mathbf{Q}^{U}\right)$, $i=1,\cdots,n_s$ and find the best feasible points $\mathbf{Q}^{U}_{\min}=\mathrm{argmin}_{i=1,\cdots,n_s}G\left(\mathbf{Q}^{U[i]}\right)$.

Step 2: Calculate the parameters for the Kriging metamodel. Find the parameters $\tilde{\rho}$ and $\tilde{\lambda}^{w,u}$, via the data $\left(\mathbf{Q}^{U[i]},G\left(\mathbf{Q}^{U[i]}\right)\right)$, $i=1,\cdots,n_s$ and a divide-and-conquer algorithm parameter estimation (PE).

Step 3: Generate the candidate points for the new sample. Uniformly select points and perturb the best point found to produce candidates, which are denoted as $\mathscr{Q}^{U[t]}\in\Theta_{\mathscr{Q}}$, $t=1,\cdots,T$, where $\Theta_{\mathscr{Q}}$ denotes the new sample.

Step 4: Search for the best candidate in $\Theta_{\mathscr{Q}}$.

Step 4–1: Estimate the objective function. Through Eq. (61) and the parameters for the Kriging metamodel, the response surface $\tilde{G}\left(\mathscr{Q}^{U[t]};\tilde{\rho},\cdots,\tilde{\lambda}^{w,u},\cdots\right)$ for every candidate $\mathscr{Q}^{U[t]}\in\Theta_{\mathscr{Q}}$ is computed.

$\tilde{G}\left(\mathscr{Q}^{U[t]}\right)=\tilde{\rho}+\tilde{\mathbf{R}}^{T}\left(\mathscr{Q}^{U[t]}\right)\tilde{\Psi}^{-1}(\mathbf{G}-1\tilde{\rho})$(61).

where $\mathbf{G}=\left(G\left(\mathbf{Q}^{U[1]}\right),G\left(\mathbf{Q}^{U[2]}\right),\cdots,G\left(\mathbf{Q}^{U[n_s]}\right)\right)^{T}$, 1 is the $n_s\times 1$ unit column and

$$\tilde{\Psi}=\begin{bmatrix}\tilde{\psi}\left(\mathbf{Q}^{U[1]},\mathbf{Q}^{U[1]}\right) & \tilde{\psi}\left(\mathbf{Q}^{U[1]},\mathbf{Q}^{U[2]}\right) & \cdots & \tilde{\psi}\left(\mathbf{Q}^{U[1]},\mathbf{Q}^{U[n_s]}\right)\\ \tilde{\psi}\left(\mathbf{Q}^{U[2]},\mathbf{Q}^{U[1]}\right) & \tilde{\psi}\left(\mathbf{Q}^{U[2]},\mathbf{Q}^{U[2]}\right) & \cdots & \tilde{\psi}\left(\mathbf{Q}^{U[2]},\mathbf{Q}^{U[n_s]}\right)\\ \vdots & \vdots & \ddots & \vdots\\ \tilde{\psi}\left(\mathbf{Q}^{U[n_s]},\mathbf{Q}^{U[1]}\right) & \tilde{\psi}\left(\mathbf{Q}^{U[n_s]},\mathbf{Q}^{U[2]}\right) & \cdots & \tilde{\psi}\left(\mathbf{Q}^{U[n_s]},\mathbf{Q}^{U[n_s]}\right)\end{bmatrix} \tag{62}$$

$$\tilde{\mathbf{R}}\left(\mathscr{Q}^{U[t]}\right)=\left(\tilde{\psi}\left(\mathscr{Q}^{U[t]},\mathbf{Q}^{U[1]}\right),\tilde{\psi}\left(\mathscr{Q}^{U[t]},\mathbf{Q}^{U[2]}\right),\cdots,\tilde{\psi}\left(\mathscr{Q}^{U[t]},\mathbf{Q}^{U[n_s]}\right)\right)^{T} \tag{63}$$

$$\tilde{\psi}\left(\mathbf{Q}^{U[i]},\mathbf{Q}^{U[j]}\right)=\exp\left[-\sum_{w\in\mathscr{W}}\sum_{u\in\mathscr{U}}\tilde{\lambda}^{w,u}\left(\hat{Q}^{w,u[i]}-\hat{Q}^{w,u[j]}\right)^{2}\right] \tag{64}$$

Step 4–2: Evaluate a weighted score. Calculate the response surface criterion $V_R\left(\mathscr{Q}^{U[t]}\right)$ and distance criterion $V_D\left(\mathscr{Q}^{U[t]}\right)$ through Eq. (65) and (66), respectively. Get the weighted score $W(t)$ for every candidate $\mathscr{Q}^{U[t]}\in\Theta_{\mathscr{Q}}$ in Eq. (67) (Regis and Shoemaker, 2007).

$$V_R\left(\mathscr{Q}^{U[t]}\right)=\begin{cases}\dfrac{\tilde{G}\left(\mathscr{Q}^{U[t]}\right)-\min\limits_{t=1,\cdots,T}\tilde{G}\left(\mathscr{Q}^{U[t]}\right)}{\max\limits_{t=1,\cdots,T}\tilde{G}\left(\mathscr{Q}^{U[t]}\right)-\min\limits_{t=1,\cdots,T}\tilde{G}\left(\mathscr{Q}^{U[t]}\right)} & \text{if }\max\limits_{t=1,\cdots,T}\tilde{G}\left(\mathscr{Q}^{U[t]}\right)\neq\min\limits_{t=1,\cdots,T}\tilde{G}\left(\mathscr{Q}^{U[t]}\right)\\ 1 & \text{otherwise}\end{cases} \tag{65}$$

$$V_D\left(\mathscr{Q}^{U[t]}\right)=\begin{cases}\dfrac{\max\limits_{t=1,\cdots,T}\Delta\left(\mathscr{Q}^{U[t]}\right)-\Delta\left(\mathscr{Q}^{U[t]}\right)}{\max\limits_{t=1,\cdots,T}\Delta\left(\mathscr{Q}^{U[t]}\right)-\min\limits_{t=1,\cdots,T}\Delta\left(\mathscr{Q}^{U[t]}\right)} & \text{if }\max\limits_{t=1,\cdots,T}\Delta\left(\mathscr{Q}^{U[t]}\right)\neq\min\limits_{t=1,\cdots,T}\Delta\left(\mathscr{Q}^{U[t]}\right)\\ 1 & \text{otherwise}\end{cases} \tag{66}$$

$$W(t)=w_R V_R\left(\chi^{[t]}\right)+w_D V_D\left(\chi^{[t]}\right) \tag{67}$$

where $w_R+w_D=1$, $w_R\geq 0$ and $w_D\geq 0$. $\Delta\left(\mathscr{Q}^{U[t]}\right)=\min\limits_{i=1,\cdots,n_s}\left\|\mathscr{Q}^{U[t]}-\mathbf{Q}^{U[i]}\right\|$

Step 4–3: Select the best candidates in the new sample. Search for T_0 candidates with the lowest W, $\mathscr{Q}^{U[t_0]}\in\Theta_{\mathscr{Q}_0}\subseteq\Theta_{\mathscr{Q}}$, $t_0=n_s+1,\cdots,n_s+T_0$.

Step 5: Update the best feasible points in the sample consisting of the best candidates $\Theta_{\mathscr{Q}_0}$.

Step 5–1: Compute the value of $G\left(\mathscr{Q}^{U[t_0]}\right)$ for every best candidate $\mathscr{Q}^{U[t_0]}$ of $\Theta_{\mathscr{Q}_0}$ in parallel, $t_0=n_s+1,\cdots,n_s+T_0$. Update the best feasible points, $\mathbf{Q}^{U}_{\min}=\min\left\{\mathbf{Q}^{U}_{\min},\mathrm{argmin}_{t_0=n_s+1,\cdots,n_s+T_0}G\left(\mathscr{Q}^{U[t_0]}\right)\right\}$;

Step 5–2: expand the sample $\Theta\left(\mathbf{Q}^{U}\right)=\Theta\left(\mathbf{Q}^{U}\right)\cup\Theta_{\mathscr{Q}_0}$ and update the sample size $n_s=n_s+T_0$.

Step 6: criterion of terminating iteration. If the sample size exceeds the maximum allowed number of function evaluations $n_{s,\max}$, i.e., $n_s > n_{s,\max}$, iteration terminates and outputs the optimal parking permit assignment scheme $\mathbf{Q}^{U*} = \mathbf{Q}^{U}_{\min}$ and the smallest Gini coefficient $G(\mathbf{Q}^{U*})$; return to **Step 2** otherwise.

The divide-and-conquer algorithm PE is presented as follows (Kleijnen, 2015; Cheng et al., 2019).

[PE].

Step 0: Parameter initialization for $\widetilde{\lambda}^{w,u}$, and compute $\widetilde{\Psi}$ defined in Eq. (62).

Step 1: Determine the value of $\widetilde{\rho}$ according to Eq. (68).

$$\widetilde{\rho} = \left(1^T \widetilde{\Psi}^{-1} 1\right)^{-1} \left(1^T \widetilde{\Psi}^{-1} \mathbf{G}\right) \tag{68}$$

Step 2: Substitute $\widetilde{\Psi}$ from **Step 0** and $\widetilde{\rho}$ obtained in **Step 1** into Eq. (69). Solve Eq. (69) and update $\widetilde{\lambda}^{w,u}$.

$$\widetilde{\lambda}^{w,u} = \underset{\widetilde{\lambda}^{w,u}>0}{\operatorname{argmax}} -\frac{1}{2}\left\{ n_s \ln\left[\frac{1}{n_s}(\mathbf{G} - 1\widetilde{\rho})^T \widetilde{\Psi}^{-1} (\mathbf{G} - 1\widetilde{\rho})\right] + \ln(\det\widetilde{\Psi})\right\} \tag{69}$$

where det represents the determinant of matrix.

Step 3: Update $\widetilde{\Psi}$ with the new $\widetilde{\lambda}^{w,u}$, and replace the updated $\widetilde{\Psi}$ into Eq. (68).

Step 4: If the convergence criterion does not reach, return to **Step 1**; otherwise, output the estimators of $\widetilde{\rho}$ and $\widetilde{\lambda}^{w,u}$.

7. Numerical examples

We conduct a numerical case study to test the performance of our proposed approaches. Particularly, we compare the current situation without a shared parking strategy, shared parking policies of paying for booked parking (SPBP) and the proposed TPPSP. The current situation without a shared parking strategy is the case when NPAs do not share parking facilities with external users. SPBP is a price-based control approach for shared parking with external users. The SPBP is a pretty good price-based control approach for shared parking and has been regarded as the comparison benchmark in relevant studies (Hepburn, 2006; Wang and Yang, 2012; Wu et al., 2012; Wang and Zhang, 2016; Li and Robusté, 2021). Therefore, we also use SPBP as the comparison model herein. Except for the equity measure Gini coefficient, the social welfare for shared parking policies is evaluated as well. Social welfare is quantified by Eq.

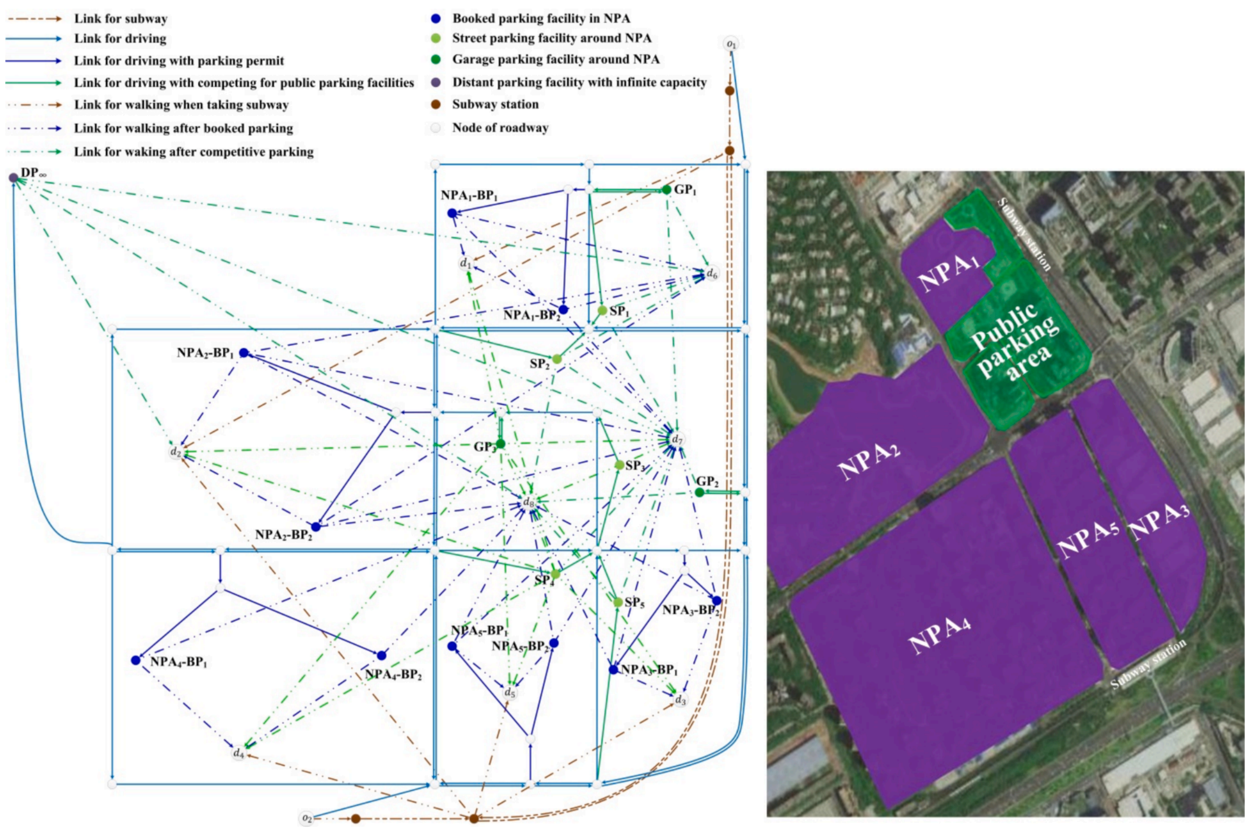

Fig. 2. Multimodal network in the studied urban area.

(70) and means the overall accumulated generalized travel costs of all vehicles. Taking Eq. (70) as the objective function, an MPEC problem MPEC-SW is formulated with the constraints (57)-(59).

$$SW(\mathbf{f}^U, \mathbf{x}^U, p; \mathbf{Q}^U) = \sum_u \sum_w Q^{w,u} E^{w,u}(\mathbf{f}^U, \mathbf{x}^U, p; \mathbf{Q}^U) \tag{70}$$

Then the optimal initial assignment schemes for TPPSP are obtained where the Gini coefficient is minimized, and the social welfare is maximized.

The used multimodal transportation network is created based on a real urban area near Auto Expo Center, Yubei District, Chongqing, China, as presented in Fig. 2. In our case study, we merely consider the subway as the representative of public transit for the sake of complexity. However, our proposed framework is absolutely scalable and applicable to any scale and type of multimodal transportation network. In the case study, there are five NPAs and three workplaces outside the NPAs. In each NPA, there are two parking facilities for booked parking, denoted by NPA_α-BP_γ where α and γ represent the NPA number and parking facility number, respectively. In the area outside the NPAs, there are five street parking facilities (denoted by SP_γ) and three garage parking facilities (denoted by GP_γ). The area can provide 3000 parking spaces in the NPAs and 1500 parking spaces in the area outside the NPAs. The number of parking spaces in each parking facility is presented in Table 1. It should be noted there is a distant parking facility $\mathrm{DP}_\infty \in \mathcal{N}_P$ with infinite capacity far away from the studied area. This facility is used to satisfy the supply constraint in Eq. (28) when the demand of external users exceeds the supply of public parking facilities in the studied urban area if the parking spaces in NPAs are insufficiently shared. When external users fail to find a space to park after a long cruise in Chongqing, they are more inclined to park at the roadside lanes or sidewalks, which will be fined by the traffic police. To describe the long-time cruising and illegal parking fines (Morillo and Campos, 2014; Nourinejad et al., 2020) in the travel cost equivalently, we set the distance between DP_∞ and the studied area to be far enough.

The VOTs for different user groups with different car ownerships are $\beta^1 = 15¥/\mathrm{h}$ (Group (1), and $\beta^2 = 20¥/\mathrm{h}$ (Group (2), where ¥ is the currency unit (CNY) in China. In the used travel scenario, the travel time of commuting by car is generally shorter than the travel time using public transit. Therefore, people with a higher value of time generally prefer to use the car for commuting. This means that the travelers in Group (2) represent the travelers with higher values of time and show more preference for using the car for commuting, and travelers in Group (1) represent the travelers with lower values of time and are captive to public transit in the urban contexts of cities. In the area outside the NPAs, the parking fee is charged by $p_n = 20¥/\mathrm{h}$ for all public parking facilities $n \in \mathcal{N}_P$. In the SPBP scheme, the prices of booking a space in NPAs are the same as the parking fee outside the NPAs, i.e. $p_n = 20¥/\mathrm{h}$ for all public parking facilities $n \in \mathcal{N}_B$. There are 3500 internal beneficiaries with 10 OD pairs in NPAs and 2500 external users with 6 OD pairs. Among these 3500 internal beneficiaries, 2000 belong to Group (2) ($\beta^2 = 20¥/\mathrm{h}$) and 1500 belong to Group (1) ($\beta^1 = 15¥/\mathrm{h}$). Table 2 presents the demands of users between OD pairs.

In price-based control SPBP, all internal beneficiaries who prefer driving (in total 2000) to commute are directly granted the right to have a parking spot in NPAs for free. External users compete and pay for 600 shared parking spaces in NPAs. For the TPPSP that shares the parking facilities in NPAs, there are 2600 parking permits assigned to 3500 internal beneficiaries. This assignment scheme is an initial one before optimization. In NPAs, an AP internal beneficiary of Group (1) can sell the permit and get monetary rewards. An AP internal beneficiary of Group (2) can book a parking space via the assigned permit, while an NP internal beneficiary of Group (2) can take the subway to commute, purchase the permit from other internal beneficiaries or compete for public parking spaces outside the NPA. All external users who drive to work are NP users, so they either purchase the permit from AP internal beneficiaries or compete for public parking spaces outside the NPA. We assume that external users in the parking competition have two levels of rationality based on their origins. The set of rationality levels is denoted by $E = \{\varepsilon^w | \varepsilon^w = 0.1, o = o_1; \varepsilon^w = 1, o = o_2\}$.

One important measure reflecting the parking condition is the occupancy of parking facilities. The occupancies of parking facilities inside and outside the NPAs are presented in Eqs.(71) and (72), respectively.

$$OC_n = \frac{x_n}{N_n^B}, \forall n \in \mathcal{N}_B \tag{71}$$

$$OC_n = \frac{\sum_{d \in \mathcal{D}} x_{nd}}{N_n^P}, \forall n \in \mathcal{N}_P \tag{72}$$

Please note that parked cars in the distant parking garage DP_∞ are counted as illegally parked cars at the roadside lanes or sidewalks. Hence, the number of these cars will be added to the numerator of Eq. (71) for street parking facilities. Another important measure reflecting the traffic condition of the road network is the link flow-capacity (V/C) ratio. The V/C ratios on the road links with

Table 1
Parking facilities in the studied urban area.

	NPA_1		NPA_2		NPA_3		NPA_4		NPA_5		Sum
Parking facility	BP_1	BP_2	BP_1	BP_2	BP_1	BP_2	BP_1	BP_2	BP_1	BP_2	
Number of parking spaces	100	190	860	580	220	80	300	400	160	110	3000
	Area outside NPAs										Sum
Parking facility	SP_1	SP_2	SP_3	SP_4	SP_5	GP_1	GP_2	GP_3			
Number of parking spaces	50	70	40	60	80	400	500	300			1500

Table 2
Demands of users between OD pairs and TPPSP assignment scheme.

OD pair		Users without cars-Group (1) $Q^{w,1}$			Users with cars-Group (2) $Q^{w,2}$			Sum
		AP user $\hat{Q}^{w,1}$	NP user $\check{Q}^{w,1}$	Sum	AP user $\hat{Q}^{w,2}$	NP user $\check{Q}^{w,2}$	Sum	
NPA_1	o_1d_1	95	45	140	110	50	160	300
	o_2d_1	95	45	140	110	50	160	300
NPA_2	o_1d_2	210	40	250	340	20	360	610
	o_2d_2	210	40	250	340	20	360	610
NPA_3	o_1d_3	45	65	110	140	20	160	270
	o_2d_3	45	65	110	140	20	160	270
NPA_4	o_1d_4	90	10	100	120	40	160	260
	o_2d_4	90	10	100	120	40	160	260
NPA_5	o_1d_5	40	110	150	110	50	160	310
	o_2d_5	40	110	150	110	50	160	310
WP_1	o_1d_6	/	/	/	/	250	250	250
	o_2d_6	/	/	/	/	250	250	250
WP_2	o_1d_7	/	/	/	/	550	550	550
	o_2d_7	/	/	/	/	550	550	550
WP_3	o_1d_8	/	/	/	/	450	450	450
	o_2d_8	/	/	/	/	450	450	450
SUM		960	540	1500	1640	2860	4500	6000

Note: WP denotes workplace.

and without street parking facilities are presented in Eqs. (73) and (74), respectively.

$$VC_{ij} = \frac{x_{ij} + x_{ik}}{S_{ij}}, \forall (i,j) \in \mathscr{A}_D, k \in \mathrm{SP}_k \tag{73}$$

$$VC_{ij} = \frac{x_{ij}}{S_{ij}}, \forall (i,j) \in \mathscr{A}_D \tag{74}$$

where S_{ij} denotes the road link capacity. Eq. (73) represents that the link flows consist of both flows traveling through the link and flows searching the street parking spaces.

7.1. Computational performance of the KMA

KMA is applied to solve the problems MPEC-GN and MPEC-SW. To show the computational performance of KMA, we take one of the typical heuristic algorithms, the genetic algorithm (GA) and one of the typical metamodel methods, the radial basis function model algorithm (RBFMA; Du et al., 2019) as comparisons. In the experiment, 600 function evaluations are allowed for both KMA and RBFMA, with 20 initial points and 20 new sample points in each iteration. For GA, 600 function evaluations are made, with 20 initial points, 19 generations for iteration, 20 populations, 80 % crossover, 35 % Pareto fraction and 2 individuals selected for elite. KMA,

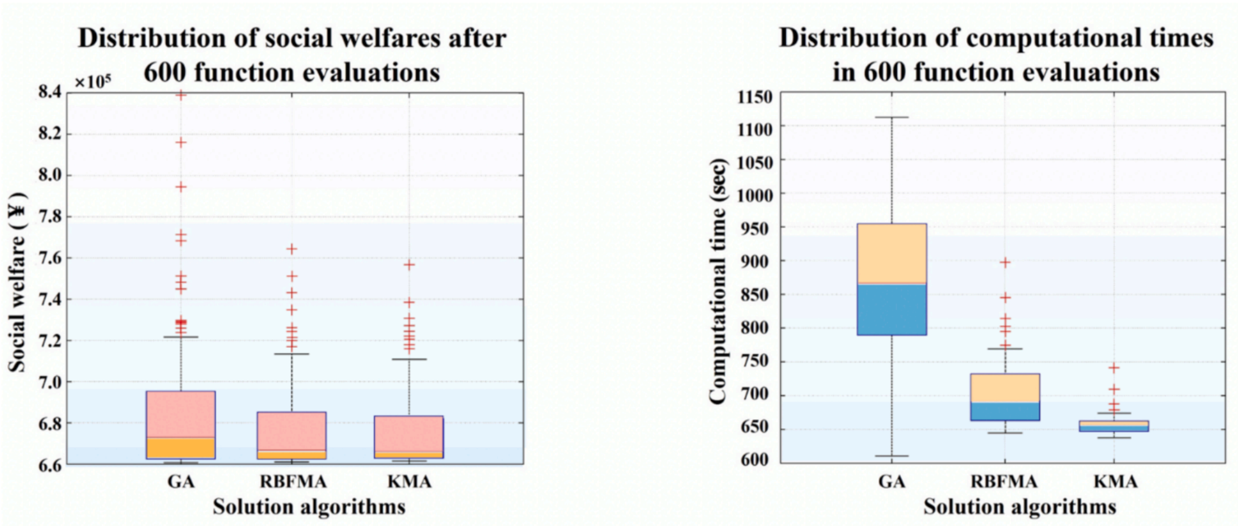

Fig. 3. Computational performances of KMA, RBFMA and GA.

RBFMA and GA share the same 20 initial points in one experiment. Therefore, we can compare KMA, RBFMA and GA within the 600 function evaluations. 100 parallel experiments are performed in MPEC-SW and the distributions of social welfare after 600 function evaluations and computational times within 600 function evaluations for KMA, RBFMA and GA are obtained, respectively, as demonstrated in Fig. 3. As the results present, both KMA and RBFMA give an impressive performance on the convergence and solution bounds. Additionally, KM is far more computationally inexpensive than both RBFMA and GA.

7.2. Comparison among no shared parking, SPBP, and TPPSP

In addition to metric such as parking occupancy and V/C ratio, this analysis also considers the number of parking spaces utilized by external users, the percentage of internal beneficiaries opting for subway travel, social welfare, the Gini coefficient, and parking fees under scenarios devoid of shared parking, SPBP, and TPPSP schemes. The findings, encapsulated in Table 3 and illustrated in Fig. 4, reveal that the provision of parking spaces within NPAs to external users significantly mitigates congestion and the excessive occupancy rates of street parking across the road network. The congestion and occupancy relief offered by both the SPBP and TPPSP schemes are comparable.

Moreover, the TPPSP scheme incentivizes a modal shift among internal beneficiaries from driving to subway usage. This shift occurs as the TPPSP scheme allocates permits to a subset of internal beneficiaries of Group (1), leaving some from Group (2) without initial permits. Those unassigned must resort to subway travel if unsuccessful in acquiring permits on the market. A nominal subset of Group (1) internal beneficiaries, originally allocated permits, opt to sell these for monetary gain, compensating for their subway travel time. Conversely, a minuscule fraction (0.2 %) of Group (2) internal beneficiaries attempt to secure public parking, deterred by the lingering congestion, the extended travel times associated with cyclic cruising for parking, and potential fines for illegal parking. Thus, TPPSP slightly outperforms SPBP in alleviating congestion due to its influence on travel mode choices.

Contrarily, the SPBP approach guarantees all Group (2) internal beneficiaries a non-transferable parking space within NPAs, leaving their travel mode preferences unchanged. The differential in travel mode choices under TPPSP, where some beneficiaries opt to sell their permit for subway use, causes the permit trade prices to reflect the equilibrium of generalized travel costs across modes, diverging from the fixed parking fees associated with the SPBP scheme.

The TPPSP and SPBP schemes both facilitate the utilization of parking spaces within NPAs to enhance social welfare. However, it is exclusively the TPPSP scheme that fosters equity among internal beneficiaries and enhances the welfare of both internal beneficiaries and external users (see Table 3). Without shared parking policies, social welfare is notably diminished, as external users are relegated to parking in less desirable locations such as roadside lanes or sidewalks, often resulting in fines after unsuccessful attempts to locate parking outside NPAs.

Upon implementing the SPBP scheme, social welfare for internal beneficiaries remains largely unaffected, given that SPBP does not induce a shift in travel modes among internal beneficiaries. Conversely, the TPPSP scheme offers comprehensive benefits across all internal beneficiary groups. Group (1) beneficiaries, initially allocated permits, can opt to share these permits in exchange for monetary compensation, offsetting the longer travel times associated with public transit usage. The introduction of TPPSP provides an opportunity for NPAs to allocate vacant parking spaces to external users, thereby generating financial returns. Furthermore, the TPPSP aims to redirect parking revenue from operators to travelers who choose alternative commuting methods, suggesting that income from external user payments should compensate internal beneficiaries, irrespective of car ownership status. This mechanism reduces the welfare disparity between car-owning and non-car-owning internal beneficiaries. In contrast, under the SPBP scheme, the financial benefits derived from selling parking permits to external users accrue to the operators rather than the internal beneficiaries, undermining equity. Consequently, compared to TPPSP, the SPBP scheme exhibits lower equity, underscoring TPPSP's superior capacity to balance welfare and equity among different internal beneficiaries.

7.3. TPPSP assignment optimization

The results of the optimal TPPSP assignment scheme are presented in Table 4 and 5. We find the optimal TPPSP assignment scheme

Table 3
Evaluation measures before and after the introduction of SPBP and TPPSP.

		CS	SPBP	TPPSP
Occupancy of street parking facilities outside the NPAs		2.388	1.070	1.062
Occupancy of parking garages outside the NPAs		0.947	1.067	1.071
Occupancy of parking facilities in NPAs		0.667	0.867	0.882
Number of parking spaces shared by external users in NPAs		0	600	647
The proportion of internal beneficiaries taking the subway		0.428	0.428	0.441
The proportion of internal beneficiaries with car ownership taking the subway		0	0	0.021
The proportion of internal beneficiaries with car ownership competing for public parking		0	0	0.002
Social welfare ($\times 10^5$¥)	External users	2.93	3.74	3.79
	Internal beneficiaries	2.70	2.67	3.09
	All users	5.63	6.41	7.08
Gini coefficient for internal beneficiaries		0.049	0.041	0.024
Parking fee: booking price in SPBP or average trade price of permits in the market equilibrium of TPPSP (¥/h)		/	20.00	11.96

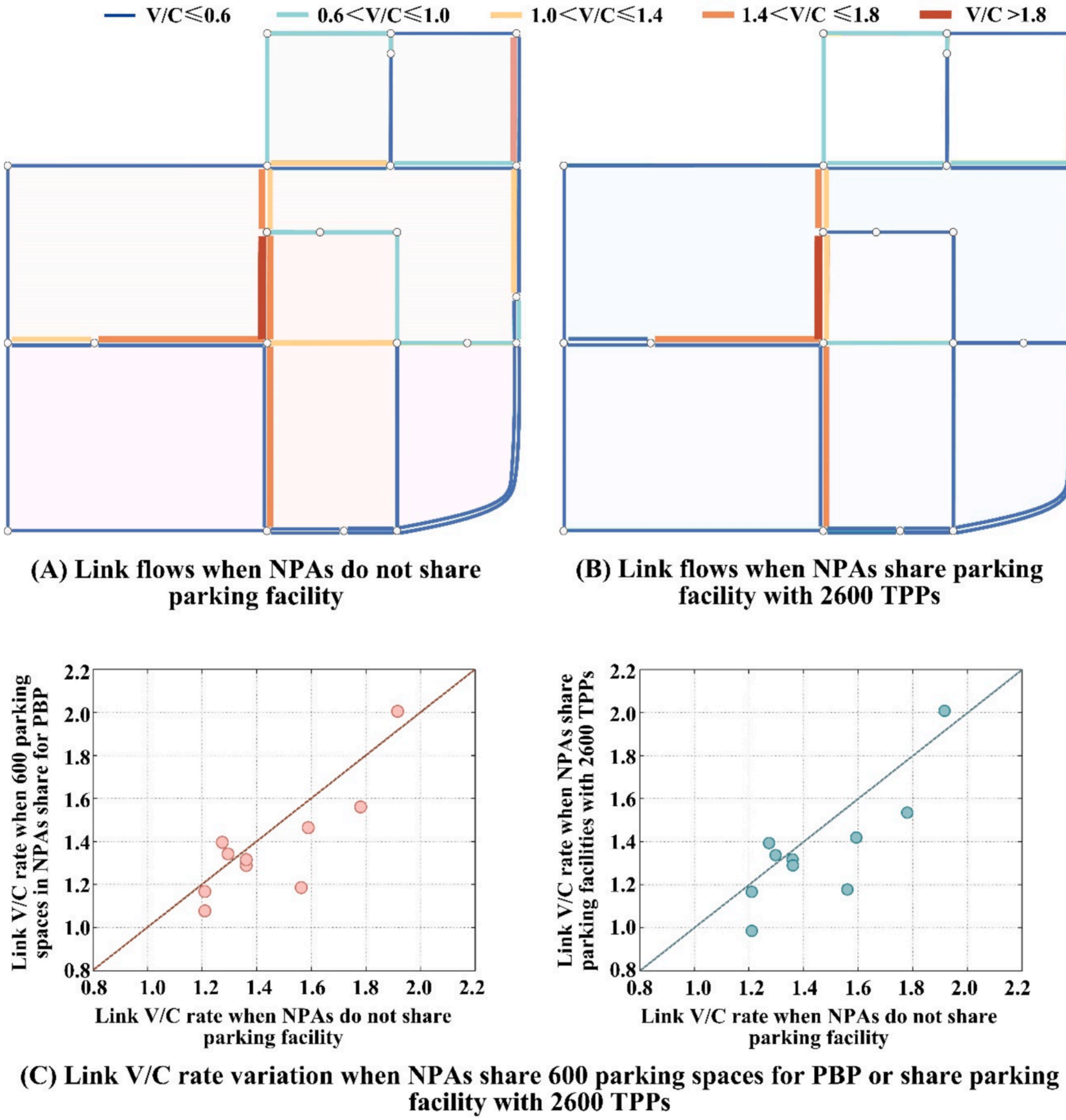

Fig. 4. Traffic flow conditions before and after the introduction of SPBP and TPPSP (PBP, pay for booked parking; TPP, tradable parking permit).

under a given number of parking permits N^{TPP}, namely 2600 in our case study. After solving MPEC-GN, the best Gini coefficient GN^* is 0.128 where the average trade price of permits in equilibrium market conditions is 13.50 ¥/h. After solving MPEC-SW, the best social welfare SW^* is ¥683578.4 where the average trade price of permits in equilibrium market conditions is 13.80 ¥/h and the corresponding results of TPPSP assignment scheme are presented in Table 5. Note that the feasible domain $\Theta(\mathbf{Q}^U)$ is associated with N^{TPP}. Therefore, we investigate the influences of N^{TPP} on GN^* or SW^*, whose results are presented in Fig. 5**(A)**. The TPPSP assignment schemes $\mathbf{Q}^{U*}$ and average trade price of permits in equilibrium market condition p^* under GN^* or SW^* are associated with N^{TPP}, respectively, as illustrated in Fig. 5 (B) and Fig. 6.

As illustrated in Fig. 5, an increase in the issuance of parking permits correlates with an enhancement in social welfare, particularly when the optimization objective encompasses maximizing social welfare. This improvement can be attributed to the escalation in reserved parking spaces, which concurrently diminishes the necessity for external users to either circulate in search of parking or resort to parking in remote facilities, denoted as DP_∞. Similarly, this increase reduces the incidence of Group (2) internal beneficiaries within NPAs needing to opt for subway transit upon their inability to secure a parking permit. Fig. 6 elucidates that the average trading price of parking permits within market equilibrium conditions diminishes as the quantity of available permits rises, under the premise of social welfare optimization. This dynamic emerges because, in scenarios where Group (2) internal beneficiaries are unsuccessful in

Table 4
Results of equity-based TPPSP assignment scheme with 2600 parking permits.

OD pair		Internal beneficiaries of Group (1) $Q^{w,1}$			Internal beneficiaries of Group (2) $Q^{w,2}$			Sum
		AP user $\widehat{Q}^{w,1}$	NP user $\check{Q}^{w,1}$	Sum	AP user $\widehat{Q}^{w,2}$	NP user $\check{Q}^{w,2}$	Sum	
NPA$_1$	o_1d_1	80	60	140	109	51	160	300
	o_2d_1	82	58	140	65	95	160	300
NPA$_2$	o_1d_2	248	2	250	305	55	360	610
	o_2d_2	234	16	250	341	19	360	610
NPA$_3$	o_1d_3	23	87	110	66	94	160	270
	o_2d_3	81	29	110	125	35	160	270
NPA$_4$	o_1d_4	47	53	100	146	14	160	260
	o_2d_4	67	33	100	119	41	160	260
NPA$_5$	o_1d_5	129	21	150	100	60	160	310
	o_2d_5	99	51	150	134	26	160	310
Sum		1090	410	1500	1510	490	2000	3500

Table 5
Results of social welfare-based TPPSP assignment scheme with 2600 parking permits.

OD pair		Internal beneficiaries of Group (1) $Q^{w,1}$			Internal beneficiaries of Group (2) $Q^{w,2}$			Sum
		AP user $\widehat{Q}^{w,1}$	NP user $\check{Q}^{w,1}$	Sum	AP user $\widehat{Q}^{w,2}$	NP user $\check{Q}^{w,2}$	Sum	
NPA$_1$	o_1d_1	29	111	140	156	4	160	300
	o_2d_1	89	51	140	143	17	160	300
NPA$_2$	o_1d_2	156	94	250	355	5	360	610
	o_2d_2	169	81	250	347	13	360	610
NPA$_3$	o_1d_3	48	62	110	138	22	160	270
	o_2d_3	58	52	110	143	17	160	270
NPA$_4$	o_1d_4	68	32	100	132	28	160	260
	o_2d_4	52	48	100	155	5	160	260
NPA$_5$	o_1d_5	20	130	150	83	77	160	310
	o_2d_5	116	34	150	143	17	160	310
Sum		805	695	1500	1795	205	2000	3500

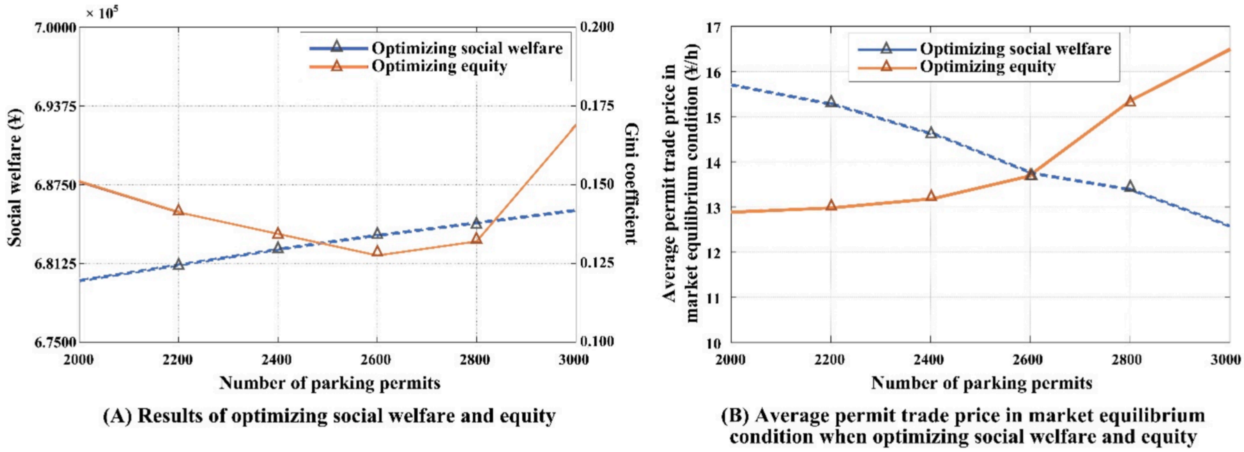

Fig. 5. Best equity and social welfare, and corresponding average permit trade price in market equilibrium with different numbers of parking permits.

acquiring a parking permit, their alternatives are limited to either subway usage or competition for public parking spots. Concurrently, external users confronted with the inability to secure a parking permit face the dilemma of either searching for alternative parking or targeting distant parking garages. Consequently, both NP internal beneficiaries and external users vie for the limited supply of transferable parking permits, catalyzing an elevation in permit trading prices. An increase in permit issuance mitigates the competition for parking permits by significantly reducing the fraction of NP internal beneficiaries without assigned permits, thus optimizing social welfare. Concurrently, this increment leads to a rise in the volume of permits traded to external users, attributable to the augmented contingent of Group (1) internal beneficiaries. As a result, the market witnesses an amplified supply of parking permits against a diminished demand, culminating in a reduction of trading prices under equilibrium when social welfare optimization is the focal point.

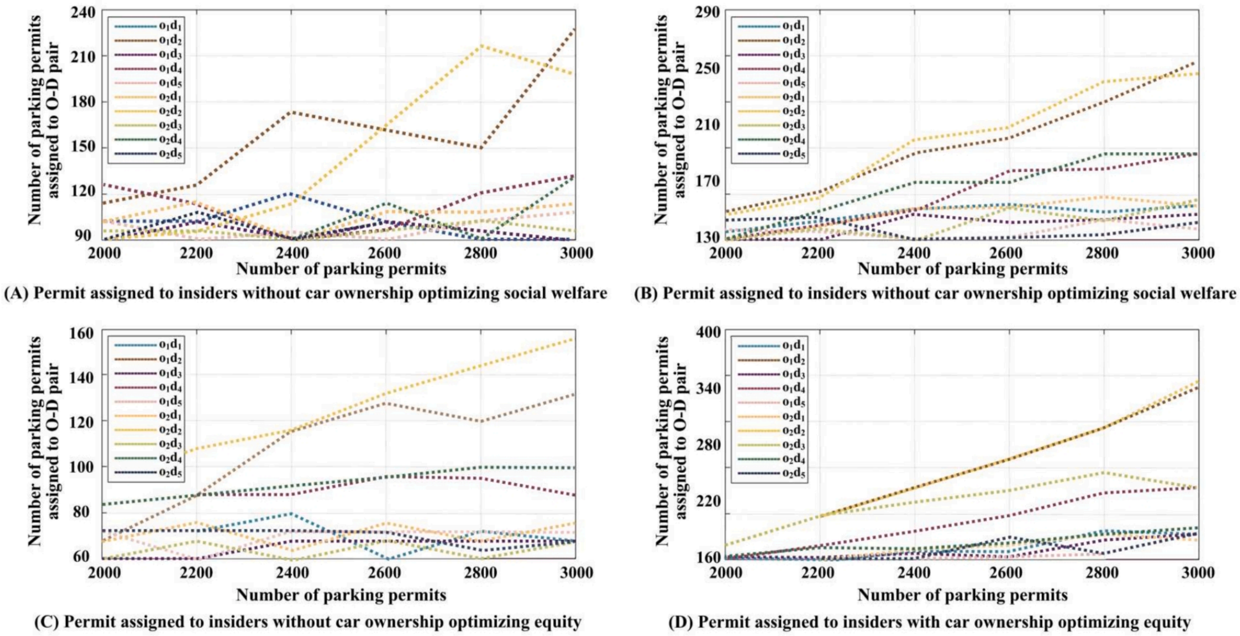

Fig. 6. Permit assignment under best equity and social welfare with different numbers of parking permits.

In contrast to strategies aimed at optimizing social welfare, approaches focused on enhancing equity tend to prioritize the allocation of parking permits to Group (1) internal beneficiaries. This preference stems from the significantly greater disutility associated with subway travel time compared to car usage under initial conditions devoid of shared parking permits. Despite Group (1) internal beneficiaries not utilizing NPA parking facilities, they are entitled to financial gains through the sale of their allocated permits to external users. It is imperative to reiterate that the equity analysis, as encapsulated by the Gini coefficient, specifically addresses the internal beneficiaries' equity—those entitled to utilize the NPAs—rather than that of external users. This emphasis underscores the internal beneficiaries as the primary stakeholders in the shared parking permit framework, warranting a focused approach on equitable permit distribution among them, particularly favoring those initially less inclined to utilize NPA facilities directly.

When optimizing equity, the average trade price increases with the number of parking permits. The final average trade price of permits depends on the numbers of NP internal beneficiaries of Group (2) and external users. Since the permits are more inclined to be assigned to internal beneficiaries of Group (1) (internal beneficiaries without car ownership) in the equity-based assignment scheme, there are more NP internal beneficiaries of Group (2) (internal beneficiaries with car ownership) competing for permits if the total number of parking permits is relatively insufficient. As the number of parking permits in the market increases, the number of NP internal beneficiaries of Group (2) reduces, and more permits are purchased by external users. The generalized travel cost (including travel time and monetary costs) of NP internal beneficiaries is lower than that of external users if they do not purchase the permits. If promoting equity among internal beneficiaries is considered as the objective (i.e., namely minimizing the gap in generalized travel costs among different internal beneficiary groups), AP internal beneficiaries of Group (1) sell the parking permit at a higher price to compensate for the long-time trip in the subway and walking to minimize their generalized travel costs, modelled by utilities in mode choice in the low-level modelling. Therefore, the trading price increases more when more external users and fewer NP internal beneficiaries of Group (2) participate in the market with more parking permits.

However, the equity among internal beneficiaries will improve even if the trade price of parking permits increases, because assigning more permits to Group (1) improves the monetary rewards through selling parking permits and reduces the generalized travel cost of Group (1), narrowing the gap in the generalized travel cost between Groups (1) and (2). Nevertheless, if the trade price further increases due to more parking permits, and it will, in turn, widen the gaps of generalized travel costs between AP internal beneficiaries and NP internal beneficiaries in Group (1) and between AP internal beneficiaries and NP internal beneficiaries who purchase the permits in Group (2). This is why we can observe the equity of internal beneficiaries decreases (i.e., larger Gini coefficient) with the number of parking permits after a certain threshold in Fig. 5. When the objective is optimizing the equity, the value of the Gini coefficient decreases first, and increases later with the number of parking permits as shown in Fig. 5. When 2600 permits are in the user market, equity reaches its maximum. This phenomenon seems counterintuitive at first glimpse but makes sense, as explained above. This is an interesting and useful finding when designing TPPSP for NPAs.

8. Conclusions

This study scrutinizes the efficacy of implementing a TPPSP within Non-Public Parking Areas. Our analytical model assesses user preferences in travel mode and route selection, offering a detailed examination of how the TPPSP impacts both the operational dynamics of road networks and parking facilities, and the equity among stakeholders characterized by diverse origin–destination matrices and levels of car ownership. To optimize equity, we introduce a mathematical programming model with equilibrium constraints, employing the Kriging metamodel algorithm to derive the optimal allocation of parking permits. Empirical analyses underscore the TPPSP's capacity to enhance social welfare by augmenting the efficiency of road networks and parking facilities external to NPAs, while concurrently fostering equity among stakeholders. Notably, the TPPSP facilitates a modal shift from private vehicle use to public transport. When contrasted with strategies primarily aimed at optimizing social welfare, equity-focused schemes are observed to preferentially allocate parking permits to carless internal beneficiaries. While the number of parking permits positively correlates with social welfare, the relationship with equity exhibits an initial increase followed by a decrement beyond a certain permit threshold. This nuanced finding highlights the TPPSP's potential in simultaneously advancing road network performance, parking facility utilization, and equitable access to parking resources.

The findings from this study offer significant insights for policymakers and urban planners aiming to enhance urban mobility, optimize the use of parking resources, and ensure equitable access to these facilities. The evidence suggests that TPPSP not only improves the efficiency of road networks and parking facilities but also promotes social welfare and equity among users with varied car ownership statuses and travel needs. Policymakers should consider equity-based allocation mechanisms for parking permits that prioritize carless internal beneficiaries within NPAs. This approach ensures a more equitable distribution of travel costs and benefits, potentially encouraging a modal shift from private vehicle use to public transport. This study indicates that the number of parking permits issued plays a crucial role in balancing social welfare and equity. Authorities could implement a dynamic permit issuance system that adjusts the number of available permits based on real-time demand and usage patterns to optimize both social welfare and equity outcomes. Given the potential of TPPSP to facilitate a shift towards public transit, policymakers could further incentivize this modal shift through improvements in transit services, subsidies for transit users, or additional benefits for carless individuals participating in TPPSP.

Even though this study makes new contributions, some further improvements can be made in the future. A more advanced dynamic method is required to well model both the process of searching for parking and parking occupancy over time. The drivers who travel through the study area are not well considered in this research, but they will affect the congestion levels of roads in the study area, which needs future work to improve. In addition, the TPPSP scheme and permit trading should also be time-dependent, evolving with parking occupancy fluctuating over time. Therefore, the dynamic network analysis-based approach to consider traffic dynamics in temporal dimension could be further developed in future research (Berghaus et al., 2024). Moreover, the users can be very diverse. Considering more user groups with different characteristics may make the problem to be much more complex but an interesting direction to explore. It is also interesting to extend the case study to be a much larger network to further validate the computation efficiency of the proposed methods and to develop other solution algorithms that are superior to Kriging metamodel algorithm we used in this study.

CRediT authorship contribution statement

Shanchuan Yu: Writing – review & editing, Writing – original draft, Visualization, Validation, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Kun Gao:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Lang Song:** Visualization, Validation, Methodology, Investigation, Conceptualization. **Yuchuan Du:** Validation, Supervision, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. -Proofs

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Lemma 1. *The pattern of flows and demands for external users and NP internal beneficiaries cruising for public parking facilities,* $\check{\mathbf{x}}^{3,U^*} \in \Omega_X\left(\check{\mathbf{q}}^{3,U}\right)$ *is in Eq. (31) under the given demands of user groups* $\check{\mathbf{q}}^{3,U}$*, if it satisfies the conditions of VI-NPP.*

Proof. *The inequality* (33) *is equivalent to.*

$$\sum_u\sum_w\sum_{ij}\left(c_{ij}^u(\mathbf{x}^*)+c_{j,P}^u\left(x_j^*\right)\delta_j^{\mathcal{N}_P}+p_i\delta_j^{\mathcal{D}}+\frac{1}{\varepsilon^w}\left(\ln\check{x}_{ij}^{w,3,u^*}-\ln\sum_k\check{x}_{kj}^{w,3,u^*}\right)\right)\check{x}_{ij}^{w,3,u}\geq$$
$$\sum_u\sum_w\sum_{ij}\left(c_{ij}^u(\mathbf{x})+c_{j,P}^u\left(x_j^*\right)\delta_j^{\mathcal{N}_P}+p_i\delta_j^{\mathcal{D}}+\frac{1}{\varepsilon^w}\left(\ln\check{x}_{ij}^{w,3,u^*}-\ln\sum_k\check{x}_{kj}^{w,3,u^*}\right)\right)\check{x}_{ij}^{w,3,u^*}, \tag{A1}$$
$$\forall\check{\mathbf{x}}^{3,U}\in\Omega_X\left(\check{\mathbf{q}}^{3,U}\right)$$

Therefore, $\widehat{x}_{ij}^{w,3,u^*}$ is the solution to the VI problem if and only if it is the optimal solution of the following linear program:

$$\underset{\mathbf{x}^U\in\Omega_P(\mathbf{q}^U)}{\mathrm{Min}}\sum_u\sum_w\sum_{ij}\left(c_{ij}^u(\mathbf{x}^*)+c_{j,P}^u\left(x_j^*\right)\delta_j^{\mathcal{N}_P}+p_i\delta_j^{\mathcal{D}}+\frac{1}{\varepsilon^w}\left(\ln\check{x}_{ij}^{w,3,u^*}-\ln\sum_k\check{x}_{kj}^{w,3,u^*}\right)\right)\check{x}_{ij}^{w,3,u} \tag{A2}$$

The Lagrangian for the program is:

$$L=\sum_u\sum_w\sum_{ij}\left(c_{ij}^u(\mathbf{x}^*)+c_{j,P}^u\left(x_j^*\right)\delta_j^{\mathcal{N}_P}+p_i\delta_j^{\mathcal{D}}+\frac{1}{\varepsilon^w}\left(\ln\check{x}_{ij}^{w,3,u^*}-\ln\sum_k\check{x}_{kj}^{w,3,u^*}\right)\right)\check{x}_{ij}^{w,3,u}$$
$$+\sum_u\sum_w\sum_n\check{\xi}_n^{w,u}\left(\sum_i\check{x}_{in}^{w,3,u}-\sum_j\check{x}_{nj}^{w,3,u}-\check{q}^{w,3,u}\delta_{n,d}+\check{q}^{w,3,u}\delta_{o,n}\right) \tag{A3}$$

where $\check{\xi}_n^{w,u}$ is the Lagrangian multipliers in constraint (13). When $\check{x}_{ij}^{w,3,u^*}>0$, the Karush-Kuhn-Tucker conditions for the program are

$$c_{ij}^u(\mathbf{x}^*)+c_{j,P}^u\left(x_j^*\right)\delta_j^{\mathcal{N}_P}+p_i\delta_j^{\mathcal{D}}+\frac{1}{\varepsilon^w}\left(\ln\check{x}_{ij}^{w,3,u^*}-\ln\sum_{k\in N_t(j)}\check{x}_{kj}^{w,3,u^*}\right)+\check{\xi}_j^{w,u^*}-\check{\xi}_i^{w,u^*}=0 \tag{A4}$$

$$\sum_{i\in\mathcal{N}_t(n)}\check{x}_{in}^{w,3,u^*}-\sum_{j\in\mathcal{N}_h(n)}\check{x}_{nj}^{w,3,u^*}-\check{q}^{w,3,u}\delta_{n,d}+\check{q}^{w,3,u}\delta_{o,n}=0 \tag{A5}$$

$$\check{\xi}_n^{w,u^*}\geq 0 \tag{A6}$$

Eq.(A4) can be written as

$$\frac{\check{x}_{ij}^{w,3,u^*}}{\sum_k\check{x}_{kj}^{w,3,u^*}}=\exp\left\{-\varepsilon^w\left[c_{ij}^u(\mathbf{x}^*)+c_{j,P}^u\left(x_j^*\right)\delta_j^{\mathcal{N}_P}+p_i\delta_j^{\mathcal{D}}+\check{\xi}_j^{w,u^*}-\check{\xi}_i^{w,u^*}\right]\right\} \tag{A7}$$

Note that the left of Eq.(A7) denotes the probability of external users that select a node i conditional on selecting a successor node j. Then, we can get

$$\Pr^{w,u}(i|j)=\frac{\check{x}_{ij}^{w,3,u^*}}{\sum_k\check{x}_{kj}^{w,3,u^*}} \tag{A8}$$

In terms of the Markov property of the stochastic traffic assignment, the probability of external users choosing the r th route in $\mathcal{R}_l^{w,3}$,$P_{r,l}^{w,3,u}$ is

$$\check{P}_{r_l}=\prod_{k_1\in\mathcal{N}_t(d)}\prod_{k_2\in\mathcal{N}_t(k_1)}\cdots\prod_{o\in\mathcal{N}_t(k_{l-1})}\Pr^{w,u}(k_1|d)^{\delta_{k_1d,r}^w}\Pr^{w,u}(k_2|k_1)^{\delta_{k_2k_1,r}^w}\cdots\Pr^{w,u}(o|k_{l-1})^{\delta_{ok_{l-1},r}^w}\quad=\exp\left[-\varepsilon^w\left(\check{\xi}_d^{w,u^*}-\check{\xi}_o^{w,u^*}\right)\right]$$
$$\exp\left[-\varepsilon_e\sum_{k_1\in N_t(d)}\sum_{k_2\in N_t(k_1)}\cdots\sum_{o\in N_t(k_{l-1})}\left(c_{k_1d}^u(\mathbf{x}^*)+p_{k_1}\right)\delta_{k_1d,r}^w+\left(c_{k_2k_1}^u(\mathbf{x})+c_{k_1,P}^u\left(x_{k_1}^*\right)\delta_{k_1}^{\mathcal{N}_P}\right)\delta_{k_2k_1,r}^w+\cdots+c_{ok_{l-1}}^u(\mathbf{x}^*)\delta_{ok_{l-1},r}^w\right]$$
$$=\exp\left[-\varepsilon^w\left(\check{\xi}_d^{w,u^*}-\check{\xi}_o^{w,u^*}\right)\right]\exp\left[-\varepsilon^w c_{r_l}^{w,3,u}(\mathbf{x}^*)\right] \tag{A9}$$

Consider the flow conservation, $\sum_{k\in\mathcal{N}_t(j)}\Pr^{w,u}(k|j)=1$, and use Eq.(A7), then we can give the following equation.

$$\exp\left(\varepsilon^w\check{\xi}_j^{w,u^*}\right)=\sum_{k\in\mathcal{N}_t(j)}\exp\left\{-\varepsilon^w\left[c_{kj}^u(\mathbf{x}^*)+c_{j,P}^u\left(x_j^*\right)\delta_j^{\mathcal{N}_P}+p_k\delta_j^{\mathcal{D}}\right]\right\}\exp\left(\varepsilon^w\check{\xi}_k^{w,u^*}\right),\ j\neq o \tag{A10}$$

Let $\left\{v_i^{w,u}\right\}$ and $\left\{h_{ij}\right\}$ be defined as

$$v_i^{w,u} = \exp\left[\varepsilon^w\left(\check{\xi}_i^{w,u*} - \check{\xi}_o^{w,u*}\right)\right], \ i \neq d \tag{A11}$$

$$h_{ij} = \begin{cases} \exp\left\{ -\varepsilon^w\left[c_{ij}^u(\mathbf{x}^*) + c_{j,P}^u\left(x_j^*\right)\delta_j^{\mathscr{N}_P} + p_i\delta_j^{\mathscr{D}}\right]\right\} & \forall(i,j) \in \mathscr{A} \\ 0 & (i,j) \text{ does not exist} \end{cases} \tag{A12}$$

Note that Eq.(A10) can be rewritten as

$$(\mathbf{I} - \mathbf{H})\mathbf{V} = \mathbf{H}_o \tag{A13}$$

where $\mathbf{V} = \left[v_1^{w,u}, v_2^{w,u}, \cdots, v_{\|\mathscr{N}\|-1}^{w,u}\right]^T$, $\mathbf{H}$ denotes a $(\|\mathscr{N}\|-1) \times (\|\mathscr{N}\|-1)$ matrix where $\text{ent}_{ji}(\mathbf{H}) = h_{ij}$ $(i \neq o$ and $j \neq o)$, $\mathbf{H}_o = \left[h_{o1}, h_{o2}, \cdots, h_{o,\|\mathscr{N}\|-1}\right]^T$ and $\|\mathscr{N}\|$ represents the amount of nodes. If $\boldsymbol{H}$ complied with the Hawkins-Simon condition, we have

$$\mathbf{V} = (\mathbf{I} - \mathbf{H})^{-1}\mathbf{H}_o = \left(\mathbf{I} + \mathbf{H} + \mathbf{H}^2 + \cdots\right)\mathbf{H}_o \tag{A14}$$

Eq.(A14) can also explain why the route set of external users who cruise for parking must involve infinite cycles. The element $h_{ij}^{[l]}$, $h_{ij}^{[l]} = \text{ent}_{ji}\left(\mathbf{H}^l\right)$ is given by

$$\begin{aligned} h_{ij}^{[l]} &= \sum_{k_1=1}^{\|\mathscr{N}\|-1}\sum_{k_2=1}^{\|\mathscr{N}\|-1}\cdots\sum_{k_{l-1}=1}^{\|\mathscr{N}\|-1}\left(h_{ik_1}h_{k_1k_2}\cdots h_{k_{l-1}j}\right) \\ &= \sum_{k_1\in\mathscr{N}_h(i)}\sum_{k_2\in\mathscr{N}_h(k_1)}\cdots\sum_{j\in\mathscr{N}_h(k_{l-1})}\exp\Big\{-\varepsilon^w\Big\{\left[c_{ik_1}^u(\mathbf{x}^*) + c_{k_1,P}^u\left(x_{k_1}^*\right)\delta_{k_1}^{\mathscr{N}_P} + p_i\delta_{k_1}^{\mathscr{D}}\right] + \left[c_{k_1k_2}^u(\mathbf{x}^*) + c_{k_2,P}^u\left(x_{k_2}^*\right)\delta_{k_2}^{\mathscr{N}_P} + p_{k_1}\delta_{k_2}^{\mathscr{D}}\right] + \cdots + \Big[c_{k_{l-1}j}^u(\mathbf{x}^*) \\ &\quad + c_{j,P}^u\left(x_j^*\right)\delta_j^{\mathscr{N}_P} + p_{k_{l-1}}\delta_j^{\mathscr{D}}\Big]\Big\}\Big\} \\ &= \sum_{r\in\mathscr{R}_l^{ij,3}}\exp\left[-\varepsilon^w c_{r_l}^{ij,3,u}(\mathbf{x}^*)\right] \end{aligned} \tag{A15}$$

where $\mathscr{R}_l^{ij,3}$ is the set of external users' routes which connect node i and j by traveling through l links. $c_{r_l}^{ij,3,u}(\bullet)$ is the travel cost of the r th route in $\mathscr{R}_l^{ij}$. $\mathscr{R}^{w,3}$ is the set of external users' routes that connect OD pair $w \in \mathscr{W}$, comprising $\left\{\mathscr{R}_1^{w,3}, \mathscr{R}_2^{w,3}, \cdots, \mathscr{R}_l^{w,3}, \cdots\right\}$. Using Eq.(A15), $v_d^{w,u}$ in Eq. (90) is written as

$$\begin{aligned} v_d^{w,u} &= h_{od} + \sum_{i\neq d}\left(\sum_{l=1}^{\infty}h_{oi}h_{id}^{[l]}\right) \\ &= 0 + \sum_{i\in\mathscr{N}_h(o)}\sum_{l=1}^{\infty}\sum_{r\in\mathscr{R}_l^{ij,3}}\exp\left\{-\varepsilon^w\left[c_{oi}^u(\mathbf{x}^*) + c_{r_l}^{id,3,u}(\mathbf{x}^*)\right]\right\} \\ &= \sum_{r\in\mathscr{R}^{w,3}}\exp\left[-\varepsilon^w c_r^{w,3,u}(\mathbf{x}^*)\right] \end{aligned} \tag{A16}$$

Using Eq.(A11), we have $v_d^{w,u} = \exp\left[\varepsilon^w\left(\check{\xi}_d^{w,u*} - \check{\xi}_o^{w,u*}\right)\right]$ and therefore, Eq.(34) holds. Combined with Eq.(A16), Eq.(A9) can be rewritten as

$$\check{P}_{r_l} = \frac{\exp\left[-\varepsilon^w c_{r_l}^{w,3,u}(\mathbf{x}^*)\right]}{\sum_{r\in\mathscr{R}^{w,3}}\exp\left[-\varepsilon^w c_r^{w,3,u}(\mathbf{x}^*)\right]}, \ \forall u \in \mathscr{U}, \ \forall w \in \mathscr{W}, \ \forall r \in \mathscr{R}_l^{w,3}, \ l = 1, 2, \cdots \tag{A17}$$

As a result, the pattern for link flows of external users cruising for public parking facilities, $\check{\mathbf{x}}^{3,U*} \in \Omega_X\left(\check{\mathbf{q}}^{3,U}\right)$ is in the heterogeneous quantal response equilibrium condition for the game in cruising for parking with infinite cyclic flows.

Lemma 2. *The pattern of users' flows and demands, $\left(\mathbf{f}^{U*}, \mathbf{x}^{U*}, \mathbf{q}^{U*}\right) \in \Theta\left(\mathbf{Q}^U\right)$ is in the equilibrium conditions (45a), (45b), (46a), (46b), (49) and the average trade prices of parking permits, $p^* \in \mathbb{R}^+$ is in the market equilibrium under the given demands of user groups $\mathbf{Q}^U$, if it is the solution of the VI problem VI-TPP.*

Proof. $\left(\mathbf{f}^{U*}, \mathbf{x}^{U*}, \mathbf{q}^{U*}\right) \in \Theta\left(\mathbf{Q}^U\right), p^* \in \mathbb{R}^+$ *is the solution to the problem VI-TPP if and only if it is the optimal solution for the following linear program* (Facchinei and Pang, 2003):

$$\underset{\left(f^U, x^U, q^U\right) \in \Theta_P\left(Q^U\right), p \in \mathbb{R}^+}{\mathrm{Min}} \mathbf{F}\left(\mathbf{f}^{U^*}, \mathbf{x}^{U^*}, \mathbf{q}^{U^*}, p^*\right) \bullet \left(\mathbf{f}^U, \mathbf{x}^U, \mathbf{q}^U, p\right)^T \tag{A18}$$

where

$$\begin{aligned}\mathbf{F}\left(\mathbf{f}^{U^*}, \mathbf{x}^{U^*}, \mathbf{q}^{U^*}, p^*\right) \bullet \left(\mathbf{f}^U, \mathbf{x}^U, \mathbf{q}^U, p\right)^T &= \sum_u \sum_w \sum_{r \in \mathscr{R}^{w,1}} \left(c_r^{w,1,u} - p^* + \frac{1}{\theta^{w,1,u}} \ln \widehat{f}_r^{w,1,u^*}\right) \widehat{f}_r^{w,1,u} \\ &+ \left(c_r^{w,1,u} + \frac{1}{\theta^{w,1,u}} \ln \check{f}_r^{w,1,u^*}\right) \check{f}_r^{w,1,u} + \sum_u \sum_w \sum_{r \in \mathscr{R}^{w,2}} \left(c_r^{w,2,u}(\mathbf{x}^*) + \frac{1}{\theta^{w,2,u}} \ln \widehat{f}_r^{w,2,u^*}\right) \widehat{f}_r^{w,2,u} \\ &+ \left(c_r^{w,2,u}(\mathbf{x}^*) + p^* + \frac{1}{\theta^{w,2,u}} \ln \check{f}_r^{w,2,u^*}\right) \check{f}_r^{w,2,u} \\ &+ \sum_u \sum_w \sum_{ij} \left(c_{ij}^u(\mathbf{x}^*) + c_{j,P}^u\left(x_j^*\right) \delta_j^{\mathscr{N}_P} + p_i \delta_j^{\mathscr{D}} + \frac{1}{\varepsilon^w}\left(\ln \check{x}_{ij}^{w,3,u^*} - \ln \sum_{k \in \mathscr{N}_t(j)} \check{x}_{kj}^{w,3,u^*}\right)\right) \check{x}_{ij}^{w,3,u} \\ &+ \sum_u \sum_{m=1}^{2} \sum_w \left(\frac{1}{\theta^{w,u}} - \frac{1}{\theta^{w,m,u}}\right) \left(\widehat{q}^{w,m,u} \ln \widehat{q}^{w,m,u^*} + \check{q}^{w,m,u} \ln \check{q}^{w,m,u^*}\right) \\ &+ \sum_u \sum_w \frac{1}{\theta^{w,u}} \check{q}^{w,3,u} \ln \check{q}^{w,3,u^*} + p \sum_{n \in \mathscr{N}_B} \left(N_n^B - x_n^*\right)\end{aligned} \tag{A19}$$

when $\widehat{f}_r^{w,m,u^*} > 0$, $\check{f}_r^{w,m,u^*}$, $\widehat{x}_{ij}^{w,3,u^*} > 0$, $\check{x}_{ij}^{w,3,u^*} > 0$ and $\check{q}^{w,m,u^*} > 0$, the Karush-Kuhn-Tucker conditions for the program are

$$c_r^{w,m,u} - p^*\left(1 - \delta_{m,2}\right) + \frac{1}{\theta^{w,m,u}} \ln \widehat{f}_r^{w,m,u^*} + \widehat{\mu}^{w,m,u^*} = 0,\ m = 1,2 \tag{A20a}$$

$$c_r^{w,m,u} + p^* \delta_{m,2} + \frac{1}{\theta^{w,m,u}} \ln \check{f}_r^{w,m,u^*} + \check{\mu}^{w,m,u^*} = 0,\ m = 1,2 \tag{A20b}$$

$$c_{ij}^u(\mathbf{x}^*) + c_{j,P}^u\left(x_j^*\right) \delta_j^{\mathscr{N}_P} + p_i \delta_j^{\mathscr{D}} + \frac{1}{\varepsilon^w}\left(\ln \check{x}_{ij}^{w,3,u^*} - \ln \sum_{k \in \mathscr{N}_t(j)} \check{x}_{kj}^{w,3,u^*}\right) + \check{\xi}_j^{w,u} - \check{\xi}_i^{w,u} = 0 \tag{A21}$$

$$\left(\frac{1}{\theta^{w,u}} - \frac{1}{\theta^{w,m,u}}\right) \ln \widehat{q}^{w,m,u^*} - \widehat{\mu}^{w,m,u^*} + \widehat{\lambda}^{w,u^*} = 0,\ m = 1,2 \tag{A22a}$$

$$\left(\frac{1}{\theta^{w,u}} - \frac{1}{\theta^{w,m,u}}\right) \ln \check{q}^{w,m,u^*} - \check{\mu}^{w,m,u^*} + \check{\lambda}^{w,u^*} = 0,\ m = 1,2 \tag{A22b}$$

$$\frac{1}{\theta^{w,u}} \ln \check{q}^{w,3,u^*} - \check{\xi}_d^{w,u^*} + \check{\xi}_o^{w,u^*} + \check{\lambda}^{w,u^*} = 0 \tag{A23}$$

$$\sum_m \widehat{q}^{w,m,u^*} - \widehat{Q}^{w,u} = 0 \tag{A24a}$$

$$\sum_m \check{q}^{w,m,u^*} - \check{Q}^{w,u} = 0 \tag{A24b}$$

$$\sum_{r \in \mathscr{R}^{w,m}} \widehat{f}_r^{w,m,u^*} - \widehat{q}^{w,m,u^*} = 0,\ m = 1,2 \tag{A25a}$$

$$\sum_{r \in \mathscr{R}^{w,m}} \check{f}_r^{w,m,u^*} - \check{q}^{w,m,u^*} = 0,\ m = 1,2 \tag{A25b}$$

$$\sum_{i \in \mathscr{N}_t(n)} \check{x}_{in}^{w,3,u^*} - \sum_{j \in \mathscr{N}_h(n)} \check{x}_{nj}^{w,3,u^*} - \check{q}^{w,3,u^*} \delta_{n,d} + \check{q}^{w,3,u^*} \delta_{o,n} = 0 \tag{A26}$$

$$\sum_{n \in \mathscr{N}_B} \left(N_n^B - x_n^*\right) \geq 0 \tag{A27}$$

$$p^* \sum_{n \in \mathscr{N}_B} \left(N_n^B - x_n^*\right) = 0 \tag{A28}$$

where, $\widehat{\mu}^{w,m,u^*}$, $\breve{\mu}^{w,m,u^*}$, $\widehat{\xi}_n^{w,u^*}$, $\breve{\xi}_n^{w,u^*}$, $\widehat{\lambda}^{w,u^*}$ and $\breve{\lambda}^{w,u^*}$ are the optimal Lagrangian multipliers in constraints (10)-(13), (6) and (7), respectively. Eq. (A27) ensures the number of permits charged for booked parking spaces is less than or equal to the number of users parked at these parking spaces. Eq.(A28) represents the equilibrium condition of the permit market (Yang and Wang, 2011).

Then, we obtain the equilibrium conditions for NP users, i.e., Eqs. (45b), (46b), (49) according to Eqs. (A20b), (A21), (A22b), (A23), (A24b), (A25b) and (A26). The derivation of equilibrium conditions for AP users, i.e., Eqs. (45a) and (46a) is the same based on Eqs. (A20a), (A22a), (A24a) and A(25a).

For travel modes 1 and 2, Eq.(A20b) can be written as

$$\breve{f}_r^{w,m,u^*} = \exp\left[-\theta^{w,m,u}\left(c_r^{w,m,u} + p^*\delta_{m,2} + \breve{\mu}^{w,m,u^*}\right)\right] \tag{A29}$$

According to Eq.(A25b), we have

$$\breve{q}^{w,m,u^*} = \sum_{r\in\mathscr{R}^{w,m}} \breve{f}_r^{w,m,u^*} = \exp(-\theta^{w,m,u}\breve{\mu}^{w,m,u^*}) \sum_{r\in\mathscr{R}^{w,m}} \exp\left[-\theta^{w,m,u}\left(c_r^{w,m,u} + p^*\delta_{m,2}\right)\right] \tag{A30}$$

and hence

$$\breve{\mu}^{w,m,u^*} = -\frac{1}{\theta^{w,m,u}} \ln \frac{\breve{q}^{w,m,u^*}}{\sum_{r\in\mathscr{R}^{w,m}} \exp\left[-\theta^{w,m,u}\left(c_r^{w,m,u} + p^*\delta_{m,2}\right)\right]} \tag{A31}$$

Given the condition that

$$\overline{c}^{w,m,u}(p^*) = -\frac{1}{\theta^{w,m,u}} \ln \sum_{r\in\mathscr{R}^{w,m}} \exp\left[-\theta^{w,m,u}\left(c_r^{w,m,u} + p^*\delta_{m,2}\right)\right] \tag{A32}$$

which is equivalent to NP users in Eqs. (48) and (49), we have

$$\breve{\mu}^{w,m,u^*} = -\frac{1}{\theta^{w,m,u}} \ln \breve{q}^{w,m,u^*} - \overline{c}^{w,m,u}(p^*) \tag{A33}$$

Associated with Eq.(A22b), we have

$$\breve{q}^{w,m,u^*} = \exp\left\{-\theta^{w,u}\left[\overline{c}^{w,m,u}(p^*) + \breve{\lambda}^{w,u^*}\right]\right\} \tag{A34}$$

For travel mode 3, according to Lemma 1 and Eq. (50), we have

$$\exp\left[\varepsilon^w\left(\breve{\xi}_d^{w,u^*} - \breve{\xi}_o^{w,u^*}\right)\right] = \sum_{r\in\mathscr{R}^{w,3}} \exp\left[-\varepsilon^w c_r^{w,3,u}(\mathbf{x}^*)\right] = \exp\left[-\varepsilon^w \overline{c}^{w,3,u}(\mathbf{x}^*)\right] \tag{A35}$$

Therefore, we have

$$\breve{\xi}_d^{w,u^*} - \breve{\xi}_o^{w,u^*} = -\overline{c}^{w,3,u}(\mathbf{x}^*, p^*) \tag{A36}$$

Associated with Eq.(A23b), we have

$$\breve{q}^{w,3,u^*} = \exp\left\{-\theta^{w,u}\left[\overline{c}^{w,3,u}(\mathbf{x}^*, p^*) + \breve{\lambda}^{w,u^*}\right]\right\} \tag{A37}$$

Combining Eqs. (A29), (A34), (A37) with Eq.(A17) in Lemma 1, we have the following relationships:

$$\begin{aligned} \breve{P}_r^{w,m,u} &= \frac{\breve{f}_r^{w,m,u^*}}{\breve{Q}^{w,u}} = \frac{\breve{f}_r^{w,m,u^*}}{\breve{q}^{w,m,u^*}} \bullet \frac{\breve{q}^{w,m,u^*}}{\breve{Q}^{w,u}} = \frac{\breve{f}_r^{w,m,u^*}}{\sum_{r\in\mathscr{R}^{w,m}} \breve{f}_r^{w,m,u^*}} \bullet \frac{\breve{q}^{w,m,u^*}}{\sum_m \breve{q}^{w,m,u^*}} \\ &= \frac{\exp\left[-\theta^{w,m,u}\left(c_r^{w,m,u} + p^*\delta_{m,2}\right)\right]}{\sum_{r\in\mathscr{R}^{w,m}} \exp\left[-\theta^{w,m,u}\left(c_r^{w,m,u} + p^*\delta_{m,2}\right)\right]} \bullet \frac{\exp\left[-\theta^{w,u}\overline{c}^{w,m,u}(p^*)\right]}{\sum_m \exp\left[-\theta^{w,u}\overline{c}^{w,m,u}(\mathbf{x}^*, p^*)\right]}, \\ &\forall u \in \mathscr{U}, \forall w \in \mathscr{W}, m = 1, 2, \cdots \end{aligned} \tag{A38}$$

$$\breve{P}_{r_l}^{w,3,u} = \breve{P}_{r_l} \bullet \frac{\breve{q}^{w,3,u^*}}{\breve{Q}^{w,u}} = \breve{P}_{r_l} \bullet \frac{\breve{q}^{w,m,u^*}}{\sum_m \breve{q}^{w,m,u^*}} = \frac{\exp\left[-\varepsilon^w c_{r_l}^{w,3,u}(\mathbf{x}^*)\right]}{\sum_{r\in\mathscr{R}^{w,3}} \exp\left[-\varepsilon^w c_r^{w,3,u}(\mathbf{x}^*)\right]} \bullet \frac{\exp\left[-\theta^{w,u}\overline{c}^{w,3,u}(\mathbf{x}^*)\right]}{\sum_m \exp\left[-\theta^{w,u}\overline{c}^{w,m,u}(\mathbf{x}^*, p^*)\right]}, \tag{A39}$$

$$\forall u \in \mathscr{U}, \forall w \in \mathscr{W}, \forall r \in \mathscr{R}_l^{w,3}, l = 1, 2, \cdots$$

The pattern of users' flows and demands, $\left(\mathbf{f}^{U^*}, \mathbf{x}^{U^*}, \mathbf{q}^{U^*}\right) \in \Theta\left(\mathbf{Q}^U\right)$ is in the equilibrium conditions (45a), (45b), (46a), (46b), (49) and the average trade prices of parking permits, $p^* \in \mathbb{R}^+$ is in the market equilibrium under the given demands of user groups $\mathbf{Q}^U$.

Data availability

The authors do not have permission to share data.

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