Criteria for the Existence of Schwartz Gabor Frames Over Rational Lattices

Ulrik Enstad[1,*], Hannes Thiel[2], and Eduard Vilalta[2]

[1]Department of Mathematics, University of Oslo, Moltke Moes vei 35, 0851 Oslo, Norway
[2]Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Chalmers Tvärgata 3, SE-412 96 Gothenburg, Sweden
*Correspondence to be sent to: e-mail: ubenstad@math.uio.no
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We give an explicit criterion for a rational lattice in the time-frequency plane to admit a Gabor frame with window in the Schwartz class. The criterion is an inequality formulated in terms of the lattice covolume, the dimension of the underlying Euclidean space, and the index of an associated subgroup measuring the degree of non-integrality of the lattice. For arbitrary lattices we also give an upper bound on the number of windows in the Schwartz class needed for a multi-window Gabor frame.

1 Introduction

Given a point $z = (x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d \cong \mathbb{R}^{2d}$, denote by $\pi(z)$ the unitary operator on $L^2(\mathbb{R}^d)$ given by

$$(\pi(z)f)(t) = e^{2\pi i \langle \omega, t \rangle} f(t - x), \qquad \text{for } f \in L^2(\mathbb{R}^d),\ t \in \mathbb{R}^d. \tag{1.1}$$

Given a lattice $\Gamma \subseteq \mathbb{R}^{2d}$ and $g \in L^2(\mathbb{R}^d)$, a sequence in $L^2(\mathbb{R}^d)$ of the form

$$\pi(\Gamma)g = (\pi(\gamma)g)_{\gamma \in \Gamma}$$

is known as a *Gabor system*, and the function g is called the *window* of the system. The question of when the Gabor system $\pi(\Gamma)g$ is a frame, that is, when there exist $A, B > 0$ such that

$$A\|f\|_2^2 \le \sum_{\gamma \in \Gamma} |\langle f, \pi(\gamma)g \rangle|^2 \le B\|f\|_2^2, \qquad \text{for } f \in L^2(\mathbb{R}^d),$$

is central to time-frequency analysis and related fields [14].

We will call a Gabor system $\pi(\Gamma)g$ *Schwartz* if the window g belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decaying smooth functions. A version of the Balian–Low Theorem [2, 10, 19] states that a Schwartz Gabor system can be a frame only if $\mathbf{covol}(\Gamma) < 1$. This paper concerns the converse of the Balian–Low Theorem as follows:

Problem A. Let Γ be a lattice in $\mathbb{R}^{2d}$ with $\mathbf{covol}(\Gamma) < 1$. Does there exist a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$ such that $\pi(\Gamma)g$ is a frame for $L^2(\mathbb{R}^d)$?

In dimension $d = 1$ Problem A always has an affirmative answer: It is a consequence of the theorems of Lyubarskii, Seip, and Wallsten [23, 29, 30] that every lattice Γ with $\mathbf{covol}(\Gamma) < 1$ admits a Schwartz Gabor frame. In fact, g can be chosen to be the Gaussian function $g(t) = e^{-\pi t^2}$.

For $d \geq 2$ the structure of Gabor systems generated by Gaussian functions is considerably more complicated; see, for example, [13, 17]. In fact, despite several known partial solutions, for every $d \geq 2$ Problem A is still unsolved in its full generality [15, p. 225, Remarks 2]. One important partial answer to the question was obtained by Jakobsen–Luef in [18, Theorem 5.4], where they showed the existence of a Schwartz Gabor frame whenever the lattice is non-rational. Here, recall that a lattice is said to be *non-rational* if $\Omega(\gamma, \gamma') \notin \mathbb{Q}$ for some $\gamma, \gamma' \in \Gamma$, where Ω denotes the standard symplectic form on $\mathbb{R}^{2d}$ given by

$$\Omega((x, \omega), (x', \omega')) = \langle x, \omega' \rangle - \langle x', \omega \rangle, \qquad \text{for } (x, \omega), (x', \omega') \in \mathbb{R}^{2d}. \tag{1.2}$$

Thus, the instance of the problem that remains open is the case of *rational* lattices, that is, those for which the symplectic forms of all pairs of lattice elements are rational.

To prove their result, Jakobsen–Luef exploited a reformulation of Problem A in terms of comparison of projections in C*-algebras [3, 20, 21], which allowed them to employ prior results due to Rieffel [26]. Building on their ideas, the purpose of the current paper is to show how comparison theory of projections in C*-algebras can be used to obtain a partial solution of Problem A for rational lattices. For this, we consider the following invariant of a lattice in $\mathbb{R}^{2d}$ which is related to the concept of symplectic complements [24, p. 38] (see also the related work [12]).

Definition B. Let $\Gamma \subseteq \mathbb{R}^{2d}$ be a lattice. Consider the subgroup of Γ given by

$$\Gamma_\Omega = \left\{ \gamma \in \Gamma : \Omega(\gamma, \gamma') \in \mathbb{Z} \text{ for all } \gamma' \in \Gamma \right\}. \tag{1.3}$$

We define $n_\Gamma = [\Gamma : \Gamma_\Omega]^{1/2}$ if the index $[\Gamma : \Gamma_\Omega]$ is finite, and $n_\Gamma = \infty$ otherwise.

We prove in Problem 2.1 that the index $[\Gamma : \Gamma_\Omega]$ is finite if and only if Γ is rational. Moreover, in this case we show in Proposition 4.2 that the noncommutative $2d$-torus associated to Γ is a n_Γ-homogeneous C*-algebra, that is, all its irreducible representations have dimension n_Γ. We therefore view n_Γ as a measure of the non-integrality of Γ, that is, the non-integrality of the numbers $\Omega(\gamma, \gamma')$ for $\gamma, \gamma' \in \Gamma$.

Our main result provides a criterion for the existence of a Schwartz Gabor frame:

Theorem C. Let $\Gamma \subseteq \mathbb{R}^{2d}$ be a lattice such that

$$\mathbf{covol}(\Gamma) < 1 - \frac{d-1}{n_\Gamma}. \tag{1.4}$$

Then there exists $g \in \mathcal{S}(\mathbb{R}^d)$ such that $\pi(\Gamma)g$ is a frame for $L^2(\mathbb{R}^d)$.

Thus, the closer the covolume of Γ is to being 1, the more non-integrality of Γ is required for (1.4) to hold. In the extreme case that Γ is non-rational, we interpret $(d-1)/n_\Gamma$ as being zero, and then Theorem C recovers the known partial solution to Problem A from [18, Theorem 5.4]. Our result also recovers the known solution for $d = 1$.

Theorem C is a special case of the following result, which gives an upper bound on the number of windows required for a Schwartz multi-window Gabor frame over any lattice $\Gamma \subseteq \mathbb{R}^d$. Due to abstract principles [20, Theorem 4.5] or alternatively a compactness argument [3, Proposition 6.4] this number is known to be finite, but the proof techniques of [3, 20] give no control on the number of windows. Theorem D on the other hand provides a concrete number in terms of $\mathbf{covol}(\Gamma)$, n_Γ, and d. In the statement below, $\lfloor x \rfloor$ denotes the biggest integer smaller than or equal to x.

Theorem D. Let $\Gamma \subseteq \mathbb{R}^{2d}$ be a lattice and set

$$k = \left\lfloor \text{covol}(\Gamma) + \frac{d-1}{n_\Gamma} \right\rfloor + 1.$$

Then there exist $g_1, \ldots, g_k \in \mathcal{S}(\mathbb{R}^d)$ such that $(\pi(\gamma)g_j)_{\gamma \in \Gamma, 1 \le j \le k}$ is a k-multi-window Gabor frame for $L^2(\mathbb{R}^d)$.

Note that Theorem D also gives a (less optimal) bound independent of $[\Gamma : \Gamma_\Omega]$, namely $k = \lfloor \text{covol}(\Gamma) \rfloor + d$.

2 Rational Lattices

Denote by $\text{GL}(2d, \mathbb{R})$ the set of real, invertible $2d \times 2d$ matrices and by $\text{GL}(2d, \mathbb{Z})$ the subset of integer matrices with determinant equal to ± 1. Let

$$J = \begin{pmatrix} 0_d & I_d \\ -I_d & 0_d \end{pmatrix}$$

denote the standard symplectic $2d \times 2d$ matrix, where 0_d denotes a $d \times d$ zero matrix and I_d denotes a $d \times d$ identity matrix. The standard symplectic form defined in (1.2) can be written as

$$\Omega(z, w) = \langle z, Jw \rangle, \qquad \text{for } z, w \in \mathbb{R}^{2d}.$$

As already stated in the introduction, a lattice $\Gamma \subseteq \mathbb{R}^{2d}$ is called *rational* if the numbers $\Omega(\gamma, \gamma')$ are rational for all $\gamma, \gamma' \in \Gamma$. We can always write a lattice as $\Gamma = M\mathbb{Z}^{2d}$ for some $M \in \text{GL}(2d, \mathbb{R})$, and we say that *M represents* Γ in this case. If $M' \in \text{GL}(2d, \mathbb{R})$ also represents Γ, then there exists $R \in \text{GL}(2d, \mathbb{Z})$ such that $M' = MR$. For any $M \in \text{GL}(2d, \mathbb{R})$ the associated matrix $\theta = M^t J M$ is skew-symmetric, that is, $\theta^t = -\theta$ where θ^t denotes the transpose of θ. If $M' = MR$ then $\theta' := (M')^t J M' = R^t \theta R$, that is, θ and θ' are *congruent*. Thus, any lattice Γ determines a real, skew-symmetric matrix θ up to congruence, where $\theta = M^t J M$ for some $M \in \text{GL}(2d, \mathbb{R})$ such that $\Gamma = M\mathbb{Z}^{2d}$. In particular, Γ is rational if and only if θ is rational, in other words, if and only if θ has rational entries.

Suppose that θ is a rational skew-symmetric $2d \times 2d$ matrix. Let r be the least natural number such that $r\theta$ is integral. We will call r the *order* of θ. We will represent $r\theta$ in its *skew normal form* (see [25, p. 57, Theorem IV.1] for an algorithm). For this, there are natural numbers $h_1, \ldots, h_d$ with $h_i \mid h_{i+1}$ for $1 \le i \le d-1$ and a matrix $R \in GL(\mathbb{Z}, 2d)$ such that

$$R^t(r\theta)R = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}, \tag{2.1}$$

where

$$B = \begin{pmatrix} h_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & h_d \end{pmatrix}.$$

The numbers $h_1, \ldots, h_d$ are unique and are called the *invariant factors* of $r\theta$.

If θ and θ' are congruent rational skew-symmetric $2d \times 2d$ matrices, say $\theta' = R^t \theta R$ for some $R \in \text{GL}(2d, \mathbb{Z})$, then they have the same order. Indeed, $r\theta$ is rational if and only if $r\theta' = R^t(r\theta)R$ is rational. Furthermore, since the matrices $r\theta$ and $r\theta'$ are congruent, they share the same skew normal form, hence they have identical invariant factors as well. For this reason it makes sense to define the *order* and *invariant factors* of a rational lattice $\Gamma \subseteq \mathbb{R}^{2d}$ respectively as the order r of θ and invariant factors of $r\theta$, where $\theta = M^t J M$ for any $M \in \text{GL}(2d, \mathbb{R})$ that represents Γ.

Proposition 2.1. Let $\Gamma \subseteq \mathbb{R}^{2d}$ be a lattice. Then Γ is rational if and only if the subgroup Γ_Ω of Γ defined in (1.3) has finite index. Furthermore, letting r be the order of Γ and $h_1, \ldots, h_d$ its invariant factors, we have that

$$\Gamma/\Gamma_\Omega \cong (\mathbb{Z}/r_1\mathbb{Z})^2 \times \cdots \times (\mathbb{Z}/r_d\mathbb{Z})^2,$$

where $r_i = r/\gcd(h_i, r)$ for $1 \le i \le d$. In particular $[\Gamma : \Gamma_\Omega] = r_1^2 \cdots r_d^2$, and thus

$$n_\Gamma = [\Gamma : \Gamma_\Omega]^{1/2} = r_1 \cdots r_d = \frac{r^d}{\gcd(h_1, r) \cdots \gcd(h_d, r)}.$$

Proof. First, note that by picking a basis $e_1, \ldots, e_{2d}$ for Γ, the numbers $\Omega(\gamma, \gamma')$ for $\gamma, \gamma' \in \Gamma$ can all be expressed as integer linear combinations of $\Omega(e_i, e_j)$ for $1 \le i, j \le 2d$. Hence, Γ is rational if and only if there exists $n \in \mathbb{N}$ such that $\Omega(\gamma, \gamma') \in n^{-1}\mathbb{Z}$ for all $\gamma, \gamma' \in \Gamma$. Equivalently $n\gamma \in \Gamma_\Omega$ for all $\gamma \in \Gamma$, which means that every element of Γ/Γ_Ω has order at most n. Since Γ/Γ_Ω is a finitely generated abelian group, this happens exactly when it has finite order, that is, when $[\Gamma : \Gamma_\Omega] < \infty$.

Suppose now that Γ is rational. Let $M \in GL(2d, \mathbb{R})$ be a matrix that represents Γ and let $\theta = M^t J M$. Since the order and invariant factors are invariant under congruence, we may assume that $r\theta$ is already of the block form in (2.1). If we identify Γ with $\mathbb{Z}^{2d}$ via $\gamma \mapsto M^{-1}\gamma$, then Γ_Ω is identified with the subgroup

$$H = \left\{ k \in \mathbb{Z}^{2d} : \Omega(Mk, Ml) \in \mathbb{Z} \text{ for all } l \in \mathbb{Z}^{2d} \right\}.$$

Note that $k = (k_1, \ldots, k_{2d}) \in H$ if and only if

$$\sum_{i=1}^{2d} k_i \theta_{i,j} = \sum_{i=1}^{d} (k_i - k_{i+d}) \frac{h_i}{r} \in \mathbb{Z}, \qquad \text{for } 1 \le j \le 2d.$$

Hence, we see that a basis for H is given by the vectors

$$r_1 \cdot e_1, \ldots, r_d \cdot e_d, r_1 \cdot e_{d+1}, \ldots, r_d \cdot e_{2d},$$

where e_i denotes the ith standard basis vector of $\mathbb{R}^{2d}$. It follows that $\mathbb{Z}^{2d}/H \cong (\mathbb{Z}/r_1\mathbb{Z})^2 \times \cdots \times (\mathbb{Z}/r_d\mathbb{Z})^2$, which finishes the proof. ∎

3 Comparison of Projections

In this section we introduce the relevant notions and results we need from the comparison theory of projections and positive elements in C*-algebras. These will later be applied to noncommutative tori.

Let A be a C*-algebra and let p and q be projections in A. We say that p is *(Murray-von Neumann) subequivalent* to q, written $p \precsim q$, if there exists an element $v \in A$ such that $v^*v = p$ and $vv^* \le q$. This notion can be extended to matrices with values in A in the following way: Denote by $M_n(A)$ the set of $n \times n$ matrices with values in A and set

$$M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A),$$

where $M_n(A)$ is identified with the top left corner of $M_{n+1}(A)$. We define subequivalence of two projections $p, q \in M_\infty(A)$ by viewing them as elements in $M_n(A)$ for n big enough and applying the definition in this C*-algebra (this is independent of the chosen n).

For projections $p \in M_m(A)$ and $q \in M_n(A)$ we set

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in M_{m+n}(A).$$

Thus, for $p, q \in M_\infty(A)$, we can define $p \oplus q \in M_\infty(A)$. For $n \in \mathbb{N}$ we also set $p^{\oplus n} = p \oplus \cdots \oplus p$ where the number of summands equals n.

A *tracial state* on a C*-algebra A is a positive linear functional $\tau \colon A \to \mathbb{C}$ such that $\|\tau\| = 1$ and $\tau(ab) = \tau(ba)$ for all $a, b \in A$. By abuse of notation we denote also by τ the function on $M_\infty(A)$ given by $a \mapsto \sum_{i=1}^n \tau(a_{i,i})$ for $a = (a_{i,j})_{i,j=1}^n \in M_n(A)$.

One of the central tools of the present paper is the *radius of comparison* of a C*-algebra A, denoted by $\mathrm{rc}(A)$, which is a number in $[0, \infty]$; see [31, Definition 6.1] for (residually) stably finite C*-algebras, and [6] for an amended definition in the general setting. It encodes the necessary size of the gap between the ranks of two positive elements in some matrix algebra over A to ensure their subequivalence in the sense of Cuntz. Since the precise definition is not relevant here, we only state a specific application to the comparison of projections.

Lemma 3.1. Let A be a unital, exact C*-algebra, and let p, q be projections in $M_\infty(A)$. Suppose that

$$\tau(p) < \tau(q) - \mathrm{rc}(A)$$

for every tracial state τ on A. Then $p \precsim q$.

Proof. This follows from the definition in [6, Section 3.1], using that for exact C*-algebras all quasitraces τ are traces, that for a projection r we have $d_\tau([r]) = \tau(r)$, and that Murray–von Neumann subequivalence and Cuntz subequivalence agree on projections. ■

A C*-algebra is said to be *n-homogeneous* if each of its irreducible representations is n-dimensional. To simplify the discussion, we restrict to the unital case. Here, the typical example is $M_n(C(X))$ for some compact, Hausdorff space X, which we can view as the algebra of continuous sections of the trivial bundle over X with fibers $M_n(\mathbb{C})$. More generally, given a locally trivial $M_n(\mathbb{C})$-bundle over a compact, Hausdorff space X, the algebra of continuous sections is a unital, n-homogeneous C*-algebra with primitive ideal space naturally homeomorphic to X. All unital n-homogeneous C*-algebras arise this way; see, for example, [4, Theorem IV.1.7.23].

We will need the following estimate of the radius of comparison of n-homogeneous C*-algebras, which is a consequence of [32, Theorem 4.6].

Proposition 3.2 (Toms). Let A be a unital n-homogeneous C*-algebra with primitive ideal space X. Then,

$$\mathrm{rc}(A) \leq \frac{\dim(X) - 2}{2n}.$$

4 Noncommutative Tori

Let θ be a real, skew-symmetric $2d \times 2d$ matrix, that is, $\theta^t = -\theta$ where θ^t denotes the transpose of θ. The *noncommutative torus* A_θ associated to θ is the universal C*-algebra generated by $2d$ unitary elements $u_1, \ldots, u_{2d}$ satisfying

$$u_j u_i = e^{2\pi i \theta_{i,j}} u_i u_j, \qquad \text{for } 1 \leq i, j \leq 2d. \tag{4.1}$$

As in Section 2, two real skew-symmetric matrices θ and θ' are congruent if there exists an $R \in \mathrm{GL}(2d, \mathbb{Z})$ such that $\theta' = R^t \theta R$. In this case the associated noncommutative tori A_θ and $A_{\theta'}$ are isomorphic [27, p. 293].

The noncommutative torus A_θ is equipped with a canonical tracial state τ which is determined by

$$\tau(u_1^{k_1} \cdots u_{2d}^{k_{2d}}) = \delta_{k_1,0} \cdots \delta_{k_{2d},0}, \quad k_1, \ldots, k_{2d} \in \mathbb{Z}.$$

For a proof of the following proposition, see [7, Lemma 2.3].

Proposition 4.1. Let p be a projection in $M_\infty(A_\theta)$ and let τ' be any tracial state on A_θ. Then $\tau'(p) = \tau(p)$ where τ denotes the canonical tracial state.

We call a noncommutative torus A_θ *rational* if the associated skew-symmetric matrix θ only has rational entries. It is known that rational noncommutative tori are homogeneous C*-algebras with spectrum homeomorphic to the torus $\mathbb{T}^{2d}$. The dimension of the irreducible representations of a rational noncommutative torus is computed in [33], but we include the computation here for completeness.

Proposition 4.2. Let θ be a rational skew-symmetric nondegenerate $2d \times 2d$ matrix of order $r \in \mathbb{N}$ and let $h_1, \ldots, h_d$ be the invariant factors of $r\theta$. Set

$$n = \frac{r^d}{\gcd(h_1, r) \cdots \gcd(h_d, r)}. \tag{4.2}$$

Then A_θ is n-homogeneous with spectrum homeomorphic to $\mathbb{T}^{2d}$.

Proof. According to the skew normal form, we can find $R \in \mathrm{Gl}(2d, \mathbb{Z})$ such that (2.1) holds, with corresponding diagonal matrix B having the invariant factors on the diagonal. It follows that

$$R^t \theta R = \begin{pmatrix} 0 & r^{-1}B \\ -r^{-1}B & 0 \end{pmatrix}.$$

Hence, $A_\theta \cong A_{R^t\theta R}$ is universally generated by unitaries $u_1, \ldots, u_{2d}$ where $u_{j+d}u_j = e^{2\pi i h_j/r} u_j u_{j+d}$ for $1 \leq j \leq d$ and where all the other unitaries commute. In other words, A_θ is isomorphic to the tensor product

$$A_\theta \cong A_{h_1/r} \otimes \cdots \otimes A_{h_d/r}$$

of d noncommutative 2-tori.

It is well known that the noncommutative 2-torus $A_{p/q}$ where $p, q \in \mathbb{Z}$, $q > 0$ and $\gcd(p, q) = 1$ is q-homogeneous. An explicit irreducible q-dimensional representation $\pi_{p,q}$ is given by $u \mapsto U$ and $v \mapsto V$, where the $q \times q$ matrices U and V are given by

$$U = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{2\pi ip/q} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi ip(q-1)/q} \end{pmatrix}.$$

We form the numbers

$$p_i = \frac{h_i}{\gcd(h_i, r)}, \quad q_i = \frac{r}{\gcd(h_i, r)}, \quad 1 \leq i \leq s,$$

so that $\gcd(p_i, q_i) = 1$. The C*-algebra $A_{h_i/r} = A_{p_i/q_i}$ is then q_i-homogeneous with spectrum $\mathbb{T}^2$.

In general, if A is a unital, n-homogeneous C*-algebra with spectrum X and B is unital, k-homogeneous with spectrum Y, then $A \otimes B$ is nk-homogeneous with spectrum $X \times Y$. Indeed, if π_A and π_B are irreducible representations of A and B, respectively, then $\pi_A \otimes \pi_B$ is an irreducible representation of $A \otimes B$, and by [4, IV.3.4.25] every irreducible representation of $A \otimes B$ arises this way up to unitary equivalence. (This works as soon as one of the C*-algebras is type I, which includes homogeneous C*-algebras.)

Using that $A_\theta \cong A_{h_1/r} \otimes \cdots \otimes A_{h_d/r}$, we obtain that A_θ is $q_1 \cdots q_d$-homogeneous with spectrum homeomorphic to $\mathbb{T}^{2d} = \mathbb{T}^2 \times \cdots \times \mathbb{T}^2$, and the result follows since

$$n = q_1 \cdots q_d = \frac{r^d}{\gcd(h_1, r) \cdots \gcd(h_d, r)}.$$

∎

Using Proposition 4.2 we can now infer the following about the radius of comparison of noncommutative tori.

Proposition 4.3. Let θ be a real, skew-symmetric $2d \times 2d$ matrix.

(1) If θ is non-rational, then A_θ has strict comparison, that is, $\mathrm{rc}(A_\theta) = 0$.
(2) If θ is rational, then

$$\mathrm{rc}(A_\theta) \leq \frac{d-1}{n},$$

where n is as in Proposition 4.2.

Proof. (1) We know from [5, Theorem 1.5] that A_θ is approximately divisible. Then, a combination of [5, Lemma 3.8] and [9, Proposition 6.2] shows that A_θ has strict comparison.

(2) By Proposition 4.2, A_θ is n-homogeneous with spectrum $\mathbb{T}^{2d}$. Applying Proposition 3.2, we get

$$\mathrm{rc}(A_\theta) \leq \frac{\dim(\mathbb{T}^{2d}) - 2}{2n} = \frac{d-1}{n},$$

as desired.

∎

5 Proofs of Main Results

The time-frequency shifts defined in (1.1) satisfy the commutation relation

$$\pi(w)\pi(z) = e^{2\pi i \Omega(z,w)} \pi(z)\pi(w), \qquad \text{for } z, w \in \mathbb{R}^{2d}. \tag{5.1}$$

Let Γ be a lattice in $\mathbb{R}^{2d}$. We consider the C*-algebra $C_\pi^*(\Gamma)$ generated by $\pi(\Gamma) = \{\pi(\gamma) : \gamma \in \Gamma\}$, that is, the smallest operator norm-closed, self-adjoint algebra of operators on $L^2(\mathbb{R}^d)$ that contains $\pi(\Gamma)$. Say that the lattice is represented by a matrix $M \in GL(2d, \mathbb{R})$ and let $\theta = M^t J M$ be the associated real skew-symmetric $2d \times 2d$ matrix. The entries of θ are given by $\theta_{i,j} = \Omega(Me_i, Me_j)$ where e_i denotes the ith standard basis vector of $\mathbb{R}^{2d}$. Hence, according to (5.1), the unitary operators $u_i = \pi(Me_i)$, $1 \leq i \leq 2d$, generating $C_\pi^*(\Gamma)$ satisfy the commutation relations (4.1). They thus define a representation of A_θ on $L^2(\mathbb{R}^d)$ which can be shown to be faithful (see, e.g., [26, Proposition 2.2]). Hence, the C*-algebra $C^*(\pi(\Gamma))$ is isomorphic to A_θ for $\theta = M^t J M$.

Note that if Γ is represented by another matrix $M' \in GL(2d, \mathbb{R})$, then $\theta' = (M')^t J M$ is congruent to θ, so the associated noncommutative torus $A_{\theta'}$ is isomorphic to A_θ. Hence, the lattice Γ uniquely determines a noncommutative torus. Furthermore, Γ is rational if and only if A_θ is rational.

In [26], Rieffel shows that the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ can be completed into a finitely generated projective module $\mathcal{E}_\Gamma$ over $C_\pi^*(\Gamma)$ where the action is implemented by time-frequency shifts. Since the module is finitely generated and projective, it is represented by a projection in a matrix algebra over $C_\pi^*(\Gamma)$. We make the following notation for this projection.

Notation 5.1. We denote by $p_\Gamma \in M_\infty(C_\pi^*(\Gamma))$ the projection representing the Heisenberg module $\mathcal{E}_\Gamma = \overline{\mathcal{S}(\mathbb{R}^d)}$ over $C_\pi^*(\Gamma)$. That is, if $p_\Gamma \in M_n(C_\pi^*(\Gamma))$ for some $n \in \mathbb{N}$, then $\mathcal{E}_\Gamma \cong C_\pi^*(\Gamma)^n p_\Gamma$.

By [26, Theorem 3.4] we have that

$$\tau(p_\Gamma) = \mathrm{covol}(\Gamma), \tag{5.2}$$

where τ denotes the canonical tracial state on $C^*_\pi(\Gamma)$.

In [20, 21] (see also [1, Theorem A (iii)]) Luef established the link between Gabor frames with windows in the Schwartz class and generators of these modules.

Proposition 5.2. The following are equivalent:

(1) Γ admits a k-multi-window Gabor frame with windows in the Schwartz space.
(2) $p_\Gamma \precsim 1^{\oplus k}_{A_\theta}$.

We are now in position to prove the theorems from the introduction.

Proof of Theorem D. Set $s = (d-1)/n_\Gamma$ if Γ is rational and $s = 0$ otherwise, so that by Proposition 4.3, we have that $\mathrm{rc}(C^*_\pi(\Gamma)) \leq s$ in both cases.

Set $k = \lfloor \mathrm{covol}(\Gamma) + s \rfloor + 1$. Then,

$$\mathrm{covol}(\Gamma) < k - s \leq k - \mathrm{rc}(C^*_\pi(\Gamma)).$$

By (5.2) and Proposition 4.1, we have

$$\tau'(p_\Gamma) = \mathrm{covol}(\Gamma) < k - \mathrm{rc}(C^*_\pi(\Gamma)) = \tau'(1^{\oplus k}) - \mathrm{rc}(C^*_\pi(\Gamma))$$

for every tracial state τ' on $C^*_\pi(\Gamma)$. We conclude from Lemma 3.1 that $p_\Gamma \precsim 1^{\oplus k}$. By Proposition 5.2 we infer that Γ admits a k-multi-window Gabor frame with windows in the Schwartz class. ∎

Proof of Theorem C. The assumption that $\mathrm{covol}(\Gamma) < 1 - (d-1)/n_\Gamma$ implies that $k = 1$, where k is as in Theorem D. Hence, the result follows. ∎

6 Applications

In this section we provide some examples where Theorem C can applied. First we reformulate Theorem C in terms of the order and invariant factors of a rational lattice.

Proposition 6.1. Let $\Gamma \subseteq \mathbb{R}^{2d}$ be a rational lattice with order r and invariant factors $h_1, \ldots, h_d$. If

$$h_1 \cdots h_d < r^d - (d-1)\gcd(r, h_1) \cdots \gcd(r, h_d), \tag{6.1}$$

then Γ admits a Schwartz Gabor frame.

Proof. Let $\Gamma = M\mathbb{Z}^{2d}, \theta = M^t J M$ and let $R \in \mathrm{GL}(2d, \mathbb{Z})$ such that $R^t(r\theta)R$ is in its skew normal form. Then

$$\mathrm{covol}(\Gamma) = \det M = (\det \theta)^{1/2} = (\det R^t \theta R)^{1/2}$$

$$= \left(\det \begin{pmatrix} 0 & r^{-1}B \\ -r^{-1}B & 0 \end{pmatrix} \right)^{1/2} = \frac{h_1 \cdots h_d}{r^d}.$$

Hence, the inequality in Theorem C translates into (6.1) when combined with Proposition 2.1. ∎

Example 6.2. As defined in, for example, [16, Definition 2], a lattice $\Gamma \subseteq \mathbb{R}^{2d}$ is *symplectic* if $\Gamma = \alpha M\mathbb{Z}^{2d}$ where $\alpha \in \mathbb{R} \setminus \{0\}$ and M is a *symplectic matrix*, that is, $M^t J M = J$. The frame property of $\pi(\Gamma)g$ is equivalent to the frame property of $\pi(\alpha\mathbb{Z}^{2d})(Tg)$ where T is the associated metaplectic operator, see [14, Section 9.4]. Since a Gabor frame can constructed over $\alpha\mathbb{Z}^{2d}$ using one-dimensional Gaussians whenever $|\alpha| < 1$ and metaplectic operators preserve the Schwartz space [11, Proposition 4.27], it follows that symplectic lattices admit Gabor frames with window in the Schwartz class whenever $|\alpha| < 1$. This can be alternatively shown using Proposition 6.1. Indeed, note that a rational symplectic lattice of covolume less than 1 is represented by a

matrix αM with M symplectic and $\alpha^2 = \frac{p}{q}$ with $p < q$ coprime. It follows that the associated skew-symmetric matrix is $\theta = (\alpha M^t)J(\alpha M) = \alpha^2 J$. In particular, the order of θ is $r = q^2$. The invariant factors $h_1, \ldots, h_d$ are all equal to p^2.

Now, since $p \le q - 1$, one gets $0 < q^{2d} - p^{2d} - (d-1)$ for every $d \ge 1$. Thus, we have

$$h_1 \cdots h_d = p^{2d} < q^{2d} - (d-1) = r^d - (d-1) \gcd(r, h_1) \cdots \gcd(r, h_d).$$

It follows from Proposition 6.1 that Γ admits a Schwartz Gabor frame whenever $\mathbf{covol}(\Gamma) < 1$.

Example 6.3. Let M be the $2d \times 2d$ matrix

$$\begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & \ddots & & 0 \\ 0 & 0 & \cdots & & 1/q \end{pmatrix}$$

where $q \in \mathbb{N}$. Then

$$\theta = M^t J M = \begin{pmatrix} 0 & B \\ -B & 0, \end{pmatrix},$$

where B is the $d \times d$ matrix with diagonal $1, \ldots, 1, 1/q$. Thus, the order of θ is $r = q$ while the invariant factors of $r\theta$ are $h_1 = 1$, $h_i = q$ for $2 \le i \le d$. Thus, by Proposition 6.1 Γ admits a Schwartz Gabor frame whenever

$$q^{d-1} < q^d - (d-1)q^{d-1},$$

that is, $q > d$. Obviously this fails to hold if d is large relative to q.

Example 6.4. Any lattice Γ in $\mathbb{R}^{2d}$ with $\mathbf{covol}(\Gamma) < 1/d$ admits a Schwartz Gabor frame. Indeed, if $\mathbf{covol}(\Gamma) < 1 - (d-1)/n_\Gamma$ then the result follows from Theorem C. Otherwise, assume that $\mathbf{covol}(\Gamma) \ge 1 - (d-1)/n_\Gamma$, which implies $n_\Gamma < d$. Further, we know that

$$\mathbf{covol}(\Gamma) = \frac{h_1 \cdots h_d}{r^d} = \frac{1}{n_\Gamma} \left(\frac{h_1 \cdots h_d}{\gcd(h_1, r) \cdots \gcd(h_d, r)} \right),$$

where note that the second factor is an integer. Thus, $\mathbf{covol}(\Gamma) \ge 1/n_\Gamma$. By our previous computations, we obtain $\mathbf{covol}(\Gamma) \ge 1/d$. This contradicts our assumption.

This result should be compared with [22, Theorem 1.1], which states that almost all lattices with $\mathbf{covol}(\Gamma) < d!/d^d$ admit a Gabor frame with a Gaussian window.

Example 6.5. Let p and q be relatively prime integers and let $a, b, c, d > 0$. Set $\sigma = \sqrt{p^2 + q^2}$ and set

$$M = \begin{bmatrix} pc/(a\sigma^2) & -qd/a & 0 & 0 \\ qc/(b\sigma^2) & pd/b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{bmatrix}.$$

In [28, Corollary 1.6] it is proved that the lattice $\Gamma = M\mathbb{Z}^4$ admits a Gabor frame with window a certain Gaussian function depending on a and b whenever either (i) $c < 1$ and $d < 1$ or (ii) $c < \sigma^2$ and $d < \sigma^{-2}$.

In what follows we interpret Theorem C for Γ when it is a rational lattice. We have that

$$M^t J M = \begin{bmatrix} 0 & S \\ -S^t & 0 \end{bmatrix} \quad \text{where} \quad S = \begin{bmatrix} pc/\sigma^2 & qc/\sigma^2 \\ -qd & pd \end{bmatrix}.$$

Write $c = c_1/c_2$ and $d = d_1/d_2$ with $c_1, c_2, d_1, d_2 \in \mathbb{Z}$. To simplify the analysis we also assume that c_1, c_2, d_1, d_2 and σ^2 have no common factors. Then the order of $M^t J M$ equals $r = \sigma^2 c_2 d_2$ and

$$rM^t JM = \begin{bmatrix} 0 & rS \\ -rS^t & 0 \end{bmatrix} \quad \text{where} \quad rS = \begin{bmatrix} pc_1 d_2 & qc_1 d_2 \\ -qd_1 c_2 \sigma^2 & pd_1 c_2 \sigma^2 \end{bmatrix}.$$

Since $rM^t JM$ is of block form with zero blocks in the upper left and lower right corners, its invariant factors will be the diagonal entries of the the Smith normal form of rS. These are given by $h_1 = \delta_1$ and $h_2 = \delta_2/\delta_1$ where

$$\delta_1 = \gcd(pc_1 d_2, qc_1 d_2, -qd_1 c_2 \sigma^2, pd_1 c_2 \sigma^2) = 1,$$

$$\delta_2 = \det(rS) = p^2 \sigma^2 c_1 c_2 d_1 d_2 + q^2 \sigma^2 c_1 c_2 d_1 d_2 = \sigma^4 c_1 c_2 d_1 d_2.$$

Thus, the inequality in Proposition 6.1 translates into $\sigma^4 c_1 c_2 d_1 d_2 < \sigma^4 c_2^2 d_2^2 - 1 \cdot \sigma^2 c_2 d_2$ or

$$\sigma^2 (c_2 d_2 - c_1 d_1) > 1.$$

This inequality is always satisfied if $c < \alpha$ and $d < \alpha^{-1}$ for some $\alpha > 0$, in particular it is satisfied under either of the assumptions (i) and (ii) from [28]. Thus, Theorem C guarantees the existence of a Schwartz Gabor frame over Γ when these assumptions are satisfied, although it does not give an explicit window like in [28].

We are currently not aware of a Schwartz Gabor frame over a lattice that violates the inequality in Theorem C.

Remark 6.1. We conclude by remarking that the abstract techniques in this paper cannot be used to completely settle Problem A. Indeed, rational noncommutative tori do not have strict comparison, that is, they have strictly positive radius of comparison. For example, it follows from [8, Corollary 1.2] (see also [6, Section 4.1]) that

$$\mathrm{rc}(C(\mathbb{T}^{2n+1})) = n - 1, \quad \text{and} \quad \mathrm{rc}(C(\mathbb{T}^{2n})) \in \{n - 2, n - 1\}$$

for each $n \in \mathbb{N}$.

More concretely, one of the main reasons why these techniques do not fully solve Problem A is that, while for this problem it is enough to show that $\tau(p_\Gamma) < \tau(1)$ implies $p_\Gamma \precsim 1$ (Problem 5.2), the radius of comparison studies tracial comparison (in the sense of Lemma 3.1) of all pairs of positive elements. Thus, a more detailed analysis of the canonical trace and the distinguished projection p_Γ is needed.

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