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Incremental Stability of Traffic Reaction Models

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Abstract

Incremental asymptotic stability is assessed for cooperative systems of ordinary differential equations (ODEs). Such systems of ODEs arise in macroscopic traffic flow modeling which is emphasized in the present thesis. If a system of ODEs is incrementally asymptotically stable then there exists a set of initial conditions from which all solutions converge to each other asymptotically and this can be exploited in a state estimation context.

It is shown that if the state space of a cooperative system of ODEs is a Cartesian product of intervals, then this system is incrementally asymptotically stable if and only if all solutions that are initially ordered, converge to each other in an appropriate sense. This fact is used to establish incremental asymptotic stability for a class Traffic Reaction Models.

The Traffic Reaction Model form a family of numerical schemes to solve scalar conservation laws, governed by partial differential equations (PDEs). For one conservation law there are several numerical schemes and if the scheme is semi-discrete it gives rise to a system of ODEs. Suitable conditions on the conservation law are provided such that a particular semi-discrete scheme gives rise to an incrementally exponentially stable system of ODEs.

Keywords: Cooperative Systems, Conservation Laws, Finite Volume Scheme, Incremental Stability, State Estimation, Traffic Reaction Model

List of Publications

This thesis is based on the following publications:

[A] **Sondre Wiersdalen**, Mike Pereira, Annika Lang, Gábor Szederkényi, Jean Auriol, and Balázs Kulcsár, “Incremental Exponential Stability of the Unidirectional Flow Model”. .

Other publications by the author, not included in this thesis, are:

[B] Stability Analysis of Compartmental and Cooperative Systems, “**Sondre Wiersdalen**, Mike Pereira, Annika Lang, Gábor Szederkényi, Jean Auriol, and Balázs Kulcsár”. *IEEE Transaction on Automatic Control, 2025 (Under Review and available on arXiv).*

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Acronyms

ODE:	Ordinary Differential Equation
PDE:	Partial Differential Equation
IAS:	Incrementally Asymptotically Stable
IES:	Incrementally Exponentially Stable

Contents

Part I

Overview

CHAPTER 1

Introduction

The Traffic Reaction Model, is a family of numerical schemes to solve certain scalar conservation laws that arise in traffic modelling [1]. A semi-discrete instance of this family, gives rise to a system of ordinary differential equations (ODEs). The present thesis is an assessment of incremental exponential stability of such systems. If a system of ODEs is incrementally exponentially stable, then some or all of its solutions converge to each other at an exponential rate, which can be useful in the context of state estimation.

State estimation is a typical problem in control engineering. In broad terms, this means inferring the internal states of a system from known inputs and outputs. If the inputs and outputs of a system are related by a linear (dynamical) system, then this problem is well understood with a well-established solution: the *Luenberger observer* [2]. For linear discrete-time systems subject to noise, we even have the celebrated Kalman filter [3] and its continuous-time analog by Kalman and Bucy [4]. For nonlinear systems, there is no one-size-fits-all solution.

Let us put state estimation in more precise terms, for the general system

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \quad x(0) = x_0 \\ y(t) = h(t, x(t)) \end{cases} \tag{1.1}$$

where $x(t)$ is the state and $y(t)$ the measurement, both vectors in general. For simplicity, we assume that $x(t) \in \mathbb{R}^n$ for all $t \geq 0$ for any given $x_0 \in \mathbb{R}^n$. The associated state estimation problem then reads: Provide an estimate $\hat{x}(t)$ of $x(t)$ given y, h, and f, such that

$$|x(t) - \hat{x}(t)|_1 \to 0, \quad t \to +\infty \tag{1.2}$$

for all $x_0 \in \mathbb{R}^n$.

It is well known that if (1.1) is a linear time-invariant system and (completely) observable, then the asymptotic convergence (1.2) can be achieved at an arbitrary exponential rate [2, Lemma 1]. For the sake of argument, let us suppose that the asymptotic convergence (1.2) alone is satisfactory. Then (1.2) can be achieved if the nonlinear system (1.1) is *incrementally asymptotically stable* [5], a property that is defined in precise terms in the next chapter.

However, this property implies that any pair of solutions of (1.1) that starts in a particular set, say $\mathcal{S}$, approaches each other asymptotically. It is then evident that if $x_0 \in \mathcal{S}$ and we define $\hat{x}$ to be the solution to (1.1) for *some* initial condition $\hat{x}_0 \in \mathcal{S}$, then (1.2) follows. This approach to state estimation is applicable if the right-hand side f of (1.1) and the set $\mathcal{S}$ is known. On the one hand, a limitation of this approach is that the convergence rate of (1.2) depends solely on f. On the other hand, no additional (formal) measurement, such as $y(t)$ in (1.1), is needed.

Research questions and contributions

Macroscopic traffic flow modeling leads, in certain cases, to the study of conservation laws governed by scalar partial differential equations in one spatial dimension. For one such conservation law, there are many Traffic Reaction Models, and each semi-discrete Traffic Reaction Model gives rise to a system of ODEs. It is these systems of ODEs that we put under the microscope in this thesis and we simply refer to these systems as Traffic Reaction Models.

With the previous exposition of state estimation and incremental asymptotic stability in mind, we pose the following research questions:

RQ1) Under what conditions (if any) is a Traffic Reaction Model incrementally exponentially stable?

RQ2) How do these conditions relate to the underlying conservation law?

RQ3) If a Traffic Reaction Model is incrementally exponentially stable, at what rate do the solutions converge to each other?

The included paper A gives an affirmative answer to RQ1) and RQ3) for the Unidirectional Flow Model. This model is a particular Traffic Reaction Model for a conservation law with a quadratic flux function and without sink and source terms. The convergence rate is shown to be exponential and explicit formulae to compute an estimate of the rate are provided. Simulation experiments suggest that the estimate is conservative.

Every semi-discrete Traffic Reaction Model is governed by a cooperative system of ODEs and has a state space given by a box (Cartesian product of intervals). We show that such systems are incrementally asymptotically stable if and only if all pairs of initially ordered solutions converges to each other in an appropriate sense. The conclusion of this result can be verified by finding an appropriate Lyapunov function. A simple test to verify nonexpansiveness is also provided for the same class of systems.

Using the main theoretical results we address RQ1) and RQ2) for a class Traffic Reaction Models that is more general than the Unidirectional Flow Model. These Traffic Reaction Models may have a sink and source term, in contrast to the Unidirectional Flow Model. The flux function of the underlying conservation law need not be quadratic, but sufficiently smooth, concave, and satisfy a positivity requirement. In the context of traffic flow modeling these conditions are quite mild.

Layout of the thesis

Chapter 2 includes basic definitions and a short exposition on cooperative systems. The exposition is followed by stability analysis of cooperative systems defined on a box. In particular, the analysis involves incremental asymptotic stability and this constitutes the main theoretical results of the thesis.

The main theoretical results are applied to Traffic Reaction Models in Chapter 3. First, an introduction to conservation laws and Traffic Reaction Models is provided. Then, a special class Traffic Reaction Models is introduced for which incremental exponential stability is established.

The concluding Chapter 5 is preceded by a summary of the included paper A. Most of the proofs are put into the appendices.

CHAPTER 2

Definitions and Preliminary Results

Notation: We denote the set of the natural numbers $1, 2, 3, \ldots$ by $\mathbb{N}$. The set of the real numbers is denoted by $\mathbb{R}$. A real number x is positive if $x > 0$ and nonnegative if $x \geq 0$. Likewise, a real number x is said to be negative if $x < 0$ and nonpositive if $x \leq 0$.

Euclidean space is denoted by $\mathbb{R}^n$, $n \in \mathbb{N}$ and we define $|\cdot|_1 : \mathbb{R}^n \to [0, +\infty)$ by $|x|_1 := \sum_{i=1}^{n} |x_i|$. The vectors $x, y \in \mathbb{R}^n$ are said to be ordered if $x - y$ is an element of $(-\infty, 0]^n$ or $[0, +\infty)^n$. If $y - x \in [0, +\infty)^n$ we write $x \leq y$.

For a set X we use the convention that $X \subset X$. Let X be a subset of $\mathbb{R}^n$ then D is a neighborhood of X if D is open and $X \subset D$. If x is a point in $\mathbb{R}^n$, then $D \subset \mathbb{R}^n$ is a neighbourhood of x if D is open and $x \in D$.

We use $\dot{y}$ to denote the derivative function of $y : Y \to \mathbb{R}^n$, $n \geq 1$, $Y \subset \mathbb{R}$.

2.1 Notions of Stability

The primary objects of study in this thesis are systems of first-order ordinary differential equations (ODEs)

$$\dot{x}(t) = f(t, x(t)). \tag{S}$$

Assume that there exists a nonempty set $\mathcal{X} \subset \mathbb{R}^n$ such that f is continuous from a neighborhood D of $[0,+\infty) \times \mathcal{X}$ to $\mathbb{R}^n$ and for every $(\tau, \xi) \in [0,\infty) \times \mathcal{X}$ there exists a unique solution $x : [\tau,+\infty) \to \mathcal{X}$ to (S) with $x(\tau) = \xi$. We adopt the following notation:

$$\begin{cases} \text{The solution to (S) passing through } \xi \text{ at} \\ \text{time } t = 0 \text{ is denoted by } t \mapsto \phi(t,\xi),\ \phi(0,\xi) = \xi. \end{cases}$$

The main notion of stability we consider for (S) is incremental asymptotic stability. This notion is characterized by $\mathcal{KL}$ functions which are composed of all functions β from $[0,+\infty)^2$ to $[0,+\infty)$ such that

i) β is continuous,

ii) for each fixed $r \geq 0$, $\beta(r,t) \to 0$ as $t \to +\infty$, and

iii) for each fixed $t \geq 0$, $r \mapsto \beta(r,t)$ is an increasing function.

In most cases, we consider an exponentially decreasing $\mathcal{KL}$ function

$$(r,t) \mapsto \gamma e^{-\lambda t} r \tag{2.1}$$

where $\gamma \geq 1$ and $\lambda > 0$ is refered to as the rate of decay.

Definition 1 (Incremental asymptotic stability)**:** The system (S) is said to be *incrementally asymptotically stable (IAS) with respect to* $\mathcal{X}$ if there exists $\beta \in \mathcal{KL}$ such that for all $\xi^1, \xi^2 \in \mathcal{X}$

$$|\phi(t,\xi^1) - \phi(t,\xi^2)|_1 \leq \beta(|\xi^1 - \xi^2|_1, t), \quad t \geq 0. \tag{2.2}$$

The system (S) is said to be *incrementally exponentially stable (IES) in* $\mathcal{X}$ if there exist constants $\lambda > 0$ and $\gamma \geq 1$ such that (2.2) holds with $\beta(r,t) \coloneqq \gamma e^{-\lambda t} r$.

A similar definition of incremental asymptotic stability is used in [6], but there, the underlying dynamical system depends on an input function. The following, is a Lyapunov approach to establish incremental exponential stability for (S), which aligns with the theory presented in [6].

Proposition 1: *Consider the system* (S) *and fix any continuously differentiable* $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. *If there exist constants* $k_1, k_2, \lambda > 0$ *such that for all*

$x, y \in \mathcal{X}$ and $t \geq 0$

$$k_1|x-y|_1 \leq V(x,y) \leq k_2|x-y|_1 \tag{2.3a}$$

$$W(t,x,y) := \frac{\partial V(x,y)}{\partial y} f(t,y) - \frac{\partial V(x,y)}{\partial x} f(t,x) \leq -\lambda V(x,y) \tag{2.3b}$$

then (S) *is IES with respect to $\mathcal{X}$, with the rate of decay λ and $\gamma := k_2/k_1$.*

Proof. Pick any two solutions $x^1, x^2 : [0, +\infty) \to \mathcal{X}$ of (S). Fix any continuously differentiable $V : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ and constants $\lambda, k_1, k_2 > 0$ such that (2.3) holds. Define $t \mapsto M(t) := V(x^1(t), x^2(t))$, then it follows by (2.3a) that

$$M(0) \leq k_2|x^1(0) - x^2(0)|_1, \quad |x^1(t) - x^2(t)|_1 \leq \frac{1}{k_1} M(t), \quad t \geq 0, \tag{2.4}$$

and it follows by (2.3b) that $\dot{M}(t) = W(t, x^1(t), x^2(t)) \leq -\lambda M(t)$ for all $t \geq 0$. By virtue of Grönwalls inequality, $M(t) \leq e^{-\lambda t} M(0)$ for all $t \geq 0$. The latter and (2.4) implies that $|x^1(t) - x^2(t)| \leq \ k_2/k_1 e^{-\lambda t}|x^1(0) - x^2(0)|_1$ for all $t \geq 0$, which completes the proof. □

Later in the thesis, we use the notion of nonexpansiveness to establish incremental asymptotic stability for a particular class ODEs and it is defined as follows.

Definition 2 (Nonexpansiveness)**:** The system (S) is said to be *nonexpansive with respect to* $\mathcal{X}$ if for all $\xi^1, \xi^2 \in \mathcal{X}$ and for all $t_1 \geq t_0 \geq 0$

$$|\phi(t_1, \xi^1) - \phi(t_1, \xi^2)|_1 \leq |\phi(t_0, \xi^1) - \phi(t_0, \xi^2)|_1. \tag{2.5}$$

If (S) is nonexpansive with respect to $\mathcal{X}$, it means that the distance (1-norm) between any pair of solutions is nonincreasing over time. In the next section, we provide a simple test to verify this property for cooperative systems (cf. Proposition 4).

2.2 Cooperative Systems

We provide here a short exposition on cooperative systems based upon Chapter 3.1 in [7]. In the next section, we provide some methods to establish incremental asymptotic stability for cooperative systems defined on a Cartesian product of n intervals of $\mathbb{R}$.

A set $D \subset \mathbb{R}^n$ is said to be *p-convex* if $ta + (1-t)b \in D$ for all $t \in [0,1]$ and $a \leq b$ in D. And a function $F : D \to \mathbb{R}^n$ is said to be *type K* on D if

$$F_i(a) \leq F_i(b), \quad i = 1, \ldots, n$$

whenever $a \leq b$ and $a_i = b_i$.

In [7], the system of ODEs

$$\dot{x}(t) = F(x(t)) \tag{2.6}$$

is said to be a *cooperative system* if $F : D \to \mathbb{R}^n$ is continuously differentiable, $D \subset \mathbb{R}^n$ is open and p-convex, and

$$\frac{\partial F_i(x)}{\partial x_j} \geq 0, \quad i \neq j, \quad x \in D. \tag{2.7}$$

The inequality (2.7) implies that F is type K on D [7, Remark 1.1]

A fundamental property of cooperative systems is the ordering of their solutions. That is, if x and y satisfy (2.6) on an interval $[t_0, t_1]$ and $x(t_0) \leq y(t_0)$, then $x(t_1) \leq y(t_1)$ [7, Proposition 1.1].

Note that there is no requirement that the solutions to a cooperative system are defined for all $t \geq 0$ and so far we have not considered the nonautonomous case. For reference, we define the following class of systems.

Definition 3: Let $\mathcal{X} \subset \mathbb{R}^n$ have nonempty interior, let f be continuous from a neighbourhood of $[0, +\infty) \times \mathcal{X}$ to $\mathbb{R}^n$, and suppose for all $(\tau, \xi) \in [0, +\infty) \times \mathcal{X}$ the system

$$\dot{x}(t) = f(t, x(t)) \tag{2.8}$$

admits a unique solution $x : [\tau, +\infty) \to \mathcal{X}$ such that $x(\tau) = \xi$. If $x \mapsto f(t,x)$ is type K on the interior of $\mathcal{X}$ for each $t \geq 0$, then (2.8) is said to be *cooperative in* $\mathcal{X}$.

In the sequel, we consider systems that are cooperative in a Cartesian product of n intervals of $\mathbb{R}$, hereby referred to as a *box*. Let $\mathcal{X}_1, \ldots, \mathcal{X}_n \subset \mathbb{R}$ be intervals and assume hereon that the system

$$\dot{x}(t) = f(t, x(t)), \tag{C}$$

is cooperative in the box $\mathcal{X} \coloneqq \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$. The following result is salient to the next section.

Proposition 2 (Ordering of solutions)**:** *Fix any* $t_1 > t_0 \geq 0$ *and suppose* (C) *is cooperative in a box* $\mathcal{X} \subset \mathbb{R}^n$. *If* x *and* y *satisfy* (C) *on* $[t_0, t_1]$, $x(t_0), y(t_0) \in \mathcal{X}$, *and* $x(t_0) \leq y(t_0)$, *then* $x(t_1) \leq y(t_1)$.

Proof. Since $x \mapsto f(t,x)$ type K on the interior of $\mathcal{X}$ for each $t \geq 0$ and $\mathcal{X}$ is convex, this result is due to Proposition 1.1, Remark 1.3, and Remark 1.4 in [7]. ☐

Stability Analysis of Cooperative Systems

In the included paper A, incremental exponential stability is established for the Unidirectional Flow Model. This system is cooperative in a box which plays an important part in the proofs presented there. We provide here some stability results that apply to generic systems that are cooperative in a box. Every statement in this subsection is proven in Appendix A. First, we state a supporting lemma which is the main enabler for the stability results.

Lemma 1: *Let* $\mathcal{X} \subset \mathbb{R}^n$ *be a box and pick any two points* $p^1, p^2 \in \mathcal{X}$. *If* $q^1, q^2 \in \mathbb{R}^n$ *are given by*

$$q_i^1 \coloneqq \min\{p_i^1, p_i^2\}, \quad q_i^2 \coloneqq \max\{p_i^1, p_i^2\}, \quad i = 1, \ldots, n, \tag{2.9}$$

then $q^1, q^2 \in \mathcal{X}$, $|q^1 - q^2|_1 = |p^1 - p^2|_1$, *and* $q^1 \leq p^k \leq q^2$ *for* $k = 1, 2$.

Fix any $t_1 > t_0 \geq 0$ *and suppose* (C) *is cooperative in the box* $\mathcal{X}$. *If* x^k, y^k *are the solutions to* (C) *such that* $x^k(t_0) = p^k$, $y^k(t_0) = q^k$ *for* $k = 1, 2$, *then*

$$|x^1(t_0) - x^2(t_0)|_1 = |y^1(t_0) - y^2(t_0)|_1 \tag{2.10a}$$

$$|x^1(t_1) - x^2(t_1)|_1 \leq |y^1(t_1) - y^2(t_1)|_1 \tag{2.10b}$$

We can now state the main theorem of this section.

Theorem 1: *Suppose the system* (C) *is cooperative in a box* $\mathcal{X} \subset \mathbb{R}^n$. *Then* (C) *is IAS with respect to* $\mathcal{X}$ *if and only if there exists* $\beta \in \mathcal{KL}$ *such that for all* $\xi^1 \leq \xi^2$ *in* $\mathcal{X}$ *and* $t \geq 0$

$$\sum_{i=1}^{n} \left(\phi_i(t, \xi^2) - \phi_i(t, \xi^1)\right) \leq \beta\Big(\sum_{i=1}^{n} \left(\xi_i^2 - \xi_i^1\right), t\Big) \tag{2.11}$$

Loosely speaking, the recent theorem says the following. A system that is cooperative in a box $\mathcal{X}$ is incrementally asymptotically stable with respect to $\mathcal{X}$, if and only if every pair of solutions that is initially ordered converges to each other asymptotically. To apply the theorem we need means to show that the ordered solutions converge to each other in the first place. This is provided by the following proposition.

Proposition 3: *Suppose* (C) *is cooperative in a box* $\mathcal{X} \subset \mathbb{R}^n$*. If there exist a continously differentiable* $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *and constants* $k_1, k_2, \lambda > 0$ *such that* (2.3a) *and* (2.3b) *hold for all* $t \geq 0$ *and* $x \leq y$ *in* $\mathcal{X}$*, then* (C) *is IES with respect to* $\mathcal{X}$*, with the rate of decay* λ *and* $\gamma \coloneqq k_2/k_1$*.*

Proposition 3 is identical to Proposition 1 except for the fact that (2.3) need only hold when x and y are ordered. If the hypotheses of Proposition 3 are true, then every pair of solutions that are initially ordered, converge to each other exponentially. In consequence, the cooperative system (C) is IES with respect to $\mathcal{X}$, owing to Theorem 1. The proof of Proposition 3 is found in Appendix A.

We close this subsection and chapter with a test to verify nonexpansiveness for systems that are cooperative in a box.

Proposition 4: *If* (C) *is cooperative in a box* $\mathcal{X} \subset \mathbb{R}^n$ *and*

$$\sum_{i=1}^{n} \big(f_i(t,y) - f_i(t,x)\big) \leq 0 \text{ for all } t \geq 0 \text{ and } x \leq y \text{ in } \mathcal{X}, \tag{2.12}$$

then (C) *is nonexpansive with respect to* $\mathcal{X}$*.*

CHAPTER 3

Incremental Stability of Traffic Reaction Models

Traffic Reaction Models are numerical schemes to solve certain 1-dimensional conservation laws arising in (vehicle) traffic modeling [1]. A Traffic Reaction Model is either fully discrete or semi-discrete. The latter, which we consider in this thesis, are governed by systems of ODEs and will be simply referred to as (semi-discrete) Traffic Reaction Models.

In the first section, we describe the underlying conservation law and Traffic Reaction Models. The second section introduces a class Traffic Reaction Models and states some basic properties of these systems as a setup for the section thereafter. There we state suitable conditions for a Traffic Reaction Model to be IES with respect to its entire state space.

This chapter can be seen as an application of the preliminary results in Section 2.2 and an extension to some of the results presented by paper A. In paper A, the Unidirectional Flow Model is shown to be IES with respect to its entire state space under simple-to-check conditions. This model can be seen as a particular Traffic Reaction Model.

3.1 Numerical Scheme

The study of vehicular traffic leads, in certain cases, to the study of conservation laws [8, Ch 11.1]. Consider a highway of infinite length and suppose the maximum (traffic) density [veh/km] on this highway is given by $\rho_{\max} > 0$. Denote by $\rho(x,t) \in [0, \rho_{\max}]$, the traffic density at time $t \geq 0$ and position $x \in \mathbb{R}$ on the highway. Then, the 1-dimensional[1] conservation law (3.1a)

$$\partial_t \rho(x,t) + \partial_x F(\rho(x,t)) = q(x,t,\rho(x,t)), \quad x \in \mathbb{R}, \quad t \geq 0, \tag{3.1a}$$

$$\rho(x,0) = d(x), \qquad x \in \mathbb{R}, \tag{3.1b}$$

also known as The Lighthill–Whitham–Richards (LWR) traffic model [9], [10], is one among several models ([11],[12],[13]) used to study vehicle traffic. We refer to $F : \mathbb{R} \to \mathbb{R}$ as a *flux function*. The term q represents on-ramps and off-ramps.

In [1], a family of numerical schemes is proposed to solve the Cauchy problem (3.1a)-(3.1b), collectively known as Traffic Reaction Models. A Traffic Reaction Model is either fully discrete (in time and space) or semi-discrete (discrete in space and continuous in time). Under suitable conditions on F, q, and d, (3.1a)-(3.1b) admits a unique *entropy solution* [14, Definition 1], and the fully discrete Traffic Reaction Models converges to this entropy solution [1, Theorem 2.4].

We describe next finite volume discretizations (in space) of (3.1a)-(3.1b) on a finite spatial domain. That is, we want to approximate $\rho(x,t)$ for $t \geq 0$ and $x \in [a,b]$ for some choice of $a < b$. For simplicity of this presentation, we assume that $q \equiv 0$. We begin with imposing some assumptions on F and d.

Assumption 1. The flux function $F : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and there exists a function $g : [0, \rho_{\max}]^2 \to \mathbb{R}$ such that

(i) g is Lipscitz continuous
(ii) $(z_1, z_2) \mapsto g(z_1, z_2)$ is nondecreasing in z_1 and nonincreasing in z_2
(iii) $g(0, z) = 0$ and $g(z, \rho_{\max}) = 0$ for all $z \in [0, \rho_{\max}]$
(iv) $F(z) = g(z, z)$ for all $z \in [0, \rho_{\max}]$.

Assumption 2. The initial data d map $\mathbb{R}$ to $[0, \rho_{\max}]$ and $\int_{\mathbb{R}} d(x) \mathrm{d}x < +\infty$.

[1] 1-dimensional because the dimension of the space variable is 1

Choose a spatial domain $[a, b] \subset \mathbb{R}$, $a < b$ and divide this domain into a collection $\{C_0, \ldots, C_{n+1}\}$, of *road segments* (closed intervals). Each with the same length and such that C_i is upstream and adjacent to C_{i+1}, i.e

$$\begin{aligned} |C_i| &= \frac{b-a}{n+2} =: \Delta x, && i = 0, \ldots, n+1. \\ \max C_i &= \min C_{i+1}, && i = 0, \ldots, n. \end{aligned} \tag{3.2}$$

Define the *boundary values* ρ_0 and ρ_{n+1} by

$$\rho_k(t) := \frac{1}{\Delta x} \int_{C_k} \rho(x, t) \mathrm{d}x, \quad t \geq 0, \quad k = 0, n+1. \tag{3.3}$$

The finite volume scheme then reads

$$\dot{\rho}_i = \frac{1}{\Delta x} \big(g(\rho_{i-1}, \rho_i) - g(\rho_i, \rho_{i+1}) \big), \qquad i = 1, \ldots, n, \tag{3.4a}$$

$$\rho_i(0) = d_i := \frac{1}{\Delta x} \int_{C_i} d(x) \mathrm{d}x \in [0, \rho_{\max}], \quad i = 1, \ldots, n, \tag{3.4b}$$

where we write $\rho_i = \rho_i(t)$ for brevity, (3.4a) is called a (semi-discrete) Traffic Reaction Model, and $\rho_i(t)$ can be seen as an approximation of the spatial mean of ρ over C_i at time $t \geq 0$, i.e

$$\rho_i(t) \approx \frac{1}{\Delta x} \int_{C_i} \rho(x, t) \mathrm{d}x, \quad t \geq 0, \quad i = 1, \ldots, n. \tag{3.5}$$

3.2 A Class Traffic Reaction Models

In the next section, we assess incremental exponential stability for a class Traffic Reaction Models. We define this class here and state some basic properties of these systems.

Given an integer $n > 1$ and a real number $\rho_{\max} > 0$, we consider the following class Traffic Reaction Models

$$\dot{\rho}_i = \rho_{i-1} h(\rho_i) - \rho_i h(\rho_{i+1}) + q_i(t, \rho_i), \quad i = 1, \ldots, n \tag{3.6}$$

for which we assume

(A1) ρ_0 and ρ_{n+1} are continuous form $\mathbb{R}$ to $[0, \rho_{\max}]$

(A2) $h : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, nonincreasing, and such that $h(\rho_{\max}) = 0$.
(A3) the $(t, z) \mapsto q_i(t, z)$ are continuous from $\mathbb{R}^2$ to $\mathbb{R}$, locally Lipschitz in z and uniformly in t, and

$$\begin{aligned} q_i(t, \rho_{\max}) \leq 0 \leq q_i(t, 0), \quad t \geq 0, \quad i = 1, \dots, n, \\ t \mapsto q_i(t, z) \text{ is nonincreasing}, \quad t \geq 0, \quad i = 1, \dots, n. \end{aligned} \tag{3.7}$$

We do not make it an explicit assumption that (3.6) *has* to be a finite volume scheme for the conservation law (3.1a). However, (3.6) can be seen as such if the flux function in (3.1a) satisfies

$$F(z) = zh(z), \quad z \in \mathbb{R}. \tag{3.8}$$

We may then choose

$$g(z_1, z_2) \coloneqq z_1 h(z_2), \quad (z_1, z_2) \in [0, \rho_{\max}]^2$$

which satisfies the Assumption 1. In the next section, we justify that several flux functions can be written as in (3.8) under relatively mild assumptions. The term q_i represents on- and off-ramps on the road segment C_i. The reader is referred to [1] to see how these terms are related to the conservation law (3.1a) and the appropriate conditions needed on the term q. Before we move on to the next section we state some preliminary properties of the system (3.6).

Proposition 5: *Consider the Traffic Reaction Model* (3.6) *with (A1)-(A3), then*

(i) for all $(\tau, \xi) \in [0, +\infty) \times [0, \rho_{\max}]^n$, (3.6) *admits a unique solution* $(\rho_1, \dots, \rho_n) : [\tau, +\infty) \to [0, \rho_{\max}]^n$*, with* $(\rho_1(0), \dots, \rho_n(0)) = \xi$*,*
(ii) (3.6) *is cooperative in* $[0, \rho_{\max}]^n$*,*
(iii) (3.6) *is nonexpansive with respect to* $[0, \rho_{\max}]^n$*.*

Proof. See Appendix C □

3.3 Stability Results

The results presented here extend a result in the included paper A. There, the Unidirectional Flow Model is considered, which is governed by

$$\dot{\rho}_i = w\rho_{i-1}(\rho_{\max} - \rho_i) - w\rho_i(\rho_{\max} - \rho_i), \quad i = 1, \dots, n \tag{3.9}$$

for some positive constant w and ρ_0, ρ_{n+1} continuous from $\mathbb{R}$ to $[0, \rho_{\max}]$. This system is a special case of the Traffic Reaction Model (3.6) with the q_i set identically zero and

$$h(z) := w(c - z), \quad z \in \mathbb{R} \tag{3.10}$$

and can be seen as a finite volume discretization of the conservation law (3.1a) with $q \equiv 0$ and the quadratic flux function $f(z) := wz(\rho_{\max} - z)$.

The Corollary 1 in paper A provides sufficient criteria for (3.9) to be IES with respect to $[0, \rho_{\max}]^n$. Since the included paper A works with a slightly different definition for incremental exponential stability than this thesis, we translate Corollary 1 in paper A, to conform with Definition 1.

Proposition 6 (Corollary 4, Paper A)**:** *If there exists* $\bar{\rho} \in (0, \rho_{\max}]$ *such that*

$$\rho_{n+1}(t) \leq \rho_{\max} - \bar{\rho}, \quad t \geq 0 \tag{3.11}$$

then the system (3.9) *is IES with respect to* $[0, \rho_{\max}]^n$*.*

Arguably, the hypothesis of Proposition 6 is simple to verify. Let us view (3.9) as a Traffic Reaction Model. Then the condition (3.11) means that the density $\rho_{n+1}(t)$ in the last road segment C_{n+1} is bounded away from $\rho_{\max}$ for all $t \geq 0$. Roughly speaking, this implies last road segment C_{n+1} is never congested and therefore the flow of vehicles from C_n to C_{n+1} is never "blocked". Note that if

$$\rho_0(t) = 0 \text{ and } \rho_{n+1}(t) = \rho_{\max} \text{ for all } t \geq 0,$$

then every solution to (3.9) satisfies

$$\sum_{i=1}^{n} \dot{\rho}_i(t) = 0, \quad \sum_{i=1}^{n} \rho_i(t) = \sum_{i=1}^{n} \rho_i(0), \quad t \geq 0, \tag{3.12}$$

which is a natural consequence of the conservation of vehicles. So, it is a necessary condition that there exists a least some $t \geq 0$ such that

$$\rho_0(t) \neq 0 \text{ or } \rho_{n+1}(t) \neq \rho_{\max}$$

for (3.9) to be incrementally exponentially stable with respect to $[0, \rho_{\max}]^n$. We can now state the main result of this chapter.

Theorem 2: *Consider the Traffic Reaction Model* (3.6) *with (A1)-(A3) and suppose*

$$h(z) > 0, \quad z \in [0, \rho_{\max}). \tag{3.13}$$

If there exists $\bar{\rho} \in (0, \rho_{\max}]$ *such that* $\rho_{n+1}(t) \leq \rho_{\max} - \bar{\rho}$*, then* (3.6) *is IES with respect to* $[0, \rho_{\max}]^n$*.*

Proof. See Appendix C. □

Granted the function h satisfies (3.13), this theorem is an extension of Proposition 6 to a larger class Traffic Reaction Models. To justify the hypothesis (3.13), we round off this chapter with the following proposition.

Proposition 7: *Fix any constant* $\rho_{\max} > 0$*. Assume that* $F : \mathbb{R} \to \mathbb{R}$ *is twice continuously differentiable, concave, and such that*

$$F(0) = F(\rho_{\max}) = 0 \tag{3.14a}$$

$$F(z) > 0, \quad z \in (0, \rho_{\max}). \tag{3.14b}$$

Then, $h : \mathbb{R} \to \mathbb{R}$ *given by*

$$h(z) := \int_0^1 \dot{F}(zy)\mathrm{d}y, \quad z \in \mathbb{R}, \tag{3.15}$$

is locally Lipschitz and nonincreasing, $h(z) > h(\rho_{\max}) = 0$ *for all* $z \in [0, \rho_{\max})$*, and*

$$F(z) = zh(z), \quad z \in \mathbb{R}. \tag{3.16}$$

Proof. See Appendix C. □

Remark: To assume that the flux function F is concave is typical in traffic flow modeling [15, pg. 27]. The assumption (3.14a) asserts that the flux of vehicles about a point x on the highway is zero if the highway is empty or congested about x. Whereas the assumption (3.14b) asserts that the flux of vehicles is positive about x if the highway is neither empty nor congested about x.

CHAPTER 4

Summary of included papers

This chapter provides a summary of the included papers.

4.1 Paper A

> **Sondre Wiersdalen**, Mike Pereira, Annika Lang, Gábor Szederkényi, Jean Auriol, and Balázs Kulcsár
> Incremental Exponential Stability of the Unidirectional Flow Model
> Submitted to IEEE Transactions on Automatic Control, Dec 2023.

This paper considers incremental stability analysis of the Unidirectional Flow Model. The model is a particular case of the class Traffic Reaction Models which are used to predict traffic density on a stretch of highway. A system is incrementally exponentially stable if all solutions that start in some set converge to each other at an exponential rate. And this stability property is considered because of its utility to state estimation problems.
Suitable conditions are provided such that the Unidirectional Flow Model is incrementally exponentially stable with respect to its entire state space. The conditions are simple to check and an estimate of the exponential conver-

gence rate is provided with explicit formulae. The estimate is tested against numerical simulations and seems to be conservative.

CHAPTER 5

Concluding Remarks and Future Work

The present thesis and its results can be divided into two parts: theory and application. The theoretical part (Section 2.2) is a self-contained analysis of cooperative systems of ODEs defined on a box. The main theoretical result is that such systems are incrementally asymptotically stable on their entire state space if and only if all initially ordered solutions converge to each other asymptotically in a $\mathcal{KL}$-sense (Theorem 1). The necessary and sufficient condition can be shown if an appropriate Lyapunov function can be found. However, no claim is made that such a Lyapunov function must exist for a cooperative system (defined on a box) to be incrementally asymptotically stable. Suitable conditions to establish nonexpansiveness for cooperative systems of ODEs defined on a box, are also provided.

In the application part, the theoretical results are applied to Traffic Reaction Models. These models are governed by cooperative systems of ODEs and are derived from finite volume discretizations (in space) of the LWR model, which is a 1-dimensional conservation law. Easy-to-check conditions are provided such that a Traffic Reaction Model is incrementally exponentially stable. These conditions were shown to be rather mild in view of the underlying conservation law. Incremental asymptotic stability has a clear use case in state estimation and this was the main motivation to assess this type of stability for Traffic Reaction Models. Future research directions include but are not limited to:

- Leverage incremental asymptotic stability of Traffic Reaction Models for controller design.
- Assess incremental asymptotic stability for systems similar to Traffic Reaction Models, that are not necessarily cooperative.
- Any of the above in a more general context, for example, a network of interconnected roads.

Appendices

Appendix A Cooperative Systems

Proof of Lemma 1

Since $\mathcal{X}$ is a box and $p^1, p^2 \in \mathcal{X}$, it follows by the definition (2.9) that $q^1, q^2 \in \mathcal{X}$ and

$$|p^1 - p^2|_1 = |q^1 - q^2|_1, \quad q^1 \leq p^k \leq p^2, \quad k = 1, 2. \tag{A.1}$$

The equality (A.1) implies (2.10a). Since (C) is assumed cooperative in the box $\mathcal{X}$ it follows by Proposition 2 and the inequality (A.1) that

$$y^1(t_1) \leq x^k(t_1) \leq y^2(t_1)$$

which implies the bound (2.10b). This completes the proof. □

Proof of Theorem 1

Fix any points $\xi^1, \xi^2 \in \mathcal{X}$ such that $\xi^1 \leq \xi^2$. Such points exist because $\mathcal{X}$ is a box. Since $\xi^1, \xi^2 \in \mathcal{X}$, (C) is cooperative in the box $\mathcal{X}$, and since $\xi^1 \leq \xi^2$, it follows from Proposition 2 that $\phi(t, \xi^1) \leq \phi(t, \xi^2)$ for all $t \geq 0$. We can thus write (2.11) as

$$|\phi(t, \xi^1) - \phi(t, \xi^2)|_1 \leq \beta(|\xi^1 - \xi^2|_1, t), \quad t \geq 0. \tag{A.2}$$

For (C) to be incrementally asymptotically stable with respect to $\mathcal{X}$, there must exist $\beta \in \mathcal{KL}$ such that bound (A.2) holds for all $\xi^1, \xi^2 \in \mathcal{X}$. Therefore (2.11), which can be written as (A.2), must hold whenever $\xi^1 \leq \xi^2$. This proves the necessity.

To prove sufficiency, we assume that there exists $\beta \in \mathcal{KL}$ such that (2.11) holds whenever $\xi^1, \xi^2 \in \mathcal{X}$ and $\xi^1 \leq \xi^2$. Fix β accordingly and let $p^1, p^2 \in \mathcal{X}$ and $t \geq 0$ be arbitrary but fixed. If

$$|\phi(t, p^1) - \phi(t, p^2)|_1 \leq \beta(|p^1 - p^2|_1, t), \tag{A.3}$$

then we are done.

Define $q^1, q^2 \in \mathcal{X}$ by (2.9) then it follows by Lemma 1 with $[t_0, t_1] \coloneqq [0, t]$, that $q^1 \leq q^2$, $|q^1 - q^2|_1 = |p^1 - p^2|_1$, and

$$|\phi(t, p^1) - \phi(t, p^2)|_1 \leq |\phi(t, q^1) - \phi(t, q^2)|_1. \tag{A.4}$$

Since (A.2) holds for for all $\xi^1 \leq \xi^2$ in $\mathcal{X}$, we can bound the right-hand side of (A.4) as follows

$$\begin{aligned} |\phi(t, q^1) - \phi(t, q^2)|_1 &\leq \beta(|q^1 - q^2|_1, t) \\ &= \beta(|p^1 - p^2|_1, t) \end{aligned} \tag{A.5}$$

where the last step follows, since $|q^1 - q^2|_1 = |p^1 - p^2|_1$. Now the bound (A.3) follows from (A.5) and (A.4), which completes the proof of the theorem. □

Proof of Proposition 3

Pick any two solutions $x^1, x^2 : [0, +\infty) \to \mathcal{X}$ of (C) such that $x^1(0) \leq x^2(0)$, then $x^1(t) \leq x^2(t)$ for all $t \geq 0$ due to Proposition 2. If $|x^1(t) - x^2(t)|_1 \leq (k_2/k_1)e^{-\lambda t}|x^1(0) - x^2(0)|_1$ for all $t \geq 0$, then the proposition follows by Theorem 1.

Fix any continuously differentiable $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and constants $\lambda, k_1, k_2 > 0$ such that (2.3) hold for all $t \geq 0$ and $x \leq y$ in $\mathcal{X}$. Besides, define $t \mapsto M(t) \coloneqq V(x^1(t), x^2(t))$. Since $x^1(t) \leq x^2(t)$ and $x^1(t), x^2(t) \in \mathcal{X}$ for all $t \geq 0$ and (2.3) hold for all $t \geq 0$ and $x \leq y$ in $\mathcal{X}$,

$$\begin{aligned} &M(0) \leq k_2|x^1(0) - x^2(0)|_1, \quad |x^1(t) - x^2(t)|_1 \leq \frac{1}{k_1}M(t), \quad t \geq 0, \\ &\dot{M}(t) = W(t, x^1(t), x^2(t)) \leq -\lambda M(t), \quad t \geq 0. \end{aligned} \tag{A.6}$$

The remainder of the proof is due to Grönwalls inequality, similar to the proof (page 9) of Proposition 1. □

Proof of Proposition 4

Recall Definition 2. Let $\xi^1, \xi^2 \in \mathcal{X}$ and $0 \leq t_0 < t_1$ be arbitrary but fixed and denote $x^k(t) \coloneqq \phi(t, \xi^k)$, $k = 1, 2$. If we can show

$$|x^1(t_1) - x^2(t_1)|_1 \leq |x^1(t_0) - x^2(t_0)|_1, \tag{A.7}$$

we are done.

For $k = 1, 2$, let $p^k \coloneqq x^k(t_0)$, let q^k be given by (2.9), and let $y^k : [t_0, t_1] \to \mathcal{X}$ be the solution to (C) such that $y^k(t_0) = q^k$. Then it follows by Lemma 1 that

$$|x^1(t_0) - x^2(t_0)|_1 = |y^1(t_0) - y^2(t_0)|_1 \tag{A.8a}$$
$$|x^1(t_1) - x^2(t_1)|_1 \leq |y^1(t_1) - y^2(t_1)|_1 \tag{A.8b}$$

since (C) is cooperative in $\mathcal{X}$.

If $|y^1(t_1) - y^2(t_1)|_1 \leq |y^1(t_0) - y^2(t_0)|_1$, then (A.7) follows due to (A.8). By the definition (2.9) of q^1, q^2 it holds that $q^1 \leq q^2$ and $q^1, q^2 \in \mathcal{X}$. Therefore, $y^1(t) \leq y^2(t)$ for $t \in [t_0, t_1]$, since (C) is cooperative in $\mathcal{X}$. We can thus write

$$|y^1(t) - y^2(t)|_1 = \sum_{i=1}^{n} \big(y_i^2(t) - y_i^1(t)\big), \quad t \in [t_0, t_1] \tag{A.9}$$

and by using the assumption (2.12) we may write

$$\sum_{i=1}^{n} \big(\dot{y}_i^2(t) - \dot{y}_i^1(t)\big) \leq 0, \quad t \in [t_0, t_1]. \tag{A.10}$$

Therefore $|y^1(t_1)-y^2(t_1)|_1 \leq |y^1(t_0)-y^2(t_0)|_1$, which completes the proof. □

Appendix B Tridiagonal Matrices

A general tridiagonal matrix $T \subset \mathbb{R}^{n\times n}$ has zero-elements above its sup-diagonal and below its sub-diagonal:

$$T := \begin{bmatrix} d_1 & b_1 & 0 & \cdots & 0 \\ a_1 & d_2 & b_2 & \cdots & 0 \\ 0 & a_2 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix} \tag{B.11}$$

For $n > 1$ we use the notation $T_n(a, b)$ denote (B.11) for some given $a \in \mathbb{R}^n$, $b \in \mathbb{R}^{n-1}$ with the diagonal elements

$$d_1 := -a_1, \quad d_i := -a_i - b_{i-1}, \quad i = 2, \dots, n. \tag{B.12}$$

That is

$$T_n(a,b) := \begin{bmatrix} -a_1 & b_1 & 0 & \cdots & 0 \\ a_1 & -a_2 - b_1 & b_2 & \cdots & 0 \\ 0 & a_2 & -a_3 - b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_n - b_{n-1} \end{bmatrix}. \tag{B.13}$$

Proposition 8: *Fix any $a \in (0,+\infty)^n$ and $b \in [0,+\infty)^{n-1}$ and consider the set*

$$\mathcal{T}_n(a,b) := \{T_n(\tilde{a},\tilde{b}) \ : \tilde{a} \geq a, \ \tilde{b} \leq b\}. \tag{B.14}$$

Then, there exist $\lambda > 0$ and $v \in (0,+\infty)^n$ such that

$$v^T A \leq -\lambda v^T \text{ for all } A \in \mathcal{T}_n(a,b). \tag{B.15}$$

Fix any $\sigma \in (0,+\infty)^n$ and define $p \in (0,+\infty)^n$ by

$$p_1 := \sigma_1, \quad p_i := \sigma_i + \frac{b_{i-1}}{a_i} p_{i-1}, \quad i = 2,\dots,n. \tag{B.16}$$

Then, (B.15) *is verified with λ and v given by*

$$\begin{aligned} v_i &:= \sum_{k=i}^{n} p_i > 0, \quad i = 1,\dots,n \\ \lambda &:= \frac{\min\{a_1\sigma_1,\dots,a_n\sigma_n\}}{\sum_{i=1}^{n} p_i} > 0. \end{aligned} \tag{B.17}$$

Proof. Since $a,\sigma \in (0,+\infty)^n$ and $b \in [0,+\infty)^{n-1}$ it follows by (B.16) that $p \in (0,\infty)^n$, which in turn implies the strict inequalities (B.17). Let $A \in \mathcal{T}(a,b)$ be arbitrary but fixed. We will verify the inequality (B.15) with λ and v defined by (B.17).

If we can show that

$$(v^T A)_i \leq -\sigma_i a_i, \quad i = 1,\dots,n, \tag{B.18}$$

then (B.15) follows by the definition (B.17) of λ and because $v_i/(p_1+\cdots+p_n) \in (0,1)$, $i = 1,\dots,n$.

Since $A \in \mathcal{T}(a, b)$ there exist $\tilde{a} \in \mathbb{R}^n$ and $\tilde{b} \in \mathbb{R}^{n-1}$ such that

$$A = T_n(\tilde{a}, \tilde{b}), \quad \tilde{a} \geq a, \quad 0 \leq \tilde{b} \leq b \tag{B.19}$$

where $T_n(\cdot, \cdot)$ is defined in (B.13). This is a consequence of the definition (B.14).

Using (B.13) we may write

$$(v^T A)_i = \begin{cases} -\tilde{a}_1(v_1 - v_2), & i = 1 \\ \tilde{b}_{i-1}(v_{i-1} - v_i) - \tilde{a}_i(v_i - v_{i+1}), & i \neq 1, n \\ \tilde{b}_{n-1}(v_{n-1} - v_n) - \tilde{a}_n v_n, & i = n \end{cases} \tag{B.20}$$

And to prove (B.18), we make use of

$$v_n = p_n, \quad v_i - v_{i+1} = p_i, \quad i = 1, \ldots, n-1 \tag{B.21}$$

which follows from the definition (B.17) of v.

We begin proving (B.18) for $i = 1$:

$$(v^T A)_1 = -\tilde{a}_1 p_1 \leq -a_1 \sigma_1,$$

where the first step is due to (B.20), (B.21) and the last step follows since $p_1 := \sigma_1 > 0$ and $\tilde{a}_1 \geq a_1 > 0$. Now we take $i \in \{2, \ldots, n-1\}$ and obtain

$$(v^T A)_i = p_{i-1}\tilde{b}_{i-1} - \tilde{a}_i p_i \leq p_{i-1} b_{i-1} - a_i p_i = -\sigma_i a_i \tag{B.22}$$

where the first step is due to (B.20), (B.21), and the second step is due to (B.19), and the last step follows by inserting the definition (B.16) of p_i into the second step. Finally, we take $i = n$ and obtain

$$(v^T A)_n = \tilde{b}_{n-1} p_{n-1} - \tilde{a}_n p_n \leq b_{n-1} p_{n-1} - a_n p_n = -\sigma_n a_n$$

which follows from the same steps as in (B.22). This completes the proof, since $A \in \mathcal{T}_n(a, b)$ was chosen arbitrarily. $\square$

Appendix C Traffic Reaction Models

Here we prove the statements in Chapter 3, most of which are related to the Traffic Reaction Model (3.6). For reference to several of the proofs given here, we denote the right-hand side of (3.6) by $f := (f_1, \ldots, f_n) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

$$f_i(t,x) := \begin{cases} \rho_0(t)h(x_1) - x_1 h(x_2) + q_1(t,x_1), & i = 1 \\ x_{i-1}h(x_i) - x_i h(x_{i+1}) + q_i(t,x_i), & i \neq 1, n \\ x_{n-1}h(x_n) - x_n h(\rho_{n+1}(t)) + q_n(t,x_n), & i = n \end{cases} \tag{C.23}$$

Proof of Proposition 5

The claim (i) is addressed in [1], so we do not provide it here.

We begin proving the claim (ii), that (3.6) is cooperative in $[0,\rho_{\max}]^n$. Recall then, Definition 3 and let f in (C.23) denote the right-hand side of (3.6). It follows from (A1)-(A3) that f is continuous from a neighborhood of $[0,+\infty) \times \mathcal{X}$ to $\mathbb{R}^n$. The interior $\operatorname{Int} \mathcal{X} := (0,\rho_{\max})^n$ of $\mathcal{X} := [0,\rho_{\max}]^n$ is nonempty. It remains to show that $x \mapsto f(t,x)$ is type K on $\operatorname{Int} \mathcal{X}$ for each $t \geq 0$.

Fix any $t \geq 0$, $i \in \{1,\ldots,n\}$, and $a \leq b$ in $\operatorname{Int} \mathcal{X}$ such that $a_i = b_i$. If $\delta_i := f_i(t,b) - f_i(t,a) \geq 0$, then we are done. First, we compute

$$\delta_i = \begin{cases} a_1(h(a_2) - h(b_2)), & i = 1 \\ h(a_i)(b_{i-1} - a_{i-1}) + a_i(h(a_{i+1}) - h(b_{i+1})), & i \neq 1, n \\ h(a_n)(b_{n-1} - a_{n-1}), & i = n \end{cases} \tag{C.24}$$

It is assumed that (cf. (A2)) h is nonincreasing and such that $h(\rho_{\max}) = 0$. The latter and the fact that $a \leq b$ are in $[0,\rho_{\max}]^n$ implies that $\delta_i \geq 0$. This shows (ii).

Lastly, we show the claim (iii), that (3.6) is nonexpansive with respect to $[0,\rho_{\max}]^n$ (cf. Definition 2), with the aid of Proposition 4. Pick any $t \geq 0$ and $a \leq b$ in $[0,\rho_{\max}]^n$, then

$$\sum_{i=1}^{n} f_i(t,b) - f_i(t,a) = \sum_{i=1}^{n} q_i(t,b_i) - q_i(t,a_i) \leq 0 \tag{C.25}$$

owing to the monotony assumption (A3) on the q_i. It now follows from Proposition 4 that (3.6) is nonexpansive with respect to $\mathcal{X}$ since (3.6) is cooperative in the box $[0, \rho_{\max}]$. This completes the proof. $\square$

Two Supporting Lemmas

Lemma 2: *Consider the Traffic Reaction Model* (3.6) *with (A1)-(A3) (page 15). If there exists* $\bar{\rho} \in (0, \rho_{\max}]$ *such that*

$$h(z) > 0, \quad z \in [0, \rho_{\max}) \text{ and } \rho_{n+1}(t) \leq \rho_{\max} - \bar{\rho}, \quad t \geq 0, \tag{C.26}$$

then for every $\tau > 0$*, there exist* $s_2, \ldots, s_n \in (0, \rho_{\max})$ *such that*

$$\phi(t, \xi) \in [0, \rho_{\max}] \times [0, s_2] \times \cdots \times [0, s_n], \quad t \geq \tau, \quad \xi \in [0, \rho_{\max}]^n \tag{C.27}$$

where $\phi(t, \xi)$*,* $\phi(0, \xi) = \xi$ *denotes the solution to* (3.6)*.*

Proof. It follows by Proposition 5 that $\phi(t, \xi) \in [0, \rho_{\max}]^n$ for all $t \geq 0$ and $\xi \in [0, \rho_{\max}]^n$. Fix any $\xi \in [0, \rho_{\max}]^n$ and $\tau > 0$ and denote $x(t) := \phi(t, \xi)$. Besides, denote $y(t) := \phi(t, \hat{\rho_{\max}})$, with $\hat{\rho_{\max}} := (\rho_{\max}, \ldots, \rho_{\max}) \in \mathbb{R}^n$. Since (3.6) is cooperative in the box $[0, \rho_{\max}]^n$ (cf. Proposition 5) it follows by Proposition 2 that

$$x(t) \leq y(t), \quad t \geq 0. \tag{C.28}$$

Therefore, it is sufficient to show that there exist $s_2, \ldots, s_n \in (0, \rho_{\max})$ such that

$$y_i(t) \leq s_i, \quad t \geq \tau, \quad i = 2, \ldots, n \tag{C.29}$$

since $x(t), y(t) \in [0, \rho_{\max}]^n$ for all $t \geq 0$.

Consider the claim

$$P_i : \quad \begin{cases} \text{there exist } s_i \in (0, \rho_{\max}) \text{ and } \tau_i \in (0, \tau) \\ \text{such that } y_i(t) \leq s_i \text{ for } t \geq \tau_i \end{cases},$$

defined for $i = 2, \ldots, n$. If P_n is true and P_{i+1} implies P_i for $i = 2, \ldots, n-1$, then we are done.

To set up the proof we define some constants. It follows by the assumption (A2) on h that there exists a constant $H \geq 0$ such that

$$h(z) \leq H(\rho_{\max} - z), \quad z \in [0, \rho_{\max}]. \tag{C.30}$$

Similarly, there exists by the assumption (A3) a constant $Q_i \geq 0$ such that

$$q_i(t,z) \leq Q_i(\rho_{\max} - z), \quad z \in [0, \rho_{\max}], \quad t \geq 0, \quad i = 2, \dots, n. \tag{C.31}$$

And by the assumption (C.26)

$$h(\rho_{n+1}(t)) \geq a_n \coloneqq h(\rho_{\max} - \bar{\rho}) > 0, \quad t \geq 0. \tag{C.32}$$

We proceed by showing P_n. In what follows, note that h is nonnegative on $[0, \rho_{\max}]$, which follows by the assumption (A2). Denote $L_n \coloneqq \rho_{\max} H + Q_n$. For each $t \geq 0$ we have

$$\begin{aligned} \dot{y}_n(t) =& y_{n-1}(t)h(y_n(t)) - y_n(t)h(\rho_{n+1}(t)) + q_n(t, y_n(t)) \\ \leq& \rho_{\max} H(\rho_{\max} - y_n(t)) - y_n(t)a_n + Q_n(\rho_{\max} - y_n(t)) \\ =& (L_n + a_n)\Big(\rho_{\max}\frac{L_n}{L_n + a_n} - y_n(t)\Big) \end{aligned} \tag{C.33}$$

where the second step follows by (C.30)-(C.32), and since $y_{n-1}(t) \in [0, \rho_{\max}]$ for all $t \geq 0$ and h is nonnegative on $[0, \rho_{\max}]$. The last step follows by definition of L_n.

Note that $b_n \coloneqq \rho_{\max}\frac{L_n}{L_n + a_n} < \rho_{\max}$. Therefore, $\dot{y}_n(t) < 0$ whenever $y_n(t) \in (b_n, \rho_{\max}]$ and $t \geq 0$. In turn, if $y_n(\tau_n) < \rho_{\max}$, then $y_n(t) \leq y_n(\tau_n)$ for all $t \geq \tau_n$. Since $y_n(0) = \rho_{\max}$, there exists $\tau_n \in (0, \tau)$ such that $y_n(t) \leq s_n \coloneqq y_n(\tau_n) < \rho_{\max}$ for all $t \geq \tau_n$. This shows P_n.

Fix any $i \in \{2, \dots, n-1\}$, assume that P_{i+1} is true. We will show that P_i follows. Since P_{i+1} holds by assumption, pick any $s_{i+1} \in (0, \rho_{\max})$ and $\tau_{i+1} \in (0, \tau)$ such that $y_{i+1}(t) \leq s_{i+1}$ for all $t \geq \tau_{i+1}$. Then,

$$h(y_{i+1}(t)) \geq h(s_{i+1}) \eqqcolon a_i > 0, \quad t \geq \tau_{i+1}. \tag{C.34}$$

Denote $L_i \coloneqq \rho_{\max} H + Q_i$, then

$$\dot{y}_i(t) \leq (L_i + a_i)\Big(\rho_{\max}\frac{L_i}{L_i + a_i} - y_i(t)\Big), \quad t \geq \tau_{i+1} \tag{C.35}$$

follows by similar steps as in the proof of (C.33).

Note that $b_i \coloneqq \rho_{\max}\frac{L_i}{L_i + a_i} < \rho_{\max}$. Therefore $\dot{y}_i(t) < 0$ whenever $y_i(t) \in (b_i, \rho_{\max}]$ and $t \geq \tau_{i+1}$. Recall that $y_i(t) \in [0, \rho_{\max}]$ for all $t \geq 0$. If $y_i(\tau_{i+1}) =$

$\rho_{\max}$, then P_i follows from the same steps as in the proof of P_n. If $y_i(\tau_{i+1}) \in [0, \rho_{\max})$, then P_i follows with $\tau_i \coloneqq \tau_{i+1} < \tau$ and any choice of s_i in $(b_i, \rho_{\max}) \cap [y_i(\tau_{i+1}), \rho_{\max})$. This completes the proof. $\square$

Lemma 3: *Consider the Traffic Reaction Model* (3.6) *with (A1)-(A3) (page 15) and let* (C.23) *denote the right-hand side of* (3.6). *Fix any* $s \in [0, \rho_{\max}] \times [0, \rho_{\max})^{n-1}$ *and define the set* $\mathcal{S} \coloneqq \{0 \leq x \leq s\} \subset [0, \rho_{\max}]^n$. *If there exists* $\bar{\rho} \in (0, \rho_{\max}]$ *such that*

$$h(z) > 0, \quad z \in [0, \rho_{\max}) \text{ and } \rho_{n+1}(t) \leq \rho_{\max} - \bar{\rho}, \quad t \geq 0, \tag{C.36}$$

then there exist $v \in (0, +\infty)^n$ *and* $\lambda > 0$ *such that*

$$\langle v, f(t, y) - f(t, x) \rangle \leq -\lambda \langle v, y - x \rangle \tag{C.37}$$

for all $t \geq 0$ *and* $x \leq y$ *in* $\mathcal{S}$.

Proof. To prove the lemma we will apply Proposition 8. First, we introduce two functions $A : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ and $\tilde{q} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$f(t, y) - f(t, x) = A(t, x, y)(y - x) + \tilde{q}(t, x, y), \quad t \geq 0, \quad x, y \in \mathcal{S}. \tag{C.38}$$

Let the vector-valued function $\tilde{q}$ (A will be defined later) be given by

$$\tilde{q}_i(t, x, y) \coloneqq \begin{cases} \rho_0(t)(h(y_1) - h(x_1)) + q_1(t, y_1) - q_1(t, x_1), & i = 1, \\ q_i(t, y_i) - q_i(t, x_i), & i = 2, \ldots, n. \end{cases} \tag{C.39}$$

By the assumptions (A1)-(A3), h is nonincreasing, $\rho_0(t) \geq 0$ for $t \geq 0$, and the q_i are nonincreasing with respect to the second argument. Therefore

$$\langle v, \tilde{q}(t, x, y) \rangle \leq 0 \text{ if } v \in (0, +\infty)^n, \; t \geq 0, \text{ and } x \leq y \text{ in } \mathcal{S}. \tag{C.40}$$

The rest of the proof is dedicated to showing that there exist $v \in (0, +\infty)^n$ and $\lambda > 0$ such that

$$v^T A(t, x, y) \leq -\lambda v^T \text{ for all } t \geq 0 \text{ and } x \leq y \text{ in } \mathcal{S}. \tag{C.41}$$

If this is true, then (C.37) follows owing to (C.40).

We move on to define the matrix-valued function A. Let $\tilde{a} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$

and $\tilde{b} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n-1}$ be given by

$$\begin{aligned} \tilde{a}_i(t,y) &:= \begin{cases} h(y_{i+1}), & i = 1,\dots,n-1 \\ h(\rho_{n+1}(t)), & i = n \end{cases} \\ \tilde{b}_i(x,y) &:= \begin{cases} x_i \frac{h(x_{i+1})-h(y_{i+1})}{y_{i+1}-x_{i+1}}, & x_{i+1} \neq y_{i+1}, \quad i = 1,\dots,n-1 \\ 0, & x_{i+1} = y_{i+1}, \quad i = 1,\dots,n-1 \end{cases} \end{aligned} \tag{C.42}$$

We then define

$$A(t,x,y) := T_n(\tilde{a}(t,y), \tilde{b}(x,y)), \quad t \geq 0, \quad x, y \in \mathcal{S} \tag{C.43}$$

where $T_n(\cdot,\cdot)$ is given by (B.13) and we leave it as an exercise to verify (C.38).

Define $a \in \mathbb{R}^n$ by

$$a_i := h(s_{i+1}), \quad i = 1,\dots,n-1, \quad a_n := h(\rho_{\max} - \bar{\rho}) \tag{C.44}$$

By the assumption (A1) and (C.26), $h(z) > 0$ for all $z \in [0, \rho_{\max})$ and $0 \leq \rho_{n+1}(t) \leq \rho_{\max} - \bar{\rho} < \rho_{\max}$ for all $t \geq 0$. Therefore

$$a \in (0, +\infty)^n \text{ and } \tilde{a}(t,y) \geq a \text{ for all } t \geq 0 \text{ and } y \in \mathcal{S} \tag{C.45}$$

since $s_2, \dots, s_n \in (0, \rho_{\max})$ and h is nonincreasing (cf. (A2)).

Recall the assumption (A2). Since $h : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, h is Lipschitz continuous on the closed interval $[0, \rho_{\max}]$ with some Lipshitz constant $L \geq 0$. Besides, h is nonincreasing and such that $h(\rho_{\max}) = 0$, hence

$$0 \leq \frac{h(x_{i+1}) - h(y_{i+1})}{y_{i+1} - x_{i+1}} \leq L, \quad 0 \leq x_{i+1} < y_{i+1} \leq s_{i+1}, \quad i = 1,\dots,n-1.$$

Therefore

$$0 \leq \tilde{b}_i(x,y) \leq b_i := s_i L, \quad i = 1,\dots,n-1, \quad x \leq y \text{ in } \mathcal{S}. \tag{C.46}$$

Let $b := (b_1, \dots, b_{n-1}) \in [0,\infty)^{n-1}$ where the b_i are given in (C.46). Recall the Proposition 8 and the set of matrices $\mathcal{T}_n(\cdot,\cdot)$ defined by (B.14). By the definition (C.43) of A and the inequalities (C.44), (C.46) it follows that

$$A(t,x,y) \in \mathcal{T}_n(a,b), \quad t \geq 0, \; x \leq y \text{ in } \mathcal{S}. \tag{C.47}$$

And since $a \in (0, +\infty)^n$ and $b \in [0, \infty)^{n-1}$, it follows by Proposition 8 that there exist a vector $v \in (0, +\infty)^n$ and constant $\lambda > 0$ such that (C.41) holds. This completes the proof. □

Proof of Theorem 2

Choose some $\tau > 0$ then it follows by Lemma 2 that there exist $s_2, \dots, s_n \in (0, \rho_{\max})$ such that

$$\phi(t, \xi) \in \mathcal{S} := [0, \rho_{\max}] \times [0, s_2] \times \cdots \times [0, s_n], \quad t \geq \tau, \quad \xi \in \mathcal{X}. \tag{C.48}$$

Besides, it follows by Lemma 3 that there exist $\lambda > 0$ and $v \in (0, +\infty)^n$ such that

$$\langle v, f(t, y) - f(t, x) \rangle \leq -\lambda \langle v, y - x \rangle, \quad t \geq 0, \quad x \leq y \text{ in } \mathcal{S}. \tag{C.49}$$

Denote

$$\gamma := e^{\lambda \tau} \frac{\max_i \{v_i\}}{\min_i \{v_i\}} \geq 1. \tag{C.50}$$

Fix any $\xi^1 \leq \xi^2$ in $[0, \rho_{\max}]^n$ and denote $x^k(t) := \phi(t, \xi^k)$, $k = 1, 2$. If we can show that

$$|x^1(t) - x^2(t)|_1 \leq \gamma e^{-\lambda t} |\xi^1 - \xi^2|_1, \quad t \geq 0, \tag{C.51}$$

then proof is complete owing to Theorem 1.

Note that (3.6) is nonexpansive with respect to $\mathcal{X}$ (cf. Proposition 5) and $\xi^1, \xi^2 \in \mathcal{X}$. By the latter and the definition (C.50) of γ, the bound (C.51) holds for $t \in [0, \tau]$ and if we can show that

$$|x^1(t) - x^2(t)|_1 \leq \frac{\max_i \{v_i\}}{\min_i \{v_i\}} e^{-\lambda (t - \tau)} |x^1(\tau) - x^2(\tau)|_1, \quad t \geq \tau \tag{C.52}$$

then (C.51) follows.

To show (C.52) consider the function $M(t) := \langle v, x^2(t) - x^1(t) \rangle$, $t \geq \tau$. Recall that (3.6) is cooperative in $\mathcal{X}$. Since $\xi^1 \leq \xi^2$ and $\xi^1, \xi^2 \in \mathcal{X}$ we claim with Proposition 2 that

$$x^1(t) \leq x^2(t), \quad t \geq \tau. \tag{C.53}$$

And it follows by (C.48) that

$$x^1(t), x^2(t) \in \mathcal{S}, \quad t \geq \tau. \tag{C.54}$$

Therefore,

$$\dot{M}(t) \leq -\lambda M(t), \quad t \geq \tau \tag{C.55}$$

by virtue of (C.49). Applying Grönwalls inequality then yields

$$M(t) \leq e^{-\lambda(t-\tau)} M(\tau), \quad t \geq \tau. \tag{C.56}$$

Since $v \in (0, +\infty)^n$ and $x^1(t) \leq x^2(t)$ for all $t \geq \tau$, the bound (C.56) implies the bound (C.52). This completes the proof of the theorem. □

Proof of Proposition 7

The equality (3.16) is due to [16, Appendix 1]. By the fundamental theorem of calculus

$$z \int_0^1 \dot{F}(zy)\mathrm{d}y = F(z) - F(0), \quad z \in \mathbb{R}. \tag{C.57}$$

Since $F(0) = 0$, the equality (3.16) follows by (C.57) and the definition (3.15) of h.

Next we show that h is nonincreasing. It is assumed that F is twice continuously differentiable, so we can compute

$$\dot{h}(z) = \int_0^1 y\ddot{F}(yz)\mathrm{d}y, \quad z \in \mathbb{R}. \tag{C.58}$$

Since $\dot{h}(z)$ is defined for all $z \in \mathbb{R}$, h is locally Lipschitz continuous. Besides it is assumed that F is concave, so $\ddot{F}(z) \leq 0$ for all $z \in \mathbb{R}$. In turn the integrand (C.58) is nonpositive for all $y \in [0, 1]$ and $z \in \mathbb{R}$. Therefore $\dot{h}$ is nonnegative on $\mathbb{R}$, which implies that h is nonincreasing.

It remains to show that $h(\rho_{\max}) = 0$ and $h(z) > 0$ for all $z \in [0, \rho_{\max})$. The former follows by the assumption (3.14a) and the equality (3.16) since $\rho_{\max} > 0$. By assumption, $F(z) > 0$ for all $z \in (0, \rho_{\max})$, therefore the equality (3.16) implies that $h(z) > 0$ for all $z \in (0, \rho_{\max})$. To show $h(0) > 0$, note that

$$h(0) = \int_0^1 \dot{F}(0)\mathrm{d}y = \dot{F}(0). \tag{C.59}$$

If $\dot{F}(0) > 0$, then we are done. Because F is concave,

$$F((1-\gamma)a + \gamma b) \geq (1-\gamma)F(a) + \gamma F(b) \tag{C.60}$$

for all $\gamma \in [0, 1]$ and $a, b \in \mathbb{R}$. Set $a \coloneqq 0$ and pick any $b \in (0, \rho_{\max})$, then

$$F(\gamma b) \geq \gamma F(b) > 0, \quad \gamma \in (0, 1] \tag{C.61}$$

and since $F(0) = 0$ we can write

$$\dot{F}(0) = \lim_{\gamma \to 0^+} \frac{F(\gamma b)}{\gamma b} \geq \lim_{\gamma \to 0^+} \frac{F(b)}{b} > 0 \tag{C.62}$$

where the last two steps follow from (C.61). This completes the proof. □

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