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RESEARCH ARTICLE

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Fully noncentral Lie ideals and invariant additive subgroups in rings

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Abstract
We prove conditions ensuring that a Lie ideal or an invariant additive subgroup in a ring contains all additive commutators. A crucial assumption is that the subgroup is fully noncentral, that is, its image in every quotient is noncentral. For a unital algebra over a field of characteristic $\neq 2$ where every additive commutator is a sum of square-zero elements, we show that a fully noncentral subspace is a Lie ideal if and only if it is invariant under all inner automorphisms. This applies in particular to zero-product balanced algebras.

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1 | INTRODUCTION

Every associative ring R carries a natural Lie ring structure with Lie product of two elements $x, y \in R$ defined as their commutator $[x, y] := xy - yx$. A *Lie ideal* in R is then an additive subgroup $L \subseteq R$ such that $[R, L] \subseteq L$. Obvious examples of Lie ideals in R are all additive subgroups that are either contained in the center $Z(R)$, or that contain the commutator subgroup $[R, R]$. (Given subsets $G, H \subseteq R$ at least one of which is an additive subgroup, we follow the standard convention that $[G, H]$ and GH denote the additive subgroups of R generated by $\{[g, h] : g \in G, h \in H\}$ and $\{gh : g \in G, h \in H\}$, respectively.)

In 1955, Herstein proved that this essentially already describes all Lie ideals in simple rings: An additive subgroup V of a simple ring R is a Lie ideal if and only if either $V \subseteq Z(R)$ or $[R, R] \subseteq V$, unless R has characteristic 2 and is four-dimensional over its center; see [16, Theorem 5] and

[17, Theorem 1.5]; see also Examples 3.6 and 3.7. This was subsequently developed into a comprehensive theory of Lie ideals in prime and semiprime rings [18, 21].

The deep results on Lie ideals in associative rings sparked the interest in results verifying that an additive subgroup in a ring is a Lie ideal. A natural necessary condition is invariance under (certain) inner automorphisms. Indeed, it is easy to see that every Lie ideal in a nonexceptional (Definition 4.1), simple ring is invariant under all inner automorphisms. Further, if a subspace in an algebra over a field of characteristic $\neq 2$ is a Lie ideal, then it is invariant under all inner automorphisms induced by square-zero elements; see Lemma 5.1.

The converse problem of when an additive subgroup that is invariant under (certain) inner automorphisms is automatically a Lie ideal was investigate by many authors, and we refer to the thorough introduction of Lanski's paper [20] for an overview. We just mention the result of Amitsur [1] that for a nonexceptional, simple algebra over a field of characteristic $\neq 2$ that contains a nontrivial idempotent, a subspace is a Lie ideal if (and only if) it is invariant under inner automorphisms induced by square-zero elements. This was extended to certain prime rings containing a nontrivial idempotent by Chuang [9], and then to certain prime rings containing sufficiently many square-zero elements by Lanski [20]. For nonexceptional, simple rings generated by their quasi-regular elements, and containing sufficiently many square-zero elements, Chuang [10] showed that an additive subgroup is a Lie ideal if (and only if) it is invariant under inner automorphisms.

Analogous questions for (closed) linear subspaces in operator algebras have also been extensively studied [6, 24, 27]. In particular, it was shown by Marcoux and Murphy that a closed subspace in a C^*-algebra is a Lie ideal if and only if it is invariant under conjugation by unitaries [25].

In this paper, we aim at conditions ensuring that an additive subgroup V satisfies $[R,R] \subseteq V$ (and therefore is a Lie ideal). We note that invariance under inner automorphisms of R is a necessary condition: Given an invertible element u in the minimal unitization of R, and $x \in V$, we have

$$uxu^{-1} = [u, xu^{-1}] + x \in [R,R] + V \subseteq V.$$

We focus on rings R for which $[R,R]$ is full. Here, we say that a subset X in R is *full* if it generates R as an ideal, that is, if it is not contained in a proper ideal of R. Rings that are generated by their commutators were studied in [2, 11, 14, 26].

In [7], Chand and Robert define a subset X of a C^*-algebra A to be "fully noncentral" if $[A,X]$ is full in the sense that it generates A as a closed ideal. We adopt this terminology to the algebraic setting.

Definition A. We say that a subset X in a ring R is *fully noncentral* if $[R,X]$ is full, that is, $[R,X]$ is not contained in a proper ideal of R.

Our main result shows that for a large class of algebras, a fully noncentral subspace is a Lie ideal if and only if it is invariant under inner automorphisms. The following is Theorem 5.2.

Theorem B. *Let A be an algebra over a field $\neq \{0,1\}$ such that every commutator in A is a sum of square-zero elements, and every proper ideal is contained in a nonexceptional prime ideal. Let $V \subseteq A$ be a fully noncentral subspace. Then, the following are equivalent.*

(1) *The subspace V is invariant under all inner automorphisms of A.*

(2) *The subspace V is invariant under all inner automorphisms induced by square-zero elements of A.*
(3) *We have $[A, A] \subseteq V$.*
(4) *The subspace V is a Lie ideal.*
(5) *The subspace V is an $[A, A]$-submodule.*

We note that for unital algebras over a field of characteristic $\neq 2$, every proper ideal is contained in a nonexceptional prime ideal. Further, if an algebra is *zero-product balanced* in the sense of [12] (a notion closely related to the concept of a *zero-product determined* algebra [5]), then every commutator is a sum of square-zero elements ([12, Theorem 5.3]). In particular, for a zero-product balanced algebra over a field of characteristic $\neq 2$, a fully noncentral subspace V is a Lie ideal if and only V is invariant under all inner automorphisms of A, if and only if $[A, A] \subseteq V$; see Corollary 5.7. The class of zero-product balanced algebras contains all algebras generated by idempotents ([12, Example 3.7]), in particular, matrix algebras as well as simple algebras containing a nontrivial idempotent.

The proof of Theorem B relies on the general result that if a subspace V of an algebra A over a field $\neq \{0, 1\}$ is invariant under all inner automorphisms, then $[x, V] \subseteq V$ for every square-zero element x; see Lemma 5.1. If one additionally assumes that the field is infinite, then one can deduce that $[x, V] \subseteq V$ for every nilpotent element x; see Lemma 5.3. As a consequence, we obtain a variation of Theorem B, where the stronger assumption of working over an infinite field allows us to relax the condition of writing every commutator as a sum of square-zero elements to a sum of nilpotent elements; see Theorem 5.5. Using that the subspace generated by nilpotent elements is invariant under inner automorphisms, we obtain as an application that additive commutators of nilpotent elements in an algebra over an infinite field are sums of nilpotent elements; see Proposition 5.4.

2 | RESULTS ABOUT GENERAL RINGS

In this section, we devise a method that, for a general additive subgroup V in a ring R, constructs an ideal $I \subseteq R$ satisfying $[R, I] \subseteq V$; see Theorem 2.3. The ideal I is built from higher order (generalized) normalizers of V (see Notation 2.1), and, in general, it may very well happen that $I = \{0\}$. If V is a Lie ideal of $[R, R]$, or more generally an $[R, R]$-submodule, then we have better control over I; see Corollary 2.6.

Given an additive subgroup V in a ring R, the normalizer (with respect to the Lie product) is $\{x \in R : [V, x] \subseteq V\}$. We consider the closely related set $\{x \in R : [R, x] \subseteq V\}$, which seems to have been considered first by Herstein in [16], see also [17, p.5], with the notation $T(V)$. This set also played a crucial role of the analysis of Lie ideals in [6], where it is denoted by $N(V)$. We follow Herstein's notation, and also introduce higher order versions:

Notation 2.1. Let R be a ring, and let $V \subseteq R$ be an additive subgroup. We set

$$T(V) := \{x \in R : [R, x] \subseteq V\}.$$

We inductively define $T^n(V)$ for $n \geqslant 1$ by setting $T^1(V) := T(V)$ and

$$T^{n+1}(V) := T(T^n(V)).$$

Note that an additive subgroup $V \subseteq R$ is a Lie ideal if and only if $V \subseteq T(V)$.

The next result is folklore and follows for example from the proof of [16, Lemma 3] or [17, Lemma 1.4]. We include the short argument for the convenience of the reader.

Lemma 2.2. *Let R be a ring, and let $V \subseteq R$ be an additive subgroup. Then $T(V)$ is a subring.*

Proof. Using the biadditivity of the Lie product, we see that $T(V)$ is an additive subgroup. To show that $T(V)$ is closed under multiplication, let $x, y \in T(V)$. Using that $[a, xy] = [ax, y] + [ya, x]$ for all $a \in R$, we have

$$[R, xy] \subseteq [R, y] + [R, x] \subseteq V + V = V,$$

and thus, $xy \in T(V)$. □

Given a ring R, we use $\widetilde{R}$ to denote its minimal unitization, given by $\widetilde{R} = R$ if R is unital, and by $\widetilde{R} = \mathbb{Z} \times R$ with coordinatewise addition and multiplication $(m, x)(n, y) = (mn, my + nx + xy)$ if R is nonunital. The map $R \to \widetilde{R}$ given by $x \mapsto (x, 0)$, identifies R with an ideal in $\widetilde{R}$.

The ideal of R generated by an additive subgroup $V \subseteq R$ is $V + RV + VR + RVR$, which agrees with $\widetilde{R}V\widetilde{R}$. Note that RVR is also an ideal of R, but if R is nonunital, then it may not contain V and therefore can be strictly smaller than $\widetilde{R}V\widetilde{R}$.

Theorem 2.3. *Let R be a ring, and let $V \subseteq R$ be an additive subgroup. Then,*

$$\widetilde{R}\big[T(V) \cap V, T^2(V) \cap T(V)\big]\widetilde{R} \subseteq V + V^2$$

and

$$\big[R, \widetilde{R}[T^2(V) \cap T(V), T^3(V) \cap T^2(V)]\widetilde{R}\big] \subseteq V.$$

In particular, if $[T^2(V) \cap T(V), T^3(V) \cap T^2(V)]$ is full, then $[R, R] \subseteq V$, and so V is a Lie ideal in R.

Proof. Let $a, b \in \widetilde{R}$, let $x \in T(V) \cap V$, and let $y \in T^2(V) \cap T(V)$. Note that $[\widetilde{R}, X] = [R, X]$ for every subset $X \subseteq R$. Using a direct computation in the first step, we get

$$\begin{aligned} a[x, y]b &= [ax, [y, b]] + [[y, b], a]x + [a[x, b], y] \\ &\quad + [y, a][x, b] + [abx, y] + [y, ab]x \\ &\in \big[R, [T^2(V), \widetilde{R}]\big] + \big[[T^2(V), \widetilde{R}], \widetilde{R}\big]V + [R, T(V)] \\ &\quad + [T(V), \widetilde{R}][T(V), \widetilde{R}] + [R, T(V)] + [T(V), \widetilde{R}]V \\ &\subseteq V + V^2. \end{aligned}$$

This verifies the first claimed inclusion.

Applying this inclusion for $T(V)$ in place of V, and using at the last step that $T(V)$ is a subring by Lemma 2.2, we get

$$\widetilde{R}\big[T^2(V)\cap T(V), T^3(V)\cap T^2(V)\big]\widetilde{R} \subseteq T(V)+T(V)^2 \subseteq T(V).$$

It follows that

$$\big[R, \widetilde{R}[T^3(V)\cap T^2(V), T^2(V)\cap T(V)]\widetilde{R}\big] \subseteq [R, T(V)] \subseteq V,$$

as desired. □

If we apply Theorem 2.3 to a Lie ideal, then we obtain the following well-known result; see, for example, [27, Lemma 1.1], [22, Lemma 2.1(ii)].

Corollary 2.4. *Let R be a ring, and let $L \subseteq R$ be a Lie ideal. Then, $T^n(L) \subseteq T^{n+1}(L)$ for all $n \geqslant 0$, and we deduce that*

$$\widetilde{R}[L,L]\widetilde{R} \subseteq L + L^2, \quad \textit{and} \quad \big[R, \widetilde{R}[L,L]\widetilde{R}\big] \subseteq L.$$

Proof. Given additive subgroups $V_1 \subseteq V_2 \subseteq R$, we have $T(V_1) \subseteq T(V_2)$. Since L is a Lie ideal, we have $L \subseteq T(L)$ by definition. Applying the above observation inductively, we get the desired inclusion. The other inclusions now follow from Theorem 2.3. □

One says that an additive subgroup V of a ring R is an $[R,R]$*-submodule* if $[[R,R],V] \subseteq V$. This includes Lie ideals in $[R,R]$, but it also allows for subgroups V that are not contained in $[R,R]$. Given an additive subgroup $V \subseteq R$, we set $V^{(0)} := V$, $V^{(1)} := [V,V]$, and $V^{(n+1)} := [V^{(n)}, V^{(n)}]$ for $n \geqslant 1$.

In the next result, we show that for an $[R,R]$-submodule V, the groups $V^{(n)}$, for $n \geqslant 1$, form a decreasing sequence. We do not claim that $V^{(1)} \subseteq V = V^{(0)}$.

Lemma 2.5. *Let $V \subseteq R$ be an $[R,R]$-submodule and let $n \geqslant 1$. Then, $V^{(n)}$ is an $[R,R]$-submodule satisfying $V^{(n)} \subseteq T^n(V)$. Further, we have $V^{(1)} \supseteq V^{(2)} \supseteq \cdots$.*

Proof. Claim 1: *Given an $[R,R]$-module $W \subseteq R$, we have $[W,W] \subseteq T(W)$.* Indeed, we have $[[R,R],W] \subseteq W$ by assumption. Applying the Jacobi identity at the first step, we get

$$[R,[W,W]] \subseteq [W,[R,W]] + [W,[W,R]] \subseteq [W,[R,R]] \subseteq W,$$

and thus, $[W,W] \subseteq T(W)$, which proves the claim.

Claim 2: *Given an $[R,R]$-module $W \subseteq R$, then $[W,W]$ is an $[R,R]$-submodule as well.* Applying the Jacobi identity at the first step, and using Claim 1 at the second step, we get

$$[[R,R],[W,W]] \subseteq [[R,[W,W]],W] + [[[W,W],R],W], \subseteq [W,W],$$

which verifies the claim.

Now, applying Claim 2 successively, we obtain that $V^{(n)}$ is an $[R, R]$-submodule for all $n \geqslant 1$. Further, we deduce that

$$V^{(n+2)} = \left[V^{(n+1)}, V^{(n+1)}\right] = \left[[V^{(n)}, V^{(n)}], V^{(n+1)}\right] \subseteq \left[[R, R], V^{(n+1)}\right] \subseteq V^{(n+1)}$$

for all $n \geqslant 0$. Thus, we have $V^{(1)} \supseteq V^{(2)} \supseteq \ldots$.

Next, we verify by induction that $V^{(n)} \subseteq T^n(V)$ for all $n \geqslant 1$. We have $V^{(1)} = [V, V] \subseteq T(V) = T^1(V)$ by Claim 1, which verifies the case $n = 1$. Assume that $V^{(n)} \subseteq T^n(V)$ for some $n \geqslant 1$. Applying Claim 1 at the second step, we get

$$\left[R, V^{(n+1)}\right] = \left[R, [V^{(n)}, V^{(n)}]\right] \subseteq V^{(n)} \subseteq T^n(V),$$

and thus, $V^{(n+1)} \subseteq T(T^n(V)) = T^{(n+1)}(V)$. □

We obtain another corollary of Theorem 2.3.

Corollary 2.6. *Let R be a ring, and let $V \subseteq R$ be an $[R, R]$-submodule. Then,*

$$V^{(n)} \subseteq \bigcap_{j=1}^{n} T^j(V)$$

for all $n \geqslant 1$, and we deduce that

$$\widetilde{R}\left[V^{(1)} \cap V, V^{(2)}\right]\widetilde{R} \subseteq V + V^2, \quad \text{and} \quad \left[R, \widetilde{R}[V^{(2)}, V^{(3)}]\widetilde{R}\right] \subseteq V.$$

In particular, if $[V^{(2)}, V^{(3)}]$ is full, then $[R, R] \subseteq V$, and so, V is a Lie ideal in R.

Proof. The first inclusion follow from Lemma 2.5. The other inclusions then follow from Theorem 2.3. □

Remark 2.7. Let $V \subseteq R$ be an $[R, R]$-submodule. In light of Corollary 2.6, it is interesting to determine when $[V^{(2)}, V^{(3)}]$ is full. By Lemma 2.5, we have

$$[V^{(2)}, V^{(3)}] \subseteq [V^{(2)}, V^{(2)}] = V^{(3)} \subseteq V^{(1)} = [V, V] \subseteq [R, V].$$

Thus, a necessary condition is that $[R, V]$ is full, that is, V is fully noncentral (Definition A). In the next section, we study rings for which full noncentrality of V is also sufficient.

3 | RINGS WHERE PROPER IDEALS ARE CONTAINED IN PRIME IDEALS

In this section, we study rings where every proper ideal is contained in a prime ideal. For a Lie ideal L in such a ring R for which $[L, L]$ is full, we show that L contains the commutator subgroup $[R, R]$ and that $R = L^2$; see Theorem 3.3.

In [22, 23], the second named author initiated the study of rings where every proper ideal is contained in a maximal ideal. This class includes all rings that are unital or just finitely generated as an ideal, as well as all rings satisfying the ascending chain condition for ideals. In Proposition 3.1 below, we clarify the relationship with the class of rings where every ideal is contained in a prime ideal. Note that there exist rings with maximal ideals that are not prime. Further, there exist rings where every proper ideal is contained in a prime ideal, but not every proper ideal is contained in a maximal ideal, for example, the commutative C^*-algebra $C_0(\mathbb{R})$ of continuous functions $\mathbb{R} \to \mathbb{C}$ that vanish at infinity. We refer the reader to [3] and [19, Chapter 4] for the basic theory of prime ideals and prime rings. We also note that the class of rings where every proper ideal is contained in a prime ideal (even a nonexceptional prime ideal) includes all unital rings, as well as every C^*-algebra [13].

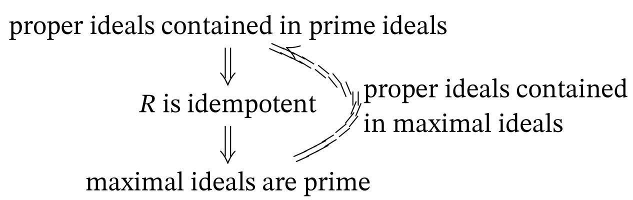
A ring R is said to be *idempotent* if $R = R^2$, that is, if every element of R is the sum of products of elements of R. This holds for every unital ring, as well as for Banach algebras with bounded approximate identity by the Cohen factorization theorem.

Proposition 3.1. *Let R be a ring. Then, the following hold.*

(1) *If every proper ideal is contained in a prime ideal, then R is idempotent.*
(2) *If R is idempotent, then every maximal ideal in R is prime.*
(3) *If every maximal ideal in R is prime, and every proper ideal is contained in a maximal ideal, then every proper ideal in R is contained in a prime ideal.*

In particular, if every proper ideal of R is contained in a maximal ideal, then every maximal ideal of R is a prime ideal if and only if R is idempotent.

The implications are shown in the following diagram:



Proof.

(1) Suppose on the contrary that $R \neq R^2$. Then, R^2 is a proper ideal, and thus, $R^2 \subseteq P$ for some prime ideal P of R. This implies that $R = P$, a contradiction.
(2) Let M be a maximal ideal of R. Since $R = R^2$, we get $(R/M)^2 \neq 0$. It follows that R/M is a simple ring and so it is a prime ring. Thus, M is a prime ideal of R.
(3) This is clear. □

For later use, we recall basic results about Lie ideals in rings. These are well known, and we include the short proofs for the convenience of the reader.

Lemma 3.2. *Let L be a Lie ideal in a ring R. Then*

(1) *We have $\widetilde{R}L\widetilde{R} = \widetilde{R}L$.*
(2) *We have $[R, L^2] \subseteq [R, L]$.*
(3) *We have $\widetilde{R}[L, L]\widetilde{R} \subseteq L + L^2$.*
(4) *We have $\widetilde{R}[L, L^2]\widetilde{R} \subseteq L^2$.*

Proof. We will use that $[\widetilde{R}, V] = [R, V]$ for every additive subgroup $V \subseteq \widetilde{R}$. In the proof of (3) and (4), we will use that $U[V, W] \subseteq [UV, W] + [W, U]V$ for additive subgroups $U, V, W \subseteq R$.

(1) The inclusion $\widetilde{R}L \subseteq \widetilde{R}L\widetilde{R}$ is clear. We therefore have

$$\widetilde{R}L\widetilde{R} \subseteq [\widetilde{R}L, \widetilde{R}] + \widetilde{R}^2L \subseteq \widetilde{R}[L, \widetilde{R}] + [\widetilde{R}, \widetilde{R}]L + \widetilde{R}^2L \subseteq \widetilde{R}L.$$

(2) Let $x, y \in L$ and $a \in R$. Then,

$$[a, xy] = axy - xya = axy - yax + yax - xya = [ax, y] + [ya, x] \in [R, L].$$

In fact, if V is any additive subgroup of R, then the same argument shows that $[a, x] \in [R, V]$ for every $a \in R$ and every x in the subring of R generated by V.

(3) This was already proved in Corollary 2.4. Let us give an alternative proof here. Since $[L, L]$ is a again a Lie ideal, we can apply (1) at the first step, and get

$$\widetilde{R}[L, L]\widetilde{R} = \widetilde{R}[L, L] \subseteq [\widetilde{R}L, L] + [L, \widetilde{R}]L \subseteq L + L^2.$$

(4) Proceeding at the first two steps as in (3), and using that $[R, L^2] \subseteq L^2$ and (2) at the third step, we get

$$\widetilde{R}[L, L^2]\widetilde{R} = \widetilde{R}[L, L^2] \subseteq [\widetilde{R}L, L^2] + [\widetilde{R}, L^2]L \subseteq L^2 + [R, L]L \subseteq L^2$$

as desired. □

Theorem 3.3. *Let R be a ring such that every proper ideal is contained in a prime ideal, and such that $[R, R]$ is full. Let $L \subseteq R$ be a Lie ideal. Then, the following are equivalent.*

(1) *The subgroup $[L, L]$ is full.*
(2) *The subgroup $[L, L^2]$ is full.*
(3) *We have $R = L^2$.*
(4) *We have $[R, R] \subseteq L$.*

Moreover, if this is the case, then $[R, R] = [R, L]$.

Proof. We first show that (1) implies (4). Assuming that $[L, L]$ is full, we apply Lemma 3.2(3) at the second step to get

$$R = \widetilde{R}[L, L]\widetilde{R} \subseteq L + L^2.$$

Using the above at the first step, and Lemma 3.2(2) at the second step, we obtain

$$[R, R] \subseteq [R, L + L^2] \subseteq [R, L] \subseteq L.$$

To show that (2) implies (3), assume that $[L, L^2]$ is full. Applying Lemma 3.2(4) at the second step, we get

$$R = \widetilde{R}[L, L^2]\widetilde{R} \subseteq L^2.$$

To show that (3) implies (4), assume that $R = L^2$. Applying Lemma 3.2(2) at the third step, we get

$$[R, R] = [L^2, L^2] = [R, L^2] \subseteq [R, L] \subseteq L.$$

Next, we show that (4) implies (1) and (2). Assume that $[R, R] \subseteq L$. To verify (1), set $I := \widetilde{R}[[R, R], [R, R]]\widetilde{R}$. If $I \neq R$, then by assumption, we obtain a prime ideal $P \subseteq R$ containing I. Then, the quotient R/I is a prime ring satisfying $[[R/I, R/I], [R/I, R/I]] = \{0\}$, which by [14, Theorem 2.3] implies that R/I is commutative. This contradicts the fact that commutators are full in R, and hence in R/I. Thus, $I = R$, which means that $[[R, R], [R, R]]$ is full, and consequently so is $[L, L]$.

Similarly, to verify (2), if the ideal J generated by $[[R, R], [R, R]^2]$ is proper, then it is contained in a prime ideal Q such that $[[Q/J, Q/J], [Q/J, Q/J]^2] = \{0\}$, which also implies that Q/J is commutative by [14, Theorem 2.3], a contradiction. This shows that $[[R, R], [R, R]^2]$ is full, and hence so is $[L, L^2]$.

Finally, we have seen in the proof of "(1)⇒(4)" that $[R, R] \subseteq [R, L]$. □

Corollary 3.4. *Let R be a ring such that every proper ideal is contained in a maximal ideal. If L is a Lie ideal of R and if $[L, L]$ is full, then $R = L^2$.*

Proof. Using that $[L, L]$ is full, it follows that R is idempotent, and thus, every maximal ideal of R is a prime ideal by Proposition 3.1. Hence, every proper ideal of R is contained in a prime ideal of R. It follows from Theorem 3.3 that $R = L^2$. □

Corollary 3.5. *Let R be a ring such that every proper ideal is contained in a prime ideal. Then, the following are equivalent.*

(1) *The subgroup $[[R, R], [R, R]]$ is full.*
(2) *The subgroup $[[R, R], [R, R]^2]$ is full.*
(3) *We have $R = [R, R]^2$.*
(4) *The commutator subgroup $[R, R]$ is full.*

Moreover, if this is the case, then $[R, R] = [R, [R, R]]$.

Proof. We note that each of the conditions (1)–(3) implies that $[R, R]$ is full. Hence, their equivalence follows from Theorem 3.3 applied for the Lie ideal $L = [R, R]$. Further, condition (3) clearly implies (4). Next, assuming that (4) holds, it also follows from Theorem 3.3 (again applied for the Lie ideal $L = [R, R]$) that $[[R, R], [R, R]]$ is full, which verifies (1). □

In the setting of Theorem 3.3, we saw conditions characterizing when $[L, L]$ is full for a Lie ideal L in a ring R. Clearly, these conditions also imply that L is fully noncentral, that is, $[R, L]$ is full. However, the following examples show that the converse does not hold.

Example 3.6. Let $\mathbb{F}$ be a field of characteristic 2, and consider the simple, unital ring $R = M_2(\mathbb{F})$ of 2-by-2 matrices over $\mathbb{F}$. Consider the subgroup

$$L = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{F} \right\}.$$

One readily sees that $[R, L] = L$, and thus, $L \nsubseteq Z(R)$, while $[L, L] = \{0\}$ and $L^2 = L \neq R$. We also have $[R, R] \nsubseteq L$. In particular, L is a fully noncentral Lie ideal that does not generate R as a ring, and that does not contain the commutator subgroup. In light of Corollary 2.6, we also compute the (higher) commutator subgroups of R as

$$R^{(1)} = [R, R] = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} : a, b, c \in \mathbb{F} \right\}, \quad R^{(2)} = Z(R), \quad \text{and} \quad R^{(3)} = \{0\}.$$

In particular, R is fully noncentral, while $[R^{(2)}, R^{(3)}] = \{0\}$.

Example 3.7. Let R be a four-dimensional central division algebra of characteristic 2. Then, there exists a Lie ideal L of R such that neither $[R, R] \subseteq L$ nor $L \subseteq Z(R)$. To see this, let K be a maximal subfield of R, and let $\{1, \mu\}$ be a basis of K over $Z(R)$. Then, $R \otimes_{Z(R)} K \cong \mathrm{M}_2(K)$.

Given a Lie ideal $\widetilde{L}$ of the K-algebra $R \otimes_{Z(R)} K$, set

$$L = \left\{ r \in R : \text{ there is } s \in R \text{ such that } r \otimes 1 + s \otimes \mu \in \widetilde{L} \right\}.$$

Clearly, L is a Lie ideal of the $Z(R)$-algebra R. Moreover, $\dim_{Z(R)} L = \dim_K \widetilde{L}$. In particular, if $\dim_K \widetilde{L} = 2$ (as in Example 3.6), then neither $[R, R] \subseteq L$ nor $L \subseteq Z(R)$.

4 | RINGS WHERE PROPER IDEALS ARE CONTAINED IN NONEXCEPTIONAL PRIME IDEALS

To extend the results of Section 3 to fully noncentral Lie ideals and $[R, R]$-submodules, we need to exclude prime rings as in Examples 3.6 and 3.7. These are exactly the prime rings where Herstein's techniques for the study of Lie ideals break down (see, e.g., [21, p.120]), and we call them *exceptional*; see Definition 4.1.

In this section, we study rings where every proper ideal is contained in a nonexceptional prime ideal. We show that an $[R, R]$-submodule V in such a ring is fully noncentral if and only if it contains $[R, R]$; see Theorem 4.4. It follows that a fully noncentral subgroup V is an $[R, R]$-submodule if and only if it is a Lie ideal, if and only if $[R, R] \subseteq V$; see Corollary 4.5.

We note that in an algebra over a field of characteristic $\neq 2$, every prime ideal is automatically nonexceptional. In particular, the class of rings where every proper ideal is contained in a nonexceptional prime ideal includes every C^*-algebra [13].

For $n \geqslant 1$, we write S_n for the permutation group on n elements. The *standard polynomial of degree n* (see, e.g., [4, Definition 6.10]) is the polynomial $s_n \in \mathbb{Z}[x_1, \dots, x_n]$ given by

$$s_n(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

We will only need the polynomial s_4. For a ring R, we let $S_4(R)$ denote the additive subgroup of R generated by the elements $s_4(x_1, \dots, x_4)$ for $x_1, \dots, x_4 \in R$. A ring R is said to *satisfy the polynomial identity* s_4 if $s_4(R) = \{0\}$.

Recall that a prime ring R is said to have *characteristic* 2 if $2x = 0$ for every $x \in R$. (Equivalently, if $2x = 0$ for some nonzero $x \in R$.)

Definition 4.1. We say that a prime ring R is *exceptional* if it has characteristic 2 and it satisfies the polynomial identity s_4. We say that a prime ideal P in a ring S is *exceptional* if the prime ring S/P is exceptional.

Let R be an exceptional prime ring. Then, R is commutative if and only if R is a field of characteristic 2. If R is noncommutative, then R is a prime PI-ring and therefore has nonzero center Z, the extended centroid of R (see [4, Definition 7.29]) agrees with the field of fractions ZZ^{-1}, and the ring of central quotients RZ^{-1} is isomorphic to the ring $M_2(ZZ^{-1})$ of two-by-two matrices over the field ZZ^{-1}; see [4, Section 7.9]. It follows that a prime ring R is exceptional if and only if it embeds into $M_2(\mathbb{F})$ for some field $\mathbb{F}$ of characteristic 2; see [4, Corollary 7.59]. We also note that a prime ring R is exceptional if and only if $2R + \widetilde{R}S_4(R)\widetilde{R} = \{0\}$.

Proposition 4.2. *Let R be a ring. Then, the following hold.*

(1) *If* $R = 2R + \widetilde{R}S_4(R)\widetilde{R}$*, then every prime ideal of R is nonexceptional.*
(2) *If every proper ideal of R is contained in a nonexceptional prime ideal, then* $R = 2R + \widetilde{R}S_4(R)\widetilde{R}$.

In particular, every proper ideal of R is contained in a nonexceptional prime ideal if and only if $R = 2R + \widetilde{R}S_4(R)\widetilde{R}$ *and every proper ideal of R is contained in a prime ideal.*

Proof.

(1) Assume that $R = 2R + \widetilde{R}S_4(R)\widetilde{R}$, and let P be a prime ideal of R. Then, $R/P = 2(R/P) + \widetilde{R/P}S_4(R/P)\widetilde{R/P} \neq \{0\}$, which implies that R/P is nonexceptional.
(2) Assume that every proper ideal of R is contained in a nonexceptional prime ideal. To reach a contradiction, assume that the ideal $I := 2R + \widetilde{R}S_4(R)\widetilde{R}$ is proper. By assumption, we obtain a nonexceptional prime ideal $P \subseteq R$ containing I. Then, $2(R/P) + \widetilde{R/P}S_4(R/P)\widetilde{R/P} = \{0\}$, which shows that R/P is exceptional, a contradiction. □

Given an additive subgroup V in a ring, recall that $V^{(n)}$ is defined inductively for $n \geqslant 0$ as $V^{(0)} := V$ and $V^{(n+1)} := [V^{(n)}, V^{(n)}]$.

Lemma 4.3. *Let R be a nonexceptional prime ring, and let* $V \subseteq R$ *be an* $[R, R]$*-submodule with* $[R, V] \neq \{0\}$*. Then,* $[V^{(m)}, V^{(n)}]$ *is not central for every* $m, n \geqslant 1$.

Proof. Let $Z := Z(R)$ denote the center of R. Since R is prime, any element $a \in R$ that satisfies $[a, R] \subseteq Z$ will automatically belong to Z. Indeed, the primeness of R implies that either $Z = \{0\}$ or Z is a domain. Assuming that $[a, R] \subseteq Z$, and given $x \in R$, we have $a[a, x] = [a, ax] \in Z$, and so, $xa[a, x] = a[a, x]x = ax[a, x]$. Thus, $[a, x][a, x] = 0$, implying $[a, x] = 0$. So, $a \in Z$.

We claim that $[R, R] \nsubseteq Z$. Assuming by contradiction that $[R, R] \subseteq Z$, the above observation implies that $R \subseteq Z$, and thus, $[R, V] = \{0\}$, which is a contradiction. Thus, we have $[R, R] \nsubseteq Z$, as desired.

Since

$$V^{(m+n+1)} = \left[V^{(m+n)}, V^{(m+n)}\right] \subseteq \left[V^{(m)}, V^{(n)}\right],$$

it suffices to verify that $V^{(n+1)}$, which equals $\left[V^{(n)}, V^{(n)}\right]$ by definition, is not central for every $n \geqslant 1$. Using induction over n, we will show that in fact $V^{(n)} \nsubseteq Z$ for every $n \geqslant 0$.

The case $n = 0$ is true by assumption. Assume that $V^{(n)} \nsubseteq Z$, and, in order to reach a contradiction, assume that $V^{(n+1)} \subseteq Z$. Applying [21, Lemma 11] for $U = [R, R]$ and $G = V^{(n)}$, we have $[U, G] \subseteq G$ (since $V^{(n)}$ is an $[R, R]$-submodule) and $[G, G] = [V^{(n)}, V^{(n)}] = V^{(n+1)} \subseteq Z$. Since $[R, R] = U \nsubseteq Z$, we deduce that $V^{(n)} = G \subseteq Z$, a contradiction. This finishes the proof. □

In [22, Theorem 1.2], it is shown that if R is a ring with $R = 2R$ such that every proper ideal is contained in a maximal ideal, and if $L \subseteq R$ is a Lie ideal such that $[R, L]$ is full, then $R = [R, L] + [R, L]^2$ and $[R, R] \subseteq L$. Next, we show that this result also holds for $[R, R]$-submodules instead of Lie ideals. Moreover, we can relax the condition that proper ideals are contained in maximal ideals to containment in prime ideals. Further, we conclude that $R = [R, L]^2$, that is, the summand $[R, L]$ is not necessary.

Theorem 4.4. *Let R be a ring such that every proper ideal is contained in a nonexceptional prime ideal, and such that $[R, R]$ is full. Let $V \subseteq R$ be an $[R, R]$-submodule. Then, the following are equivalent.*

(1) *The subgroup V is fully noncentral.*
(2) *The subgroup $V^{(n)}$ is full for some (equivalently, every) $n \geqslant 1$.*
(3) *The subgroup $[V^{(m)}, V^{(n)}]$ is full for some (equivalently all) $m, n \geqslant 1$*
(4) *We have $[R, R] \subseteq V$ (and, in particular, V is a Lie ideal).*

Moreover, if V is fully noncentral, then $[R, R] = [R, V] = [V, V] = V^{(n)} = R^{(n)}$ for every $n \geqslant 1$, and $R = V^2$.

Proof. We begin by showing that (3) implies (1). Given $m, n \geqslant 1$, we can apply Lemma 2.5 as in Remark 2.7 to see that

$$[V^{(m)}, V^{(n)}] \subseteq [V^{(1)}, V^{(1)}] = V^{(2)} \subseteq V^{(1)} = [V, V] \subseteq [R, V].$$

Thus, if $[V^{(m)}, V^{(n)}]$ is full for some $m, n \geqslant 1$, then V is fully noncentral.

Conversely, assume that $[R, V]$ is full, and let $m, n \geqslant 1$. Set $I := \widetilde{R}[V^{(m)}, V^{(n)}]\widetilde{R}$ and let us verify that $I = R$. To reach a contradiction, assume that $I \neq R$. By assumption, we obtain some nonexceptional prime ideal $P \subseteq R$ such that $I \subseteq P$. Then, R/P is a nonexceptional prime ring. Let W denote the image of V in R/P. Then, W is an additive subgroup of R/P with $[[R/P, R/P], W] \subseteq W$. Since $[R, V]$ is full, so is $[R/P, W]$. However, since $I \subseteq P$, we have $[W^{(m)}, W^{(n)}] = \{0\}$, which contradicts Lemma 4.3. This shows that $I = R$, as desired. Thus, (1) and (3) are equivalent.

Using Lemma 2.5, one shows that (2) and (3) are equivalent. By Corollary 2.6, (3) implies (4). Conversely, to see that (4) implies (1), assume that $[R, R] \subseteq V$. Using the last statement in Corollary 3.5 at the first step, we get

$$[R, R] = [R, [R, R]] \subseteq [R, V],$$

and thus, $[R, V]$ is full. Thus, (1)–(4) are equivalent.

It remains to show the last statement. We first verify that $[R, R] = R^{(n)}$ for every $n \geqslant 1$. By Corollary 3.5, we have $[R, [R, R]] = [R, R]$, which shows that $[R, R]$ is fully noncentral. Given $n \geqslant 1$, applying the equivalence of (1) and (3) for $V = [R, R]$, we get that $[R^{(1)}, R^{(n)}]$ is full.

Since $[R^{(1)}, R^{(n)}] \subseteq [R, R^{(n)}]$, we deduce that $R^{(n)}$ is fully noncentral, and thus, $[R,R] \subseteq R^{(n)}$. The converse inclusion always holds, which shows that $[R,R] = R^{(n)}$.

Next, assume that V is fully noncentral. Using (4), it follows that

$$[R,R] = R^{(2)} = [[R,R],[R,R]] \subseteq [V,V],$$

and thus, $[R,R] = [R,V] = [V,V]$. This implies that $[R,R] = R^{(n)} = V^{(n)}$ for all $n \geqslant 1$, and since V is a Lie ideal, we also get $R = L^2$ by Theorem 3.3. □

Corollary 4.5. *Let R be a ring such that every proper ideal is contained in a nonexceptional prime ideal, and let $V \subseteq R$ be a fully noncentral, additive subgroup. Then, the following are equivalent.*

(1) *We have $[R,R] \subseteq V$.*
(1) *The subgroup V is a Lie ideal.*
(3) *The subgroup V is an $[R,R]$-submodule.*

In particular, if V is a fully noncentral Lie ideal in $[R,R]$, then $V = [R,R]$.

Proof. It is clear that (1) implies (2), and that (2) implies (3). To see that (3) implies (1), we note that $[R,R]$ is full (since $[R,V]$ is full), and thus, the fact that (1) implies (4) in Theorem 4.4 gives the result. □

Remark 4.6. In the setting of Theorem 4.4, it is not clear that $R = V^2$ implies that V is fully noncentral. This should be contrasted with Theorem 3.3, where under fewer assumptions on the ring R, it was shown that a Lie ideal L is fully noncentral whenever $R = L^2$. We do not know if a similar result holds in the context of Theorem 4.4, since it is unclear if an $[R,R]$-submodule V satisfying $R = V^2$ is a Lie ideal.

Corollary 4.7. *Let R be a ring such that every proper ideal is contained in a nonexceptional prime ideal. Then, the following are equivalent.*

(1) *The subgroup $[R,R]$ is full.*
(2) *The subgroup $R^{(n)}$ is full for some (equivalently, every) $n \geqslant 1$.*
(3) *We have $R = [R,R]^2$.*

Moreover, if this is the case, then $[R,R] = [R^{(m)}, R^{(n)}]$ for all $m, n \geqslant 0$, and in particular $[R,R] = R^{(n)}$ for all $n \geqslant 1$.

Proof. It is clear that (2) implies (1), since $R^{(n)} \subseteq [R,R]$. The equivalence of (1) and (3) was shown in Corollary 3.5. By applying Theorem 4.4 with $V = [R,R]$, it follows that $[R,R] = R^{(n)}$ for all $n \geqslant 1$. This shows that (1)–(3) are equivalent.

Assume that $[R,R]$ is full. As noted above, we then have $[R,R] = R^{(n)}$ for all $n \geqslant 1$. Given $m, n \geqslant 0$, without loss of generality, assume that $m \leqslant n$. Then,

$$[R,R] = R^{(n+1)} = [R^{(n)}, R^{(n)}] \subseteq [R^{(m)}, R^{(n)}].$$

The converse inclusion is clear. □

We also obtain the following result for pairs of Lie ideals, which in the setting of C^*-algebras was obtained in [13, Theorem 3.6].

Theorem 4.8. *Let R be a ring such that every proper ideal is contained in a nonexceptional prime ideal, and let $K, L \subseteq R$ be Lie ideals. Then, the following are equivalent.*

(1) *The Lie ideal $[K, L]$ is full.*
(2) *The Lie ideals $[K, K]$ and $[L, L]$ are full.*
(3) *The Lie ideals $[R, K]$ and $[R, L]$ are full.*

Moreover, if this is the case, then $[R, R] = [K, L] = [K, K] = [L, L]$.

Proof. It is clear that (1) implies (3) and that (2) implies (3). To show that (3) implies (1) and (2), assume that K and L are fully noncentral. By Theorem 4.4, we get $[R, R] \subseteq K$ and $[R, R] \subseteq L$. Using that $R^{(2)} = R^{(3)}$ at the first step (which is true by Corollary 4.7), we get

$$[R, R] = [[R, R], [R, R]] \subseteq [K, L],$$

and thus, $[R, R] = [K, L]$. Similarly, we get $[R, R] = [K, K] = [L, L]$. This shows that $[K, L]$, $[K, K]$ and $[L, L]$ are full, and equal to each other. □

The results of this section are applicable in many concrete cases of interest. We point out the following special case, which seems to not have appeared in the literature.

Corollary 4.9. *Let R be a unital algebra over a field of characteristic $\neq 2$, and let $V \subseteq R$ be a fully noncentral $[R, R]$-submodule. Then, $R = V^2$ and $[R, R] \subseteq V$. In particular, V is a Lie ideal.*

Proof. Since R is unital, every proper ideal is contained in a proper (and maximal) ideal. Further, since R is an algebra over a field of characteristic $\neq 2$, no prime quotient of R is exceptional. The result then follows from Theorem 4.4. □

5 | FULLY NONCENTRAL, INVARIANT SUBSPACES

Given a fully noncentral subspace V in an algebra A over a field $\mathbb{F}$, we study the connection between the invariance of V under inner automorphisms of A and the Lie ideal property for V. We do this in two different settings: we assume that every commutator is a sum of either square-zero elements (Theorem 5.2), or of nilpotent elements (Theorem 5.5).

As a first step, we connect invariance of V under inner automorphism induced by square-zero elements to the condition $[x, V] \subseteq V$ for $x^2 = 0$; see Lemma 5.1. For nilpotent elements instead of square-zero elements, this is done in Lemma 5.3. We then combine these lemmas with the results from Section 4 to obtain the main results Theorem 5.2 and Theorem 5.5.

The condition that every commutator is a sum of square-zero elements is automatically satisfied in a number of cases of interest, for example, for the class of zero-product balanced algebras introduced in [12]; see Corollary 5.7.

Given an algebra A, we say that an automorphism $\alpha : A \to A$ is inner if there exists an invertible element $v \in \widetilde{A}$ such that $\alpha(a) = vav^{-1}$ for all $a \in A$. We actually only consider automorphisms of the form $a \mapsto (1 + x + x^2 + \dots + x^{k-1})a(1 - x)$ induced by a nilpotent element $x \in A$ with $x^k = 0$.

We set $N_2(A) := \{x \in A : x^2 = 0\}$, the set of square-zero elements in A. We let $N_2(A)^+$ denote the subspace of A generated by the square-zero elements. We will say that a subset $V \subseteq A$ is $N_2(A)$-*invariant*, if it is invariant under all inner automorphisms induced by elements from $N_2(A)$. Equivalently, if $a \in V$ and $x^2 = 0$ imply $(1 + x)a(1 - x) \in V$.

The following result is probably known, but we could not locate it in the literature.

Lemma 5.1. *Let A be an algebra over a field $\mathbb{F}$, and let $V \subseteq A$ be a subspace. Then, the following statements hold:*

(1) *If $\mathbb{F} \neq \{0, 1\}$ and V is $N_2(A)$-invariant, then $[N_2(A)^+, V] \subseteq V$.*
(2) *If $\mathrm{char}(\mathbb{F}) \neq 2$ and $[N_2(V)^+, V] \subseteq V$, then V is $N_2(A)$-invariant.*

Proof.

(1) Let $a \in V$ and $x \in N_2(A)$. We need to verify $[x, a] \in V$.
We have $(1 + x)V(1 - x) \subseteq V$, and thus,

$$[x, a] - xax = (1 + x)a(1 - x) - a \in V.$$

Since $\mathbb{F}$ contains at least three elements, we can choose $\lambda \in F$ with $\lambda \neq 0, 1$. Then, λx is also a square-zero element, and applying the above to λx instead of x, we obtain

$$\lambda[x, a] - \lambda^2 xax = (1 + \lambda x)a(1 - \lambda x) - a \in V.$$

Since V is a subspace, we obtain that $\lambda^2([x, a] - xax) \in V$, and thus,

$$(\lambda^2 - \lambda)[x, a] = \lambda^2([x, a] - xax) - \lambda[x, a] + \lambda^2 xax$$

also belongs to V. Since $(\lambda^2 - \lambda) \neq 0$, and using again that V is a subspace, we get $[x, a] \in V$.
(2) Let $a \in V$ and $x \in N_2(A)$. We need to verify $(1 + x)a(1 - x) \in V$. We have

$$(1 + x)a(1 - x) = a + [x, a] - xax.$$

Since $[x, a] \in V$, it suffices to verify that $xax \in V$. To see this, note that

$$-2xax = [x, [x, a]]$$

belongs to V, since $[x, a] \in V$. Since $\mathrm{char}(\mathbb{F}) \neq 2$ and since V is a subspace, we get $xax \in V$. □

Theorem 5.2. *Let A be an algebra over a field $\neq \{0, 1\}$ such that every commutator in A is a sum of square-zero elements, and every proper ideal is contained in a nonexceptional prime ideal. Let $V \subseteq A$ be a fully noncentral subspace. Then, the following are equivalent.*

(1) *The subspace V is invariant under all inner automorphisms of A.*

(2) *The subspace V is invariant under all inner automorphisms induced by square-zero elements of A.*
(3) *We have $[A, A] \subseteq V$.*
(4) *The subspace V is a Lie ideal.*
(5) *The subspace V is an $[A, A]$-submodule.*

Proof. It is clear that (1) implies (2). By Corollary 4.5, (3)–(5) are equivalent. Let us show that (2) implies (5). By Lemma 5.1, we have $[N_2(A)^+, V] \subseteq V$, and since $[A, A] \subseteq N_2(A)^+$ by assumption, we get $[[A, A], V] \subseteq V$.

Finally, to show that (3) implies (1), let u be an invertible element in $\widetilde{A}$, and let $x \in V$. We have $[\widetilde{A}, \widetilde{A}] = [A, A] \subseteq V$, and therefore,

$$uxu^{-1} = [u, xu^{-1}] + x \in [\widetilde{A}, \widetilde{A}] + V \subseteq V.$$

This shows that $uVu^{-1} \subseteq V$, as desired. □

For an algebra A, we write $N(A)$ for the set of its nilpotent elements, and $N(A)^+$ for the subspace it generates. The next result is a variation of Lemma 5.1, using nilpotent elements instead of square-zero elements.

Lemma 5.3. *Let A be an algebra over a field $\mathbb{F}$, and let $V \subseteq A$ be a subspace. Then, the following statements hold.*

(1) *If $\mathbb{F}$ is infinite and V is $N(A)$-invariant, then $[N(A)^+, V] \subseteq V$.*
(2) *If $\mathrm{char}(\mathbb{F}) = 0$ and $[N(A)^+, V] \subseteq V$, then V is $N(A)$-invariant.*

Proof.

(1) Let $a \in V$ and $k \geqslant 2$. We need to verify that the commutator $[x, a]$ belongs to V whenever $x \in A$ satisfies $x^k = 0$. Given such an element x, we have $(1 + x + x^2 + \cdots + x^{k-1})V(1 - x) \subseteq V$, and thus,

$$[x, a] + x[x, a] + \cdots + x^{k-1}[x, a] = (1 + x + x^2 + \cdots + x^{k-1})a(1 - x) - a \in V.$$

Given a nonzero $\lambda \in \mathbb{F}$, using that $(\lambda x)^k = 0$, we get

$$\begin{aligned}[x, a] &+ \lambda x[x, a] + \cdots + \lambda^{k-1}x^{k-1}[x, a] \\ &= \lambda^{-1}\big([\lambda x, a] + \lambda x[\lambda x, a] + \cdots + (\lambda x)^{k-1}[\lambda x, a]\big) \in V.\end{aligned}$$

Using that $\mathbb{F}$ is infinite, we can choose elements $\lambda_1, \ldots, \lambda_{k-1} \in F$ that are pairwise distinct and nonzero. Then, the corresponding Vandermonde matrix

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{k-1} \\ \vdots & & & & \vdots \\ 1 & \lambda_{k-1} & \lambda_{k-1}^2 & \cdots & \lambda_{k-1}^{k-1} \end{pmatrix}$$

is invertible and using the entries in the first column of the inverse matrix, we obtain $\mu_1, \dots, \mu_{k-1} \in \mathbb{F}$ such that

$$[x, a] = \sum_{j=1}^{k-1} \mu_j \Big([x, a] + \lambda_j x[x, a] + \dots + \lambda_j^{k-1} x^{k-1}[x, a] \Big) \in V.$$

This finishes the proof.

(2) Let $x \in A$ satisfy $x^{k+1} = 0$ and let $a \in V$. Set $b = (1 - x)a(1 + x + \dots + x^k)$. Then,

$$\begin{aligned} b &= (1 - x)a(1 + x + \dots + x^k) \\ &= a - [x, a] - xa(x + \dots + x^k) + a(x^2 + \dots + x^k) \\ &= a - [x, a] - \sum_{i=1}^{k} [x, a]x^j. \end{aligned}$$

We aim to prove that $b \in V$. For this, it suffices to show that $[x, a]x^j \in V$ for all $j = 1, \dots, k$. Since $[x, [x, a]]$ belongs to V as $[N(A), V] \subseteq V$, we deduce that also $x^2a - 2xax + ax^2$ belongs to V. Also, $x^2a - ax^2 \in V$. We get

$$ax^2 - xax = [a, x]x \in V. \tag{5.1}$$

Note that

$$[x, [x, [x, a]]] = x^3a - 3x^2ax + 3xax^2 - ax^3 = [x^3, a] - 3x[x, a]x$$

belongs to V as $[x, [x, a]] \in V$ and $[N(A), V] \subseteq V$, and thus, $x[x, a]x \in V$. Note that

$$[x, [a, x]x] \overset{(5.1)}{=} x[a, x]x - [a, x]x^2$$

belongs to V, and thus,

$$[a, x]x^2 \in V. \tag{5.2}$$

Let $x \in N(A)$. Given $n \geqslant 2$, we assume that $[a, x]x^j \in V$ for all $j = 1, \dots, n$. We claim that $[a, x]x^{n+1} \in V$. Since $x + x^n$ is also nilpotent, replacing x by $x + x^n$ in (5.2) gives $[a, x + x^n](x + x^n)^2 \in V$. Using this, we get

$$[a, x + x^n](x + x^n)^2 = [a, x + x^n](x^2 + 2x^{n+1} + x^{2n}) \in V,$$

which implies that

$$[a, x](2x^{n+1} + x^{2n}) + [a, x^n](x^2 + 2x^{n+1}) \in V.$$

Hence,

$$2[a, x]x^{n+1} + [a, x^n]x^2 \in V. \tag{5.3}$$

We compute

$$
\begin{aligned}
[a, x^n]x^2 &= \sum_{j=0}^{n-1} x^j[a,x]x^{n-j-1}x^2 \\
&= \sum_{j=1}^{n-1} x^j[a,x]x^{n-j+1} + [a,x]x^{n+1} \\
&= \sum_{j=1}^{n-1} \left[x^j, [a,x]x^{n-j+1}\right] + (n-1)[a,x]x^{n+1} + [a,x]x^{n+1} \\
&= \sum_{j=1}^{n-1} \left[x^j, [a,x]x^{n-j+1}\right] + n[a,x]x^{n+1}.
\end{aligned}
$$

Using (5.3) at the last step, we get

$$
2[a,x]x^{n+1} + [a,x^n]x^2 = \sum_{j=1}^{n-1} \left[x^j, [a,x]x^{n-j+1}\right] + (n+2)[a,x]x^{n+1} \in V.
$$

Since $n - j + 1 \leqslant n$ for $j = 1, \dots, n-1$, we get $\left[x^j, [a,x]x^{n-j+1}\right] \in V$, and so,

$$
[a,x]x^{n+1} \in V,
$$

as desired. □

Proposition 5.4. *Let A be an algebra over an infinite field. Then, given nilpotent elements $x, y \in A$, the commutator $[x, y]$ is a finite sum of nilpotent elements.*

Proof. This follows from part (1) of Lemma 5.3 applied to $V = N(A)^+$, since the set consisting of finite sums of nilpotent elements is a subspace that is invariant under all (inner) automorphisms. □

The next result is a variation of Theorem 5.2. By strengthening the assumption on the field, we can relax the assumption that every commutator is a sum of square-zero elements to allow nilpotent elements instead.

Theorem 5.5. *Let A be an algebra over an infinite field such that every commutator in A is a sum of nilpotent elements, and such that every proper ideal is contained in a nonexceptional prime ideal. Let $V \subseteq A$ be a fully noncentral subspace. Then, the following are equivalent.*

1. *The subspace V is invariant under all inner automorphisms of A.*
2. *The subspace V is invariant under all inner automorphisms induced by nilpotent elements of A.*
3. *We have $[A, A] \subseteq V$.*
4. *The subspace V is a Lie ideal.*
5. *The subspace V is an $[A, A]$-submodule.*

Proof. The result is proved analogous to Theorem 5.2, with the only difference that for the implication "(2)⇒(5)," we use that $[A, A] \subseteq N(A)^+$ by assumption, that $[N(A)^+, V] \subseteq V$ by Lemma 5.3, and consequently, $[[A, A], V] \subseteq V$. □

The next result can also be deduced from [10, Theorem 2].

Example 5.6. Let R be a noncommutative, simple ring with infinite center and such that every commutator is a sum of nilpotent elements. Assume that R is not an exceptional prime ring (i.e., R does not embed into $M_2(\mathbb{F})$ for a field $\mathbb{F}$ of characteristic 2). Let $V \subseteq R$ be an additive subgroup that is invariant under all inner automorphisms of R. Then, either $V \subseteq Z(R)$ or $[R, R] \subseteq V$. In either case, V is a Lie ideal. Indeed, if $V \nsubseteq Z(R)$, then V is fully noncentral because R is simple, and the result follows from Theorem 5.5.

An algebra A over a field $\mathbb{F}$ is said to be *zero-product balanced* if for all $x, y, z \in A$, the element $xy \otimes z - x \otimes yz \in A \otimes_{\mathbb{F}} A$ belongs to the subspace of $A \otimes_{\mathbb{F}} A$ generated by $\{v \otimes w : vw = 0\}$; see [12, Definition 2.6]. This is closely related to the concept of a *zero-product determined* algebra [5, 12, Section 2]. In particular, a unital algebra is zero-product balanced if and only if it is zero-product determined.

Corollary 5.7. *Let A be a zero-product balanced, unital algebra over a field of characteristic* $\neq 2$. *For a fully noncentral subspace $V \subseteq A$, the following are equivalent.*

(1) *The subspace V is a Lie ideal.*
(2) *The subspace V is invariant under all inner automorphisms of A.*
(3) *We have* $[A, A] \subseteq V$.

Proof. Since A is unital and zero-product balanced (hence zero-product determined), every commutator in A is a sum of square-zero elements by [5, Theorem 9.1]; see also [12, Theorem 5.3]. Further, since A is unital, every proper ideal is contained in a maximal ideal, and maximal ideals in unital algebras are prime; see Proposition 3.1. Finally, since A is an algebra over a field of characteristic $\neq 2$, every prime ideal in A is nonexceptional. Hence, the result follows from Theorem 5.2. □

Example 5.8. Let A be a unital algebra over a field of characteristic $\neq 2$, let $n \geqslant 2$, and let $V \subseteq M_n(A)$ be a fully noncentral subspace. Then, V is a Lie ideal if and only if it is invariant under all inner automorphisms of $M_n(A)$, and if and only if $[M_n(A), M_n(A)] \subseteq V$. Indeed, by [12, Theorem 3.8], $M_n(A)$ is zero-product balanced (see also [5, Corollary 2.4]), whence the result follows from Corollary 5.7.

6 | COMMUTATORS AS SUMS OF SQUARE-ZERO ELEMENTS

Many interesting rings have the property that every commutator is a sum of square-zero elements. In this section, we show that if this is the case, then under mild additional assumptions, the commutator subgroup admits a precise description as the additive subgroup generated by a special class of square-zero elements; see Theorem 6.2. It remains open if the same holds under the weaker assumption that every commutator is a sum of nilpotent elements; see Question 6.7.

Given a ring R, recall that we set $N_2(R) := \{x \in R : x^2 = 0\}$. We say that $x \in R$ is an *orthogonally factorizable square-zero element* if there exist $y, z \in R$ such that $x = yz$ and $zy = 0$. We denote the set of such elements by $FN_2(R)$; see [12, Definition 5.1]. We have $FN_2(R) \subseteq [R, R]$, since if $x = yz$ and $zy = 0$, then $x = [y, z]$.

Given a subset B of R, we use B^+ to denote the additive subgroup of R generated by B. Recall that a subset in a ring is *full* if it is not contained in any proper ideal.

Lemma 6.1. *Let R be a ring such that every proper ideal is contained in a prime ideal. Then, the following are equivalent.*

(1) *The set $N_2(R)$ is full.*
(2) *The set $FN_2(R)$ is full.*
(3) *The set $FN_2(R)$ is fully noncentral.*

Proof. Since $[R, FN_2(R))]$ is contained in the ideal generated by $FN_2(R)$, we see that (3) implies (2). It is clear that (2) implies (1).

To show that (1) implies (3), let I denote the ideal of R generated by $[R, FN_2(R)]$. Assuming that $I \neq R$, choose a prime ideal $P \subseteq R$ such that $I \subseteq P$. We will show that every square-zero element is contained in P.

Let $x \in N_2(R)$. Note that $xRx \subseteq FN_2(R)$, and thus, $[R, xRx] \subseteq P$. Thus, for each $a \in R$, the image of xax in R/P is a central square-zero element. Since R/P is a prime ring, its center is either zero or a domain. Consequently, every central square-zero element in R/P is zero, and we deduce that $xRx \subseteq P$, which implies $x \in P$, as desired. □

Theorem 6.2. *Let R be a ring such that every proper ideal is contained in a nonexceptional prime ideal. Assume that $[R, R] \subseteq N_2(R)^+$, and that $N_2(R)$ is full. Then,*

$$[R, R] = FN_2(R)^+.$$

Proof. Set $V := FN_2(R)^+$. The inclusion $V \subseteq [R, R]$ holds in general. We will apply Corollary 4.5 to obtain the converse inclusion $[R, R] \subseteq V$.

Since $N_2(R)$ is full, it follows from Lemma 6.1 that V is fully noncentral. Next, we show that $[N_2(R)^+, V] \subseteq V$. Let $x \in N_2(R)$ and $w \in V$. Since V is invariant under all automorphisms of R, we have $(1 + x)w(1 - x) \in V$. Further, we have $w \in V$ and $xwx \in FN_2(R) \subseteq V$, and therefore,

$$[x, w] = (1 + x)w(1 - x) - w - xwx \in V.$$

By assumption, we have $[R, R] \subseteq N_2(R)^+$, and it follows that V is an $[R, R]$-submodule. Applying Corollary 4.5, we get $[R, R] \subseteq V$. □

If $R = M_2(\mathbb{F})$ for a field $\mathbb{F}$ of characteristic 2, then $[R, R]$ agrees with the additive subgroup generated by $FN_2(R)$. This suggests that the answer to the following question might be positive.

Question 6.3. Does Theorem 6.2 also hold for rings where every proper ideal is contained in a (possibly exceptional) prime ideal?

Corollary 6.4. *Let R be a ring such that every proper ideal is contained in a nonexceptional prime ideal, and assume that $[R,R]$ is full. Then, the following are equivalent.*

(1) *We have* $[R,R] \subseteq N_2(R)^+$.
(2) *We have* $[R,R] = FN_2(R)^+$.

Proof. It is clear that (2) implies (1). Conversely, assume that every commutator is a sum of square-zero elements. Since $[R,R]$ is full, it follows that the set of square-zero elements is full, and now (2) follows from Theorem 6.2. □

Remarks 6.5.

(1) If R is a zero-product balanced, idempotent ring, then $[R,R] = FN_2(R)^+$ by [12, Theorem 5.3]. This includes all rings generated by idempotents [12, Example 3.7], in particular, all simple rings that contain a nontrivial idempotent. It also includes all unital C^*-algebras that have no one-dimensional irreducible representations [15]. The famous Jiang–Su algebra is a unital, simple C^*-algebra that contains no idempotents other than zero and one. This algebra has no one-dimensional irreducible representations and is therefore zero-product balanced.
(2) There exist simple rings where $FN_2(R)^+$ is a proper subset of $[R,R]$. Indeed, by [8, Theorem 10], there exists a simple ring R such that the nilpotent elements in R form a subring W with $\{0\} \neq W \neq R$. Then, $R = [R,R]^2$ (see Corollary 3.5), but R is not generated by $FN_2(R)$ as a ring.

Problem 6.6. Find rings R such that $FN_2(R)^+$ is a proper subset of $N_2(R)^+$.

We end the paper with a short discussion of the relationship between nilpotent elements, square-zero elements, and commutators. In C^*-algebras, it is known that every nilpotent element is a sum of commutators ([27, Lemma 2.1]), and for many C^*-algebras, it is known that every commutator is a sum of square-zero elements ([27, Theorem 4.2], [15]), although it remains open if this holds for every C^*-algebra ([27, Question 2.5], [13, Question 4.1]). This raises the question if every nilpotent element in a C^*-algebra is a sum of square-zero elements ([13, Question 4.5]).

Of course, in general rings, it is not true that nilpotent elements are sums of square-zero elements. Nevertheless, it is conceivable that the answer to the following question may be positive.

Question 6.7. Let R be a ring such that every proper ideal of R is contained in a nonexceptional prime ideal, and such that $[R,R]$ is full. Assume that every commutator in R is a sum of nilpotent elements. Does it follow that every commutator in R is a sum of square-zero elements?

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