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Symmetries in Complex-Valued Spherical Harmonic Processing of Real-Valued Signals

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Abstract

Spherical harmonics (SHs) are widely used in audio and acoustics to represent sound fields and their spatial information. While SH expansions of real-valued functions can equivalently employ real- or complex-valued definitions of SH basis functions, real-valued definitions reduce computational load and storage needs. Researchers, however, often prefer a complex definition for its concise handling of operations like rotations and translations, which comes at the expense of partially redundant computations in practical implementations. To mitigate this downside, this work examines symmetries in expansions of real signals in terms of complex SHs in both the time and frequency domain and identifies redundancies akin to Hermitian symmetry in Fourier transforms. The resulting collection of the symmetry properties of SHs and circular harmonics (CHs) can be leveraged to limit computations to the non-redundant coefficients, significantly reducing the computational complexity and storage requirements in algorithms using complex-valued SHs.

Introduction

Spherical harmonics (SHs) play a central role in audio and acoustics for analyzing and synthesizing the directional and spatial properties of sound fields [1]. Applications span a wide range of domains, including microphone array processing [2, 3], sound field synthesis [4], source directivity modeling [5, 6, 7], and spatial audio recording, mixing, and rendering [8].

Acoustic problems typically involve real-valued functions of time, most commonly representing sound pressure in space. For such cases, SH expansions can use either complex- or real-valued SH definitions. This is still valid in the frequency domain: the complex-valued Fourier spectrum of a real-valued time-domain signal can be expanded using either real- or complex-valued SHs. Real-valued SHs have the advantage of reducing computational complexity and storage requirements, making them a preferred choice in the Ambisonics framework [8]. However, complex-valued SHs remain prevalent in many signal processing algorithms due to their mathematical convenience, as operations such as rotations and translations, as well as recurrence relations are more compactly expressed in the complex domain [9, 10, 11, 12, 13, 14].

As a result, mathematical relations using complex-valued SHs are widely available but the corresponding relations based on real-valued SHs are often unavailable in the literature and need to be meticulously derived [15, 16][17, App. A]. However, while the theoretical basis for complex SHs is well-established, certain inherent symmetries in SH expansions of real-valued signals are often overlooked in practical implementations. These symmetries arise naturally from the properties of SH basis functions and, when leveraged, can reduce computational and storage demands, bringing the efficiency of complex SH-based algorithms close to that of real-valued ones.

This work provides a collection of the symmetry properties of complex-valued SH and CH expansions for real signals in both the time and frequency domain and highlights redundancies similar to Hermitian symmetry in Fourier transforms. By identifying and formalizing these redundancies, we enable more efficient implementations of signal processing algorithms that rely on complex SHs. Additionally, we offer conversion relations between real-valued and complex-valued SH representations, further bridging the gap between the two approaches. To support practical adoption, the paper is accompanied by MATLAB code demonstrating the application of the symmetry relations[1].

Conjugate Symmetry of Time-Domain SH Coefficients

SHs form a complete set of orthogonal functions on the sphere and compose the angular part of the solution of the wave equation in spherical coordinates. The SHs of azimuth angle ϕ and zenith angle θ can be defined as [1, Eq. (6.20)]

$$Y_n^m(\phi,\theta) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\, P_n^m(\cos(\theta))\, e^{\mathrm{i}m\phi}\,, \quad (1)$$

where $P_n^m(\cdot)$ are the associated Legendre functions[2], $n \in \mathbb{N}$ is the order, and $m \in \mathbb{Z}$ is the degree. Real-valued SH definitions alternatively replace the complex exponential term with sine and cosine terms, depending on the degree m [19, 20]. In either case, the definitions can also vary concerning normalization of the coefficients. This paper's orthonormal definitions of SHs are commonly used within the audio and acoustics communities, for example, in the Spherical Harmonic Transform Library[3] [19].

Let $f(\phi,\theta)$ be a real-valued function defined on the surface of a sphere. The function can be expanded in terms of

[1] `https://github.com/thomasdeppisch/sh-symmetries`

[2] We define $P_n^m(\cdot)$ including the Condon-Shortley phase $(-1)^m$ as in [18, Eq. (2.1.20)-(2.1.21)] and also used in MATLAB and SciPy:

$$P_n^m(\mu) = (-1)^m (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_n(\mu), \quad \forall n \geq 0,\ m \geq m,$$

$$P_n(\mu) = \frac{1}{2^n n!} \frac{d^n}{d\mu^n} (\mu^2-1)^n, \quad \forall n \geq 0.$$

[3] `https://github.com/polarch/Spherical-Harmonic-Transform`

complex SHs $Y_n^m(\phi, \theta)$,

$$f(\phi,\theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm}\, Y_n^m(\phi,\theta)\,, \tag{2}$$

where f_{nm} are the function coefficients in the SH domain. The coefficients f_{nm} are found by projecting the function $f(\phi, \theta)$ onto the SH basis,

$$f_{nm} = \int_0^{2\pi} \int_0^{\pi} f(\phi,\theta)\, Y_n^{m*}(\phi,\theta)\, \sin\theta\, d\theta\, d\phi\,. \tag{3}$$

$Y_n^{m*}(\phi, \theta)$ is the complex conjugate of $Y_n^m(\phi, \theta)$, satisfying [1, Eq. (6.44)]

$$Y_n^{m*}(\phi,\theta) = (-1)^m\, Y_n^{-m}(\phi,\theta), \tag{4}$$

if the definition (1) is employed. This means that for a given order n, the SH corresponding to the negative degree $-m$ is related to the complex conjugate of the SH with degree m.

Now, consider that $f(\phi, \theta)$ is real-valued. Taking the complex conjugate of the SH expansion of $f(\phi, \theta)$ in (2), we get

$$f^*(\phi,\theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm}^*\, Y_n^{m*}(\phi,\theta)\,. \tag{5}$$

Using the SH conjugation property (4), this becomes

$$f^*(\phi,\theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm}^*\, (-1)^m\, Y_n^{-m}(\phi,\theta)\,. \tag{6}$$

Since $f(\phi, \theta)$ is real, the original sum in (2) and its complex conjugate in (6) must be identical. Comparing the two expansions, we find that the SH expansion coefficients must satisfy the same conjugation property as the SH basis functions,

$$f_{nm}^* = (-1)^m\, f_{n,-m}\,. \tag{7}$$

This equation expresses the complex conjugate symmetry of the SH coefficients for a real-valued function: The coefficients for negative m are determined by the complex conjugates of the positive m coefficients, with a factor of $(-1)^m$. Just as in the Fourier transform of real-valued signals, where only the positive frequencies need to be retained (since the negative frequencies are just the complex conjugates of the positives), in the SH domain, only the coefficients for $m \geq 0$ need to be computed and stored. The coefficients for $m < 0$ are redundant and can be recovered from the symmetry relation. The number of non-redundant coefficients up to order N, counting real and imaginary parts separately[4], is $(N+1)^2$, which is the same number of coefficients a real-valued SH expansion requires.

[4]Note that $m = 0$ coefficients are always real-valued.

	Space Domain	SH Domain
Time Domain	$f(t,\phi,\theta)$	$f_{n,-m}(t) = (-1)^m f_{nm}^*(t)$
Frequency Domain	$\hat{f}(-\omega,\phi,\theta) = \hat{f}^*(\omega,\phi,\theta)$	$\hat{f}_{n,-m}(\omega) = (-1)^m \hat{f}_{nm}^*(-\omega)$

Figure 1: Conjugate symmetries of real-valued signals $f(t, \phi, \theta)$ in complex frequency and SH domains.

Conjugate Symmetry of Frequency-Domain SH Coefficients

For a time-dependent real-valued function $f(\phi, \theta, t)$, the SH coefficients $f_{nm}(t)$ can be transformed to the frequency domain using the Fourier transform $\mathcal{F}\{\cdot\}$,

$$\mathcal{F}\{f_{nm}(t)\} =: \hat{f}_{nm}(\omega) = \int_{-\infty}^{\infty} f_{nm}(t)\, e^{-\mathrm{i}\omega t}\, dt\,. \tag{8}$$

For a real-valued time-domain signal $f(t)$, the Fourier transform $\hat{f}(\omega)$ satisfies Hermitian symmetry

$$\hat{f}^*(\omega) = \hat{f}(-\omega)\,. \tag{9}$$

Using the conjugate symmetry of SH coefficients for real-valued functions from (7), the Fourier transform of the conjugate of $f_{nm}(t)$ is

$$\mathcal{F}\{f_{n,-m}(t)\} = \mathcal{F}\{(-1)^m f_{nm}^*(t)\}\,, \tag{10}$$

$$= (-1)^m \mathcal{F}\{f_{nm}^*(t)\}\,. \tag{11}$$

Since the Fourier transform of the complex conjugate of a function is

$$\mathcal{F}\{f_{nm}^*(t)\} = \hat{f}_{nm}^*(-\omega)\,, \tag{12}$$

the frequency-domain SH coefficients of negative degree m follow a similar symmetry,

$$\hat{f}_{n,-m}(\omega) = (-1)^m\, \hat{f}_{nm}^*(-\omega)\,. \tag{13}$$

This shows that the frequency-domain SH coefficients for negative m of positive frequency are related to the coefficients for positive m of negative frequency by a complex conjugation and a sign factor depending on m. The calculation of frequency-domain SH coefficients can thus be limited to either positive degrees $m \geq 0$ or to positive frequencies, yielding again the same number of non-redundant coefficients as for a real-valued SH expansion. The relations are summarized in Fig. 1.

Practical Example

Theoretical descriptions of sound fields and their corresponding signal processing algorithms are often formulated in the frequency domain, with the final result converted back to the time domain using the inverse (discrete) Fourier transform. When the sound field is represented as a real-valued function in the time domain and the SH expansion follows a complex-valued definition, the frequency-domain implementation must satisfy the symmetry relation from (13). Consequently, the algorithm only needs to compute the positive-frequency coefficients (for both positive and negative degrees m), while the negative-frequency coefficients can be recovered using

```
f(n,m,-k) = (-1)^m * conj(f(n,-m,k))
```

where `k` is the frequency bin index and `-k` the index of the corresponding negative-frequency bin. Alternatively, only coefficients for $m \geq 0$ can be computed for both positive and negative frequencies, and the coefficients for $m < 0$ can be retrieved from the symmetry. Numerically validated MATLAB code for such conversions is available in the online repository[1].

Conversion Between Complex- and Real-Valued SH Coefficients

An alternative to explicitly considering symmetries is to convert complex SH coefficients to their real counterparts before further processing. The methodology for this conversion is detailed in [15]. For completeness, we state conversion relations that are compatible with the SH definitions in this paper.

In the acoustics and audio communities, a common orthonormal real-valued SH definition is given by

$$R_n^m(\phi,\theta) = (-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\, P_n^{|m|}(\cos(\theta)) \times \begin{cases} \sqrt{2}\sin(|m|\phi) & \text{if } m<0 \\ 1 & \text{if } m=0 \\ \sqrt{2}\cos(|m|\phi) & \text{if } m>0 \end{cases}, \tag{14}$$

where the factor $(-1)^m$ cancels out the Condon-Shortley phase from the Legendre polynomials. A semi-normalized version (SN3D) thereof is used in Ambisonics [21].

The conversion relations can be obtained from the SH definitions in (1) and (14):

$$R_n^m = \begin{cases} \sqrt{2}(-1)^m \Im\{Y_n^{-m}\} = \frac{\mathrm{i}}{\sqrt{2}}(Y_n^m - (-1)^m Y_n^{-m}) & \text{if } m<0 \\ Y_n^m & \text{if } m=0 \\ \sqrt{2}(-1)^m \Re\{Y_n^m\} = \frac{1}{\sqrt{2}}((-1)^m Y_n^m + Y_n^{-m}) & \text{if } m>0 \end{cases} \tag{15}$$

$$Y_n^m = \begin{cases} \frac{1}{\sqrt{2}}(R_n^{-m} - \mathrm{i}R_n^m) & \text{if } m<0 \\ R_n^m & \text{if } m=0 \\ \frac{(-1)^m}{\sqrt{2}}(R_n^m + \mathrm{i}R_n^{-m}) & \text{if } m>0 \end{cases}. \tag{16}$$

Similar relations apply for complex- and real-valued SH coefficients f_{nm} and r_{nm}:

$$r_{nm} = \begin{cases} -\sqrt{2}(-1)^m \Im\{f_{n,-m}\} = -\frac{\mathrm{i}}{\sqrt{2}}(f_{nm} - (-1)^m f_{n,-m}) & \text{if } m<0 \\ f_{nm} & \text{if } m=0 \\ \sqrt{2}(-1)^m \Re\{f_{nm}\} = \frac{1}{\sqrt{2}}((-1)^m f_{nm} + f_{n,-m}) & \text{if } m>0 \end{cases} \tag{17}$$

$$f_{nm} = \begin{cases} \frac{1}{\sqrt{2}}(r_{n,-m} + \mathrm{i}\, r_{nm}) & \text{if } m<0 \\ r_{nm} & \text{if } m=0 \\ \frac{(-1)^m}{\sqrt{2}}(r_{nm} - \mathrm{i}\, r_{n,-m}) & \text{if } m>0 \end{cases}. \tag{18}$$

These coefficient conversions can also be applied in the frequency domain; however, only the second terms without the $\Re$ and $\Im$ operators are valid in this context. Numerical validation of these relations is included in the online repository[1].

Implications on Circular Harmonics Expansions

CHs form an orthogonal set of functions on the circle. They describe the azimuthal dependency of the SHs and a complex-valued CH definition is given by

$$C_m(\phi) = e^{\mathrm{i}m\phi}. \tag{19}$$

Due to their close relationship to the SHs, CHs exhibit similar properties. The above symmetries for SH expansions of real-valued signals can similarly be derived for CHs and are:

- CHs of positive and negative degree m are conjugate symmetric:
$$C_m^*(\phi) = C_{-m}(\phi) \tag{20}$$
- CH coefficients g_m of positive and negative degree m are conjugate symmetric:
$$g_m^* = g_{-m} \tag{21}$$
- Frequency-domain CH coefficients for negative m are conjugates of the coefficients for positive m and negative frequency:
$$\hat{g}_{-m}(\omega) = \hat{g}_m^*(-\omega). \tag{22}$$

Real-valued CHs can be defined as

$$D_m(\phi) = \begin{cases} \sqrt{2}\sin(|m|\phi) & \text{if } m<0 \\ 1 & \text{if } m=0 \\ \sqrt{2}\cos(|m|\phi) & \text{if } m>0 \end{cases}. \tag{23}$$

Complex- and real-valued CHs can be converted via:

$$D_m = \begin{cases} \sqrt{2}\Im\{C_{-m}\} = \frac{\mathrm{i}}{\sqrt{2}}(C_m - C_{-m}) & \text{if } m<0 \\ C_m & \text{if } m=0 \\ \sqrt{2}\Re\{C_m\} = \frac{1}{\sqrt{2}}(C_m + C_{-m}) & \text{if } m>0 \end{cases} \tag{24}$$

$$C_m = \begin{cases} \frac{1}{\sqrt{2}}(D_{-m} - \mathrm{i}\, D_m) & \text{if } m<0 \\ D_m & \text{if } m=0 \\ \frac{1}{\sqrt{2}}(D_m + \mathrm{i}\, D_{-m}) & \text{if } m>0 \end{cases}. \tag{25}$$

For complex- and real-valued coefficients g_m and d_m follows:

$$d_m = \begin{cases} -\sqrt{2}\Im\{g_{-m}\} = -\frac{\mathrm{i}}{\sqrt{2}}(g_m - g_{-m}) & \text{if } m<0 \\ g_m & \text{if } m=0 \\ \sqrt{2}\Re\{g_m\} = \frac{1}{\sqrt{2}}(g_m + g_{-m}) & \text{if } m>0 \end{cases} \tag{26}$$

$$g_m = \begin{cases} \frac{1}{\sqrt{2}}(d_{-m} + \mathrm{i}\, d_m) & \text{if } m<0 \\ d_m & \text{if } m=0 \\ \frac{1}{\sqrt{2}}(d_m - \mathrm{i}\, d_{-m}) & \text{if } m>0 \end{cases}. \tag{27}$$

As before, these coefficient conversions can also be applied in the frequency domain but only the second terms without the $\Re$ and $\Im$ operators are valid in this context.

Conclusion

This work analyzed the symmetries in SH and CH expansions of real-valued signals, highlighting redundancies that arise when using complex-valued SH and CH definitions. It identified symmetry properties in both the time and frequency domains that enable a reduction in computational complexity and storage requirements. The work provides a comprehensive collection of these symmetry properties and conversion relations, along with an online repository containing numerically validated code to facilitate their practical use.

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