String theory amplitudes and partial fractions

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Abstract
We give rigorous proofs and generalizations of partial fraction expansions for string amplitudes that were recently discovered by Saha and Sinha.

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1 Introduction

Veneziano [8] interpreted the function

$$B(x_1, x_2) + B(x_1, x_3) + B(x_2, x_3)$$

as a scattering amplitude for strongly interacting mesons. Here,

$$B(x_1, x_2) = \frac{\Gamma(x_1)\Gamma(x_2)}{\Gamma(x_1 + x_2)}$$

is the beta function. Soon afterwards, Virasoro [9] and Shapiro [7] found other combinations of beta functions with similar properties, such as

$$\frac{\Gamma(x_1)\Gamma(x_2)\Gamma(x_3)}{\Gamma(1 - x_1)\Gamma(1 - x_2)\Gamma(1 - x_3)}, \tag{1}$$

where $x_1 + x_2 + x_3 = 1$. These findings were crucial for the early development of string theory, where the Veneziano and Virasoro–Shapiro amplitudes are associated with open and closed bosonic strings, respectively. Related expressions appear in supersymmetric string theory; for instance, tree level scattering in type II theories is related to (1) with $x_1 + x_2 + x_3 = 0$ [2, Eq. (7.4.56)].

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As a function of x_1, the beta function has poles at the non-positive integers and can be written as an infinite partial fraction

$$B(x_1, x_2) = \sum_{k=0}^{\infty} \frac{(1-x_2)_k}{k!} \cdot \frac{1}{x_1+k}, \qquad \mathrm{Re}(x_2) > 0. \tag{2}$$

The identity (2) is a special case of Gauss' summation formula for the hypergeometric function ${}_2F_1$ [1, Thm. 2.2.2]. Here we use the standard notation

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+k-1), & k \geq 0, \\ (a-1)^{-1}(a-2)^{-1}\cdots(a+k)^{-1}, & k < 0. \end{cases}$$

An unattractive feature of (2) is that the variables x_1 and x_2 play a different role on the right-hand side. In [5], Saha and Sinha used physics arguments to obtain more symmetric expansions of string amplitudes. To give an example, the case $\alpha = \beta = p = 0$ of [5, Eq. (4)] can be written

$$B(x_1, x_2) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(1 - \lambda + \frac{(\lambda - x_1)(\lambda - x_2)}{\lambda + k}\right)_k \left(\frac{1}{x_1+k} + \frac{1}{x_2+k} - \frac{1}{\lambda+k}\right) \tag{3}$$

Here, λ is a "field redefinition parameter", subject only to the convergence condition $\mathrm{Re}(\lambda) > 0$. If $\lambda = x_2$, the symmetry between x_1 and x_2 is broken and we recover (2). Another interesting case is the limit $\lambda \to \infty$. Since

$$1 - \lambda + \frac{(\lambda - x_1)(\lambda - x_2)}{\lambda + k} = 1 - k - x_1 - x_2 + \frac{(x_1+k)(x_2+k)}{\lambda + k},$$

this limit is

$$B(x_1, x_2) = \sum_{k=0}^{\infty} \frac{(-1)^k (x_1+x_2)_k}{k!} \left(\frac{1}{x_1+k} + \frac{1}{x_2+k}\right).$$

This is a special case of a well-known ${}_4F_3$ summation [1, Cor. 3.5.3].

As noted by Saha and Sinha, special cases of their identities give intriguing new formulas for π and other mathematical constants. For instance, the case $x_1 = x_2 = 1/2$ of (3) is

$$\pi = \sum_{k=0}^{\infty} \frac{1}{k!} \left(1 - \lambda + \frac{(\lambda - 1/2)^2}{\lambda + k}\right)_k \left(\frac{4}{2k+1} - \frac{1}{\lambda+k}\right).$$

For $\lambda = \infty$, this is the well-known Madhava series

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1},$$

whereas the case $\lambda = 1/2$ is the identity

$$\frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{k!2^k(2k+1)},$$

which can be obtained by letting $x = 1$ in the Taylor series for $\arcsin(x)$.

Saha and Sinha also gave several expansions related to closed strings. For instance, the case $\alpha = 1/3$, $\beta = 2/3$ of [5, Eq. (A3)] gives the following expansion of the Virasoro–Shapiro amplitude (1):

$$\frac{\Gamma(x_1)\Gamma(x_2)\Gamma(x_3)}{\Gamma(1-x_1)\Gamma(1-x_2)\Gamma(1-x_3)} = \sum_{k=0}^{\infty} \frac{(\eta_k)_k^2}{(k!)^2} \left(\sum_{j=1}^{3} \frac{1}{x_j + k} - \frac{1}{\frac{1}{3} + k} \right). \tag{4}$$

Here, $x_1 + x_2 + x_3 = 1$ and

$$\eta_k = \frac{1-k}{2} + \sqrt{\frac{\left(\frac{1}{3}+k\right)^2}{4} + \frac{(x_1 - \frac{1}{3})(x_2 - \frac{1}{3})(x_3 - \frac{1}{3})}{\frac{1}{3} + k}}.$$

Since

$$\begin{aligned} \left(\frac{1-k}{2} + \sqrt{a}\right)_k^2 &= (-1)^k \left(\frac{1-k}{2} + \sqrt{a}\right)_k \left(\frac{1-k}{2} - \sqrt{a}\right)_k \\ &= \prod_{j=0}^{k-1} \left(a - \left(\frac{1-k}{2} + j\right)^2 \right), \end{aligned} \tag{5}$$

the terms in (4) are rational functions of the parameters.

The purpose of the present work is to give independent and rigorous proofs of the mathematical results of [5]. We also give generalizations, which we hope may have some interest for physicists. For instance, when $x_1 + x_2 + x_3 = s$ is an integer and $\mathrm{Re}(\lambda + 1) > 0$, we show that

$$\frac{\Gamma(x_1)\Gamma(x_2)\Gamma(x_3)}{\Gamma(1-x_1)\Gamma(1-x_2)\Gamma(1-x_3)} = \sum_{k=0}^{\infty} \frac{(-1)^{s+1}(\xi_k)_{k+s-1}^2}{(k!)^2} \left(\sum_{j=1}^{3} \frac{1}{x_j + k} - \frac{1}{\lambda + k} \right), \tag{6}$$

where

$$\xi_k = \frac{2-s-k}{2} + \sqrt{\left(\frac{s+k-2\lambda}{2}\right)^2 + \frac{(x_1-\lambda)(x_2-\lambda)(x_3-\lambda)}{\lambda + k}}.$$

As mentioned above, the cases $s = 0$ and $s = 1$ are of special relevance in physics. In [5], (6) is only obtained for the fixed value $\lambda = s/3$. In particular, the case $s = 1$ and $\lambda = 1/3$ is (4).

Our method is to obtain infinite partial fractions as limit cases of finite partial fractions for symmetric rational functions. One source of inspiration was Schlosser's work [6, §7], which is the only place where I have seen series remotely similar to (3). The matrix inversions used by Schlosser are in fact closely related to partial fractions [4], so there may be more direct connections that remain to be explored.

2 Partial fraction expansions

As a warm-up, we consider the identity (2), where we write $x_1 = x$, $x_2 = a$. One way to derive it is to start from the finite truncation

$$\frac{\Gamma(x)}{\Gamma(a+x)} \cdot \frac{\Gamma(a+x+n)}{\Gamma(x+n+1)} = \frac{(a+x)_n}{(x)_{n+1}}.$$

It has the partial fraction expansion

$$\frac{(a+x)_n}{(x)_{n+1}} = \sum_{k=0}^{n} \frac{B_k}{x+k}, \tag{7}$$

where

$$B_k = \lim_{x \to -k} (x+k) \frac{(a+x)_n}{(x)_{n+1}} = \frac{(-1)^k (a-k)_n}{k!(n-k)!}.$$

The identity (7) can be recognized as a special case of the Pfaff–Saalschütz ${}_3F_2$ summation [1, Thm. 2.2.6].

We now let $n \to \infty$ in (7). Using that

$$\frac{\Gamma(a+n)}{\Gamma(b+n)} \sim n^{a-b}, \qquad n \to \infty, \tag{8}$$

it is straight-forward to formally obtain (2) in the limit. To make this rigorous one can apply Tannery's theorem if $\mathrm{Re}(x_2) > 1$, and then extend the identity analytically to the larger domain $\mathrm{Re}(x_2) > 0$. This is discussed for more general series in the Appendix.

To obtain an analogous proof of (3), we start from

$$\frac{\Gamma(x_1)\Gamma(x_2)}{\Gamma(x_1+x_2)} \cdot \frac{\Gamma(x_1+x_2+n)}{\Gamma(x_1+n+1)\Gamma(x_2+n+1)} = \frac{(x_1+x_2)_n}{(x_1)_{n+1}(x_2)_{n+1}}.$$

We then apply a partial fraction expansion for symmetric rational functions in two variables. For (3), we need to work with three variables. In the remainder of this section, we discuss the general case of r variables.

Recall that the ring of symmetric polynomials is generated by the elementary symmetric polynomials, which may be defined by the generating function

$$\prod_{j=1}^{r}(t+x_j)=\sum_{m=0}^{r}t^{r-m}e_m(\mathbf{x}). \tag{9}$$

Thus, to compute the value of a symmetric polynomial at a point $\mathbf{x}=(x_1,\dots,x_r)$, it suffices to know $e_m(\mathbf{x})$ for $1\le m\le r$. With this in mind, we can formulate the following rather general partial fraction expansion.

Theorem 1 *Let P be a symmetric polynomial of r variables, which is of degree at most $n+1$ in each variable. Let $Q_0,\dots,Q_n$ be symmetric polynomials that are of degree exactly 1 in each variable. Then,*

$$\frac{P(\mathbf{x})}{\prod_{j=0}^{n}Q_j(\mathbf{x})}-\frac{P(\mathbf{y})}{\prod_{j=0}^{n}Q_j(\mathbf{y})}=\sum_{k=0}^{n}\frac{P(\mathbf{t}_k)}{\prod_{j=0,\,j\neq k}^{n}Q_j(\mathbf{t}_k)}\left(\frac{1}{Q_k(\mathbf{x})}-\frac{1}{Q_k(\mathbf{y})}\right), \tag{10}$$

where $\mathbf{t}_k$ denotes the vector defined up to permutations by

$$e_m(\mathbf{t}_k)=\frac{Q_k(\mathbf{y})e_m(\mathbf{x})-Q_k(\mathbf{x})e_m(\mathbf{y})}{Q_k(\mathbf{y})-Q_k(\mathbf{x})}. \tag{11}$$

By (9), we can equivalently write (11) as

$$(Q_k(\mathbf{y})-Q_k(\mathbf{x}))\prod_{j=1}^{r}(u-t_j)=Q_k(\mathbf{y})\prod_{j=1}^{r}(u-x_j)-Q_k(\mathbf{x})\prod_{j=1}^{r}(u-y_j), \tag{12}$$

where $\mathbf{t}_k=(t_1,\dots,t_r)$. That is, the components of $\mathbf{t}_k$ are the solutions to

$$Q_k(\mathbf{x})\prod_{j=1}^{r}(u-y_j)=Q_k(\mathbf{y})\prod_{j=1}^{r}(u-x_j), \tag{13}$$

considered as an equation in u.

Proof Since we are proving a rational function identity, we may assume that all parameters are generic. Define λ_m and μ_m by

$$e_m(\mathbf{x})=\lambda_m e_1(\mathbf{x})+\mu_m, \tag{14a}$$

$$e_m(\mathbf{y})=\lambda_m e_1(\mathbf{y})+\mu_m, \tag{14b}$$

that is,

$$\lambda_m=\frac{e_m(\mathbf{x})-e_m(\mathbf{y})}{e_1(\mathbf{x})-e_1(\mathbf{y})},\qquad \mu_m=\frac{e_1(\mathbf{x})e_m(\mathbf{y})-e_m(\mathbf{x})e_1(\mathbf{y})}{e_1(\mathbf{x})-e_1(\mathbf{y})}.$$

Considering λ_m and μ_m as constants, we can view any symmetric polynomial in $\mathbf{x}$ as a function of the single variable $e_1(\mathbf{x})$.

For fixed k, let us write $Q_k = Q$ and $\mathbf{t}_k = \mathbf{t}$. Note that

$$Q(\mathbf{x}) = \sum_{m=0}^{r} A_m e_m(\mathbf{x}) \tag{15}$$

for some coefficients A_m. Multiplying (11) with A_m and summing over m gives

$$Q(\mathbf{t}) = 0. \tag{16}$$

It is also clear from (11) and (14) that

$$e_m(\mathbf{t}) = \lambda_m e_1(\mathbf{t}) + \mu_m. \tag{17}$$

The equations (16) and (17) determine $\mathbf{t}$ uniquely. Hence, $\mathbf{t}$ depends on $\mathbf{x}$ and $\mathbf{y}$ only through the parameters λ_m and μ_m. To make this explicit, we write

$$0 = \sum_{m=0}^{r} A_m e_m(\mathbf{t}) = \sum_{m=0}^{r} A_m(\lambda_m e_1(\mathbf{t}) + \mu_m).$$

Solving this for $e_1(\mathbf{t})$ and using again (17) gives

$$e_m(\mathbf{t}) = -\lambda_m \frac{\sum_{j=0}^{r} A_j \mu_j}{\sum_{j=0}^{r} A_j \lambda_j} + \mu_m. \tag{18}$$

We are now reduced to a partial fraction expansion in one variable. Let L and R denote the two sides of (10) and $S = \prod_{j=0}^{n} Q_j$. Using (14a), we can write SL as a polynomial in $e_1(\mathbf{x})$ of degree at most $n+1$. By (18), the coefficient $P(\mathbf{t}_k)/\prod_{j\neq k} Q_j(\mathbf{t}_k)$ can be treated as a constant. Hence, SR is also a polynomial in $e_1(\mathbf{x})$ of degree at most $n+1$.

To complete the proof, it suffices to check that SL and SR agree for $n+2$ independent values of $\mathbf{x}$, subject to (14a). By (14b) and (17), we can take these as $\mathbf{t}_0, \dots, \mathbf{t}_n, \mathbf{y}$. By (16), if $\mathbf{x} = \mathbf{t}_k$, only the first term on the left and the k-th term on the right contribute to the value of SR. It is then straight-forward to check that $SL = SR$. The identity for $\mathbf{x} = \mathbf{y}$ is obvious. This completes the proof. □

Let us now consider the limit case of Theorem 1 when $y_1, \dots, y_l$ are fixed and $y_{l+1}, \dots, y_r \to \infty$, where $0 \leq l \leq r-1$. If P is of degree at most n in each variable, then all terms in (10) that explicitly depend on $\mathbf{y}$ tend to zero. In particular, the left-hand side is independent of $y_1, \dots, y_l$, so we can consider these as free parameters. If Q is as in (15), we write

$$\hat{Q}(y_1, \dots, y_l) = \lim_{y_{l+1},\dots,y_r\to\infty} \frac{Q(y_1, \dots, y_r)}{y_{l+1}\cdots y_r} = \sum_{m=0}^{l} A_{m+r-l}\, e_m(y_1, \dots, y_l).$$

Then, the equation (11) degenerates to

$$e_m(\mathbf{t}_k) = e_m(\mathbf{x}) - \frac{Q_k(\mathbf{x})}{\hat{Q}_k(\mathbf{y})} e_{m+l-r}(\mathbf{y}), \tag{19}$$

where we should interpret $e_k(\mathbf{y})$ as 0 for $k < 0$.

Corollary 2 *Let P and Q_j be as in Theorem 1, but assume that P is of degree at most n in each variable. Let $y_1, \ldots, y_l$ be generic scalars and let $\mathbf{t}_k$ be defined up to permutations by (19). Then,*

$$\frac{P(\mathbf{x})}{\prod_{j=0}^{n} Q_j(\mathbf{x})} = \sum_{k=0}^{n} \frac{P(\mathbf{t}_k)}{\prod_{j=0, j\neq k}^{n} Q_j(\mathbf{t}_k)} \cdot \frac{1}{Q_k(\mathbf{x})}. \tag{20}$$

Returning to Theorem 1, suppose that $Q_k(\mathbf{x}) = \prod_{m=1}^{r}(x_m - b_k)$ for some scalars b_k. Then, (13) reduces to

$$\prod_{j=1}^{r}(b_k - x_j)(u - y_j) = \prod_{j=1}^{r}(b_k - y_j)(u - x_j),$$

which has one obvious solution $u = b_k$. We write the full vector of solutions as $\mathbf{t}_k = (b_k, b_k', \ldots, b_k^{(r-1)})$. Differentiating (12) in u and then substituting $u = b_k$ gives after simplification

$$\frac{1}{Q_k(\mathbf{x})} - \frac{1}{Q_k(\mathbf{y})} = \frac{1}{\prod_{j=1}^{r-1}(b_k^{(j)} - b_k)} \left(\sum_{j=1}^{r} \frac{1}{x_j - b_k} - \sum_{j=1}^{r} \frac{1}{y_j - b_k} \right).$$

Thus, (10) can be written

$$\frac{P(\mathbf{x})}{\prod_{j=0}^{n}\prod_{m=1}^{r}(x_m - b_j)} - \frac{P(\mathbf{y})}{\prod_{j=0}^{n}\prod_{m=1}^{r}(y_m - b_j)}$$
$$= \sum_{k=0}^{n} \frac{P(b_k, b_k', \ldots, b_k^{(r-1)})}{\prod_{j=0, j\neq k}^{n}(b_k - b_j)\prod_{j=0}^{n}\prod_{m=1}^{r-1}(b_k^{(m)} - b_j)} \left(\sum_{j=1}^{r} \frac{1}{x_j - b_k} - \sum_{j=1}^{r} \frac{1}{y_j - b_k} \right). \tag{21}$$

Taking as before the limit $y_{l+1}, \ldots, y_r \to \infty$, when P has degree at most n in each variable, gives

$$\frac{P(\mathbf{x})}{\prod_{j=0}^{n}\prod_{m=1}^{r}(x_m - b_j)}$$
$$= \sum_{k=0}^{n} \frac{P(b_k, b_k', \ldots, b_k^{(r-1)})}{\prod_{j=0, j\neq k}^{n}(b_k - b_j)\prod_{j=0}^{n}\prod_{m=1}^{r-1}(b_k^{(m)} - b_j)} \left(\sum_{j=1}^{r} \frac{1}{x_j - b_k} - \sum_{j=1}^{l} \frac{1}{y_j - b_k} \right). \tag{22}$$

Here, $\mathbf{t}_k = (b_k, b_k', \dots, b_k^{(r-1)})$ is given by (19), which in this case reads

$$e_m(\mathbf{t}_k) = e_m(\mathbf{x}) - \frac{\prod_{j=1}^{r}(x_j - b_k)}{\prod_{j=1}^{l}(y_j - b_k)} e_{m+l-r}(\mathbf{y}) \tag{23}$$

or, equivalently,

$$\prod_{j=1}^{r}(b_k^{(j)} - u) = \prod_{j=1}^{r}(x_j - u) - \prod_{j=1}^{l} \frac{y_j - u}{y_j - b_k} \prod_{j=1}^{r}(x_j - b_k). \tag{24}$$

We will only need the cases $(r, l) = (2, 1)$ and $(r, l) = (3, 1)$ of the results above. However, we find it instructive and potentially useful to state them in greater generality.

3 Expansions of open string amplitudes

To prove (3) and some related results, we start from the case $(r, l) = (2, 1)$ of (22). Writing $y_1 = \lambda$, it has the form

$$\frac{P(x_1, x_2)}{\prod_{j=0}^{n}(x_1 - b_j)(x_2 - b_j)} = \sum_{k=0}^{n} \frac{P(b_k, b_k')}{\prod_{j=0,\, j\neq k}^{n}(b_k - b_j)\prod_{j=0}^{n}(b_k' - b_j)} \left(\frac{1}{x_1 - b_k} + \frac{1}{x_2 - b_k} - \frac{1}{\lambda - b_k} \right). \tag{25}$$

By the case $u = \lambda$ of (24),

$$(b - \lambda)(b' - \lambda) = (x_1 - \lambda)(x_2 - \lambda),$$

which gives

$$b' = \lambda - \frac{(\lambda - x_1)(\lambda - x_2)}{\lambda - b}.$$

We now specialize to the case $b_k = -k$ and $P(x_1, x_2) = (x_1 + x_2 + a)_n$. After a straight-forward computation, we arrive at the rational function identity

$$\frac{(x_1 + x_2 + a)_n}{(x_1)_{n+1}(x_2)_{n+1}} = \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \frac{\left(\lambda - \frac{(\lambda - x_1)(\lambda - x_2)}{\lambda + k} + a - k\right)_n}{\left(\lambda - \frac{(\lambda - x_1)(\lambda - x_2)}{\lambda + k}\right)_{n+1}} \left(\frac{1}{x_1 + k} + \frac{1}{x_2 + k} - \frac{1}{\lambda + k} \right). \tag{26}$$

Next, we multiply both sides of (26) by $\Gamma(n+2-a)$ and let $n \to \infty$. Formally using (8), it is straight-forward to obtain the following result. However, this involves an interchange of limit and summation that needs to be justified. We provide the details in the Appendix.

Corollary 3 *Assuming* $\operatorname{Re}(a+\lambda) > 0$,

$$\frac{\Gamma(x_1)\Gamma(x_2)}{\Gamma(x_1+x_2+a)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \frac{\Gamma\left(\lambda - \frac{(\lambda-x_1)(\lambda-x_2)}{\lambda+k}\right)}{\Gamma\left(\lambda - \frac{(\lambda-x_1)(\lambda-x_2)}{\lambda+k} + a - k\right)} \left(\frac{1}{x_1+k} + \frac{1}{x_2+k} - \frac{1}{\lambda+k}\right). \tag{27}$$

When a is an integer, (27) can be written

$$\frac{\Gamma(x_1)\Gamma(x_2)}{\Gamma(x_1+x_2+a)} = \sum_{k=0}^{\infty} \frac{(-1)^a \left(1-\lambda+\frac{(\lambda-x_1)(\lambda-x_2)}{\lambda+k}\right)_{k-a}}{k!} \left(\frac{1}{x_1+k} + \frac{1}{x_2+k} - \frac{1}{\lambda+k}\right).$$

This is [5, Eq. (4)], although it is only stated there under the additional assumption $a \le 1$. The case $a = 0$ is the example (3) given in the introduction.

If one is interested in formulas for π one can specialize $x_1 = 1/2+m, x_2 = 1/2+n$ and use

$$\Gamma\left(\frac{1}{2}+m\right) = (1/2)_m\, \Gamma(1/2) = \frac{(2m-1)!!}{2^m}\sqrt{\pi}.$$

This gives a family of identities

$$\pi = \frac{(-1)^a (m+n+a)!2^{m+n}}{(2m-1)!!(2n-1)!!} \sum_{k=0}^{\infty} \frac{1}{k!} \left(1-\lambda+\frac{(\lambda-\frac{1}{2}-m)(\lambda-\frac{1}{2}-n)}{\lambda+k}\right)_{k-a} \times \left(\frac{2}{2m+2k+1} + \frac{2}{2n+2k+1} - \frac{1}{\lambda+k}\right),$$

parametrized by three integers m, n and a with $m \ge 0$, $n \ge 0$ and $m+n+a \ge 0$ and a continuous parameter λ with $\operatorname{Re}(\lambda) > -a$.

4 Asymmetric expansions of closed string amplitudes

Saha and Sinha gave two types of expansions for amplitudes related to closed strings. We start with the first type [5, Eq. (A2)], which is symmetric in the variables x_1 and x_2

but where x_3 plays a different role. Taking the Virasoro–Shapiro amplitude (1) as an example, we will treat x_1 and x_2 as variables, but x_3 as determined from the relation $x_1+x_2+x_3=1$. The left-hand side then has some poles of the form $x_1=x_2=-k$ (for k a non-negative integer) and others of the form $x_1+x_2=k+1$. We will therefore start from symmetric rational functions of the form

$$\frac{P(x_1,x_2)}{\prod_{j=0}^n(x_1-b_j)(x_2-b_j)(c_j-x_1-x_2)}.$$

We will apply (20) with $(r,l)=(2,1)$ and write $y_1=\lambda$. We then have $n+1$ terms that can be written as in (25), with the additional denominator factors $\prod_{j=0}^n(c_j-b_k-b_k')$. Let us write the vector $\mathbf{t}$ corresponding to a factor $Q=c-x_1-x_2$ as (c^+,c^-). The equations (19) reduce to

$$c^++c^-=c,\qquad c^+c^-=x_1x_2+(c-x_1-x_2)\lambda.$$

Hence,

$$c^\pm=\frac{c^++c^-}{2}\pm\sqrt{\frac{(c^++c^-)^2-4c^+c^-}{4}}=\frac{c}{2}\pm\sqrt{\frac{c^2}{4}+\lambda(x_1+x_2-c)-x_1x_2}.$$

This leads to the identity

$$\begin{aligned}&\frac{P(x_1,x_2)}{\prod_{j=0}^n(x_1-b_j)(x_2-b_j)(c_j-x_1-x_2)}\\&=\sum_{k=0}^n\frac{P(b_k,b_k')}{\prod_{j=0,\,j\neq k}^n(b_k-b_j)\prod_{j=0}^n(b_k'-b_j)(c_j-b_k-b_k')}\left(\frac{1}{x_1-b_k}+\frac{1}{x_2-b_k}-\frac{1}{\lambda-b_k}\right)\\&+\sum_{k=0}^n\frac{P(c_k^+,c_k^-)}{\prod_{j=0}^n(c_k^+-b_j)(c_k^--b_j)\prod_{j=0,\,j\neq k}^n(c_j-c_k)}\cdot\frac{1}{c_k-x_1-x_2},\end{aligned}$$

which holds for P a symmetric polynomial of degree at most $2n+1$ in each variable.

Specializing $b_k=-k$, $c_k=s+k$ and $P(x_1,x_2)=(t-x_1)_n(t-x_2)_n(u+x_1+x_2)_n$ gives after simplification

$$\begin{aligned}&\frac{(t-x_1)_n(t-x_2)_n(u+x_1+x_2)_n}{(x_1)_{n+1}(x_2)_{n+1}(s-x_1-x_2)_{n+1}}\\&=\sum_{k=0}^n\frac{(-1)^k}{k!(n-k)!}\frac{(t+k)_n\left(t-\lambda+\frac{(\lambda-x_1)(\lambda-x_2)}{\lambda+k}\right)_n\left(u+\lambda-\frac{(\lambda-x_1)(\lambda-x_2)}{\lambda+k}-k\right)_n}{\left(\lambda-\frac{(\lambda-x_1)(\lambda-x_2)}{\lambda+k}\right)_{n+1}\left(s-\lambda+\frac{(\lambda-x_1)(\lambda-x_2)}{\lambda+k}+k\right)_{n+1}}\\&\quad\times\left(\frac{1}{x_1+k}+\frac{1}{x_2+k}-\frac{1}{\lambda+k}\right)\\&\quad+\sum_{k=0}^n\frac{(-1)^k}{k!(n-k)!}\frac{(t-c_k^+)_n(t-c_k^-)_n(u+s+k)_n}{(c_k^+)_{n+1}(c_k^-)_{n+1}}\cdot\frac{1}{s+k-x_1-x_2}.\end{aligned}\tag{28}$$

If we formally let $n \to \infty$, we obtain the following result. To make this rigorous, we again need some estimates that are given in the Appendix.

Corollary 4 *Assuming* $\operatorname{Re}(2\lambda - s + t + u) > 0$, *we have*

$$\begin{aligned}
&\frac{\Gamma(x_1)\Gamma(x_2)\Gamma(s - x_1 - x_2)}{\Gamma(t - x_1)\Gamma(t - x_2)\Gamma(u + x_1 + x_2)} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma\left(\lambda - \frac{(\lambda - x_1)(\lambda - x_2)}{\lambda + k}\right)\Gamma\left(s - \lambda + \frac{(\lambda - x_1)(\lambda - x_2)}{\lambda + k} + k\right)}{\Gamma(t + k)\Gamma\left(t - \lambda + \frac{(\lambda - x_1)(\lambda - x_2)}{\lambda + k}\right)\Gamma\left(u + \lambda - \frac{(\lambda - x_1)(\lambda - x_2)}{\lambda + k} - k\right)} \\
&\quad \times \left(\frac{1}{x_1 + k} + \frac{1}{x_2 + k} - \frac{1}{\lambda + k}\right) \\
&\quad + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(c_k^+)\Gamma(c_k^-)}{\Gamma(t - c_k^+)\Gamma(t - c_k^-)\Gamma(u + s + k)} \cdot \frac{1}{s + k - x_1 - x_2},
\end{aligned} \tag{29}$$

where

$$c_k^{\pm} = \frac{s + k}{2} \pm \sqrt{\frac{(s + k)^2}{4} + \lambda(x_1 + x_2 - s - k) - x_1 x_2}.$$

As an example, let $s = t = 1$ and $u = 0$ in (29) and reintroduce the variable $x_3 = 1 - x_1 - x_2$. After simplification, this gives the following asymmetric expansion of the Virasoro–Shapiro amplitude (1):

$$\begin{aligned}
&\frac{\Gamma(x_1)\Gamma(x_2)\Gamma(x_3)}{\Gamma(1 - x_1)\Gamma(1 - x_2)\Gamma(1 - x_3)} \\
&= \sum_{k=0}^{\infty} \frac{\left(1 - \lambda + \frac{(\lambda - x_1)(\lambda - x_2)}{\lambda + k}\right)_k^2}{(k!)^2} \left(\frac{1}{x_1 + k} + \frac{1}{x_2 + k} - \frac{1}{\lambda + k}\right) \\
&\quad + \sum_{k=0}^{\infty} \frac{\left(\frac{1-k}{2} + \sqrt{\frac{(1+k)^2}{4} - \lambda(x_3 + k) - x_1 x_2}\right)_k^2}{(k!)^2} \cdot \frac{1}{x_3 + k},
\end{aligned}$$

where $x_1 + x_2 + x_3 = 1$ and $\operatorname{Re}(\lambda) > 0$. By (5), the terms are rational functions of the parameters. If we instead let $s = 0$ and $t = u = 1$ we recover [5, Eq. (A2)].

5 Symmetric expansions of closed string amplitudes

To obtain symmetric expansions for closed string amplitudes, such as (4), we start from

$$\frac{(u - x_1)_n (u - x_2)_n (u - x_3)_n}{(x_1)_{n+1} (x_2)_{n+1} (x_3)_{n+1}}$$

and apply (22) with $(r, l) = (3, 1)$. We will write $y_1 = \lambda$ and t^+, t^- instead of t', t''. It follows from (23) that

$$t + t^+ + t^- = x_1 + x_2 + x_3$$

and from the case $u = \lambda$ of (24) that

$$(t - \lambda)(t^+ - \lambda)(t^- - \lambda) = (x_1 - \lambda)(x_2 - \lambda)(x_3 - \lambda).$$

This leads to the explicit formula

$$\begin{aligned} t^\pm &= \frac{t^+ + t^-}{2} \pm \sqrt{\left(\frac{t^+ + t^- - 2\lambda}{2}\right)^2 - (t^+ - \lambda)(t^- - \lambda)} \\ &= \frac{e_1(\mathbf{x}) - t}{2} \pm \sqrt{\frac{(e_1(\mathbf{x}) - t - 2\lambda)^2}{4} - \frac{\prod_{j=1}^3 (x_j - \lambda)}{t - \lambda}}. \end{aligned}$$

The resulting special case of (22) is

$$\begin{aligned} &\frac{(u - x_1)_n (u - x_2)_n (u - x_3)_n}{(x_1)_{n+1} (x_2)_{n+1} (x_3)_{n+1}} \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \frac{(u + k)_n (u - \eta_k^+)_n (u - \eta_k^-)_n}{(\eta_k^+)_{n+1} (\eta_k^-)_{n+1}} \left(\sum_{j=1}^{3} \frac{1}{x_j + k} - \frac{1}{\lambda + k} \right), \end{aligned} \tag{30}$$

where we write $\eta_k^\pm = (-k)^\pm$. If we let $n \to \infty$, we obtain the following result. As before, we provide some further details in the Appendix.

Corollary 5 *If* $\operatorname{Re}(\lambda + u - e_1(\mathbf{x})) > 0$ *then*

$$\begin{aligned} &\frac{\Gamma(x_1)\Gamma(x_2)\Gamma(x_3)}{\Gamma(u - x_1)\Gamma(u - x_2)\Gamma(u - x_3)} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(\eta_k^+)\Gamma(\eta_k^-)}{\Gamma(u + k)\Gamma(u - \eta_k^+)\Gamma(u - \eta_k^-)} \left(\sum_{j=1}^{3} \frac{1}{x_j + k} - \frac{1}{\lambda + k} \right), \end{aligned} \tag{31}$$

where

$$\eta_k^\pm = \frac{e_1(\mathbf{x}) + k}{2} \pm \sqrt{\frac{(e_1(\mathbf{x}) + k - 2\lambda)^2}{4} + \frac{\prod_{j=1}^3 (x_j - \lambda)}{\lambda + k}}.$$

If we assume that $u - e_1(\mathbf{x}) = a$ is an integer, (31) can be written as

$$\frac{\Gamma(x_1)\Gamma(x_2)\Gamma(x_3)\Gamma(a + s)}{\Gamma(a + s - x_1)\Gamma(a + s - x_2)\Gamma(a + s - x_3)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(1-\eta_k^+)_{k-a}(1-\eta_k^-)_{k-a}}{(a+s)_k} \left(\sum_{j=1}^{3} \frac{1}{x_j+k} - \frac{1}{\lambda+k} \right), \tag{32}$$

where $x_1 + x_2 + x_3 = s$. The special case $\lambda = s/3$ of (32) is [5, Eq. (A3)]. The case $a + s = 1$ is the identity (6) given in the introduction.

As another example, the case $x_1 = x_2 = x_3 = 1/2$, $a = 0$ of (32) can be written

$$\frac{\pi^2}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k (\xi_k^+)_k (\xi_k^-)_k}{k!(3/2)_k} \left(\frac{6}{2k+1} - \frac{1}{\lambda+k} \right), \qquad \mathrm{Re}(\lambda) > 0,$$

where

$$\xi_k^{\pm} = \frac{1}{4} - \frac{k}{2} \pm \frac{2k+1}{4} \sqrt{\frac{k+2-3\lambda}{k+\lambda}}.$$

When $\lambda = 1/2$, this reduces to Euler's identity

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

The series obtained by formally letting $\lambda \to \infty$ is divergent.

The reader may ask why we did not use (22) in the more general case $l = 2$. We would then replace (30) with

$$\frac{(u-x_1)_n(u-x_2)_n(u-x_3)_n}{(x_1)_{n+1}(x_2)_{n+1}(x_3)_{n+1}} = \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \frac{(u+k)_n(u-(-k)')_n(u-(-k)'')_n}{((-k)')_{n+1}((-k)'')_{n+1}} \left(\sum_{j=1}^{3} \frac{1}{x_j+k} - \sum_{j=1}^{2} \frac{1}{y_j+k} \right). \tag{33}$$

The problem is that, when $n \to \infty$, the left-hand side behaves as $n^{3u-2e_1(\mathbf{x})-3}$ and the terms on the right as $n^{3u-2e_1(\mathbf{t})-3}$, where $\mathbf{t} = (-k, (-k)', (-k)'')$. Hence, to take a termwise limit, we need $e_1(\mathbf{x}) = e_1(\mathbf{t})$. However, (23) gives

$$e_1(\mathbf{t}) = e_1(\mathbf{x}) - \frac{\prod_{j=1}^{3}(x_j+k)}{\prod_{j=1}^{2}(y_j+k)}.$$

This forces y_1 or $y_2 \to \infty$, which is precisely the case $l = 1$ studied above. We also note that, in the special case $y_1 = x_1$, $y_2 = x_2$, it follows from (24) that $\mathbf{t} = (-k, x_1, x_2)$. Then, (33) reduces to

$$\frac{(u-x_3)_n}{(x_3)_{n+1}} = \sum_{k=0}^{n} \frac{(-1)^k (u+k)_n}{k!(n-k)!} \cdot \frac{1}{x_3+k},$$

which is just a restatement of (7) (with $a = 1 - u - n$). In hypergeometric notation, the series on the right is a multiple of

$$ {}_3F_2\left(\begin{matrix} -n, u+n, x_3 \\ u, x_3+1 \end{matrix}; 1\right). $$

The asymptotics as $n \to \infty$ of this type of series is well studied, see e.g. [3, §7.4]. It is conceivable that a similar analysis of the series (33) would lead to some interesting asymptotic expansions of string amplitudes.

Appendix A Limit transitions

We have obtained our main results, Corollary 3, Corollary 4 and Corollary 5, by a limit transition from finite to infinite series. It requires some work to make this rigorous. In the present Appendix, we explain this in detail for Corollary 3 and then briefly discuss the necessary modifications for the other two results.

We start from Stirling's formula

$$ \Gamma(z) \sim \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z}, $$

which holds if $|z| \to \infty$ with $\arg(z) < \pi - \delta$ for any fixed $\delta > 0$. It implies that (8) can be strengthened to

$$ \frac{\Gamma(z + a(z))}{\Gamma(z + b(z))} \sim z^{a(z)-b(z)}, \tag{A1} $$

which holds in the same sense, assuming that $a(z)$ and $b(z)$ are bounded.

As we have already noted, if we multiply both sides of (26) with $\Gamma(n + 2 - a)$ and then let $n \to \infty$ we obtain (27), provided that we are allowed to interchange limit and summation on the right-hand side. To justify this, we will use Tannery's theorem, which states that if $|a_{kn}| < M_k$, where M_k is independent of n and $\sum_{k=0}^{\infty} M_k < \infty$, then

$$ \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{kn} = \sum_{k=0}^{\infty} \lim_{n\to\infty} a_{kn}. $$

We need to find such an estimate when a_{kn} is supported on $k \le n$ and given by

$$ a_{kn} = \frac{(-1)^k \Gamma(n+2-a)}{k!(n-k)!} \frac{(\lambda + a - \varepsilon_k - k)_n}{(\lambda - \varepsilon_k)_{n+1}} \left(\frac{1}{x_1 + k} + \frac{1}{x_2 + k} - \frac{1}{\lambda + k} \right), $$

where

$$ \varepsilon_k = \frac{(\lambda - x_1)(\lambda - x_2)}{\lambda + k}. $$

Note that $\varepsilon_k \to 0$ quickly enough so that

$$k^{\varepsilon_k} \to 1, \qquad k \to \infty. \tag{A2}$$

It will be useful to write

$$a_{kn} = \frac{\Gamma(\lambda - \varepsilon_k)}{\Gamma(\lambda + a - \varepsilon_k)\Gamma(1 - \lambda - a + \varepsilon_k)} \cdot \frac{\Gamma(n + 2 - a)\Gamma(\lambda + a - \varepsilon_k + n)}{\Gamma(n + 1)\Gamma(\lambda + 1 - \varepsilon_k + n)}$$
$$\times \frac{\Gamma(1 - \lambda - a + \varepsilon_k + k)}{\Gamma(k + 1)} \cdot \frac{(-n)_k}{(1 - \lambda - a + \varepsilon_k - n)_k} \left(\frac{1}{x_1 + k} + \frac{1}{x_2 + k} - \frac{1}{\lambda + k} \right). \tag{A3}$$

The first factor in (A3) has a finite limit as $k \to \infty$ and is hence bounded by a constant. By (8), the second factor is bounded by a constant times $n^{1-a}(n - \varepsilon_k)^{a-1}$. However, this is itself a bounded quantity, so the second factor is bounded. By (A1) and (A2), the third factor is

$$\frac{\Gamma(1 - \lambda - a + \varepsilon_k + k)}{\Gamma(k + 1)} = \mathcal{O}(k^{-\lambda - a + \varepsilon_k}) = \mathcal{O}(k^{-\operatorname{Re}(\lambda + a)}). \tag{A4}$$

We write the fourth factor as

$$\frac{(-n)_k}{(1 - \lambda - a + \varepsilon_k - n)_k} = \prod_{j=0}^{k-1} \frac{(n - j)}{(\lambda + a - \varepsilon_k - 1 + n - j)}. \tag{A5}$$

Assuming that $\operatorname{Re}(\lambda + a) > 1$, we have $\operatorname{Re}(\lambda + a - \varepsilon_k - 1) > 0$ for large enough k. Then, each factor in (A5) is bounded in modulus by 1. Hence, (A5) is bounded by a constant. The final factor in (A3) is $\mathcal{O}(k^{-1})$. This all shows that

$$|a_{kn}| \leq C k^{-1 - \operatorname{Re}(\lambda + a)},$$

with C independent of k and n. We can then apply Tannery's theorem and deduce Corollary 3 under the assumption $\operatorname{Re}(\lambda + a) > 1$.

To weaken the assumption to $\operatorname{Re}(\lambda + a) > 0$, we consider the series in (27) as a function of a. If we can show that it converges locally uniformly in $\operatorname{Re}(\lambda + a) > 0$, the general case of Corollary 3 follows by analytic continuation. The terms in the relevant series are formed by the first, third and last factor in (A3). It is clear that the bound on the first factor can be made locally uniform in a. By (A4), if $\operatorname{Re}(\lambda + a) > \varepsilon > 0$, the second factor is $\mathcal{O}(k^{-\varepsilon})$. Since the final factor is $\mathcal{O}(k^{-1})$, each term can be estimated with $Ck^{-\varepsilon-1}$, where C is locally bounded as a function of a. This completes the proof of Corollary 3.

Let us now turn to Corollary 4. Formally, (29) is obtained by multiplying (28) with $n^{s-2t-u+3}$ and letting $n \to \infty$. In the first sum on the right of (28), the resulting terms can be factored as

$$\frac{\Gamma(\lambda - \varepsilon_k)}{\Gamma(\lambda + u - \varepsilon_k)\Gamma(1 - \lambda - u + \varepsilon_k)\Gamma(t - \lambda + \varepsilon_k)}$$

$$\times \frac{n^{s-2t-u+3}\Gamma(n+t)\Gamma(\lambda+u-\varepsilon_k+n)\Gamma(t-\lambda+\varepsilon_k+n)}{\Gamma(n+1)\Gamma(\lambda+1-\varepsilon_k+n)\Gamma(s+1-\lambda+\varepsilon_k+n)}$$
$$\times \frac{\Gamma(1-\lambda-u+\varepsilon_k+k)\Gamma(s-\lambda+\varepsilon_k+k)}{\Gamma(k+1)\Gamma(t+k)} \cdot \frac{(-n)_k}{(1-\lambda-u+\varepsilon_k-n)_k}$$
$$\times \frac{(t+n)_k}{(s+1-\lambda+\varepsilon_k+n)_k} \cdot \left(\frac{1}{x_1+k}+\frac{1}{x_2+k}-\frac{1}{\lambda+k}\right). \tag{A6}$$

In particular, we have a factor of the form (A5) but with a replaced by u. Under the assumption $\mathrm{Re}(\lambda+u) > 1$, it can be estimated by 1, for large enough k. In order to estimate

$$\frac{(t+n)_k}{(s+1-\lambda+\varepsilon_k+n)_k} = \prod_{j=0}^{k-1} \frac{t+n+j}{s+1-\lambda+\varepsilon_k+n+j},$$

we will assume that $\mathrm{Re}(\lambda-s+t) > 1$. Note that if $\mathrm{Re}(a) > \mathrm{Re}(b)$, then $|n+a| > |n+b|$ if n is large enough. If we choose k large enough so that $\mathrm{Re}(\lambda-s+t-\varepsilon_k) > 1$ and then n large enough so that $|t+n+j| > |s+1-\lambda+\varepsilon_k+n+j|$ for all j, we have

$$\left|\frac{(t+n)_k}{(s+1-\lambda+\varepsilon_k+n)_k}\right| \leq \left|\frac{(t+n)_n}{(s+1-\lambda+\varepsilon_k+n)_n}\right|$$
$$= \left|\frac{\Gamma(t+2n)\Gamma(s+1-\lambda+\varepsilon_k+n)}{\Gamma(t+n)\Gamma(s+1-\lambda+\varepsilon_k+2n)}\right| \sim 2^{\mathrm{Re}(\lambda-s+t-\varepsilon_k)-1}, \qquad n\to\infty.$$

Hence, this factor can be estimated by a constant. The remaining factors in (A6) can be treated exactly as in (A3).

Turning to the second term on the right-hand side of (28), we note that

$$c_k^{\pm} = \frac{s+k}{2} \pm \left(\frac{s+k}{2} - \lambda + \mathcal{O}(k^{-1})\right), \qquad k\to\infty.$$

Hence,

$$c_k^{+} = s-\lambda+\delta_k+k, \qquad c_k^{-} = \lambda-\delta_k,$$

where $k^{\delta_k} \to 1$ as $k\to\infty$. We write the terms as

$$\frac{\Gamma(\lambda-\delta_k)}{\Gamma(t-\lambda+\delta_k)\Gamma(t-s+\lambda-\delta_k)\Gamma(1+s-t-\lambda+\delta_k)}$$
$$\times \frac{\Gamma(s+u+n)\Gamma(t-\lambda+\delta_k+n)\Gamma(t-s+\lambda-\delta_k+n)n^{s-2t-u+3}}{\Gamma(n+1)\Gamma(1+s-\lambda+\delta_k+n)\Gamma(1+\lambda-\delta_k+n)}$$
$$\times \frac{\Gamma(s-\lambda+\delta_k+k)\Gamma(1+s-t-\lambda+\delta_k+k)}{\Gamma(k+1)\Gamma(s+u+k)}$$
$$\times \frac{(-n)_k}{(1+s-t-\lambda+\delta_k-n)_k} \cdot \frac{(s+u+n)_k}{(1+s-\lambda+\delta_k+n)_k} \cdot \frac{1}{s+k-x_1-x_2},$$

where each factor can be treated as before, under the same assumptions $\mathrm{Re}(\lambda+u) > 1$ and $\mathrm{Re}(\lambda - s + t) > 1$. The analytic continuation to $\mathrm{Re}(2\lambda - s + t + u) > 1$ works as for Corollary 3.

Finally, we turn to Corollary 5. Writing $s = e_1(\mathbf{x})$, we have

$$\eta_k^+ = s - \lambda + \delta_k + k, \qquad \eta_k^- = \lambda - \delta_k,$$

where $k^{\delta_k} \to 1$, $k \to \infty$. We need to multiply (30) with $n^{2s+3-3u}$ before letting $n \to \infty$. The terms on the right can be factored as

$$\begin{aligned}&\frac{\Gamma(\lambda-\delta_k)}{\Gamma(u-\lambda+\delta_k)\Gamma(u-s+\lambda-\delta_k)\Gamma(1+s-u-\lambda+\delta_k)}\\ &\times \frac{\Gamma(u+n)\Gamma(u-\lambda+\delta_k+n)\Gamma(u-s+\lambda-\delta_k+n)n^{2s+3-3u}}{\Gamma(n+1)\Gamma(1+s-\lambda+\delta_k+n)\Gamma(1+\lambda-\delta_k+n)}\\ &\times \frac{\Gamma(s-\lambda+\delta_k+k)\Gamma(1+s-u-\lambda+\delta_k+k)}{\Gamma(k+1)\Gamma(u+k)}\\ &\times \frac{(-n)_k}{(1+s-u-\lambda+\delta_k-n)_k}\cdot\frac{(u+n)_k}{(1+s-\lambda+\delta_k+n)_k}\cdot\left(\prod_{j=1}^{3}\frac{1}{x_j+k}-\frac{1}{s+k}\right).\end{aligned}$$

We then proceed exactly as for Corollary 4.

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