On a Hardy–Morrey inequality

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On a Hardy–Morrey inequality

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ABSTRACT

Morrey's classical inequality implies the Hölder continuity of a function whose gradient is sufficiently integrable. Another consequence is the Hardy-type inequality

$$\lambda\left\|\frac{u}{d_\Omega^{1-n/p}}\right\|_\infty^p \leq \int_\Omega |Du|^p\, dx$$

for any open set $\Omega \subsetneq \mathbb{R}^n$. This inequality is valid for functions supported in Ω and with λ a positive constant independent of u. The crucial hypothesis is that the exponent p exceeds the dimension n. This paper aims to develop a basic theory for this inequality and the associated variational problem. In particular, we study the relationship between the geometry of Ω, sharp constants, and the existence of a nontrivial u which saturates the inequality.

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1. Introduction and main results

The topic of this paper concerns a geometric Hardy inequality in the setting of a Sobolev space for which the associated exponent p is larger than the dimension of the ambient space. Specifically, the inequality states that if Ω is a proper open subset of $\mathbb{R}^n$ and $p > n$, there exists a constant $\lambda > 0$ such that

$$\lambda \left\| \frac{u}{d_\Omega^{1-n/p}} \right\|_\infty^p \leq \int_\Omega |Du|^p \, dx \tag{1.1}$$

for all $u \in C_c^\infty(\Omega)$. Here and in what follows $d_\Omega(x)$ denotes the distance from x to the complement of Ω. The inequality extends to u being an element of the Sobolev space $\mathcal{D}_0^{1,p}(\Omega)$ as discussed in Section 2 below.

The one-dimensional case of this inequality previously appeared in [35] and the general case in [37]. This inequality has also recently been considered in [5] where it occurs as an endpoint case of a family of inequalities interpolating between Sobolev, Morrey, and Hardy inequalities. Nevertheless, we will explain below that the existence of a constant λ such that (1.1) holds is a direct consequence of Morrey's classical inequality (see also [37]). As a result, it is natural to refer to (1.1) as a *Hardy–Morrey* inequality. We also acknowledge that this terminology has been used to describe related inequalities in [15,36,37].

In this note, we turn our attention to the variational problem associated to (1.1). Namely, we define

$$\mathcal{R}_p(\Omega, u) = \left\| \frac{u}{d_\Omega^{1-n/p}} \right\|_\infty^{-p} \|Du\|_p^p \quad \text{for } u \in \mathcal{D}_0^{1,p}(\Omega)$$

and observe that the sharp constant λ in (1.1) can be characterized as

$$\lambda_p(\Omega) = \inf_{u \in \mathcal{D}_0^{1,p}(\Omega) \setminus \{0\}} \mathcal{R}_p(\Omega, u) \,. \tag{1.2}$$

Here $\|\cdot\|_p$ denotes the $L^p(\Omega)$ norm. Whenever the infimum (1.2) is attained by a nontrivial $u \in \mathcal{D}_0^{1,p}(\Omega)$, we say that u is an *extremal* and that Ω has an extremal.

The questions addressed in this paper are:

(1) How does $\lambda_p(\Omega)$ depend on the geometry of Ω?
(2) When does Ω have an extremal?

As it turns out, both of these questions are subtle. To some extent, we shall see that this subtlety can be traced back to the fact that $\lambda_p(\Omega)$ is invariant under orthogonal transformations, translations, and dilations of Ω.

1.1. Main results

Our first result provides sharp upper and lower bounds for $\lambda_p(\Omega)$. The upper bound involves the halfspace $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$.

Theorem A. *Assume $p > n \geq 1$. If $\Omega \subsetneq \mathbb{R}^n$ is open, then*

$$C_{n,p}^{-p} = \lambda_p(\mathbb{R}^n \setminus \{0\}) \leq \lambda_p(\Omega) \leq \lambda_p(\mathbb{R}^n_+) = 2^{n-1} C_{n,p}^{-p} \,.$$

Here $C_{n,p}$ is the sharp constant in Morrey's inequality (see (2.3)).

A natural question to ask is whether equality is attained in the bounds of the theorem only when $\Omega = \mathbb{R}^n_+$ or $\Omega = \mathbb{R}^n \setminus \{0\}$ up to the natural symmetries of the problem (see Section 2). It turns out that this is not the case.

Theorem B. *Suppose $p > n \geq 2$.*

(1) If $\Omega \subsetneq \mathbb{R}^n$ is convex and open, then

$$\lambda_p(\Omega) = \lambda_p(\mathbb{R}^n_+)$$

and Ω has an extremal if and only if Ω is a halfspace.

(2) If $\Omega \subset \mathbb{R}^n$ is open and $x_0 \in \Omega$, then

$$\lambda_p(\Omega \setminus \{x_0\}) = \lambda_p(\mathbb{R}^n \setminus \{0\})$$

and $\Omega \setminus \{x_0\}$ has an extremal if and only if $\Omega = \mathbb{R}^n$.

(3) If $K \subset \mathbb{R}^n$ is compact, then

$$\lambda_p(\mathbb{R}^n \setminus K) = \lambda_p(\mathbb{R}^n \setminus \{0\})$$

and $\mathbb{R}^n \setminus K$ has an extremal if and only if K is a singleton.

According to this theorem, the infimum (1.2) is attained for a halfspace and a punctured whole space. More generally, we will show that this holds whenever Ω^c is a closed convex cone. See Section 7.

In view of our remarks above, one might suspect that extremals exist only in rather special geometries. However, the following two theorems assert that this is far from the case. As will be elaborated on later, the first result is a consequence of a more general compactness threshold-result in the spirit of the work of Brezis and Nirenberg [8] (see Proposition 3.4). In what follows, we will simply say that Ω is C^k if $\partial\Omega$ is C^k-regular.

Theorem C. *Fix $p > n \geq 2$. If $\Omega \subset \mathbb{R}^n$ is bounded, open, and C^1 with*

$$\lambda_p(\Omega) < \lambda_p(\mathbb{R}^n_+)\,, \tag{1.3}$$

then Ω has an extremal.

We will verify the above claim by establishing that any minimizing sequence $\{u_k\}_{k\geq 1} \subset \mathcal{D}^{1,p}_0(\Omega)$ for (1.2) with $\|Du_k\|_p = 1$ for all k is precompact in $\mathcal{D}^{1,p}_0(\Omega)$. In particular, we will show that $\{u_k\}_{k\geq 1}$ has a subsequence which converges to an extremal.

We will say that a C^2 subset $\Omega \subset \mathbb{R}^n$ is *mean convex* provided that the mean curvature at each point of $\partial\Omega$ is nonnegative. We will always measure mean curvature with respect to the outward unit normal so that convex shapes have nonnegative mean curvature. The central assertion of our work is that a bounded Ω which is not mean convex admits an extremal.

Theorem D. *Let $p > n \geq 2$. If $\Omega \subset \mathbb{R}^n$ is bounded, open, C^2, and not mean convex, then (1.3) holds. Therefore, Ω has an extremal.*

In addition to the above theorems, we provide various examples where our results give detailed knowledge about $\lambda_p(\Omega)$; see Section 7 and 11. These examples encompass for instance concave cones, polygons and piecewise C^1 sets, epigraphs and examples that indicate the instability $\lambda_p(\Omega)$ with respect to small changes in Ω. It is also worth noting that we prove that any $\lambda \in [\lambda_p(\mathbb{R}^n \setminus \{0\}), \lambda_p(\mathbb{R}^n_+)]$ is realized as $\lambda = \lambda_p(\Omega)$ for some open $\Omega \subsetneq \mathbb{R}^n$; refer to Theorem 10.1 below. The last section of this article also includes a short list of open problems.

1.2. Related results

Inequality (1.1) can be seen as the limiting case as $q \to \infty$ of the family of Hardy-type inequalities

$$\lambda_{p,q}(\Omega)\left\|\frac{u}{d_\Omega^\gamma}\right\|_q^p \leq \int_\Omega |Du|^p\,dx\,, \tag{1.4}$$

where $\gamma = n/q + 1 - n/p$ and $n < p \leq q < \infty$. The special case when $p = q$

$$\lambda_{p,p}(\Omega)\left\|\frac{u}{d_\Omega}\right\|_p^p \leq \int_\Omega |Du|^p\,dx \tag{1.5}$$

has been the topic of many studies. Making use of Hölder's inequality, it is straightforward to conclude that (1.4) follows from (1.1) and (1.5), which can be seen as the endpoints of this family of inequalities. This has recently been observed in [5]. However, the theory for (1.1) and (1.4) in general is still in its infancy.

As previously mentioned, the one-dimensional version of (1.1) appears in Chapter 1.5 of [35] and (1.4) is treated in Chapters 2 and 3. The n-dimensional version of (1.1) as well as (1.4) is mentioned in [34]; see page 22 in Paper A. Other versions of inequality (1.1) also appear in Section 2.1.6 in [33] and Section 2.1.0 in [38].

Hardy's inequality (1.5) was first proved in the one dimensional setting by Hardy (cf. [21] and [22]) even though a special case was perhaps known as early as 1907 (see [4]). For an overview of Hardy's inequality (1.5) and its rich history, we refer the reader to [3,18,27,35]. The validity of such an inequality for $p \leq n$ is a rather delicate matter. However, in the case $p > n$ that we are concerned with, (1.5) holds for any open set Ω (see [2], [29], [30], and [42]). Since (1.1) is also valid for general open sets, Hölder's inequality implies that the same is true for (1.4).

There is also a well established connection between the geometry of Ω, the optimal constant, and the existence of extremals for inequality (1.5). These results are very much in the spirit of what we accomplish in Theorems B, C and D. For instance, it has been proved that $\lambda_{p,p}(\Omega) \leq c_p$ with

$$c_p = \left(1 - \frac{1}{p}\right)^{\frac{1}{p}}$$

for bounded and sufficiently regular Ω and $\lambda_{p,p}(\Omega) = c_p$ for convex Ω. Moreover, for sufficiently smooth bounded sets, $\lambda_{p,p}(\Omega) = c_p$ if and only if there is no extremal (see [30], [31], [32], and [28]). We also remark that a parallel theory for Hardy's inequality (1.5) on exterior domains has been established (as described in [28], [9], and [12]).

For $1 < p \leq n$ and $q \in [2, \frac{2n}{n-2}]$ the inequality in (1.4) was recently considered in [41], and its validity established under the assumption that Ω has Lipschitz-regular boundary.

For the case $p = 2, n \geq 2$ and $q \in (2, \frac{2n}{n-2})$, results similar to those proved in this paper concerning the attainability of the sharp constant in (1.4) were obtained in [39,41].

Another result in the spirit of Theorem D was discovered by Ghoussoub and Robert. They considered the Hardy–Sobolev inequality

$$\mu_s(\Omega)\left(\int_\Omega \frac{|u|^{2^*}}{|x|^s}\,dx\right)^{2/2^*} \leq \int_\Omega |Du|^2\,dx \tag{1.6}$$

for a smooth and bounded domain Ω with $0 \in \partial\Omega$. Here $n \geq 3$, $s \in (0,2)$,

$$2^* = \frac{2(n-s)}{n-2},$$

and the admissible functions u belong to the Sobolev space $H_0^1(\Omega)$.

Ghoussoub and Robert showed that if $\partial\Omega$ has negative mean curvature at 0, then inequality (1.6) has an extremal [19]. That is, equality holds in (1.6) for a nontrivial $u \in H_0^1(\Omega)$. This extended earlier work by Egnell [13] who showed that extremals of (1.6) exist for certain conical domains and by Ghoussoub and Kang [16] who verified existence when $\partial\Omega$ is negatively curved at 0. For other results along these lines, see [10,11,17,20,40].

1.3. Outline of the strategy

Our approach rests heavily upon the fundamental property that λ_p is invariant under translations, rotations and dilations (cf. Section 2.2). We follow two main strategies which we now briefly explain.

The first strategy consist of obtaining information about the value of λ_p and possible extremals by transplanting a competitor defined in Ω into the corresponding minimization problem for a different set Ω'. In certain situations, this will allow us to deduce interesting bounds for λ_p. In particular, we obtain bounds by finding an appropriate exhaustion of a set Ω and using Lemma 2.5, or by touching Ω from outside with a cleverly chosen larger set Ω' and using Proposition 5.3. These ideas lead up to the proof of Theorems A and B in Section 6.

The second line of our analysis, which is the more technical, consists of studying the nature of sequences of trial functions which in a certain sense either concentrate at a boundary point or move of towards infinity. In order to explain this idea, it is convenient to first state a few properties of minimizers and introduce some notation. A first important observation is that any extremal u satisfies

$$-\Delta_p u = 0 \quad \text{in } \Omega \setminus \{x_0\}$$

for some $x_0 \in \Omega$; see Proposition 3.1. This leads to the useful idea (cf. Proposition 3.3) that it is enough to study solutions of this PDE for a given $x_0 \in \Omega$ which satisfy $u(x_0) = 1$ and $u|_{\partial\Omega} = 0$ when searching for an extremal. We call such functions potentials.

The above observations lead to Proposition 3.4, where we deduce that the following statements are equivalent:

(1) Ω has an extremal.
(2) There exists a sequence $\{x_k\}_{k\geq 1} \subset \Omega$ such that $\lim_{k\to\infty} x_k \in \Omega$ and the corresponding sequence of potentials is a minimizing sequence.

It therefore becomes crucial to understand when a sequence of points x_k related to a minimizing sequence of potentials stays inside Ω, approaches the boundary or escapes to infinity. The background for this is developed in Section 3 and these questions are then pursued in detail in Sections 8 and 9.

In Section 8, we are able to obtain estimates in terms of λ_p for certain global prototype sets that locally or globally approximate Ω. The main outcome of this analysis is Theorem C. The local analysis amounts to performing a blow-up. Despite the substantial difference in how this approach is carried out, the underlying idea is similar to ideas used also in the context of Hardy's inequality (1.5). See for instance Brezis–Marcus [7] and Marcus–Mizel–Pinchover [31] for more.

In Section 9 we perform an analysis in the spirit of a domain variation. This is done to prove that if the boundary has a point of negative mean curvature, the sequence of points x_k corresponding to a minimizing sequence of potentials cannot approach the boundary. This is one of the main ideas used to prove Theorem D.

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2. Preliminaries

Throughout the paper, we will assume $n \in \mathbb{N}$ and $p > n$. We will denote by $B_r(x)$ the ball of radius r centered at x. In the case when $x = 0$, we will simply write B_r. Unless otherwise stated, Ω will always be a proper, nonempty, open subset of $\mathbb{R}^n$, and

$$d_\Omega(x) = \inf_{y\in\Omega^c} |x-y|\,, \quad x \in \mathbb{R}^n\,.$$

Note that for $x \in \Omega$, d_Ω is the distance to the boundary of $\partial\Omega$.

Our work below will concern functions in the homogeneous Sobolev space

$$\mathcal{D}^{1,p}(\mathbb{R}^n) = \{u \in L^1_{\text{loc}}(\mathbb{R}^n) : u_{x_1}, \dots, u_{x_n} \in L^p(\mathbb{R}^n)\}$$

equipped with the seminorm $u \mapsto \|Du\|_p$. As usual, u_{x_i} denotes the weak partial derivative with respect to x_i. When $p > n$ any $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ has a Hölder continuous representative u^*, and we will identify u with u^* going forward.

In what follows, we shall mostly consider functions $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ which vanish on the complement of an open set $\Omega \subsetneq \mathbb{R}^n$. We will denote this space of functions by

$$\mathcal{D}^{1,p}_0(\Omega) = \{u \in \mathcal{D}^{1,p}(\mathbb{R}^n) : u(x) = 0 \text{ for each } x \notin \Omega\}.$$

By Morrey's inequality (2.1), $u \mapsto \|Du\|_p$ is a norm on the restricted space $\mathcal{D}^{1,p}_0(\Omega)$. Moreover, the space $\mathcal{D}^{1,p}_0(\Omega)$ is a Banach space which can be identified as the completion of $C^\infty_c(\Omega)$ (see Lemma 2.3).

We will make use of *Morrey's inequality*, which asserts that there is $C > 0$ depending on n, p such that

$$\sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{1-n/p}} \le C \left(\int_{\mathbb{R}^n} |Du|^p \, dz \right)^{1/p} \tag{2.1}$$

for each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. Morrey's inequality is a consequence of *Morrey's estimate* that posits that there is another constant $c = c(n, p)$ such that if B is a ball of radius r and $x, y \in B$, then

$$|u(x) - u(y)| \le c r^{1-n/p} \left(\int_B |Du|^p \, dz \right)^{1/p} \tag{2.2}$$

(see [14, Theorem 4.10]). We note that (2.1) holds with $C = c$. Let us denote by

$$C_{n,p}, \text{ the smallest } C > 0 \text{ for which Morrey's inequality holds.} \tag{2.3}$$

We will now show that for any $\Omega \subsetneq \mathbb{R}^n$ inequality (1.1) holds with some constant λ. In what follows, the quotient $u/d_\Omega^{1-n/p}$ should be interpreted as zero in the complement of Ω for $u \in \mathcal{D}^{1,p}_0(\Omega)$.

Proposition 2.1. *For any $u \in \mathcal{D}^{1,p}_0(\Omega)$,*

$$C_{n,p}^{-p} \left\| \frac{u}{d_\Omega^{1-n/p}} \right\|_\infty^p \le \int_\Omega |Du|^p \, dx.$$

Proof. Let $u \in \mathcal{D}_0^{1,p}(\Omega) \subset \mathcal{D}^{1,p}(\mathbb{R}^n)$. By Morrey's inequality

$$\frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \leq C_{n,p}\left(\int_\Omega |Du|^p\, dz\right)^{1/p}$$

for distinct $x, y \in \mathbb{R}^n$. In particular, if we choose $x \in \Omega$ and $y \in \partial\Omega$ such that $d_\Omega(x) = |x-y|$, then

$$\frac{|u(x)|}{d_\Omega(x)^{1-n/p}} \leq C_{n,p}\left(\int_\Omega |Du|^p\, dz\right)^{1/p}.$$

We conclude by taking the supremum over $x \in \Omega$. □

By the previous proposition, $\mathcal{R}(\Omega, u) \geq C_{n,p}^{-p}$ for any $u \in \mathcal{D}_0^{1,p}(\Omega)$ with $u \not\equiv 0$. Since $\lambda_p(\Omega)$ is defined in (1.2) as the infimum of all such quotients,

$$\lambda_p(\Omega) \geq C_{n,p}^{-p}\,. \tag{2.4}$$

In particular, $\lambda_p(\Omega)$ is positive and the above lower bound is independent of Ω.

Next we recall some technical results concerning functions in $\mathcal{D}_0^{1,p}(\Omega)$.

Lemma 2.2. *Suppose $u \in \mathcal{D}_0^{1,p}(\Omega)$.*

(i) If $y \in \partial\Omega$, then

$$\lim_{\substack{x\to y\\ x\in\Omega}} \frac{u(x)}{d_\Omega(x)^{1-n/p}} = 0\,.$$

(ii) If Ω is unbounded,

$$\lim_{\substack{|x|\to\infty\\ x\in\Omega}} \frac{u(x)}{d_\Omega(x)^{1-n/p}} = 0\,.$$

(iii) The function $|u|/d_\Omega^{1-n/p}$ achieves its supremum within Ω.

Proof. In this proof, we will exploit the following limits, which were established in Section 6 of [26]. For any $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$,

$$\lim_{|x-y|\to 0} \frac{|v(x)-v(y)|}{|x-y|^{1-n/p}} = 0 \tag{2.5}$$

and

$$\lim_{|x|+|y|\to\infty} \frac{|v(x)-v(y)|}{|x-y|^{1-n/p}} = 0\,. \tag{2.6}$$

Claim (i) follows from (2.5) and (ii) follows from (2.6). As for (iii), we may select a maximizing sequence $\{x_k\}_{k\geq 1} \subset \Omega$ for $|u|/d_\Omega^{1-n/p}$. If $\{x_k\}_{k\geq 1}$ has a cluster point at the boundary of Ω or if $\{x_k\}_{k\geq 1}$ is unbounded, then $|u|/d_\Omega^{1-n/p}$ vanishes identically by (i) and (ii). Otherwise, since $x \mapsto |u(x)|/d_\Omega(x)^{1-n/p}$ is continuous in Ω its maximum is attained at cluster points of $\{x_k\}_{k\geq 1}$ in Ω. $\square$

2.1. Approximation results

In this section, we recall some facts about approximation of functions in $\mathcal{D}_0^{1,p}(\Omega)$ by smooth functions and list some consequences. The first result is the following lemma, which will be important in many of our constructions.

Lemma 2.3. *Suppose $u \in \mathcal{D}_0^{1,p}(\Omega)$ and $\epsilon > 0$. There exists $v \in C_c^\infty(\Omega)$ such that*

$$\|Du - Dv\|_p \leq \epsilon \|Du\|_p\,.$$

Note that we did not assume any regularity or boundedness of the set Ω in the above lemma. This is a feature which is special to the supercritical setting $p > n$. As the proof is standard, yet somewhat lengthy, it is deferred to Appendix A.

An important consequence of Lemma 2.3 is that in the infimum (1.2) defining $\lambda_p(\Omega)$ the space of test functions $\mathcal{D}_0^{1,p}(\Omega)$ can be exchanged with $C_c^\infty(\Omega)$.

Lemma 2.4. *The infimum (1.2) is also given by*

$$\lambda_p(\Omega) = \inf_{u \in C_c^\infty(\Omega)\setminus\{0\}} \mathcal{R}_p(\Omega, u)\,.$$

Proof. Let $\epsilon > 0$. There exists a $v \in \mathcal{D}_0^{1,p}(\Omega)$ such that

$$\mathcal{R}_p(\Omega, v) \leq \lambda_p(\Omega) + \epsilon\,.$$

According to Lemma 2.2, there is $x_0 \in \Omega$ so that

$$\frac{|v(x_0)|}{d_\Omega(x_0)^{1-n/p}} = \left\| \frac{v}{d_\Omega^{1-n/p}} \right\|_\infty .$$

In view of Lemma 2.3, we may select $u \in C_c^\infty(\Omega)$ such that $\|Du - Dv\|_p \leq \epsilon' \|Dv\|_p$ for every $\epsilon' > 0$. Note $v(x_0) = u(x_0) + O(\epsilon')$ by the Hardy–Morrey inequality.

As a result,

$$\mathcal{R}_p(\Omega, u) = \left\| \frac{u}{d_\Omega^{1-n/p}} \right\|_\infty^{-p} \|Du\|_p^p$$

$$
\begin{aligned}
&\leq \frac{d_\Omega(x_0)^{p-n}\|Du\|_p^p}{|u(x_0)|^p} \\
&\leq \frac{d_\Omega(x_0)^{n-p}\|Dv\|_p^p}{|v(x_0)|^p}(1+o_{\epsilon'\to 0}(1)) \\
&= \mathcal{R}_p(\Omega, v)(1+o_{\epsilon'\to 0}(1)) \\
&\leq \lambda_p(\Omega) + o_{\epsilon'\to 0}(1) + \epsilon\,.
\end{aligned}
$$

Therefore,

$$
\lambda_p(\Omega) \leq \inf_{u\in C_c^\infty(\Omega)} \mathcal{R}_p(\Omega, u) \leq \lambda_p(\Omega) + o_{\epsilon'\to 0}(1) + \epsilon\,.
$$

We conclude after sending ϵ' and then ϵ to zero. □

We will say that Ω is *exhausted* by $\{\Omega_j\}_{j\geq 1}$ whenever every $\Omega_j \subset \mathbb{R}^n$ is open, $\Omega_j \subset \Omega_{j+1}$ for each $j \geq 1$, and $\Omega = \bigcup_{j\geq 1}\Omega_j$. It turns out that λ_p is upper semicontinuous with respect to exhaustions.

Lemma 2.5. *If Ω is exhausted by $\{\Omega_j\}_{j\geq 1}$, then*

$$
\limsup_{j\to\infty} \lambda_p(\Omega_j) \leq \lambda_p(\Omega)\,.
$$

Proof. Fix $\epsilon > 0$. By Lemma 2.4, there exists $u \in C_c^\infty(\Omega)$ such that

$$
\mathcal{R}_p(\Omega, u) \leq \lambda_p(\Omega) + \epsilon\,.
$$

Since u supported in a compact subset of Ω and $\{\Omega_j\}_{j\geq 1}$ is an open cover of Ω, $\mathrm{supp}(u) \subset \Omega_j$ for all sufficiently large j. Moreover, $\Omega_j \subset \Omega$ implies that $d_{\Omega_j}(x) \leq d_\Omega(x)$ for all $x \in \Omega_j$. Hence,

$$
\lambda_p(\Omega_j) \leq \mathcal{R}_p(\Omega_j, u) \leq \mathcal{R}_p(\Omega, u) \leq \lambda_p(\Omega) + \epsilon
$$

for all large enough j. Since ϵ was arbitrary, this proves the lemma. □

2.2. Similarity invariance

For $Q \in O(n)$, $r > 0$ and $y \in \mathbb{R}^n$, we define the similarity transform

$$
T_{r,Q,y}\colon \mathbb{R}^n \to \mathbb{R}^n\,;\ x \mapsto rQx + y\,.
$$

For a set $U \subset \mathbb{R}^n$, we write

$$
T_{r,Q,y}U = rQU + y = \{rQx + y : x \in U\}
$$

to denote the image of U under the similarity transform. Note that

$$T_{r,Q,y}^{-1} = T_{1/r,Q^{-1},-Q^{-1}y/r}\,.$$

If the parameters r, Q, y are understood, we may write simply T.

Since $T_{r,Q,y}$ is a diffeomorphism of $\mathbb{R}^n$, it must be that

$$\partial(T_{r,Q,y}\Omega) = T_{r,Q,y}\partial\Omega\,.$$

It follows that

$$d_{T_{r,Q,y}\Omega}(T_{r,Q,y}x) = r d_\Omega(x)\,, \tag{2.7}$$

which will be useful below.

An all important property of the best constant $\lambda_p(\Omega)$ is that it is invariant under similarity transformations. We will sometimes refer to this as *similarity invariance.*

Lemma 2.6. *If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a similarity transform and $\Omega \subsetneq \mathbb{R}^n$ is open, then*

$$\lambda_p(T\Omega) = \lambda_p(\Omega)$$

and

$$\mathcal{R}_p(\Omega, u) = \mathcal{R}_p(T\Omega, u \circ T^{-1}) \quad \text{for all } u \in \mathcal{D}_0^{1,p}(\Omega)\,.$$

Proof. Let $T = T_{r,Q,y}$. Suppose $u \in \mathcal{D}_0^{1,p}(\Omega)$ and set $v = u \circ T^{-1}$. Then v vanishes in $T\Omega^c$ and by a change of variables

$$\int\limits_{T\Omega} |Dv(z)|^p\,dz = r^n \int\limits_{\Omega} |Dv(Tx)|^p\,dx = r^{n-p} \int\limits_{\Omega} |Du(x)|^p\,dx\,.$$

Consequently, $v \in \mathcal{D}_0^{1,p}(T\Omega)$. Moreover, by (2.7),

$$\sup_{z\in T\Omega} \frac{|v(z)|^p}{d_{T\Omega}(z)^{p-n}} = \sup_{x\in\Omega} \frac{|v(Tx)|^p}{d_{T\Omega}(Tx)^{p-n}} = \sup_{x\in\Omega} \frac{|u(x)|^p}{d_\Omega(x)^{p-n}} r^{n-p}\,.$$

Therefore,

$$\mathcal{R}_p(T\Omega, v) = \mathcal{R}_p(\Omega, u)\,.$$

This proves the second claim.

For any $\epsilon > 0$, there exists $u \in \mathcal{D}_0^{1,p}(\Omega)$ such that

$$\lambda_p(\Omega) \geq \mathcal{R}_p(\Omega, u) - \epsilon\,.$$

By the above, the function $u \circ T^{-1} \in \mathcal{D}_0^{1,p}(T\Omega)$ satisfies $\mathcal{R}_p(\Omega, u) = \mathcal{R}_p(T\Omega, u \circ T^{-1})$ and thus

$$\lambda_p(\Omega) \geq \mathcal{R}_p(T\Omega, u \circ T^{-1}) - \epsilon \geq \lambda_p(T\Omega) - \epsilon \,.$$

Since ϵ was arbitrary it follows that $\lambda_p(\Omega) \geq \lambda_p(T\Omega)$. Switching the roles of Ω and $T\Omega$ in this argument gives the reverse inequality and completes the proof. $\square$

3. Extremals and potentials

In this section, we focus on properties satisfied by extremals and more generally to properties of potentials. We recall that a function $u \in \mathcal{D}_0^{1,p}(\Omega)$ with $u \not\equiv 0$ is an *extremal* provided that $\lambda_p(\Omega) = \mathcal{R}(\Omega, u)$. That is,

$$\lambda_p(\Omega) \left\| \frac{u}{d_\Omega^{1-n/p}} \right\|_\infty^p = \int_\Omega |Du|^p \, dx \,.$$

Below, δ_{x_0} is the Dirac delta distribution at x_0.

Proposition 3.1. *Let $\Omega \subsetneq \mathbb{R}^n$ be an open set. A function $u \in \mathcal{D}_0^{1,p}(\Omega) \setminus \{0\}$ is an extremal if and only if there is $x_0 \in \Omega$ for which u is a weak solution of*

$$\begin{cases} -\Delta_p u = \lambda_p(\Omega) \dfrac{|u(x_0)|^{p-2} u(x_0)}{d_\Omega(x_0)^{p-n}} \delta_{x_0} & \text{in } \Omega \,, \\ \quad u = 0 & \text{on } \partial\Omega \,. \end{cases} \tag{3.1}$$

Proof. Suppose u is a weak solution of (3.1). Then

$$\int_\Omega |Du|^{p-2} Du \cdot Dv \, dx = \lambda_p(\Omega) \frac{|u(x_0)|^{p-2} u(x_0)}{d_\Omega(x_0)^{p-n}} v(x_0)$$

for each $v \in \mathcal{D}_0^{1,p}(\Omega)$. Choosing $v = u$ gives

$$\int_\Omega |Du|^p \, dx = \lambda_p(\Omega) \frac{|u(x_0)|^p}{d_\Omega(x_0)^{p-n}} \leq \lambda_p(\Omega) \left\| \frac{u}{d_\Omega^{1-n/p}} \right\|_\infty^p .$$

We conclude that u is an extremal.

Assume that $u \in \mathcal{D}_0^{1,p}(\Omega)$ is an extremal. By part (*iii*) of Lemma 2.2, there is $x_0 \in \Omega$ with

$$\left\| \frac{u}{d_\Omega^{1-n/p}} \right\|_\infty = \frac{|u(x_0)|}{d_\Omega(x_0)^{1-n/p}} \,.$$

For $t > 0$ and $v \in \mathcal{D}_0^{1,p}(\Omega)$,

$$\begin{aligned}\int_\Omega \left(\frac{|Du + tDv|^p - |Du|^p}{pt}\right) dx &= \frac{1}{pt}\int_\Omega |Du + tDv|^p\, dx - \frac{1}{pt}\int_\Omega |Du|^p\, dx \\ &\geq \frac{\lambda_p(\Omega)}{pt}\left\|\frac{u + tv}{d_\Omega^{1-n/p}}\right\|_\infty^p - \frac{\lambda_p(\Omega)}{pt}\left\|\frac{u}{d_\Omega^{1-n/p}}\right\|_\infty^p \\ &\geq \frac{\lambda_p(\Omega)}{pt}\frac{|u(x_0) + tv(x_0)|^p}{d_\Omega(x_0)^{p-n}} - \frac{\lambda_p(\Omega)}{pt}\frac{|u(x_0)|^p}{d_\Omega(x_0)^{p-n}} \\ &= \frac{\lambda_p(\Omega)}{d_\Omega(x_0)^{p-n}}\left(\frac{|u(x_0) + tv(x_0)|^p - |u(x_0)|^p}{pt}\right).\end{aligned}$$

By routine estimates, we find

$$\int_\Omega |Du|^{p-2}Du \cdot Dv\, dx \geq \lambda_p(\Omega)\frac{|u(x_0)|^{p-2}u(x_0)}{d_\Omega(x_0)^{p-n}}v(x_0)$$

in the limit as $t \to 0^+$. Replacing v by $-v$ gives equality. Thus, u is a weak solution of the boundary value problem (3.1). $\square$

Based on our characterization of extremals, it is natural to consider the following family of functions. Given $y \in \Omega$ we call the unique weak solution $w_y^\Omega \in \mathcal{D}_0^{1,p}(\Omega)$ of the equation

$$\begin{cases} -\Delta_p w_y^\Omega = 0 & \text{in } \Omega \setminus \{y\}\,, \\ w_y^\Omega = 0 & \text{on } \partial\Omega\,, \\ w_y^\Omega(y) = 1\,, \end{cases} \tag{3.2}$$

a *potential* in Ω. That w_y^Ω is a weak solution of (3.2) is equivalent to it being a weak solution of the equation

$$\begin{cases} -\Delta_p w_y^\Omega = \|Dw_y^\Omega\|_p^p \delta_y & \text{in } \Omega\,, \\ w_y^\Omega = 0 & \text{on } \partial\Omega\,. \end{cases}$$

In addition,

$$\|Dw_y^\Omega\|_p \leq \|Dv\|_p$$

among all $v \in \mathcal{D}_0^{1,p}(\Omega)$ which satisfy $v(y) = 1$. Furthermore, w_y^Ω is the unique function in $\mathcal{D}_0^{1,p}(\Omega)$ with this property. We will refer to this variational characterization of w_y^Ω several times below and call any $v \in \mathcal{D}_0^{1,p}(\Omega)$ which satisfies $v(y) = 1$ a *competitor* for w_y^Ω.

Corollary 3.2. *If $\Omega \subsetneq \mathbb{R}^n$ and $y \in \Omega$, then w_y^Ω is identically zero in each connected component of Ω except the one which contains y where w_y^Ω is everywhere positive. In particular, if $u \in \mathcal{D}_0^{1,p}(\Omega)$ is an extremal, then u vanishes identically in all but one component of Ω where it is either everywhere positive or everywhere negative.*

Proof. If Ω' is a connected component of Ω and $y \notin \Omega'$, then $-\Delta_p w_y^\Omega = 0$ in Ω' with $w_y^\Omega|_{\partial\Omega'} = 0$, so $w_y^\Omega \equiv 0$ in Ω'. Let Ω_0 be the connected component of Ω containing y. Then $w_y^\Omega(y) = 1$, $-\Delta_p w_y^\Omega \geq 0$ in Ω_0 and $w_y^\Omega|_{\partial\Omega_0} = 0$. Therefore, $w_y^\Omega > 0$ in Ω_0 by the strong minimum principle. By Proposition 3.1, there exists $y \in \Omega, c \neq 0$ such that $u = cw_y^\Omega$. The assertion for u follows. □

As remarked in our proof above, any extremal is a non-zero multiple of a potential. The next result tells us that not only must any extremal be a potential, but it is in fact sufficient to consider the infimum defining $\lambda_p(\Omega)$ restricted to potentials (even in the case extremals do not exist). Recall that Lemma 2.2 implies that $|u|/d_\Omega^{1-n/p}$ attains a maximum in Ω provided $u \in \mathcal{D}_0^{1,p}(\Omega)$.

Proposition 3.3. *If $u \in \mathcal{D}_0^{1,p}(\Omega) \setminus \{0\}$ and $x_0 \in \Omega$ satisfies*

$$\frac{|u(x_0)|}{d_\Omega(x_0)^{1-n/p}} = \left\| \frac{u}{d_\Omega^{1-n/p}} \right\|_\infty, \tag{3.3}$$

then

$$\mathcal{R}_p(\Omega, u) \geq d_\Omega(x_0)^{p-n} \|Dw_{x_0}^\Omega\|_p^p \geq \mathcal{R}_p(\Omega, w_{x_0}^\Omega). \tag{3.4}$$

Equality holds in the first inequality if and only if $u = u(x_0)w_{x_0}^\Omega$ in which case equality holds also in the second.

Proof. As $u(x_0)^{-1}u$ is a competitor for $w_{x_0}^\Omega$,

$$\|Dw_{x_0}^\Omega\|_p \leq \|D(u(x_0)^{-1}u)\|_p = |u(x_0)|^{-1}\|Du\|_p. \tag{3.5}$$

Moreover, equality holds if and only if $u = u(x_0)w_{x_0}^\Omega$.

Using

$$\left\| \frac{w_{x_0}^\Omega}{d_\Omega^{1-n/p}} \right\|_\infty \geq \frac{w_{x_0}^\Omega(x_0)}{d_\Omega(x_0)^{1-n/p}} = \frac{1}{d_\Omega(x_0)^{1-n/p}},$$

together with (3.3) and (3.5), we find

$$\mathcal{R}_p(\Omega, w_{x_0}^\Omega) \leq d_\Omega(x_0)^{p-n}\|Dw_{x_0}^\Omega\|_p^p \leq \frac{\|Du\|_p^p}{\left(\dfrac{|u(x_0)|}{d_\Omega(x_0)^{1-n/p}}\right)^p} = \mathcal{R}_p(\Omega, u).$$

In addition, equality holds in the second inequality if and only if $u = u(x_0)w^{\Omega}_{x_0}$. In this case, equality also holds in the first inequality by (3.3). $\square$

A direct consequence of Proposition 3.3 is that

$$\lambda_p(\Omega) = \inf_{x\in\Omega} \mathcal{R}_p(\Omega, w^{\Omega}_x)\,. \tag{3.6}$$

Since $\mathcal{R}_p(\Omega, w^{\Omega}_x)$ is a continuous function of $x \in \Omega$, the infimum above is a minimum if the value at some interior point is smaller than any limit either as x approaches the boundary or $|x|$ tends to infinity. Along such sequences, it is useful to study a quantity that is slightly larger than $\mathcal{R}_p(\Omega, w^{\Omega}_x)$ but has the key property that its limit is the same along minimizing sequences.

In order to formalize this idea we introduce the following notation. For a given Ω, define $\mathcal{Y}_\Omega$ as the collection of sequences $\{x_k\}_{k\geq 1} \subset \Omega$ satisfying

$$\liminf_{k\to\infty} |x_k| = \infty \quad \text{or} \quad \limsup_{k\to\infty} d_\Omega(x_k) = 0\,.$$

That is, $\mathcal{Y}_\Omega$ is the set of sequences which eventually leave every compact subset of Ω. Define

$$\Lambda_p(\Omega) := \inf\left\{\liminf_{k\to\infty} d_\Omega(x_k)^{p-n}\|Dw^{\Omega}_{x_k}\|_p^p : \{x_k\}_{k\geq 1} \in \mathcal{Y}_\Omega\right\}. \tag{3.7}$$

By a standard diagonalization argument, it follows that the infimum defining Λ_p is actually a minimum (see Appendix B). The quantity $\Lambda_p(\Omega)$ is analogous to the "Hardy constant at infinity", which is central to the study of the existence of extremals for inequality (1.5) [7,9,12,28,32].

In view of (3.4) and (3.6),

$$\lambda_p(\Omega) \leq \Lambda_p(\Omega)\,.$$

Next, we show that the only way that Ω can lack an extremal is if it is favorable for minimizing sequences to concentrate at the boundary or move away to infinity.

Proposition 3.4. *If*

$$\lambda_p(\Omega) < \Lambda_p(\Omega)\,, \tag{3.8}$$

then Ω has an extremal. Furthermore, all minimizing sequences $\{u_k\}_{k\geq 1}$ for (1.2) with $\|Du_k\|_p = 1$ for all k are precompact in $\mathcal{D}^{1,p}_0(\Omega)$.

Proof. Let $\{u_k\}_{k\geq 1} \subset \mathcal{D}^{1,p}_0(\Omega)$ be a minimizing sequence for $\lambda_p(\Omega)$ with $\|Du_k\|_p = 1$ for each $k \geq 1$. Since $\{u_k\}_{k\geq 1}$ is bounded in $\mathcal{D}^{1,p}_0(\Omega)$ there exists a subsequence that converges weakly in $\mathcal{D}_0^{1,p}(\Omega)$. Upon renaming this subsequence, we assume that the full sequence $\{u_k\}_{k\geq 1}$ converges weakly to u in $\mathcal{D}_0^{1,p}(\Omega)$. In particular, this implies that $u_k \to u$ in $C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n)$ for any $0 < \alpha < 1 - n/p$. It then suffices to show that $u_k \to u$ in $\mathcal{D}_0^{1,p}(\Omega)$ and u is an extremal. We will use (3.8) to exclude the possibility that $\{u_k\}$ either concentrates at the boundary or moves off to infinity.

By Lemma 2.2, there exists a sequence $\{x_k\}_{k\geq 1} \subset \Omega$ such that

$$\left\| \frac{u_k}{d_\Omega^{1-n/p}} \right\|_\infty = \frac{|u_k(x_k)|}{d_\Omega(x_k)^{1-n/p}}\,.$$

We claim that no subsequence of $\{x_k\}_{k\geq 1}$ can belong to $\mathcal{Y}_\Omega$. Indeed, if a subsequence of $\{x_k\}_{k\geq 1}$ belonged to $\mathcal{Y}_\Omega$ the corresponding subsequence of the sequence of potentials $\{w^\Omega_{x_k}\}_{k\geq 1} \in \mathcal{D}_0^{1,p}(\Omega)$ is admissible in the definition of $\Lambda_p(\Omega)$. In view of Proposition 3.3, the existence of such a subsequence would imply

$$\Lambda_p(\Omega) \leq \limsup_{k\to\infty} d_\Omega(x_k)^{p-n} \|Dw^\Omega_{x_k}\|_p^p \leq \lim_{k\to\infty} \mathcal{R}_p(\Omega, u_k) = \lambda_p(\Omega)\,.$$

This contradicts our assumption and proves the claim. Consequently, $\limsup\limits_{k\to\infty} |x_k| < \infty$ and $\liminf\limits_{k\to\infty} d_\Omega(x_k) > 0$. Therefore, $\{x_k\}_{k\geq 1}$ is precompact in Ω.

Passing to another subsequence if necessary, we may assume $\lim_{k\to\infty} x_k = x_0 \in \Omega$. Thus,

$$\lim_{k\to\infty} \left\| \frac{u_k}{d_\Omega^{1-n/p}} \right\|_\infty = \lim_{k\to\infty} \frac{|u_k(x_k)|}{d_\Omega(x_k)^{1-n/p}} = \frac{|u(x_0)|}{d_\Omega(x_0)^{1-n/p}} \leq \left\| \frac{u}{d_\Omega^{1-n/p}} \right\|_\infty .$$

Since $\|Du_k\|_p = 1$ for all k and $\{u_k\}_{k\geq 1}$ is a minimizing sequence, $u(x_0) \neq 0$. We also have

$$\|Du\|_p \leq \lim_{k\to\infty} \|Du_k\|_p = 1 \tag{3.9}$$

by weak convergence. Therefore,

$$\lambda_p(\Omega) = \lim_{k\to\infty} \mathcal{R}_p(\Omega, u_k) = \lim_{k\to\infty} \frac{d_\Omega(x_k)^{p-n}}{|u_k(x_k)|^p} = \frac{d_\Omega(x_0)^{p-n}}{|u(x_0)|^p} \geq \left\| \frac{u}{d_\Omega^{1-n/p}} \right\|_\infty^{-p} \|Du\|_p^p\,.$$

As $\lambda_p(\Omega) \geq \mathcal{R}_p(\Omega, u)$ and $u \in \mathcal{D}_0^{1,p}(\Omega)$, u is an extremal. In addition, equality must hold in (3.9), from which we conclude that $u_k \to u$ in $\mathcal{D}_0^{1,p}(\Omega)$, as weak convergence together with convergence of the L^p-norm implies strong convergence (see [6, Proposition 3.32]). □

4. A complete picture in one dimension

We can now fully describe what happens in the case $n = 1$.

Lemma 4.1. *Assume $n = 1$. Then*

(1) $\lambda_p(\Omega) = 1$, and
(2) Ω has an extremal if and only if Ω contains an unbounded interval.

Proof. Fix $y \in \Omega$ and consider the potential $w_y^\Omega \in \mathcal{D}_0^{1,p}(\Omega)$. By Corollary 3.2, w_y^Ω vanishes in all connected components of Ω except the one containing y. We may assume that the connected component of Ω which contains y is given by (a, b) with $-\infty \leq a < b \leq \infty$ and either $-\infty < a$ or $b < \infty$. Routine computations lead us to the following observations.

(1) If $-\infty < a < b < \infty$, then

$$w_y^\Omega(x) = \begin{cases} 0 & \text{if } x \notin (a,b)\,, \\ \dfrac{x-a}{y-a} & \text{if } a < x \leq y\,, \\ \dfrac{b-x}{b-y} & \text{if } y < x < b\,, \end{cases} \quad \text{and} \quad \|Dw_y^\Omega\|_p^p = \frac{1}{(y-a)^{p-1}} + \frac{1}{(b-y)^{p-1}}\,.$$

(2) If $a = -\infty$, then

$$w_y^\Omega(x) = \begin{cases} 1 & \text{if } x \leq y\,, \\ \dfrac{b-x}{b-y} & \text{if } y < x < b\,, \\ 0 & \text{if } x \geq b\,, \end{cases} \quad \text{and} \quad \|Dw_y^\Omega\|_p^p = \frac{1}{(b-y)^{p-1}}\,.$$

(3) if $b = \infty$, then

$$w_y^\Omega(x) = \begin{cases} 0 & \text{if } x \leq a\,, \\ \dfrac{x-a}{y-a} & \text{if } a < x < y\,, \\ 1 & \text{if } x \geq b\,, \end{cases} \quad \text{and} \quad \|Dw_y^\Omega\|_p^p = \frac{1}{(y-a)^{p-1}}\,.$$

Note that in the cases of an unbounded interval, $\mathcal{R}_p(\Omega, w_y^\Omega) = 1$, while in the bounded case,

$$\mathcal{R}_p(\Omega, w_y^\Omega) = 1 + \left(\frac{\min\{y-a, b-y\}}{\max\{y-a, b-y\}}\right)^{p-1} > 1\,.$$

Nevertheless, this expression for $\mathcal{R}_p(\Omega, w_y^\Omega)$ can be made arbitrarily close to 1 by letting y approach either a or b. In view of Proposition 3.3, we conclude that in all cases $\lambda_p(\Omega) = 1$. Furthermore, $\Omega \ni y \mapsto \mathcal{R}_p(\Omega, w_y^\Omega)$ attains the value 1 if and only if Ω contains an unbounded interval. $\square$

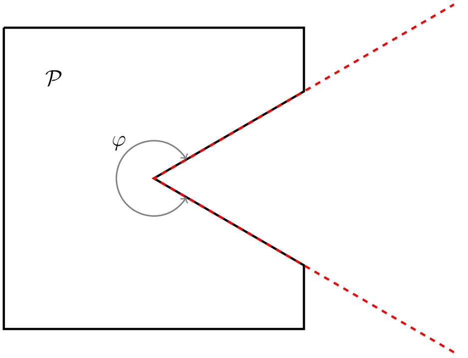

Fig. 1. A non-convex polygon $\mathcal{P}$ which is fully supported by an infinite sector with opening angle φ with one such supporting sector depicted in red. Equivalently, $\mathcal{P}$ satisfies a uniform (infinite) exterior cone condition. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

5. Universal bounds via supporting sets

In this section, we turn to the question of universal upper and lower bounds for $\lambda_p(\Omega)$. We first notice that if $\Omega \subseteq \Omega'$ then any $u \in \mathcal{D}_0^{1,p}(\Omega)$ also belongs to $\mathcal{D}_0^{1,p}(\Omega')$. In this case, we also have $d_\Omega \leq d_{\Omega'}$ so that

$$\left\| \frac{u}{d_{\Omega'}^{1-n/p}} \right\|_\infty \leq \left\| \frac{u}{d_{\Omega}^{1-n/p}} \right\|_\infty .$$

Therefore, $\mathcal{R}_p(\Omega, u) \leq \mathcal{R}_p(\Omega', u)$ provided that $u \not\equiv 0$. In certain situations we can in fact conclude that $\mathcal{R}_p(\Omega, u) = \mathcal{R}_p(\Omega', u)$. To this end, we introduce the following notion.

Definition 5.1. Suppose $\Omega, \Omega' \subsetneq \mathbb{R}^n$. We say that Ω' *supports* Ω at $x \in \partial\Omega$ if

$$\Omega \subseteq \Omega' \qquad \text{and} \qquad x \in \partial\Omega' .$$

We say that Ω is *fully supported* by Ω', if for each $x \in \partial\Omega$ there exists a similarity transformation T so that $T\Omega'$ supports Ω at x.

Remark 5.2. A set Ω is fully supported by $\mathbb{R}^n_+$ if and only if Ω is convex. In fact, the notion of supporting sets is intended as a generalization of this property of convex sets. An example of a set fully supported by an infinite sector is depicted in Fig. 1.

As we shall see, the following proposition can be useful both in proving upper and lower bounds for λ_p.

Proposition 5.3. *Assume Ω' supports Ω at $y_0 \in \partial\Omega$. If $u \in \mathcal{D}_0^{1,p}(\Omega)$ and $x_0 \in \Omega$ satisfies*

$$\left\| \frac{u}{d_{\Omega}^{1-n/p}} \right\|_\infty = \frac{|u(x_0)|}{d_\Omega(x_0)^{1-n/p}} \quad \textit{and} \quad |x_0 - y_0| = d_\Omega(x_0) ,$$

then

$$\mathcal{R}_p(\Omega, u) = \mathcal{R}_p(\Omega', u)\,.$$

Furthermore, if Ω is fully supported by Ω' then

$$\lambda_p(\Omega) \geq \lambda_p(\Omega')\,.$$

Proof. As $u \in \mathcal{D}_0^{1,p}(\Omega')$ and $d_\Omega \leq d_{\Omega'}$,

$$\frac{|u(x)|}{d_\Omega(x)^{1-n/p}} \geq \frac{|u(x)|}{d_{\Omega'}(x)^{1-n/p}} \quad \text{for all } x \in \Omega\,. \tag{5.1}$$

Since $\Omega \subseteq \Omega'$, $y_0 \in \partial\Omega \cap \partial\Omega'$, and $d_\Omega(x_0) = |x_0 - y_0|$ it follows that $d_{\Omega'}(x_0) = |x_0 - y_0|$. Therefore, equality holds in (5.1) for $x = x_0$. Consequently,

$$\frac{|u(x_0)|}{d_{\Omega'}(x_0)^{1-n/p}} = \frac{|u(x_0)|}{d_\Omega(x_0)^{1-n/p}} = \left\| \frac{u}{d_\Omega^{1-n/p}} \right\|_\infty = \left\| \frac{u}{d_{\Omega'}^{1-n/p}} \right\|_\infty .$$

We conclude that $\mathcal{R}_p(\Omega, u) = \mathcal{R}_p(\Omega', u)$.

If Ω is fully supported by Ω', then for each $u \in \mathcal{D}_0^{1,p}(\Omega)$ we can apply Lemma 2.2 and the above reasoning to deduce that there exists a similarity transform T_u such that $u \in \mathcal{D}_0^{1,p}(T_u\Omega')$ and $\mathcal{R}_p(\Omega, u) = \mathcal{R}_p(T_u\Omega', u)$. By Lemma 2.6, $\mathcal{R}_p(T_u\Omega', u) = \mathcal{R}_p(\Omega', u \circ T_u^{-1})$. Therefore,

$$\lambda_p(\Omega') \leq \inf_{u \in \mathcal{D}_0^{1,p}(\Omega)\setminus\{0\}} \mathcal{R}_p(\Omega', u \circ T_u^{-1}) = \inf_{u \in \mathcal{D}_0^{1,p}(\Omega)\setminus\{0\}} \mathcal{R}_p(\Omega, u) = \lambda_p(\Omega)\,. \quad \square$$

We obtain the following corollaries which follow directly from Proposition 5.3.

Corollary 5.4. *For any open $\Omega \subsetneq \mathbb{R}^n$,*

$$\lambda_p(\mathbb{R}^n \setminus \{0\}) \leq \lambda_p(\Omega) \leq \lambda_p(B_1)\,.$$

Proof. The lower bound follows from the translation invariance of λ_p and noting that if $x \in \partial\Omega$ then $\mathbb{R}^n \setminus \{x\}$ supports Ω at x. This implies that every $\Omega \subsetneq \mathbb{R}^n$ is fully supported by $\mathbb{R}^n \setminus \{0\}$. The upper bound follows by the similarity invariance combined with the observation that B_1 is fully supported by any set Ω. Indeed, for any $y_0 \in \Omega$, Ω supports the ball $B_{d_\Omega(y_0)}(y_0)$ at x_0 where $|x_0 - y_0| = d_\Omega(y_0)$. Since B_1 is rotationally invariant, we conclude that B_1 is fully supported by Ω (Fig. 2). $\square$

Corollary 5.5. *If $\Omega \subsetneq \mathbb{R}^n$ is convex then*

$$\lambda_p(\mathbb{R}^n_+) \leq \lambda_p(\Omega) \leq \lambda_p(B_1)\,.$$

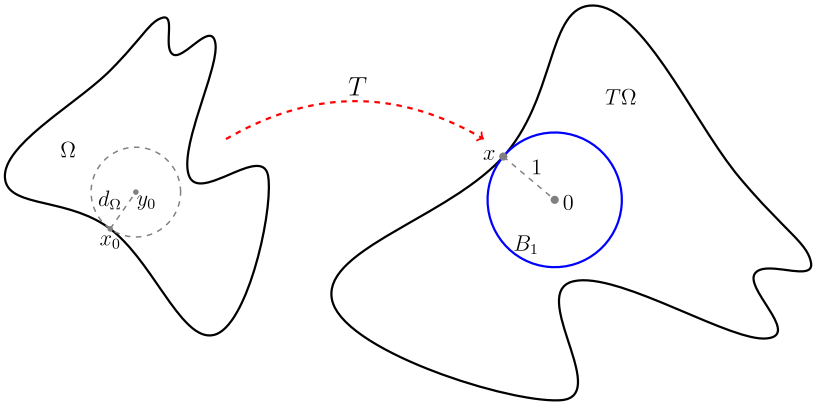

Fig. 2. A schematic description of the fact that every open set Ω fully supports B_1. Given $\Omega, y_0 \in \Omega, x \in \partial B_1$ a similarity transform T is constructed satisfying that $Ty_0 = 0$ and $T\Omega$ supports B_1 at x.

Proof. The upper bound was proved above, so we focus on the lower bound. As noted in Remark 5.2, Ω is fully supported by $\mathbb{R}^n_+$. We can therefore conclude that $\lambda_p(\Omega) \geq \lambda_p(\mathbb{R}^n_+)$. $\square$

6. Proof of Theorem A and Theorem B

Our next aim is to prove Theorem A and Theorem B. We first establish the lemma below, where we study the particular cases $\Omega = \mathbb{R}^n_+$ and $\Omega = \mathbb{R}^n \setminus \{0\}$. It turns out that both the value of $\lambda_p(\Omega)$ and extremals for these domains can be expressed in terms of the sharp constant in Morrey's inequality and Morrey extremals, respectively. Here we call $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ a *Morrey extremal* if v is not constant throughout $\mathbb{R}^n$ and equality holds in (2.1) with $C = C_{n,p}$.

Lemma 6.1. *The following assertions hold.*

(i) $\lambda_p(\mathbb{R}^n_+) = 2^{n-1}C_{n,p}^{-p}$.

(ii) If $y \in \mathbb{R}^n_+$*, then* $w_y^{\mathbb{R}^n_+}$ *is an extremal for* $\lambda_p(\mathbb{R}^n_+)$*. Furthermore, the odd extension of* $w_y^{\mathbb{R}^n_+}$ *through the hyperplane* $x_n = 0$ *is a Morrey extremal.*

(iii) $\lambda_p(\mathbb{R}^n \setminus \{0\}) = C_{n,p}^{-p}$.

(iv) If $y \in \mathbb{R}^n \setminus \{0\}$*, then* $w_y^{\mathbb{R}^n\setminus\{0\}}$ *is an extremal for* $\lambda_p(\mathbb{R}^n \setminus \{0\})$ *and a Morrey extremal.*

Proof. *Part 1: (i) and (ii).* We claim that $\mathcal{R}_p(\mathbb{R}^n_+, w_y^{\mathbb{R}^n_+})$ is independent of $y \in \mathbb{R}^n_+$. Note that if $y = (y', y_n) \in \mathbb{R}^n_+$, then

$$\tilde{w}(x) = w_y^{\mathbb{R}^n_+}((y', 0) + y_n x) \in \mathcal{D}_0^{1,p}(\mathbb{R}^n_+)$$

satisfies

$$\begin{cases} -\Delta_p \tilde{w} = 0 & \text{in } \mathbb{R}^n_+ \setminus \{e_n\}\,, \\ \tilde{w} = 0 & \text{on } \partial\mathbb{R}^n_+\,, \\ \tilde{w}(e_n) = 1\,. \end{cases} \tag{6.1}$$

That is, $\tilde{w} = w_{e_n}^{\mathbb{R}^n_+}$. Lemma 2.6 then implies that $\mathcal{R}_p(\mathbb{R}^n_+, w_y^{\mathbb{R}^n_+}) = \mathcal{R}_p(\mathbb{R}^n_+, w_{e_n}^{\mathbb{R}^n_+})$. By (3.6), $\lambda_p(\mathbb{R}^n_+) = \mathcal{R}_p(\mathbb{R}^n_+, w_y^{\mathbb{R}^n_+})$. We conclude that $w_y^{\mathbb{R}^n_+}$ is an extremal.

Again fix $y = (y', y_n) \in \mathbb{R}^n_+$. By [26, Theorem 2.4], there exists a Morrey extremal u satisfying

$$u(y) = 1\,, \quad u((y', -y_n)) = -1\,, \quad \text{and} \quad [u]_{1-n/p} = \frac{|u(y) - u((y', -y_n))|}{|y - (y', -y_n)|^{1-n/p}}\,. \tag{6.2}$$

Here $[u]_{1-n/p}$ denotes the $1 - n/p$ Hölder seminorm of u. Moreover, $-\Delta_p u = 0$ in $\mathbb{R}^n \setminus \{y, (y', -y_n)\}$. By [25, Theorem 1.1], u is antisymmetric across the hyperplane $x_n = 0$. In particular, u vanishes on this hyperplane. It follows that $u|_{\mathbb{R}^n_+} = w_y^{\mathbb{R}^n_+}$. Therefore, the odd extension of $w_y^{\mathbb{R}^n_+}$ through this hyperplane is u.

Let us write $w = w_{e_n}^{\mathbb{R}^n_+}$ and u for the odd reflection of w through $x_n = 0$. As noted above, u is a Morrey extremal satisfying (6.2) with $y = e_n$. As a result,

$$\begin{aligned} 1 &= \frac{|w(e_n)|}{d_{\mathbb{R}^n_+}(e_n)^{1-n/p}} \\ &\le \left\| \frac{w}{d_{\mathbb{R}^n_+}^{1-n/p}} \right\|_\infty \\ &= \sup_{x \in \mathbb{R}^n_+} \frac{|w(x)|}{|x_n|^{1-n/p}} \\ &= 2^{-n/p} \sup_{x \in \mathbb{R}^n_+} \frac{|u(x) - u((x', -x_n))|}{|x - (x', -x_n)|^{1-n/p}} \\ &\le 2^{-n/p} [u]_{1-n/p} \\ &= 1 \end{aligned}$$

and

$$\lambda_p(\mathbb{R}^n_+) = \left\| \frac{w}{d_{\mathbb{R}^n_+}^{1-n/p}} \right\|_\infty^{-p} \|Dw\|_p^p = 2^{n-1} [u]_{1-n/p}^{-p} \|Du\|_p^p = 2^{n-1} C_{n,p}^{-p}\,.$$

Part 2: (iii) and (iv). As we argued above, we can deduce that $\mathcal{R}_p(\mathbb{R}^n \setminus \{0\}, w_y^{\mathbb{R}^n \setminus \{0\}})$ does not depend on $y \in \mathbb{R}^n \setminus \{0\}$ by showing

$$w_{e_n}^{\mathbb{R}^n \setminus \{0\}}(x) = w_y^{\mathbb{R}^n \setminus \{0\}}(|y| Q x)$$

for any $Q \in O(n)$ with $Qe_n = y/|y|$. Again, equation (3.6) implies that $\lambda_p(\mathbb{R}^n \setminus \{0\}) = \mathcal{R}_p(\mathbb{R}^n \setminus \{0\}, w_y^{\mathbb{R}^n \setminus \{0\}})$. It follows that $w_y^{\mathbb{R}^n \setminus \{0\}}$ is an extremal.

Let u be a Morrey extremal with

$$u(0) = 0\,, \quad u(y) = 1\,, \quad \text{and} \quad [u]_{1-n/p} = \frac{|u(y)|}{|y|^{1-n/p}}\,.$$

As $u \in \mathcal{D}_0^{1,p}(\mathbb{R}^n \setminus \{0\})$ and $-\Delta_p u = 0$ in $\mathbb{R}^n \setminus \{0, y\}$, $u = w_y^{\mathbb{R}^n \setminus \{0\}}$. It follows that

$$\lambda_p(\mathbb{R}^n \setminus \{0\}) = \left\| \frac{u}{d_{\mathbb{R}^n \setminus \{0\}}^{1-n/p}} \right\|_\infty^{-p} \|Du\|_p^p = [u]_{1-n/p}^{-p} \|Du\|_p^p = C_{n,p}^{-p}\,. \quad \square$$

We will now prove Theorems A and B by applying various of the assertions derived above.

Proof of Theorem A. We first observe that $\mathbb{R}_+^n$ can be exhausted by $\{B_j(je_n)\}_{j\geq 1}$. Therefore, by Lemma 2.5 and the similarity invariance

$$\lambda_p(B_1) = \lim_{j\to\infty} \lambda_p(B_j(je_n)) \leq \lambda_p(\mathbb{R}_+^n)\,.$$

The theorem now follows from this inequality together with Lemma 6.1 and Corollary 5.4. $\square$

Proof of Theorem B. (1) Assume Ω is convex. By Theorem A and Corollary 5.5,

$$\lambda_p(\mathbb{R}_+^n) \leq \lambda_p(\Omega) \leq \lambda_p(B_1) \leq \lambda_p(\mathbb{R}_+^n)\,.$$

That is, $\lambda_p(\Omega) = \lambda_p(\mathbb{R}_+^n)$.

Now suppose $u \in \mathcal{D}_0^{1,p}(\Omega)$ is an extremal. By Corollary 3.2, we may assume that $u > 0$ in Ω. Let us also choose $x_0 \in \Omega$ with

$$\left\| \frac{u}{d_\Omega^{1-n/p}} \right\|_\infty = \frac{u(x_0)}{d_\Omega(x_0)^{1-n/p}}$$

and $y_0 \in \partial\Omega$ with $d_\Omega(x_0) = |x_0 - y_0|$. Since Ω is convex, there exists a halfspace Π such that

$$\Omega \subseteq \Pi \quad \text{and} \quad y_0 \in \partial\Pi\,. \tag{6.3}$$

Then $u \in \mathcal{D}_0^{1,p}(\Pi)$ and $d_\Pi(x_0) = |x_0 - y_0|$. By Proposition 5.3,

$$\mathcal{R}_p(\Pi, u) = \mathcal{R}_p(\Omega, u) = \lambda_p(\Omega) = \lambda_p(\mathbb{R}_+^n) = \lambda_p(\Pi)\,.$$

Hence, u is also an extremal for Π. If $\Omega \subsetneq \Pi$, then u vanishes in the open set $\Pi \setminus \overline{\Omega}$. This contradicts Corollary 3.2 since Π is connected. As a result, Ω does not admit an extremal unless it is a halfspace.

(2) Suppose $\Omega \subset \mathbb{R}^n$ is open. By translation invariance, we may assume $x_0 = 0$. We have already established that $\lambda_p(\Omega \setminus \{0\}) \geq \lambda_p(\mathbb{R}^n \setminus \{0\})$ in Theorem A. As for the upper bound, $\mathbb{R}^n \setminus \{0\}$ is exhausted by $\{j\Omega \setminus \{0\}\}_{j\geq 1}$, therefore Lemma 2.5 implies that $\lambda_p(\Omega \setminus \{0\}) \leq \lambda_p(\mathbb{R}^n \setminus \{0\})$. Arguing as in (1), we can show that if $u \in \mathcal{D}_0^{1,p}(\Omega \setminus \{0\})$ is an extremal, it is also an extremal in $\mathbb{R}^n \setminus \{0\}$ which vanishes at any point in the complement of $\Omega \setminus \{0\}$. As $\mathbb{R}^n \setminus \{0\}$ is connected, u would have to vanish identically unless $\Omega = \mathbb{R}^n$. We conclude that no such extremal exists for $\Omega \setminus \{0\}$ unless $\Omega = \mathbb{R}^n$.

(3) Let $K \subset \mathbb{R}^n$ be compact and nonempty. By translation invariance, we may assume $0 \in K$. According to Theorem A, $\lambda_p(\mathbb{R}^n \setminus K) \geq \lambda_p(\mathbb{R}^n \setminus \{0\})$. Since $\mathbb{R}^n \setminus \{0\}$ is exhausted by $\{j^{-1}(\mathbb{R}^n \setminus K)\}_{j\geq 1}$, Lemma 2.5 gives $\lambda_p(\mathbb{R}^n \setminus K) \leq \lambda_p(\mathbb{R}^n \setminus \{0\})$. Similar to how we reasoned in (2) and (3), we may conclude that the only way for $\mathbb{R}^n \setminus K$ to have an extremal is if $K = \{0\}$. $\square$

Remark 6.2. The method used to prove the nonexistence of extremals in Theorem B can also be used to show: if Ω has an extremal and $\lambda_p(\Omega) = \lambda_p(\mathbb{R}^n \setminus \{0\})$, then $\Omega = \mathbb{R}^n \setminus \{x_0\}$ for some x_0. The key observation here is that Ω is fully supported by $\mathbb{R}^n \setminus \{0\}$. We leave the details to the reader.

Remark 6.3. Modulo symmetry, the only connected sets whose extremals are restrictions of Morrey extremals are those covered by Lemma 6.1. Indeed, if $\Omega \subsetneq \mathbb{R}^n$ is a connected set and $u \in \mathcal{D}_0^{1,p}(\Omega)$ is an extremal for $\lambda_p(\Omega)$ which is the restriction of a Morrey extremal v, then $\partial\Omega$ is the zero level set of this Morrey extremal. By [26], each level set of a Morrey extremal is either a bounded convex set, a halfspace, or the complement of a bounded convex set. If Ω is convex and not a halfspace then Ω does not admit an extremal, contradicting that $u = v|_\Omega$ is an extremal. Similarly if Ω is the complement of a compact set then Ω admits an extremal if and only if this compact set is a singleton. Therefore, halfspaces and $\mathbb{R}^n \setminus \{x_0\}$ are the only connected sets having extremals given by restrictions of Morrey extremals.

7. Dilation invariant domains

We will say $\mathcal{C} \subset \mathbb{R}^n$ is a *cone* if

$$t\mathcal{C} = \mathcal{C} \text{ for each } t > 0\,.$$

That is, $\mathcal{C}$ is a cone if it is dilation invariant. It is evident that the complement of a cone is also a cone. It is also easy to check that if a cone is convex, then it is closed under vector addition. In this section, we will give sufficient conditions under which a cone admits an extremal. We will assume throughout that $\Omega \subsetneq \mathbb{R}^n$ is open and nonempty and $n \geq 2$.

Proposition 7.1. *Suppose $\Omega \subsetneq \mathbb{R}^n$ is a cone. Further assume there is a compact*

$$K \subset \{x \in \Omega : d_\Omega(x) = 1\}$$

such that for each $x \in K$ there is $y \in \mathbb{R}^n$ with

$$\begin{cases} x + y \in K\,, \\ y + \Omega \subset \Omega\,. \end{cases}$$

Then Ω admits an extremal u for which $|u|/d_\Omega^{1-n/p}$ attains its maximum in K.

Proof. *Step 1.* Let $\{u_k\}_{k\geq 1} \subset \mathcal{D}_0^{1,p}(\Omega)$ with

$$\lambda_p(\Omega) = \lim_{k\to\infty} \mathcal{R}_p(\Omega, u_k)$$

and choose $\{z_k\}_{k\geq 1} \subset \Omega$ which satisfies

$$\left\| \frac{u_k}{d_\Omega^{1-n/p}} \right\|_\infty = \frac{|u_k(z_k)|}{d_\Omega(z_k)^{1-n/p}}$$

for each $k \geq 1$. Since Ω is a cone, we may define $v_k \in \mathcal{D}_0^{1,p}(\Omega)$ by setting

$$v_k(x) = \frac{u_k(d_\Omega(z_k)x)}{d_\Omega(z_k)^{1-n/p}}\,.$$

By Lemma 2.6, $\mathcal{R}_p(\Omega, v_k) = \mathcal{R}_p(\Omega, u_k)$. Therefore, $\{v_k\}_{k\geq 1}$ is also a minimizing sequence.

Next we set

$$x_k = \frac{z_k}{d_\Omega(z_k)}$$

and note that

$$\left\| \frac{v_k}{d_\Omega^{1-n/p}} \right\|_\infty = \frac{|v_k(x_k)|}{d_\Omega(x_k)^{1-n/p}}\,.$$

By (2.7), d_Ω is positively homogeneous as Ω is dilation invariant. It follows that $d_\Omega(x_k) = 1$, so $|v_k|/d_\Omega^{1-n/p}$ attains its maximum in the set $\{x \in \Omega : d_\Omega(x) = 1\}$.

Step 2. By assumption, there exists $y_k \in \mathbb{R}^n$ for each k with

$$\begin{cases} x_k + y_k \in K\,, \\ y_k + \Omega \subset \Omega\,. \end{cases}$$

Setting

$$w_k(x) = v_k(x - y_k)\,,$$

we observe that $w_k \in \mathcal{D}_0^{1,p}(\Omega)$ and

$$\|Dw_k\|_p = \|Dv_k\|_p\,. \tag{7.1}$$

Moreover, since $y_k + \Omega \subseteq \Omega$

$$d_\Omega(x - y_k) = d_{y_k+\Omega}(x) \leq d_\Omega(x)$$

for all $x \in y_k + \Omega$. Thus,

$$\frac{|w_k(x)|}{d_\Omega(x)^{1-n/p}} \leq \frac{|w_k(x)|}{d_\Omega(x-y_k)^{1-n/p}} = \frac{|v_k(x-y_k)|}{d_\Omega(x-y_k)^{1-n/p}} \leq \left\|\frac{v_k}{d_\Omega^{1-n/p}}\right\|_\infty \tag{7.2}$$

for all $x \in \operatorname{supp}(w_k) \subseteq y_k + \Omega$ with equality when $x = x_k + y_k \in K$. Combining (7.1) and (7.2) leads to

$$\mathcal{R}_p(\Omega, w_k) = \mathcal{R}_p(\Omega, v_k)\,.$$

It follows that $\{w_k\}_{k\geq 1}$ is a minimizing sequence satisfying that $|w_k|/d_\Omega^{1-n/p}$ is attained in K for each $k \geq 1$.

Step 3. Since K is compact, $\{x_k + y_k\}_{k\geq 1}$ has a subsequence which converges to a limit $x \in K$. Arguing as at the end of the proof of Proposition 3.4, we deduce that along this subsequence w_k converges locally uniformly and in $\mathcal{D}_0^{1,p}(\Omega)$ to a function u which is an extremal for $\lambda_p(\Omega)$. As $|w_k|/d_\Omega^{1-n/p}$ was maximized in K for each $k \geq 1$, it follows that this property remains true for $|u|/d_\Omega^{1-n/p}$. We leave the details to the reader. □

We will verify that Ω satisfies the hypotheses of Proposition 7.1 whenever its complement is a convex cone. First we will need to justify the following lemma.

Lemma 7.2. *Assume Ω^c is a convex cone, $x_0 \in \Omega$, and $y_0 \in \partial\Omega$ with $d_\Omega(x_0) = |x_0 - y_0|$. Then*

$$-y_0 + \Omega \subset \Omega \quad \textit{and} \quad d_\Omega(x_0 - y_0) = |x_0 - y_0|\,.$$

Proof. Let $x \in \Omega$. We aim to prove that $-y_0 + x \in \Omega$. If this were not the case, then $x = (-y_0 + x) + y_0 \in \Omega^c$ since Ω^c is a convex cone. However, this would be a contradiction. As $x \in \Omega$ was arbitrary, we conclude that $-y_0 + \Omega \subset \Omega$.

Since $0 \in \partial\Omega$,

$$d_\Omega(x_0 - y_0) \leq |x_0 - y_0|\,.$$

We are left to verify the opposite inequality. To this end, we recall that

$$(x_0 - y_0) \cdot (x - y_0) \leq 0$$

for all $x \in \Omega^c$ as Ω^c is closed and convex [6, Theorem 5.2]. Choosing $x = 2y_0$ and $x = 0$ gives $(x_0 - y_0) \cdot y_0 = 0$ and hence

$$\Omega^c \subset \Pi := \{x \in \mathbb{R}^n : (x_0 - y_0) \cdot x \leq 0\}.$$

It follows that

$$d_\Omega(x_0 - y_0) = \inf_{w \in \Omega^c} |x_0 - y_0 - w| \geq \inf_{w \in \Pi} |x_0 - y_0 - w|.$$

And as $(x_0 - y_0) \cdot w \leq 0$ for $w \in \Pi$,

$$|x_0 - y_0 - w|^2 = |x_0 - y_0|^2 + |w|^2 - 2(x_0 - y_0) \cdot w \geq |x_0 - y_0|^2 + |w|^2 \geq |x_0 - y_0|^2.$$

We conclude that

$$d_\Omega(x_0 - y_0) \geq \inf_{w \in \Pi} |x_0 - y_0 - w| = |x_0 - y_0|. \quad \square$$

Theorem 7.3. *Assume that Ω^c is a convex cone. Then Ω has an extremal.*

Proof. Note that

$$K = \{x \in \Omega : d_\Omega(x) = 1\} \cap \overline{B_1}$$

is a compact subset of $\{x \in \Omega : d_\Omega(x) = 1\}$. If $x \in \Omega$ and $y \in \partial\Omega$ with $d_\Omega(x) = |x - y| = 1$, the previous lemma shows that

$$\begin{cases} x - y \in K, \\ -y + \Omega \subset \Omega. \end{cases}$$

We conclude the proof by appealing to Proposition 7.1. $\square$

Theorem 7.3 implies the existence of extremals proved in Lemma 6.1 since the complement of $\mathbb{R}^n_+$ and $\mathbb{R}^n \setminus \{0\}$ are convex cones. We will consider another family of examples below.

Example 7.4. For $\varphi \in (0, \pi)$, set

$$\mathcal{C}^n_\varphi = \left\{x = (x', x_n) \in \mathbb{R}^n : x_n > \cot(\varphi)|x'|\right\}.$$

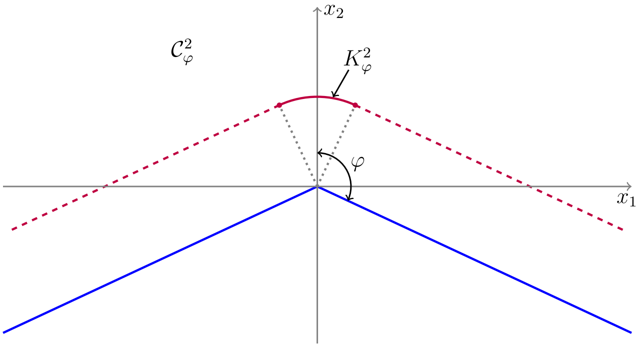

Fig. 3. A domain $\mathcal{C}_\varphi^2 \subset \mathbb{R}^2$ for $\varphi \in (\pi/2, \pi]$. The domain is the region of the plane above the blue curve. The purple curve indicates the set $d_{\mathcal{C}_\varphi^2} = 1$, which is the union of three curves; two rays (dashed) and a circular arc K_φ^2 (solid). In higher dimensions the set $\mathcal{C}_\varphi^n$ can be obtained by rotation of $\mathcal{C}_\varphi^2$ around the axis of symmetry.

We will also consider the limiting case as the angle φ tends to π

$$\mathcal{C}_\pi^n = \mathbb{R}^n \setminus \{x = (x', x_n) \in \mathbb{R}^n : |x'| = 0, x_n \leq 0\}\,.$$

The set $\mathcal{C}_\varphi^n$ is a circular cone around the positive x_n-axis with opening angle φ measured relative to the x_n-axis (see Fig. 3). If $\varphi < \pi/2$, then $\mathcal{C}_\varphi^n$ is a convex cone which is not a halfspace. By Theorem B, $\lambda_p(\mathcal{C}_\varphi^n) = \lambda_p(\mathbb{R}_+^n)$, and no extremals exist. Alternatively, if $\varphi \geq \pi/2$, the set $\mathcal{C}_\varphi^n$ is the complement of a convex cone. Theorem 7.3 therefore implies that $\mathcal{C}_\varphi^n$ has an extremal for every $\varphi \in [\pi/2, \pi]$.

The family $\mathcal{C}_\varphi^n$ is of particular interest as it forms a natural class of limiting profiles that can occur in the analysis developed in Sections 8 and 11 (especially, for $n = 2$). With this in mind, we note for future reference that $[\pi/2, \pi] \ni \varphi \mapsto \lambda_p(\mathcal{C}_\varphi^n)$ is strictly decreasing.

Lemma 7.5. *For $\pi/2 \leq \varphi_1 < \varphi_2 \leq \pi$,*

$$\lambda_p(\mathcal{C}_{\varphi_2}^n) < \lambda_p(\mathcal{C}_{\varphi_1}^n)\,.$$

Proof. Given $x_0 \in \partial\mathcal{C}_{\varphi_1}^n$ we observe that $\mathcal{C}_{\varphi_1}^n \subset x_0 + \mathcal{C}_{\varphi_2}^n$ and $x_0 \in \partial(x_0 + \mathcal{C}_{\varphi_2}^n)$. Therefore, $\mathcal{C}_{\varphi_1}^n$ is fully supported by $\mathcal{C}_{\varphi_2}^n$; Proposition 5.3 implies that $\lambda_p(\mathcal{C}_{\varphi_2}^n) \leq \lambda_p(\mathcal{C}_{\varphi_1}^n)$. By Proposition 7.1, $\mathcal{C}_{\varphi_1}^n$ has an extremal $u \in \mathcal{D}_0^{1,p}(\mathcal{C}_{\varphi_1}^n)$. If $\lambda_p(\mathcal{C}_{\varphi_2}^n) = \lambda_p(\mathcal{C}_{\varphi_1}^n)$, then u would be an extremal also in $\mathcal{C}_{\varphi_2}^n$. As u vanishes in $\mathcal{C}_{\varphi_2}^n \setminus \overline{\mathcal{C}_{\varphi_1}^n} \neq \emptyset$, this contradicts Corollary 3.2. Thus, $\lambda_p(\mathcal{C}_{\varphi_2}) < \lambda_p(\mathcal{C}_{\varphi_1})$. $\square$

Remark 7.6. It is also possible to show that $[\pi/2, \pi] \ni \varphi \mapsto \lambda_p(\mathcal{C}_\varphi^n)$ is continuous.

8. Local and global analysis of $\Lambda_p(\Omega)$

In this section, we will focus on understanding how Λ_p compares to the value of λ_p in simpler model sets. The typical model sets are dilation invariant sets such as halfspaces or cones, which were discussed in Section 7. Our aim is to prove upper bounds on $\Lambda_p(\Omega)$, and in turn on $\lambda_p(\Omega)$, in terms of blow-up/blow-down limits of Ω. We will also establish lower bounds on $\Lambda_p(\Omega)$, by studying the asymptotic behavior of $\mathcal{R}_p(\Omega, w_{x_k}^{\Omega})$ when $\{x_k\}_{k\geq 1} \subset \Omega$ does not have a limit in Ω. The combination of these lower bounds and Proposition 3.4 will be one our main tools in proving the existence of extremals for $\lambda_p(\Omega)$.

For a given $\Omega \subsetneq \mathbb{R}^n$, we will deduce upper bounds on $\Lambda_p(\Omega)$ by making the following observation: if we can locally approximate a dilation invariant set $\mathcal{C}$ to arbitrary precision by a sequence of dilations and translations acting on Ω, then a trial sequence for $\Lambda_p(\Omega)$ can be constructed from an almost minimizer of $\lambda_p(\mathcal{C})$. There are two important cases to have in mind. The first is zooming in at a boundary point of Ω, in which case $\mathcal{C}$ is the blow-up of Ω at this point (an example is shown in Fig. 4). The second case occurs when zooming out so far that only asymptotic features of Ω remain visible and are described by a cone $\mathcal{C}$.

Lemma 8.1. *Let $\mathcal{C} \subsetneq \mathbb{R}^n$ be an open cone. Assume for all small enough $\delta \in (0,1)$ there are $t = t(\delta) > 0$ and $y = y(\delta) \in \mathbb{R}^n$ such that*

$$(t\Omega - y) \cap B_1 \supset \{x \in \mathcal{C} : d_{\mathcal{C}}(x) > \delta\} \cap B_1$$

and for each $x \in \mathcal{C} \cap B_{1/2}$

$$\lim_{\delta\to 0} d_{t\Omega - y}(x) = d_{\mathcal{C}}(x) \, .$$

Then

$$\Lambda_p(\Omega) \leq \lambda_p(\mathcal{C}) \, .$$

Proof. Fix $\epsilon > 0$. By Lemma 2.4, there exists an $r > 0$ and a function $u \in C_c^\infty(\mathcal{C} \cap B_r)$ so that

$$\mathcal{R}_p(\mathcal{C}, u) \leq \lambda_p(\mathcal{C}) + \epsilon \, .$$

Since both $\mathcal{C}$ and the Rayleigh quotient are invariant under dilations, we can ensure that $r = 1/2$. By Lemma 2.2, there is $x^* \in \mathcal{C} \cap B_{1/2}$ such that

$$\frac{|u(x^*)|}{d_{\mathcal{C}}(x^*)^{1-n/p}} = \left\| \frac{u}{d_{\mathcal{C}}^{1-n/p}} \right\|_\infty .$$

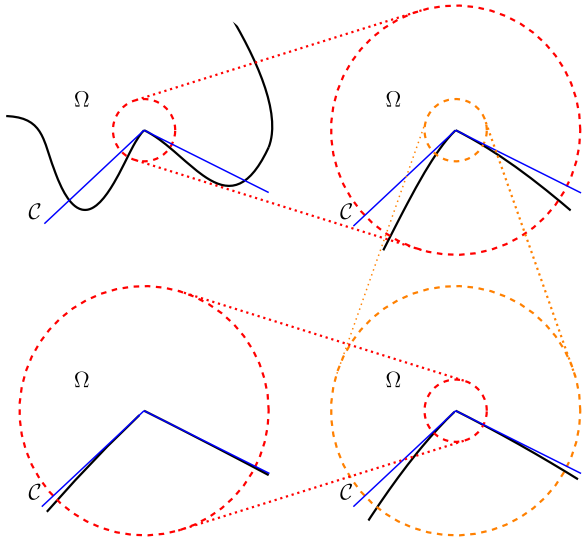

Fig. 4. A sequence of four blow-ups around a point on the boundary of a domain Ω with a limiting profile $\mathcal{C}$ depicted in blue.

For $\delta > 0$, define

$$\mathcal{C}_\delta = \{x \in \mathcal{C} : d_{\mathcal{C}}(x) > \delta\}.$$

By assumption, there are $t > 0, y \in \mathbb{R}^n$ such that $(t\Omega - y) \cap B_1 \supset \mathcal{C}_\delta \cap B_1$. As the support of u is a compact subset of $\mathcal{C} \cap B_{1/2}$, $\mathrm{supp}(u) \subset \mathcal{C}_\delta \cap B_{1/2}$ for sufficiently small δ. Consequently, v_δ defined by $v_\delta(x) = u(tx - y)$ belongs to $\mathcal{D}_0^{1,p}(\Omega)$ for all small enough δ.

Set $x_\delta = t^{-1}(x^* + y)$. By similarity invariance,

$$d_\Omega(x_\delta)^{p-n} \frac{\|Dv_\delta\|_p^p}{|v_\delta(x_\delta)|^p} = d_{t\Omega-y}(x^*)^{p-n} \frac{\|Du\|_p^p}{|u(x^*)|^p}.$$

It follows from the definition of u and x^* that

$$d_{t\Omega-y}(x^*)^{p-n} \frac{\|Du\|_p^p}{|u(x^*)|^p} = \frac{d_{t\Omega-y}(x^*)^{p-n}}{d_{\mathcal{C}}(x^*)^{p-n}} \mathcal{R}_p(\mathcal{C}, u) \le \frac{d_{t\Omega-y}(x^*)^{p-n}}{d_{\mathcal{C}}(x^*)^{p-n}} (\lambda_p(\mathcal{C}) + \epsilon).$$

By hypothesis

$$\lim_{\delta\to 0} d_{t\Omega-y}(x^*) = d_{\mathcal{C}}(x^*), \tag{8.1}$$

which gives

$$\limsup_{\delta\to 0} d_\Omega(x_\delta)^{p-n} \frac{\|Dv_\delta\|_p^p}{|v_\delta(x_\delta)|^p} \le \lambda_p(\mathcal{C}) + \epsilon. \tag{8.2}$$

Moreover, the variational characterization of $w^{\Omega}_{x_\delta}$ implies that

$$\limsup_{\delta\to 0} d_\Omega(x_\delta)^{p-n}\|Dw^{\Omega}_{x_\delta}\|_p^p \le \lambda_p(\mathcal{C}) + \epsilon\,. \tag{8.3}$$

We split the remainder of this proof into two cases depending on the asymptotic behavior of t and y as $\delta \to 0$.

Case 1: If along a sequence $\{\delta_k\}_{k\ge 1}$ with $\lim_{k\to\infty}\delta_k = 0$ it holds that

$$\limsup_{k\to\infty} t(\delta_k) = \infty\,, \quad \liminf_{k\to\infty} t(\delta_k) = 0\,, \quad \text{or} \quad \limsup_{k\to\infty} |y(\delta_k)| = \infty\,, \tag{8.4}$$

then we claim that a subsequence of $\{x_{\delta_k}\}_{k\ge 1}$ belongs to $\mathcal{Y}_\Omega$. Thus along this subsequence $w^{\Omega}_{x_{\delta_k}}$ is admissible in the definition of $\Lambda_p(\Omega)$, which when combined with (8.3) implies

$$\Lambda_p(\Omega) \le \lambda_p(\mathcal{C}) + \epsilon\,.$$

Since ϵ was arbitrary, we would then conclude our proof in this case.

To prove the claim, we argue as follows. By (8.1),

$$d_\Omega(x_{\delta_k}) = t(\delta_k)^{-1} d_{t(\delta_k)\Omega - y(\delta_k)}(x^*) = t(\delta_k)^{-1}(d_{\mathcal{C}}(x^*) + o(1))$$

as $k \to \infty$. Therefore, if $\liminf_{k\to\infty} t(\delta_k) = 0$, then $\limsup_{k\to\infty} |x_{\delta_k}| = \infty$, and a subsequence of $\{x_{\delta_k}\}_{k\ge 1}$ belongs to $\mathcal{Y}_\Omega$. If instead $\limsup_{k\to\infty} t(\delta_k) = \infty$, then $\liminf_{k\to\infty} d_\Omega(x_{\delta_k}) = 0$ and again a subsequence of $\{x_{\delta_k}\}_{k\ge 1}$ belongs to $\mathcal{Y}_\Omega$. Finally if there exist $c, C > 0$ so that $c < t(\delta_k) < C$ for all k but $\limsup_{k\to\infty} |y(\delta_k)| = \infty$, then

$$\limsup_{k\to\infty} |x_{\delta_k}| = \limsup_{k\to\infty} |t(\delta_k)^{-1}(x^* + y(\delta_k))| = \infty\,.$$

Again we deduce that $\{x_{\delta_k}\}_{k\ge 1}$ belongs to $\mathcal{Y}_\Omega$.

Case 2: If (8.4) fails, then there exists $t_0 > 0, y_0 \in \mathbb{R}^n$, and a sequence $\{\delta_k\}_{k\ge 1}$ such that

$$\lim_{k\to\infty} \delta_k = 0\,, \quad \lim_{k\to\infty} t(\delta_k) = t_0\,, \quad \text{and} \quad \lim_{k\to\infty} y(\delta_k) = y_0\,.$$

As $\lim_{\delta\to 0} d_{t\Omega - y}(x) = d_{\mathcal{C}}(x)$ for $x \in \mathcal{C} \cap B_{1/2}$,

$$\lim_{k\to\infty} d_{t(\delta_k)\Omega - y(\delta_k)}(x) = d_{t_0\Omega - y_0}(x) = d_{\mathcal{C}}(x) \qquad \text{for } x \in \mathcal{C} \cap B_{1/2}\,.$$

Here we used that d_Ω is continuous and $d_{t\Omega - y}(x) = t d_\Omega(t^{-1}(x + y))$. As a result, we have that $\mathcal{C} \cap B_{1/2} \subset t_0\Omega - y_0$.

Since $\mathcal{C}$ is dilation invariant, the function defined by $v_s(x) = u((t_0 x - y_0)/s)$ belongs to $\mathcal{D}^{1,p}_0(\Omega)$ for any $s \in (0, 1)$. Moreover, we claim that

$$\left\|\frac{v_s}{d_\Omega^{1-n/p}}\right\|_\infty = \frac{|v_s(x_s)|}{d_\Omega(x_s)^{1-n/p}} \quad \text{with} \quad x_s = \frac{sx^* + y_0}{t_0}$$

for $s \in (0,1)$. To see this, recall that $d_\mathcal{C} = d_{t_0\Omega - y_0}$ in $B_{1/2} \cap \mathcal{C}$, $\operatorname{supp}(u(\cdot/s)) \subset \mathcal{C} \cap B_{s/2}$, and $d_\mathcal{C}(x/s) = s^{-1} d_\mathcal{C}(x)$ for all $x \in \operatorname{supp}(u(\cdot/s))$. It follows that

$$\begin{aligned}\frac{|v_s(x)|}{d_\Omega(x)^{1-n/p}} &= t_0^{1-n/p} \frac{|u((t_0x - y_0)/s)|}{d_{t_0\Omega - y_0}((t_0x - y_0))^{1-n/p}} \\ &= (t_0/s)^{1-n/p} \frac{|u((t_0x - y_0)/s)|}{d_\mathcal{C}((t_0x - y_0)/s)^{1-n/p}} \\ &\leq (t_0/s)^{1-n/p} \frac{|u(x^*)|}{d_\mathcal{C}(x^*)^{1-n/p}}\,,\end{aligned}$$

with equality for $x = x_s$. Thus,

$$\mathcal{R}_p(\Omega, v_s) = \mathcal{R}_p(\mathcal{C}, u) \leq \lambda_p(\mathcal{C}) + \epsilon\,. \tag{8.5}$$

Notice that $y_0/t_0 = \lim_{s\to 0} x_s \in \partial\Omega$ since

$$\lim_{s\to 0} d_\Omega(x_s) = t_0^{-1} \lim_{s\to 0} d_{t_0\Omega - y_0}(sx^*) = t_0^{-1} \lim_{s\to 0} d_\mathcal{C}(sx^*) = t_0^{-1} \lim_{s\to 0} s d_\mathcal{C}(x^*) = 0\,.$$

Therefore, along any sequence $\{s_k\}_{k\geq 1}$ with $s_k \to 0$ the sequence $\{x_{s_k}\}_{k\geq 1} \in \mathcal{Y}_\Omega$. Consequently, the potentials $\{w^\Omega_{x_{s_k}}\}_{k\geq 1}$ are admissible in the definition of $\Lambda_p(\Omega)$. Combining this observation with Proposition 3.3 and (8.5), we have

$$\Lambda_p(\Omega) \leq \liminf_{k\to\infty} d_\Omega(x_{s_k})^{p-n} \|Dw^\Omega_{x_{s_k}}\|_p^p \leq \liminf_{k\to\infty} \mathcal{R}_p(\Omega, v_{s_k}) \leq \lambda_p(\mathcal{C}) + \epsilon\,.$$

Since ϵ was arbitrary, this completes our proof. □

The most evident case when Lemma 8.1 can be applied is if the boundary has at least one point at which it is differentiable. More generally it holds when $\partial\Omega$ is asymptotically a cone at some boundary point.

Corollary 8.2. *Assume*

$$\Omega \cap B_r = \{x = (x', x_n) \in B_r : x_n > f(x')\}$$

for some $r > 0$, where $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is continuous with $f(0) = 0$. Further suppose the limit

$$F(x') = \lim_{t\to\infty} t f(x'/t)$$

exists locally uniformly. Then $\Lambda_p(\Omega) \leq \lambda_p(\mathcal{C})$, where

$$\mathcal{C} = \{x = (x', x_n) \in \mathbb{R}^n : x_n > F(x')\}\,.$$

Proof. As F is positively homogeneous and continuous, $\mathcal{C}$ is an open cone. Therefore, it suffices to check the two hypotheses of the previous lemma for each $\delta \in (0,1)$.

By assumption, there is $t > r^{-1}$ with $F(x') - \delta < tf(x'/t) < F(x') + \delta$ uniformly in $|x'| \leq 1$. Since

$$t\Omega \cap B_1 = \{(x', x_n) \in B_1 : x_n > tf(x'/t)\},$$

it follows that

$$(\mathcal{C} + \delta e_n) \cap B_1 \subset t\Omega \cap B_1 \subset (\mathcal{C} - \delta e_n) \cap B_1. \tag{8.6}$$

Observe that if $d_{\mathcal{C}}(x) > \delta$, then $\overline{B_\delta}(x) \subset \mathcal{C}$. In this case, $x - \delta e_n \in \overline{B_\delta}(x) \subset \mathcal{C}$ and thus $x \in \mathcal{C} + \delta e_n$. In view of the first inclusion in (8.6),

$$\{x \in \mathbb{R}^n : d_{\mathcal{C}}(x) > \delta\} \cap B_1 \subset t\Omega \cap B_1 \, .$$

This verifies the first hypothesis of the lemma.

For the remainder of this proof, fix $x \in \mathcal{C}$ with $|x| < 1/2$. We claim that

$$d_{t\Omega\cap B_1}(x) = d_{t\Omega}(x)$$

To see this, we note $d_{t\Omega\cap B_1}(x) \leq |x| < 1/2$ since $0 \notin t\Omega$. Moreover, if the distance from x to the complement is realized for some $y \notin B_1$, then $d_{t\Omega\cap B_1}(x) = |x-y| \geq 1 - |x| > 1/2$. The claim follows. As $0 \notin \mathcal{C} + \delta e_n$, we also conclude

$$d_{(\mathcal{C}+\delta e_n)\cap B_1}(x) = d_{(\mathcal{C}+\delta e_n)}(x) \, .$$

Based on the first inclusion in (8.6) and the observations just made,

$$d_{t\Omega}(x) \geq d_{\mathcal{C}+\delta e_n}(x) \geq d_{\mathcal{C}}(x) - \delta \, .$$

Moreover, by the second inclusion in (8.6)

$$d_{t\Omega}(x) \leq d_{(\mathcal{C}-\delta e_n)\cap B_1}(x) \leq d_{\mathcal{C}-\delta e_n}(x) \leq d_{\mathcal{C}}(x) + \delta \, .$$

We deduce that the second hypothesis of the lemma holds as $|d_{t\Omega}(x) - d_{\mathcal{C}}(x)| \leq \delta$. $\square$

Remark 8.3. The corollary holds under the weaker assumption that

$$F(x') = \lim_{k\to\infty} t_k f(x'/t_k)$$

locally uniformly for some sequence $t_k \to \infty$ provided that F is positively homogeneous.

Remark 8.4. In the above corollary, if f is differentiable at 0, then $F(x') = Df(0)\cdot x'$. In this case, $\mathcal{C}$ is a halfspace. We conclude that $\Lambda_p(\Omega) \leq \lambda_p(\mathbb{R}^n_+)$ whenever $\partial\Omega$ has at least one point where it is differentiable.

Now we turn our attention to a lower bound on Λ_p and the behavior of sequences of potentials associated with sequences belonging to $\mathcal{Y}_\Omega$. In the statement below, we will use the notation $T_{r,Q,y}(x) = rQx + y$ for a similarity transformation as discussed in subsection 2.2.

Proposition 8.5. *Assume $\{x_k\}_{k\geq 1} \subset \Omega$ and $\{y_k\}_{k\geq 1} \subset \partial\Omega$ satisfy*

$$|y_k - x_k| = d_\Omega(x_k)$$

and choose $Q_k \in O(n)$ with $Q_k(e_n) = (x_k - y_k)/d_\Omega(x_k)$ for each $k \geq 1$. Then

$$v_k = w^\Omega_{x_k} \circ T_{d_\Omega(x_k),Q_k,y_k}\,,$$

defines a bounded sequence in $\mathcal{D}^{1,p}_0(\mathbb{R}^n \setminus \{0\})$. If $v_k \rightharpoonup v$ in $\mathcal{D}^{1,p}_0(\mathbb{R}^n \setminus \{0\})$ then the set $\{v > 0\} \subset \mathbb{R}^n$ is open,

$$B_1(e_n) \subset \{v > 0\} \subset \mathbb{R}^n \setminus \{0\}\,,$$

and

$$\liminf_{k\to\infty} d_\Omega(x_k)^{p-n}\|Dw^\Omega_{x_k}\|^p_p \geq \|Dv\|^p_p \geq \lambda_p(\{v > 0\})\,.$$

Proof. As $w(x) = (1 - |x - x_k|/d_\Omega(x_k))_+$ is a competitor for $w^\Omega_{x_k}$,

$$\|Dw^\Omega_{x_k}\|^p_p \leq |B_1| d_\Omega(x_k)^{n-p}\,.$$

We also have

$$\|Dv^\Omega_k\|^p_p = d_\Omega(x_k)^{p-n}\|Dw^\Omega_{x_k}\|^p_p \geq \mathcal{R}_p(\Omega, w^\Omega_{x_k}) \geq \lambda_p(\Omega)$$

by Proposition 3.3. Therefore, the sequence $\{v_k\}_{k\geq 1}$ is bounded in $\mathcal{D}^{1,p}_0(\mathbb{R}^n \setminus \{0\})$ with

$$\lambda_p(\Omega)^{1/p} \leq \|Dv_k\|_p \leq |B_1|^{1/p}\,.$$

Assume that $v_k \rightharpoonup v$ in $\mathcal{D}^{1,p}_0(\mathbb{R}^n \setminus \{0\})$. By the weak lower semi-continuity of the Dirichlet energy,

$$\liminf_{k\to\infty} d_\Omega(x_k)^{p-n}\|Dw^\Omega_{x_k}\|^p_p = \liminf_{k\to\infty} \|Dv_k\|^p_p \geq \|Dv\|^p_p\,.$$

This proves the first inequality in the proposition.

Since we always identify $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ with its Hölder continuous representative the set $\{v > 0\}$ is open. To see that $B_1(e_n) \subset \{v > 0\} \subset \mathbb{R}^n \setminus \{0\}$ we observe that for each $k \geq 1$, v_k is p-superharmonic in $B_1(e_n)$, nonnegative and satisfies $v_k(e_n) = 1$. By passing to a subsequence if necessary, we may assume $v_k \to v$ uniformly on compact subsets, by the Arzelà–Ascoli theorem and Morrey's inequality. Therefore, the above properties hold true also for v. The strong minimum principle implies that $v > 0$ in $B_1(e_n)$.

As $0 \in \partial\{v > 0\}$ and $B_1(e_n) \subset \{v > 0\}$, it follows that $d_{\{v>0\}}(e_n) = 1$. By combining these two observations,

$$\|Dv\|_p^p = \left(\frac{v(e_n)}{d_{\{v>0\}}(e_n)^{1-n/p}}\right)^{-p} \|Dv\|_p^p \geq \left\|\frac{v}{d_{\{v>0\}}^{1-n/p}}\right\|_\infty^{-p} \|Dv\|_p^p \geq \lambda_p(\{v > 0\})\,.$$

This completes the proof of the proposition. □

The application of the previous result gives the following lemma.

Lemma 8.6. *Assume $n \geq 2$. Suppose further that $\{x_k\}_{k\geq1} \in \mathcal{Y}_\Omega, x_0 \in \partial\Omega$ and $r > 0$ are such that*

$$\lim_{k\to\infty} x_k = x_0$$

and that $\partial\Omega \cap B_r(x_0)$ is C^1 regular or $\Omega \cap B_r(x_0)$ is convex. Then

$$\liminf_{k\to\infty} d_\Omega(x_k)^{p-n} \|Dw^\Omega_{x_k}\|_p^p \geq \lambda_p(\mathbb{R}^n_+)\,.$$

Proof. Let $\{x_k\}_{k\geq1}$ be as in the statement of the lemma. By passing to a subsequence we may without loss of generality assume that

$$\liminf_{k\to\infty} d_\Omega(x_k)^{p-n} \|Dw^\Omega_{x_k}\|_p^p = \lim_{k\to\infty} d_\Omega(x_k)^{p-n} \|Dw^\Omega_{x_k}\|_p^p\,.$$

Let $\{y_k\}_{k\geq1} \subset \partial\Omega$ be a sequence such that

$$|x_k - y_k| = d_\Omega(x_k) \quad \text{for each } k \geq 1\,,$$

and $\{Q_k\}_{k\geq1} \in O(n)$ be a sequence of rotations such that $Q_k(e_n) = \frac{x_k - y_k}{d_\Omega(x_k)}$. Define

$$v_k = w^\Omega_{x_k} \circ T_{d_\Omega(x_k),Q_k,y_k}$$

as in Proposition 8.5. We will analyze $\{v > 0\}$, where v is a subsequential limit of v_k.

Performing a translation and rotation, we can assume that $x_0 = 0$ without loss of generality. By assumption, there exists a function $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ with

$$\Omega \cap B_r = \{x = (x', x_n) \in \mathbb{R}^n : x_n > f(x')\} \cap B_r\,.$$

If $\Omega \cap B_r$ is convex, f is convex. While if $\partial\Omega \cap B_r$ is C^1, then f is C^1. In either case, we may assume that f is Lipschitz.

Since $\{x_k\}_{k\geq 1}$ converges and $\lim_{k\to\infty} d_\Omega(x_k) = \lim_{k\to\infty} |x_k - y_k| = 0$, it follows that $\lim_{k\to\infty} y_k = x_0 = 0$. In particular, $|y_k| < r/2$ for k large enough and

$$(y_k)_n = f(y_k') \,.$$

Therefore, $x \in \Omega \cap B_r$ if and only if $|x| < r$ and

$$(x - y_k)_n > f(x') - f(y_k') \,.$$

Consequently, if $z \in B_R$ with $R < r/(2d_\Omega(x_k))$ then $y_k + d_\Omega(x_k)Q_k(z) \in \Omega$ if and only if

$$(Q_k(z))_n > \frac{f(d_\Omega(x_k)(Q_k(z))' + y_k') - f(y_k')}{d_\Omega(x_k)} \,.$$

Since $\{v_k\}_{k\geq 1} \subset \mathcal{D}_0^{1,p}(\mathbb{R}^n \setminus \{0\})$ is bounded we may by passing to a subsequence assume that $v_k \rightharpoonup v$ in $\mathcal{D}_0^{1,p}(\mathbb{R}^n \setminus \{0\})$ and $v_k \to v$ uniformly on compact sets. Since $O(n)$ is compact we may also pass to a further subsequence such that $Q_k \to Q_0$. In particular, this implies that $\tilde{v}_k := w_{x_k}^\Omega(d_\Omega(x_k) \cdot + y_k) \rightharpoonup v(Q_0^{-1} \cdot) := \tilde{v}$. We claim that $\{\tilde{v} > 0\} = Q_0\{v > 0\}$ is given by the region above a Lipschitz graph. We first observe that since $w_{x_k}^\Omega$ is a potential

$$\begin{aligned}
\tilde{v}_k(z) = 0 \quad \text{if } z_n \leq \frac{f(d_\Omega(x_k)z' + y_k') - f(y_k')}{d_\Omega(x_k)} \,, \\
\tilde{v}_k(z) > 0 \quad \text{if } z_n > \frac{f(d_\Omega(x_k)z' + y_k') - f(y_k')}{d_\Omega(x_k)} \,,
\end{aligned} \tag{8.7}$$

for $z \in B_R$. Moreover, $\tilde{v}(Q_0 e_n) = 1$ and since the p-Laplacian is invariant under the action of $O(n)$, $\tilde{v}$ is p-superharmonic whenever $\tilde{v} > 0$.

Since f is Lipschitz, the functions

$$f_k(z') := \frac{f(d_\Omega(x_k)z' + y_k') - f(y_k')}{d_\Omega(x_k)}$$

are uniformly bounded and Lipschitz for $|z'| < R$ for any fixed $R > 0$ provided k is large enough (with a Lipschitz constant independent of R). Therefore, we can pass to a subsequence along which f_k converges uniformly to a limit $f_{0,R}$ in $|z'| < R$. It follows that $\tilde{v}$ is nonnegative, $\tilde{v} = 0$ in $\{z_n < f_{0,R}(z')\} \cap B_R$ and $\tilde{v}$ is p-superharmonic in $\{z_n > f_{0,R}(z')\} \cap B_R$. Since, $\tilde{v}(Q_0 e_n) = 1$, the strong minimum principle implies

$$\{z \in \mathbb{R}^n : \tilde{v}(z) > 0, |z| < R\} = \{z \in \mathbb{R}^n : z_n > f_{0,R}(z'), |z| < R\}.$$

As $\{\tilde{v} > 0\}$ is independent of R, $f_{0,R}$ is the restriction of some globally Lipschitz function f_0 and $\{\tilde{v} > 0\} = \{z_n > f_0(z')\}$.

If f was convex, then f_0 is convex; so $\{\tilde{v} > 0\}$ and thus $\{v > 0\}$ is convex. If f was C^1 at 0, then $f_0(z') = Df(0) \cdot z'$; so $\{\tilde{v} > 0\}$ and hence $\{v > 0\}$ is a halfspace. In either case $\lambda_p(\{v > 0\}) = \lambda_p(\mathbb{R}^n_+)$ by Theorem B. Appealing to Proposition 8.5, we finally conclude that

$$\lim_{k\to\infty} d_\Omega(x_k)^{p-n} \|Dw^\Omega_{x_k}\|_p^p \geq \lambda_p(\mathbb{R}^n_+) . \quad \square$$

Remark 8.7. In the above proof, we only used the regularity assumption on $\partial\Omega \cap B_r(x_0)$ when concluding that $\lambda_p(\{v > 0\}) = \lambda_p(\mathbb{R}^n_+)$. Whenever $\partial\Omega \cap B_r(x_0)$ is given by a Lipschitz graph it follows in the same manner that $\{v > 0\}$ is the region above a Lipschitz graph.

Combining the results of this section we are able to determine $\Lambda_p(\Omega)$ if Ω is regular.

Corollary 8.8. *Assume $n \geq 2$. If Ω is bounded and C^1, then $\Lambda_p(\Omega) = \lambda_p(\mathbb{R}^n_+)$.*

Proof. Fix a sequence $\{x_k\}_{k\geq 1} \in \mathcal{Y}_\Omega$ such that

$$\lim_{k\to\infty} d_\Omega(x_k)^{p-n} \|Dw^\Omega_{x_k}\|_p^p = \Lambda_p(\Omega) .$$

Since $\overline{\Omega}$ is compact, we may assume that $\{x_k\}_{k\geq 1}$ converges to some limit x_0. As $\{x_k\}_{k\geq 1} \in \mathcal{Y}_\Omega$, it follows that $x_0 \in \partial\Omega$. By Lemma 8.6, $\Lambda_p(\Omega) \geq \lambda_p(\mathbb{R}^n_+)$. In view of Corollary 8.2 (and Remark 8.4), we also have $\Lambda_p(\Omega) \leq \lambda_p(\mathbb{R}^n_+)$. $\square$

Remark 8.9. While the equality in Corollary 8.8 is not true in general, it is possible to prove an analogue of Theorem A for Λ_p. Namely, for any $\Omega \subsetneq \mathbb{R}^n$,

$$\lambda_p(\mathbb{R}^n \setminus \{0\}) \leq \Lambda_p(\Omega) \leq \lambda_p(\mathbb{R}^n_+) .$$

Indeed, the lower bound follows directly from the general fact that $\lambda_p(\Omega) \leq \Lambda_p(\Omega)$ and the lower bound in Theorem A. The upper bound can be obtained by transplanting a sequence of potentials realizing $\Lambda_p(B_1) = \lambda_p(\mathbb{R}^n_+)$ into Ω in the spirit of our proof of Corollary 5.4.

We are now ready to prove Theorem C by combining our previous results.

Proof of Theorem C. By Corollary 8.8, $\Lambda_p(\Omega) = \lambda_p(\mathbb{R}^n_+)$. Therefore, the assumption of this theorem implies that $\lambda_p(\Omega) < \Lambda_p(\Omega)$. Proposition 3.4 yields the desired conclusion. $\square$

9. Refined analysis at points of negative mean curvature

The aim of this section is to prove that if $\partial\Omega$ has a point x_0 where the mean curvature is negative then $\lambda_p(\Omega) < \lambda_p(\mathbb{R}^n_+)$. The idea is to take a sequence of potentials whose singular point approaches x_0 from within Ω and show that along this sequence $\mathcal{R}_p$ is eventually smaller than when it is evaluated on a potential in $\mathbb{R}^n_+$. We begin by considering a model situation in which $\partial\Omega$ is a parabola in a neighborhood of x_0. We will assume throughout this section that $p > n \geq 2$ and $\Omega \subsetneq \mathbb{R}^n$ is open.

Let $K \in \mathbb{R}^{(n-1)\times(n-1)}$ be a real symmetric matrix and denote its operator norm by $\|K\|$. Define

$$\Omega_K = \{x = (x', x_n) \in \mathbb{R}^n : x_n > x' \cdot Kx', |x| < 1\},$$

and set $u_{K,\epsilon} = w^{\Omega_K}_{\epsilon e_n}$ for $\epsilon \in (0, 1/2)$. Recall that $u_{K,\epsilon} \in \mathcal{D}^{1,p}_0(\Omega_K)$ is the solution of

$$\begin{cases} -\Delta_p u_{K,\epsilon} = 0 & \text{in } \Omega_K \setminus \{\epsilon e_n\}, \\ u_{K,\epsilon} = 0 & \text{on } \partial\Omega_K, \\ u_{K,\epsilon}(\epsilon e_n) = 1. \end{cases} \tag{9.1}$$

We will also write $u_0 = w^{\mathbb{R}^n_+}_{e_n}$ and use that $u_0 \in \mathcal{D}^{1,p}_0(\mathbb{R}^n_+)$ is characterized as the solution of

$$\begin{cases} -\Delta_p u_0 = 0 & \text{in } \mathbb{R}^n_+ \setminus \{e_n\}, \\ u_0 = 0 & \text{on } \partial\mathbb{R}^n_+, \\ u_0(e_n) = 1. \end{cases} \tag{9.2}$$

The crucial ingredient in our proof of Theorem D is the following proposition.

Proposition 9.1. *Assume $K \in \mathbb{R}^{(n-1)\times(n-1)}$ is symmetric with $\|K\| < 1/2$. There are constants $C > 0$ and $\gamma \in (0, 1]$ so that*

$$\epsilon^{p-n}\|Du_{K,\epsilon}\|_p^p \leq \|Du_0\|_p^p + \epsilon\,\mathrm{tr}(K)\frac{p-1}{n-1}\int_{\mathbb{R}^{n-1}} |Du_0(x', 0)|^p |x'|^2\,dx' + C\epsilon^{1+\gamma}$$

for $\epsilon \in (0, 1/8)$.

With this proposition in hand we can prove the following theorem. An illustration of the geometric assumption made in this theorem is shown in Fig. 5.

Theorem 9.2. *Assume that there exist $x_0 \in \partial\Omega, Q \in O(n), r > 0$, and a symmetric matrix $K \in \mathbb{R}^{(n-1)\times(n-1)}$ with $\mathrm{tr}(K) < 0$ such that*

$$\{x = (x', x_n) \in \mathbb{R}^n : x_n > x' \cdot Kx'\} \cap B_r \subset Q(\Omega - x_0)\,.$$

Then $\lambda_p(\Omega) < \lambda_p(\mathbb{R}^n_+)$.

Proof. Since $\lambda_p(\Omega) = \lambda_p(Q(\Omega - x_0))$, we may assume without loss of generality that $x_0 = 0$ and $Q = \mathrm{Id}$. Also observe that if the assumption of the theorem holds for some $r > 0$, then it is also valid for any smaller value of r. As such, we may assume that

$$r\|K\| < \frac{1}{2}\,. \tag{9.3}$$

It suffices to construct $w \in \mathcal{D}^{1,p}_0(\Omega)$ which satisfies $\mathcal{R}_p(\Omega, w) < \lambda_p(\mathbb{R}^n_+)$. With this in mind, we define $w_\epsilon \in \mathcal{D}^{1,p}_0(r\Omega_{rK})$ via $w_\epsilon(x) = u_{rK,\epsilon/r}(x/r)$. As

$$r\Omega_{rK} = \{x = (x', x_n) \in \mathbb{R}^n : x_n > x' \cdot Kx'\} \cap B_r \subset \Omega\,,$$

we also have $w_\epsilon \in \mathcal{D}^{1,p}_0(\Omega)$. Moreover, $\|Dw_\epsilon\|_p = r^{n/p-1}\|Du_{rK,\epsilon/r}\|_p$. And since $w_\epsilon(\epsilon e_n) = 1$ for $\epsilon < r$ and $0 \in \partial\Omega$,

$$\Big\|\frac{w_\epsilon}{d_\Omega^{1-n/p}}\Big\|_\infty \geq \frac{|w_\epsilon(\epsilon e_n)|}{d_\Omega(\epsilon e_n)^{1-n/p}} \geq \epsilon^{n/p-1}\,.$$

Consequently,

$$\mathcal{R}_p(\Omega, w_\epsilon) \leq (\epsilon/r)^{p-n}\|Du_{rK,\epsilon/r}\|_p^p\,.$$

In view of (9.3), $\|rK\| < 1/2$, so Proposition 9.1 can be applied. As $\|Du_0\|_p^p = \lambda_p(\mathbb{R}^n_+)$,

$$\mathcal{R}_p(\Omega, w_\epsilon) \leq \lambda_p(\mathbb{R}^n_+) + \epsilon\,\mathrm{tr}(K)\frac{p-1}{n-1}\int\limits_{\mathbb{R}^{n-1}} |Du_0(x',0)|^p|x'|^2\,dx' + C(\epsilon/r)^{1+\gamma}$$

for all ϵ sufficiently small. By Lemma 6.1, u_0 is the restriction of a Morrey extremal to $\mathbb{R}^n_+$. It then follows that $|Du_0(x',0)| > 0$ for $x' \in \mathbb{R}^{n-1}$ [26, Proposition 3.6]. As a result,

$$\int\limits_{\mathbb{R}^{n-1}} |Du_0(x',0)|^p|x'|^2\,dx' > 0\,.$$

In addition, this integral is finite by the decay estimate proved in [23]. Finally, as $\mathrm{tr}(K) < 0$ and $\gamma > 0$, we can choose ϵ sufficiently small so that $\mathcal{R}_p(\Omega, w_\epsilon) < \lambda_p(\mathbb{R}^n_+)$. □

We are now ready to prove Theorem D. Let us briefly recall the formula for the mean curvature of a surface in $\mathbb{R}^n$ given by the graph of C^2 function $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$. The mean curvature H at $x = (x', f(x'))$ is given by

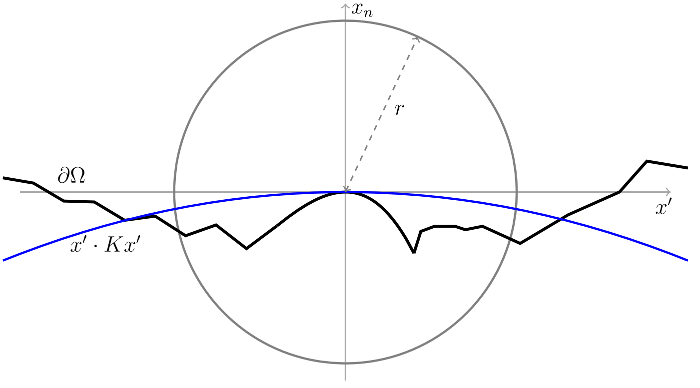

Fig. 5. A depiction of the geometric assumptions in Theorem 9.2. Here Ω is the set above the black curve, x_0 is the origin, Q the identity, and $\partial\Omega$ can be touched at x_0 from the inside with a negatively curved parabola (blue).

$$(n-1)H = \operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}}\right).$$

This is the mean curvature with respect to the unit normal $(Df,-1)/\sqrt{1+|Df|^2}$ at x, and the derivatives of f are evaluated at x'. Also note that $(n-1)H = \Delta f$ at $x = (x', f(x'))$ whenever $Df(x') = 0$.

Proof of Theorem D. By assumption, Ω is C^2 and there is $x_0 \in \partial\Omega$ which has negative mean curvature. After translating and rotating Ω, we may assume $x_0 = 0$ and

$$\{x = (x', x_n) \in \mathbb{R}^n : x_n > f(x')\} \cap B_r = \Omega \cap B_r \tag{9.4}$$

for some $r > 0$. Here $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is C^2 when $|x'| < r$ and satisfies that $f(0) = |Df(0)| = 0$, and $\Delta f(0) < 0$.

Fix ϵ so small that $\operatorname{tr}(K) < 0$, where

$$K = \frac{1}{2} D^2 f(0) + \epsilon\,\mathrm{Id}'.$$

Here Id' is the $(n-1)\times(n-1)$ identity matrix. Reducing r if necessary, $f(x') \le Kx' \cdot x'$ for $|x'| < r$ by Taylor's theorem. In view of (9.4),

$$\{x = (x', x_n) \in \mathbb{R}^n : x_n > x' \cdot Kx'\} \cap B_r \subset \Omega.$$

We conclude that $\lambda_p(\Omega) < \lambda_p(\mathbb{R}^n_+)$ by Theorem 9.2. Theorem C in turn implies that Ω has an extremal. $\square$

Remark 9.3. The assumptions in Theorem D can be weakened significantly. It suffices to assume that

(1) Ω is bounded,
(2) that after an appropriate rotation and translation $0 \in \partial\Omega$ and there exists $r > 0$ so that

$$\Omega \cap B_r \supset \{x = (x', x_n) \in \mathbb{R}^n : x_n > x' \cdot Kx'\} \cap B_r$$

where $K \in \mathbb{R}^{(n-1)\times(n-1)}$ is a symmetric matrix with $\operatorname{tr}(K) < 0$, and
(3) that for every $x \in \partial\Omega$ there exists $r > 0$ so that either

 (a) $\Omega \cap B_r(x)$ is convex, or
 (b) $\partial\Omega \cap B_r(x)$ is C^1-regular.

Indeed, under these assumptions Lemma 8.6 implies that $\Lambda_p(\Omega) \geq \lambda_p(\mathbb{R}^n_+)$. By Theorem 9.2, $\lambda_p(\Omega) < \lambda_p(\mathbb{R}^n_+)$. Then Proposition 3.4 yields the existence of an extremal.

9.1. Proof of Proposition 9.1

Proposition 9.1 is a direct consequence of the following two lemmas.

Lemma 9.4. *Assume $K \in \mathbb{R}^{(n-1)\times(n-1)}$ is symmetric with $\|K\| < 1/2$. There are constants $C > 0$ and $\gamma \in (0, 1]$ so that*

$$\begin{aligned}&\epsilon^{p-n}\|Du_{K,\epsilon}\|_p^p \\ &\leq \|Du_0\|_p^p - \epsilon\frac{2p}{n-1}\operatorname{tr}(K)\int_{\mathbb{R}^n_+} |Du_0(x)|^{p-2}\partial_{x_n}u_0(x)Du_0(x)\cdot(x',0)\,dx + C\epsilon^{1+\gamma}\end{aligned}$$

for $\epsilon \in (0, 1/8)$.

Lemma 9.5. *The equality*

$$\int_{\mathbb{R}^n_+} |Du_0(x)|^{p-2}\partial_{x_n}u_0(x)Du_0(x)\cdot(x',0)\,dx = -\frac{p-1}{2p}\int_{\mathbb{R}^{n-1}} |Du_0(x',0)|^p|x'|^2\,dx'$$

holds and both integrals are convergent.

Proof of Lemma 9.4. We prove this lemma by exploiting the variational characterization of $u_{K,\epsilon}$. In particular, we will derive the desired inequality by constructing an appropriate competitor for $u_{K,\epsilon}$.

Step 1. For $y = (y', y_n) \in \mathbb{R}^n$, define

$$\Phi(y) = (y', y_n - y' \cdot Ky'). \tag{9.5}$$

This mapping is a diffeomorphism from $\mathbb{R}^n$ to itself, $\Phi(te_n) = te_n$ for all $t \in \mathbb{R}$,

$$\Phi(\{y \in \mathbb{R}^n : y_n > y' \cdot Ky'\}) = \mathbb{R}^n_+ , \quad \text{and} \quad \Phi^{-1}(x) = (x', x_n + x' \cdot Kx') . \tag{9.6}$$

As

$$D\Phi(y) = \begin{pmatrix} \mathrm{Id} & 0 \\ -2(Ky')^\top & 1 \end{pmatrix}, \tag{9.7}$$

it also follows that $\det D\Phi(y) = 1$ for all $y \in \mathbb{R}^n$.

In addition, we claim

$$|\Phi(y)| \geq 1/2 \text{ for all } y \in B_1^c . \tag{9.8}$$

We first note that if $|y| \geq 1$ and $|y'| \geq 1/2$, then

$$|\Phi(y)|^2 = |y'|^2 + (y_n - y' \cdot Ky')^2 \geq |y'|^2 \geq 1/4 .$$

Next, we observe that if $|y| \geq 1$ and $|y'| < 1/2$, then

$$|y_n|^2 = |y|^2 - |y'|^2 \geq 3/4 .$$

As $\|K\| < 1/2$ and $t^2 - t/4$ is increasing for $t \geq \sqrt{3}/2$,

$$\begin{aligned} |\Phi(y)|^2 &= |y'|^2 + (y_n - y' \cdot Ky')^2 \\ &\geq |y_n|^2 - 2y_n y' \cdot Ky' \\ &\geq |y_n|^2 - \frac{1}{4}|y_n| \\ &\geq \left(\frac{3}{4}\right) - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) \\ &> \frac{1}{4} . \end{aligned}$$

We conclude that (9.8) holds.

Set

$$\phi(s) = \begin{cases} 0 & \text{if } s \leq 0 , \\ s & \text{if } 0 < s < 1 , \\ 1 & \text{if } s \geq 1 , \end{cases}$$

and define

$$w(y) = \phi\big(2 - 4|\Phi(y)|\big) u_0(\Phi(y)/\epsilon)$$

for $\epsilon \in (0, 1/8)$ and $y \in \mathbb{R}^n$. Since $|\Phi(y)| \geq 1/2$ whenever $|y| \geq 1$, the first factor of w vanishes for $|y| \geq 1$. The second factor of w vanishes for all y with $y_n < y' \cdot Ky'$. Indeed Φ maps this set of points to $(\mathbb{R}^n_+)^c$, where u_0 vanishes. Therefore, w is supported in Ω_K.

Since the first factor of w is Lipschitz with compact support and the second factor belongs to $\mathcal{D}^{1,p}(\mathbb{R}^n)$, $w \in \mathcal{D}^{1,p}_0(\Omega_K)$. Further, $w(\epsilon e_n) = \phi(2-4\epsilon)u_0(e_n) = 1$ as $0 < \epsilon < 1/4$. It follows that

$$\|Du_{K,\epsilon}\|_p^p \leq \|Dw\|_p^p$$

by the variational characterization of $u_{K,\epsilon}$. We now wish to estimate $\|Dw\|_p^p$ in terms of u_0 and ϵ.

Step 2. In view of (9.7),

$$D\Phi(\Phi^{-1}(x))^\top z = z - z_n(2Kx', 0) \quad \text{and} \quad |D\Phi(\Phi^{-1}(x))^\top z| \leq |z|(1 + |x'|) \tag{9.9}$$

for $x, z \in \mathbb{R}^n$. Using the change of variables $y = \Phi^{-1}(x)$ in the integral below leads to

$$\begin{aligned}
&\int_{\mathbb{R}^n} |Dw(y)|^p \, dy \\
&= \int_{B_{1/2}} \left| D\Phi(\Phi^{-1}(x))^\top D\big(\phi(2-4|x|)u_0(x/\epsilon)\big)\right|^p dx \\
&= \epsilon^{-p} \int_{B_{1/4}} \left| D\Phi(\Phi^{-1}(x))^\top Du_0(x/\epsilon)\right|^p dx \\
&\quad + \int_{B_{1/2}\setminus B_{1/4}} \left| D\Phi(\Phi^{-1}(x))^\top \Big(-4u_0(x/\epsilon)\frac{x}{|x|} + \frac{1}{\epsilon}(2-4|x|)Du_0(x/\epsilon)\Big)\right|^p dx \\
&\leq \epsilon^{-p} \int_{B_{1/4}} |Du_0(x/\epsilon) - \partial_{x_n} u_0(x/\epsilon)(2Kx', 0)|^p \, dx \\
&\quad + (3/2)^p \int_{B_{1/2}\setminus B_{1/4}} \left| -4u_0(x/\epsilon)\frac{x}{|x|} + \frac{1}{\epsilon}(2-4|x|)Du_0(x/\epsilon)\right|^p dx \\
&\leq \epsilon^{-p} \int_{B_{1/4}} |Du_0(x/\epsilon) - \partial_{x_n} u_0(x/\epsilon)(2Kx', 0)|^p \, dx \\
&\quad + (3/2)^p 4^p 2^{p-1} \int_{B_{1/2}\setminus B_{1/4}} |u_0(x/\epsilon)|^p \, dx + (3/2)^p \epsilon^{-p} 2^{p-1} \int_{B_{1/2}\setminus B_{1/4}} |Du_0(x/\epsilon)|^p \, dx\,.
\end{aligned} \tag{9.10}$$

Note that the first inequality is due to (9.9) and the second follows from the elementary inequality: $(a+b)^p \le 2^{p-1}a^p + 2^{p-1}b^p$ for $a, b \ge 0$.

We shall also utilize the estimate

$$|z+h|^p \le |z|^p + p|z|^{p-2} z \cdot h + \frac{p(p-1)}{2}|z|^{p-2}|h|^2$$

for $z, h \in \mathbb{R}^n$ with $|h| \le |z|$; see Lemma 10.2.1 in [1], for example. With $z = Du_0(x/\epsilon)$ and $h = -\partial_{x_n} u_0(x/\epsilon)(2Kx', 0)$, the above inequality and that $|Kx'| \le \|K\||x'| < |x'|/2$ gives

$$\begin{aligned}
&\int\limits_{B_{1/4}} |Du_0(x/\epsilon) - \partial_{x_n} u_0(x/\epsilon)(2Kx', 0)|^p \, dx \\
&\quad \le \int\limits_{B_{1/4}} |Du_0(x/\epsilon)|^p \, dx - 2p \int\limits_{B_{1/4}} |Du_0(x/\epsilon)|^{p-2} \partial_{x_n} u_0(x/\epsilon) Du_0(x/\epsilon) \cdot (Kx', 0) \, dx \\
&\qquad + \frac{p(p-1)}{2} \int\limits_{B_{1/4}} |Du_0(x/\epsilon)|^p |x'|^2 \, dx \\
&\quad \le \int\limits_{B_{1/4}} |Du_0(x/\epsilon)|^p \, dx - 2p \int\limits_{\mathbb{R}^n_+} |Du_0(x/\epsilon)|^{p-2} \partial_{x_n} u_0(x/\epsilon) Du_0(x/\epsilon) \cdot (Kx', 0) \, dx \\
&\qquad + \frac{p(p-1)}{2} \int\limits_{B_{1/4}} |Du_0(x/\epsilon)|^p |x'|^2 \, dx \\
&\qquad + 2p \int\limits_{B^c_{1/4}} |Du_0(x/\epsilon)|^{p-2} \partial_{x_n} u_0(x/\epsilon) Du_0(x/\epsilon) \cdot (Kx', 0) \, dx \\
&\quad \le \int\limits_{\mathbb{R}^n_+} |Du_0(x/\epsilon)|^p \, dx - 2p \int\limits_{\mathbb{R}^n_+} |Du_0(x/\epsilon)|^{p-2} \partial_{x_n} u_0(x/\epsilon) Du_0(x/\epsilon) \cdot (Kx', 0) \, dx \\
&\qquad + \frac{p(p-1)}{2} \int\limits_{B_{1/4}} |Du_0(x/\epsilon)|^p |x'|^2 \, dx + p \int\limits_{B^c_{1/4}} |Du_0(x/\epsilon)|^p |x'| \, dx \, .
\end{aligned}$$

Changing variables $x \mapsto \epsilon x$ in the integrals above and combining this inequality with the upper bound on $\|Dw\|_p^p$ in (9.10) gives

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |Dw(y)|^p \, dy \\
&\leq \epsilon^{n-p} \int_{\mathbb{R}^n_+} |Du_0(x)|^p \, dx - 2p\epsilon^{n+1-p} \int_{\mathbb{R}^n_+} |Du_0(x)|^{p-2} \partial_{x_n} u_0(x) Du_0(x) \cdot (Kx', 0) \, dx \\
&\quad + C_0 \epsilon^{n+2-p} \int_{B_{1/(4\epsilon)}} |Du_0(x)|^p |x'|^2 \, dx + C_0 \epsilon^{n+1-p} \int_{B^c_{1/(4\epsilon)}} |Du_0(x)|^p |x'| \, dx \\
&\quad + C_0 \epsilon^n \int_{B_{1/(2\epsilon)} \setminus B_{1/(4\epsilon)}} |u_0(x)|^p \, dx + C_0 \epsilon^{n-p} \int_{B_{1/(2\epsilon)} \setminus B_{1/(4\epsilon)}} |Du_0(x)|^p \, dx \,.
\end{aligned}
\tag{9.11}
$$

Here C_0 is a constant which only depends on p.

Since u_0 is the restriction of a Morrey extremal, u_0 is axially-symmetric about the x_n-axis [25, Theorem 1.1]. Therefore,

$$
\partial_{x_n} u_0(r\theta, x_n) = \partial_{x_n} u_0(r\theta_0, x_n) \quad \text{and} \quad (Du_0(r\theta, x_n))' = \theta |(Du_0(r\theta_0, x_n))'|
$$

for any $\theta, \theta_0 \in \mathbb{S}^{n-2}, r > 0, x_n > 0$. Furthermore, if $\theta_0 \in \mathbb{S}^{n-2}$ is fixed, then

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+} |Du_0(x)|^{p-2} \partial_{x_n} u_0(x) Du_0(x) \cdot (Kx', 0) \, dx \\
&= \int_0^\infty \int_0^\infty \int_{\mathbb{S}^{n-2}} |Du_0(r\theta_0, x_n)|^{p-2} \partial_{x_n} u_0(r\theta_0, x_n) |Du_0(r\theta_0, x_n)'| (\theta \cdot K\theta) r^{n-1} \, d\sigma(\theta) dr dx_n \\
&= \frac{\operatorname{tr}(K)}{n-1} \int_0^\infty \int_0^\infty \int_{\mathbb{S}^{n-2}} |Du_0(r\theta_0, x_n)|^{p-2} \partial_{x_n} u_0(r\theta_0, x_n) |Du_0(r\theta_0, x_n)'| |\theta|^2 r^{n-1} \, d\sigma(\theta) dr dx_n \\
&= \frac{\operatorname{tr}(K)}{n-1} \int_{\mathbb{R}^n_+} |Du_0(x)|^{p-2} \partial_{x_n} u_0(x) Du_0(x) \cdot (x', 0) \, dx.
\end{aligned}
\tag{9.12}
$$

Here σ denotes the surface measure and we used the identity

$$
\frac{1}{\sigma(\mathbb{S}^{n-2})} \int_{\mathbb{S}^{n-2}} \theta \cdot K\theta \, d\sigma(\theta) = \frac{\operatorname{tr}(K)}{n-1} \,.
$$

Combining (9.11), (9.12), we arrive at the estimate

$$\epsilon^{p-n}\|Du_{K,\epsilon}\|_p^p \leq \|Du_0\|_p^p - \epsilon\frac{2p\,\mathrm{tr}(K)}{n-1}\int_{\mathbb{R}^n_+}|Du_0(x)|^{p-2}\partial_{x_n}u_0(x)Du_0(x)\cdot(x',0)\,dx$$
$$+ C_0\epsilon^2\int_{B_{1/(4\epsilon)}}|Du_0(x)|^p|x'|^2\,dx + C_0\epsilon\int_{B^c_{1/(4\epsilon)}}|Du_0(x)|^p|x'|\,dx$$
$$+ C_0\epsilon^p\int_{B_{1/(2\epsilon)}\setminus B_{1/(4\epsilon)}}|u_0(x)|^p\,dx + C_0\int_{B_{1/(2\epsilon)}\setminus B_{1/(4\epsilon)}}|Du_0(x)|^p\,dx\,. \tag{9.13}$$

It remains is to bound the last four terms.

Step 3. It follows from [23] that for any

$$0<\beta<\beta_0(p):=-\frac{1}{3}+\frac{2}{3(p-1)}+\sqrt{\left(-\frac{1}{3}+\frac{2}{3(p-1)}\right)^2+\frac{1}{3}} \tag{9.14}$$

there is a constant C_1 depending on β and p such that

$$|u_0(x)|\leq C_1|x|^{-\beta} \quad\text{and}\quad |Du_0(x)|\leq C_1|x|^{-\beta-1} \tag{9.15}$$

for all $|x|\geq 2$. Note that $[2,\infty)\ni p\mapsto\beta_0(p)$ is decreasing. As $\lim_{p\to\infty}\beta_0(p)=\frac{1}{3}$, it must be that $\beta_0(p)>1/p$ for all $p>3$. Moreover, the monotonicity of β_0 implies that

$$\beta_0(p)\geq\beta_0(3)=\frac{1}{\sqrt{3}}>\frac{1}{2}\geq\frac{1}{p} \text{ for all } p\in[2,3]\,.$$

Therefore, $\beta_0(p)>1/p$ for all $p\in[2,\infty)$.

We now fix

$$\frac{1}{p}<\beta<\beta_0(p),$$

and use the decay estimates (9.15) to bound the four error terms in (9.13). To estimate the integral of $|Du_0(x)|^p|x'|^2$ over $B_{1/(4\epsilon)}$ we split it into two parts; one away from the singularity where we can employ the decay estimates, and one near to the singularity where we simply use that $|Du_0|\in L^p$.

Observe that

$$\epsilon\int_{B^c_{1/(4\epsilon)}}|Du_0(x)|^p|x'|\,dx\leq\sigma(\mathbb{S}^{n-1})C_1\epsilon\int_{1/(4\epsilon)}^{\infty}r^{-(\beta+1)p}r^n\,dr\leq C_2\epsilon^{p-n+\beta p}\,,$$

$$\epsilon^2 \int_{B_2} |Du_0(x)|^p |x'|^2 \, dx \le 2^2 \epsilon^2 \int_{B_2} |Du_0(x)|^p \, dx \le C_2 \epsilon^2 \,,$$

$$\epsilon^2 \int_{B_{1/(4\epsilon)} \setminus B_2} |Du_0(x)|^p |x'|^2 \, dx \le \sigma(\mathbb{S}^{n-1}) C_1 \epsilon^2 \int_2^{1/(4\epsilon)} r^{-(\beta+1)p+n+1} \, dr \le C_2 \epsilon^{p-n+\beta p} \,,$$

$$\epsilon^p \int_{B_{1/(2\epsilon)} \setminus B_{1/(4\epsilon)}} |u_0(x)|^p \, dx \le \sigma(\mathbb{S}^{n-1}) C_1 \epsilon^p \int_{1/(4\epsilon)}^{1/(2\epsilon)} r^{-\beta p+n-1} \, dr \le C_2 \epsilon^{p-n+\beta p} \,,$$

$$\int_{B_{1/(2\epsilon)} \setminus B_{1/(4\epsilon)}} |Du_0(x)|^p \, dx \le \sigma(\mathbb{S}^{n-1}) C_1 \int_{1/(4\epsilon)}^{1/(2\epsilon)} r^{-(\beta+1)p+n-1} \, dr \le C_2 \epsilon^{p-n+\beta p} \,.$$

Here C_2 is a constant which only depends on p, n, and β. Combining these estimates with (9.13), we deduce

$$\epsilon^{p-n} \|Du_{K,\epsilon}\|_p^p \le \|Du_0\|_p^p - \epsilon \frac{2p \operatorname{tr}(K)}{n-1} \int_{\mathbb{R}^n_+} |Du_0(x)|^{p-2} \partial_{x_n} u_0(x) Du_0(x) \cdot (x', 0) \, dx$$
$$+ C_3 (\epsilon^{p-n+\beta p} + \epsilon^2)$$

for some constant C_3. Choosing

$$\gamma = \min\{p - n + \beta p - 1, 1\}$$

completes the proof since $\gamma > 0$ and $C_3(\epsilon^{p-n+\beta p} + \epsilon^2) \le 2C_3 \epsilon^{1+\gamma}$. □

Proof of Lemma 9.5. We first note that $|Du_0|^p |x'| \in L^1(\mathbb{R}^n_+)$ by the decay estimate proved in [23]. It follows that the integral on the left-hand side of the lemma is well-defined. Therefore, dominated convergence implies that

$$\int_{\mathbb{R}^n_+} |Du_0(x)|^{p-2} \partial_{x_n} u_0(x) Du_0(x) \cdot (x', 0) dx$$
$$= \lim_{\delta \to 0^+} \int_{\mathbb{R}^n_+ \setminus B_\delta(e_n)} |Du_0(x)|^{p-2} \partial_{x_n} u_0(x) Du_0(x) \cdot (x', 0) dx \,.$$

Next observe that since u_0 is smooth and p-harmonic in $\mathbb{R}^n_+ \setminus \{e_n\}$,

$$|Du_0(x)|^{p-2} Du_0(x) \cdot DV(x)^\top Du_0(x) = \operatorname{div}\big(|Du_0(x)|^{p-2} Du_0(x) Du_0(x) \cdot V(x)\big)$$
$$- \frac{1}{p} D\big(|Du_0(x)|^p\big) \cdot V(x)$$

$$
\begin{aligned}
&- \underbrace{\operatorname{div}\big(|Du_0(x)|^{p-2}Du_0(x)\big)}_{\Delta_p u_0=0} Du_0(x)\cdot V(x) \\
&= \operatorname{div}\big(|Du_0(x)|^{p-2}Du_0(x)Du_0(x)\cdot V(x)\big) \\
&\quad - \frac{1}{p} D\big(|Du_0(x)|^p\big)\cdot V(x)
\end{aligned}
$$

for any $x \in \mathbb{R}^n_+ \setminus \{e_n\}$ and $V \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

Employing this identity with $V(x) = (0,\dots,0,|x'|^2/2)$ and integrating by parts gives

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^n_+\setminus B_\delta(e_n)} |Du_0(x)|^{p-2}\partial_{x_n}u_0(x)Du_0(x)\cdot(x',0)\,dx \\
&\qquad = \int\limits_{\mathbb{R}^n_+\setminus B_\delta(e_n)} \operatorname{div}\big(|Du_0(x)|^{p-2}Du_0(x)Du_0(x)\cdot V(x)\big)\,dx \\
&\qquad\quad - \frac{1}{p}\int\limits_{\mathbb{R}^n_+\setminus B_\delta(e_n)} D\big(|Du_0(x)|^p\big)\cdot V(x)\,dx \\
&\qquad = \int\limits_{\partial(\mathbb{R}^n_+\setminus B_\delta(e_n))} |Du_0(x)|^{p-2}Du_0(x)\cdot V(x)Du_0(x)\cdot\nu(x)\,d\sigma(x) \\
&\qquad\quad + \frac{1}{p}\int\limits_{\mathbb{R}^n_+\setminus B_\delta(e_n)} |Du_0(x)|^p \operatorname{div}(V(x))\,dx \\
&\qquad\quad - \frac{1}{p}\int\limits_{\partial(\mathbb{R}^n_+\setminus B_\delta(e_n))} |Du_0(x)|^p V(x)\cdot\nu(x)\,d\sigma(x)\,,
\end{aligned}
$$

where ν is the outward unit normal to the surface and σ is the surface measure. The expression above simplifies as $\operatorname{div}(V) \equiv 0$ for our choice of V. Moreover, $Du_0(x) = -\nu\partial_{x_n}u_0(x)$ for $x \in \partial\mathbb{R}^n_+$ since u_0 vanishes on $\partial\mathbb{R}^n_+$. As a result, the equality above reduces to

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^n_+\setminus B_\delta(e_n)} |Du_0(x)|^{p-2}\partial_{x_n}u_0(x)Du_0(x)\cdot(x',0)\,dx \\
&\qquad = -\frac{p-1}{2p}\int\limits_{\partial\mathbb{R}^n_+} |Du_0(x)|^p|x'|^2\,dx' \\
&\qquad\quad + \frac{1}{2}\int\limits_{\partial B_\delta(e_n)} |Du_0(x)|^{p-2}\partial_{x_n}u_0(x)|x'|^2 Du_0(x)\cdot\frac{e_n-x}{|e_n-x|}\,d\sigma(x) \\
&\qquad\quad - \frac{1}{2p}\int\limits_{\partial B_\delta(e_n)} |Du_0(x)|^p|x'|^2\frac{1-x_n}{|e_n-x|}\,d\sigma(x)\,.
\end{aligned}
\tag{9.16}
$$

Since $|x'| \leq \delta$ on $\partial B_\delta(e_n)$ and $|Du_0(x)| \leq C|x - e_n|^{-\frac{n-1}{p-1}}$ (by, e.g., [24, Proposition 2.8]), the integrals above over $\partial B_\delta(e_n)$ can be estimated as

$$\left| \int_{\partial B_\delta(e_n)} |Du_0(x)|^{p-2}\partial_{x_n} u_0(x)|x'|^2 Du_0(x) \cdot \frac{e_n - x}{|e_n - x|}\, d\sigma(x) \right| \leq \sigma(\mathbb{S}^{n-1}) C \delta^{n+1-p\frac{n-1}{p-1}}$$

and

$$\left| \int_{\partial B_\delta(e_n)} |Du_0(x)|^p |x'|^2 \frac{1 - x_n}{|e_n - x_n|}\, d\sigma(x) \right| \leq \sigma(\mathbb{S}^{n-1}) C \delta^{n+1-p\frac{n-1}{p-1}}\,.$$

Our assumption $p > n$ gives

$$n + 1 - p\frac{n-1}{p-1} = 1 + \frac{p-n}{p-1} > 1\,.$$

We conclude upon sending $\delta \to 0$ in (9.16). □

10. On the range of $\boldsymbol{\lambda_p(\Omega)}$

Theorem A asserts that

$$\lambda_p(\mathbb{R}^n \setminus \{0\}) \leq \lambda_p(\Omega) \leq \lambda_p(\mathbb{R}^n_+)$$

for any open $\Omega \subsetneq \mathbb{R}^n$. In this section, we shall use Theorem D to deduce that every point in the interval

$$\left[\lambda_p(\mathbb{R}^n \setminus \{0\}), \lambda_p(\mathbb{R}^n_+)\right]$$

equals $\lambda_p(\Omega)$ for some Ω. In fact, we shall prove that it suffices to consider the annular regions

$$A_{r_1,r_2} = \{x \in \mathbb{R}^n : r_1 < |x| < r_2\} \quad \text{for } 0 < r_1 < r_2\,.$$

Theorem 10.1. *Assume $p > n \geq 2$. For any*

$$\lambda_p(\mathbb{R}^n \setminus \{0\}) < \lambda < \lambda_p(\mathbb{R}^n_+)$$

there exists a unique $\delta \in (0,1)$ such that $\lambda = \lambda_p(A_{\delta,1})$.

Proof. If suffices to show the function $(0,1) \ni \delta \mapsto \lambda_p(A_{\delta,1})$ is increasing, continuous, and satisfies

$$\lim_{\delta\to 0^+} \lambda_p(A_{\delta,1}) = \lambda_p(\mathbb{R}^n \setminus \{0\}) \quad \text{and} \quad \lim_{\delta\to 1^-} \lambda_p(A_{\delta,1}) = \lambda_p(\mathbb{R}^n_+)\,.$$

First we will make a few preliminary observations.

By Theorem D, $A_{\delta,1}$ admits an extremal u_δ and $\lambda_p(A_{\delta,1}) < \lambda_p(\mathbb{R}^n_+)$ for each $\delta \in (0,1)$. As $A_{\delta,1}$ is rotationally symmetric, we may assume

$$u_\delta = w^{A_{\delta,1}}_{r_\delta e_n}$$

for some $r_\delta \in (\delta, 1)$. If $1 - r_\delta \le r_\delta - \delta$, then $d_{A_{\delta,1}}(r_\delta e_n) = d_{B_1}(r_\delta e_n)$. It would then follow from Proposition 5.3 that

$$\lambda_p(A_{\delta,1}) = \mathcal{R}_p(A_{\delta,1}, u_\delta) = \mathcal{R}_p(B_1, u_\delta) \ge \lambda_p(B_1) = \lambda_p(\mathbb{R}^n_+)\,,$$

which is a contradiction. Therefore,

$$r_\delta - \delta < 1 - r_\delta\,.$$

That is, $r_\delta e_n$ is closer to the inner boundary sphere of $A_{\delta,1}$ than to its outer boundary sphere.

Part 1 (Monotonicity): Suppose $0 < \delta_1 < \delta_2 < 1$. Observe that $A_{\delta_1,1}$ fully supports $A_{\delta_2,1}$. Indeed, if $y_0 \in \partial A_{\delta_2,1}$ and $|y_0| = 1$, then $A_{\delta_1,1}$ supports $A_{\delta_2,1}$ at y_0. If $|y_0| = \delta_2$ instead, then $TA_{\delta_1,1}$ supports $A_{\delta_2,1}$ at y_0, where $T(x) = (\delta_2/\delta_1)x$. By Proposition 5.3,

$$\lambda_p(A_{\delta_1,1}) \le \lambda_p(A_{\delta_2,1})\,.$$

We claim that this inequality is strict.

Note that $\tilde{w}(x) = w^{A_{\delta_2,1}}_{r_{\delta_2} e_n} \in \mathcal{D}^{1,p}_0(A_{\delta_2,\delta_2/\delta_1})$. And as $r_{\delta_2} - \delta_2 < 1 - r_{\delta_2} < \delta_2/\delta_1 - r_{\delta_2}$,

$$\lambda_p(A_{\delta_1,1}) = \lambda_p(A_{\delta_2,\delta_2/\delta_1}) \le \mathcal{R}_p(A_{\delta_2,\delta_2/\delta_1}, \tilde{w}) = \mathcal{R}_p(A_{\delta_2,1}, \tilde{w}) = \lambda_p(A_{\delta_2,1})\,.$$

Consequently, if $\lambda_p(A_{\delta_1,1}) = \lambda_p(A_{\delta_2,1})$, $\tilde{w}$ is an extremal for $\lambda_p(A_{\delta_2,\delta_2/\delta_1})$. However, this would contradict Corollary 3.2 since $\tilde{w}$ vanishes in $A_{1,\delta_2/\delta_1} \ne \emptyset$. Therefore, $\lambda_p(A_{\delta_1,1}) < \lambda_p(A_{\delta_2,1})$.

Part 2 (Continuity): We will argue that $\delta \mapsto \lambda_p(A_{\delta,1})$ is both right and left continuous at a fixed but arbitrary $\delta_0 \in (0,1)$. By the monotonicity from Part 1, the function $\delta \mapsto \lambda_p(A_{\delta,1})$ has both left and right limits at δ_0. By Lemma 2.5,

$$\lim_{\delta\to\delta_0^+} \lambda_p(A_{\delta,1}) \le \lambda_p(A_{\delta_0,1})\,.$$

Also note that the monotonicity proved above implies $\lambda_p(A_{\delta_0,1}) \le \lambda_p(A_{\delta,1})$ for $\delta > \delta_0$. Therefore,

$$\lambda_p(A_{\delta_0,1}) \leq \lim_{\delta\to\delta_0^+} \lambda_p(A_{\delta,1})\,.$$

This verifies right continuity at δ_0.

Monotonicity also gives

$$\lim_{\delta\to\delta_0^-} \lambda_p(A_{\delta,1}) \leq \lambda_p(A_{\delta_0,1})\,.$$

In order to conclude the left continuity at δ_0, we need to verify that

$$\lim_{\delta\to\delta_0^-} \lambda_p(A_{\delta,1}) \geq \lambda_p(A_{\delta_0,1})\,. \tag{10.1}$$

To this end, we choose an increasing sequence $\{\delta_k\}_{k\geq 1}$ of positive numbers tending to δ_0. Passing to a subsequence of $\{\delta_k\}_{k\geq 1}$ if necessary, we may also assume that r_{δ_k} converges. As $r_{\delta_k} \geq \delta_k$ for all k,

$$\lim_{k\to\infty} r_{\delta_k} = \delta_0 \tag{10.2}$$

or

$$\lim_{k\to\infty} r_{\delta_k} > \delta_0\,. \tag{10.3}$$

We will consider both cases separately below.

Case 1. We claim that (10.2) cannot occur. If (10.2) holds, we will show that

$$\lim_{k\to\infty} \lambda_p(A_{\delta_k,1}) = \lambda_p(\mathbb{R}^n_+)\,. \tag{10.4}$$

As $\delta \mapsto \lambda_p(A_{\delta,1})$ is increasing, (10.2) would imply that $\lambda_p(A_{\delta,1}) > \lambda_p(\mathbb{R}^n_+)$ for $\delta > \delta_0$ but this contradicts Theorem A. As a result, we can focus on proving that (10.2) implies (10.4).

Assume that (10.2) holds. Consider the rescaled sequence of extremals defined by

$$w_k(y) = w^{A_{\delta_k,1}}_{r_{\delta_k}e_n}\big((r_{\delta_k}-\delta_k)y + \delta_k e_n\big)\,.$$

Notice that $w_k = w^{\Omega_k}_{e_n}$, where Ω_k is the annulus

$$\Omega_k = \left\{ y \in \mathbb{R}^n : \frac{\delta_k}{r_{\delta_k}-\delta_k} < \left| y + \frac{\delta_k e_n}{r_{\delta_k}-\delta_k} \right| < \frac{1}{r_{\delta_k}-\delta_k} \right\}.$$

As

$$\lambda_p(A_{\delta_k,1}) = (\delta_k - r_{\delta_k})^{p-n} \| D w^{A_{\delta_k,1}}_{r_{\delta_k}e_n} \|_p^p = \| D w_k \|_p^p\,,$$

$\{w_k\}_{k\geq 1} \subset \mathcal{D}_0^{1,p}(\mathbb{R}^n \setminus \{0\})$ is bounded. By passing to a subsequence if needed, we may assume $w_k \rightharpoonup w_0$ in $\mathcal{D}_0^{1,p}(\mathbb{R}^n \setminus \{0\})$ and $w_k \to w_0$ uniformly in B_2. It follows that

$$\lim_{k\to\infty} \lambda_p(A_{\delta_k,1}) = \liminf_{k\to\infty} \|Dw_k\|_p^p \geq \|Dw_0\|_p^p.$$

Recall that $B_{s_k}(-s_k e_n)$ is an exhaustion of $\{y \in \mathbb{R}^n : y_n < 0\}$ for any increasing sequence $s_k \to \infty$. Since

$$B_{\frac{\delta_k}{r_{\delta_k}-\delta_k}}\Big(-\frac{\delta_k e_n}{r_{\delta_k}-\delta_k}\Big) \subset \Omega_k^c$$

and $\delta_k/(r_{\delta_k}-\delta_k) \to \infty$, $B_s(-se_n) \subset \Omega_k^c$ for any $s > 0$ provided k is large enough. It follows that w_0 vanishes in $B_s(-se_n)$ for every $s > 0$. This implies $w_0(y) = 0$ whenever $y_n \leq 0$. Therefore, $w_0 \in \mathcal{D}_0^{1,p}(\mathbb{R}^n_+)$. By uniform convergence in B_2, we also have $w_0(e_n) = 1$. Thus,

$$\lim_{k\to\infty} \lambda_p(A_{\delta_k,1}) \geq \|Dw_0\|_p^p \geq \mathcal{R}_p(\mathbb{R}^n_+, w_0) \geq \lambda_p(\mathbb{R}^n_+)\,.$$

By Theorem A, $\lambda_p(A_{\delta_k,1}) \leq \lambda_p(\mathbb{R}^n_+)$ for each k and thus we conclude that (10.2) implies (10.4).

Case 2. Now we assume that (10.3) holds. By passing to a subsequence, we may assume that the sequence $w_k = w^{A_{\delta_k,1}}_{r_{\delta_k}e_n}$ converges weakly to some $w_0 \in \mathcal{D}_0^{1,p}(A_{\delta_0,1})$ and uniformly to w_0 on compact sets. In particular, $w_0(r_0 e_n) = 1$. We deduce that

$$\begin{aligned}\lim_{k\to\infty} \lambda_p(A_{\delta_k,1}) &= \lim_{k\to\infty} (\delta_k - r_{\delta_k})^{p-n}\|Dw_k\|_p^p \\ &\geq (\delta_0 - r_0)^{p-n}\|Dw_0\|_p^p \\ &\geq \mathcal{R}_p(A_{\delta_0,1}, w_0) \\ &\geq \lambda_p(A_{\delta_0,1})\,.\end{aligned}$$

Therefore, (10.1) holds. This concludes the proof that $\delta \mapsto \lambda_p(A_{\delta,1})$ is continuous.

Part 3 (Endpoint limits): By Theorem A, $\lambda_p(A_{\delta,1}) \geq \lambda_p(\mathbb{R}^n \setminus \{0\})$ for any $\delta \in (0,1)$. Lemma 2.5 yields that

$$\lim_{\delta\to 0^+} \lambda_p(A_{\delta,1}) \leq \lambda_p(B_1 \setminus \{0\})\,.$$

Therefore, in view of Theorem B,

$$\lim_{\delta\to 0^+} \lambda_p(A_{\delta,1}) = \lambda_p(B_1 \setminus \{0\}) = \lambda_p(\mathbb{R}^n \setminus \{0\})\,.$$

As for the limit

$$\lim_{\delta\to 1^+} \lambda_p(A_{\delta,1}) = \lambda_p(\mathbb{R}^n_+)\,,$$

we can adapt our argument in Case 1 of Part 2 by choosing a sequence of positive numbers δ_k tending to 1 from below. In this case, we also have $r_{\delta_k} \in (\delta_k, 1)$ for each k. Therefore, it must be that $r_{\delta_k} \to 1$. As a result, the argument used in Case 1 of Part 2 translates directly to establish the above limit. □

11. Examples

In Corollary 8.8, we computed $\Lambda_p(\Omega) = \lambda_p(\mathbb{R}^n_+)$ under the assumption that all boundary points of Ω are regular. The key to this result was that we were able to characterize all limiting geometries as halfspaces. The existence of well-defined limits may hold under weaker regularity assumptions on $\partial\Omega$. However, it is in general difficult both to characterize all possible limits and even more so to compute the value of λ_p at these limits. In this section, we shall illustrate how the analysis can be carried for certain classes of domains.

11.1. Polygonal domains

We will now proceed to study the best constant λ_p in polygonal domains. Specifically, we say that a domain $\Omega \subset \mathbb{R}^2$ is *polygonal* if $\partial\Omega$ is nonempty and consists of the union of finitely many line segments or rays $\{\Gamma_j\}_{j=1}^K$ only intersecting at their endpoints. We'll also require that each endpoint belongs to exactly two of the Γ_j's. We will denote by $\mathcal{S}_\Omega$ the collection of corners of a polygonal domain Ω, which are defined as the set of endpoints of the Γ_j. To each $y \in \mathcal{S}_\Omega$ we can associate the corresponding interior angle $\varphi_y \in (0, 2\pi)$. We emphasize that we do not exclude $\varphi_y = \pi$ even though such an angle could be removed by merging the two line segments that meet at y.

The following claim is a corollary of Lemma 8.1. It basically asserts that the family of cones $\mathcal{C}^2_\varphi$ for $\varphi \in (0, \pi)$ constitutes the natural class of local model sets for polygonal domains.

Lemma 11.1. *If $\Omega \subsetneq \mathbb{R}^2$ is polygonal with $\mathcal{S}_\Omega \neq \emptyset$, then*

$$\Lambda_p(\Omega) \le \min_{y \in \mathcal{S}_\Omega} \lambda_p(\mathcal{C}^2_{\varphi_y/2}) = \lambda_p(\mathcal{C}^2_{\max\limits_{y \in \mathcal{S}_\Omega} \varphi_y/2}) \,.$$

Furthermore, if Ω is bounded equality holds.

Proof. Each $y \in \mathcal{S}_\Omega$ is isolated, so there exists an $r > 0$ small enough so that up to translation and rotation $B_r(y) \cap \Omega$ agrees with $B_r \cap \mathcal{C}^2_{\varphi_y/2}$. Therefore the inequality follows by applying Lemma 8.1 at each of the corners and recalling that $\varphi \mapsto \lambda_p(\mathcal{C}^2_\varphi)$ is non-increasing by Lemma 7.5.

To deduce that we have equality in the case when Ω is bounded we argue as in Corollary 8.8. Take a sequence $\{x_k\}_{k\ge1} \in \mathcal{Y}_\Omega$ such that

$$\lim_{k\to\infty} d_\Omega(x_k)^{p-n} \|Dw^\Omega_{x_k}\|_p^p = \Lambda_p(\Omega) \,.$$

Since $\{x_k\}_{k\geq 1} \in \mathcal{Y}_\Omega$ and $\overline{\Omega}$ is compact, $\{x_k\}_{k\geq 1}$ has a convergent subsequence whose limit x_0 belongs to $\partial\Omega$. If $x_0 \notin \mathcal{S}_\Omega$, then $\partial\Omega$ is C^1-regular in a neighborhood of x_0 so $\Lambda_p(\Omega) \geq \lambda_p(\mathbb{R}^n_+)$ by Lemma 8.6. Alternatively, if $x_0 \in \mathcal{S}_\Omega$, then following the argument in Lemma 8.6 one proves that $\Lambda_p(\Omega) \geq \lambda_p(\mathcal{C}^2_{\varphi_{x_0}/2})$ if the approach of $\{x_k\}_{k\geq 1}$ to x_0 is non-tangential while $\Lambda_p(\Omega) \geq \lambda_p(\mathbb{R}^n_+)$ if the approach is tangential. Since we established above that $\Lambda_p(\Omega) \leq \min_{y\in\mathcal{S}_\Omega} \lambda_p(\mathcal{C}^2_{\varphi_y/2})$, it follows from Lemma 7.5 and Theorem A that either

$$\Lambda_p(\Omega) = \Lambda_p(\mathcal{C}^2_{\varphi_{x_0}/2}) = \min_{y\in\mathcal{S}_\Omega} \lambda_p(\mathcal{C}^2_{\varphi_y/2}) \quad \text{or} \quad \Lambda_p(\Omega) = \lambda_p(\mathbb{R}^n_+) = \min_{y\in\mathcal{S}_\Omega} \lambda_p(\mathcal{C}^2_{\varphi_y/2}).$$

We note that the latter case can only happen if $\max_{y\in\mathcal{S}_\Omega} \varphi_y \leq \pi$. In either case, we have proved the desired equality. □

Recall that Theorem B asserts that convex domains in $\mathbb{R}^n$ which are not halfspaces do not admit extremals. By applying Proposition 5.3 there is a similar criterion for a class of polygonal domains. We say that a polygonal domain Ω is *fully supported* if Ω is fully supported by $\mathcal{C}^2_{\varphi^*/2}$ with $\varphi^* = \max_{y\in\mathcal{S}_\Omega} \varphi_y$. In the case $\varphi^* \leq \pi$, Ω is a convex polygon, while for $\varphi^* > \pi$, Ω satisfies a uniform (infinite) outer cone condition with opening angle $2\pi - \varphi^*$.

Lemma 11.2. *Suppose $\Omega \subset \mathbb{R}^2$ is a fully supported polygonal domain with $\varphi^* = \max_{y\in\mathcal{S}_\Omega} \varphi_y$. Then*

$$\lambda_p(\Omega) = \lambda_p(\mathcal{C}^2_{\varphi^*/2}).$$

Moreover, Ω does not admit an extremal unless $\Omega = x + Q\mathcal{C}^2_{\varphi^*/2}$ for some $x \in \mathbb{R}^2$ and $Q \in O(2)$.

Remark 11.3. In Fig. 6, four polygonal domains are shown together with the implications of Lemmas 11.1 and 11.2 concerning their respective values of λ_p.

Proof. That $\lambda_p(\Omega) = \lambda_p(\mathcal{C}^2_{\varphi^*/2})$ is a direct consequence of Proposition 5.3 and Lemma 11.1. To see that Ω does not admit an extremal, we can argue that the existence of an extremal would contradict Corollary 3.2 (as in the proof of Theorem B). □

Lemmas 11.1 and 11.2 can be directly extended to planar domains whose boundary is C^1 outside of a finite set of corners at which the boundary is given by two simple C^1 curves that meet at a common endpoint. In this setting it is natural to allow for corners with interior angles equal to 0 or 2π to allow for boundaries with cusps. Lemmas 11.1 and 11.2 can also be generalized to polytopes in higher dimension by following the argument given above almost verbatim. However, the statements obtained become more complicated as the class of relevant model sets is much larger and our understanding of λ_p for these sets is limited. Nevertheless, in the next subsection, we will provide a family

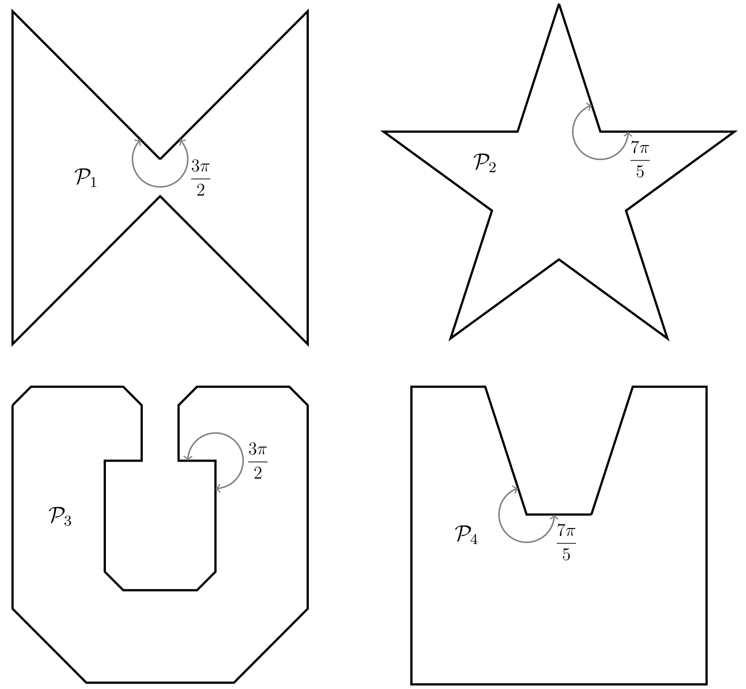

Fig. 6. Four polygonal domains. For $\mathcal{P}_1, \mathcal{P}_3$ the largest interior angle is $\frac{3\pi}{2}$ and for $\mathcal{P}_2, \mathcal{P}_4$ the largest interior angle is $\frac{7\pi}{5}$. The polygons $\mathcal{P}_1, \mathcal{P}_2$ are fully supported while $\mathcal{P}_3, \mathcal{P}_4$ are not. Therefore, by Lemmas 11.1 and 11.2, we know that $\lambda_p(\mathcal{P}_1) = \lambda_p(\mathcal{C}^2_{3\pi/4})$, $\lambda_p(\mathcal{P}_2) = \lambda_p(\mathcal{C}^2_{7\pi/10})$, $\lambda_p(\mathcal{P}_3) \le \lambda_p(\mathcal{C}^2_{3\pi/4})$, $\lambda_p(\mathcal{P}_4) \le \lambda_p(\mathcal{C}^2_{7\pi/10})$.

of bounded, non-smooth, and non-convex domains in arbitrary dimension where we can determine the value of λ_p.

11.2. Examples of non-smooth domains in higher dimensions

Using the strategy discussed in the previous subsection, we can construct examples of bounded simply connected domains Ω whose boundary is regular except at a single point and $\lambda_p(\Omega) = \lambda_p(\mathcal{C}^n_\varphi)$ for every $n \ge 2$ and $\varphi \in (0, \pi]$. Specifically one can construct such domains by starting from $B_1 \cap \mathcal{C}^n_\varphi$ and regularizing the boundary in a small neighborhood of $\partial B_1 \cap \partial \mathcal{C}^n_\varphi$ in such a manner that the resulting set remains fully supported by $\mathcal{C}^n_\varphi$. Two such domains are depicted in Fig. 7. The resulting set Ω satisfies that $\lambda_p(\Omega) = \lambda_p(\mathcal{C}^n_\varphi)$ by combining Proposition 5.3 and Lemma 8.1 with a blow-up around the singular boundary point.

11.3. Epigraphs

In much of our analysis we have utilized Lemma 8.1 and Proposition 8.5 by looking at the local geometry near some point on the boundary. In this subsection we demon-

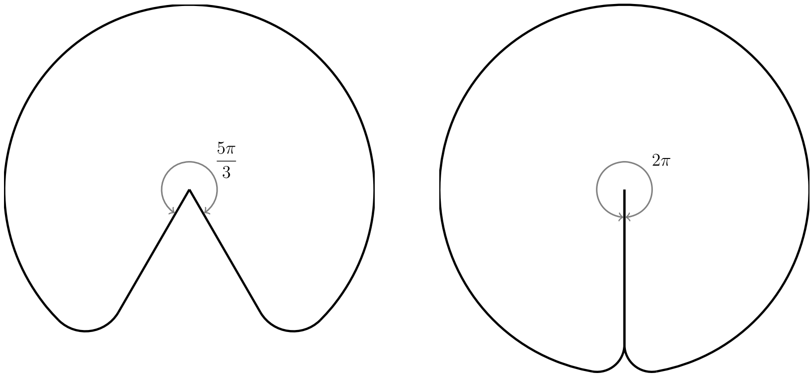

Fig. 7. Two planar domains constructed as described. The value of λ_p for domain on the left is $\lambda_p(\mathcal{C}^2_{5\pi/6})$ and for the domain to the right $\lambda_p(\mathcal{C}^2_\pi)$. Examples for $n > 2$ can be obtained by rotation around the axis of symmetry.

strate how these results can yield interesting information when instead zooming out. An argument of this form was applied in the proof of case (3) in Theorem B but the idea is interesting enough to deserve including a second example. Specifically, we shall consider sets given by the region that lies above the graph of a function.

Let $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ be a continuous function with $f(0) = 0$ and such that along a sequence $\{t_k\}_{k\geq 1} \subset (0,\infty)$ with $\lim_{k\to\infty} t_k = 0$ it holds that $F_k\colon \mathbb{R}^{n-1} \to \mathbb{R}$ defined by $z \mapsto t_k f(z/t_k)$ converges locally uniformly to a one-homogeneous function $F\colon \mathbb{R}^{n-1} \to \mathbb{R}$. If we define

$$\Omega_f = \{x = (x', x_n) \in \mathbb{R}^n : x_n > f(x')\}$$

and the dilation invariant set

$$\Omega_F = \{x \in (x', x_n) \in \mathbb{R}^n : x_n > F(x')\}$$

then it holds that $\Lambda_p(\Omega_f) \leq \lambda_p(\Omega_F)$. Indeed, by the uniform convergence when zooming further and further out the set Ω_f locally converges to Ω_F allowing us to apply Lemma 8.1. Indeed, given $\delta > 0$ there is k sufficiently large so that

$$(\Omega_F + \delta e_n) \cap B_1 \subset t_k\Omega_f \cap B_1 \subset (\Omega_F - \delta e_n) \cap B_1 \,.$$

Thus by arguing as in the proof of Corollary 8.2, the assumptions of Lemma 8.1 are fulfilled.

A similar argument can be made to work even if the limit of $tf(\theta/t)$ is infinite on some set. However, in this situation one needs to replace the locally uniform convergence in a suitable manner which is in general difficult. But in special cases there are natural ways to do this. For instance if $f\colon \mathbb{R}^n \to \mathbb{R}$ is such that $f(0) = 0$ and $f(x) \leq -c|x|^\alpha$ if $|x| \geq R$ for some constants $c > 0, \alpha > 1$, and $R > 0$, then arguing as above one proves

$$\Lambda_p(\Omega_f) \leq \lambda_p(\mathcal{C}^n_\pi)\,.$$

In fact, since Ω_f (for any f) is fully supported by $\mathcal{C}^n_\pi$ the prescribed asymptotic behavior implies that $\lambda_p(\Omega_f) = \lambda_p(\mathcal{C}^n_\pi)$ by Proposition 5.3. Indeed, the picture to keep in mind is that if viewing Ω_f from farther and farther away the asymptotic behavior of f will lead to the epigraph more and more resembling $\mathcal{C}^n_\pi$.

11.4. Instability under small perturbations

Here we provide a few examples of domains which are almost identical but in which the value of λ_p and the existence/non-existence of extremals are different.

Example 11.4. Let $\Omega \subsetneq \mathbb{R}^n$ be an open set satisfying the assumptions of Theorem D. Then $\lambda_p(\Omega) < \lambda_p(\mathbb{R}^n_+)$ and Ω admits an extremal. Assume that $\lambda_p(\Omega) > \lambda_p(\mathcal{C}^n_\varphi)$ for some $\varphi \in (\pi/2, \pi]$. Given any $r > 0$ and $x_0 \in \partial\Omega$ we can construct a set Ω' such that $\Omega\Delta\Omega' \subset B_r(x_0)$ and $\lambda_p(\Omega') \leq \lambda_p(\mathcal{C}^n_\varphi) < \lambda_p(\Omega)$. The idea is to remove a small conical piece of Ω near x_0, see Fig. 8.

By translation and rotation we can without loss of generality assume that $x_0 = 0$ and that the outward unit pointing normal to $\partial\Omega$ at x_0 is $(0, \ldots, -1)$. The regularity of $\partial\Omega$ ensures that there is an $0 < r' < r$ so that $\partial\Omega \cap B_{r'}$ is contained in a $r'/4$ neighborhood of the hyperplane $x_n = 0$. We can then take Ω' as

$$(\Omega \cap B^c_{r'}) \cup (\Omega \cap (\mathcal{C}^n_\varphi + \tfrac{r'}{2} e_n))\,.$$

That is we locally remove a conical piece of Ω in such a manner that we create a singular boundary point matching that of $\mathcal{C}^n_\varphi$. That $\lambda_p(\Omega') \leq \lambda_p(\mathcal{C}^n_\varphi)$ now follows directly from Lemma 8.1.

Example 11.5. Let $\Omega \subsetneq \mathbb{R}^n$ be an open, bounded, convex set with C^2-regular boundary. Part (1) of Theorem B implies that $\lambda_p(\Omega) = \lambda_p(\mathbb{R}^n_+)$ and that Ω does not admit an extremal. Given $r > 0, x_0 \in \partial\Omega$ we can construct $\Omega' \subset \mathbb{R}^n$ so that $\Omega\Delta\Omega' \subset B_r(x_0)$ and, either

(1) $\lambda_p(\Omega') = \lambda_p(\mathcal{C}^n_\varphi)$ for any $\varphi \in (\pi/2, \pi]$ and Ω' does not admit an extremal, or
(2) $\lambda_p(\Omega') < \lambda_p(\mathbb{R}^n_+)$ and Ω' admits an extremal.

The construction for (1) is identical to that in the previous example we consider $\Omega' = \Omega \cap (Q\mathcal{C}^n_\varphi + y_0)$ for a suitably chosen $y_0 \in \Omega \cap B_r(x_0)$ and $Q \in O(n)$. That $\lambda_p(\Omega') \geq \lambda_p(\mathcal{C}^n_\varphi)$ follows by noting that Ω being convex implies that Ω' is fully supported by $\mathcal{C}^n_\varphi$. By Lemma 8.1 and a blow-up at y_0 we also have $\lambda_p(\Omega') \leq \Lambda_p(\Omega') \leq \lambda_p(\mathcal{C}^n_\varphi)$. Non-existence of an extremal follows by observing that if an extremal existed it would

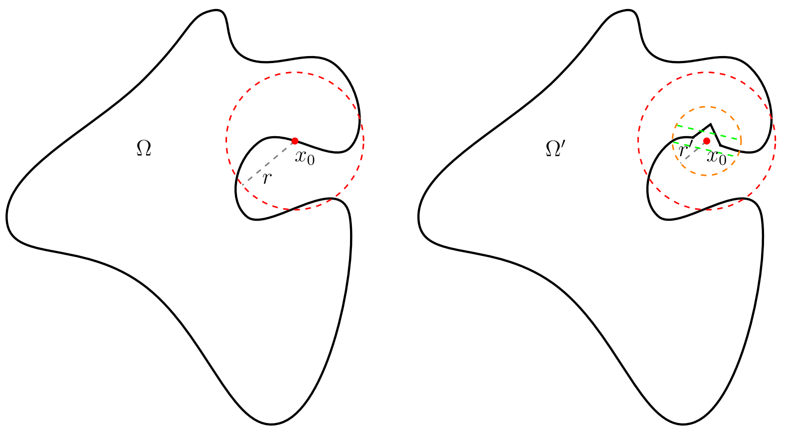

Fig. 8. A pictorial description of Example 11.4. Given Ω, x_0, r we construct the modified set Ω' by locally removing a conical piece from Ω.

be extremal also for some rotated and translated copy of $\mathcal{C}_\varphi^n$ and so Corollary 3.2 would imply that Ω' coincides with $\mathcal{C}_\varphi^n$ up to translation and rotation which is impossible.

The construction for (2) is similar but instead of introducing a singular point at the boundary we make a smooth indentation and apply Theorem D. Since Ω is convex there exists an $0 < r' < r$ so that $\partial\Omega \cap B_{r'}(x_0)$ can be represented as the graph of a C^2 convex function. By rotation and translating we may assume that $x_0 = 0$ and that there is a convex function $f\colon \mathbb{R}^{n-1} \to [0,\infty)$ such that $f(0) = |Df(0)| = 0$ and

$$\Omega \cap B_{r'} = \{x = (x', x_n) \in \mathbb{R}^n : x_n > f(x')\} \cap B_{r'}\,.$$

If we set

$$\phi_\delta(x') = \begin{cases} e^{1-1/(1-|x'|^2/\delta^2)} & \text{for } |x'| < \delta\,, \\ 0 & \text{otherwise,} \end{cases}$$

then we can define Ω' by letting

$$\Omega' \cap B_{r'} = \Big\{x = (x', x_n) \in \mathbb{R}^n : x_n > f(x') + \frac{r'}{2}\phi_\delta(x')\Big\} \cap B_{r'}$$

for some $0 < \delta < r'$. Provided δ is chosen sufficiently small the mean curvature of the boundary at $(0, \ldots, 0, r'/2)$ will be negative and we can apply Theorem D to draw the desired conclusion.

Example 11.6. Fix a nontrivial $f \in C^2(\mathbb{R}^{n-1})$ which has compact support and define

$$\Omega_f = \{x = (x', x_n) \in \mathbb{R}^n : x_n > f(x')\}\,.$$

Then Ω_f admits an extremal and $\lambda_p(\Omega_f) < \lambda_p(\mathbb{R}^n_+)$. To see this we first argue that $\Lambda_p(\Omega_f) = \lambda_p(\mathbb{R}^n_+)$. By Remark 8.4, $\Lambda_p(\Omega_f) \le \lambda_p(\mathbb{R}^n_+)$. For the reverse inequality, let $\{x_k\}_{k\ge1} \in \mathcal{Y}_{\Omega_f}$ realize the infimum defining $\Lambda_p(\Omega_f)$. Then up to passing to a subsequence we may assume that either $\liminf_{k\to\infty} |x_k| = \infty$ or $\limsup_{k\to\infty} d_{\Omega_f}(x_k) = 0$. In either case, as f is C^1 and has compact support, the set $\{v > 0\}$ in Proposition 8.5 will be a halfspace so $\Lambda_p(\Omega_f) \ge \lambda_p(\mathbb{R}^n_+)$. Since

$$\int_{\mathbb{R}^{n-1}} \operatorname{div}\left(\frac{Df(x')}{\sqrt{1+|Df(x')|^2}}\right) dx' = 0$$

it follows that the mean curvature of $\partial\Omega_f$ must be negative somewhere. Therefore, Theorem 9.2 implies that $\lambda_p(\Omega_f) < \lambda_p(\mathbb{R}^n_+)$. The desired conclusion thus follows from Proposition 3.4.

12. Open problems

We bring this paper to an end by listing a few open problems that appear as natural possible extensions of the results we obtained.

Open problem 1. Assume that $\Omega \subsetneq \mathbb{R}^n$ is connected, that $\lambda_p(\Omega) = \lambda_p(\mathbb{R}^n_+)$, and that Ω admits an extremal. Is it true that Ω is a halfspace?

Open problem 2. Assume that $\Omega \subsetneq \mathbb{R}^n$ is mean-convex. Is it true that $\lambda_p(\Omega) = \lambda_p(\mathbb{R}^n_+)$?

Open problem 3. Assume that $\Omega \subsetneq \mathbb{R}^n$ is bounded and has boundary homeomorphic to $\mathbb{S}^{n-1}$. Is it true that $\lambda_p(\Omega) \ge \lambda_p(\mathcal{C}^n_\pi)$?

Open problem 4. If $\Omega \subsetneq \mathbb{R}^n$ admits an extremal, is the extremal unique up to multiplication by constants and similarity transforms that leave Ω invariant?

Remark on open problem 1. By Theorem B, the only convex domains with an extremal are halfspaces. We also recall that if $\Omega \subset \mathbb{R}^2$ is a bounded C^2 domain, then either Ω is convex, Ω does not have an extremal, and $\lambda_p(\Omega) = \lambda_p(\mathbb{R}^2_+)$ or Ω is not convex, Ω has an extremal, and $\lambda_p(\Omega) < \lambda_p(\mathbb{R}^2_+)$.

Remark on open problem 2. We saw that if Ω is bounded and fails to be mean-convex, then $\lambda_p(\Omega) < \lambda_p(\mathbb{R}^n_+)$. Furthermore, the hypothesis of this problem implies that every connected component of Ω is convex when $n = 2$. It follows that in the plane, the answer is yes by Theorem B. It is also worth noting that for Hardy's inequality (1.5), the sharp constant in any bounded, C^2, and mean-convex domain coincides with the constant of the halfspace [30]. That is, the analog of open problem 2 is settled for inequality (1.5).

Remark on open problem 3. The motivation behind this problem is to further understand to what degree λ_p is governed by local and/or global geometric properties. The assumptions entail that Ω is topologically very simple and the suggested lower bound is motivated by the fact that $\mathcal{C}^n_\pi$ should be the blow-up that gives the lowest value of λ_p

possible under the assumptions. We note that Proposition 5.3 yields the desired conclusion if Ω is additionally assumed to be fully supported by $\mathcal{C}^n_\pi$. In particular, this includes the case when Ω is a star-shaped domain. Indeed, suppose Ω is star-shaped with respect to the $0 \in \Omega$ and $x \in \partial\Omega$. Then $tx \in \Omega$ for $t \in [0,1)$ and $tx \notin \Omega$ for $t \geq 1$. It follows that there is $Q \in O(n)$ with $Q(\Omega - x) \subset \mathcal{C}^n_\pi$. As $x \in \partial\Omega$ was arbitrary, Ω is fully supported by $\mathcal{C}^n_\pi$.

Remark on open problem 4. The answer is yes when Ω is a halfspace or a punctured wholespace for $n \geq 2$. This follows from Proposition 6.1 and the uniqueness of Morrey extremals up to similarity transformations (Section 3 of [26]).

Declaration of competing interest

The authors declare that there are no conflicts of interest.

Appendix A. Approximation

This appendix is dedicated to proving Lemma 2.3. To this end, we suppose $u \in \mathcal{D}^{1,p}_0(\Omega)$ and $\epsilon > 0$. Recall that our goal is to find $v \in C^\infty_c(\Omega)$ with

$$\|Du - Dv\|_p \leq \epsilon \|Du\|_p\,. \tag{A.1}$$

By translating Ω if necessary, we may assume that $0 \in \partial\Omega$. And to ease notation in the following proof, we will write $d(x)$ for $d_\Omega(x)$.

Step 1. First, we choose a non-increasing $\eta \in C^\infty([0,\infty))$ with $0 \leq \eta \leq 1$, $\eta(1/2) = 1$, and $\eta(1) = 0$. Next, we set

$$f(x) = (1 - \eta(d(x)/\delta))\eta(|x|/r) \tag{A.2}$$

for $x \in \Omega$ and $r, \delta > 0$. It is evident that f is supported in $\{x \in \Omega : d(x) \geq \delta/2, |x| \leq r\}$,

$$f = 1 \text{ in } \{x \in \Omega : d(x) \geq \delta, |x| \leq r/2\}\,, \tag{A.3}$$

and f is Lipschitz continuous. It follows that

$$v_1 = fu \in \mathcal{D}^{1,p}_0(\Omega)$$

and

$$\operatorname{supp}(v_1) \subset \{x \in \Omega : d(x) \geq \delta/2, |x| \leq r\}\,. \tag{A.4}$$

Direct computation gives

$$Dv_1(x) = f(x)Du(x) - \delta^{-1}\eta'(d(x)/\delta)\eta(|x|/r)u(x)Dd(x) \\ + r^{-1}(1-\eta(d(x)/\delta))\eta'(|x|/r)u(x)\frac{x}{|x|}\,.$$

Employing (A.3) and recalling $|Dd| \leq 1$ almost everywhere, we also find

$$\|Du - Dv_1\|_p \leq \|\chi_{\{d<\delta\}}Du\|_p + \|\chi_{\mathbb{R}^n\setminus B_{r/2}}Du\|_p + \delta^{-1}\|\eta'\|_\infty\|\chi_{\{d<\delta\}}u\|_p \\ + r^{-1}\|\eta'\|_\infty\|\chi_{B_r\setminus B_{r/2}}u\|_p\,.$$

By dominated convergence,

$$\|\chi_{\{d<\delta\}}Du\|_p + \|\chi_{\mathbb{R}^n\setminus B_{r/2}}Du\|_p \leq \frac{\epsilon}{4}\|Du\|_p\,,$$

provided we choose δ sufficiently small and r sufficiently large. To see that the remaining terms $\delta^{-1}\|\eta'\|_\infty\|\chi_{\{d<\delta\}}u\|_p$ and $r^{-1}\|\eta'\|_\infty\|\chi_{B_r\setminus B_{r/2}}u\|_p$ can also be made small, we argue as follows.

Step 2. By Morrey's estimate (2.2), there is a constant c such that

$$|u(x)|^p \leq cd(x)^{p-n}\int\limits_{B_{d(x)}(x)}|Du(w)|^p\,dw$$

for all $x \in \Omega$. Therefore,

$$\delta^{-p}\int\limits_\Omega \chi_{\{d<\delta\}}(x)|u(x)|^p\,dx \leq c\delta^{-p}\int\limits_\Omega \chi_{\{d<\delta\}}(x)d(x)^{p-n}\int\limits_{B_{d(x)}(x)}|Du(w)|^p\,dwdx \\ = c\delta^{-p}\int\limits_\Omega\int\limits_\Omega \chi_{\{d<\delta\}}(x)\chi_{B_{d(x)}(x)}(w)d(x)^{p-n}|Du(w)|^p\,dwdx\,.$$

It is not hard to see that if $d(x) < \delta$ and $w \in B_{d(x)}(x)$, then $d(w) < 2\delta$ and $x \in B_\delta(w)$. As a result,

$$d(x)^{p-n}\chi_{\{d<\delta\}}(x)\chi_{B_{d(x)}(x)}(w) \leq \delta^{p-n}\chi_{\{d<\delta\}}(x)\chi_{B_{d(x)}(x)}(w) \\ \leq \delta^{p-n}\chi_{\{d<2\delta\}}(w)\chi_{B_\delta(w)}(x)\,.$$

Thus, there is a constant C with

$$\delta^{-p}\int\limits_\Omega \chi_{\{d<\delta\}}(x)|u(x)|^p\,dx \leq c\delta^{-n}\int\limits_\Omega\int\limits_\Omega \chi_{\{d<2\delta\}}(w)\chi_{B_\delta(w)}(x)|Du(w)|^p\,dwdx$$

$$\leq C \int_{\Omega} \chi_{\{d<2\delta\}}(w)|Du(w)|^p \, dw\,.$$

By dominated convergence,

$$\delta^{-1}\|\eta'\|_\infty \|\chi_{\{d<\delta\}} u\|_p \leq \frac{\epsilon}{4}\|Du\|_p$$

provided $\delta > 0$ is small enough.

Step 3. In order to estimate $r^{-1}\|u\chi_{B_r\setminus B_{r/2}}\|_p$, we first change variables in the integral

$$r^{-p} \int_{B_r\setminus B_{r/2}} |u(x)|^p \, dx = \int_{B_1\setminus B_{1/2}} |r^{n/p-1}u(ry)|^p \, dy\,.$$

Recall that we assumed that $0 \in \Omega^c$. Thus for $y \in B_1 \setminus B_{1/2}$ such that $ry \in \mathrm{supp}(v) \subset \Omega$,

$$|r^{n/p-1}u(ry)| = \frac{d(ry)^{1-n/p}}{r^{1-n/p}} \frac{|u(ry)|}{d(ry)^{1-n/p}} \leq |y|^{1-n/p} \frac{|u(ry)|}{d(ry)^{1-n/p}} \leq \frac{|u(ry)|}{d(ry)^{1-n/p}}\,.$$

Consequently,

$$r^{-1}\|\chi_{B_r\setminus B_{r/2}} u\|_p \leq \left(\int_{B_1\setminus B_{1/2}} \frac{|u(ry)|^p}{d(ry)^{p-n}} \, dy \right)^{1/p}.$$

By (1.1), the functions $B_1 \ni y \mapsto \frac{|u(ry)|}{d(ry)^{1-n/p}}$ are bounded uniformly in $r > 0$; and by Lemma 2.2 (*ii*), they tend to zero pointwise in the limit as $r \to \infty$. Therefore,

$$r^{-1}\|\eta'\|_\infty \|\chi_{B_r\setminus B_{r/2}} u\|_p \leq \frac{\epsilon}{4}\|Du\|_p$$

provided $r > 0$ is large enough. In summary, for $r > 0$ sufficiently large and $\delta > 0$ sufficiently small

$$\|Du - Dv_1\|_p \leq \frac{3\epsilon}{4}\|Du\|_p\,.$$

Step 4. Select any $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\mathrm{supp}(\psi) \in B_1$ and $\int_{\mathbb{R}^n} \psi(x)\, dx = 1$. Also define $\psi_\tau = \tau^{-n}\psi(\cdot/\tau)$ and note that $\mathrm{supp}(\psi_\tau) \in B_\tau$. Using (A.4), it is straightforward to verify

$$v = \psi_\tau * v_1 \in C_c^\infty(\Omega)$$

for $\tau < \delta/2$. Note that

$$\|Du - Dv\|_p \leq \|Du - Dv_1\|_p + \|Dv_1 - Dv\|_p \leq \frac{3\epsilon}{4}\|Du\|_p + \|Dv_1 - \psi_\tau * Dv_1\|_p\,.$$

Since $|Dv_1| \in L^p(\mathbb{R}^n)$, it follows from standard results on mollification that the last term is smaller than $\frac{\epsilon}{4}\|Du\|_p$ for $\tau > 0$ chosen sufficiently small (see, e.g. [14]). This concludes the proof of (A.1).

Appendix B. $\Lambda_p(\Omega)$ is attained

In this appendix, we argue that for any open $\Omega \subsetneq \mathbb{R}^n$ there exists a sequence $\{x_k\}_{k\geq 1} \subset \mathcal{Y}_\Omega$ which realizes the infimum defining $\Lambda_p(\Omega)$. Namely, we will show

$$\liminf_{k\to\infty} d_\Omega(x_k)^{p-n}\|Dw^\Omega_{x_k}\|_p^p = \Lambda_p(\Omega)\,.$$

With this goal in mind, we let $\{\epsilon_j\}_{j\geq 1} \subset (0,1)$ satisfy $\lim_{j\to\infty}\epsilon_j = 0$. For each j, there exists a sequence $\{x^j_k\}_{k\geq 1} \in \mathcal{Y}_\Omega$ with

$$\liminf_{k\to\infty} d_\Omega(x^j_k)^{p-n}\|Dw^\Omega_{x^j_k}\|_p^p \leq \Lambda_p(\Omega) + \epsilon_j\,. \tag{B.1}$$

Let $A \subset \mathbb{N}$ be the subset of indexes j such that $\liminf_{k\to\infty}|x^j_k| = \infty$. Recall that for $j \in A^c$, $\limsup_{k\to\infty} d_\Omega(x^j_k) = 0$. Notice that at most one of A and A^c is a finite set.

First suppose that A is infinite, and consider the subsequence of the sequence of sequences $\{x^j_k\}_{k\geq 1}$ for which $j \in A$. Relabeling the subsequence if necessary, we obtain a collection of sequences $\{x^j_k\}_{k\geq 1} \in \mathcal{Y}_\Omega$ which satisfy (B.1) for some sequence of positive numbers $\{\epsilon_j\}_{j\geq 1}$ with $\lim_{j\to\infty}\epsilon_j = 0$ and for which $\liminf_{k\to\infty}|x^j_k| = \infty$ for each j.

We now iteratively construct a new sequence $\{x^*_k\}_{k\geq 1} \in \mathcal{Y}_\Omega$ as follows. Let $x^*_1 = x^1_1$. Given $\{x^*_k\}_{k=1}^{N-1}$ for $N \geq 2$, choose $x^*_N = x^N_l$, where l is the first index so that

$$|x^N_l| \geq |x^*_{N-1}| + 1 \quad \text{and} \quad d_\Omega(x^N_l)^{p-n}\|Dw^\Omega_{x^N_l}\|_p^p \leq \Lambda_p(\Omega) + 2\epsilon_N\,.$$

The construction implies that

$$\liminf_{k\to\infty}|x^*_k| \geq \liminf_{k\to\infty}(k-1) = \infty$$

and

$$\liminf_{k\to\infty} d_\Omega(x^*_k)^{p-n}\|Dw^\Omega_{x^*_k}\|_p^p \leq \liminf_{k\to\infty}\Lambda_p(\Omega) + 2\epsilon_k = \Lambda_p(\Omega)\,.$$

Thus $\{x^*_k\}_{k\geq 1} \in \mathcal{Y}_\Omega$ and

$$\liminf_{k\to\infty} d_\Omega(x^*_k)^{p-n}\|Dw^\Omega_{x^*_k}\|_p^p = \Lambda_p(\Omega)\,.$$

The case when A is finite can be treated similarly except the first criteria when choosing x^*_N is replaced with $d_\Omega(x^N_l) \leq d_\Omega(x^*_{N-1})/2$.

Data availability

No data was used for the research described in the article.

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