Generalized classical Yang-Baxter equation and regular decompositions

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Generalized classical Yang-Baxter equation and regular decompositions

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Abstract

The focus of the paper is on constructing new solutions of the generalized classical Yang-Baxter equation (GCYBE) that are not skew-symmetric. Using regular decompositions of finite-dimensional simple Lie algebras, we construct Lie algebra decompositions of $\mathfrak{g}((x)) \times \mathfrak{g}[x]/x^m\mathfrak{g}[x]$. The latter decompositions are in bijection with the solutions to the GCYBE. Under appropriate regularity conditions, we obtain a partial classification of such solutions. The paper is concluded with the presentations of the Gaudin-type models associated to these solutions.

Keywords Yang-Baxter equations · Infinite-dimensional Lie algebras · Lie algebra decompositions · Integrable systems

Mathematics Subject Classification 17B38 · 17B80

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1 Introduction

The r-matrix approach is a fundamental method for constructing integrable systems; see, e.g. [6, 7, 10]. Consider a mechanical system with a phase space M and a Hamiltonian H. If this system admits a Lax representation, i.e. the equations of motions with respect to H are equivalent to

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [P, L]$$

for some functions P and L on M with values in some Lie algebra $\mathfrak{L}$, invariant polynomials of this Lie algebra automatically define constants of motion of the system. The involutivity property for these constants of motion is equivalent to the fact that Lax matrix L satisfies the relation

$$\{L \otimes L\} = [L \otimes 1, r] - [1 \otimes L, r^{21}] \tag{1}$$

for some function r on M with values in $\mathfrak{L} \otimes \mathfrak{L}$. The statements we just made can be adjusted in a way so that they remain valid even for some infinite-dimensional Lie algebras $\mathfrak{L}$. For example, we consider one of such cases when $\mathfrak{L} = \mathfrak{g}((x))^{\oplus n}$, where $\mathfrak{g}((x))$ is the Lie algebra of formal Laurent power series with coefficients in a finite-dimensional simple complex Lie algebra $\mathfrak{g}$. In this case, $L = L(x)$ and $r = r(x, y)$ depend on the formal parameters.

The Jacobi identity for the Poisson bracket in Eq. (1) imposes some constraints on the corresponding function r. If r is constant along M a sufficient condition that these constraints are satisfied is given by the generalized classical Yang-Baxter equation (GCYBE)

$$[r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] + [r^{32}(x_3, x_2), r^{13}(x_1, x_3)] = 0. \tag{2}$$

A key idea of the r-matrix method is to start with a meromorphic solution to GCYBE, having a simple pole along the diagonal, and construct a mechanical system possessing many integrals of motion. For example, to every such solution, one can associate a classical integrable system by considering the Poisson-commuting Hamiltonians

$$H_i := \sum_{k\neq i} r(u_k, u_i)^{(ki)} + \frac{1}{2}(g(u_i, u_i)^{(ii)} + \tau(g(u_i, u_i))^{(ii)}). \tag{3}$$

Here, u_i are points inside the domain of the definition of r, g is the regular part of r and Poisson-commuting means $\{H_i, H_j\} = 0$ for the linear Poisson structure $\{\cdot,\cdot\}$ on $\mathfrak{g}^*$. Expression Eq. (3) can be understood as a quadratic polynomial on $\mathfrak{g}^{*,\oplus n}$ and therefore as an element of the symmetric algebra $S(\mathfrak{g}^{\oplus n}) \cong S(\mathfrak{g})^{\otimes n}$. If one replaces the space of functions $S(\mathfrak{g})^{\otimes n}$ by the universal enveloping algebra $U(\mathfrak{g})^{\otimes n}$, one obtains a quantum integrable system which generalizes the Gaudin models; see [23, 24].

This motivates the search for solutions to the GCYBE. One well-known strategy is to consider certain Lie algebra decompositions into two subalgebras. For example, a decomposition of the form

$$\mathfrak{g}(\!(x)\!) = \mathfrak{g}[\![x]\!] \oplus W$$

for a subalgebra $W \subseteq \mathfrak{g}(\!(x)\!)$ leads to a generalized r-matrix of the form

$$r(x, y) = \frac{\Omega}{x-y} + g(x, y),$$

where Ω is the quadratic Casimir element of $\mathfrak{g}$ and $g(x, y) \in (\mathfrak{g}\otimes\mathfrak{g})[\![x, y]\!]$. This idea is further developed in [4]. More formally, we can consider solutions

$$r(x, y) = \frac{y^m\Omega}{x-y} + g(x, y) \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!] \tag{4}$$

of the GCYBE.

Such an r now corresponds uniquely to a Lie algebra decomposition

$$L_m := \mathfrak{g}(\!(x)\!) \times \mathfrak{g}[x]/x^m\mathfrak{g}[x] = \mathfrak{D} \oplus W, \tag{5}$$

where $\mathfrak{D} = \{(f, [f]) \mid f \in \mathfrak{g}[\![x]\!]\}$ is the diagonal embedding of $\mathfrak{g}[\![x]\!]$ into L_m and W is a complementary subalgebra determined by r.

Remark 1.1 Instead of Eq. (4) we can start with a solution in a more general form

$$r(x, y) = \frac{h(x, y)\Omega}{x-y} + g(x, y)$$

for some $h(x, y) \in \mathbb{C}[\![x, y]\!]$ such that $h(y, y) \neq 0$. The last condition implies

$$h(x, y) = h(y, y) + (x-y)\widetilde{h}(x, y)$$

for some $\widetilde{h}(x, y) \in \mathbb{C}[\![x, y]\!]$. Writing $h(y, y) = y^m s(y)$ for some $s(y) \in \mathbb{C}^*[\![y]\!]$ and substituting this expression into the r-matrix above, we get

$$r(x, y) = \frac{y^m s(y)\Omega}{x - y} + \widetilde{g}(x, y).$$

Multiplication in the y-component of $r(x, y)$ by invertible series preserves the property of solving GCYBE. For that reason, we can, without loss of generality, restrict ourselves to the solutions of the form Eq. (4). $\Diamond$

Decompositions of the form (5) give another point of view on the construction of the generalized Gaudin models defined by the Hamiltonians (3). Taking the difference of the projections onto the two components $R := P_{\mathfrak{D}} - P_W$, we can define another Lie algebra structure on L_m, by letting

$$[a, b]_R := [Ra, b] + [a, Rb], \quad \forall a, b \in L_m. \tag{6}$$

The second Lie algebra structure makes L_m into what is now known as a Lie dialgebra structure [21]. Both Lie algebra structures $[-, -]$ and $[-, -]_R$ give rise to two linear Poisson brackets on the space of polynomial functions $S(L'_m)$ and its n-point adaptation $S(L'_m)^{\otimes n}$. Evaluating at n chosen points we recover the generalized Gaudin Hamiltonians Eq. (3) and obtain the integrability of the associated system.

In general, one can consider a skew-symmetric bilinear form (6) in an arbitrary Lie algebra $\mathfrak{L}$ with an endomorphism R. It satisfies the Jacobi identity, and hence turns $\mathfrak{L}$ into a Lie dialgebra, if and only if the following equality holds

$$[B_R(a, b), c] + [B_R(b, c), a] + [B_R(c, a), b] = 0, \quad \forall a, b \in \mathfrak{L}, \tag{7}$$

where

$$B_R(a, b) := [Ra, Rb] - R([Ra, b] + [a, Rb]).$$

One obvious sufficient condition for Eq. (7) to hold is $B_R = 0$. Furthermore, if the endomorphism R can be identified with a tensor r, using a non-degenerate invariant symmetric bilinear form on $\mathfrak{L}$, condition $B_R = 0$ transforms into the GCYBE. In particular, if $\mathfrak{L} = L_m$, we restore Eq. (2).

In this paper, we consider the problem of classification of generalized classical r-matrices. This problem in its full generality is too wild, so we impose some natural additional restrictions. Formally, we consider those generalized r-matrices for which the corresponding $W \subseteq L_m$ satisfies the following three restrictions:

1. $(x^{-1}, 0)W \subseteq W$ and $(0, [x])W \subset W$;
2. $[(h, h), W] \subseteq W$ for any h in a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$;
3. $W_+ \subseteq x^N \mathfrak{g}[x^{-1}]$ for some positive integer N, where $W_+ := (1, 0)W$ is the projection of W onto the left component $\mathfrak{g}(\!(x)\!)$ of L_m.

The first two restrictions are motivated by the utility of the resulting integrable systems [13, 15, 23] and the last condition allows us to apply the theory of maximal orders, developed in [26, 27]. Subalgebras $W \subseteq L_m$ with the above-mentioned properties are called regular.

In [16], the authors proposed a method of constructing Lie algebra decompositions $\mathfrak{g}((x)) = \mathfrak{g}[[x]] \oplus W$ satisfying $x^{-1}W \subseteq W \subseteq \mathfrak{g}[x^{-1}]$ by utilizing decompositions

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$$

satisfying $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_i \oplus \mathfrak{g}_j$ for all $1 \leqslant i, j \leqslant \ell$. In this paper, we exploit this idea under the additional assumption of Cartan invariance. This allows to construct subalgebras of $W \subseteq L_m$ with $m \geqslant 1$ with the desired properties 1.–3. above.

The classification of regular subalgebras is further split into several classification subproblems. We were able to completely resolve some of them and to construct a vast set of examples for the remaining ones. The key idea that was used in all the cases is the reduction of regular decompositions $L_m = \mathfrak{D} \oplus W$ to regular partitions $\Delta = \Delta_1 \oplus \cdots \oplus \Delta_\ell$ of the underlying irreducible root system of $\mathfrak{g}$.

Additionally, we introduce a technique for constructing weakly regular subalgebras $W \subseteq L_m$, i.e. those for which condition 2. above is replaced with a weaker one

2'. $[h, W_\pm] \subseteq W_\pm$.

The construction is based on a generalization of Belavin-Drinfeld triples. The corresponding r-matrices are written explicitly.

The Lie dialgebras constructed in this paper are amenable to several quantizations, other than the above-mentioned ones of Gaudin-type.

The first approach to quantization comes from [14]. There, the solutions of the GCYBE associated to Lie algebra decompositions $\mathfrak{g}((x)) = \mathfrak{g}[[x]] \oplus W$ such that $x^{-1}W \subseteq W \subseteq \mathfrak{g}[x^{-1}]$ are quantized to solutions of the quantum Yang-Baxter equation. The key idea is a transition from $\mathfrak{g}$ to the Lie algebra $\mathfrak{g} \ltimes \mathfrak{g}^*$ with Lie bracket

$$[x + f, y + g] := [x, y] + \mathrm{ad}^*(x)g - \mathrm{ad}^*(y)f, \quad \forall x, y \in \mathfrak{g}, \ \forall f, g \in \mathfrak{g}^*. \tag{8}$$

This procedure is related to the one in [5], where to any Lie algebra decomposition $\mathfrak{g}((x)) = \mathfrak{g}[[x]] \oplus W$ the authors associate a Lie bialgebra structure on $(\mathfrak{g} \ltimes \mathfrak{g}^*)[[z]]$, which turns out to admit a natural quantization resembling the monoidal structure of sheaves on a certain double quotient of formal groups. The same scheme could be applied to Lie algebra decompositions of L_m for $m > 0$.

The second way of quantizing the constructed decomposition emerges from [12]. The paper states that every Lie quasi-bialgebra can be quantized. Therefore, if we extend one of the Lie quasi-bialgebra structures on $\mathfrak{D} \cong \mathfrak{g}[[x]]$ (see [4]) to L_m and quantize it according to [12], we get a quasi-Hopf algebra $U_\hbar(L_m)$ and a quasi-Hopf subalgebra $U_\hbar(\mathfrak{D}) \subseteq U_\hbar(L_m)$. A quantization of a Lie algebra decomposition $L_m = \mathfrak{D} \oplus W$ could now be defined as a (not necessarily quasi-Hopf) subalgebra $U_\hbar(W) \subseteq U_\hbar(L_m)$ such that

$$U_\hbar(L_m) = U_\hbar(\mathfrak{D}) \otimes U_\hbar(W)$$

and $U_\hbar(W)/\hbar U_\hbar(W) = U(W)$.

1.1 Structure of the paper

Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$. A decomposition of $\mathfrak{g}$ of the form

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$$

is called regular if

1. All $\mathfrak{g}_i$ and $\mathfrak{g}_i \oplus \mathfrak{g}_j$ are Lie subalgebras of $\mathfrak{g}$ and
2. $[\mathfrak{h}, \mathfrak{g}_i] \subseteq \mathfrak{g}_i$ for all $1 \leqslant i \leqslant \ell$.

We consider such decompositions as building blocks for regular decompositions $L_m = \mathfrak{D} \oplus W$. Note that the invariance under the adjoint action of Cartan subalgebra implies that each $\mathfrak{g}_i$ has the form

$$\mathfrak{g}_i = \mathfrak{s}_i \bigoplus_{\alpha \in \Delta_i} \mathfrak{g}_\alpha,$$

for some vector subspace $\mathfrak{s}_i \subseteq \mathfrak{h}$ and a subset Δ_i of roots of $\mathfrak{g}$. Forgetting the Cartan part of $\mathfrak{g}$ in Eq. (1.1) we obtain a partition of the corresponding root system

$$\Delta = \Delta_1 \sqcup \cdots \sqcup \Delta_\ell,$$

where Δ_i and $\Delta_i \sqcup \Delta_j$ are closed with respect to root addition. Such a partition of a root system is again called regular. Regular decompositions of simple Lie algebras $\mathfrak{g}$ and regular partition of the corresponding root systems are completely classified in [11, 19]. We present this classification in Sect. 2.1.

The theory of maximal orders, that allows us to split the classification of regular decompositions $L_m = \mathfrak{D} \oplus W$ further into smaller subproblems, is presented in Sect. 2.2. More precisely, the theory states that, up to a certain equivalence, every regular subalgebra $W \subseteq L_m$ satisfies

$$W \subseteq \mathfrak{P}_i \times \mathfrak{g}[x]/x^m\mathfrak{g}[x],$$

where $\mathfrak{P}_i$ is the parabolic subalgebra of $\mathfrak{g}[x, x^{-1}] \subseteq \mathfrak{g}((x))$ corresponding to the simple root α_i with $0 \leqslant i \leqslant \operatorname{rank}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}[x, x^{-1}]$. Moreover, each parabolic subalgebra $\mathfrak{P}_i$ is associated with an integer k_i, called the type of $\mathfrak{P}_i$. Consequently, we have a reduction of the original problem to the classification of regular subalgebras $W \subseteq L_m$ with $m \in \{0, 1, 2\}$ and different types $0 \leqslant k \leqslant 6$.

We complete Sect. 2 with an explanation of the relation between regular decompositions $L_m = \mathfrak{D} \oplus W$ and solutions of the GCYBE.

A full classification of regular subalgebras $W \subseteq L_0$ of type 0 is obtained in Sect. 3. Precisely, we prove the following result.

Theorem A *Let $W \subseteq \mathfrak{g}((x))$ be a regular subalgebra of type 0. Then we can find a regular partition of the root system of $\mathfrak{g}$*

$$\Delta = \bigsqcup_{i=1}^{n} \Delta_i$$

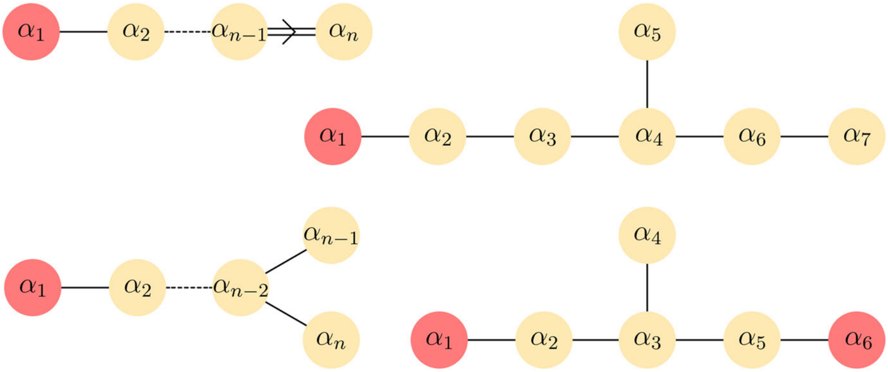


Fig. 1 Regular subalgebras contained in the parabolic subalgebras corresponding to the red roots are completely classified

and constants $a_1, \dots, a_n \in F$ such that

$$W = W_\phi \bigoplus_{i=1}^{n} \bigoplus_{\alpha \in \Delta_i} (x^{-1} - a_i)\mathfrak{g}_\alpha[x^{-1}],$$

where $W_\phi \subseteq \mathfrak{h}[x^{-1}]$ is defined by a linear map $\phi \colon \mathfrak{h} \to \mathfrak{h}$ compatible with the regular partition of Δ; see Theorem 3.11. The converse direction is also true.

The solution of the GCYBE associated to the subalgebra above is given by

$$r(x, y) = \frac{\Omega}{x - y} + \left(\frac{\phi}{y\phi - 1} \otimes 1\right)\Omega_\mathfrak{h} + \sum_{i=1}^{n} \frac{a_i \Omega_i}{a_i y - 1},$$

where $\Omega \in (\mathfrak{h} \otimes \mathfrak{h}) \bigoplus_{\alpha \in \Delta} (\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha})$ is the quadratic Casimir element and $\Omega_\mathfrak{h} \in \mathfrak{h} \otimes \mathfrak{h}$ as well as $\Omega_i \in \bigoplus_{\alpha \in \Delta_i} \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$ are the corresponding components of Ω.

In Sect. 3.3 we look at regular subalgebras $W \subseteq \mathfrak{g}((x))$ of type 1. We give a complete description of regular subalgebras contained inside the parabolic subalgebras corresponding to the roots presented in Fig. 1. The description is given in terms of regular decompositions $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and some additional data.
In order to present this result, we need to introduce the following subsets of roots:

$$\Delta_\pm^{<m\alpha_i} := \left\{ \alpha \in \Delta_\pm \mid \alpha = \pm \sum_{i=1}^{n} c_i \alpha_i,\ 0 \leqslant c_i < m \right\},$$

$$\Delta_\pm^{\geqslant m\alpha_i} := \left\{ \alpha \in \Delta_\pm \mid \alpha = \pm \sum_{i=1}^{n} c_i \alpha_i,\ c_i \geqslant m \right\}.$$

Then the precise statement for the result mentioned above is the following.

Theorem B *Let Δ be the irreducible root system corresponding to $\mathfrak{g}$. Assume α_i is a simple root of Δ with $k_i = 1$. For any partition $\Delta^{<\alpha_i} = \Delta_1 \sqcup \Delta_2$ into two closed subsets and two constants $a_1, a_2 \in F^\times$ the formula*

$$W = W_\phi \bigoplus_{\alpha\in\Delta_1} (x^{-1} - a_1)\mathfrak{g}_\alpha[x^{-1}] \bigoplus_{\alpha\in\Delta_2} (x^{-1} - a_1)\mathfrak{g}_\alpha^{a_2}[x^{-1}]$$
$$\bigoplus_{\beta\in\Delta_+^{\geqslant\alpha_i}} x(x^{-1} - a_1)(x^{-1} - a_2)\mathfrak{g}_\beta[x^{-1}] \bigoplus_{\alpha\in\Delta_-^{\geqslant\alpha_i}} x^{-1}\mathfrak{g}_\alpha[x^{-1}],$$

defines a regular subalgebra of type 1*. Here $W_\phi \subseteq \mathfrak{h}[x^{-1}]$ is a subalgebra defined by a certain linear map $\phi\colon \mathfrak{h} \to \mathfrak{h}$; see Example 3.15. Moreover, every regular $W \subseteq L_0$ of type* 1 *associated to a node from Fig. 1 is of this form.*

The corresponding solution of the GCYBE is given by

$$r(x, y) = \frac{\Omega}{x-y} + \left(\frac{\phi}{y\phi - 1} \otimes 1\right)\Omega_{\mathfrak{h}} + \frac{a_1\Omega_1}{a_1 y - 1} + \frac{a_2\Omega_2}{a_2 y - 1}$$
$$+ \frac{a_1a_2(x+y) - a_1 - a_2}{(a_1y-1)(a_2y-1)}\Omega_{\mathfrak{c}},$$

where $\Omega \in (\mathfrak{h}\otimes\mathfrak{h}) \bigoplus_{\alpha\in\Delta}(\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha})$ is the quadratic Casimir element and

$$\Omega_i \in (\oplus_{\alpha\in\Delta_i}\mathfrak{g}_\alpha)\otimes\mathfrak{g}, \quad \Omega_{\mathfrak{c}} \in (\oplus_{\alpha\in\Delta_+^{\geqslant\alpha_i}}\mathfrak{g}_\alpha)\otimes\mathfrak{g}$$

are components of Ω lying in the corresponding subspaces.

In Sect. 3.3 we also present explicit constructions of regular subalgebras in $\mathfrak{g}(\!(x)\!)$ for the remaining type 1 cases starting with a regular decomposition of $\mathfrak{g}$.

Regular subalgebras in $\mathfrak{g}(\!(x)\!)$ of type $k \geqslant 2$ are considered in Sect. 3.4. The structure of such subalgebras is even wilder and their classification seems unfeasible. However, we present a general algorithm for constructing such objects. It is demonstrated in Example 3.28.

Section 4 is devoted to regular decompositions $L_m = \mathfrak{D} \oplus W$ with $m \geqslant 1$. We prove that regular subalgebras of L_1 of type 0 can be reduced to regular subalgebras of $\mathfrak{g}\times\mathfrak{g}$. The latter are classified in Sect. 4.1.

Theorem C *Let $\mathfrak{w} \subseteq \mathfrak{g}\times\mathfrak{g}$ be a subalgebra such that*

- $[(h, h), \mathfrak{w}] \subseteq \mathfrak{w}$ *for all $h \in \mathfrak{h}$ and*
- $\Delta \oplus \mathfrak{w} = \mathfrak{g}\times\mathfrak{g}$.

Then there is a regular partition $\Delta = S_+ \sqcup S_-$ and subspaces $\mathfrak{s}_\pm = \mathfrak{t}_\pm \oplus \mathfrak{r}_\pm \subseteq \mathfrak{h}$, having the properties $\mathfrak{h} = \mathfrak{s}_+ + \mathfrak{s}_-$ and $\mathfrak{t}_+ \cap \mathfrak{t}_- = \{0\}$, such that

$$\mathfrak{w} = \left((\mathfrak{t}_+ \bigoplus_{\alpha\in S_+}\mathfrak{g}_\alpha)\times\{0\}\right) \oplus \left(\{0\}\times(\mathfrak{t}_- \bigoplus_{\beta\in S_-}\mathfrak{g}_\beta)\right) \oplus \mathrm{span}_F\{(h, \phi(h)) \mid h \in \mathfrak{r}_+\},$$

where $\phi\colon \mathfrak{r}_+ \to \mathfrak{r}_-$ is a vector space isomorphism with no nonzero fixed points. The converse direction is also true. In other words, all regular subalgebras are described by partitions $\Delta = S_1 \sqcup S_2$ and the extra datum $(\mathfrak{t}_\pm, \mathfrak{r}_\pm, \phi)$ describing the gluing of the Cartan parts.

Similar to the L_0 case, in Sect. 4.2, we classify regular subalgebras $W \subseteq L_1$ of type 0 in terms of regular subalgebras of $\mathfrak{g} \times \mathfrak{g}$ and some additional datum.

Theorem D *Let $W \subseteq L_1$ be a regular subalgebra of type 0. Then*

$$W = \mathfrak{w} \oplus \left(\left(W_\psi \bigoplus_{i=0}^{n} \bigoplus_{\alpha \in \Delta_i} (x^{-1} - a_i)\mathfrak{g}_\alpha[x^{-1}] \right) \times \{0\} \right),$$

where

- *$\mathfrak{w} \subseteq \mathfrak{g} \times \mathfrak{g}$ is described by the datum $\Delta = S_+ \sqcup S_-$ and $(\mathfrak{t}_\pm, \mathfrak{r}_\pm, \phi)$;*
- *$\Delta = \sqcup_{i=0}^{n} \Delta_i$ is a regular decomposition with $S_+ \subseteq \Delta_0$ and $a_0, \dots, a_n \in F$ are distinct constants such that $a_0 = 0$;*
- *$\psi : \mathfrak{h} \to \mathfrak{h}$ is a linear map defining W_ψ and compatible with the previous root space data; see Theorem 4.5 for details.*

The converse holds as well.

Furthermore, the r-matrix of W above is

$$r(x, y) = \frac{y\Omega}{x - y} + r_{(S_\pm, \mathfrak{t}_\pm, \mathfrak{r}_\pm, \phi)} + \frac{1}{2}\Omega + \left(\frac{\psi}{y\psi - 1} \otimes 1 \right) \Omega_{\mathfrak{h}} + \sum_{i=1}^{n} \frac{a_i \Omega_i}{a_i y - 1},$$

where $\Omega_i \in \bigoplus_{\alpha \in \Delta_i} (\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha})$ is the component of the Casimir element and $r_{(S_\pm, \mathfrak{t}_\pm, \mathfrak{r}_\pm, \phi)}$ is a certain tensor in $\mathfrak{g} \otimes \mathfrak{g}$ determined by $\mathfrak{m}$ in Eq. (41).

Replacing the condition $[(h, h), W] \subseteq W$ with the weaker condition $[h, W_\pm] \subseteq W_\pm$, i.e. only the projections of W are required to be $\mathfrak{h}$-invariant, we obtain the notions of a weakly regular subalgebra of L_m. We show in Sect. 4.3 that such subalgebras can be effectively constructed using a generalized version of Belavin-Drinfeld triples combined with Theorem D.

In the remaining part of Sect. 4 we prove that regular subalgebras $W \subseteq L_m$ with $m > 0$ of type $k > 0$ admit a certain standard form. Since all regular subalgebras $W \subseteq L_0$ are trivially included into the set of regular subalgebras of L_m, with $m \geqslant 1$, the classification of the latter objects is even wilder. However, we present methods for constructing such non-trivial subalgebras. See Table 1 for a short summary of the results.

The paper is concluded with Sect. 5, where we relate the decompositions mentioned above to Gaudin models through the corresponding solutions to the GCYBE. We give some explicit examples of new generalized Gaudin Hamiltonians.

In Appendix A, we placed a table with the most used notation to simplify the reading.

Table 1 Overview of classification results

$W \subseteq L_m$		
$m = 0$	type $= 0$	Classified by regular partitions of Δ and compatible linear maps $\phi\colon \mathfrak{h} \to \mathfrak{h}$.
$m = 0$	type > 0	Cases from Fig. 1 are completely classified using decompositions $\Delta = \Delta_1 \sqcup \Delta_2$ and $\phi\colon \mathfrak{h} \to \mathfrak{h}$. For the remaining cases explicit constructions are presented.
$m = 1$	type $= 0$	Classified by regular decompositions of $\mathfrak{g} \times \mathfrak{g}$ and compatible linear maps $\phi, \psi\colon \mathfrak{h} \to \mathfrak{h}$.
$m > 0$	type $k_i > 0$	Always of the form $W = W_{\mathfrak{h}} \oplus (I_+ \times \{0\}) \oplus (\{0\} \times I_-)$ for $I_+ \subseteq \mathfrak{P}_i$, $I_- \subseteq \mathfrak{g}[x]/x^m\mathfrak{g}[x]$ and $W_{\mathfrak{h}} \subseteq \mathfrak{h}[x] \times \mathfrak{h}[x]/x^m\mathfrak{h}[x]$. Explicit constructions are presented.

2 Preliminaries

We fix once and for all an algebraically closed field F of characteristic 0. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over F with a Cartan subalgebra $\mathfrak{h}$. We write Δ for the set of (nonzero) roots and $\pi = \{\alpha_1, \dots, \alpha_n\} \subset \Delta$ for a chosen set of simple roots. The choice of simple roots gives us the polarization $\Delta = \Delta_+ \sqcup \Delta_-$. Later we use α_0 to denote the maximal root of Δ. Furthermore, we fix a basis

$$\{H_{\alpha_i}, E_{\pm\alpha} \mid 1 \leqslant i \leqslant n,\ \alpha \in \Delta_+\}$$

of $\mathfrak{g}$, such that $\kappa(E_\alpha, E_{-\alpha}) = 1$ and $\kappa(H_\alpha, H) = \alpha(H)$, where κ is the Killing form of $\mathfrak{g}$. In particular, we have $[E_\alpha, E_{-\alpha}] = H_\alpha = \sum_{i=1}^n c_i H_{\alpha_i}$ for any $\alpha = \sum_1^n c_i\alpha_i \in \Delta_+$.

2.1 Regular decompositions of simple Lie algebras

An *m-regular decomposition of* $\mathfrak{g}$ is a decomposition of the form

$$\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i,\ m \geqslant 2,$$

satisfying the conditions:

1. All $\mathfrak{g}_i$ as well as $\mathfrak{g}_i \oplus \mathfrak{g}_j$ are Lie subalgebras of $\mathfrak{g}$;
2. Each $\mathfrak{g}_i$ has the form $\mathfrak{s}_i \oplus_{\alpha\in\Delta_i} \mathfrak{g}_\alpha$ for some subspace $\mathfrak{s}_i \subseteq \mathfrak{h}$ and some subset $\Delta_i \subseteq \Delta$.

Subalgebras of the form described in 2. are called *regular*, motivating the name. Equivalently, one can say that $\mathfrak{g}_i$ and $\mathfrak{g}_i \oplus \mathfrak{g}_j$ are subalgebras of $\mathfrak{g}$ invariant under the action of $\mathfrak{h}$.

We say that a subset $S \subseteq \Delta$ is *closed* if for all $\alpha, \beta \in S$ the containment $\alpha + \beta \in \Delta$ implies $\alpha + \beta \in S$. Regular decompositions of simple Lie algebras are closely related with *regular partitions* of irreducible root systems of finite type, i.e. partitions

$$\Delta = \bigsqcup_{i=1}^{m} \Delta_i, \ \ m \geqslant 2, \tag{9}$$

with the property that all Δ_i and $\Delta_i \sqcup \Delta_j$ are closed. More precisely, the results of [19] give us the following correspondence.

Proposition 2.1 *Given an m-regular partition $\Delta = S_1 \sqcup \cdots \sqcup S_m$ with $m \geqslant 2$, we can find subspaces $\mathfrak{s}_1, \dots, \mathfrak{s}_m \subseteq \mathfrak{h}$ (not unique in general) such that*

$$\mathfrak{g} = \bigoplus_{i=1}^{m} \left(\mathfrak{s}_i \bigoplus_{\alpha \in S_i} \mathfrak{g}_\alpha \right)$$

is a regular decomposition. Conversely, by forgetting the Cartan part of an m-regular decomposition of $\mathfrak{g}$ we obtain an m-regular partition of Δ.

Regular partitions of irreducible finite root systems into two parts were classified in [11]. Later in [19], it was shown that m-regular partitions with $m \geqslant 3$ exist only for root systems of type A_n, $n \geqslant 2$. Such partitions were completely described in the same paper.

We now shortly recall these classifications starting with the 2-regular case. Let S be a subset of simple roots π. We denote by $\Delta^S = \Delta^S_+ \sqcup \Delta^S_-$ the root subsystem of Δ generated by S. Given two subsets $S, T \subseteq \pi$ we define

$$P(S, T) := \Delta^S \cup (\Delta_+ \backslash \Delta^T_+).$$

The following theorem describes all 2-partitions.

Theorem 2.2 ([11], Theorem 4) *Let $S \subseteq T \subseteq \pi$ be two subsets of simple roots such that S is orthogonal to $T \setminus S$. Then*

$$\Delta = P(S, T) \sqcup (\Delta \backslash P(S, T))$$

is a 2-regular partition of Δ. Moreover, up to the action of the Weyl group $W(\Delta)$, any 2-regular partition is of this form.

Let us order the simple roots $\pi = \{\alpha_1, \dots, \alpha_n\}$ of the system A_n in such a way, that

$$\beta_i := \alpha_1 + \cdots + \alpha_i \ \text{ and } \ \beta_i - \beta_j$$

are roots for all $1 \leqslant i \neq j \leqslant n$. For convenience, we put $\beta_0 := 0$.

Theorem 2.3 ([19], Theorem A) *If $\Delta = \Delta_1 \sqcup \cdots \sqcup \Delta_m$, $m \geqslant 3$ is a regular partition, then Δ is necessarily of type A_n. Moreover, the following statements are true:*

1. *Up to swapping positive and negative roots, re-numbering elements Δ_i of the partition and action of $W(A_n)$, there is a unique maximal $(n+1)$-partition with*

$$\Delta_i = \{-\beta_i + \beta_j \mid 0 \leqslant i \neq j \leqslant n\},\ \ 0 \leqslant i \leqslant n;$$

2. *Any other $(3 \leqslant m < n+1)$-regular partition is obtained from the maximal one above by combining several subsets Δ_i together;*
3. *Up to equivalences mentioned above, all m-regular partitions are described by m-partitions $\lambda = (\lambda_1, \dots, \lambda_m)$ of $n+1$.*

To pass from a regular partition $\Delta = \sqcup_{i=1}^m \Delta_i$ to a regular decomposition $\mathfrak{g} = \oplus_{i=1}^m \mathfrak{g}_i$, we let

$$\mathfrak{g}_i := \mathfrak{s}_i \oplus \operatorname{span}_F\{E_\alpha \mid \alpha \in \Delta_i\}$$

for some (non-unique) disjoint subspaces $\mathfrak{s}_i$ of $\mathfrak{h}$. In case $m = 2$, we first put

$$\mathfrak{s}_i' := \operatorname{span}_F\{H_\alpha \mid \pm\alpha \in \Delta_i\}.$$

Then we represent $\mathfrak{h} = (\mathfrak{s}_1' \oplus \mathfrak{s}_2') \oplus \mathfrak{h}'$ and, finally, obtain $\mathfrak{s}_i$ by distributing $\mathfrak{h}'$ into $\mathfrak{s}_i'$ in an arbitrary way. In case $m \geqslant 3$ there is even less freedom.

Theorem 2.4 ([19], Theorem B) *Let $\mathfrak{sl}(n+1, F) = \oplus_{i=1}^m \mathfrak{g}_i$ be an m-regular partition. Up to swapping positive and negative roots, re-numbering $\mathfrak{g}_i$'s and the action of $W(A_n)$, it has one of the following forms*

1. $$\mathfrak{g}_1 = \operatorname{span}_F\{E_{\beta_i} \mid 1 \leqslant i \leqslant n\},$$ $$\mathfrak{g}_\ell = \operatorname{span}_F\left\{E_{-\beta_i+\beta_j}, H_{\beta_i} \,\middle|\, \sum_{t=1}^{\ell-2} \lambda_t < i \leqslant \sum_{t=1}^{\ell-1} \lambda_t,\ 0 \leqslant j \neq i \leqslant n\right\},$$ *where $2 \leqslant \ell \leqslant k+1$ and $(\lambda_1, \dots, \lambda_k)$ is a k-partition of n;*
2. $$\mathfrak{g}_1 = \operatorname{span}_F\{E_{-\beta_i+\beta_j}, H_{\beta_i}, X \mid 0 \leqslant i \leqslant \lambda_1,\ 0 \leqslant j \neq i \leqslant n\},$$ $$\mathfrak{g}_\ell = \operatorname{span}_F\left\{E_{-\beta_i+\beta_j}, H_{\beta_i} - X \,\middle|\, \sum_{t=1}^{\ell-1} \lambda_t < i \leqslant \sum_{t=1}^{\ell} \lambda_t,\ 0 \leqslant j \neq i \leqslant n\right\},$$ *where $2 \leqslant \ell \leqslant k$, $(\lambda_1, \dots, \lambda_k)$ is a k-partition of n and X is an arbitrary vector in*

$$(FH_{\beta_1} \oplus \cdots \oplus FH_{\beta_{\lambda_1}}) \cup \left\{H_{\beta_p} \,\middle|\, 2 \leqslant m \leqslant k,\ \lambda_m > 1,\ \sum_{t=1}^{m-1} \lambda_t < p \leqslant \sum_{t=1}^{m} \lambda_t\right\}.$$

Remark 2.5 The decompositions in the theorem above have an intuitive description: the Lie algebra $\mathfrak{gl}(n+1, F)$ can be decomposed into $(n+1)$ subalgebras, consisting of $(n+1) \times (n+1)$-matrices with a single nonzero row in position i. The projection of $\mathfrak{gl}(n+1, F)$ onto $\mathfrak{sl}(n+1, F)$ preserves this decomposition outside the diagonal. The Cartan part, however, has to be re-distributed. Two different descriptions of $\mathfrak{g}_i$ come from two essentially different possibilities to do that. ◊

2.2 Maximal upper-bounded subalgebras in $\mathfrak{g}((x))$

Subalgebras W of $\mathfrak{g}((x))$ are called *upper-bounded*, if they satisfy the condition

$$W \subseteq x^N \mathfrak{g}[x^{-1}] \ \text{ for some } \ N \in \mathbb{Z}_+,$$

and *bounded*, if they satisfy the stronger condition

$$x^{-N} \mathfrak{g}[x^{-1}] \subseteq W \subseteq x^N \mathfrak{g}[x^{-1}] \ \text{ for some } \ N \in \mathbb{Z}_+. \tag{10}$$

Among all upper-bounded algebras, we distinguish those that are both maximal, with respect to inclusion, and sum up with $\mathfrak{g}[\![x]\!]$ to the whole $\mathfrak{g}((x))$, i.e.

$$\mathfrak{g}[\![x]\!] + W = \mathfrak{g}((x)). \tag{11}$$

Example 2.6 The trivial maximal bounded algebra is given by

$$\mathfrak{P} := \mathfrak{P}_0 := \mathfrak{g}[x^{-1}].$$

It is clearly a maximal proper subalgebra of $\mathfrak{g}((x))$ satisfying Eqs. (10) and (11) with $N = 1$. ♢

Example 2.7 For each simple root $\alpha_i \in \{\alpha_1, \dots, \alpha_n\}$, there is a maximal bounded algebra

$$\mathfrak{P}_i := \mathfrak{h}[x^{-1}] \bigoplus_{\alpha \in \Delta} x^{\lfloor \alpha(h_i) \rfloor} \mathfrak{g}_\alpha[x^{-1}], \tag{12}$$

where $h_i \in \mathfrak{h}$ are defined by $\alpha_i(h_j) = \delta_{ij}/k_i$ with $\alpha_0 = \sum k_i \alpha_i, k_i \in \mathbb{Z}_+$. The notation is motivated by the fact that $\mathfrak{P}_i$ are the standard maximal parabolic subalgebras of $\mathfrak{g}((x))$, if the latter is considered as a completed affine Lie algebra modulo its center. ♢

Maximal bounded subalgebras, which sometimes are also called maximal orders, were thoroughly studied in [27], where the following result was proven.

Proposition 2.8 *Let $W \subseteq \mathfrak{g}((x))$ be a proper bounded subalgebra such that*

$$\mathfrak{g}[\![x]\!] + W = \mathfrak{g}((x)).$$

Then there is an automorphism $\varphi \in \mathrm{Aut}_{F[\![x]\!]\text{-LieAlg}}(\mathfrak{g}[\![x]\!])$ such that $\varphi(W) \subseteq \mathfrak{P}_i$ for some $0 \leqslant i \leqslant n$.

Remark 2.9 To be precise, paper [27] studies subalgebras $W \subseteq \mathfrak{g}((x^{-1}))$ with the property

$$x^{-N} \mathfrak{g}[\![x^{-1}]\!] \subseteq W \subseteq x^N \mathfrak{g}[\![x^{-1}]\!].$$

These are called orders. It is straight-forward to transport the results from [27] to our setting, since both our setting and the one in [27] can be reduced to the study of bounded subalgebras of $\mathfrak{g}[x, x^{-1}]$. ♢

It turns out, that upper-bounded subalgebras W of $\mathfrak{g}((x))$ can also be placed into the maximal bounded ones.

Proposition 2.10 *Let W be a subalgebra of $\mathfrak{g}((x))$ such that*

$$W \subseteq x^N \mathfrak{g}[x^{-1}] \text{ and } \dim(x^N \mathfrak{g}[x^{-1}]/W) < \infty$$

for some $N \in \mathbb{Z}_+$. Then there exists $g \in G(F[x, x^{-1}])$, where G is the connected semisimple affine algebraic group associated to $\mathfrak{g}$, such that

$$\operatorname{Ad}(g)W \subseteq \mathfrak{P}_i$$

for some $0 \leqslant i \leqslant n$.

Proof By enlarging W to $F[x^{-1}]W$, we can assume without loss of generality that

$$F[x^{-1}]W \subseteq W.$$

Let $I := x^{-2N-1}W \subseteq W$, then I is an ideal in W and $I \subseteq x^{-N-1}\mathfrak{g}[x^{-1}]$. Consider the algebra $W \oplus Fc$ as a subalgebra of the affine Lie algebra $\widehat{\mathfrak{g}}$, i.e. the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[x, x^{-1}] \oplus Fc, \tag{13}$$

endowed with the bracket

$$[ax^k, bx^\ell]_{\widehat{\mathfrak{g}}} = [a, b]x^{k+\ell} + \delta_{k,-\ell} k\kappa(a, b)c.$$

Because $W \subseteq x^N \mathfrak{g}[x^{-1}]$ and $I \subseteq x^{-N-1}\mathfrak{g}[x^{-1}]$, the algebra I is also an ideal in $W \oplus Fc \subseteq \widehat{\mathfrak{g}}$. Furthermore, I has finite codimension in the subalgebra

$$\mathfrak{n}_- \oplus x^{-1}\mathfrak{g}[x^{-1}] \subseteq \widehat{\mathfrak{g}},$$

since $W \subseteq x^N \mathfrak{g}[x^{-1}]$ is of finite codimension by assumption. Therefore, copying the proof from [2, Proposition 3.10], we see that [17, Proposition 2.8] implies the existence of $g \in G(F[x, x^{-1}])$ such that $\operatorname{Ad}(g)W \subseteq \mathfrak{P}_i$ for some $0 \leqslant i \leqslant n$. □

Adding the extra condition on W to sum up with $\mathfrak{g}[[x]]$ to the whole $\mathfrak{g}((x))$, the result of Proposition 2.10 can be refined as follows.

Corollary 2.11 *Let W be a subalgebra of $\mathfrak{g}((x))$ with the properties*

1. *$W \subseteq x^N \mathfrak{g}[x^{-1}]$ for some $N \in \mathbb{Z}_+$ and*
2. *$\mathfrak{g}[[x]] + W = \mathfrak{g}((x))$.*

Then there exists $g \in G(F[x])$ such that

$$\operatorname{Ad}_g(W) \subseteq \mathfrak{P}_i$$

for some $0 \leqslant i \leqslant n$.

Proof By virtue of Proposition 2.10, we have $\mathrm{Ad}_g(W) \subseteq \mathfrak{P}_i$ for some element $g \in G(F[x,x^{-1}])$. The Kac-Moody group $\widehat{G}$ associated to the affine Lie algebra $\widehat{\mathfrak{g}}$ (see Eq. (13)) is the central extension of $G(F[x,x^{-1}])$ by $F^\times$ via the rescaling of x; see [20]. Let $B_\pm := HU_\pm \subseteq \widehat{G}$, where the subgroups $U_\pm$ have the property

$$\mathrm{Ad}(U_\pm) = \langle e^{x^{\pm k}\mathrm{ad}(a)} \mid a \in \mathfrak{g}_\alpha, \alpha \in \Delta_\pm, k \in \mathbb{Z}_{\geqslant 0}\rangle \subseteq \mathrm{Ad}(G(F[x,x^{-1}]))$$

and H is a maximal torus of $\widehat{G}$. We note that $\mathrm{Ad}(H)$ stabilizes all root spaces and the Cartan subalgebra of $\widehat{\mathfrak{g}}$. By [20, Corollary 2] we can decompose g inside $\widehat{G}$ as

$$g = b_+ w b_-,$$

for some $b_\pm \in B_\pm$ and an element w inside the Weyl group W of $\widehat{\mathfrak{g}}$. Using that $\mathrm{Ad}(b_-^{-1})\mathfrak{P}_i = \mathfrak{P}_i$ we obtain

$$\mathrm{Ad}(b_+)W \subseteq \mathrm{Ad}(w^{-1})\mathfrak{P}_i.$$

Applying $\mathrm{Ad}(w^{-1})$ to the equality $\mathfrak{g}[\![x]\!] + \mathfrak{P}_i = \mathfrak{g}(\!(x)\!)$ we get $\mathfrak{g}[\![x]\!] + \mathrm{Ad}(w^{-1})\mathfrak{P}_i = \mathfrak{g}(\!(x)\!)$. The argument right before [1, Lemma 9.2.2] shows that this is possible if and only if

$$\mathrm{Ad}(w^{-1})\mathfrak{P}_i = \mathrm{Ad}(w')\mathfrak{P}_j$$

for some (possibly different) integer $0 \leqslant j \leqslant n$ and some element w' of the Weyl group of $\mathfrak{g}$, viewed as an element of G. Taking $g = w'^{,-1}b_+$ and changing i to j if necessary, we get the desired statement. □

2.3 Regular decompositions

Motivated by [4, 13, 13, 15], we focus on Lie algebras

$$L_m := \mathfrak{g}(\!(x)\!) \times \mathfrak{g}[x]/x^k\mathfrak{g}[x],\ m \geqslant 0$$

and their Lie algebra decompositions. More precise, we study *regular decompositions*, i.e. decompositions of the form $L_m = \mathfrak{D} \oplus W$, where

- Δ is the diagonal embedding of $\mathfrak{g}[\![x]\!]$ into L_m with a complementary subalgebra $W \subseteq L_m$;
- W is invariant under the adjoint action by $\{(h,h) \mid h \in \mathfrak{h}\}$;
- W is invariant under the multiplication by $(x^{-1},0)$ and $(0,[x])$ and
- The projection W_+ of W onto the left component $\mathfrak{g}(\!(x)\!)$ is upper-bounded.

We call a subalgebra $W \subseteq L_m$ itself *regular* if $L_m = \mathfrak{D} \oplus W$ is a regular decomposition.

Projecting a regular subalgebra W onto the left component $\mathfrak{g}(\!(x)\!)$ of L_m gives an upper-bounded subalgebra $W_+ \subseteq \mathfrak{g}(\!(x)\!)$ such that $W_+ + \mathfrak{g}[\![x]\!] = \mathfrak{g}(\!(x)\!)$. By Corollary 2.11 we can find an automorphism φ such that

$$\varphi(W_+) \subset \mathfrak{P}_i,\ 0 \leqslant i \leqslant \mathrm{rank}(\mathfrak{g}).$$

Extending this automorphism to L_m we get the inclusion

$$(\varphi \times [\varphi])W \subseteq \mathfrak{P}_i \times \mathfrak{g}[x]/x^k\mathfrak{g}[x]. \tag{14}$$

Allowing such equivalences, we can replace the last requirement on the projection of W with the requirement on W to be contained in $\mathfrak{P}_i \times \mathfrak{g}[x]/x^k\mathfrak{g}[x]$ for some $0 \leqslant i \leqslant \mathrm{rank}(\mathfrak{g})$. When the latter inclusion is satisfied, we say that W is regular and has *type* k_i. For consistency, we let $k_0 := 0$.

Remark 2.12 The inclusion Eq. (14) restricts the set of integers m we need to consider. Formally, since $\mathfrak{P}_i \subseteq x\mathfrak{g}[x^{-1}]$, combining Eq. (14) with $L_m = \mathfrak{D} \oplus W$ we get

$$\{0\} \times x^2\mathfrak{g}[x]/x^m\mathfrak{g}[x] \subseteq (\varphi \times [\varphi])W.$$

Therefore, $(\varphi \times [\varphi])W$ is completely determined by its image in

$$L_m/(\{0\} \times x^2\mathfrak{g}[x]/x^m\mathfrak{g}[x]) = L_2$$

and we can assume without loss of generality that $0 \leqslant m \leqslant 2$. $\Diamond$

2.4 Connection to the classical Yang-Baxter equation

In view of [4], subalgebras $W \subseteq L_m$ complementary to $\mathfrak{D}$ are in bijection with formal power series r of the form

$$r(x, y) = \frac{y^k\Omega}{x - y} + p(x, y) \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!],$$

where $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the quadratic Casimir element of $\mathfrak{g}$, which satisfy the *generalized classical Yang-Baxter equation* (GCYBE)

$$[r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] + [r^{32}(x_3, x_2), r^{13}(x_1, x_3)] = 0. \tag{15}$$

Such a series can be viewed as a generating series of W. For a regular subalgebra $W \subseteq L_m$, the upper-boundedness of W guarantees that p is a polynomial in $(\mathfrak{g} \otimes \mathfrak{g})[x, y]$ and $\mathfrak{h}$-invariance gives the identity

$$[h \otimes 1 + 1 \otimes h, p(x, y)] = 0$$

for all $h \in \mathfrak{h}$. The fact that W is an $F[x^{-1}]$-module, on the other hand, has no nice interpretation on the side of r.

3 Regular decompositions $\mathfrak{g}(\!(x)\!) = \mathfrak{g}[\![x]\!] \oplus W$

We start the study of regular subalgebras $W \subseteq L_m$ with the simplest case, namely $m = 0$ and hence $L_0 \cong \mathfrak{g}(\!(x)\!)$.

3.1 Decompositions $\mathfrak{g}((x)) = \mathfrak{g}[\![x]\!] \oplus W$

Following [13], let us first consider decompositions $\mathfrak{g}((x)) = \mathfrak{g}[\![x]\!] \oplus W$ for subalgebras W of $\mathfrak{g}((x))$ which satisfy $x^{-1}W \subseteq W$ but are not-necessarily $\mathfrak{h}$-invariant. It is not hard to see that in this case we can write

$$W = W_A := A(x^{-1}\mathfrak{g}[x^{-1}])$$

for some series

$$A = 1 + Rx + Sx^2 + \dots \tag{16}$$

which is considered as an $F((x))$-linear map $A\colon \mathfrak{g}((x)) \to \mathfrak{g}((x))$. Clearly, the subspace W_A for an arbitrary series Eq. (16) is not a subalgebra in general. The next statement provides necessary and sufficient conditions on A for W_A to be a subalgebra.

Lemma 3.1 ([13]) *The vector space $W_A \subseteq \mathfrak{g}((x))$ is a subalgebra if and only if for all $a, b \in \mathfrak{g}$*

$$[Aa, Ab] = A([a, b] + x([Ra, b] + [a, Rb] - R[a, b])) \tag{17}$$

holds.

Proof Clearly, W_A is a subalgebra if Eq. (17) is satisfied. On the other hand, if W_A is a subalgebra, we have that

$$x^{-2}[Aa, Ab] = Ac$$

for some unique $c \in x^{-2}\mathfrak{g}[\![x]\!]$. On one side, the form of endomorphism A gives the containment

$$x^{-2}[Aa, Ab] \in x^{-2}[a, b] + x^{-1}([Ra, b] + [a, Rb]) + \mathfrak{g}[\![x]\!]. \tag{18}$$

On the other side, since $W_A \cap \mathfrak{g}[\![x]\!] = \{0\}$, the coefficients in front of x^{-2} and x^{-1} determine completely the $\mathfrak{g}[\![x]\!]$ part of the element Eq. (18). Therefore, $c = c_2x^{-2} + c_1x^{-1}$ for

$$c_2 = [a, b] \text{ and } c_1 + Rc_2 = [Ra, b] + [a, Rb]$$

and Eq. (17) holds true. □

We can see from Proposition 2.10 that after applying a polynomial automorphism, every upper-bounded subalgebra W complementary to $\mathfrak{g}[\![x]\!]$ is contained in $\mathfrak{P}_i \subseteq x\mathfrak{g}[x^{-1}]$. Looking at coefficients in front of higher powers of x in Eq. (17) we get the following statement.

Lemma 3.2 ([13]) *The vector subspace $W_A \subseteq \mathfrak{g}((x))$ is an upper-bounded subalgebra if and only if, after applying an $F[x]$-linear automorphism of $\mathfrak{g}[x]$, the following conditions hold for all $a, b \in \mathfrak{g}$:*

1. $A = 1 + Rx + Sx^2$;
2. $R([Ra, b] + [a, Rb] - R[a, b]) - [Ra, Rb] = [Sa, b] + [a, Sb] - S[a, b]$;

3. $[Sa, Sb] = 0$ *and*
4. $[Sa, Rb] + [Ra, Sb] = S([Ra, b] + [a, Rb] - R[a, b])$.

When the subalgebra W_A is $\mathfrak{h}$-invariant, we have $[\mathfrak{h}, A] = 0$ and therefore $A(x^{-1}\mathfrak{g}_\alpha) \subseteq f_\alpha \mathfrak{g}_\alpha$ for some $f_\alpha \in x^{-1} + F[\![x]\!]$. In particular, Lemma 3.1 specializes to $\mathfrak{h}$-invariant decompositions as follows.

Lemma 3.3 *Let $\mathfrak{g}(\!(x)\!) = \mathfrak{g}[\![x]\!] \oplus W$ be a decomposition for a subalgebra W satisfying $x^{-1}W \subseteq W$ and $[\mathfrak{h}, W] \subseteq W$. Then we can write*

$$W = W_{\mathfrak{h}} \bigoplus_{\alpha\in\Delta} f_\alpha \, \mathfrak{g}_\alpha[x^{-1}],$$

where $W_{\mathfrak{h}}$ is an $F[x^{-1}]$-invariant subspace of $\mathfrak{h}(\!(x)\!)$ and $f_\alpha \in x^{-1} + F[\![x]\!]$ satisfy the relations

$$f_\alpha f_\beta = (x^{-1} + c_{\alpha\beta}) f_{\alpha+\beta}, \quad \forall\alpha, \beta, \alpha + \beta \in \Delta, \tag{19}$$

for some constants $c_{\alpha\beta} \in F$.

Remark 3.4 Any Lie algebra automorphism A of $\mathfrak{g}(\!(x)\!)$ given by Eq. (16) satisfy Eq. (17), since for such automorphisms $[Ra, b] + [a, Rb] - R[a, b] = 0$. In this case $W_A \cong x^{-1}\mathfrak{g}[x^{-1}]$.

In the setting of Lemma 3.3 this observation can be interpreted as follows. There is a trivial choice of series f_α satisfying the relations Eq. (19). More precisely, we take arbitrary series $f_{\alpha_i} \in x^{-1} + F[\![x]\!]$ for simple roots α_i and for any root of the form $\alpha = \sum_{i=1}^n k_i\alpha_i$ put

$$f_\alpha := x^{k-1}\prod_{i=1}^n f_{\alpha_i}^{k_i} \text{ and } f_{-\alpha} := x^{-k-1}\prod_{i=1}^n f_{\alpha_i}^{-k_i}, \tag{20}$$

where $k := \sum_{i=1}^n k_i$. In this case, the relations Eq. (19) hold true with $c_{\alpha,\beta} = 0$. However, this choice is not interesting, because the assignment $E_{\alpha_i} \mapsto x f_{\alpha_i} E_{\alpha_i}$ already defines an automorphism A of $\mathfrak{g}(\!(x)\!)$ of the form Eq. (16) that maps W_0 to W defined by the f_α as above. $\diamondsuit$

Remark 3.5 Following [9, 13, 23, 24], let $\mathfrak{g}$ be matrix Lie algebra and D be a matrix such that $Y \mapsto DY + YD$ defines an endomorphism of $\mathfrak{g}$. Then one can define a series A by

$$A(Y) := \sqrt{1 + Dx}\; Y \sqrt{1 + Dx}.$$

This endomorphism of $\mathfrak{g}(\!(x)\!)$ satisfies the condition of Lemma 3.1 with

$$R(Y) = \frac{1}{2}(DY + YD).$$

Therefore, W_A is an $F[x^{-1}]$-invariant subalgebra of $\mathfrak{g}(\!(x)\!)$ complementary to $\mathfrak{g}[\![x]\!]$.

When $[\mathrm{ad}(h), R] = 0$ for all $h \in \mathfrak{h}$, the corresponding W_A is additionally $\mathfrak{h}$-invariant. For instance, take $\mathfrak{g} = \mathfrak{sp}(2n)$ whose Cartan subalgebra is given by diagonal matrices. Then one can take $D = \mathrm{diag}(a_1, \dots, a_n, a_1, \dots, a_n)$. A similar construction is possible for other classical Lie algebras.

Note that the series A, in general, has more than 3 nonzero terms and hence do not give rise to an upper-bounded subalgebra of $\mathfrak{g}((x))$. It is unclear however if these subalgebras are gauge equivalent to upper-bounded ones. $\diamondsuit$

Combining previous results, we can derive a standard form for regular subalgebras $W \subseteq L_0$ of any type $k \geqslant 0$. To simplify the presentation of this normal form, we introduce the following subsets of roots

$$\Delta_{\pm}^{<m\alpha_i} := \left\{\alpha \in \Delta_{\pm} \mid \alpha = \pm \sum_{i=1}^{n} c_i \alpha_i,\ 0 \leqslant c_i < m\right\},$$
$$\Delta_{\pm}^{\geqslant m\alpha_i} := \left\{\alpha \in \Delta_{\pm} \mid \alpha = \pm \sum_{i=1}^{n} c_i \alpha_i,\ c_i \geqslant m\right\}$$

and let

$$\mathfrak{l} := \bigoplus_{\alpha \in \Delta_{+}^{<k_i\alpha_i}} \mathfrak{g}_\alpha \bigoplus_{\alpha \in \Delta_{-}^{<\alpha_i}} \mathfrak{g}_\alpha, \quad \mathfrak{c} := \bigoplus_{\alpha \in \Delta_{+}^{\geqslant k_i\alpha_i}} \mathfrak{g}_\alpha \text{ and } \mathfrak{r} := \bigoplus_{\alpha \in \Delta_{-}^{\geqslant \alpha_i}} \mathfrak{g}_\alpha. \tag{21}$$

Furthermore, we define the following subspaces of $\mathfrak{g}((x))$

$$V^a := (x^{-1} - a)V[x^{-1}],$$
$$V^{a,b} := x(x^{-1} - a)(x^{-1} - b)V[x^{-1}]$$

for any subspace $V \subseteq \mathfrak{g}$ and $a, b \in F$.

If $\alpha_0 = \sum_{i=1}^{n} k_i \alpha_i$ is the decomposition of the maximal root into a sum of simple roots, then the parabolic $\mathfrak{P}_i$ (see Eq. (12)) can be written in the form

$$\mathfrak{P}_i = (\mathfrak{h} \oplus \mathfrak{l} \oplus x\mathfrak{c} \oplus x^{-1}\mathfrak{r})[x^{-1}] \tag{22}$$

and by Lemma 3.3 any regular subalgebra $W \subseteq \mathfrak{P}_i$ is necessarily of the form

$$W = W_{\mathfrak{h}} \bigoplus_{\alpha \in \Delta_{+}^{<k_i\alpha_i}} \mathfrak{g}_\alpha^{a_\alpha} \bigoplus_{\alpha \in \Delta_{-}^{<\alpha_i}} \mathfrak{g}_\alpha^{a_\alpha} \bigoplus_{\alpha \in \Delta_{+}^{\geqslant k_i\alpha_i}} \mathfrak{g}_\alpha^{c_\alpha, d_\alpha} \bigoplus_{\alpha \in \Delta_{-}^{\geqslant \alpha_i}} \mathfrak{g}_\alpha^{0}. \tag{23}$$

In other words, defining a regular subalgebra $W \subseteq \mathfrak{P}_i$ is equivalent to defining constants $a_\alpha, c_\alpha, d_\alpha$ in Eq. (23) in a consistent way and finding a compatible $W_{\mathfrak{h}} \subseteq \mathfrak{h}[x^{-1}]$.

If we write

$$W_{\mathfrak{h}} = W_\phi := \{x^{-n}(x^{-1}h - \phi(h)) \mid h \in \mathfrak{h}, n \in \mathbb{Z}_{\geqslant 0}\}$$

for some linear map $\phi\colon \mathfrak{h} \to \mathfrak{h}$, the associated $A = 1 + Rx + Sx^2$ is given by

$$Rv = \begin{cases} -\phi(v) & v \in \mathfrak{h}, \\ -a_\alpha v & v \in \mathfrak{g}_\alpha,\, \alpha \in \Delta_+^{<k_i\alpha_i} \cup \Delta_-^{<\alpha_i}, \\ -(c_\alpha + d_\alpha)v & v \in \mathfrak{g}_\alpha,\, \alpha \in \Delta_+^{\geqslant k_i\alpha_i}, \\ 0 & v \in \mathfrak{g}_\alpha,\, \alpha \in \Delta_-^{\geqslant \alpha_i} \end{cases} \quad \text{and}$$

$$Sv = \begin{cases} c_\alpha d_\alpha v & v \in \mathfrak{g}_\alpha,\, \alpha \in \Delta_+^{\geqslant k_i\alpha_i} \\ 0 & v \in \mathfrak{h} \bigoplus_{\alpha \in \Delta \setminus \Delta_+^{\geqslant k_i\alpha_i}} \mathfrak{g}_\alpha, \end{cases}$$

According to [23, Theorem 3.1], the r-matrix associated to W_A is given by

$$\begin{aligned} r(x, y) &= \frac{(A(x) \otimes \overline{A}(y))\Omega}{x - y} = \frac{((A(x)A(y)^{-1} - 1) \otimes 1 + 1 \otimes 1)\Omega}{x - y} \\ &= \frac{\Omega}{x - y} + \left(\frac{A(x) - A(y)}{x - y} A(y)^{-1} \otimes 1 \right) \Omega, \end{aligned} \tag{24}$$

where $\overline{A}$ is uniquely determined by $\kappa(Aa, \overline{A}b) = \kappa(a, b)$. To obtain the expression above we have used $(1 \otimes \overline{A})\Omega = (A^{-1} \otimes 1)\Omega$.

Therefore, if $A = 1 + Rx + Sx^2$ we obtain

$$r(x, y) = \frac{\Omega}{x - y} + \left((R + (x + y)S)(1 + Ry + Sy^2)^{-1} \otimes 1 \right) \Omega.$$

Here we used that

$$(1 + Tx)^{-1} = \sum_{k=0}^{n} (-1)^n T^n x^n.$$

holds for all linear maps $T\colon \mathfrak{g}[\![x]\!] \to \mathfrak{g}[\![x]\!]$.

Summarized, the r-matrix of the subspace Eq. (23) is given by

$$\begin{aligned} r(x, y) &= \frac{\Omega}{x - y} + \left(\frac{\phi}{y\phi - 1} \otimes 1 \right) \Omega_{\mathfrak{h}} + \sum_{\alpha \in \Delta_+^{<k_i\alpha_i} \cup \Delta_-^{<\alpha_i}} \frac{a_\alpha}{a_\alpha y - 1} E_\alpha \otimes E_{-\alpha} \\ &\quad + \sum_{\alpha \in \Delta_+^{\geqslant k_i\alpha_i}} \frac{c_\alpha d_\alpha (x + y) - c_\alpha - d_\alpha}{(c_\alpha y - 1)(d_\alpha y - 1)} E_\alpha \otimes E_{-\alpha}. \end{aligned} \tag{25}$$

Note that $\frac{\phi}{y\phi - 1} = \phi(y\phi - 1)^{-1} = (y\phi - 1)^{-1}\phi$ is unambiguous since ϕ and $y\phi - 1$ commute.

3.2 Regular decompositions $\mathfrak{g}(\!(x)\!) = \mathfrak{g}[\![x]\!] \oplus W$ of type 0

Let us now discuss regular subalgebras $\mathfrak{g}(\!(x)\!) = \mathfrak{g}[\![x]\!] \oplus W$ of type 0, i.e. such that $W \subseteq \mathfrak{g}[x^{-1}]$. Using Lemma 3.2 with $S = 0$ yields the following result.

Corollary 3.6 *The vector space $W_A = A(x^{-1}\mathfrak{g}[x^{-1}]) \subseteq \mathfrak{g}(\!(x)\!)$ is a subalgebra satisfying $W_A \subseteq \mathfrak{g}[x^{-1}]$ if and only if $A = 1 + Rx$ and the so-called Nijenhuis tensor*

$$N_R(a,b) := R([Ra,b] + [a,Rb] - R[a,b]) - [Ra,Rb]$$

of R vanishes: $N_R = 0$.

The vanishing condition for the Nijenhuis tensor can be expressed using the Jordan decomposition of R.

Proposition 3.7 *Let $R\colon \mathfrak{g} \to \mathfrak{g}$ be a linear map, $R = R_s + R_n$ be its Jordan decomposition and $\lambda_1, \ldots, \lambda_m \in F$ be the eigenvalues of the semi-simple linear map R_s with associated eigenspaces $\mathfrak{g}_i := \mathrm{Ker}(R_s - \lambda_i)$. Then*

1. *$N_{R_s} = 0$ if and only if $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_i + \mathfrak{g}_j$;*
2. *$N_R = 0$ if and only if $N_{R_s} = 0$ and*

$$\begin{aligned}\pi_i\big(N_{R_n}(a,b) - (\lambda_i - \lambda_j)(R_n[a,b] - [R_n a, b])\big) = 0,\\ \pi_j\big(N_{R_n}(a,b) + (\lambda_i - \lambda_j)(R_n[a,b] - [a, R_n b])\big) = 0\end{aligned} \tag{26}$$

holds for all $a \in \mathfrak{g}_i, b \in \mathfrak{g}_j$ and $1 \leqslant i, j \leqslant m$, where $\pi_k\colon \mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i \to \mathfrak{g}_k$ is the canonical projection.

Proof By direct calculation we see that the equality $N_R(a,b) = 0$ implies

$$\begin{aligned}(R-\mu)(R-\lambda)[a,b] = (R-\mu)[(R-\lambda)a,b] + (R-\lambda)[a,(R-\mu)b]\\ - [(R-\lambda)a,(R-\mu)b]\end{aligned} \tag{27}$$

for all $\lambda, \mu \in F$. Assume λ and μ are two eigenvalues of R with generalized eigenvectors v and w of ranks r and t, respectively. We prove now by induction that the equalities $(R-\lambda)^r v = (R-\mu)^t w = 0$ imply the equality

$$(R-\lambda)^r (R-\mu)^t [v,w] = 0.$$

If v and w are eigenvectors, then, using Eq. (27), we get the base case

$$(R-\lambda)(R-\mu)[v,w] = 0.$$

Assume the statement is true for $r = 1$ and $1 \leqslant t \leqslant k-1$. Then,

$$\begin{aligned}(R-\lambda)(R-\mu)^k[v,w] &= (R-\mu)^{k-1}(R-\lambda)(R-\mu)[v,w]\\ &= (R-\lambda)(R-\mu)^{k-1}[v,(R-\mu)w]\\ &= 0,\end{aligned}$$

where the last equality follows from the induction hypothesis and the fact that the vector $(R-\mu)w$ has rank $k-1$. Assume now that the statement is true for all $1 \leqslant r \leqslant k-1$ and $t \geqslant 1$. We then have

$$\begin{aligned}(R-\lambda)^k(R-\mu)^t[v,w] &= (R-\lambda)^{k-1}(R-\mu)^{t-1}(R-\lambda)(R-\mu)[v,w]\\ &= (R-\lambda)^{k-1}(R-\mu)^{t-1}\{(R-\mu)[(R-\lambda)v,w]\\ &\quad + (R-\lambda)[v,(R-\mu)w]\\ &\quad -[(R-\lambda)v,(R-\mu)w]\}\\ &= (R-\lambda)^{k-1}(R-\mu)^{t}[(R-\lambda)v,w]\\ &\quad + (R-\lambda)^{k-1}(R-\lambda)(R-\mu)^{t-1}[v,(R-\mu)w]\\ &\quad + (R-\lambda)^{k-1}(R-\mu)^{t-1}[(R-\lambda)v,(R-\mu)w]\\ &= 0.\end{aligned}$$

This shows that $[\mathfrak{g}_i,\mathfrak{g}_j]\subseteq\mathfrak{g}_i+\mathfrak{g}_j$ for $\lambda=\lambda_i$ and $\mu=\lambda_j$, since $\mathfrak{g}_i=\bigcup_{n=1}^{\infty}\operatorname{Ker}((R-\lambda_i)^n)$ and $(R-\lambda_i)|_{\mathfrak{g}_j}$ has trivial kernel for all $i\neq j$.

It is now easy to see from Eq. (27) that $[\mathfrak{g}_i,\mathfrak{g}_j]\subseteq\mathfrak{g}_i+\mathfrak{g}_j$ implies $N_{R_s}=0$. In particular, we see that these two conditions are equivalent. And if one of these equivalent conditions is satisfied, Eq. (27) for $\lambda=\lambda_i$ and $\mu=\lambda_j$ implies $N_R(a,b)=N_{R_n}(a,b)$ for $a,b\in\mathfrak{g}_i$ and

$$\begin{aligned}N_R(a,b) = (R-\lambda_i)(R-\lambda_j)[a,b]-(R-\lambda_i)[a,R_nb]\\ -(R-\lambda_j)[R_na,b]+[R_na,R_nb]\end{aligned}$$

for $a\in\mathfrak{g}_i$, $b\in\mathfrak{g}_j$. The latter can be rewritten as Eq. (26) since

$$\begin{aligned}(R-\lambda_i)v &= (R_s+R_n-\lambda_i)v=(R_s-\lambda_i)v+R_nv=R_nv+(R_s-\lambda_i)\pi_j(v)\\ &= R_nv+(\lambda_j-\lambda_i)\pi_j(v)\end{aligned}$$

and similarly

$$(R-\lambda_j)v=R_nv+(\lambda_i-\lambda_j)\pi_i(v)$$

holds for all $v\in\mathfrak{g}_i\oplus\mathfrak{g}_j$, so

$$\begin{aligned}(R-\lambda_i)(R-\lambda_j)v &= (R-\lambda_i)(R_nv-(\lambda_i-\lambda_j)\pi_i(v))\\ &= R_n^2v-(\lambda_i-\lambda_j)R_n\pi_j(v)+(\lambda_i-\lambda_j)R_n\pi_i(v).\end{aligned}$$

□

Remark 3.8 Observe that in particular, the Nijenhuis tensor of $R_n|_{\mathfrak{g}_i}$ vanishes for all $1\leqslant i\leqslant m$. Therefore, the construction of all linear maps satisfying $N_R=0$ can be split into two steps: first find a decomposition $\mathfrak{g}=\bigoplus_{i=1}^m\mathfrak{g}_i$ satisfying $[\mathfrak{g}_i,\mathfrak{g}_j]\subseteq\mathfrak{g}_i+\mathfrak{g}_j$ and then find nilpotent linear maps $\{R_{n,i}:\mathfrak{g}_i\to\mathfrak{g}_i\}_{i=1}^m$ satisfying Eq. (26). ◊

The following corollary, which is equivalent to the first part of Proposition 3.7, was already noticed in [16].

Corollary 3.9 *Let $R\colon \mathfrak{g} \to \mathfrak{g}$ be a diagonalizable linear map and $a_1, \dots, a_m \in F$ be its eigenvalues with associated eigenspaces $\mathfrak{g}_i := \operatorname{Ker}(R - a_i)$. Then $N_R = 0$ if and only if $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_i + \mathfrak{g}_j$ for $1 \leqslant i, j \leqslant m$.*

Remark 3.10 In view of Corollary 3.9, we can construct a solution R to $N_R = 0$ by first taking a decomposition $\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i$ with the property that $\mathfrak{g}_i$ and $\mathfrak{g}_i \oplus \mathfrak{g}_j$ are subalgebras of $\mathfrak{g}$ and then defining

$$R := \sum_{i=1}^m \lambda_i \pi_i,$$

where π_i is the projection of $\mathfrak{g}$ onto $\mathfrak{g}_i$ and $\lambda_i \in F$ is an arbitrary constant. In particular, we can start with a regular decomposition of $\mathfrak{g}$ defined in Sect. 2.1. $\diamondsuit$

Let us now specify Proposition 3.7 for regular partitions.

Theorem 3.11 *Let*

$$\Delta = \bigsqcup_{i=1}^n \Delta_i$$

be a regular partition, $a_1, \dots, a_n$ be some constants in F and $\phi\colon \mathfrak{h} \to \mathfrak{h}$ be a linear map such that

$$(\phi - a_i)(\phi - a_j)H_\alpha = 0 \ \textit{for}\ \alpha \in \Delta_i,\ -\alpha \in \Delta_j. \tag{28}$$

Then

$$W := W_\phi \bigoplus_{i=1}^n \bigoplus_{\alpha \in \Delta_i} \mathfrak{g}_\alpha^{a_i}, \tag{29}$$

is a regular subalgebra of type 0, where

$$W_\phi := \{x^{-n}(x^{-1}h - \phi(h)) \mid h \in \mathfrak{h}, n \in \mathbb{Z}_{\geqslant 0}\} \tag{30}$$

This assignment is a bijection between the set of data $(\Delta = \bigsqcup_{i=1}^n \Delta_i, \{a_1, \dots, a_n\}, \phi)$ and regular decompositions $\mathfrak{g}((x)) = \mathfrak{g}[\![x]\!] \oplus W$ of type 0.

Furthermore, the r-matrix of W in Eq. (29) has the form

$$r(x, y) = \frac{\Omega}{x - y} + \left(\frac{\phi}{y\phi - 1} \otimes 1\right)\Omega_\mathfrak{h} + \sum_{i=1}^n \frac{a_i \Omega_i}{a_i y - 1},$$

where $\Omega_i = \sum_{\alpha \in \Delta_i} E_\alpha \otimes E_{-\alpha}$.

Proof Let $W = W_A$ for $A = 1 + Rx$, $R = R_s + R_n$ be the Jordan decomposition of R and $\lambda_1, \dots, \lambda_m \in F$ be the eigenvalues of the semi-simple part R_s with associated eigenspaces $\mathfrak{g}_i := \operatorname{Ker}(R_s - \lambda_i)$. By virtue of Corollary 3.6, $W \subseteq \mathfrak{g}((x))$ is a subalgebra if and only if $N_R = 0$.

Let us observe that W is $\mathfrak{h}$-invariant if and only if $[\mathrm{ad}(\mathfrak{h}), R] = 0$. Therefore, for every $i \in \{1, \dots, m\}$ we get $\mathfrak{g}_i = \mathfrak{s}_i \bigoplus_{\alpha\in\Delta_i} \mathfrak{g}_\alpha$ for some $\Delta_i \subseteq \Delta$. The decomposition $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i$ is regular if and only if $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_i + \mathfrak{g}_j$. However, by Proposition 3.7 the latter is equivalent to $N_{R_s} = 0$, which is a necessary condition for $N_R = 0$.

Since $\mathfrak{g}_\alpha$ is one-dimensional for all $\alpha \in \Delta \cup \{0\}$, we have $R|_{\mathfrak{g}_\alpha} = R_s|_{\mathfrak{g}_\alpha}$. This implies that $N_R(a, b) = N_{R_s}(a, b) = 0$ for all $a \in \mathfrak{g}_\alpha$, $b \in \mathfrak{g}_\beta$ with $\alpha, \beta \in \Delta \cup \{0\}$ and $\alpha + \beta \neq 0$. Here, $\mathfrak{g}_0 = \mathfrak{h}$ was used. Moreover, $R(\mathfrak{h}) \subseteq \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{h}] = 0$ implies $N_R(a, b) = 0$ for all $a, b \in \mathfrak{h}$.

Put $\phi := R|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}$ and assume $\alpha \in \Delta_i$ and $-\alpha \in \Delta_j$. Then

$$N_R(E_\alpha, E_{-\alpha}) = (-\phi^2 + (a_i + a_j)\phi - a_i a_j)H_\alpha = -(\phi - a_i)(\phi - a_j)H_\alpha$$

holds. In particular, we see that $N_R = 0$ if and only if Eq. (28) holds.

The form of the r-matrix follows immediately from Eq. (25). □

Remark 3.12 At first glance, it is unclear whether the system of equations Eq. (28) is consistent. However, by Proposition 2.1 for any regular partition $\Delta = \bigsqcup_{i=1}^n \Delta_i$ we can find a decomposition $\mathfrak{h} = \bigoplus_{i=1}^n \mathfrak{s}_i$ such that $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{s}_i \bigoplus_{\alpha\in\Delta_i} \mathfrak{g}_\alpha$ is a regular decomposition. Then we can define ϕ satisfying Eq. (28) by letting $\phi(v) = a_i v$ for $v \in \mathfrak{s}_i$. ♢

Remark 3.13 Note that solutions $r \in \mathfrak{g} \otimes \mathfrak{g}$ to the constant generalized classical Yang-Baxter equation

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{32}, r^{13}] = 0 \tag{31}$$

are in bijection with subalgebras $\mathfrak{w} \subseteq \mathfrak{g}[x^{-1}]/x^{-2}\mathfrak{g}[x^{-1}]$, such that $\mathfrak{g} \oplus \mathfrak{w} = \mathfrak{g}[x^{-1}]/x^{-2}\mathfrak{g}[x^{-1}]$. Indeed, this follows from the fact that tensors $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfying Eq. (31) are in bijection with solutions of the GCYBE of the form $\frac{\Omega}{x-y} + r$ and [4].

We could also consider the classification of such tensors r under the assumption of $\mathfrak{h}$-invariance. It turns out that this classification is trivial: these tensors are precisely arbitrary tensors in $\mathfrak{h} \otimes \mathfrak{h}$. Let us briefly explain why. The $\mathfrak{h}$-invariance is again equivalent to the $\mathfrak{h}$-invariance of $\mathfrak{w}$ and it is easy to see that there exists a linear map $\phi: \mathfrak{h} \to \mathfrak{h}$ such that

$$\mathfrak{w} = \mathrm{span}_F\{x^{-1}h - \phi(h) \mid h \in \mathfrak{h}\} \oplus \sum_{\alpha\in\Delta} (x^{-1} - a_\alpha)\mathfrak{g}_\alpha$$

for a set $\{a_\alpha\}_{\alpha\in\Delta} \subset F$. Assume that some $a_\alpha \neq 0$. Then

$$(x^{-1} - a_\alpha)(x^{-1} - a_{-\alpha})(x^{-1} - a_\alpha)\mathfrak{g}_\alpha \subseteq (x^{-1} - a_\alpha)\mathfrak{g}_\alpha.$$

This implies that $2a_\alpha a_{-\alpha} + a_\alpha^2 = \lambda$ and $a_\alpha^2 a_{-\alpha} = \lambda a_\alpha$ so $a_\alpha a_{-\alpha} = \lambda$ and $2\lambda + a_\alpha^2 = \lambda$, so $a_\alpha^2 = -\lambda$. Plugging this back into $a_\alpha^2 a_{-\alpha} = \lambda a_\alpha$ gives $a_\alpha = -a_{-\alpha}$. But this implies that

$$[(x^{-1} - a_\alpha)\mathfrak{g}_\alpha, (x^{-1} + a_\alpha)\mathfrak{g}_{-\alpha}] = F H_\alpha \subseteq \mathfrak{w} \cap \mathfrak{g}$$

which is a contradiction. We conclude $a_\alpha = 0$ for all $\alpha \in \Delta$. Therefore, every $\mathfrak{h}$-invariant subalgebra $\mathfrak{w} \subseteq \mathfrak{g}[x^{-1}]/x^{-2}\mathfrak{g}[x^{-1}]$ complementary to $\mathfrak{g}$ is of the form

$$\mathfrak{w} = \operatorname{span}_F\{x^{-1}h - \phi(h) \mid h \in \mathfrak{h}\} \oplus x^{-1}(\mathfrak{n}_+ \oplus \mathfrak{n}_-)$$

for some linear map $\phi\colon \mathfrak{h} \to \mathfrak{h}$. The r-matrix corresponding to $\mathfrak{w}$ is of the form

$$r = \phi(h_i) \otimes h_i \in \mathfrak{h} \otimes \mathfrak{h}$$

for an orthonormal basis $\{h_i\}_{i=1}^{\ell}$ of $\mathfrak{h}$. Since there are no restrictions on ϕ, r can be an arbitrary tensor in $\mathfrak{h} \otimes \mathfrak{h}$. $\lozenge$

3.3 Regular decompositions $\mathfrak{g}((x)) = \mathfrak{g}[[x]] \oplus W$ of type 1

Let us now consider regular splittings $\mathfrak{g}((x)) = \mathfrak{g}[[x]] \oplus W$ of type 1. In view of Eq. (22), we have

$$W \subseteq \mathfrak{P}_i = (\mathfrak{h} \oplus \mathfrak{l} \oplus x\mathfrak{c} \oplus x^{-1}\mathfrak{r})[x^{-1}] \tag{32}$$

for $i \in \{1, \dots, n\}$ such that the simple root α_i has multiplicity $k_i = 1$. In this case the inclusion $[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{h} \oplus \mathfrak{l}$ holds. Using this observation we can immediately concoct the following example.

Example 3.14 For two different constants $a_1, a_2 \in F^\times$ define

$$W = (\mathfrak{h} \oplus \mathfrak{l})^{a_1} \oplus \mathfrak{c}^{a_1,a_2} \oplus \mathfrak{r}^0.$$

Clearly, it is a regular subalgebra of $\mathfrak{P}_i$ of type 1. By Lemma 3.2, it must be of the form $W = (1 + Rx + Sx^2)x^{-1}\mathfrak{g}[x^{-1}]$ for some endomorphisms R and S of $\mathfrak{g}$. Indeed, take

$$Rv = \begin{cases} -a_1 v & v \in \mathfrak{h} \oplus \mathfrak{l}, \\ -(a_1 + a_2)v & v \in \mathfrak{c}, \\ 0 & v \in \mathfrak{r} \end{cases} \quad \text{and} \quad Sv = \begin{cases} a_1 a_2 v & v \in \mathfrak{c}, \\ 0 & v \in \mathfrak{h} \oplus \mathfrak{l} \oplus \mathfrak{r}. \end{cases}$$

In particular, by Lemma 3.2, these endomorphisms solve $N_R = dS$. We do not allow a_1 or a_2 be equal to 0, because then the type of W becomes 0. $\lozenge$

We can also make an example of a regular subalgebra of type 1 using a 2-regular partition of Δ.

Example 3.15 Let $\Delta^{<\alpha_i}$ be the root system one obtains from Δ by removing all the roots containing α_i, with $k_i = 1$. In general, it is a union of two irreducible root systems. A partition

$$\Delta^{<\alpha_i} = \Delta_1 \sqcup \Delta_2$$

into two closed subsets gives another two-parameter example of a type 1 regular subalgebra:

$$W = W_\phi \bigoplus_{\alpha\in\Delta_1} \mathfrak{g}_\alpha^{a_1} \bigoplus_{\alpha\in\Delta_2} \mathfrak{g}_\alpha^{a_2} \oplus \mathfrak{c}^{a_1,a_2} \oplus \mathfrak{r}^0,$$

where $a_1, a_2 \in F^\times$ and W_ϕ is given by the same formula Eq. (30), but we require the linear map $\phi\colon \mathfrak{h} \to \mathfrak{h}$ to satisfy

$$\begin{aligned}(\phi - a_i)(\phi - a_j)H_\alpha &= 0 \text{ for } \alpha \in \Delta_i,\ -\alpha \in \Delta_j,\\ (\phi - a_1)(\phi - a_2)H_\alpha &= 0 \text{ for } \alpha \in \Delta_+^{\geqslant \alpha_i}.\end{aligned}$$

Indeed, this follows from Lemma 3.2 with endomorphisms R and S given by

$$Rv = \begin{cases} -a_i v & v \in \bigoplus_{\alpha\in\Delta_i} \mathfrak{g}_\alpha,\\ -(a_1+a_2)v & v \in \mathfrak{c},\\ 0 & v \in \mathfrak{r},\\ -\phi(v) & v \in \mathfrak{h}\end{cases} \quad \text{and}\quad Sv = \begin{cases} a_1a_2 v & v \in \mathfrak{c},\\ 0 & v \in \mathfrak{h}\oplus\mathfrak{l}\oplus\mathfrak{r}.\end{cases}$$

Observe, that by letting $a_1 = a_2$ and $\phi = a_1$ we obtain a specification of Example 3.14. ◊

We now present more sophisticated constructions using m-regular decompositions. As we will see, we can construct type-1 regular subalgebras with an arbitrary number of parameters $a_i \in F$ for systems of types A_n, C_n and D_n. In all other cases, regular subalgebras of type 1 are necessarily of the form presented in Example 3.15.

Type A_n. If we want to obtain a regular subalgebra of type 1 with $m \geqslant 3$ parameters, it is natural to start with an m-regular decomposition of $\mathfrak{g}$ and try extending it. The only irreducible root systems, admitting m-regular partitions with $m \geqslant 3$, are precisely A_n, $n \geqslant 2$. Therefore, the following construction will be the main building block for the latter examples.

Assume $W \subseteq \mathfrak{P}_i$ is a type-1 regular subalgebra. Denote by R_1 and R_2 the subsystems of Δ generated by $\{\alpha_1, \dots, \alpha_{i-1}\}$ and $\{\alpha_{i+1}, \dots, \alpha_n\}$. When $i = 1$ or n, we allow R_1 or R_2 respectively to be the empty set. Such a W must have the following form

$$W = W_\mathfrak{h} \overbrace{\bigoplus_{\alpha\in R_1} \mathfrak{g}_\alpha^{a_\alpha} \bigoplus_{\alpha\in R_2} \mathfrak{g}_\alpha^{b_\alpha}}^{\text{partition of } R_1 \text{ and } R_2} \bigoplus_{\alpha\in\Delta_+^{\geqslant\alpha_i}} \mathfrak{g}_\alpha^{c_\alpha, d_\alpha} \bigoplus_{\alpha\in\Delta_-^{\geqslant\alpha_i}} \mathfrak{g}_\alpha^{0}. \tag{33}$$

Note that if we group together the roots in R_j having the same constant a_α, we obtain a regular partition of R_j in the sense of Eq. (9).

Define $c := c_{\alpha_i}$, $d = d_{\alpha_i}$ and let $\Lambda := \{a_\alpha \mid \alpha \in \pi\backslash\{\alpha_i\}\} \cup \{c, d\}$ be the set of constants in Eq. (33) associated with simple roots. The following statement says, that these constants determine constants of all other roots.

Lemma 3.16 *Let $C := \{a_\alpha \mid \alpha \in \Delta_\pm^{<\alpha_i}\} \cup \{c_\beta, d_\beta \mid \beta \in \Delta_+^{\geqslant\alpha_1}\}$ be the set of all constants in Eq. (33). Then $C \subseteq \Lambda$.*

Proof If γ is a positive root in R_1, then it can be written as $\gamma = \alpha_{k_1} + \alpha_{k_2} + \dots + \alpha_{k_m}$ for some simple roots $\alpha_{k_j} \in \{\alpha_1, \dots, \alpha_{i-1}\}$. Consequently, the constant a_γ lies in the set $\{a_{\alpha_{k_j}} \mid 1 \leqslant j \leqslant m\} \subseteq \Lambda$. The same argument shows that $a_\gamma \in \Lambda$ for any positive $\gamma \in R_2$.

Any root $\gamma \in \Delta_+^{\geqslant \alpha_i}$ is necessarily of the form $\gamma = \alpha + \alpha_i + \beta$, for some non-negative roots $\alpha \in R_1 \cup \{0\}$ and $\beta \in R_2 \cup \{0\}$. Assume first, that $\alpha \neq 0$ and $\beta = 0$. Then by commuting

$$\mathfrak{g}_{\alpha_i}^{c,d} = x(x^{-1} - c)(x^{-1} - d)\mathfrak{g}_{\alpha_i}[x^{-1}] \ \text{ with } \ \mathfrak{g}_{\alpha}^{a_\alpha} = (x^{-1} - a_\alpha)\mathfrak{g}_\alpha[x^{-1}]$$

we get the equality

$$(x^{-1} - q)(x^{-1} - c_\gamma)(x^{-1} - d_\gamma) = (x^{-1} - c)(x^{-1} - d)(x^{-1} - a_\alpha)$$

for some $q \in F$. Consequently $\{c_\gamma, d_\gamma\} \subseteq \{a_\alpha, c, d\} \subseteq \Lambda$. Similarly, for $\alpha = 0$, $\beta \neq 0$ we have $c_\gamma, d_\gamma \in \Lambda$. Finally, writing an arbitrary root $\gamma \in \Delta_+^{\geqslant \alpha_i}$ as $\gamma = \alpha + (\alpha_i + \beta)$ and using the previous results we get the containments $c_\gamma, d_\gamma \in \Lambda$.

It now remains to prove, that $a_{-\gamma} \in \Lambda$ for negative $-\gamma \in R_i$. Assume $-\alpha_k \in R_1$, then we can write it as

$$-\alpha_k = \underbrace{(\alpha_{k+1} + \cdots + \alpha_i)}_{\in \Delta_+^{\geqslant \alpha_i}} + \underbrace{(-\alpha_k - \cdots - \alpha_i)}_{\in \Delta_-^{\geqslant \alpha_i}}.$$

By commuting the corresponding subalgebras of W we get $a_{-\alpha_k} \in \Lambda$. Similarly, $a_{-\alpha_k} \in \Lambda$ for any $-\alpha_k \in R_2$. Since any negative root $-\gamma \in R_i$ is a sum of $-\alpha_k \in R_i$ we get the desired statement. □

Corollary 3.17 *In case of type A_n, a regular subalgebra $W \subseteq \mathfrak{P}_i$ can have at most $n+1$ parameters. In other words, $|C| \leq |\Lambda| = n + 1$.*

The upper bound $n + 1$ can always be achieved as it is seen from the following examples.

Example 3.18 Fix an integer $n \geqslant 3$ and a simple root α_i with $1 < i < n$. To simplify the description, we introduce the following notation:

$$\begin{aligned} \beta_k &:= \alpha_1 + \cdots + \alpha_k, \ 1 \leqslant k \leqslant n, \\ \overrightarrow{\beta_k} &:= \alpha_{i+1} + \cdots + \alpha_{i+k}, \ 1 \leqslant k \leqslant n - i, \\ \overleftarrow{\beta_k} &:= \alpha_{i-1} + \cdots + \alpha_{i-k}, \ 1 \leqslant k \leqslant i - 1. \end{aligned} \tag{34}$$

For convenience we also set $\beta_0 = \overrightarrow{\beta_0} = \overleftarrow{\beta_0} = 0$. Both R_1 and R_2 are subsystems of type A; see Fig. 2.

By Theorem 2.3, up to equivalences, they have unique finest partitions. One representative for the finest partition of R_1 is

$$\begin{aligned} \Delta_1^0 &= \left\{ -\overleftarrow{\beta_k} \mid 1 \leqslant k \leqslant i - 1 \right\}, \\ \Delta_1^m &= \left\{ \overleftarrow{\beta_m}, \overleftarrow{\beta_m} - \overleftarrow{\beta_k} \mid 1 \leqslant m \neq k \leqslant i - 1 \right\}. \end{aligned}$$

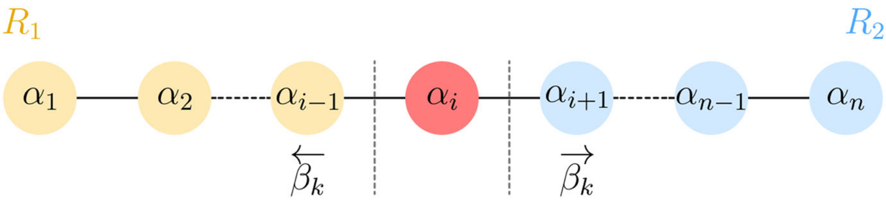


Fig. 2 $\overleftarrow{\beta_k}$ and $\overrightarrow{\beta_k}$ sum k simple roots to the left and right from α_i respectively

Similarly,

$$\Delta_2^0 = \left\{-\overrightarrow{\beta_k} \mid 1 \leqslant k \leqslant n-i\right\},$$
$$\Delta_2^j = \left\{\overrightarrow{\beta_j}, \overrightarrow{\beta_j} - \overrightarrow{\beta_k} \mid 1 \leqslant j \neq k \leqslant n-i\right\}$$

is one of the finest partitions of R_2. Define $H_0 := 0$. For all $0 \leqslant j \leqslant n-i$ and $0 \leqslant m \leqslant i-1$ we define the following subalgebras of $\mathfrak{g}$:

$$\mathfrak{g}_{1,m} := FH_{\overleftarrow{\beta_m}} \bigoplus_{\alpha \in \Delta_1^m} \mathfrak{g}_\alpha,$$
$$\mathfrak{g}_{2,j} := FH_{\overrightarrow{\beta_j}} \bigoplus_{\alpha \in \Delta_2^j} \mathfrak{g}_\alpha.$$

These subalgebras can be now "glued" into the following regular subalgebra of $\mathfrak{g}((x))$:

$$W = \bigoplus_{m=0}^{i-1} \left(\mathfrak{g}_{1,m}^{a_m} \oplus \mathfrak{g}_{\overleftarrow{\beta_m}+\alpha_i}^{a_m,b_0}\right) \bigoplus_{j=0}^{n-i} \left(\mathfrak{g}_{2,j}^{b_j} \oplus \mathfrak{g}_{\alpha_i+\overrightarrow{\beta_j}}^{a_0,b_j}\right) \bigoplus_{m=1}^{i-1} \bigoplus_{j=1}^{n-i} \mathfrak{g}_{\overleftarrow{\beta_m}+\alpha_i+\overrightarrow{\beta_j}}^{a_m,b_j} \bigoplus_{\alpha \in \Delta_-^{\geqslant \alpha_i}} \mathfrak{g}_\alpha^0$$

where a_m and b_j are distinct elements in F.

The corresponding endomorphisms R and S of $\mathfrak{g}$ are given by

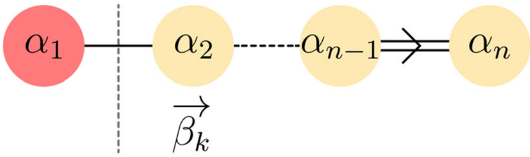
$$Rv = \begin{cases} -av & (Fv)^a \subseteq W, \\ -(a+b)v & (Fv)^{a,b} \subseteq W, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad Sv = \begin{cases} abv & (Fv)^{a,b} \subseteq W, \\ 0 & \text{otherwise}, \end{cases} \tag{35}$$

where $v \in \{H_{\overleftarrow{\beta_m}}, H_{\overrightarrow{\beta_j}}, E_{\pm\alpha}\}$. ♢

Example 3.19 When $n \geqslant 2$ and $i = 1$ or n, we can use the same approach as in Example 3.18, keeping only R_2 or R_1, respectively. More precisely, when $i = 1$ the above construction leads to the following regular subalgebra

$$W = \bigoplus_{j=0}^{n-1} \left(\mathfrak{g}_j^{a_j} \oplus \mathfrak{g}_{\alpha_i+\overrightarrow{\beta_j}}^{a_j,b}\right) \bigoplus_{\alpha \in \Delta_-^{\geqslant \alpha_i}} \mathfrak{g}_\alpha^0,$$

Fig. 3 Unique vertex with $k_i = 1$



where

$$\Delta_0 = \{-\overrightarrow{\beta_k} \mid 1 \leqslant k \leqslant n-1\}, \quad \Delta_j = \{\overrightarrow{\beta_j}, \overrightarrow{\beta_j} - \overrightarrow{\beta_k} \mid 1 \leqslant j \neq k \leqslant n-1\}$$

and

$$\mathfrak{g}_j := FH_{\overrightarrow{\beta_j}} \bigoplus_{\alpha \in \Delta_j} \mathfrak{g}_\alpha, \; 0 \leqslant j \leqslant n-1$$

with the convention $H_0 = 0$.

The endomorphisms R and S are given by the same equations Eq. (35), but with $v \in \{H_{\overrightarrow{\beta}_j}, E_{\pm\alpha}\}$. ♢

Type B_n. There is only one vertex with degree 1, namely α_1 The remaining simple roots $\{\alpha_2, \dots, \alpha_n\}$ generate a subsystem $R \subseteq \Delta$ of type B_{n-1}; see Fig. 3.
Similarly to Eq. (33), any regular subalgebra of type 1 can then be written in the form

$$W = W_\mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha^{a_\alpha} \bigoplus_{\alpha \in \Delta_+^{\geqslant \alpha_1}} \mathfrak{g}_\alpha^{c_\alpha, d_\alpha} \bigoplus_{\alpha \in \Delta_-^{\geqslant \alpha_1}} \mathfrak{g}_\alpha^0. \tag{36}$$

Since there are no regular partitions of R into more than two parts [19], we can immediately conclude that $|\{a_\alpha \mid \alpha \in R\}| \leqslant 2$. When there are exactly two different constants, call them a and b, we have a regular partition $R = R_1 \sqcup R_2$ of the root system R. Note that a system of type B_n has the following property: for any two $\eta, \gamma \in \Delta_+^{\geqslant \alpha_1}$ we can find two roots $\mu, \nu \in R \sqcup \{0\}$ such that $\eta = (\gamma + \mu) + \nu$. Consequently, by fixing a pair $\{c_\gamma, d_\gamma\}$ of constants for an arbitrary $\gamma \in \Delta_+^{\geqslant \alpha_1}$, we fix all other constants as well. From this easy observation follows, that we cannot get more than 4 different parameters in Eq. (36). The next result implies, that actually the number of parameters is at most 2.

Lemma 3.20 *There is a root* $\gamma \in \Delta_+^{\geqslant \alpha_1}$ *and two roots* $\mu, \nu \in \Delta_-^{\geqslant \alpha_1}$ *such that* $\gamma + \mu \in R_1$ *and* $\gamma + \nu \in R_2$.

Proof Assume the statement is false. Then the roots

$$\beta_n - \beta_{n-1} = \alpha_n \text{ and } \beta_n - (\beta_n + \alpha_n) = -\alpha_n$$

always lie together either in R_1 or R_2. Here, we used the notation defined in Eq. (34). Without loss of generality let $\pm\alpha_n \in R_1$. Similarly, all three roots

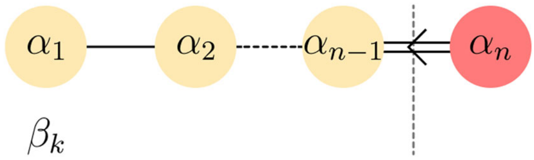


Fig. 4 Subdiagram A_{n-1} inside C_n. Here β_k is the sum of k consecutive simple roots starting from α_1

$$\begin{aligned} -\alpha_n &= \beta_{n-1} - \beta_n, \\ \alpha_{n-1} &= \beta_{n-1} - \beta_{n-2}, \\ -\alpha_{n-1} - 2\alpha_n &= \beta_{n-1} - (\beta_n + \alpha_n + \alpha_{n-1}) \end{aligned}$$

must be contained in one of the sets R_1 or R_2. Since $-\alpha_n \in R_1$, the same is true for the other two roots. Consequently, $\pm\alpha_{n-1} \in R_1$. Continuing in this way, we prove that all $\pm\alpha_i$, $2 \leqslant i \leqslant n$ lie inside R_1. This contradicts the fact that we started with a 2-regular partition $R = R_1 \sqcup R_2$. □

Taking $\gamma \in \Delta_+^{\geqslant\alpha_1}$, satisfying the condition of Lemma 3.20, we get the inclusions

$$\{c_\eta, d_\eta \mid \eta \in \Delta_+^{\geqslant\alpha_1}\} \subseteq \{c_\gamma, d_\gamma\} \cup \{a, b\} \subseteq \{a, b\}.$$

Note that none of the constants can be 0, because this would imply the type of W being 0.

In case $\{a_\alpha \mid \alpha \in R\} = \{a \neq 0\}$, we know that $a \in \{c_\gamma, d_\gamma\}$. If $c_\gamma = a$, then the remaining constant $d_\gamma \neq 0$ can be chosen arbitrarily. Summarizing everything above, we get the following statement.

Proposition 3.21 *Let $\mathfrak{g}$ be of type B_n, $n \geqslant 2$. Then any regular subalgebra $W \subseteq \mathfrak{P}_1$ is necessarily of the form presented in Example 3.15, where $\Delta^{<\alpha_1}$ is a root system of type B_{n-1} and $\Delta^{<\alpha_1} = \Delta_1 \sqcup \Delta_2$ is its 2-regular partition.*

Type C_n. As one can see from Fig. 4, removing the vertex of degree one from the Dynkin diagram of type C_n leads to a subdiagram of type A_{n-1}.
This suggests that we can expect more than just 2 parameters. The following example shows that we can obtain a regular subalgebra $W \subseteq \mathfrak{P}_n$ with n parameters.

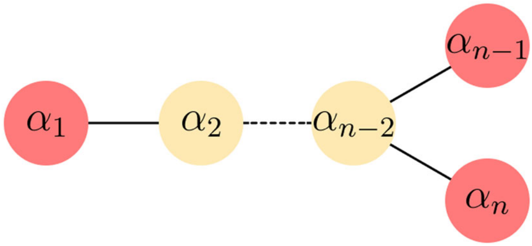
Example 3.22 Start with one of the finest regular decompositions of A_{n-1}, namely

$$\Delta_0 := \{\beta_i \mid 1 \leqslant i \leqslant n-1\} \text{ and } \Delta_j := \{-\beta_j, -\beta_j + \beta_i \mid 1 \leqslant i \neq j \leqslant n-1\},$$

where $\beta_k := \alpha_1 + \cdots + \alpha_k$ and $\beta_0 = 0$. As before, we put $H_0 = 0$ and for $0 \leqslant j \leqslant n-1$ define

$$\mathfrak{g}_j := FH_{\beta_j} \bigoplus_{\alpha \in \Delta_j} \mathfrak{g}_\alpha.$$

Fig. 5 Roots of degree 1 in D_n



We can extend this regular splitting of $\mathfrak{sl}(n, F)$ to a regular subalgebra $W \subseteq \mathfrak{P}_n$ in the following way

$$W := \bigoplus_{i=0}^{n-1} \left(\mathfrak{g}_i^{a_i} \oplus \mathfrak{g}_{\beta_n - \beta_i}^{a_i, a_{n-1}} \right) \bigoplus_{0 \leqslant i \leqslant j < n-1} \mathfrak{g}_{\beta_n + \beta_{n-1} - \beta_j - \beta_i}^{a_i, a_j} \bigoplus_{\alpha \in \Delta_-^{\geqslant \alpha_n}} \mathfrak{g}_\alpha^0$$

where $a_i \in F$ are arbitrary constants, such that at least one of the pairs (a_i, a_{n-1}), (a_i, a_j) or (a_0, a_0) has both nonzero entries. The corresponding endomorphisms R and S are given using the same formula Eq. (35) with $v \in \{H_{\beta_j}, E_{\pm\alpha} \mid 1 \leqslant j \leqslant n, \alpha \in \Delta\}$. $\Diamond$

Type D_n. When $\mathfrak{g}$ is of type D_n, we have three different roots of degree 1 and, essentially, two different cases to consider; see Fig. 5. Removing α_1 in case $n \geqslant 5$ leads to a subsystem of type D_{n-1}. On the other hand, removing α_1 when $n = 4$ and removing α_{n-1} or α_n gives a subdiagram of type A_{n-1}.
Consequently, in the first case we anticipate a rigid system with only two parameters, while in the second case, we can expect to find a regular subalgebra with many parameters.

Let us first consider the case $i = 1$ and $n \geqslant 5$. It is similar to B_n-type case. We define R to be the subsystem of Δ generated by $\{\alpha_2, \dots, \alpha_n\}$. It is a subsystem of type D_{n-1} and hence it has no regular partitions into more than 2 parts. Therefore, for any regular subalgebra $W \subseteq \mathfrak{P}_1$ we can find two constants $a, b \in F$ and a regular partition $R = R_1 \sqcup R_2$ such that

$$W = W_{\mathfrak{h}} \bigoplus_{\alpha \in R_1} \mathfrak{g}_\alpha^a \bigoplus_{\alpha \in R_2} \mathfrak{g}_\alpha^b \bigoplus_{\alpha \in \Delta_+^{\geqslant \alpha_1}} \mathfrak{g}_\alpha^{c_\alpha, d_\alpha} \bigoplus_{\alpha \in \Delta_-^{\geqslant \alpha_1}} \mathfrak{g}_\alpha^0$$

Assuming that R_1 and R_2 are non-trivial, we can again prove a similar result.

Lemma 3.23 *There is a root $\gamma \in \Delta_+^{\geqslant \alpha_1}$ and two roots $\mu, \nu \in \Delta_-^{\geqslant \alpha_1}$ such that $\gamma + \mu \in R_1$ and $\gamma + \nu \in R_2$.*

Proof Assume the opposite. Then for any fixed $\gamma \in \Delta_+^{\geqslant \alpha_1}$ the set

$$O_\gamma := \{\gamma + \mu \in \Delta \mid \mu \in \Delta_-^{\geqslant \alpha_1}\} \subseteq R,$$

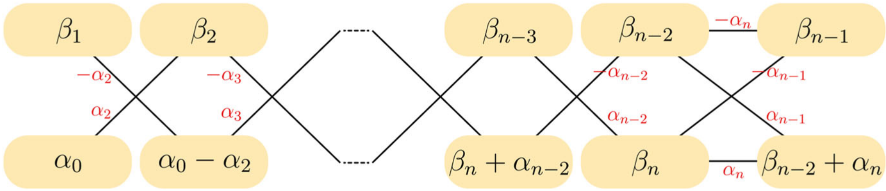


Fig. 6 Orbits related by simple roots

called the orbit of γ, must be contained entirely in one of the sets R_1 or R_2. If two orbits have a nonzero intersection $O_\gamma \cap O_\eta \neq \emptyset$, then both O_γ and O_η are contained in the same R_i. We say that two orbits O_γ and O_η are connected by a root $\beta \in R$, if there are $\mu, \nu \in \Delta_-^{\geqslant \alpha_1}$ such that $\gamma + \mu = \eta + \nu$. Connected orbits, in particular, are contained in the same closed set R_1 or R_2. Consider the following graph

1. Vertices are the roots in $\Delta_+^{\geqslant \alpha_1}$;
2. There is an edge between γ and η if there is a simple root $\alpha_i \in R$ such that α_i or $-\alpha_i$ is contained in $O_\gamma \cap O_\eta$. We label such an edge with α_i or $-\alpha_i$, respectively.

It is not hard to see that in our particular case the graph has the form presented in Fig. 6.

Consequently, all orbits are connected and they contain the set $\{\pm\alpha_2, \dots, \pm\alpha_n\}$. This means that either $R \subseteq R_1$ or $R \subseteq R_2$ contradicting our assumption that R_1 and R_2 were non-trivial. $\square$

Note that, as in the case of B_n, any two roots $\gamma, \eta \in \Delta_+^{\geqslant \alpha_1}$ can be connected by a chain of roots $\mu_1, \dots, \mu_\ell \in R$, i.e. we can write $\eta = \gamma + \mu_1 + \cdots + \mu_\ell$. Repeating the argument proceeding Proposition 3.21 we obtain the following statement.

Proposition 3.24 *Let $\mathfrak{g}$ be of type D_n, $n \geqslant 5$. Then any regular subalgebra $W \subseteq \mathfrak{P}_1$ is necessarily of the form given in Example 3.15. The system $\Delta^{<\alpha_1}$ is a system of type D_{n-1} and $\Delta^{<\alpha_1} = \Delta_1 \sqcup \Delta_2$ is its regular partition.*

Now we turn our attention to the case $i = n$. As it was mentioned before, the subsystem $R \subseteq \Delta$ generated by $\{\alpha_1, \dots, \alpha_{n-1}\}$ has type A_{n-1}. We now present a construction of a regular subalgebra $W \subseteq \mathfrak{P}_n$ with n parameters.

Example 3.25 We start exactly as in Example 3.22 with the finest partition of R:

$$\Delta_0 := \{\beta_i \mid 1 \leqslant i \leqslant n-1\} \text{ and } \Delta_j := \{-\beta_j, -\beta_j + \beta_i \mid 1 \leqslant i \neq j \leqslant n-1\},$$

where $\beta_k := \alpha_1 + \cdots + \alpha_k$ and $\beta_0 = 0$. We put $H_0 = 0$ and define

$$\mathfrak{g}_j := F H_{\beta_j} \bigoplus_{\alpha \in \Delta_j} \mathfrak{g}_\alpha$$

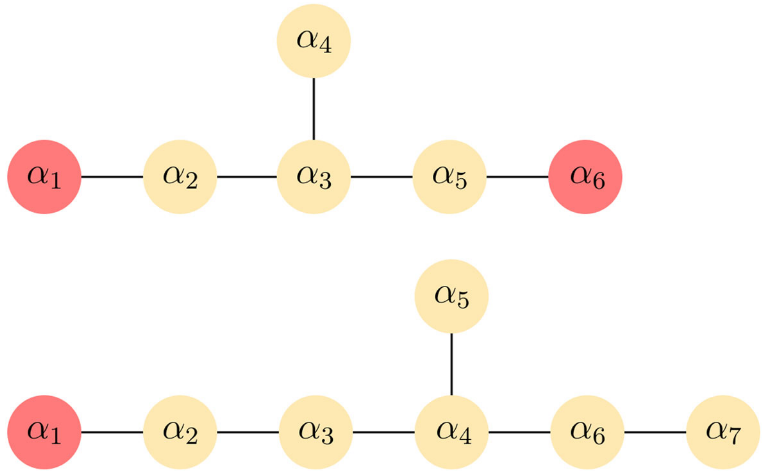


Fig. 7 Vertices of degree 1 in E_6 and E_7

for each $0 \leqslant j \leqslant n-1$. Such a decomposition can be glued to a regular subalgebra of $\mathfrak{P}_n$ in the following way:

$$W := \bigoplus_{i=0}^{n-1} \mathfrak{g}_i^{a_i} \bigoplus_{i=0}^{n-3} \mathfrak{g}_{\beta_n-\beta_i}^{a_i,a_{n-2}} \bigoplus_{i=0}^{n-2} \mathfrak{g}_{\beta_n-\alpha_{n-1}-\beta_i}^{a_i,a_{n-1}} \bigoplus_{0\leqslant i<j<n-3} \mathfrak{g}_{\beta_n+\beta_{n-2}-\beta_i-\beta_j}^{a_i,a_j} \bigoplus_{\alpha\in\Delta_-^{\geqslant\alpha_n}} \mathfrak{g}_\alpha^0,$$

where $a_i \in F$. $\Diamond$

The endomorphisms R and S are obtained using Eq. (35) with the restriction $v \in \{H_{\beta_j}, E_{\pm\alpha} \mid 1 \leqslant j \leqslant n, \alpha \in \Delta\}$.

Type E_6, E_7. There are only two vertices of degree 1 in E_6. Removing any of them leads to a subsystem of type D_5; see Fig. 7.

Furthermore, removing the only node of degree 1 from E_7 gives rise to a subsystem of type E_6. In both cases, we can repeat the arguments for D_n and prove the following.

Proposition 3.26 *Let $\mathfrak{g}$ be of type E_6 or E_7. Then any regular subalgebra $W \subseteq \mathfrak{P}_1$ (and $\mathfrak{P}_6$ in E_6 case) is necessarily of the form given in Example 3.15.*

Remark 3.27 The only difference between proofs for Propositions 3.24 and 3.26 is that in the latter case we need a more cumbersome calculation to show that all orbits intersect and contain all the roots $\{\pm\alpha_1, \ldots, \pm\alpha_n\}\backslash\{\pm\alpha_i\}$. However, this can still be done effectively by hand, using the corresponding Hasse diagrams. $\Diamond$

3.4 Regular decompositions $\mathfrak{g}((x)) = \mathfrak{g}[[x]] \oplus W$ of type > 1

When the degree k_i of a simple root α_i is greater than 1, we can still use the same two building blocks Examples 3.18 and 3.19 to construct examples of regular subalgebras

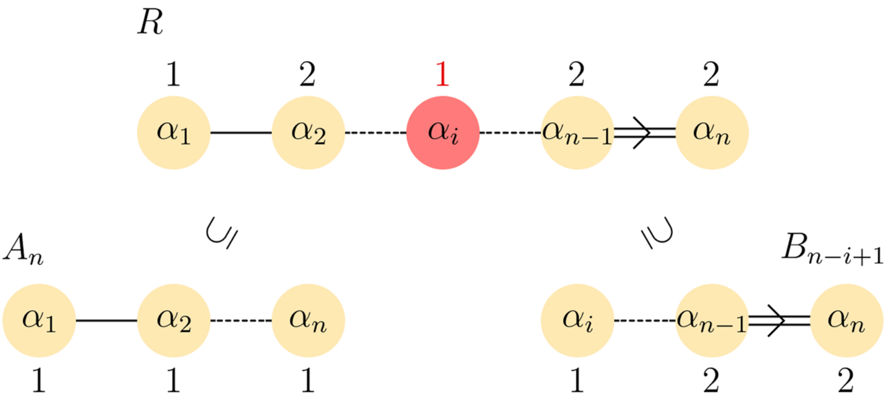


Fig. 8 Sets of type A_n and B_{n-i+1} inside R

$W \subseteq \mathfrak{P}_i$. The difference lies only in the "gluing" complexity. We consider $\mathfrak{g}$ of type B_n as an example.

Example 3.28 Let $\mathfrak{g}$ be a simple Lie algebra of type B_n, $n \geqslant 3$, and fix some $1 < i < n$. By Lemma 3.3 any regular subalgebra $W \subseteq \mathfrak{P}_i$ is of the form

$$W = W_{\mathfrak{h}} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha^{a_\alpha} \bigoplus_{\alpha \in \Delta_+^{\geqslant k_i \alpha_i}} \mathfrak{g}_\alpha^{c_\alpha, d_\alpha} \bigoplus_{\alpha \in \Delta_-^{\geqslant \alpha_i}} \mathfrak{g}_\alpha^0,$$

where $R = \Delta_+^{<k_i\alpha_i} \sqcup \Delta_-^{<\alpha_i}$. Note that R is not a subsystem in general, because it may not be closed under root addition. However, R does contain *sets of type* A_n and B_{n-i+1}; see Fig. 8.

By that we mean that $R_1 := \{\beta_k - \beta_j \mid 0 \leqslant k \neq j \leqslant n\}$ is a subset of R and for any two roots $\mu, \nu \in R_1$ we have $\mu + \nu \in R_1$ if and only if $\mu + \nu$ is a root in A_n. Here we again used the notation Eq. (34). Similarly, the set

$$R_2 := \{\beta_k - \beta_j, \pm(2\beta_n - \beta_\ell - \beta_p) \mid i \leqslant k \neq j \leqslant n,\ i - 1 \leqslant p < \ell \leqslant n - 1\}$$

has the property $\mu + \nu \in R_2$ for $\mu, \nu \in R_2$ if and only if $\mu + \nu$ is a root of B_{n-i+1}. A regular subalgebra W must induce regular partitions of the subsets R_1 and R_2, respectively. In building an example, we take the opposite path: we decompose the subsets R_1 and R_2 and then "glue" them into a regular subalgebra $W \subseteq \mathfrak{P}_i$.

Start with a finest partition of R_1:

$$\Delta_0' := \{\beta_k \mid 1 \leqslant k \leqslant n\} \text{ and } \Delta_j' := \{-\beta_j, -\beta_j + \beta_k \mid 1 \leqslant k \neq j \leqslant n\}.$$

Now we extend these sets with the remaining roots of R in the following way

$$\begin{aligned}\Delta_0 &:= \Delta_0' \sqcup \{2\beta_n - \beta_k \mid i \leqslant k \leqslant n - 1\}\\ \Delta_j &:= \Delta_j' \sqcup \{2\beta_n - \beta_j - \beta_k \mid i < k \leqslant n - 1\},\ 1 \leqslant j \leqslant i - 1,\end{aligned}$$

and

$$\Delta_i := (\Delta_+^{\geqslant 2\alpha_n} \cap \Delta_+^{<\alpha_i}) \sqcup (\Delta_-^{\geqslant 2\alpha_n} \cap \Delta_-^{<2\alpha_i}) \bigsqcup_{j=i}^{n} \Delta'_j.$$

In this way, we get a partition $R = \bigsqcup_{k=0}^{i} \Delta_k$ with the property, that all the roots of R_2 lie within two subsets Δ_{i-1} and Δ_i. We now add the Cartan part of $\mathfrak{g}$:

$$\begin{aligned}
\mathfrak{g}_0 &:= FH_{\beta_n} \bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_\alpha, \\
\mathfrak{g}_j &:= FH_{\beta_n - \beta_j} \bigoplus_{\alpha \in \Delta_j} \mathfrak{g}_\alpha, \\
\mathfrak{g}_i &:= \bigoplus_{k=i}^{n-1} FH_{\beta_n - \beta_k} \bigoplus_{\alpha \in \Delta_i} \mathfrak{g}_\alpha
\end{aligned}$$

and extend everything to a regular subalgebra $W \subseteq \mathfrak{P}_i$

$$W := \bigoplus_{k=0}^{i-1} \mathfrak{g}_k^{a_k} \bigoplus_{0 \leqslant \ell < m \leqslant i-1} \mathfrak{g}_{2\beta_n - \beta_m - \beta_\ell}^{a_m, a_\ell} \bigoplus \mathfrak{g}_i^0 \bigoplus_{\alpha \in \Delta_-^{\geqslant 2\alpha_i}} \mathfrak{g}_\alpha^0.$$

with i parameters. The corresponding endomorphisms R and S are given by Eq. (35) with $v \in \{H_{\beta_n - \beta_k}, E_{\pm\alpha} \mid 0 \leqslant k \leqslant n-1,\ \alpha \in \Delta\}$. $\Diamond$

Therefore, a general approach to constructing regular subalgebras of type > 1 is:

1. Find a subset $R_1 \subseteq R$ of type A_n and consider its finest partition $R_1 = \sqcup_{i=0}^{n} \Delta'_i$ (see Theorem 2.3);
2. Complete sets Δ'_i with the remaining roots of $R \setminus R_1$ in such a way that any subset $R_2 \subseteq R$ of a type different from A_n is contained in at most two elements of the partition;
3. Add Cartan part and extend the resulting decomposition of $\mathfrak{g}$ to a regular subalgebra W.

4 Regular decompositions $L_m = \mathfrak{D} \oplus W$ for $m > 0$

In view of Remark 2.12, to study regular subalgebras $W \subseteq L_m$ it is enough to consider cases $m \leqslant 2$. Lie algebra L_0 was considered in the previous section. Now we turn our attention to $m = 1$ and $m = 2$.

It turns out, that regular $W \subseteq L_1$ of type 0 admit a complete classification. Similar to Theorem 3.11, the classification is obtained by reducing the problem to decompositions of $\mathfrak{g} \times \mathfrak{g}$.

4.1 Regular decompositions $\mathfrak{g} \times \mathfrak{g} = \mathfrak{d} \oplus \mathfrak{w}$

We say that $\mathfrak{w} \subseteq \mathfrak{g} \times \mathfrak{g}$ is regular if it is invariant under $\{(h,h) \mid h \in \mathfrak{h}\}$ and $\mathfrak{w} \oplus \mathfrak{d} = \mathfrak{g} \times \mathfrak{g}$, where $\mathfrak{d} = \{(a,a) \mid a \in \mathfrak{g}\}$. Regular subalgebras are completely described in the following proposition.

Theorem 4.1 *Let $\mathfrak{w} \subseteq \mathfrak{g} \times \mathfrak{g}$ be a regular subalgebra. Then there is a 2-regular partition $\Delta = S_+ \sqcup S_-$ and subspaces of the Cartan subalgebra $\mathfrak{s}_\pm = \mathfrak{t}_\pm \oplus \mathfrak{r}_\pm$, having the properties $\mathfrak{h} = \mathfrak{s}_+ + \mathfrak{s}_-$ and $\mathfrak{t}_+ \cap \mathfrak{t}_- = \{0\}$, such that*

$$\mathfrak{w} = \left(\left(\left(\mathfrak{t}_+ \bigoplus_{\alpha \in S_+} \mathfrak{g}_\alpha\right) \times \{0\}\right) \oplus \left(\{0\} \times \left(\mathfrak{t}_- \bigoplus_{\beta \in S_-} \mathfrak{g}_\beta\right)\right)\right) \oplus \mathrm{span}_F\{(h, \phi(h)) \mid h \in \mathfrak{r}_+\}, \tag{37}$$

where $\phi \colon \mathfrak{r}_+ \to \mathfrak{r}_-$ is a vector space isomorphism with no nonzero fixed points. In other words, all regular subalgebras are described by 2-regular partitions of Δ and the extra datum $(\mathfrak{t}_\pm, \mathfrak{r}_\pm, \phi)$.

Proof Consider a regular subalgebra $\mathfrak{w} \subseteq \mathfrak{g} \times \mathfrak{g}$. Write $p_\pm$ for the canonical projections of $\mathfrak{w}$ onto the first and second components of $\mathfrak{g} \times \mathfrak{g}$, respectively. Define the following spaces:

$$\mathfrak{w}_\pm := p_\pm(\mathfrak{w}), \quad I_+ := \mathfrak{w} \cap (\mathfrak{g} \times \{0\}) \text{ and } I_- := \mathfrak{w} \cap (\{0\} \times \mathfrak{g}).$$

It is clear that $I_\pm$ is an ideal in $\mathfrak{w}_\pm$. This allows to define an isomorphism φ of Lie algebras:

$$\begin{aligned} \varphi \colon \mathfrak{w}_+/I_+ &\to \mathfrak{w}_-/I_-, \\ [a_+] &\mapsto [a_-] := [(p_- \circ p_+^{-1})(a_+)]. \end{aligned} \tag{38}$$

The invariance of $\mathfrak{w}$ under the adjoint action of $\{(h,h) \mid h \in \mathfrak{h}\}$ implies the invariance of $\mathfrak{w}_\pm$ under the adjoint action of $\mathfrak{h}$. Indeed, applying projections $p_\pm$ to the element

$$[(h,h),(a_+,a_-)] = ([h,a_+],[h,a_-]) \in W$$

we see that $[h, a_\pm] \in \mathfrak{w}_\pm$ for all $h \in \mathfrak{h}$ and all $(a_+, a_-) \in W$, giving the desired invariance. Moreover, since $[\mathfrak{h}, I_\pm] \subseteq I_\pm$, the isomorphism φ is an $\mathfrak{h}$-module isomorphism. Consequently, we can write

$$\mathfrak{w}_\pm = \mathfrak{s}_\pm \bigoplus_{\alpha \in S_\pm} \mathfrak{g}_\alpha \text{ and } I_\pm = \mathfrak{t}_\pm \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$$

for some subspaces $\mathfrak{t}_\pm \subseteq \mathfrak{s}_\pm \subseteq \mathfrak{h}$ and subsets $R_\pm \subseteq S_\pm \subseteq \Delta$. Note that since $I_+ \cap I_- = \{0\}$ we have $\mathfrak{t}_+ \cap \mathfrak{t}_- = \{0\}$ and $R_+ \cap R_- = \emptyset$. The isomorphism φ then means

$$\mathfrak{w}_+/I_+ \cong \mathfrak{s}_+/\mathfrak{t}_+ \bigoplus_{\alpha \in S_+ \setminus R_+} \mathfrak{g}_\alpha \cong \mathfrak{s}_-/\mathfrak{t}_- \bigoplus_{\alpha \in S_- \setminus R_-} \mathfrak{g}_\alpha \cong \mathfrak{w}_-/I_-.$$

Since φ intertwines the $\mathfrak{h}$-action, it must act as multiplication by a nonzero scalar on each nonzero root space and hence $S_+ \backslash R_+ = S_- \backslash R_- \subseteq S_+ \cap S_-$. Furthermore, since $\mathfrak{w}_+ + \mathfrak{w}_- = \mathfrak{g}$ we get

$$\Delta = S_+ \cup S_- = R_+ \cup (S_+ \backslash R_+) \cup R_- \cup (S_- \backslash R_-) = R_+ \cup R_- \cup (S_+ \backslash R_+).$$

Assume $S_+ \setminus R_+ \neq S_+ \cap S_-$, meaning that there is a root $\alpha \in R_+ \cap (S_+ \cap S_-)$ and

$$\alpha \notin S_+ \backslash R_+ = S_- \backslash R_-.$$

Then we must have the containment $\alpha \in R_-$, leading to a contradiction $R_+ \cap R_- \neq \emptyset$. Therefore, $S_+ \backslash R_+ = S_+ \cap S_-$ and $R_\pm = S_\pm \setminus S_\mp$.

Let us write $\mathfrak{s}_\pm = \mathfrak{t}_\pm \oplus \mathfrak{r}_\pm$ for some vector subspaces $\mathfrak{r}_\pm \subseteq \mathfrak{s}_\pm$. Combining the results above, we can write

$$\mathfrak{w} = \left(\left(\left(\mathfrak{t}_+ \bigoplus_{\alpha \in \Delta_1} \mathfrak{g}_\alpha\right) \times \{0\}\right) \oplus \left(\{0\} \times \left(\mathfrak{t}_- \bigoplus_{\beta \in \Delta_2} \mathfrak{g}_\beta\right)\right)\right) \bigoplus_{\gamma \in S_+ \cap S_-} \operatorname{span}_F\{(E_\gamma, \lambda_\gamma E_\gamma)\}$$
$$\oplus \operatorname{span}_F\{(h_1, h_2) \in \mathfrak{r}_+ \times \mathfrak{r}_- \mid \phi([h_1]) = [h_2]\},$$

where we put $\Delta_1 := S_+ \backslash S_-$ and $\Delta_2 := S_- \backslash S_+$. To complete the proof we need to show that the intersection $S_+ \cap S_-$ is empty. For that assume the opposite and take $\gamma \in S_+ \cap S_-$. Then either $-\gamma \in S_+ \cap S_-$ or $-\gamma \notin S_+ \cap S_-$. In the first case the following elements must be contained in $\mathfrak{w}$:

$$(E_{\pm\gamma}, \lambda_{\pm\gamma} E_{\pm\gamma}), \quad (H_\gamma, \lambda_\gamma \lambda_{-\gamma} H_\gamma),$$
$$(E_\gamma, \lambda_\gamma^2 \lambda_{-\gamma} E_\gamma), \quad (E_{-\gamma}, \lambda_\gamma \lambda_{-\gamma}^2 E_{-\gamma}).$$

This is possible if and only if $\lambda_\gamma \lambda_{-\gamma} = 1$. However, this would imply $(H_\gamma, H_\gamma) \in \mathfrak{w} \cap \mathfrak{d}$, which is a contradiction. Therefore, we can without loss of generality assume $-\gamma \in S_+ \setminus S_-$. Then, by taking appropriate commutators inside $\mathfrak{w}$, we see that the following elements are also in $\mathfrak{w}$:

$$(E_\gamma, \lambda_\gamma E_\gamma), \quad (E_{-\gamma}, 0),$$
$$(H_\gamma, 0), \quad (\gamma(H_\gamma) E_\gamma, 0),$$

giving rise to the contradiction $\gamma \in S_+ \setminus S_-$. We can now conclude that such a γ does not exist implying $S_+ \cap S_- = \emptyset$.

The isomorphism φ cannot have fixed points, because otherwise we get a non-trivial intersection of $\mathfrak{w}$ with the diagonal. Let $\phi \colon \mathfrak{r}_+ \to \mathfrak{r}_-$ be the isomorphism that maps $h_1 \in \mathfrak{r}_+$ to a representative of $\varphi([h_1]) \in \mathfrak{s}_-/\mathfrak{t}_-$ inside $\mathfrak{r}_-$. This completes the proof. □

Example 4.2 One straight-forward example of a regular subalgebra of $\mathfrak{g} \times \mathfrak{g}$ is

$$\mathfrak{w} = (\mathfrak{n}_+ \times \{0\}) \oplus (\{0\} \times \mathfrak{n}_-) \oplus \{(h, \phi(h)) \mid h \in \mathfrak{h}\},$$

where ϕ is defined on simple roots by $\phi(H_{\alpha_i}) = \lambda_{\alpha_i} H_{\alpha_i}$ for $1 \neq \lambda_{\alpha_i} \in F^{\times}$. ◊

Example 4.3 The example above can be easily generalized to an arbitrary 2-regular partition. More precisely, if $\Delta = \Delta_1 \sqcup \Delta_2$, then we can define

$$\mathfrak{g}_i := \mathrm{span}_F\{E_\alpha, H_\beta \mid \alpha \in \Delta_i, \beta \in \Delta_i \cap (-\Delta_i)\}.$$

It is not hard to see that $\mathfrak{h} \not\subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$. In other words, there is always a missing Cartan part $\mathfrak{h}'$. This allows to define

$$\mathfrak{w} = (\mathfrak{g}_1 \times \{0\}) \oplus (\{0\} \times \mathfrak{g}_2) \oplus \{(h, \phi(h)) \mid h \in \mathfrak{h}'\},$$

where ϕ is any linear isomorphism without nonzero fixed points. ◊

Remark 4.4 Let us make a connection of the proposition above to the GCYBE similar to Remark 3.13. It is not hard to see that any subspace $\mathfrak{w} \subseteq \mathfrak{g} \times \mathfrak{g}$ complementary to $\mathfrak{d}$ can be written in the form

$$\mathfrak{w} = \{(Ra + a, Ra - a) \mid a \in \mathfrak{g}\},$$

for some unique endomorphism $R \in \mathrm{End}_F(\mathfrak{g})$. The property of $\mathfrak{w}$ being a subalgebra is equivalent to the relation

$$[Ra, Rb] - R([Ra, b] + [a, Rb]) = -[a, b] \text{ for all } a, b \in \mathfrak{g}. \tag{39}$$

The Killing form κ on $\mathfrak{g}$ allows us to identify $\mathfrak{g} \otimes \mathfrak{g}$ with $\mathrm{End}_F(\mathfrak{g})$, by sending $x \otimes y$ to $\kappa(-, y)x$. Under this identification, the Eq. (39) becomes

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{32}, r^{13}] = -[\Omega^{12}, \Omega^{13}], \tag{40}$$

where $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the quadratic Casimir element of $\mathfrak{g}$.

By Theorem 4.1, all $\mathfrak{h}$-invariant solutions to Eq. (40) are classified by 2-regular partitions and additional datum $(S_\pm, \mathfrak{t}_\pm, \mathfrak{r}_\pm, \phi)$. Explicitly, consider the tensor

$$\begin{aligned}\widetilde{r} = &\sum_{\alpha \in S_+} (E_\alpha, 0) \otimes (E_{-\alpha}, E_{-\alpha}) - \sum_{\alpha \in S_-} (0, E_\alpha) \otimes (E_{-\alpha}, E_{-\alpha}) \\ &+ \sum_{i=1}^{n} (\phi(h_i) + h_i, \phi(h_i) - h_i) \otimes (h_i, h_i)\end{aligned}$$

where $\{h_i\}_{i=1}^n \subseteq \mathfrak{h}$ is an orthonormal basis and $\phi \colon \mathfrak{h} \to \mathfrak{h}$ is the unique extension of ϕ satisfying $\phi(\mathfrak{t}_+ \oplus \mathfrak{t}_-) = \{0\}$. The element $\widetilde{r}$ corresponds to the map $(a, a) \mapsto (Ra + a, Ra - a)$ and hence the solution of Eq. (40) associated with R is given by

$$r_{(S_\pm, \mathfrak{t}_\pm, \mathfrak{r}_\pm, \phi)} := \frac{1}{2}(r_+ + r_-)$$

$$= \frac{1}{2}\left(\sum_{\alpha\in S_+} E_\alpha \otimes E_{-\alpha} - \sum_{\alpha\in S_-} E_\alpha \otimes E_{-\alpha}\right) + \sum_{i=1}^{n} \phi(h_i)\otimes h_i, \quad (41)$$

where $p_\pm\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, $(a_+, a_-) \mapsto a_\pm$ are the canonical projections and $r_\pm := (p_\pm \otimes p_\pm)\widetilde{r}$. $\Diamond$

4.2 Regular decomposition $L_m = \mathfrak{D} \oplus W$ with $m > 0$ of type 0

Let us start with a regular decomposition

$$L_2 = \mathfrak{g}((x)) \times \mathfrak{g}[x]/x^2\mathfrak{g}[x] = \mathfrak{D} \oplus W.$$

By definition, the projection W_+ of W onto the left component is contained in a maximal order $\mathfrak{P}$; see Eq. (22). When W is of type 0, we have $W_+ \subseteq \mathfrak{P}_0 = \mathfrak{g}[x^{-1}]$ and, consequently,

$$0 \times x\mathfrak{g} \subseteq W.$$

Therefore, we can quotient W by the ideal $0 \times x\mathfrak{g}$ and reduce the problem to a regular decomposition of L_1 of type 0.

Consider now a regular subalgebra $W \subseteq L_1$ of type 0. This means that $W \subseteq \mathfrak{g}[x^{-1}] \times \mathfrak{g}$. Intersecting it with $\mathfrak{g}\times\mathfrak{g}$ we get a regular subalgebra $\mathfrak{w} \subseteq \mathfrak{g}\times\mathfrak{g}$. The latter subalgebras were classified in Theorem 4.1. The following result shows that we can also extend regular subalgebras of $\mathfrak{g}\times\mathfrak{g}$ back to type 0 subalgebras of L_1.

Theorem 4.5 *Let $\mathfrak{g}((x)) \times \mathfrak{g} = \mathfrak{D} \oplus W$ be a regular decomposition such that $W \subseteq \mathfrak{g}[x^{-1}] \times \mathfrak{g}$, i.e. W has type 0. Then*

$$W = \mathfrak{w} \oplus \left(\left(W_\psi \bigoplus_{i=0}^{n} \bigoplus_{\alpha\in\Delta_i} \mathfrak{g}_\alpha^{a_i}\right) \times \{0\}\right),$$

where

- *$\mathfrak{w} \subseteq \mathfrak{g}\times\mathfrak{g}$ is given by Eq. (37) with a 2-splitting $\Delta = S_+ \sqcup S_-$ and the additional datum $(\mathfrak{t}_\pm, \mathfrak{r}_\pm, \phi)$;*
- *$\Delta = \sqcup_{i=0}^{n}\Delta_i$ is a regular decomposition such that $S_+ \subseteq \Delta_0$ and $S_+ + \Delta_i$ are closed;*
- *$a_0, \ldots, a_n \in F$ are distinct constants with $a_0 = 0$;*
- *$\psi : \mathfrak{h} \to \mathfrak{h}$ satisfies $\psi(\mathfrak{r}_+ \oplus \mathfrak{t}_+) = \{0\}$ and*

$$\begin{cases} (\psi - a_i)H_\alpha = 0 & \alpha \in S_+, -\alpha \in \Delta_i, \\ (\psi - a_i)(\psi - a_j)H_\alpha = 0 & \alpha \in \Delta_i, -\alpha \in \Delta_j. \end{cases}$$

Moreover, the above datum always defines a regular subalgebra $W \subseteq L_1$ of type 0.

The r-matrix of W has the form

$$r(x,y)=\frac{y\Omega}{x-y}+r_{(S_\pm,\mathfrak{t}_\pm,\mathfrak{r}_\pm,\phi)}+\frac{1}{2}\Omega+\left(\frac{\psi}{y\psi-1}\otimes 1\right)\Omega_{\mathfrak{h}}+\sum_{i=1}^{n}\frac{a_i\Omega_i}{a_iy-1},$$

where $\Omega_i=\sum_{\alpha\in\Delta_i}E_\alpha\otimes E_{-\alpha}$ *and* $r_{(S_\pm,\mathfrak{t}_\pm,\mathfrak{r}_\pm,\phi)}$ *is given by formula Eq.* (41).

Proof Let $W\subseteq L_1$ be a regular subalgebra contained in $\mathfrak{g}[x^{-1}]\times\mathfrak{g}$. Define

$$\mathfrak{w}:=W\cap(\mathfrak{g}\times\mathfrak{g}).$$

Regularity of W implies regularity of $\mathfrak{w}$. By Theorem 4.1 we can find

1. A regular partition $\Delta=S_+\sqcup S_-$;
2. Cartan subspaces $\mathfrak{s}_\pm=\mathfrak{t}_\pm\oplus\mathfrak{r}_\pm$ such that $\mathfrak{h}=\mathfrak{s}_++\mathfrak{s}_-$ and $\mathfrak{t}_+\cap\mathfrak{t}_-=\{0\}$;
3. A linear isomorphism $\phi\colon\mathfrak{r}_+\to\mathfrak{r}_-$ without nonzero fixed points

such that

$$\mathfrak{w}=\left(\left(\left(\mathfrak{t}_+\bigoplus_{\alpha\in S_+}\mathfrak{g}_\alpha\right)\times\{0\}\right)\oplus\left(\{0\}\times\left(\mathfrak{t}_-\bigoplus_{\beta\in S_-}\mathfrak{g}_\beta\right)\right)\right)\oplus\mathrm{span}_F\{(h,\phi(h))\mid h\in\mathfrak{r}_+\}.$$

By $(x^{-1},0)$-invariance of W we have

$$x^{-k}\left(\mathfrak{s}_+\bigoplus_{\alpha\in S_+}\mathfrak{g}_\alpha\right)\times\{0\}\subseteq W\quad\text{for all}\quad k\geqslant 1.$$

Furthermore, for each $\alpha\in S_-$ we can find unique $f\in\mathfrak{g}[\![x]\!]$ and $w\in W$ such that

$$(x^{-1}E_\alpha,0)=(f,[f])+w.$$

Since W is of type 0 we necessarily have the containment $f\in\mathfrak{g}$ forcing $(x^{-1}E_\alpha-f,-[f])\in W$. The $\mathfrak{h}$-invariance of W now implies that $f=a_\alpha E_\alpha$. Consequently,

$$\bigoplus_{\alpha\in S_-}(x^{-1}-a_\alpha)\mathfrak{g}_\alpha[x^{-1}]\times\{0\}\subseteq W.$$

Let $\{a_\alpha\mid\alpha\in S_-\}=\{a_1,\dots,a_m\}$ and define $\Delta_i:=\{\alpha\in S_-\mid a_\alpha=a_i\}$. Regularity of W implies that $S_-=\sqcup_i\Delta_i$ is a regular partition of S_-. Summarizing, W must have the following form

$$W = \left(\left(\mathfrak{t}_+ \bigoplus_{\alpha\in S_+} \mathfrak{g}_\alpha\right) \times \{0\}\right) \oplus \left(\{0\} \times \left(\mathfrak{t}_- \bigoplus_{\alpha\in S_-} \mathfrak{g}_\alpha\right)\right) \oplus \operatorname{span}_F\{(h, \phi(h)) \mid h \in \mathfrak{r}_+\}$$
$$\oplus \left(\bigoplus_{\alpha\in S_+} \mathfrak{g}_\alpha[x^{-1}] \times \{0\}\right) \oplus \left(\bigoplus_{i=1}^{m} \bigoplus_{\alpha\in\Delta_i} \mathfrak{g}_\alpha^{a_i} \times \{0\}\right) \oplus W_\psi,$$

for some $W_\psi \subseteq \mathfrak{h}[x^{-1}] \times \mathfrak{h}$. To conclude the regularity of $\Delta = S_+ \sqcup_i \Delta_i$ we need to prove the closedness of $S_+ \sqcup \Delta_i$. For that choose two roots $\alpha \in S_+$ and $\beta \in \Delta_i$ such that $\alpha + \beta \in \Delta_j$. This, in particular, would imply the following inclusion

$$[\mathfrak{g}_\alpha \times \{0\}, \mathfrak{g}_\beta^{a_i} \times \{0\}] = \mathfrak{g}_{\alpha+\beta}^{a_i} \times \{0\} \subseteq \mathfrak{g}_{\alpha+\beta}^{a_j} \times \{0\} \subseteq W,$$

which is possible if and only if $a_i = a_j$ and hence $i = j$. Therefore, $\Delta = S_+ \sqcup_i \Delta_i$ is a regular partition of Δ.

It now remains to understand the Cartan part $W_\psi = \{x^{-n}(x^{-1}h - \psi(h), 0) \mid h \in \mathfrak{h}\}$. The arguments above imply that

$$x^{-1}\mathfrak{s}_+[x^{-1}] \times \{0\} \subseteq W_\mathfrak{h}$$

giving $\psi(\mathfrak{s}_+) = 0$. Now take an arbitrary root $\alpha \in \Delta$ and consider three cases:

1. For $\pm\alpha \in S_+$ we must have $H_\alpha \in \mathfrak{s}_+$ and hence $\psi(H_\alpha) = 0$;
2. When $\alpha \in S_+$ and $-\alpha \in \Delta_i$ the relation $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}^{a_i}] = (x^{-1} - a_i)FH_\alpha$ implies $\psi(H_\alpha) = a_i$;
3. Finally, when $\alpha \in \Delta_i$ and $-\alpha \in \Delta_j$ the equality

$$[\mathfrak{g}_\alpha^{a_i}, \mathfrak{g}_{-\alpha}^{a_j}] = (x^{-1} - a_i)(x^{-1} - a_j)FH_\alpha$$

leads to $(\psi - a_i)(\psi - a_j)H_\alpha = 0$.

Retracing the arguments above also shows that W defined by $\mathfrak{w}$, ψ and $\Delta = \bigsqcup_{i=1}^n \Delta_i$ as above provides a regular subalgebra of L_1 of type 0. The formula for the r-matrix is deduced similarly to Eq. (25). □

Example 4.6 The subalgebra $W = W_+ \times W_-$ with

$$W_+ = (\mathfrak{h} \oplus \mathfrak{l} \oplus x^{-1}\mathfrak{r})[x^{-1}] \oplus \bigoplus_{\alpha\in\Delta_+^{\geqslant k_i\alpha_i}} \mathfrak{g}_\alpha^{a_\alpha}$$

and $W_- = \mathfrak{r}$ defines a regular decomposition $L_1 = \Delta \oplus W$ of type $k_i \geqslant 0$. ◊

4.3 Weakly regular decompositions for L_1

For subalgebras inside $L_m, m \geqslant 1$, there exists another natural notion of h-invariance. Precisely, instead of assuming the invariance of W with respect to the action of (h, h), one can require the projections W_+ and W_- onto $\mathfrak{g}(\!(x)\!)$ and $\mathfrak{g}[x]/x^m\mathfrak{g}[x]$ respectively to be $\mathfrak{h}$-invariant. For simplicity, we call upper-bounded and $F[x^{-1}]$-invariant subalgebras $W \subseteq L_m$ with the requirement above *weakly regular*. Using a slight generalization of Belavin-Drinfeld construction [8] we can create examples of weakly regular subalgebras $W \subseteq L_1$. The construction can be extended to loop algebras to produce weakly regular subalgebras in $L_m, m \geqslant 1$; See Remark 4.8.

Let $A = (a_{i,j})_{i,j=1}^n$ be the Cartan matrix of $\mathfrak{g}$. We call $(\Gamma_1, \Gamma_2, \tau)$ *a generalized Belavin-Drinfeld triple* if $\Gamma_1, \Gamma_2 \subseteq \pi$ are subsets of simple roots and $\tau\colon \Gamma_1 \to \Gamma_2$ is a bijection such that:

1. $a_{\tau(i),\tau(j)} = a_{i,j}$ for all $\alpha_i, \alpha_j \in \Gamma_1$. In other words, τ preserves the entries of the Cartan matrix corresponding to the roots in Γ_1, i.e.
$$\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \frac{\langle \tau(\alpha_i), \tau(\alpha_j) \rangle}{\langle \tau(\alpha_i), \tau(\alpha_i) \rangle};$$
2. $\tau(\alpha), \dots, \tau^{k-1}(\alpha) \in \Gamma_2$ but $\tau^k(\alpha) \notin \Gamma_1$ holds for every $\alpha \in \Gamma_1$ and some $k \in \mathbb{N}$.

Then τ defines an isomorphism $\theta_\tau\colon \mathfrak{g}_{\Gamma_1} \to \mathfrak{g}_{\Gamma_2}$ of the Lie subalgebras $\mathfrak{g}_{\Gamma_1}, \mathfrak{g}_{\Gamma_2} \subseteq \mathfrak{g}$ generated by $\{E_\alpha, E_{-\alpha} \mid \alpha \in \Gamma_1\}$ and $\{E_\alpha, E_{-\alpha} \mid \alpha \in \Gamma_2\}$ respectively. Precisely, one defines the isomorphism by $E_{\pm\alpha} \mapsto E_{\pm\tau(\alpha)}$.

Let us write Δ_{Γ_i} for the root subsystem of Δ generated by Γ_i and put $\Delta_{\Gamma_i}^{\pm} := \Delta_{\Gamma_i} \cap \Delta_\pm$. We denote the restriction of θ_τ to $\bigoplus_{\alpha\in\Delta_{\Gamma_1}^{\pm}} \mathfrak{g}_\alpha$ with θ_τ^+ and the restriction of its inverse θ_τ^{-1} to $\bigoplus_{\alpha\in\Delta_{\Gamma_2}^{\pm}} \mathfrak{g}_\alpha$ with θ_τ^-. Both restrictions can be extended by 0 on the remaining root vectors to produce two endomorphisms $\theta_\tau^{\pm}\colon \mathfrak{n}_\pm \to \mathfrak{n}_\pm$, that we denote with the same letters. Due to condition 2. on τ, the endomorphisms $\theta_\tau^{\pm}$ are nilpotent. Consequently,
$$\rho_\tau^{\pm} := \theta_\tau^{\pm}/(1-\theta_\tau^{\pm}) := \sum_{k=1}^{\infty} \theta_\tau^{\pm,k}\colon \mathfrak{n}_\pm \to \mathfrak{n}_\pm$$
are well-defined endomorphisms as well.

We say that $\phi\colon \mathfrak{h} \to \mathfrak{h}$ is compatible with $(\Gamma_1, \Gamma_2, \tau)$ if
$$\{(h_\alpha, h_{\tau(\alpha)}) \mid \alpha \in \Gamma_1\} \subseteq \{(\phi(h) + h, \phi(h) - h) \mid h \in \mathfrak{h}\}. \tag{42}$$

Given a generalized Belavin-Drinfeld triple with a compatible linear map ϕ, we can construct a weakly regular subalgebra $W \subseteq L_1$ and the associated r-matrix in a way similar to Theorem 4.5. To simplify the notation, let us define

$$r_{(\Gamma_1,\Gamma_2,\tau,\phi)} := (\phi\otimes 1)\Omega_{\mathfrak{h}} + 2\left(\sum_{\alpha\in\Delta_+}(\rho_\tau^+ - 1)(E_\alpha)\otimes E_{-\alpha} - \sum_{\alpha\in\Delta_+}(\rho_\tau^- - 1)(E_{-\alpha})\otimes E_\alpha\right). \tag{43}$$

This is a solution to Eq. (40).

Proposition 4.7 *For every linear isomorphism $\phi\colon \mathfrak{h}\to\mathfrak{h}$ compatible with a generalized Belavin-Drinfeld triple (Γ_1,Γ_2,τ) we can define*

$$\mathfrak{w}_{(\Gamma_1,\Gamma_2,\tau,\phi)} := \left(\bigoplus_{\alpha\in\Delta_+\setminus\Delta_{\Gamma_1}}\mathfrak{g}_\alpha\times\{0\}\right)\oplus\left(\bigoplus_{\alpha\in\Delta_-\setminus\Delta_{\Gamma_2}}\{0\}\times\mathfrak{g}_\alpha\right)$$
$$\oplus\{(E_\alpha, E_{\tau(\alpha)})\mid \alpha\in\Delta_{\Gamma_1}\}\oplus\{(\phi(h)+h,\phi(h)-h)\mid h\in\mathfrak{h}\}.$$

Then

$$W := \mathfrak{w}_{(\Gamma_1,\Gamma_2,\tau,\phi)}\oplus\left(\left(W_\psi\bigoplus_{i=0}^{n}\bigoplus_{\alpha\in\Delta_i}\mathfrak{g}_\alpha^{a_i}\right)\times\{0\}\right),$$

where

- $\Delta = \sqcup_{i=0}^n\Delta_i$ *is a regular partition such that* $\Delta_{\Gamma_1}\subseteq\Delta_0$ *and* $\Delta_{\Gamma_1}+\Delta_i$ *are closed;*
- $a_0,\dots,a_n\in F$ *are distinct constants with* $a_0=0$*;*
- $\psi\colon\mathfrak{h}\to\mathfrak{h}$ *is a linear map satisfying* $\psi(\mathrm{Im}(\phi+1))=\{0\}$ *and*

$$\begin{cases}(\psi-a_i)H_\alpha = 0 & \alpha\in\Delta_{\Gamma_1}, -\alpha\in\Delta_i,\\ (\psi-a_i)(\psi-a_j)H_\alpha = 0 & \alpha\in\Delta_i, -\alpha\in\Delta_j,\end{cases}$$

is a weakly regular subalgebra of L_1.

Furthermore, the associated r-matrix is given by

$$r(x,y) = \frac{y\Omega}{x-y} + r_{(\Gamma_1,\Gamma_2,\tau,\phi)} + \frac{1}{2}\Omega + \left(\frac{\psi}{y\psi-1}\otimes 1\right)\Omega_{\mathfrak{h}} + \sum_{i=1}^{n}\frac{a_i\Omega_i}{a_iy-1},$$

where $\Omega_i = \sum_{\alpha\in\Delta_i}E_\alpha\otimes E_{-\alpha}$ *and* $r_{(\Gamma_1,\Gamma_2,\tau,\phi)}$ *is given by Eq.* (43).

Remark 4.8 It was shown in [3, 18] that quasi-trigonometric r-matrices, i.e. formal solutions of the classical Yang-Baxter equation of the form

$$r(x,y) = \frac{y\Omega}{x-y} + p(x,y)$$

for some $p\in(\mathfrak{g}\otimes\mathfrak{g})[x,y]$, are in bijection with Lagrangian Lie subalgebras of L_1 with respect to a certain form. Furthermore, in [3] these solutions were classified in terms of triples (Γ_1,Γ_2,τ), where

- Γ_i are subsets of simple roots Π of the loop algebra $\mathfrak{g}[x,x^{-1}]$;

- The minimal root α_0 is not in Γ_1 and
- $\tau\colon \Gamma_1 \to \Gamma_2$ is a bijection such that $\langle\alpha,\beta\rangle = \langle\tau(\alpha),\tau(\beta)\rangle$ and $\tau^k(\alpha) \notin \Gamma_1$ for all $\alpha,\beta \in \Gamma_1$ and some $k \in \mathbb{N}$.

If τ does not preserve the form, but only the extended affine Cartan matrix, i.e. $a_{\tau(i),\tau(j)} = a_{i,j}$ for $\alpha_i \in \Gamma_1$, we can use the same construction with minimal adjustments to produce a weakly regular subalgebra of L_1 of type 0 or 1, respectively.

More precisely, if $\alpha_0 \notin \Gamma_2$, then we get the generalized Belavin-Drinfeld described in the beginning of Sect. 4.3. When $\alpha_0 \in \Gamma_2$, we can use the same formulas for $r_{(\Gamma_1,\Gamma_2,\tau,\phi)}$ and $\mathfrak{m}_{(\Gamma_1,\Gamma_2,\tau,\phi)}$, where we understand $\Delta_\pm$ and Δ_{Γ_i} as subsystems of the affine root system Δ of $\mathfrak{g}[x,x^{-1}]$. The subalgebra $\mathfrak{m}_{(\Gamma_1,\Gamma_2,\tau,\phi)}$ obtained in this way will in general be a subalgebra of $\mathfrak{g}[x,x^{-1}]\times\mathfrak{g}[x,x^{-1}]$ complementary to the diagonal embedding of $\mathfrak{g}[x,x^{-1}]$ satisfying the inclusion

$$\mathfrak{m}_{(\Gamma_1,\Gamma_2,\tau,\phi)} \subseteq \mathfrak{g}[x,x^{-1}] \times \mathfrak{g}[x].$$

Projecting $\mathfrak{m}_{(\Gamma_1,\Gamma_2,\tau,\phi)}$ onto $\mathfrak{g}[x,x^{-1}] \times \mathfrak{g}$ we get a weakly regular subalgebra of L_1. $\Diamond$

Remark 4.9 More generally, trigonometric solutions of the classical Yang-Baxter equation were classified by Belavin-Drinfeld triples in [8] and later in [2, 3]. Moreover, it was shown in [3] that such solutions are in bijection with Lie algebra decompositions

$$\mathfrak{L} \times \mathfrak{L} = \mathfrak{D} \oplus W,$$

where $\mathfrak{L}$ is a twisted loop algebra, $\mathfrak{D}$ is the diagonal embedding of it into $\mathfrak{L}\times\mathfrak{L}$ and W is a subalgebra of the product complementary to $\mathfrak{D}$ and Lagrangian with respect to a certain bilinear form. If we allow the bijection $\tau\colon\Gamma_1\to\Gamma_2$ from a BD triple (Γ_1,Γ_2,τ) to preserve only the Cartan matrix, as in Remark 4.8, repeating the construction from [2, 3] we get a Lie algebra decomposition $\mathfrak{L}\times\mathfrak{L} = \mathfrak{D}\oplus W$, where W is a non-Lagrangian subalgebra. The latter decomposition corresponds to a non-skew-symmetric analog of a trigonometric r-matrix. We leave these statements unproven because we will not pursue this direction further in the paper. $\Diamond$

Example 4.10 Assume Δ is of exceptional type G_2. There are two simple roots $\{\alpha,\beta\}$ and, without loss of generality, we assume that α is the shortest root. Put $\Gamma_1 = \{\alpha\}$ and $\Gamma_2 = \{\beta\}$. The bijection τ given by $\tau(\alpha) = \beta$ trivially satisfies conditions 1. and 2. above. Consequently, (Γ_1,Γ_2,τ) is a generalized Belavin-Drinfeld triple. Let $\{X_{\pm\alpha}, X_{\pm\beta}, H_\alpha, H_\beta\}$ be the Chevalley basis for $\mathfrak{g}(\Delta)$, then

$$\mathfrak{h}_{\Gamma_1}^{\perp} = \mathrm{span}_F\{\underbrace{3H_\alpha + 2H_\beta}_{=:H_1}\}, \quad \mathfrak{h}_{\Gamma_2}^{\perp} = \mathrm{span}_F\{\underbrace{2H_\alpha + H_\beta}_{=:H_2}\}.$$

Consequently, by choosing the trivial ψ and a nonzero $\lambda \in F$ we get the following weakly regular subalgebra of L_1:

$$\begin{aligned}W &:= \{X_\beta, X_{\alpha+\beta}, X_{2\alpha+\beta}, X_{3\alpha+\beta}, X_{3\alpha+2\beta}\} \times \{0\}\\ &\oplus \{0\} \times \{X_{-\alpha}, X_{-\alpha-\beta}, X_{-2\alpha-\beta}, X_{-3\alpha-\beta}, X_{-3\alpha-2\beta}\}\\ &\oplus \{(X_\alpha, X_\beta), (H_\alpha, H_\beta), (H_1, \lambda H_2)\}\\ &\oplus x^{-1}\mathfrak{g}[x^{-1}] \times \{0\}.\end{aligned}$$

We can see that this subalgebra is not regular in our sense, because

$$(H_\alpha, H_\alpha) \cdot (X_\alpha, X_\beta) = (a_{11}X_\alpha, a_{12}X_\beta) = (2X_\alpha, -3X_\beta)$$

which is not in W. $\diamondsuit$

Example 4.11 The same construction can be applied to the orthogonal Lie algebra $\mathfrak{o}(5)$. Let α and β be two simple roots of B_2, such that α is shorter than β, and $\{X_{\pm\alpha}, X_{\pm\beta}, H_\alpha, H_\beta\}$ be again the Chevalley basis of $\mathfrak{o}(5)$. Then

$$\begin{aligned}W &:= \{X_\beta, X_{\alpha+\beta}, X_{2\alpha+\beta}\} \times \{0\}\\ &\oplus \{0\} \times \{X_{-\alpha}, X_{-\alpha-\beta}, X_{-2\alpha-\beta}\}\\ &\oplus \{(X_\alpha, X_\beta), (H_\alpha, H_\beta), (H_\alpha + H_\beta, \lambda(2H_\alpha + H_\beta))\}\\ &\oplus x^{-1}\mathfrak{g}[x^{-1}] \times \{0\}.\end{aligned}$$

is again a weakly regular subalgebra of L_1 for any nonzero $\lambda \in F$. $\diamondsuit$

4.4 Regular decomposition $L_m = \mathfrak{D} \oplus W$ with $m > 0$ of type $k > 0$

The classification of regular subalgebras $W \subseteq L_m$ of type $k > 0$ is overly convoluted to be formulated in detail. However, we can observe that regular subalgebras of L_m always admit the following standard form.

Theorem 4.12 *Let $L_m = \Delta \oplus W$ be a regular decomposition for $m > 0$. Then*

$$W = W_\mathfrak{h} \oplus (I_+ \times \{0\}) \oplus (\{0\} \times I_-), \tag{44}$$

where $W_\mathfrak{h} \subseteq \mathfrak{h}[x^{-1}] \times \mathfrak{h}[x]/x^m\mathfrak{h}[x]$ is an appropriate subspace and

$$\begin{aligned}I_+ \times \{0\} &:= W \cap (x\mathfrak{g}[x^{-1}] \times \{0\}),\\ \{0\} \times I_- &:= W \cap (\{0\} \times \mathfrak{g}[x]/x^m\mathfrak{g}[x]).\end{aligned}$$

Proof As it was explained in the beginning of Sect. 4, we can restrict our attention to cases $m = 1$ and 2. However, for any regular $W \subseteq L_1$ we can define a regular subalgebra

$$\widetilde{W} := W \oplus (\{0\} \times x\mathfrak{g}[x]) \subseteq L_2.$$

If $\widetilde{W}$ has the form Eq. (44), then so does W. For this reason, it is sufficient to prove the statement for $m = 2$.

Let $W \subseteq L_2$ be a regular subalgebra. Define a map $\varphi\colon W_+/I_+ \to W_-/I_-$ as in Eq. (38). It is again a Lie algebra isomorphism having no nonzero fixed points and intertwining the action of the Cartan subalgebra $\mathfrak{h}$. Consequently, we can write

$$\begin{aligned} W = W_{\mathfrak{h}} &\oplus (I_+ \times \{0\}) \oplus (\{0\} \times I_-) \\ &\oplus \underbrace{\operatorname{span}_F\{(p_\alpha E_\alpha, [q_\alpha]E_\alpha) \mid p_\alpha E_\alpha + I_+ \in W_+/I_+\}}_{D:=}. \end{aligned} \tag{45}$$

Here $W_{\mathfrak{h}} \subseteq \mathfrak{h}[x^{-1}] \times \mathfrak{h}[x]/x^2\mathfrak{h}[x]$, $p_\alpha \in F[x, x^{-1}]$, $q_\alpha \in F[x]$ and the elements in D are glued using φ, namely

$$\varphi(p_\alpha E_\alpha + I_+) = [q_\alpha]E_\alpha + I_-.$$

The $\mathfrak{h}$-invariance of W implies the $\mathfrak{h}$-invariance of $W_\pm$. Using Lemma 3.3 we can decompose

$$W_+ = W_{\mathfrak{h},+} \bigoplus_{\alpha \in \Delta} f_\alpha \mathfrak{g}_\alpha[x^{-1}],$$

for some nonzero $f_\alpha = a_\alpha x^{-1} + b_\alpha + c_\alpha x$ and $W_{\mathfrak{h},+} \subseteq \mathfrak{h}[x^{-1}]$. Similarly,

$$W_- = W_{\mathfrak{h},-} \bigoplus_{\alpha \in \Delta} [g_\alpha]E_\alpha,$$

for $g_\alpha \in \{0, 1, x\}$ and $W_{\mathfrak{h},-} \subseteq \mathfrak{h}[x]/x^2\mathfrak{h}[x]$. The invariance of W under multiplications by $(x^{-1}, 0)$ and $(0, [x])$ implies that we can choose the representatives p_α and $[q_\alpha]$ in

$$(p_\alpha E_\alpha, [q_\alpha]E_\alpha) \in D$$

in a way that (after a re-scaling of f_α and g_α by nonzero constants) we get $p_\alpha = f_\alpha$ and $[q_\alpha] = [g_\alpha]$.

By assumption, W_+ is contained in a maximal order $\mathfrak{P}_i$ corresponding to a simple root α_i. Decompose

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l} \oplus \mathfrak{c} \oplus \mathfrak{r}$$

as described in Eq. (21). Let us consider an arbitrary root vector $E_\alpha \in \mathfrak{g}$. Since W is complementary to $\mathfrak{D}$, we can find unique $A_\alpha \in \mathfrak{g}$, $B_\alpha \in \mathfrak{c}$ such that

$$w = (A_\alpha + B_\alpha x, A_\alpha + (B_\alpha + E_\alpha)[x]) \in W.$$

For $E_\alpha \notin \mathfrak{c}$ we have $f_\alpha = a_\alpha x^{-1} + b_\alpha$ and the only way to decompose the element w into a sum Eq. (45) is

$$w = (A_\alpha + B_\alpha x, A_\alpha + B_\alpha[x]) + (0, E_\alpha[x]).$$

As a consequence $A_\alpha = B_\alpha = 0$ and

$$\{0\} \times (\mathfrak{h} \oplus \mathfrak{l} \oplus \mathfrak{r})[x] \subseteq W \tag{46}$$

For an $E_\alpha \in \mathfrak{c}$, the element w above lies in W only if

$$B_\alpha = \lambda_1 E_\alpha \text{ and } A_\alpha = \lambda_2 E_\alpha,$$

for $\lambda_1, \lambda_2 \in F$. In case $\lambda_2 \neq 0$, we get $(0, [x]) \cdot w = (0, \lambda_2 E_\alpha[x]) \in W$. If $\lambda_2 \neq 0$ for all $E_\alpha \in \mathfrak{c}$, then $\{0\} \times x\mathfrak{g}[x]/x^2\mathfrak{g}[x] \subseteq W$ and we can again reduce everything to L_1 and Theorem 4.5. Otherwise, there is at least one $E_\alpha \in \mathfrak{c}$ for which $\lambda_2 = 0$, $\lambda_1 \neq 0$ and

$$(\lambda_1 E_\alpha x, (\lambda_1 + 1)E_\alpha x) \in W. \tag{47}$$

For the negative root $-\alpha \in \mathfrak{r}$ we have

$$(x^{-1}E_{-\alpha}, 0) = w' - (g, [g])$$

for some unique $g \in \mathfrak{g}[\![x]\!]$. Therefore,

$$w' = (x^{-1}E_{-\alpha} + g, g) \in W.$$

the inclusion $W_+ \subseteq \mathfrak{P}_i$ and the explicit form Eq. (22) of $\mathfrak{P}_\mathfrak{i}$ implies that $g = 0$ and hence

$$x^{-1}\mathfrak{r}[x^{-1}] \times \{0\} \subseteq W.$$

Commuting Eq. (47) with $(x^{-1}E_{-\alpha}, 0)$ we obtain the containments: $(\lambda_1 H_\alpha, 0)$, $(\lambda_1^2 x E_\alpha, 0) \in W$. From this we can conclude that for any $E_\alpha \in \mathfrak{c}$ we have either

$$(E_\alpha x, 0) \in W \text{ or } (0, E_\alpha[x]) \in W.$$

Take an arbitrary $E_\alpha \in \mathfrak{g}$ and assume

$$(f_\alpha E_\alpha, [g_\alpha]E_\alpha) \in D.$$

Then $f_\alpha = a_\alpha x + b_\alpha + c_\alpha x^{-1}$ with $a_\alpha = 0$ for $E_\alpha \notin \mathfrak{c}$. Moreover, the arguments above show that $[g_\alpha] \in F^\times$. Indeed, if $E_\alpha \notin \mathfrak{c}$, then $f_\alpha \in F[x^{-1}]$ by the form of the maximal order and $[g_\alpha] \in F^\times$ due to the inclusion Eq. (46). For $E_\alpha \in \mathfrak{c}$ we have either $(E_\alpha x^k, 0) \in W$, $k \leqslant 1$, leading to the contradiction $f_\alpha E_\alpha \in I_+$, or $(0, E_\alpha[x]) \in W$ and hence $[g_\alpha] \in F^\times$.

Our next step is to prove that the constant a_α vanishes even for $E_\alpha \in \mathfrak{c}$. Assume $E_\alpha \in \mathfrak{c}$ is such that

$$(f_\alpha E_\alpha, E_\alpha) = ((ax + b + cx^{-1})E_\alpha, E_\alpha) \in D$$

with $a \neq 0$, then by commuting this element with $(x^{-1}E_{-\alpha}, 0) \in W$ we get

$$(x^{-1} f_\alpha H_\alpha, 0), (x^{-1} f_\alpha^2 E_\alpha, 0) \in W.$$

Therefore

$$((x^{-1} f_\alpha^2 - a f_\alpha)E_\alpha, E_\alpha) \in W,$$

where the highest power of x in $(x^{-1}f_\alpha^2 - af_\alpha)$ is strictly smaller than in f_α. This means that either

$$(x^{-1}f_\alpha^2 - af_\alpha) = 0 \text{ or } (x^{-1}f_\alpha^2 - af_\alpha) = pf_\alpha$$

for a polynomial $p \in x^{-1}F[x^{-1}]$. In the first case we immediately have $(0, E_\alpha) \in W$. In the second case

$$(p, 0) \cdot (f_\alpha E_\alpha, E_\alpha) = ((x^{-1}f_\alpha^2 - af_\alpha)E_\alpha, 0) \in W$$

which again forces $(0, E_\alpha) \in W$. This contradiction implies that there are no pairs in D of the form $((ax + b + cx^{-1})E_\alpha, E_\alpha)$ with $a \neq 0$. In other words,

$$D \subseteq \mathfrak{g}[x^{-1}] \times \mathfrak{g}.$$

Let $w := ((ax^{-1} - b)E_\alpha, E_\alpha) \in D$ with $a \neq 0$. Then $(0, E_\alpha[x]) \in W$ and we can find unique $A \in \mathfrak{g}$ and $B \in \mathfrak{c}$ such that

$$((E_\alpha + A) + Bx, A + B[x]) \in W.$$

Again, this is possible only if $A = \lambda E_\alpha$ and $B = \mu E_\alpha$ for some $\lambda, \mu \in F$:

$$((1 + \lambda + \mu x)E_\alpha, (\lambda + \mu[x])E_\alpha) \in W.$$

Since $f_\alpha = ax^{-1} - b$, we must have $\mu = 0$ and hence

$$((1 + \lambda)E_\alpha, \lambda E_\alpha) \in W.$$

For $\lambda = 0$ or -1 we get the inclusions $(E_\alpha, 0)$ or $(0, E_\alpha) \in W$. Since both inclusions contradict our choice of $w \in D$, we have $\lambda \notin \{0, -1\}$ and

$$(\lambda' E_\alpha, E_\alpha) \in W$$

for some nonzero $\lambda' \in F$. Subtracting this element from w we get

$$((ax^{-1} - b - \lambda')E_\alpha, 0) \in W.$$

This means that we must be able to find a polynomial $p \in F[x^{-1}]$ such that $(ax^{-1} - b - \lambda') = p(ax^{-1} - b)$. This is possible only in the case $p = 1$ and $\lambda' = 0$, which is a contradiction.

In other words, we have shown that

$$D \subseteq \mathfrak{g} \times \mathfrak{g}.$$

Now arguing precisely as in the proof of Theorem 4.1 we see that $D = 0$. $\square$

Example 4.13 The proposition above provides us with a strategy of constructing more examples of regular subalgebras $W \subseteq L_1$ of type $k \geqslant 1$.

Start with a regular subalgebra $V_+ \subseteq \mathfrak{g}((x))$ of type $k \geqslant 1$, a 2-regular partition $\Delta = S_+ \sqcup S_-$ of the root system of $\mathfrak{g}$ and subspaces $\mathfrak{s}_\pm = \mathfrak{t}_\pm \oplus \mathfrak{r}_\pm$ of the Cartan subalgebra $\mathfrak{h}$ subject to the following conditions

1. $\mathfrak{h} = \mathfrak{s}_+ + \mathfrak{s}_-$;
2. $\mathfrak{t}_+ \cap \mathfrak{t}_- = \{0\}$;
3. $\mathfrak{v}_\pm := \mathfrak{t}_\pm \bigoplus_{\alpha \in S_\pm} \mathfrak{g}_\alpha$ is a regular subalgebra of $\mathfrak{g}$ and
4. $[\mathfrak{v}_+, V_+] \subseteq \mathfrak{v}_+ \oplus V_+$

Then the following subspace is a regular subalgebra of L_1 of the same type $k \geqslant 1$:

$$W := W_\phi \oplus ((\mathfrak{v}_+ \oplus V_+) \times \{0\}) \oplus (\{0\} \times \mathfrak{v}_-)\,,$$

where $W_\phi := \{(h, \phi(h)) \mid h \in \mathfrak{r}_+\}$ for any linear isomorphism $\phi \colon \mathfrak{r}_+ \to \mathfrak{r}_-$ without nonzero fixed points. $\diamondsuit$

Example 4.14 Write $\mathfrak{h} = FH_{\alpha_0} \oplus \mathfrak{h}'$, where α_0 is the maximal root of $\mathfrak{g}$. Then we can define

$$\begin{aligned} W_+ &:= (x^{-1}\mathfrak{g} \oplus \mathfrak{n}_+ \oplus FH_{\alpha_0} \oplus x\mathfrak{g}_{\alpha_0})[x^{-1}] \\ W_- &:= \mathfrak{h}' \oplus \mathfrak{n}_- \oplus x\left(\mathfrak{n}_- \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+ \setminus \{\alpha_0\}} \mathfrak{g}_\alpha\right). \end{aligned} \tag{48}$$

The product $W = W_+ \times W_-$ is a regular (bounded) subalgebra of L_2. $\diamondsuit$

Example 4.15 There are also examples of unbounded regular subalgebras of L_2. For example, for $\mathfrak{g} = \mathfrak{sl}(3, F)$ we can define the following spaces

$$\begin{aligned} W_+ := {} & \mathrm{span}_F\{xE_{\alpha+\beta}, x(x^{-1} - a)E_\alpha\}[x^{-1}] \oplus \mathrm{span}_F\{E_\beta, H_{\alpha+\beta}\}[x^{-1}] \\ & \oplus \mathrm{span}_F\{(x^{-1} - a)H_\alpha, (x^{-1} - a)E_{-\beta}\}[x^{-1}] \\ & \oplus \mathrm{span}_F\{x^{-1}E_{-\alpha}, x^{-1}E_{-\alpha-\beta}\}[x^{-1}], \\ W_- := {} & \mathrm{span}_F\{H_\alpha, E_{-\alpha}, E_{-\beta}, E_{-\alpha-\beta}, xE_\alpha, xE_\beta, xH_\beta\}[x]/x^2\mathfrak{g}[x] \end{aligned}$$

and set $W := W_+ \times W_-$. Here, $\pi = \{\alpha, \beta\} \subseteq \Delta$ are the simple roots and $\alpha + \beta \in \Delta$ is the only non-simple positive root. $\diamondsuit$

5 Connection to Gaudin models

In this section, $F = \mathbb{C}$. At this point, we have constructed Lie algebra decompositions $L_m = \mathfrak{D} \oplus W$ with additional compatibility with a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and stability under multiplication by x^{-1}. Moreover, we have given explicit formulas for the associated generalized r-matrices. It is known that for $m = 0$ these generalized r-matrices give rise to Gaudin integrable models; see [23]. Moreover, these models are particularly well-behaved if the generalized r-matrix satisfies the additional compatibility with the Cartan subalgebra; see [22] for $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$.

Let r be a generalized r-matrix of the form

$$r(x,y)=\frac{y^m\Omega}{x-y}+g(x,y) \tag{49}$$

which converges to some meromorphic function in some open disk around the origin. Then for any points $u_1,\dots,u_n$ in the domain of definition of r, we consider

$$H_i:=\sum_{k\neq i} r(u_k,u_i)^{(ki)}+\frac{1}{2}(g(u_i,u_i)^{(ii)}+\tau(g(u_i,u_i))^{(ii)})\in U(\mathfrak{g})^{\otimes n}, \tag{50}$$

where $(a\otimes b)^{(ij)}:=\iota_i(a)\iota_j(b)$ for the canonical embedding $\iota_i\colon U(\mathfrak{g})\to U(\mathfrak{g})^{\otimes n}$, which inserts 1 into every tensor factor except the i-th one where it inserts a.

Elements H_i commute inside $U(\mathfrak{g})$ and define a quantum integrable system, which is a generalization of the usual Gaudin models. The commutativity of H_i's is proven in Sect. 5.1. In the general scheme of integrability, H_i correspond to quadratic invariant functions on $\mathfrak{g}^{\oplus n}$. To obtain integrability of the model, one would need to consider the higher degree invariant functions as well.

The classical limit of such Gaudin models, called classical Gaudin models, simply replaces $U(\mathfrak{g})$ with the symmetric algebra $S(\mathfrak{g})$, which can be understood as the space of regular functions on $\mathfrak{g}^*$. In this way we obtain a classical integrable system.

Remark 5.1 Note that in Eq. (50) one of u_k's can be equal 0. However, such a point is a special point in the sense of [25], because the r-matrix becomes degenerate in that point. In this case, the corresponding model is not really a Gaudin model, but a "reduced" Gaudin model. $\diamondsuit$

Example 5.2 Taking $n=1$ and $u=u_1$, we obtain a quantum integrable system defined by one Hamiltonian $H=\frac{1}{2}m(g(u,u)+\tau(g(u,u)))$, where $m(a,b)=ab$ is the multiplication map of $U(\mathfrak{g})$ (resp. $S(\mathfrak{g})$ in the classical limit).

Let us assume $a_\alpha,b_\alpha,c_\alpha\in\mathbb{C}$ are chosen in such a way that Eq. (23) defines a subalgebra $W\subseteq\mathfrak{g}(\!(x)\!)$. The corresponding r-matrix Eq. (25) gives rise to the Hamiltonian

$$H=\frac{1}{2}\left(\psi_u(h_i)h_i+h_i\psi_u(h_i)+\sum_{\alpha\in\Delta_+^{<k_i\alpha_i}\cup\Delta_-^{<\alpha_i}}\frac{a_\alpha(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)}{a_\alpha u-1}\right.$$
$$\left.-\sum_{\alpha\in\Delta_+^{\geqslant k_i\alpha_i}}\frac{c_\alpha+d_\alpha-2c_\alpha d_\alpha u}{(u-c_\alpha)(u-d_\alpha)}(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha)\right),$$

where $\psi_u:=\frac{\phi}{u\phi-1}$.

Let us explicitly calculate this Hamiltonian for the regular decomposition of

$$\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})=\mathrm{Span}_{\mathbb{C}}\{E,H,F\}=(\mathbb{C}E\oplus\mathbb{C}H)\oplus\mathbb{C}F.$$

For two constants $a, b \in \mathbb{C}$ we get

$$
\begin{aligned}
H &= \frac{a\left(\frac{1}{2}H^2 + EF + FE\right)}{2(au-1)} + \frac{b(EF+FE)}{2(bu-1)} \\
&= \frac{a\left(\frac{1}{2}H^2 + (E+F)^2 - (E+F)(E-F) - H\right)}{2(au-1)} \\
&\quad + \frac{b((E+F)^2 - (E+F)(E-F) - H)}{2(bu-1)}.
\end{aligned}
$$

$\Diamond$

5.1 Commutativity of the generalized Gaudin Hamiltonians

For completeness, we present a proof that the Hamiltonians Eq. (50) commute. This is a coordinate-free rework of the proof from [23], where m is equal to 0. Such an approach shows the commutativity of Hamiltonians for any $m \geqslant 0$ and in all the points, including the special one; see Remark 5.1.

Fix $1 \leqslant i < j \leqslant n$ and consider

$$
\begin{aligned}
[H_i, H_j] = &\underbrace{\sum_{\substack{k=1\\k\neq i}}^{n}\sum_{\substack{\ell=1\\\ell\neq j}}^{n}[r(u_k,u_i)^{(ki)}, r(u_\ell,u_j)^{(\ell j)}]}_{S_1:=} \\
&+ \underbrace{\frac{1}{2}[r(u_j,u_i)^{(ji)}, g(u_j,u_j)^{(jj)} + \tau(g(u_j,u_j))^{(jj)}]}_{S_2:=} \\
&+ \underbrace{\frac{1}{2}[g(u_i,u_i)^{(ii)} + \tau(g(u_i,u_i))^{(ii)}, r(u_\ell,u_j)^{(ij)}]}_{S_3:=} \\
&+ \underbrace{\frac{1}{4}[g(u_i,u_i)^{(ii)} + \tau(g(u_i,u_i))^{(ii)}, g(u_j,u_j)^{(jj)} + \tau(g(u_j,u_j))^{(jj)}]}_{S_4:=}.
\end{aligned}
$$

The summand S_4 is equal to 0 because $i \neq j$. Terms in S_1 with $k \neq j, k \neq \ell$ and $\ell \neq i$ are equal 0. The remaining terms are

$$
\begin{aligned}
S_1 = &\sum_{\substack{k=1\\k\neq i,j}}^{n}[r(u_k,u_i)^{(ki)}, r(u_k,u_j)^{(kj)}] + \sum_{\substack{\ell=1\\\ell\neq i,j}}^{n}[r(u_j,u_i)^{(ji)}, r(u_\ell,u_j)^{(\ell j)}] \\
&+ \sum_{\substack{k=1\\k\neq i}}^{n}[r(u_k,u_i)^{(ki)}, r(u_i,u_j)^{(ij)}]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{k=1\\k\neq i\\k\neq j}}^{n} \mathrm{GCYB}^{ijk}(r(u_k,u_i)) + [r(u_j,u_i)^{(ji)}, r(u_i,u_j)^{(ij)}]\\
&= [r(u_j,u_i)^{(ji)}, r(u_i,u_j)^{(ij)}]\\
&= \frac{1}{2}\Big(m_{(0j)}([r(u_j,u_i)^{(0i)}, r(u_i,u_j)^{(ij)}] + [r(u_j,u_i)^{(ji)}, r(u_i,u_j)^{(i0)}])\\
&\quad + m_{(0i)}([r(u_j,u_i)^{(ji)}, r(u_i,u_j)^{(0j)}] + [r(u_j,u_i)^{(j0)}, r(u_i,u_j)^{(ij)}])\Big) \qquad (51)
\end{aligned}
$$

where we added an auxiliary copy of $U(\mathfrak{g})$ at tensor factor in position 0, defined

$$
\begin{aligned}
m_{(0k)}\colon U(\mathfrak{g})^{\otimes(n+1)} &\longrightarrow U(\mathfrak{g})^{\otimes n}\\
a_0\otimes a_1\otimes\cdots\otimes a_n &\longmapsto a_1\otimes\cdots\otimes a_{k-1}\otimes a_0a_k\otimes a_k\otimes\cdots\otimes a_n
\end{aligned}
$$

and used the identity

$$
\begin{aligned}
&\sum_{k,\ell=1}^{n} [(a_k\otimes b_k)^{(ji)}, (a_\ell\otimes b_\ell)^{(ij)}]\\
&\quad= \sum_{k,\ell=1}^{n} ([b_k,a_\ell]\otimes a_kb_\ell + a_\ell b_k\otimes[a_k,b_\ell])^{(ij)}\\
&\quad= \frac{1}{2}\sum_{k,\ell=1}^{n} ([b_k,a_\ell]\otimes(a_kb_\ell+b_\ell a_k+[a_k,b_\ell])+(a_\ell b_k+b_ka_\ell+[a_\ell,b_k])\otimes[a_k,b_\ell])^{(ij)}\\
&\quad= \frac{1}{2}\sum_{k,\ell=1}^{n} ([b_k,a_\ell]\otimes(a_kb_\ell+b_\ell a_k)+(a_\ell b_k+b_ka_\ell)\otimes[a_k,b_\ell])^{(ij)}\\
&\quad= \frac{1}{2}\sum_{k,\ell=1}^{n} \Big(m_{(0j)}\left((a_k\otimes[b_k,a_\ell]\otimes b_\ell + b_\ell\otimes[b_k,a_\ell]\otimes a_k)^{(0ij)}\right)\\
&\qquad + m_{(0i)}\left((a_\ell\otimes b_k\otimes[a_k,b_\ell] + b_k\otimes a_\ell\otimes[a_k,b_\ell])^{(0ij)}\right)\Big).
\end{aligned}
$$

Now let us again take a look at the GCYBE:

$$
\begin{aligned}
&[r(u_0,u_i)^{(0i)}, r(u_0,u_j)^{(0j)}] + [r(u_j,u_i)^{(ji)}, r(u_0,u_j)^{(0j)}] +\\
&\quad [r(u_0,u_i)^{(0i)}, r(u_i,u_j)^{(ij)}] = 0.
\end{aligned}
$$

Using the explicit form of the r-matrix and using the invariance of the quadratic Casimir element Ω we can rewrite the equation above as follows

$$
\begin{aligned}
&\left[r(u_0,u_i)^{(0i)} - r(u_j,u_i)^{(0i)}, \frac{u_j^m\Omega^{(0j)}}{u_0-u_j}\right] + [r(u_0,u_i)^{(0i)} + r(u_j,u_i)^{(ji)}, g(u_0,u_j)^{(0j)}]\\
&\quad + [r(u_0,u_i)^{(0i)}, r(u_i,u_j)^{(ij)}] = 0.
\end{aligned}
$$

Taking the limit $u_0 \to u_j$ we get

$$u_j^m[\partial_{u_j} r(u_j, u_i)^{(0i)}, \Omega^{(0j)}] + [r(u_j, u_i)^{(0i)} + r(u_j, u_i)^{(ji)}, g(u_j, u_j)^{(0j)}] \\ + [r(u_j, u_i)^{(0i)}, r(u_i, u_j)^{(ij)}] = 0.$$

Swapping factors j and 0 in the equality above we get

$$u_j^m[\partial_{u_j} r(u_j, u_i)^{(ji)}, \Omega^{(0j)}] + [r(u_j, u_i)^{(0i)} + r(u_j, u_i)^{(ji)}, \tau(g(u_j, u_j))^{(0j)}] \\ + [r(u_j, u_i)^{(ji)}, r(u_i, u_j)^{(i0)}] = 0.$$

Summing the last two equations and applying the multiplication $m_{(0j)}$ gives

$$\begin{aligned} S_2 &= \frac{1}{2}[r(u_j, u_i)^{(ji)}, g(u_j, u_j)^{(jj)} + \tau(g(u_j, u_j))^{(jj)}] \\ &= -\frac{1}{2} m_{(0j)}([r(u_j, u_i)^{(0i)}, r(u_i, u_j)^{(ij)}] + [r(u_j, u_i)^{(ji)}, r(u_i, u_j)^{(i0)}]) \end{aligned} \tag{52}$$

Similarly, rewriting GCYBE in the form

$$\left[\frac{u_i^m \Omega^{(0i)}}{u_0 - u_i}, r(u_0, u_j)^{(0j)} - r(u_i, u_j)^{(0j)}\right] + [g(u_0, u_i)^{(0i)}, r(u_0, u_j)^{(0j)} + r(u_i, u_j)^{(ij)}] \\ + [r(u_j, u_i)^{(ji)}, r(u_0, u_j)^{(0j)}] = 0,$$

taking the limit $u_0 \to u_i$

$$u_i^m[\Omega^{(0i)}, \partial_{u_i} r(u_i, u_j)^{(0j)}] + [g(u_i, u_i)^{(0i)}, r(u_i, u_j)^{(0j)} + r(u_i, u_j)^{(ij)}] \\ + [r(u_j, u_i)^{(ji)}, r(u_i, u_j)^{(0j)}] = 0$$

and swapping i and 0 factors results in

$$u_i^m[\Omega^{(0i)}, \partial_{u_i} r(u_i, u_j)^{(ij)}] + [\tau(g(u_i, u_j))^{(0i)}, r(u_i, u_j)^{(0j)} + r(u_i, u_j)^{(ij)}] \\ [r(u_j, u_i)^{(j0)}, r(u_i, u_j)^{(ij)}] = 0.$$

Summing these two terms and applying m_{0i} we get

$$\begin{aligned} S_3 &= \frac{1}{2}[g(u_i, u_i)^{(ii)} + \tau(g(u_i, u_i))^{(ii)}, r(u_i, u_j)^{(ij)}] \\ &= -\frac{1}{2} m_{0i}([r(u_j, u_i)^{(ji)}, r(u_i, u_j)^{(0j)}] + [r(u_j, u_i)^{(j0)}, r(u_i, u_j)^{(ij)}]). \end{aligned} \tag{53}$$

Combining Eqs. (51)–(53) we conclude

$$[H_i, H_j] = S_1 + S_2 + S_3 = 0.$$

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Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

A List of notations

Symbol	Meaning
F	Algebraically closed field of characteristic 0
$\mathfrak{g}$	Finite-dimensional simple Lie algebra over F
κ	The Killing form on $\mathfrak{g}$
Ω	The quadratic Casimir element in $\mathfrak{g} \otimes \mathfrak{g}$
$\mathfrak{h}$	Fixed Cartan subalgebra of $\mathfrak{g}$
$\Delta = \Delta_+ \sqcup \Delta_-$	Polarized root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$
$H_\alpha, E_{\pm\alpha}$	$E_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ for $\alpha \in \Delta_+$ are chosen such that $\kappa(E_\alpha, E_{-\alpha}) = 1$ and $H_\alpha = [E_\alpha, E_{-\alpha}]$
$\pi = \{\alpha_1, \dots, \alpha_n\}$	Simple roots of $\Delta = \Delta_+ \sqcup \Delta_-$
$\alpha_0 = \sum_{i=1}^n k_i \alpha_i$	Maximal root in Δ and its expansion into simple roots
V^*	Dual of a vector space V
$V[x]$	Polynomials in one variable with coefficients in a vector space V
$V[\![x]\!]$	Formal Taylor power series in one variable with coefficients in a vector space V
$V(\!(x)\!)$	Formal Laurent power series in one variable with coefficients in a vector space V
$V^a, V^{a,b}$	For $a, b \in F$, $V^a = (x^{-1} - a)V[x^{-1}]$ and $V^{a,b} = x(x^{-1} - a)(x^{-1} - b)V[x^{-1}]$
L_m	The Lie algebra $\mathfrak{g}(\!(x)\!) \times \mathfrak{g}[x]/x^m\mathfrak{g}[x]$
$\mathfrak{d}$	The subalgebra $\{(a, a) \mid a \in \mathfrak{g}\}$ of $\mathfrak{g} \times \mathfrak{g}$
$\mathfrak{D}$	The subalgebra $\{(f, [f]) \mid f \in \mathfrak{g}[\![x]\!]\}$ of L_m
$\Delta_\pm^{<m\alpha_i}$	$\{\alpha \in \Delta_\pm \mid \alpha = \pm\sum_{i=1}^n c_i\alpha_i,\ 0 \leqslant c_i < m\} \subseteq \Delta_\pm$
$\Delta_\pm^{\geqslant m\alpha_i}$	$\{\alpha \in \Delta_\pm \mid \alpha = \pm\sum_{i=1}^n c_i\alpha_i,\ c_i \geqslant m\} \subseteq \Delta_\pm$
$\mathfrak{P}_i$	Maximal parabolic subalgebra of $\mathfrak{g}(\!(x)\!)$ corresponding to $\alpha_i \in \{\alpha_0, \dots, \alpha_n\}$
W_ϕ	For a linear map $\phi\colon \mathfrak{h} \to \mathfrak{h}$ it is $\{x^{-n}(x^{-1}h - \phi(h)) \mid h \in \mathfrak{h}, n \in \mathbb{Z}_{\geqslant 0}\}$

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