Free Energy and Quark Potential in Ising Lattice Gauge Theory via Cluster Expansions

OXFORD

International Mathematics Research Notices, 2025, **2025**(12), 1–27
https://doi.org/10.1093/imrn/rnaf163
Article

Free Energy and Quark Potential in Ising Lattice Gauge Theory via Cluster Expansions

Malin P. Forsström[1] and Fredrik Viklund[2]

[1]Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE-412 96 Gothenburg, Sweden
[2]KTH Royal Institute of Technology, Department of Mathematics SE-100 44 Stockholm, Sweden
*Correspondence to be sent to: e-mail: palo@chalmers.se
Communicated by Prof. Rick Kenyon

We revisit the cluster expansion for Ising lattice gauge theory on $\mathbb{Z}^m$, $m \geqslant 3$, with Wilson's action, at a fixed inverse temperature β in the low-temperature regime. We prove existence and analyticity of the infinite volume limit of the free energy and compute the first few terms in its expansion in powers of $e^{-\beta}$. We further analyze Wilson loop expectations and derive an estimate that shows how the lattice scale geometry of a loop is reflected in the large β asymptotic expansion. Specializing to axis parallel rectangular loops $\gamma_{T,R}$ with side-lengths T and R, we consider the limiting function $V_\beta(R) := \lim_{T\to\infty} -\frac{1}{T}\log\langle W_{\gamma_{T,R}}\rangle_\beta$, known as the static quark potential in the physics literature. We verify the existence of the limit (with an estimate on the convergence rate) and compute the first few terms in the expansion in powers of $e^{-\beta}$. As a consequence, a strong version of the perimeter law follows. We also treat $-\log\langle W_{\gamma_{T,R}}\rangle_\beta/(T+R)$ as T, R tend to infinity simultaneously and give analogous estimates.

1 Introduction

Given a hypercubic lattice $\mathbb{Z}^m$ and a choice of structure group G, a (pure) lattice gauge theory models a random discretized connection form on a principal G-bundle on an underlying discretized m-dimensional smooth manifold. More concretely, after restricting to a finite box, it is a Gibbs probability measure on gauge fields, that is, G-valued discrete 1-forms σ defined on edges of the lattice. The probability measure is defined relative to the product Haar measure on G. The action can be taken to be of the form $S(\sigma) = -\sum_p A_p(\sigma)$, where for some choice of representation ρ, $A_p(\sigma) = \operatorname{Re}\operatorname{tr}(\rho(\sigma_{e_1}\sigma_{e_2}\sigma_{e_3}\sigma_{e_4}))$ captures the microscopic holonomy around the plaquette p whose boundary consists of the edges $e_1, \ldots, e_4$. The coupling parameter β acts as the inverse temperature. In a formal scaling limit, one recovers the Yang–Mills action while the model enjoys exact gauge symmetry on the discrete level. In contrast to the corresponding continuum Yang–Mills theories, the discrete measure is defined rigorously, and its analysis becomes a problem in statistical mechanics. Lattice gauge theories were introduced independently by Wegner and Wilson in the 1970s [20, 21].

Received: January 18, 2024. **Revised:** October 21, 2024. **Accepted:** May 21, 2025

Despite the presence of local symmetries, lattice gauge theories can exhibit non-trivial phase structure, but one has to consider non-local observables. Given a nearest-neighbor lattice loop γ, the Wilson loop variable W_γ records the random holonomy of the gauge field as γ is traversed. Starting with the original paper of Wilson [21], it has been argued in the physics literature that the decay rate of its expectation $\langle W_\gamma\rangle_\beta$ (in an infinite volume limit) as the loop grows encodes information about whether or not "static quarks" are "confined" in the model; see, for example, [17, Sect. 3.5] for a textbook discussion. Let $\gamma_{T,R}$ be a rectangular loop with axis parallel sides and, taking its existence for granted, consider the limit

$$V_\beta(R) = -\lim_{T\to\infty} \frac{1}{T} \log \big|\langle W_{\gamma_{T,R}}\rangle_\beta\big|.$$

The function $V_\beta(R)$ is called the static quark potential and is interpreted as the energy required to separate a static quark-antiquark pair to distance R; see, for example, [17, Sect. 3.2]. Wilson's criterion for quark confinement can then be formulated as follows: confinement occurs at β if and only if the energy $V_\beta(R)$ diverges as $R \to \infty$. However, except in the special case of planar theories, it seems that detailed mathematical proofs of such statements are not available in the literature, even for finite abelian G. Instead, the two phases are rigorously separated via estimates: confinement occurs at β if there exists some function $V(R)$, unbounded as $R \to \infty$, such that $\liminf_{T\to\infty} -\frac{1}{T}\log|\langle W_{\gamma_{R,T}}\rangle_\beta| \geqslant V(R)$, and in this case, Wilson loop expectations are said to follow the area law. If, on the other hand, there is a constant $c > 0$ independent of R such that $\limsup_{T\to\infty} -\frac{1}{T}\log|\langle W_{\gamma_{R,T}}\rangle_\beta| < c$, the Wilson loop expectations are said to follow the perimeter law. (The terminology comes from the expectation that a priori bounds of the form $e^{-cRT} \lesssim |\langle W_{\gamma_{R,T}}\rangle_\beta| \lesssim e^{-C(R+T)}$ should be essentially saturated in the two phases.) See [6] for a precise formulation of a condition for confinement and a general discussion from a probabilistic perspective, and Section 1.3 below for a brief discussion of other related work.

Here we will consider lattice gauge theory with structure group $G = \mathbb{Z}_2$ on $\mathbb{Z}^m$, $m \geqslant 3$, also known as Ising lattice gauge theory, for β in the subcritical regime. See Section 1.1 for the precise definition. This model was first studied by Wegner [20] and can be viewed as a version of the standard Ising model on $\mathbb{Z}^m$, where the global spin-flip symmetry has been "upgraded" to a local symmetry. We employ a cluster expansion to study the free energy, static quark potential, and related quantities. This classical method has been used in the past to analyze lattice gauge theories; see Section 1.3. While we only work with $G = \mathbb{Z}_2$, we believe our results can be generalized to any choice of finite abelian structure group with minor modifications.

To state our main results, we need to provide some definitions.

1.1 Ising lattice gauge theory and Wilson loop expectations

Let $m \geqslant 3$. The lattice $\mathbb{Z}^m$ has a vertex at each point $x \in \mathbb{Z}^m$ with integer coordinates and a non-oriented edge between each pair of nearest neighbors. To each non-oriented edge $\bar e$ in $\mathbb{Z}^m$ we associate two oriented edges e_1 and $e_2 = -e_1$ with the same endpoints as $\bar e$ and opposite orientations.

Let $\mathbf{e}_1 := (1, 0, 0, \ldots, 0)$, $\mathbf{e}_2 := (0, 1, 0, \ldots, 0)$, $\ldots$, $\mathbf{e}_m := (0, \ldots, 0, 1)$ be oriented edges corresponding to the unit vectors in $\mathbb{Z}^m$.

If $v \in \mathbb{Z}^m$ and $j_1 < j_2$, then $p = (v + \mathbf{e}_{j_1}) \wedge (v + \mathbf{e}_{j_2})$ is a positively oriented 2-cell, also known as a positively oriented plaquette. We let B_N denote the set $[-N, N]^m$ of $\mathbb{Z}^m$, and we let V_N, E_N, and P_N denote the sets of oriented vertices, edges, and plaquettes, respectively, whose end-points are all in B_N.

We let $\Omega^1(B_N, \mathbb{Z}_2)$ denote the set of all $\mathbb{Z}_2$-valued 1-forms σ on E_N, that is, the set of all $\mathbb{Z}_2$-valued functions $\sigma : e \mapsto \sigma_e$ on E_N such that $\sigma_e = -\sigma_{-e}$ for all $e \in E_N$. We write $\rho : \mathbb{Z}_2 \to \mathbb{C}$, $g \mapsto e^{\pi i g}$ for the natural representation of $\mathbb{Z}_2$.

When $\sigma \in \Omega^1(B_N, \mathbb{Z}_2)$ and $p \in P_N$, we let ∂p denote the four edges in the oriented boundary of p and define

$$(d\sigma)_p := \sum_{e\in\partial p} \sigma_e.$$

Elements $\sigma \in \Omega^1(B_N, \mathbb{Z}_2)$ are referred to as *gauge field configurations*.

The Wilson action functional for pure gauge theory is defined by (see, e.g., [21])

$$S(\sigma) := -\sum_{p\in P_N} \rho\big((d\sigma)_p\big), \quad \sigma \in \Omega^1(B_N, \mathbb{Z}_2).$$

The Ising lattice gauge theory probability measure on gauge field configurations is defined by

$$\mu_{\beta,N}(\sigma) := Z_{\beta,N}^{-1} e^{-\beta S(\sigma)}, \quad \sigma \in \Omega^1(B_N, \mathbb{Z}_2).$$

Here for $N \in \mathbb{N}$,

$$Z_{\beta,N} = \sum_{\sigma \in \Omega^1(B_N,\mathbb{Z}_2)} e^{-\beta S(\sigma)}$$

is the partition function and while we only consider the probability measure for positive β, the partition function is defined for $\beta \in \mathbb{C}$ when $N < \infty$. For $\beta \geqslant 0$, the corresponding expectation is written $\mathbb{E}_{\beta,N}$. Let γ be a nearest neighbor loop on $\mathbb{Z}^m$ contained in B_N. Given $\sigma \in \Omega^1(B_N, \mathbb{Z}_2)$, the Wilson loop variable for Ising lattice gauge theory is defined by

$$W_\gamma = \rho\big(\sigma(\gamma)\big) = \prod_{e \in \gamma} \rho\big(\sigma(e)\big) = e^{\pi i \sum_{e \in \gamma} \sigma(e)}.$$

For $\beta \geqslant 0$, let $\langle W_\gamma \rangle_\beta$ denote the infinite volume limit of its expected value:

$$\langle W_\gamma \rangle_\beta := \lim_{N \to \infty} \mathbb{E}_{\beta,N}[W_\gamma].$$

See, for example, [9] for a proof of the existence of this limit.

1.2 Main results

Our first result concerns the free energy for free boundary conditions. For $m \geqslant 3$, we let $\beta_0 := \beta_0(m) > 0$ be defined by (5). Let $|P_N^+|$ be the number of positively oriented plaquettes in the restriction of $\mathbb{Z}^m$ to the set $[-N, N]^m$. Note that $|P_N^+| \sim \binom{m}{2}(2N)^m$ as $N \to \infty$.

Theorem 1.1 (Free energy). Suppose $m \geqslant 3$ and $\operatorname{Re} \beta > \beta_0(m)$. Then

$$F(\beta) := \lim_{N \to \infty} \frac{1}{|P_N^+|} \log Z_{\beta,N}$$

defines an analytic function, and

$$F(\beta) = \frac{2}{m-1} e^{-8(m-1)\beta} + \frac{12(m-1)-8}{2(m-1)-1} e^{-4(4(m-1)-2)\beta} + O(e^{-16(m-1)\operatorname{Re}\beta}). \tag{1}$$

We next consider Wilson loop expectations. Given a loop γ let $\ell := |\operatorname{supp}\gamma|$ be its length, that is, the number of edges of γ. Further, let $\ell_c := \ell_c(\gamma)$ denote the number of pairs of non-parallel edges that are both in the boundary of some common plaquette (corners), and let $\ell_b := \ell_b(\gamma)$ denote the number of pairs (e, e') of parallel edges that are both in the boundary of some common plaquette (bottlenecks). Set

$$\upsilon_\beta := 2e^{-8(m-1)\beta} + 12(m-1)e^{-4(4(m-1)-2)\beta}.$$

Theorem 1.2. Suppose $m \geqslant 3$ and $\beta > \beta_0(m)$. There exists $C < \infty$ depending only on m such that for any loop γ with length ℓ, ℓ_c corner edges, and ℓ_b bottleneck edges,

$$\left| -\frac{1}{\ell} \log \langle W_\gamma \rangle_\beta - \left(\upsilon_\beta - 4\frac{\ell_c + \ell_b}{\ell} e^{-4(4(m-1)-2)\beta} \right) \right| \leqslant Ce^{-16(m-1)\beta}. \tag{2}$$

Notice how the lattice scale geometry of the loop enters into the estimate (2). Given a continuum loop, we see that the expansion is sensitive to the way the loop is embedded and discretized. For instance, the term $(\ell_c + \ell_b)/\ell$ is very different for an axis-parallel square compared to the natural discretization of the same square rotated by 45°.

The main idea of the proofs of Theorem 1.1 and 1.2 is to use a cluster expansion to rewrite $\log Z_{\beta,N}$ and $\log \langle W_\gamma \rangle_\beta$ as a sum over vortex clusters and the interaction of vortex clusters with the loop, respectively. We then show that the main contribution to these sums comes from very small clusters, and in fact, the logarithm of the contribution of a cluster of vortices of a certain size is proportional to its size. Hence,

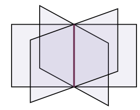
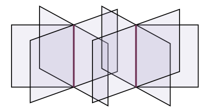
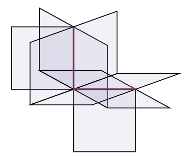
	$(\mathrm{supp}\,\sigma)^+$	$(\mathrm{supp}\,d\sigma)^+$	$\lvert(\mathrm{supp}\,\omega)^+\rvert$
(a)			$2(m-1)$
(b)			$4(m-1)-2$
(c)			$4(m-1)-2$

Fig. 1. The above table shows projections of the supports of the non-trivial and irreducible plaquette configurations in $\mathbb{Z}^4$ which has the smallest support (up to translations and rotations).

the main contribution comes from very small clusters, and the smallest such cluster consists of exactly one vortex with small support. These smallest clusters, which give the main contribution, are the so-called minimal vortices, and the next-order contribution comes from clusters of the second smallest possible size (see Figure 1). The terms appearing in (1) and (2) hence correspond to the smallest vortices that can appear in the model. In particular, the term $2e^{-8(m-1)\beta}$ in v_β correspond to minimal vortices (having support of size $2 \cdot 2(m-1)$), and the term $12(m-1)e^{-4(4(m-1)-2)\beta}$ corresponds to the smallest vortices that are not minimal vortices (having support of size $2 \cdot (4(m-1)-2)$); see Figure 1. The main effort of the proof is to confirm this picture and also to give upper bounds on the contribution of larger clusters of vortices.

Remark 1.3. Using the methods of the proof of Theorem 1.2, it is, in principle, straightforward to obtain estimates with higher precision in terms of the expansion in powers of $e^{-\beta}$. If higher-order terms are included in (2), the constants of the corresponding polynomial in $e^{-\beta}$ will further depend on the lattice scale geometry of the loop.

Our next result concerns the static quark potential $V_\beta(R)$.

Theorem 1.4 (Quark potential and perimeter law). Suppose $m \geqslant 3$ and $\beta > \beta_0(m)$. There exists a function $V_\beta(\cdot)$ and a constant $C < \infty$ such that the following holds. Let $R \geqslant 2$ be an integer and for $T = 1, 2, \ldots$ let $\gamma_{R,T}$ be a rectangular loop with side lengths R and T and axis-parallel sides. Then,

$$\left| -\frac{1}{T} \log \langle W_{\gamma_{R,T}} \rangle_\beta - V_\beta(R) \right| \leqslant \frac{C}{T}.$$

The limit $V_\beta := \lim_{R\to\infty} V_\beta(R)$ exists and

$$V_\beta = 4e^{-8(m-1)\beta} + 24(m-1)e^{-4(4(m-1)-2)\beta} + O(e^{-16(m-1)\beta}).$$

By the theorem, $V_\beta(R)$ exists for all sufficiently large β and is bounded as $R \to \infty$ so we obtain a proof of the perimeter law. Moreover, using the convergence rate estimate we also obtain the up-to-constants estimate

$$\langle W_{\gamma_{R,T}} \rangle_\beta \asymp e^{-TV_\beta(R)}, \quad T \to \infty.$$

Remark 1.5. At fixed R, we have the expansion as $\beta \to \infty$

$$V_\beta(R) = 4e^{-8(m-1)\beta} + 24(m-1)e^{-4(4(m-1)-2)\beta} + O_R(1)e^{-16(m-1)\beta}.$$

Remark 1.6. We use a cluster expansion to prove Theorem 1.4, including the existence part. Alternatively, one could prove the existence of $V_\beta(R)$ using Griffith's second inequality to deduce subadditivity and then appeal to Fekete's lemma. However, this method would give no quantitative information as in the theorem. Moreover, it cannot be used to obtain Propositions 7.3 and 7.4, which shows the existence of the limit in some generality and is needed for Theorem 1.7 below.

Our next result is a version of Theorem 1.4 in the setting where the two sides of the loop grow uniformly.

Theorem 1.7. Suppose $m \geqslant 3$ and $\beta > \beta_0(m)$, let $r, t \geqslant 1$ be integers and for $n = 1, 2, \dots$ let γ_n be an axis-parallel rectangular loop with side lengths $R_n = rn$ and $T_n = tn$. Then

$$\lim_{n\to\infty} -\frac{1}{R_n + T_n} \log \langle W_{\gamma_n}\rangle_\beta = V_\beta,$$

where $V_\beta = \lim_{R\to\infty} V_\beta(R)$.

Note that $V_\beta = 2v_\beta + O(e^{-16(m-1)\beta})$. We have chosen to state Theorem 1.7 for a rectangle but with small modifications the proof is also valid for any fixed loop for which the proportion of corners and bottlenecks in the scaled loop tend to zero as $n \to \infty$ and the corresponding V_β is the same.

Remark 1.8. It would be interesting to relate the confinement phase transition to analyticity properties of the functions $\beta \mapsto F(\beta)$ and $\beta \mapsto V_\beta$.

1.3 Related work and further comments

We refer to [4] for a thorough discussion of classical works on area and perimeter law estimates in various settings, including [12, 14–16, 18, 19]. Among more recent results, we mention [13], which considers the 4D $U(1)$ theory with Villain action. In the perimeter law regime, for sufficiently regular loops, it was shown that

$$\frac{C_0}{2\beta}(1 + C\beta e^{-2\pi^2\beta})(1 + o(1)) \leqslant -\frac{1}{|\gamma|}\log|\mathbb{E}^{\mathrm{Vil}}_\beta[W_\gamma]| \leqslant \frac{C_0}{2\beta}(1 + \epsilon(\beta))(1 + o(1)),$$

where the upper bound was due to Fröhlich and Spencer [12]. Here, C_0 is a constant related to the discrete Gaussian free field. The infinite volume free energy for this model was also considered in [13], and an upper bound was obtained for the "internal energy", that is, its derivative with respect to β.

In the important paper [5], Chatterjee studied 4-dimensional Ising lattice gauge theory. Using a resampling argument, it was proved that

$$\langle W_\gamma\rangle_\beta \leqslant e^{-\frac{1-\ell_c/\ell}{1+e^{-4(m-1)\beta}} 2\ell e^{-4(m-1)\beta}}. \tag{3}$$

(We caution that 2β in the present paper is equal to the parameter β used in [5].) This estimate is valid for all $\beta > 0$. Using (3), the inequalities

$$\left|\langle W_\gamma\rangle_\beta - e^{-2\ell e^{-8(m-1)\beta}}\right| \leqslant C e^{\frac{-2}{1+e^{-16(m-1)\beta}}\ell e^{-8(m-1)\beta}}\left(e^{-8\beta} + \sqrt{\ell_c/\ell}\right) \tag{4}$$

and

$$\left|\langle W_\gamma\rangle_\beta - e^{-2\ell e^{-8(m-1)\beta}}\right| \leqslant C_1\left(e^{-8\beta} + \sqrt{\ell_c/\ell}\right)^{C_2}$$

were obtained. The ideas introduced in [5] spurred several recent works, and analogous estimates have now been given in more general settings, including for arbitrary finite structure groups and for corresponding lattice Higgs models; see [1, 3, 8–10]. The methods used in these papers produce error terms that will generally be larger than the estimate for $\langle W_\gamma \rangle$ if one does not have a relation of the type $\ell e^{-2\ell e^{-8(m-1)\beta}} \ll \infty$. That is, one needs the size of the loop to tend to infinity at a rate tuned to $\beta \to \infty$. (Of course, one sees different exponents for different choices of structure group G; this case corresponds to Ising lattice gauge theory.) As a consequence, it is not clear (to us) how to use those methods to prove a perimeter law (lower bound) estimate at fixed β or, for example, how to analyze limits such as the one defining the quark potential. Moreover, we do not know how they can easily be modified to obtain higher precision, even if the loop grows with β at an appropriate rate.

Here we instead carry out the analysis based on the cluster expansion of the partition function, which provides information on $\log \langle W_\gamma \rangle_\beta$. One still needs β to be sufficiently large, but it does not need to grow with ℓ for the error bounds to be small, and the drawbacks discussed above can be circumvented. The method yields, in principle, arbitrary precision for the logarithm of the Wilson loop expectation and also allows to quantify the behavior of $\langle W_\gamma \rangle$ when $\ell e^{-8(m-1)\beta} \ll \infty$. This partly resolves one of the open problems in [5]. However, the work here does not directly imply the results of [5] but does give alternative proofs of several of the key lemmas therein.

The use of cluster expansions in the context of lattice gauge theories is certainly not new; see, in particular, Seiler's monograph [19] (and the references therein), where, for example, perimeter law estimates were obtained to first order for large β. However, besides basic facts about the cluster expansion as presented in the recent textbook of Friedly and Velenik [11] and some results from [8, 9], our discussion is self-contained, and we carry out all the needed estimates here. We mention that using these estimates, the cluster expansions used in this paper can also be used to get an alternative and very short proof of exponential decay of correlations (see [7] and [2]). However, to keep this paper short, we do not include such an argument here.

We only consider $G = \mathbb{Z}_2$ in this paper. We expect that one can extend the results to any finite cyclic group $G = \mathbb{Z}_k$, $k \geqslant 3$, without too much additional effort, as well as to the corresponding lattice Higgs models. The cluster expansion based on vortices crucially relies on the fact that gauge field configurations can be decomposed into discrete components. Therefore, we do not expect the methods in this paper to work in the general case of compact subgroups of $U(N)$.

2 Preliminaries

Although we will later work with $G = \mathbb{Z}_2$, in this section, we allow G to be a general finite abelian group, as this entails no additional work.

2.1 Notation and standing assumptions

In the rest of this paper, we assume that $m \geqslant 3$ is given. We define the dimension-dependent constant

$$M = M(m) := 10(m-2).$$

Next, we define

$$\beta_0(m) := \frac{1}{2}(B_0(m) + \log M), \tag{5}$$

where

$$B_0(m) := \inf\{B > 0 \colon \exists \alpha \in (0, B) \text{ s.t.} e^{4(\alpha - B)(m-1)} < \alpha(1 - e^{2(\alpha - B)})\}.$$

2.2 Discrete exterior calculus

To keep the presentation concise, and since these definitions have been introduced in several recent papers, we will refer to [9] for details on some of the basic notions of discrete exterior calculus that are useful in the present context.

- We will work with the square lattice $\mathbb{Z}^m$, where we assume that the dimension $m \geqslant 3$ throughout. We write $B_N = [-N, N]^m \cap \mathbb{Z}^m$.

- We write $C_k(B_N)$ and $C_k(B_N)^+$ for the set of unoriented and positively oriented k-cells, respectively (see [9, Sect. 2.1.2]). Note that in the introduction, we used $V_N = C_0(B_N)$, $E_N = C_1(B_N)$, and $P_N = C_2(B_N)$. An oriented 2-cell is called a plaquette.
- Formal sums of positively oriented k-cells with integer coefficients are called k-chains, and the space of k-chains is denoted by $C_k(B_N, \mathbb{Z})$ (see [9, Sect. 2.1.2]).
- Let $k \geqslant 2$ and $c = \frac{\partial}{\partial x^{j_1}}\big|_a \wedge \cdots \wedge \frac{\partial}{\partial x^{j_k}}\big|_a \in C_k(B_N)$. The *boundary* of c is the $(k-1)$-chain $\partial c \in C_{k-1}(B_N, \mathbb{Z})$ defined as the formal sum of the $(k-1)$-cells in the (oriented) boundary of c. The definition is extended to k-chains by linearity. See [9, Sect. 2.1.4].
- If $k \in \{0, 1, \ldots, n-1\}$ and $c \in C_k(B_N)$ is an oriented k-cell, we define the *coboundary* $\hat\partial c \in C_{k+1}(B_N)$ of c as the $(k+1)$-chain $\hat\partial c := \sum_{c' \in C_{k+1}(B_N)} (\partial c'[c]) c'$. See [9, Sect. 2.1.5].
- We let $\Omega^k(B_N, G)$ denote the set of G-valued (discrete differential) k-forms (see [9, Sect 2.3.1]); the exterior derivative $d : \Omega^k(B_N, G) \to \Omega^{k+1}(B_N, G)$ is defined for $0 \leqslant k \leqslant m-1$ (see [9, Sect. 2.3.2]); and $\Omega^k_0(B_N, G)$ denotes the set of closed k-forms, that is, $\omega \in \Omega^k(B_N, G)$ such that $d\omega = 0$.
- We write $\mathbf{supp}\,\omega = \{c \in C_k(B_N) : \omega(c) \neq 0\}$ for the support of a k-form ω. Similarly, we write $(\mathbf{supp}\,\omega)^+ = \{c \in C_k(B_N)^+ : \omega(c) \neq 0\}$.
- A 1-chain $\gamma \in C_1(B_N, \mathbb{Z})$ with finite support $\mathbf{supp}\,\gamma$ is called a *loop* if for all $e \in \Omega^1(B_N)$, we have that $\gamma[e] \in \{-1, 0, 1\}$, and $\partial\gamma = 0$. We write $|\gamma| = |\mathbf{supp}\,\gamma|$. (In [9] this object was called a generalized loop.)
- Let $\gamma \in C_1(B_N, \mathbb{Z})$ be a loop. A 2-chain $q \in C_2(B_N, \mathbb{Z})$ is an *oriented surface* with *boundary* γ if $\partial q = \gamma$.

2.3 Plaquette adjacency graph

Let $\mathcal{G}_2$ be the graph with vertex set $C_2(B_N)^+$ and an edge between two distinct vertices $p_1, p_2 \in C_2(B_N)^+$ if and only if $\mathbf{supp}\,\hat\partial p_1 \cap \mathbf{supp}\,\hat\partial p_2 \neq \emptyset$.

Since any plaquette $p \in C_2(B_N)^+$ in B_N is in the boundary of at most $2(m-2)$ 3-cells, and any such 3-cell has exactly five plaquettes in its boundary that are not equal to p, it follows that there are at most $5 \cdot 2(m-2) = 10(m-2) = M$ plaquettes $p' \in C_2(B_N)^+ \smallsetminus \{p\}$ with $\mathbf{supp}\,\hat\partial p \cap \mathbf{supp}\,\hat\partial p' \neq \emptyset$. Therefore, it follows that each vertex in $\mathcal{G}_2$ has degree at most M.

2.4 Vortices

Definition 2.1 (Vortex). A closed 2-form $\nu \in \Omega^2_0(B_N, G)$ is said to be a *vortex* if $(\mathbf{supp}\,\nu)^+$ induces a connected subgraph of $\mathcal{G}_2$.

The set of all vortices in $\Omega^2(B_N, G)$ is denoted by Λ. We note that the definition of vortex we use here is not exactly the same as the definition used in [7–10], but agrees with the definition used in [3, 5].

When $\omega, \nu \in \Omega^2(B_N, G)$, we say that ν is a vortex in ω if ν is a vortex and $\mathbf{supp}\,\nu$ corresponds to a connected subgraph of the subgraph of $\mathcal{G}_2$ induced by $\mathbf{supp}\,\omega$.

Lemma 2.2 (The Poincaré lemma, Lemma 2.2 in [5]). Let $k \in \{1, \ldots, m\}$ and let B be a box in $\mathbb{Z}^m$. Then the exterior derivative d is a surjective map from the set $\Omega^{k-1}(B \cap \mathbb{Z}^m, G)$ to $\Omega^k_0(B \cap \mathbb{Z}^m, G)$. Moreover, if G is finite, then this map is an $|\Omega^{k-1}_0(B \cap \mathbb{Z}^m, G)|$-to-1 correspondence. Lastly, if $k \in \{1, 2, \ldots, m-1\}$ and $\omega \in \Omega^k_0(B \cap \mathbb{Z}^m, G)$ vanishes on the boundary of B, then there is a $(k-1)$-form $\omega' \in \Omega^{k-1}(B \cap \mathbb{Z}^m, G)$ that also vanishes on the boundary of B and satisfies $d\omega' = \omega$.

Lemma 2.3 (Lemma 2.4 of [7]). Let $\omega \in \Omega^2_0(B_N, G)$. If $\omega \neq 0$ and the support of ω does not contain any boundary plaquettes of B_N, then either $|(\mathbf{supp}\,\omega)^+| = 2(m-1)$, or $|(\mathbf{supp}\,\omega)^+| \geqslant 4(m-1) - 2$.

In [7], we proved Lemma 2.3 only in the case $m = 4$, but since the proof for general $m \geqslant 2$ is analogous we do not include it here.

In Figure 1, we illustrate the only two possibilities for $(\mathbf{supp}\,\omega)^+$ if $|(\mathbf{supp}\,\omega)^+| = 4(m-1) - 1$ when $m = 4$. For general $m \geqslant 2$, the situation is analogous.

Lemma 2.4 (Lemma 4.6 in [9]). Let $\omega \in \Omega^2_0(B_N, G)$. If the support of ω does not contain any boundary plaquettes of B_N and $|(\mathbf{supp}\,\omega)^+| = 2(m-1)$, then there is an edge $e \in C_1(B_N)$ and $g \in G \smallsetminus \{0\}$ such that

$$\omega = d(g\,\mathbf{1}_e - g\mathbf{1}_{-e}). \tag{6}$$

If $\omega \in \Omega^2(B_N, G)$ is such that (6) holds for some $e \in C_1(B_N)$ and $g \in G \smallsetminus \{0\}$, then we say that ω is a *minimal vortex around* e.

Lemma 2.5. Let $\omega \in \Omega_0^2(B_N, G)$, and assume that the support of ω does not contain any boundary plaquettes of B_N and $|(\operatorname{supp} \omega)^+| = 4(m-1) - 2$. Then there are two distinct edges $e, e' \in C_1(B_N)$ with $(\hat\partial e)^+ \cap (\hat\partial e')^+ \neq \emptyset$ and $\sigma \in \Omega^1(B_N, G)$ with $(\operatorname{supp} \sigma)^+ = \{e, e'\}$ such that $d\sigma = \omega$.

For a proof of Lemma 2.5, see the proof of [7, Lemma 2.4].

Lemma 2.6. Let q be an oriented surface with $\partial q = \gamma$. Further, let $\omega \in \Omega^2(B_N, G)$ be such that $d\omega = 0$ and $\omega(q) \neq 0$. Then any box which contains $\operatorname{supp} \omega$ must intersect an edge in $\operatorname{supp} \gamma$.

Proof. Let B be a box that contains $\operatorname{supp} \omega$. Since $d\omega = 0$, by the Poincaré lemma (see, e.g., [9, Lemma 2.2] there is $\sigma \in \Omega^1(B_N, G)$ whose support is contained in B such that $d\sigma = \omega$. Moreover, we have $\omega(q) = \sigma(\gamma)$ (see, e.g., [9, Section 2.4]). Consequently, if B does not intersect $\operatorname{supp} \gamma$, then $\omega(q) = \sigma(\gamma) = 0$, a contradiction. ■

Lemma 2.7. Let $\nu \in \Omega^2(B_N, G)$ satisfy $d\nu = 0$, let B be a box that contains the support of ν and let $p \in \operatorname{supp} \nu$. Then there is at least one 1-cell in $\operatorname{supp} \partial p$ that is not in the boundary of B.

Proof. Assume for contradiction that all edges in $\operatorname{supp} \partial p$ are in the boundary of B. Then there is a 3-cell $c \in \hat\partial p$ that is not contained in B. Since B is a box, p must be the only plaquette in $\operatorname{supp} \partial c$ that is in B. Since the support of ν is contained in B, it follows that

$$d\nu(c) = \sum_{p' \in \partial c} \nu(p') = \nu(p) \neq 0.$$

Since this contradicts the assumption that $d\nu = 0$, the desired conclusion follows. ■

The following lemma is elementary.

Lemma 2.8. There is a constant $C_m > 0$ such that for any $e \in C_1(B_N)^+$ and $j \geqslant 0$, we have

$$\big|\{p \in C_2(B_N)^+ \colon \operatorname{dist}(p, e) = j\}\big| \leqslant C_m \max(1, j)^{m-1}.$$

Lemma 2.9. Let $j \geqslant 1$, let $p \in C_0(B_N)$ and let B be a box with side lengths $s_1, s_2, \dots, s_m$ that contains p and is such that every face of the box contains at least one point on distance at least j from p. Then $\sum_{i=1}^m s_i \geqslant jm/(m-1)$.

Proof. Without loss of generality we can assume that the box B has corners at $(0, 0, \dots 0)$ and $(s_1, s_2, \dots, s_m)$, that $p = (x_1, x_2, \dots, x_m)$, and that $0 \leqslant x_i \leqslant s_i/2$ for $i = 1, 2, \dots, m$. Then the assumption on B is equivalent to that

$$x_i + \sum_{k \neq i} (s_k - x_k) \geqslant j$$

for each $i \in \{1, 2, \dots, m\}$. Summing over i, we obtain

$$\sum_{i=1}^m \Big(x_i + \sum_{k \neq i} (s_k - x_k) \Big) \geqslant mj \Leftrightarrow \sum_{i=1}^m x_i + (m-1) \sum_{i=1}^m s_i - (m-1) \sum_{i=1}^m x_i \geqslant mj$$
$$\Leftrightarrow (m-1) \sum_{i=1}^m s_i \geqslant mj + (m-2) \sum_{i=1}^m x_i.$$

From this the desired conclusion immediately follows. ■

Lemma 2.10. There is a constant $\hat{C}_m > 0$ such that for any oriented surface q, any let $j \geqslant 1$, and any $\nu \in \Lambda$ with $\nu(q) = 1$ and $\mathbf{dist}(\mathbf{supp}\,\nu, \gamma) = j$, we have $|(\mathbf{supp}\,\nu)^+| \geqslant \hat{C}_m(j+1)$.

Proof. Let B be the (unique) smallest box that contains the support of ν, and assume that the side lengths of B are $s_1, \ldots, s_m$. Since $\nu(q) = 1$, it follows from Lemma 2.6 that B intersects an edge of $\mathbf{supp}\,\gamma$. Consequently, there is some edge in γ whose both endpoints are contained in B. Fix one such edge e. Note that, by assumption, we have $\mathbf{dist}(\mathbf{supp}\,\nu, e) \geqslant j$. Since B is a minimal box containing $\mathbf{supp}\,\nu$, there must be one edge on each face of the box which is contained in the boundary of some plaquette in $\mathbf{supp}\,\nu$. At the same time, by Lemma 2.7, since $d\nu = 0$, no plaquette in $\mathbf{supp}\,\nu$ can be in the boundary of B. This implies in particular that each plaquette in $p \in (\mathbf{supp}\,\nu)^+$ must have an edge in its boundary that is not in the boundary of B. Since for each such plaquette we must have $\mathbf{dist}(p, e) \geqslant j$, it follows from Lemma 2.9 that

$$\sum_{i=1}^{m}(s_i - 2) \geqslant \frac{(j+1)m}{m-1} \Leftrightarrow \sum_{i=1}^{m} s_i \geqslant \frac{(j+1)m}{m-1} + 2m.$$

Since $\omega \in \Lambda$, the set $(\mathbf{supp}\,\omega)^+$ induces a connected subgraph of G_2. Since each face of B contains at least one edge that is in the boundary of some plaquette in $(\mathbf{supp}\,\omega)^+$, the desired conclusion immediately follows. ■

We end this section by introducing some additional notation for vortices. Recall that Λ denotes the set of vortices in $\Omega_0^2(B_N, G)$. For $p \in C_2(B_N)$, $q \in C^2(B_N, \mathbb{Z})$, and $j \geqslant 1$, we define

$$\Lambda_j := \{\nu \in \Lambda : |(\mathbf{supp}\,\nu)^+| = j\} \quad \text{and} \quad \Lambda_{j^+} := \{\nu \in \Lambda : |(\mathbf{supp}\,\nu)^+| \geqslant j\},$$

$$\Lambda_{j,p} := \{\nu \in \Lambda_j : p \in \mathbf{supp}\,\nu\} \quad \text{and} \quad \Lambda_{j^+,p} := \{\nu \in \Lambda_{j^+} : p \in \mathbf{supp}\,\nu\},$$

and

$$\Lambda_{j,q} := \{\nu \in \Lambda_j : \nu(q) \neq 0\} \quad \text{and} \quad \Lambda_{j^+,q} := \{\nu \in \Lambda_{j^+} : \nu(q) \neq 0\}.$$

Further, we let

$$\Lambda_{j^+} := \{\nu \in \Lambda : |(\mathbf{supp}\,\nu)^+| \geqslant j\} \quad \text{and} \quad \Lambda_{j^-} := \Lambda \setminus \Lambda_{j^+}.$$

We will always use the notation p for a single plaquette and q for a 2-chain, so the notation above should not cause confusion when used below. We note that the sets defined above depend on N but usually suppress this in the notation. When we want to emphasize this we write $\Lambda_j(B_N)$, $\Lambda_{j+}(B_N)$, etc.

2.5 Vortex clusters

If $\nu_1, \nu_2 \in \Lambda$, we write $\nu_1 \sim \nu_2$ if there is $p_1 \in (\mathbf{supp}\,\nu_1)^+$ and $p_2 \in (\mathbf{supp}\,\nu_2)^+$ such that $p_1 \sim p_2$ in $\mathcal{G}_2$.

Consider a multiset

$$\mathcal{V} = \{\underbrace{\nu_1, \ldots, \nu_1}_{n_{\mathcal{V}}(\nu_1)\text{ times}}, \underbrace{\nu_2, \ldots, \nu_2}_{n_{\mathcal{V}}(\nu_2)\text{ times}}, \ldots, \underbrace{\nu_k, \ldots, \nu_k}_{n_{\mathcal{V}}(\nu_k)\text{ times}}\} = \{\nu_1^{n(\nu_1)}, \ldots, \nu_k^{n(\nu_k)}\},$$

where $\nu_1, \ldots, \nu_k \in \Lambda$ are distinct and $n(\nu) = n_{\mathcal{V}}(\nu)$ denotes the number of times ν occurs in $\mathcal{V}$. Following [11, Chapter 3], we say that $\mathcal{V}$ is *decomposable* if there exist non-empty and disjoint multisets $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}$ such that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ and such that for each pair $(\nu_1, \nu_2) \in \mathcal{V}_1 \times \mathcal{V}_2$, we have $\nu_1 \nsim \nu_2$. If $\mathcal{V}$ is not decomposable, it is by definition a *vortex cluster*. We stress that a vortex cluster is unordered and may contain several copies of the same vortex.

Given a vortex cluster $\mathcal{V}$, let us define

$$|\mathcal{V}| = \sum_{\nu\in\Lambda} n_{\mathcal{V}}(\nu)\big|(\mathbf{supp}\,\nu)^+\big|, \quad n(\mathcal{V}) = \sum_{\nu\in\Lambda} n_{\mathcal{V}}(\nu), \quad \text{and} \quad \mathbf{supp}\,\mathcal{V} = \bigcup_{\nu\in\mathcal{V}} \mathbf{supp}\,\nu.$$

For a 2−chain $q \in C_2(B_N, \mathbb{Z})$, we define

$$\mathcal{V}(q) = \sum_{\nu \in \Lambda} n_{\mathcal{V}}(\nu)\nu(q).$$

We write Ξ for the set of all vortex clusters of Λ. To simplify notation, for $p \in C_2(B_N)$, $q \in C^2(B_N, \mathbb{Z})$, and $j \geqslant 1$, we define

$$\Xi_j := \{\mathcal{V} \in \Xi : |\mathcal{V}| = j\} \quad \text{and} \quad \Xi_{j^+} := \{\mathcal{V} \in \Xi : |\mathcal{V}| \geqslant j\},$$

$$\Xi_{j,p} := \{\mathcal{V} \in \Xi_j : p \in \operatorname{supp} \mathcal{V}\} \quad \text{and} \quad \Xi_{j^+,p} := \{\mathcal{V} \in \Xi_{j^+} : p \in \operatorname{supp} \mathcal{V}\},$$

and

$$\Xi_{j,q} := \{\mathcal{V} \in \Xi_j : \mathcal{V}(q) \neq 0\} \quad \text{and} \quad \Xi_{j_+,q} := \{\mathcal{V} \in \Xi_{j_+} : \mathcal{V}(q) \neq 0\}.$$

Further, we let

$$\Xi_{j^+} := \{\mathcal{V} \in \Xi : |\mathcal{V}| \geqslant j\} \quad \text{and} \quad \Xi_{j^-} := \Xi_{1^+} \smallsetminus \Xi_{j^+}.$$

As before, the sets defined above depend on N but we usually suppress this. When we want to emphasize the dependence we write $\Xi_j(B_N)$, $\Xi_{j+}(B_N)$, etc.

The following lemma gives an upper bound on the number of vortices of a given size that contains a given plaquette p. This lemma will later be used to upper bound the total weight of all such clusters.

Lemma 2.11. Let $k \geqslant 1$ and let $p \in C_2(B_N)^+$. Then,

$$|\Lambda_{k,p}| \leqslant M^{2k-1}.$$

Proof. Let $\mathcal{P}$ be the set of all paths in $\mathcal{G}_2$ that starts at p and has length $2k-1$. Since each vertex in $\mathcal{G}_2$ has degree at most $10(m-2) = M$, we have $|\mathcal{P}| \leqslant M^{2k-1}$.

For $\nu \in \Lambda_{k,p}$, let G_ν be the subgraph of $\mathcal{G}_2$ induced by the set $(\operatorname{supp} \nu)^+$. Then G_ν is connected, and hence G_ν has a spanning path $T_\nu \in \mathcal{P}$ of length $2|(\operatorname{supp} \nu)^+| - 1 = 2k - 1$ which starts at p. Since the map $\nu \mapsto T_\nu$ is an injective map from $\Lambda_{k,p}$ to $\mathcal{P}$ and $|\mathcal{P}| \leqslant M^{2k-1}$, the desired conclusion immediately follows. ■

2.6 The activity

For $\beta \geqslant 0$ and $g \in G$ with a unitary, one-dimensional representation ρ, we set

$$\phi_\beta(g) := e^{\beta \operatorname{Re}(\rho(g) - \rho(0))}.$$

Since ρ is unitary, for any $g \in G$ we have $\rho(g) = \overline{\rho(-g)}$, and hence $\operatorname{Re} \rho(g) = \operatorname{Re} \rho(-g)$. In particular, for any $g \in G$

$$\phi_\beta(g) = e^{\beta(\operatorname{Re} \rho(g) - \rho(0))} = e^{\beta(\operatorname{Re} \rho(-g) - \rho(0))} = \phi_\beta(-g). \tag{7}$$

For $\omega \in \Omega^2(B_N, G)$ and $\beta \geqslant 0$ we define the *activity* of ω by

$$\phi_\beta(\omega) := \prod_{p \in C_2(B_N)} \phi_a\big(\omega(p)\big).$$

Note that for $\sigma \in \Omega^1(B_N, G)$, the Wilson action lattice gauge theory probability measure can be written

$$\mu_{\beta,N}(\sigma) = \frac{\phi_\beta(d\sigma)}{\sum_{\sigma \in \Omega^1(B_N,G)} \phi_\beta(d\sigma)}. \tag{8}$$

Moreover, in the case when $G = \mathbb{Z}^2$ for $\omega \in \Omega^2(B_N, \mathbb{Z}_2)$, we have

$$\phi_\beta(\omega) = \prod_{p \in C_2(B_N)} \phi_\beta\big(\omega(p)\big) = \prod_{p \in C_2(B_N)} e^{-2\beta \mathbb{1}\left(\omega(p)=1\right)} = e^{-2\beta \sum_{p \in C_2(B_N)} \mathbb{1}\left(\omega(p)=1\right)} = e^{-2\beta |\operatorname{supp} \omega|}.$$

We note that, by definition, if $\omega, \nu \in \Omega^2(B_N, G)$ and ν is a vortex in ω, then $\phi_\beta(\omega) = \phi_\beta(\nu)\phi_\beta(\omega - \nu)$.

We extend the notion of activity to vortex clusters $\mathcal{V} \in \Xi$ by letting

$$\phi_\beta(\mathcal{V}) = \prod_{\nu \in \Lambda} \phi_\beta(\nu)^{n_{\mathcal{V}}(\nu)} = e^{-4\beta|\mathcal{V}|}.$$

3 Low Temperature Cluster Expansion

In this section we review the cluster expansion for the relevant Ising lattice gauge theory partition functions defined on a finite box B_N. The material here is for the most part well known. See [11] for a text book presentation for the standard Ising model and [19] for a discussion in the context of lattice gauge theories.

3.1 Ursell function

We will work with the Ursell function corresponding to the choice of vortices as polymers and hard core interaction: two vortices are compatible if and only if they correspond to separate components in the graph $\mathcal{G}_2$. Before defining the Ursell function, we need some additional notation. For $k \geqslant 1$, we write $G \in \mathcal{G}^k$ if G is a connected graph with vertex set $V(G) = \{1, 2, \ldots, k\}$. Let $E(G)$ be the (undirected) edge set of G. Recall that we write $\nu_1 \sim \nu_2$ if there is $p_1 \in (\operatorname{supp} \nu_1)^+$ and $p_2 \in (\operatorname{supp} \nu_2)^+$ such that $p_1 \sim p_2$ in $\mathcal{G}_2$, where $\mathcal{G}_2$ was defined in Section 2.3.

Definition 3.1 (The Ursell function). For $k \geqslant 1$ and $\nu_1, \nu_2, \ldots, \nu_k \in \Lambda$, we define

$$U(\nu_1, \ldots, \nu_k) := \frac{1}{k!} \sum_{G \in \mathcal{G}^k} (-1)^{|E(\mathcal{G})|} \prod_{(i,j) \in E(\mathcal{G})} \mathbb{1}(\nu_i \sim \nu_j).$$

Note that this definition is invariant under permutations of the vortices $\nu_1, \nu_2, \ldots, \nu_k$.

For $\mathcal{V} \in \Xi$ with $n(\mathcal{V}) = k$ and any enumeration $\nu_1, \ldots, \nu_k$ (with multiplicities) of the vortices in $\mathcal{V}$, we define

$$U(\mathcal{V}) = k!\ U(\nu_1, \ldots, \nu_k). \tag{9}$$

Note that for any $\mathcal{V} \in \Xi$ with $n(\mathcal{V}) = 1$, we have $U(\mathcal{V}) = 1$, and for any $\mathcal{V} \in \Xi$ with $n(\mathcal{V}) = 2$, we have $U(\mathcal{V}) = -1$.

3.2 Partition functions

The partition function for Ising lattice gauge theory, viewed as a model for plaquette configurations can be written as follows:

$$Z_{\beta,N} = \sum_{\omega \in \Omega_0^2(B_N, G)} e^{\beta \sum_{p \in C_2(B_N)} \operatorname{Re}(\rho(\omega(p)) - \rho(0))} = \sum_{\omega \in \Omega_0^2(B_N, G)} \phi_\beta(\omega).$$

See, for example, Section 3 of [9]. This is a finite sum and the definition extends to $\beta \in \mathbb{C}$. An alternative representation for $Z_{\beta,N}$ is given by the *vortex partition function* which is defined by the following (formal) expression:

$$Z^{\nu}_{\beta,N} = \exp\left(\sum_{\mathcal{V} \in \Xi} \Psi_\beta(\mathcal{V})\right), \tag{10}$$

where for $\mathcal{V} \in \Xi$, we define

$$\Psi_\beta(\mathcal{V}) := U(\mathcal{V})\phi_\beta(\mathcal{V}),$$

and U is the Ursell function defined in Definition 3.1.

Recall the definition of $\beta_0 = \beta_0(m)$ from (5). It is not obvious that the series in the exponent of (10) is convergent but this follows from the next lemma, assuming $\operatorname{Re}\beta > \beta_0$. We verify below that, with this assumption, $\log Z_{\beta,N} = \log Z^\nu_{\beta,N}$. In addition, the lemma below gives us an upper bound on a sum over clusters that will be crucial in all of the other estimates of this type throughout the paper. The conclusion of the lemma follows from [11, Theorem 5.4] after verifying that the assumptions of this theorem hold, and doing this is the main step of the proof.

Lemma 3.2. Let $G = \mathbb{Z}_2$. Suppose $\operatorname{Re}\beta > \beta_0(m)$. Then there is $\alpha > 0$ such that for any $\nu \in \Lambda$, we have

$$\sum_{\mathcal{V}\in\Xi:\, \mathcal{V}\ni\nu} \left|\Psi_\beta(\mathcal{V})\right| \leqslant e^{\alpha|\operatorname{supp}\nu|}\phi_\beta(\nu).$$

Moreover, the series in (10) is absolutely convergent.

Proof. Let $\alpha > 0$ be such that

$$M^2 e^{-2(2\operatorname{Re}\beta-\alpha)} < 1$$

and

$$\frac{\left(M^2 e^{-2(2\operatorname{Re}\beta-\alpha)}\right)^{2(m-1)}}{1 - M^2 e^{-2(4\operatorname{Re}\beta-\alpha)}} \leqslant \alpha.$$

Note that by the choice of β_0, such α exists. We have, for each $\nu \in \Lambda$ we have

$$\sum_{\nu'\in\Lambda} |\phi_\beta(\nu')| e^{\alpha|\operatorname{supp}\nu'|}\mathbb{1}(\nu \sim \nu') \leqslant \alpha|\operatorname{supp}\nu|.$$

Given this, the conclusion follows from Theorem 5.4 of [11] by choosing $a(\nu) := \alpha|\operatorname{supp}\nu|$. To this end, let $\nu \in \Lambda$. Then,

$$\begin{aligned}\sum_{\nu'\in\Lambda} |\phi_\beta(\nu')| e^{\alpha|\operatorname{supp}\nu'|}\mathbb{1}(\nu \sim \nu') &= \sum_{\nu'\in\Lambda:\, \nu\sim\nu'} |\phi_\beta(\nu')| e^{\alpha|\operatorname{supp}\nu'|} \\ &= \sum_{\nu'\in\Lambda:\, \nu\sim\nu'} e^{-(2\operatorname{Re}\beta-\alpha)|\operatorname{supp}\nu'|} \\ &\leqslant \sum_{p\in(\operatorname{supp}\nu)^+} \sum_{\substack{p'\in C_2(B_N)^+:\\ p'\sim p}} \sum_{j=2(m-1)}^{\infty} |\Lambda_{j,p'}| e^{-(2\operatorname{Re}\beta-\alpha)2j}.\end{aligned}$$

Using Lemma 2.11 and the definition of M, it follows that

$$\begin{aligned}\sum_{\nu'\in\Lambda} |\phi_\beta(\nu')| e^{\alpha|\operatorname{supp}\nu'|}\mathbb{1}(\nu \sim \nu') &\leqslant \left|(\operatorname{supp}\nu)^+\right| \sum_{j=2(m-1)}^{\infty} M^{2j} e^{-(2\operatorname{Re}\beta-\alpha)2j} \\ &= |(\operatorname{supp}\nu)^+| \frac{\left(M^2 e^{-2(2\operatorname{Re}\beta-\alpha)}\right)^{2(m-1)}}{1 - M^2 e^{-2(2\beta-\alpha)}}.\end{aligned}$$

The desired conclusion now follows from the choice of α. ∎

The next lemma, Lemma 3.3, is the key lemma which connects the two partition functions $Z_{\beta,N}$ and $Z^{\nu}_{\beta,N}$.

Lemma 3.3. Let $G = \mathbb{Z}_2$. Suppose $\mathbf{Re}\,\beta > \beta_0(m)$. Then

$$\log Z_{\beta,N} = \log Z^{\nu}_{\beta,N} = \sum_{\mathcal{V}\in\Xi} \Psi_\beta(\mathcal{V}), \tag{11}$$

and $\log Z_{\beta,N}$ is an analytic function of β.

Proof. Let A be the set of all subsets Λ' of Λ with the property that the vortices in Λ' are not connected in $\mathcal{G}_2$. Then set $\Omega^2_0(B_N, G)$ is in bijection with A. Therefore, we can write

$$Z_{\beta,N} = \sum_{\Lambda'\subset\Lambda} \phi_\beta(\Lambda') \prod_{\{\nu,\nu'\}\subset\Lambda'} \mathbb{1}(\nu \nsim \nu')$$

and this holds for any choice of β. On the other hand, if $\mathbf{Re}\,\beta > \beta_0(m)$, we can apply Proposition 5.3 of [11] to see that the right-hand side in the last display equals $\log Z^{\nu}_{\beta,N}$ as defined in (10). ■

We now assume β is real. When $\beta > \beta_0$ we write $Z_{\beta,N}$ also for the vortex partition function $Z^{\nu}_{\beta,N}$ (since in this case, they are equal by Lemma 3.3). We wish to express the Wilson loop expectation using the logarithm of the partition function. For this, we fix a loop γ and an oriented surface q such that $\gamma = \partial q$ and recall the following fact; see [9, Section 3].

Lemma 3.4. Let $G = \mathbb{Z}_2$. Let $\beta \geqslant 0$ and let q be an oriented surface with $\partial q = \gamma$. Then for all N such that $\mathbf{supp}\, q \subseteq B_N$

$$\mathbb{E}_{\beta,N}[W_\gamma] = Z^{-1}_{\beta,N} \sum_{\omega\in\Omega^2_0(B_N,G)} \phi_\beta(\omega)\rho\big(\omega(q)\big).$$

Consider now the weighted vortex partition function

$$Z_{\beta,N}[q] := \exp\left(\sum_{\mathcal{V}\in\Xi} \Psi_{\beta,q}(\mathcal{V})\right), \tag{12}$$

where

$$\Psi_{\beta,q}(\mathcal{V}) := \Psi_\beta(\mathcal{V})\rho\big(\mathcal{V}(q)\big) = U(\mathcal{V})\phi_\beta(\mathcal{V})\rho\big(\mathcal{V}(q)\big).$$

The series on the right-hand side of (12) is absolutely convergent when $\beta > \beta_0(m)$ by the proof of Lemma 3.2 since $|\rho\big(\mathcal{V}(q)\big)| = 1$ for each $\mathcal{V} \in \Xi$. As in the proof of Lemma 3.3, using [11, Proposition 5.3], replacing the weight $\phi_\beta(\mathcal{V})$ by $\phi_\beta(\mathcal{V})\rho\big(\mathcal{V}(q)\big)$, we have

$$\log Z_{\beta,N}[q] = \sum_{\omega\in\Omega^2_0(B_N,G)} \phi_\beta(\omega)\rho\big(\omega(q)\big).$$

The following result is the main reason that cluster expansions are helpful to us, as it enables us to express the logarithm of the Wilson loop observable as a sum over clusters interacting with the loop. The rest of the paper is then concerned with understanding the sum in the right-hand side of (13).

Proposition 3.5. Let $G = \mathbb{Z}_2$. Let $\beta > \beta_0(m)$ and let q be an oriented surface with $\partial q = \gamma$. Then for all N such that $\mathbf{supp}\, q \subseteq B_N$,

$$-\log \mathbb{E}_{\beta,N}[W_\gamma] = \sum_{\mathcal{V}\in\Xi} \big(\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V})\big) = \sum_{\mathcal{V}\in\Xi} \Psi_\beta(\mathcal{V})\Big(1 - \rho\big(\mathcal{V}(q)\big)\Big). \tag{13}$$

Proof. Using Lemma 3.4 and then Lemma 3.3 and (12) we conclude that

$$\log \mathbb{E}_{\beta,N}[W_\gamma] = \log \frac{Z_{\beta,N}[q]}{Z_{\beta,N}},$$

which is what we wanted to prove. ∎

Remark 3.6. Notice that Proposition 3.5 implies that $\mathbb{E}_{\beta,N}[W_\gamma] \in (0,1]$ when $\beta > \beta_0(m)$. This fact is not clear from the start since $W_\gamma \in \{-1,1\}$ for every $\sigma \in \Omega^1(B_N, \mathbb{Z}_2)$. The positivity of $\mathbb{E}_{\beta,N}[W_\gamma]$ was pointed out in [5] and proved there as a consequence of duality. Here we obtain the conclusion as a result of the convergence of the cluster expansion.

4 Estimates for the Cluster Expansion

Throughout this section, we assume that γ is a simple loop, and that q is an oriented surface with $\partial q = \gamma$. Recall that we use the notation $\ell = |\gamma|$. Recall also that ℓ_c denotes the number of corners of γ, that is, pairs of non-parallel edges in γ that are both in the boundary of some common plaquette, and that ℓ_b denotes the number of bottlenecks in γ, that is, pairs (e,e') of parallel edges in γ that are both in the boundary of some common plaquette. From now on, we also assume $G = \mathbb{Z}_2$.

The main goal of this section is to provide proofs of the three propositions below, which all deal with parts of the sum $\sum_{\mathcal{V}\in\Xi}\big(\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V})\big)$, whose relevance stems from (13) of Proposition 3.5. The first of the three propositions, Proposition 4.1 below, gives upper and lower bounds for the contribution to the sum caused by clusters that contain exactly one vortex.

Proposition 4.1. Let $\epsilon > 0$. There exists $D_1 < \infty$ such that for any $\beta > \beta_0(m) + \epsilon$,

$$0 \leqslant \sum_{\mathcal{V}\in\Xi\colon n(\mathcal{V})=1} \big(\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V})\big) - \ell H_{m,\beta}(\gamma) \leqslant D_1 \ell e^{-16(m-1)\beta},$$

where

$$H_{m,\beta}(\gamma) = e^{-8(m-1)\beta} + \big(6(m-1) - \frac{2\ell_c - 2\ell_b}{\ell}\big)e^{-4(4(m-1)-2)\beta}.$$

The second main result of this section, Proposition 4.2 below, gives an upper bound to the contribution to the sum $\sum_{\mathcal{V}\in\Xi}\big(\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V})\big)$ from clusters which contain at least two vortices.

Proposition 4.2. Let $\epsilon > 0$. There exists $D_1 < \infty$ such that for any $\beta > \beta_0(m) + \epsilon$,

$$\sum_{\mathcal{V}\in\Xi\colon n(\mathcal{V})\geqslant 2} \Big|\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V})\Big| \leqslant 2D_1 \ell e^{-16(m-1)\beta}.$$

The last main result of this section, Proposition 4.3, gives an upper bound to the contribution to $\sum_{\mathcal{V}\in\Xi}\big(\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V})\big)$ from clusters whose support is larger than some integer k.

Proposition 4.3. Let $\epsilon > 0$. For any $\beta > \beta_0(m) + \epsilon =: \beta^*$ and $k \geqslant 1$,

$$\sum_{\mathcal{V}\in\Xi_{k+}} \big|\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V})\big| \leqslant C_m C_{\beta^*} \ell \sum_{j=0}^{\infty} \max(1,j)^{m-1} e^{-4(\beta-\beta^*)\max(k,\hat{C}_m(j+1))},$$

where C_{β^*} is defined in (14).

Before giving the proofs of Propositions 4.1, 4.2, and 4.3, we need several auxiliary results. The first of these results is the following lemma, which exactly quantifies the total contribution of minimal vortices to the sum on the right-hand side of (13). Since all other types of vortices are larger, these will give the leading order term in the sum $\sum_{\mathcal{V}\in\Xi}\big(\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V})\big)$, and hence in $\log\langle W_\gamma\rangle_\beta$. We further note that this leading order term will exactly match the leading order term of Wilson loop observables found in, for example, [3, 5, 9].

Lemma 4.4. Let $\beta \geqslant 0$. Then,

$$\sum_{\nu \in \Lambda_{2(m-1),q}} \phi_\beta(\nu) = \ell e^{-8(m-1)\beta}.$$

Proof. Let $\nu \in \Lambda_{2(m-1)}$. Then $\phi_\beta(\nu) = e^{-8(m-1)\beta}$. By Lemma 2.4, we have $\nu(q) \neq 1$ if and only if ν is a minimal vortex around some edge $e \in \gamma$. Combining these observations, the conclusion immediately follows. ■

The next lemma is similar to Lemma 4.4 above, but instead of considering minimal vortices, consider the contribution to the right-hand side of (13) of the vortices that are as small as possible while not being minimal.

Lemma 4.5. Let $\beta \geqslant 0$. Then,

$$\sum_{\nu \in \Lambda_{4(m-1)-2,q}} \phi_\beta(\nu) = \big(6(m-1)\ell - 2\ell_c - 2\ell_b\big) e^{-4(4(m-1)-2)\beta}.$$

Proof. Let $\nu \in \Lambda$ be such that $|(\mathbf{supp}\,\nu)^+| = 4(m-1) - 2$. Then $\phi_\beta(\nu) = e^{-4(4(m-1)-2)\beta}$. By Lemma 2.5, there is $\sigma \in \Omega^1(B_N, G)$ such that $d\sigma = \nu$ and $(\mathbf{supp}\,\sigma)^+ = \{e_1, e_2\}$, where e_1 and e_2 are distinct edges and $\mathbf{supp}\,\hat\partial e_1 \cap \mathbf{supp}\,\hat\partial e_2 \neq \emptyset$. Since

$$\nu(q) = \sigma(\gamma) = \sum_{e \in \gamma} \sigma(e),$$

and $G = \mathbb{Z}_2$, it follows that

$$\nu(q) = \begin{cases} 1 & \text{if } \big|\mathbf{supp}\,\gamma \cap \{e_1, e_2\}\big| = 1 \\ 0 & \text{else.} \end{cases}$$

From this it follows that $\nu(q) \neq 1$ if and only if one of the following hold:

(i) The edges e_1 and e_2 are parallel (see Figure 1(b)), and exactly one of e_1 and e_2 are in $\mathbf{supp}\,\partial q = \mathbf{supp}\,\gamma$.
(ii) The edges e_1 and e_2 are not parallel (see Figure 1(c)), and exactly one of e_1 and e_2 are in $\mathbf{supp}\,\partial q = \mathbf{supp}\,\gamma$.

We will now count the number of vortices $\nu \in \Lambda$ with $|(\mathbf{supp}\,\nu)^+| = 4(m-1) - 2$ which correspond to each of the two cases (i) and (ii). To this end, consider first the set A_1 of all vortices $\nu \in \Lambda$ with $|(\mathbf{supp}\,\nu)^+| = 4(m-1) - 2$ that corresponds to the case (i). By the above, this set is in bijection with the set of all pairs $\{e_1, e_2\}$ of distinct parallel edges that are both in the boundary of some common plaquette, exactly one of which is in γ. For each $e \in \gamma$, there are exactly $2(m-1)$ positively oriented plaquettes that has e in its boundary. Hence, given $e \in \gamma$, there are exactly $2(m-1)$ positively oriented edges, distinct from e, that are parallel to e and are in the boundary of some plaquette that also has e in its boundary. Hence, if γ has no bottleneck edges, meaning that $\ell_b = 0$ then the cardinality of A_1 is $2(m-1)\ell$. If $\ell_b \neq 0$, then the cardinality of A_1 is $2(m-1)\ell - 2\ell_b$ as the above argument counts each pair $\{e_1, e_2\}$ corresponding the two parallel edges in a bottleneck twice. By an analogous argument, the cardinality of the of all vortices $\nu \in \Lambda$ with $|(\mathbf{supp}\,\nu)^+| = 4(m-1) - 2$ that corresponds to the case (ii) and for which exactly one of the edges e_1, e_2 is in γ is $4(m-1)\ell - 2\ell_c$. From this, the desired conclusion immediately follows. ■

The next lemma gives an upper bound on the total weight of all sufficiently large clusters with a given plaquette in its support. This lemma will be important in proving our main results, as it will provide upper bounds on the error terms that appear.

Lemma 4.6. Let $\beta > \beta_0(m)$, $k \geqslant 1$ and $p \in C_2(B_N)$. Further, let $\beta^* \in \big(\beta_0(m), \beta\big)$. Then

$$\sum_{\mathcal{V} \in \Xi_{k^+,p}} \big|\Psi_\beta(\mathcal{V})\big| \leqslant C_{\beta^*} e^{-4(\beta-\beta^*)k},$$

where C_{β^*} is defined by

$$C_{\beta^*} := \sup_{N \geqslant 1} \sum_{\mathcal{V} \in \Xi_{1^+,p}} \big|\Psi_{\beta^*}(\mathcal{V})\big| < \infty. \tag{14}$$

Proof. By Lemma 3.2, for any $\nu \in \Lambda$, we have

$$\sum_{\mathcal{V} \in \Xi \colon \mathcal{V} \ni \nu} \Psi_\beta(\mathcal{V}) \leqslant e^{|\operatorname{supp} \nu|/3} \phi_\beta(\nu),$$

and hence

$$\sum_{\mathcal{V} \in \Xi_{1^+,p}} \big|\Psi_\beta(\mathcal{V})\big| \leqslant \sum_{\nu \in \Lambda_{1^+,p}} \sum_{\mathcal{V} \in \Xi \colon \mathcal{V} \ni \nu} \big|\Psi_\beta(\mathcal{V})\big| \leqslant \sum_{\nu \in \Lambda_{1^+,p}} e^{|\operatorname{supp} \nu|/3} \phi_\beta(\nu) = \sum_{\nu \in \Lambda_{1^+,p}} e^{-(2\beta-1/3)|\operatorname{supp} \nu|}. \tag{15}$$

Using Lemma 2.3, it follows that

$$\begin{aligned} \sum_{\nu \in \Lambda_{1^+,p}} e^{-(4\beta-1/3)|\operatorname{supp} \nu|} &= \sum_{k=2(m-1)}^{\infty} \sum_{\nu \in \Lambda_{k,p}} e^{-(2\beta-1/3)|\operatorname{supp} \nu|} \\ &= \sum_{k=2(m-1)}^{\infty} |\Lambda_{k,p}| e^{-(2\beta-1/3)2k}. \end{aligned} \tag{16}$$

Combining (15) and (16) and using Lemma 2.11, we obtain

$$\sum_{\mathcal{V} \in \Xi_{1^+,p}} \big|\Psi_\beta(\mathcal{V})\big| \leqslant \sum_{k=2(m-1)}^{\infty} M^{2k-1} e^{-2(2\beta-1/3)k}.$$

In particular, this implies that $C_{\beta^*} < \infty$, and hence C_{β^*} is well defined.

For any $\mathcal{V} \in \Xi$, we have $\phi_\beta(\mathcal{V}) = e^{-2\beta|\mathcal{V}|}$ and $\Psi_\beta(\mathcal{V}) = U(\mathcal{V})\psi_\beta(\mathcal{V})$, where $U(\mathcal{V})$ does not depend on β, and hence

$$\Psi_\beta(\mathcal{V}) = e^{-2(\beta-\beta^*)|\mathcal{V}|} \Psi_{\beta^*}(\mathcal{V}).$$

Using this observation, we obtain

$$\sum_{\mathcal{V} \in \Xi_{k^+,p}} \big|\Psi_\beta(\mathcal{V})\big| \leqslant e^{-4(\beta-\beta^*)k} \sum_{\mathcal{V} \in \Xi_{k^+,p}} \big|\Psi_{\beta^*}(\mathcal{V})\big| \leqslant e^{-4(\beta-\beta^*)k} \sum_{\mathcal{V} \in \Xi_{1^+,p}} \big|\Psi_{\beta^*}(\mathcal{V})\big|.$$

This concludes the proof. ■

The next lemma is similar to Lemma 4.6 above, but instead gives an upper bound on the total weight of sufficiently large clusters that interact with a given oriented surface q.

Lemma 4.7. Let $\beta > \beta_0(m)$, and let $k \geqslant 1$. Further, let $\beta^* \in (\beta_0(m), \beta)$. Then

$$\sum_{\mathcal{V}\in\Xi_{k^+,q}} \left|\Psi_\beta(\mathcal{V})\right| \leqslant C_m C_{\beta^*} \ell \sum_{j=0}^{\infty} \max(1,j)^{m-1} e^{-4(\beta-\beta^*)\max(k,\hat{C}_m(j+1))},$$

where C_{β^*} is defined in (14), C_m is defined in (2.8), and $\hat{C}_m$ is defined in (2.10).

Proof. Note first that

$$\sum_{\mathcal{V}\in\Xi_{k^+,q}} |\Psi_\beta(\mathcal{V})| = \sum_{j=0}^{\infty} \sum_{\substack{\mathcal{V}\in\Xi_{k^+,q} \\ \text{dist}(\text{supp}\,\mathcal{V},\gamma)=j}} |\Psi_\beta(\mathcal{V})|. \tag{17}$$

Using Lemma 2.10, we can write

$$\sum_{j=0}^{\infty} \sum_{\substack{\mathcal{V}\in\Xi_{k^+,q} \\ \text{dist}(\text{supp}\,\mathcal{V},\gamma)=j}} |\Psi_\beta(\mathcal{V})| \leqslant \sum_{j=0}^{\infty} \sum_{\substack{p\in C_2(B_N)^+: \\ \text{dist}(p,\gamma)=j}} \sum_{\mathcal{V}\in\Xi_{\max(k,\hat{C}_m(j+1))^+,p}} |\Psi_\beta(\mathcal{V})|. \tag{18}$$

By using Lemma 2.8 and Lemma 4.6, the right-hand side of the previous equation can be bounded from above by

$$\sum_{j=0}^{\infty} C_m C_{\beta^*} \ell \max(1,j)^{m-1} e^{-4(\beta-\beta^*)\max(k,\hat{C}_m(j+1))}.$$

Combining this observation with (17) and (18), we obtain the desired conclusion. ■

Remark 4.8. Lemmas 4.6 and 4.7 remain valid for complex β, assuming $\text{Re}\,\beta > \beta_0(m)$. Indeed, the proofs work verbatim replacing β by $\text{Re}\,\beta$.

We are now ready to give the proof of Proposition 4.1, which gives the leading order term and error estimates for the contribution to the sum $\sum_{\mathcal{V}\in\Xi}(\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V}))$ from clusters $\mathcal{V}$ that consist of exactly one vortex. The proof has two main steps. The first step is to note that the contribution from clusters with large support is very small (using Lemma 4.7). The second step of the proof is to carefully look at the contributions of the smallest vortices (see Figure 1). Doing this, we obtain the term $\ell H_{m,\beta}(\gamma)$. Combining the two steps, we obtain the desired conclusion.

Proof of Proposition 4.1. Set $\beta^* = \beta_0 + \epsilon$. Note first that

$$\sum_{\substack{\mathcal{V}\in\Xi: \\ n(\mathcal{V})=1}} \big(\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V})\big) = \sum_{\nu\in\Lambda} U(\{\nu\})\Psi_\beta(\nu)\big(1-\rho\big(\nu(q)\big)\big) = 2\sum_{\nu\in\Lambda_{1^+,q}} \Psi_\beta(\nu) = 2\sum_{\nu\in\Lambda_{1^+,q}} \phi_\beta(\nu). \tag{19}$$

Using Lemma 2.3, we get

$$\sum_{\nu\in\Lambda_{1^+,q}} \phi_\beta(\nu) = \sum_{\nu\in\Lambda_{2(m-1),q}} \phi_\beta(\nu) + \sum_{\nu\in\Lambda_{4(m-1)-2,q}} \phi_\beta(\nu) + \sum_{\nu\in\Lambda_{4(m-1)^+,q}} \phi_\beta(\nu). \tag{20}$$

Now note that if $\mathcal{V} \in \Xi_1$, then $\Psi(\mathcal{V}) = \phi_\beta(\mathcal{V}) > 0$. Using Lemma 4.7, applied with $k = 4(m-1)$, we thus obtain

$$0 \leqslant \sum_{\nu\in\Lambda_{4(m-1)^+,q}} \phi_\beta(\nu) \leqslant \sum_{\mathcal{V}\in\Xi_{4(m-1)^+,q}} \left|\Psi_\beta(\mathcal{V})\right| \leqslant D_1 \ell e^{-16(m-1)\beta}, \tag{21}$$

where

$$D_1 := C_m C_{\beta^*} e^{16(m-1)\beta^*} \sum_{j=0}^{\infty} \max(1,j)^{m-1} e^{-4(\beta-\beta^*)\max(0,\hat{C}_m(j+1)-4(m-1))}. \tag{22}$$

We see that $D_1 = O_\beta(1)$. At the same time, by combining Lemma 4.4 and Lemma 4.5, we have

$$\begin{aligned}\sum_{\nu\in\Lambda_{2(m-1),q}} \phi_\beta(\nu) + \sum_{\nu\in\Lambda_{4(m-1)-2,q}} \phi_\beta(\nu)\\ = \ell e^{-8(m-1)\beta} + \big(6(m-1)\ell - 2\ell_c - 2\ell_b\big)e^{-4(4(m-1)-2)\beta} = \ell H_{m,\beta}(\gamma).\end{aligned} \tag{23}$$

Combining (19), (20), (21), and (23), we obtain the desired conclusion. ■

We now prove Proposition 4.2. The main idea of the proof is to first note that any cluster with $\mathcal{V}(q) = 0$ makes a zero contribution to the sum on the right-hand side of (13). Next, we make the observation that if a cluster contains at least two vortices, then it must necessarily have support at least $4(m-1)$. From this, the desired conclusion will follow by using Lemma 4.7.

Proof of Proposition 4.2. Set $\beta^* = \beta_0 + \epsilon$. Note first that given $\mathcal{V} \in \Xi$, we have $\rho\big(\mathcal{V}(q)\big) \neq 1$ if and only if $\rho\big(\mathcal{V}(q)\big) = -1$ and hence $\mathcal{V}(q) \neq 0$. This implies in particular that

$$\sum_{\substack{\mathcal{V}\in\Xi:\\ n(\mathcal{V})\geqslant 2}} \big(\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V})\big) = \sum_{\substack{\mathcal{V}\in\Xi:\\ n(\mathcal{V})\geqslant 2}} \Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q)\big)\big) = 2\sum_{\substack{\mathcal{V}\in\Xi_{1^+,q}:\\ n(\mathcal{V})\geqslant 2}} \Psi_\beta(\mathcal{V}).$$

Consequently, using Lemma 2.3, we find that

$$\sum_{\substack{\mathcal{V}\in\Xi:\\ n(\mathcal{V})\geqslant 2}} \Big|\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V})\Big| \leqslant 2\sum_{\substack{\mathcal{V}\in\Xi_{1^+,q}:\\ n(\mathcal{V})\geqslant 2}} \big|\Psi_\beta(\mathcal{V})\big| \leqslant 2\sum_{\mathcal{V}\in\Xi_{4(m-1)^+,q}} \big|\Psi_\beta(\mathcal{V})\big|. \tag{24}$$

Using Lemma 4.7, applied with $k = 4(m-1)$, we thus obtain

$$\sum_{\substack{\mathcal{V}\in\Xi:\\ n(\mathcal{V})\geqslant 2}} \Big|\Psi_\beta(\mathcal{V}) - \Psi_{\beta,q}(\mathcal{V})\Big| \leqslant 2D_1\ell e^{-16(m-1)\beta}, \tag{25}$$

where D_1 is given by (22). This concludes the proof. ■

We now prove Proposition 4.3, which follows almost immediately from Lemma 4.7.

Proof of Proposition 4.3. By Lemma 2.3, without loss of generality, we can assume that $k \geqslant 2(m-1)$.
Given $\mathcal{V} \in \Xi$, if $\rho\big(\mathcal{V}(q)\big) \neq 1$ then $\rho\big(\mathcal{V}(q)\big) = -1$ and $\mathcal{V}(q) \neq 0$. This implies in particular that

$$\sum_{\mathcal{V}\in\Xi_{k^+}} \big|\Psi(\mathcal{V}) - \Psi_q(\mathcal{V})\big| = \sum_{\mathcal{V}\in\Xi_{k^+}} \big|\Psi_\beta(\mathcal{V})\big|\big(1-\rho\big(\mathcal{V}(q)\big)\big) = 2\sum_{\mathcal{V}\in\Xi_{k^+,q}} \big|\Psi_\beta(\mathcal{V})\big|. \tag{26}$$

Applying Lemma 4.7, the desired conclusion immediately follows. ■

5 Proof of Theorem 1.1

This section proves Theorem 1.1. First, however, we need two additional lemmas. These are then combined with results from previous sections to yield a proof of Theorem 1.1.

To simplify the notation in this section, we fix some $p_0 \in C_2(B_1)^+$. Note that $C_2(B_1)$ is contained in $C_2(B_j)$ for every $j \geqslant 1$. For $\mathbf{Re}\,\beta > \beta_0(m)$ and $N \geqslant 1$, we define

$$F_N(\beta) := \sum_{\mathcal{V}\in\Xi_{1^+,p_0}(B_N)} \frac{\Psi_\beta(\mathcal{V})}{|\mathcal{V}|}.$$

Lemma 5.1. Let $\mathbf{Re}\,\beta > \beta_0(m)$. Then

$$\lim_{N\to\infty} \left| \frac{\log Z_{\beta,N}}{|C_2(B_N)^+|} - F_N(\beta) \right| = 0$$

locally uniformly in β.

Proof. Write $\Xi_{1^+,p_0} = \Xi_{1^+,p_0}(B_N)$. By Lemma 3.3 (using also Lemma 2.3), we have

$$\log Z_{\beta,N} = \sum_{\mathcal{V}\in\Xi} \Psi_\beta(\mathcal{V}) = \sum_{p\in C_2(B_N)^+} \sum_{\mathcal{V}\in\Xi_{2(m-1)^+,p}} \frac{\Psi_\beta(\mathcal{V})}{|\mathcal{V}|}.$$

Let $k \geqslant 2(m-1)$ be arbitrary. Without loss of generality, can assume that N is large enough to ensure that $\mathbf{dist}(p_0, \partial B_N) > k$. Then, by Lemma 4.6, using also Remark 4.8, it follows that for any $p \in C_2(B_N)$, we have

$$\sum_{\mathcal{V}\in\Xi_{k^+,p}} \frac{|\Psi_\beta(\mathcal{V})|}{|\mathcal{V}|} \leqslant \frac{1}{k} \sum_{\mathcal{V}\in\Xi_{k^+,p}} |\Psi_\beta(\mathcal{V})| \leqslant \varepsilon(\beta, m, k) := C_{\beta^*} k^{-1} e^{-4(\mathbf{Re}\,\beta-\beta^*)k}. \tag{27}$$

Note that

$$\lim_{k\to\infty} \varepsilon(\beta, m, k) = 0$$

uniformly on compact subsets of the halfplane $\mathbf{Re}\,\beta > \beta_0(m)$. Now fix some $p \in C_2(B_N)^+$ with $\mathbf{dist}(p, \partial B_N) > k$. Since $\mathbf{dist}(p_0, \partial B_N) > k$ and $\mathbf{dist}(p, \partial B_N) > k$, for each $i < k$ there is a bijection $\Xi_{i,p_0} \to \Xi_{i,p}$ which maps each $\mathcal{V} \in \Xi_{i,p}$ to a translation of $\mathcal{V}$ in $\Xi_{i,p}$. Consequently,

$$\sum_{\mathcal{V}\in\Xi_{,k^-,p}} \frac{\Psi_\beta(\mathcal{V})}{|\mathcal{V}|} = \sum_{\mathcal{V}\in\Xi_{k^-,p_0}} \frac{\Psi_\beta(\mathcal{V})}{|\mathcal{V}|},$$

and hence

$$\left| \sum_{\mathcal{V}\in\Xi_{1^+,p}} \frac{\Psi_\beta(\mathcal{V})}{|\mathcal{V}|} - F_N(\beta) \right| = \left| \sum_{\mathcal{V}\in\Xi_{1^+,p}} \frac{\Psi_\beta(\mathcal{V})}{|\mathcal{V}|} - \sum_{\mathcal{V}\in\Xi_{1^+,p_0}} \frac{\Psi_\beta(\mathcal{V})}{|\mathcal{V}|} \right| \leqslant 2\varepsilon(\beta, m, 1).$$

Finally, we note that there is a constant C'_m such that

$$\left| \{ p \in C_2(B_N)^+ : \ \mathbf{dist}(p, \partial B_N) \leqslant k \} \right| \leqslant C'_m k N^{m-1}. \tag{28}$$

We now combine the above observations as follows. By (27) and (28), we have

$$\left| \log Z_{\beta,N} - \sum_{\substack{p\in C_2(B_N)^+ : \\ \mathbf{dist}(p,\partial B_N) > k}} \sum_{\mathcal{V}\in\Xi_{1^+,p}} \frac{\Psi_\beta(\mathcal{V})}{|\mathcal{V}|} \right| \leqslant C'_m k N^{m-1} \varepsilon(\beta, m, 1).$$

Using (27), it follows that

$$\left| \log Z_{\beta,N} - \sum_{\substack{p\in C_2(B_N)^+ : \\ \mathbf{dist}(p,\partial B_N) > k}} F_N(\beta) \right| \leqslant 2\varepsilon(\beta, m, k) \left| C_2(B_N)^+ \right| + C'_m k N^{m-1} \varepsilon(\beta, m, 1).$$

Again using (27) and (28), we get

$$\left|\log Z_{\beta,N} - |C_2(B_N)^+|F_N(\beta)\right| \leqslant 2\varepsilon(\beta,m,k)|C_2(B_N)^+| + 2C'_m k N^{m-1}\varepsilon(\beta,m,1).$$

Dividing both sides by $|C_2(B_N)^+|$ and letting $N\to\infty$, we finally obtain

$$\lim_{N\to\infty}\left|\frac{\log Z_{\beta,N}}{|C_2(B_N)^+|} - F_N(\beta)\right| \leqslant 2\varepsilon(\beta,m,k)$$

and this bound is decreasing in $\operatorname{Re}\beta > \beta_0$. Since k was arbitrary, the desired conclusion follows. ■

Lemma 5.2. Let $\operatorname{Re}\beta > \beta_0(m)$. Then $F_N(\beta)$ converges as $N\to\infty$ locally uniformly.

Proof. Let $k \geqslant 1$. Then we can write

$$F_N(\beta) = \sum_{\mathcal{V}\in\Xi_{,k^-,p_0}(B_N)} \frac{\Psi_\beta(\mathcal{V})}{|\mathcal{V}|} + \sum_{\mathcal{V}\in\Xi_{k^+,p_0}(B_N)} \frac{\Psi_\beta(\mathcal{V})}{|\mathcal{V}|}.$$

Note that if $j \leqslant N$, then we have $\Xi_{j,p_0}(B_N) = \Xi_{j,p_0}(B_j)$. By Lemma 4.6 and Remark 4.8, we have

$$\left|\sum_{\mathcal{V}\in\Xi_{k^+,p_0}(B_N)} \frac{\Psi_\beta(\mathcal{V})}{|\mathcal{V}|}\right| \leqslant \frac{1}{k}\sum_{\mathcal{V}\in\Xi_{k^+,p_0}(B_N)} |\Psi_\beta(\mathcal{V})| \leqslant C_{\beta^*}k^{-1}e^{4(\operatorname{Re}\beta-\beta^*)k}.$$

Note in particular that this upper bound does not depend on N, is decreasing in $\operatorname{Re}\beta$, and tends to zero as $k\to\infty$. From this the desired conclusion immediately follows. ■

Proof of Theorem 1.1. As in Lemma 5.2, write $F_N(\beta) = \sum_{\mathcal{V}\in\Xi_{1^+,p_0}(B_N)} \Psi_\beta(\mathcal{V})/|\mathcal{V}|$ which for each N, by Lemma 3.3, is analytic in the half plane $\operatorname{Re}\beta > \beta_0(m)$. By Lemma 5.2 F_N converges locally uniformly as $N\to\infty$ to a limiting function which is also analytic. By Lemma 5.1, $\log Z_{\beta,N}$ also converges locally uniformly to the same limit. On the other hand, if $q = 1\cdot p$ then, using Lemma 4.4, we have

$$\sum_{\substack{\mathcal{V}\in\Xi_{2(m-1),p}(B_N):\\ n(\mathcal{V})=1}} \phi_\beta(\mathcal{V}) = \sum_{\nu\in\Lambda_{2(m-1),p}(B_N)} \phi_\beta(\nu) = \sum_{\nu\in\Xi_{2(m-1),q}} \phi_\beta(\nu) = 4e^{-8(m-1)\beta}, \tag{29}$$

for all N sufficiently large. Similarly, using Lemma 4.5, we have

$$\sum_{\substack{\mathcal{V}\in\Xi_{1,4(m-1)-2,p}(B_N):\\ n(\mathcal{V})=1}} \phi_\beta(\mathcal{V}) = \sum_{\nu\in\Lambda_{4(m-1)-2,q}(B_N)} \phi_\beta(\mathcal{V}) = \big(24(m-1)-16\big)e^{-4(4(m-1)-2)\beta}. \tag{30}$$

At the same time, by Lemma 4.6, and Remark 4.8 we have

$$\sum_{\mathcal{V}\in\Xi_{4(m-1)^+,p}(B_N)} |\Psi_\beta(\mathcal{V})| \leqslant C_{\beta^*}e^{-4(\beta-\beta^*)k} \tag{31}$$

for all N. We conclude by combining (29), (30), and (31). ■

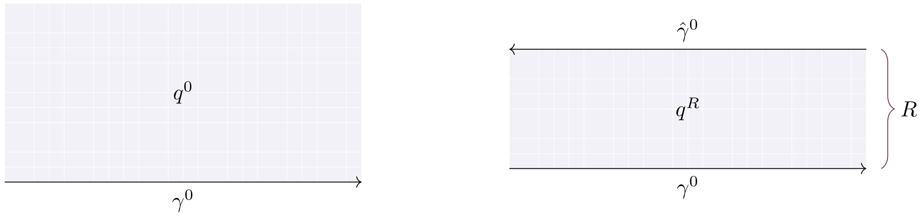
6 Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2, which will essentially follow immediately by combining Proposition 4.1 and Proposition 4.2.

Proof of Theorem 1.2. By combining Proposition 4.1 and Proposition 4.2, we obtain

$$-3D_1\ell e^{-16(m-1)\beta} \leqslant \ell H(\ell) + \sum_{\mathcal{V}\in\Xi}\big(\Psi_{\beta,q}(\mathcal{V}) - \Psi_\beta(\mathcal{V})\big) \leqslant 2D_1\ell e^{-16\beta(m-1)}.$$

Using Proposition 3.5, the proof is complete. ■



(A) The path γ^0 and the oriented surface q^0 (purple plaquettes).

(B) The 1-chain $\gamma^R = \gamma^0 + \hat{\gamma}^0$ and the oriented surface q^R (purple plaquettes).

Fig. 2. In the figures above, we illustrate γ^0 and γ^R, as well as the oriented surfaces q^0 and q^R.

(A) The loop γ.

(B) The path $\gamma_{c,j}$.

Fig. 3. In the figures above, we draw a loop γ and a corresponding path $\gamma_{c,j}$.

7 Proof of Theorem 1.4 and Theorem 1.7

In this section, we state and prove Proposition 7.3 and Proposition 7.4, which are the more technical versions of the two main results Theorem 1.4 and Theorem 1.7. The main tool in the proofs of these results is Lemma 7.1 below.

We now introduce some additional notation. Fix a mapping $\nu \mapsto \sigma^\nu$ from Λ to $\Omega^1(B_N, G)$ which satisfies the following:

1. For each $\nu \in \Lambda$, we have $d\sigma^\nu = \nu$.
2. For each $\nu \in \Lambda$, the support of σ^ν is contained in the smallest box B_ν that contains the support of ν.
3. If τ is a translation or rotation of the lattice with the property that $\operatorname{supp} \nu \circ \tau \subseteq C_2(B_N)$, then $\sigma^{\nu \circ \tau} = \sigma^\nu \circ \tau$.

Note that such a mapping exists by Lemma 2.2.

Given $\mathcal{V} \in \Xi$, let $E_{\mathcal{V}} = \bigcup_{\nu \in \mathcal{V}} \operatorname{supp} \sigma^\nu$. Given an edge $e \in C_1(B_N)^+$ and $m \geqslant 1$, let

$$\Xi_{m,e} := \{\mathcal{V} \in \Xi_m : e \in E_{\mathcal{V}}\}.$$

Define Ξ_e and $\Xi_{m+,e}$ as before. Finally, we let $\Xi_{m-,e} = \Xi_{1^+,e} \setminus \Xi_{,m+,e}$.

Fix any $e_0 \in C_1(B_N)$ and let γ^0 be the bi-infinite line through e_0. Given $R \geqslant 1$, let $\hat{\gamma}^0$ be an axis-parallel translation of $-\gamma^0$ such that the distance between γ^0 and $\hat{\gamma}^0$ is R, and let $\gamma^R = \gamma^0 + \hat{\gamma}^0$. Let q^R be the bi-infinite strip with boundary γ^R, and let q^0 be a half-plane with boundary γ^0 (see Figure 2). We use the same notations for the restrictions of γ^0, γ^R, q^0, and q^R to $C_1(B_N)$ and $C_2(B_N)$, respectively.

Given a rectangular loop γ and $j \geqslant 1$, let $\gamma_{c,j}$ be the restriction of γ to the set of edges that are on distance at most j from a corner of γ (see Figure 3). Note that if γ is a rectangular loop, then $|\gamma_{c,j}| \leqslant 8j$.

We now describe the main idea of the proofs of Theorem 1.4 and Theorem 1.7. From Proposition 3.5, we have an expansion of $\log \mathbb{E}_{\beta,N}[W_\gamma]$ as a convergent infinite sum over vortex clusters. From the estimates in Section 4, it follows that most of the contribution to this sum comes from vortex clusters of finite (fixed) size. In this sum, there are many vortex clusters that either translations or rotations of each other. For this reason, we pick any vortex cluster $\mathcal{V}$. Then $\mathcal{V}(q)$ depends on q only through the geometry of $\gamma = \partial q$ close to the support of $\mathcal{V}$. As $\max(R_n, T_n) \to \infty$, close to the support of $\mathcal{V}$ the rectangular loop γ_{R_n,T_n} will look like a straight path (or as two straight paths if only one of the sides of γ_n grows with n). At the same time, the number of translations and rotations of $\mathcal{V}$ that affect $\mathcal{V}(q)$ grows linearly with γ. The main work of the proof is to show that vortex clusters whose support is close to the corners of γ_{R_n,T_n} has little effect on the sum in the limit. This is done in Lemma 7.1. Together with the technical Lemma 7.2, we then give proofs of Proposition 7.3 and Proposition 7.4, which are the more technical versions of the two main results Theorem 1.4 and Theorem 1.7.

Lemma 7.1. Let $\beta > \beta_0(m)$. Let γ be a rectangular loop with axis-parallel sides with lengths R and T, respectively, where $R \leqslant T$. Let q be the unique flat oriented surface with $\partial q = \gamma$. Let $k \leqslant R$ and $k' \leqslant T$. Assume that N is large enough to ensure that $\operatorname{supp} \gamma \subset C_1(B_{N-k})$. Then,

$$\Big| \sum_{\mathcal{V} \in \Xi_{k'-}} \Psi_\beta(\mathcal{V}) 1 - \rho\big(\mathcal{V}(q)\big) \tag{32}$$

$$- |\gamma| \sum_{\mathcal{V} \in \Xi_{k-,e_0}} \big|E_{\mathcal{V}} \cap \operatorname{supp} \gamma^0\big|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q^0)\big)\big) \tag{33}$$

$$- 2R \sum_{\mathcal{V} \in \Xi_{k'-,e_0} \smallsetminus \Xi_{k-,e_0}} \big|E_{\mathcal{V}} \cap \operatorname{supp} \gamma^0\big|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q^0)\big)\big) \tag{34}$$

$$- 2T \sum_{\mathcal{V} \in \Xi_{k'-,e_0} \smallsetminus \Xi_{k-,e_0}} \big|E_{\mathcal{V}} \cap \operatorname{supp} \gamma^R\big|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q^R)\big)\big) \Big| \tag{35}$$

$$\leqslant 4 \sum_{j=0}^{k'-1} |\gamma_{c,j}| \sum_{\mathcal{V} \in \Xi_{j,e_0}} \big|\Psi_\beta(\mathcal{V})\big|.$$

Before proving Lemma 7.1, we explain the interpretation of the sums in (32)–(35). The sum in (32) describes the contribution to $\sum_{\mathcal{V} \in \Xi} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q)\big)\big)$ from clusters whose support is smaller than k'. The sums in (33) and (34) essentially describe the contribution to this sum of the vortices that are small enough, and at the same time far enough from the corners of γ, to interact with exactly one of the sides of γ. Finally, the sum in (33) describes the contribution of the vortices that are small enough not be able to interact with two different sides of the rectangle on distance T, but possibly large enough to interact with two of the opposite sides of the rectangle on distance R. The right-hand side of the same equation then corresponds to the contribution to $\sum_{\mathcal{V} \in \Xi} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q)\big)\big)$ from vortices which are very close to one of the corners of the loop.

Proof of Lemma 7.1. Since γ is a rectangular loop and q is a flat oriented surface with boundary γ, we have $\operatorname{supp} q \subset C_2(B_{N-k})$. Let γ_R be the restriction of γ to the two sides of γ that are on distance R, and let $\gamma_T = \gamma - \gamma_R$ be the restriction of γ to the two sides of the rectangle that are on distance T. Note that $|\operatorname{supp} \gamma_R| = 2T$ and that $|\operatorname{supp} \gamma_T| = 2R$.

Fix $j \in \{1, 2, \ldots, k'-1\}$. Then,

$$\begin{aligned} \sum_{\mathcal{V} \in \Xi_j} \Psi_\beta(\mathcal{V}) 1 - \rho\big(\mathcal{V}(q)\big) &= \sum_{e \in \gamma} \sum_{\mathcal{V} \in \Xi_{j,e}} |E_{\mathcal{V}} \cap \operatorname{supp} \gamma|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q)\big)\big) \\ &= \sum_{e \in \gamma_{c,k'}} \sum_{\mathcal{V} \in \Xi_{j,e}} |E_{\mathcal{V}} \cap \operatorname{supp} \gamma|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q)\big)\big) \\ &\quad + \sum_{e \in \gamma \smallsetminus \gamma_{c,k'}} \sum_{\mathcal{V} \in \Xi_{j,e}} \big|E_{\mathcal{V}} \cap \operatorname{supp} \gamma\big|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q)\big)\big). \end{aligned} \tag{36}$$

Since $j < k' \leqslant T$,for any $e \in \gamma_R \smallsetminus \gamma_{c,j}$, we have

$$\begin{aligned} &\sum_{\mathcal{V} \in \Xi_{j,e}} \big|E_{\mathcal{V}} \cap \operatorname{supp} \gamma\big|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q)\big)\big) \\ &\quad = \sum_{\mathcal{V} \in \Xi_{j,e_0}} \big|E_{\mathcal{V}} \cap \operatorname{supp} \gamma^R\big|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q^R)\big)\big), \end{aligned} \tag{37}$$

and for any $e \in \gamma_T \smallsetminus \gamma_{c,j}$, we have

$$\begin{aligned} &\sum_{\mathcal{V} \in \Xi_{j,e}} \big|E_{\mathcal{V}} \cap \operatorname{supp} \gamma\big|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q)\big)\big) \\ &\quad = \sum_{\mathcal{V} \in \Xi_{j,e_0}} \big|E_{\mathcal{V}} \cap \operatorname{supp} \gamma^R\big|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q^0)\big)\big). \end{aligned} \tag{38}$$

Since $k \leqslant R$, for any $\mathcal{V} \in \Xi_{k-,e_0}$ we have

$$\left|E_{\mathcal{V}} \cap \mathbf{supp}\,\gamma^R\right| = \left|E_{\mathcal{V}} \cap \mathbf{supp}\,\gamma^0\right| \quad \text{and} \quad \mathcal{V}(q^R) = \mathcal{V}(q^0). \tag{39}$$

Combining the above equations, we obtain the desired conclusion. ■

Lemma 7.2. For any $k \geqslant 1$ and any $e \in C_1(B_N)^+$, we have

$$\sum_{\mathcal{V} \in \Xi_{k+,e}} \left|\Psi_\beta(\mathcal{V})\right| \leqslant 2C_{\beta^*}\binom{m}{2}\sum_{j=k}^{\infty}(j+1)^m e^{-4(\beta-\beta^*)j} \tag{40}$$

and

$$\sum_{j=0}^{k-1} j \sum_{\mathcal{V} \in \Xi_{j,e}} \left|\Psi_\beta(\mathcal{V})\right| \leqslant 2C_{\beta^*}\binom{m}{2}\sum_{j=2(m-1)}^{k-1} j \sum_{i=j}^{\infty}(i+1)^m e^{-4(\beta-\beta^*)i}. \tag{41}$$

Proof. Let $j \geqslant k$, and let P_j be the set of all positively oriented plaquettes that are at distance at most j from e. If $\mathcal{V} \in \Xi_{j,e}$, then we must have $\mathbf{supp}\,\mathcal{V} \cap P_j \neq \emptyset$. From this it follows that

$$\sum_{\mathcal{V} \in \Xi_{k+,e}} \left|\Psi_\beta(\mathcal{V})\right| \leqslant \sum_{j=k}^{\infty}\sum_{p \in P_j}\sum_{\mathcal{V} \in \Xi_{j+,p}} \left|\Psi_\beta(\mathcal{V})\right|.$$

Note that $|P_j| \leqslant \binom{m}{2}2(j+1)^m$. Using Lemma 2.8 and Lemma 4.6, we thus obtain (40). Finally, using first Lemma 2.3, we note that.

$$\sum_{j=0}^{k-1} j \sum_{\mathcal{V} \in \Xi_{j,e}} \left|\Psi_\beta(\mathcal{V})\right| \leqslant \sum_{j=2(m-1)}^{k-1} j \sum_{\mathcal{V} \in \Xi_{j+,e}} \left|\Psi_\beta(\mathcal{V})\right|.$$

Using (40), we obtain (41) as desired. This concludes the proof. ■

We now state and prove Proposition 7.3 and Proposition 7.4, which are the more technical versions of the two main results Theorem 1.4 and Theorem 1.7.

Proposition 7.3. Let $\beta > \beta_0(m)$. Let $(R_n)_{n\geqslant 1}$ and $(T_n)_{n\geqslant 1}$ be non-decreasing sequences of positive integers with $\mathbf{lim}_{n\to\infty}\,\mathbf{min}(R_n, T_n) = \infty$. For each $n \geqslant 1$, let γ_n be a rectangular loop with axis-parallel sides with lengths R_n and T_n, respectively. Then the limit $\mathbf{lim}_{n\to\infty} -\mathbf{log}\langle W_{\gamma_n}\rangle_\beta/|\gamma_n|$ exists and is given by

$$\hat{V}_\beta := V_\beta/2 := \lim_{N\to\infty} \sum_{\mathcal{V} \in \Xi_{1+,e_0}} \left|E_{\mathcal{V}} \cap \mathbf{supp}\,\gamma^0\right|^{-1}\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q^0)\big)\big).$$

Moreover, for any $n \geqslant 1$, we have

$$\left|\frac{-\mathbf{log}\langle W_{\gamma_n}\rangle_\beta}{|\gamma_n|} - \hat{V}_\beta\right| \leqslant 32m^2|\gamma_n|^{-1}\sum_{j=2(m-1)}^{\infty}(3M)^{2j-1}(j+1)^{m+1}2^{j/(2(m-1))}e^{-4\beta j}$$
$$+ 4m^2 \sum_{j=\mathbf{min}(R_n,T_n)}^{\infty}(3M)^{2j-1}(j+1)^m 2^{j/(2(m-1))}e^{-4\beta j}.$$

Proof. Note that for any $k \geqslant 1$ and N large enough to ensure that $\mathbf{dist}(e_0, \partial B_N) > k$, we have $\Xi_{,k,e_0}(B_N) = \Xi_{k,e_0}(B_k)$. By Lemma 7.2, it follows that the sum

$$\hat{V}_{\beta,N} := \sum_{\mathcal{V} \in \Xi_{1^+,e_0}(B_N)} \big|E_{\mathcal{V}} \cap \operatorname{supp} \gamma^0\big|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q^0)\big)\big)$$

is well defined and absolute convergent, uniformly in N, and hence $\hat{V}_\beta$ is well defined.

Fix $n \geqslant 1$. Let q_n be the unique 2-form with $\partial q_n = \gamma_n$ that minimizes $|\operatorname{supp} q_n|$. By Proposition 3.5, we have

$$-\log \mathbb{E}_{\beta,N}[W_{\gamma_n}] = \sum_{\mathcal{V} \in \Xi} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q_n)\big)\big) = \sum_{j=1}^{\infty} \sum_{\mathcal{V} \in \Xi_j} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q_n)\big)\big).$$

Let $(k_n)_{n \geqslant 1}$ be a sequence of non-negative integers such that for each $n \geqslant 1$, $k_n \leqslant \min(R_n, T_n)$, and $\lim_{n\to\infty} k_n/|\gamma_n| = 0$. Then, for each $n \geqslant 1$, by applying Lemma 7.1 with $k = k' = k_n$, we obtain

$$\big|-\log \mathbb{E}_{\beta,N}[W_{\gamma_n}] - |\gamma|\hat{V}_{\beta,N}\big| \leqslant 4 \sum_{j=1}^{k_n - 1} |\gamma_{c,j}| \sum_{\mathcal{V} \in \Xi_{j,e_0}} \big|\Psi_\beta(\mathcal{V})\big| + 4|\gamma| \sum_{\mathcal{V} \in \Xi_{k_n+,e_0}} \big|\Psi_\beta(\mathcal{V})\big|.$$

Using Lemma 2.3, we note that for any $j \leqslant 2(m-1)$, we have $\Xi_{j-,e_0} = \emptyset$. Also, we note that since γ_n is rectangular for each $n \geqslant 1$, we have $|\gamma_{c,j}| \leqslant 8j$ for each $j \geqslant 1$. Using Lemma 7.2, we thus obtain

$$\begin{aligned}
&\left| \frac{-\log \mathbb{E}_{\beta,N}[W_{\gamma_n}]}{|\gamma_n|} - \sum_{\mathcal{V} \in \Xi_{1^+,e_0}} \big|E_{\mathcal{V}} \cap \operatorname{supp} \gamma^0\big|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q^0)\big)\big) \right| \\
&\leqslant 4|\gamma_n|^{-1} \sum_{j=2(m-1)}^{\infty} |\gamma_{c,j}| \sum_{\mathcal{V} \in \Xi_{j,e_0}} \big|\Psi_\beta(\mathcal{V})\big| + 4 \sum_{\mathcal{V} \in \Xi_{k_n+,e_0}} \big|\Psi_\beta(\mathcal{V})\big| \\
&\leqslant 32|\gamma_n|^{-1} \sum_{j=2(m-1)}^{\infty} j \sum_{i=j}^{\infty} \binom{m}{2} 2(i+1)^m C_{\beta^*} e^{-4(\beta-\beta^*)i} \\
&\quad + 4 \sum_{j=k_n}^{\infty} \binom{m}{2} 2(j+1)^m C_{\beta^*} e^{-4(\beta-\beta^*)j}.
\end{aligned}$$

Letting first N and then n tend to infinity, the desired conclusion immediately follows. ■

Proposition 7.4. Let $\beta > \beta_0(m)$. Let $R \geqslant 1$, and let $(T_n)_{n \geqslant 1}$ be a non-decreasing sequence of positive integers with $T_n \geqslant R$ and $\lim_{n\to\infty} T_n = \infty$. For each $n \geqslant 1$, let γ_n be a rectangular loop with axis-parallel sides with lengths R and T_n, respectively. Then the limit $\lim_{n\to\infty} -\log\langle W_{\gamma_n}\rangle_\beta/|\gamma_n|$ exists and is given by

$$\begin{aligned}
\hat{V}_\beta(R) := V_\beta(R)/2 := \lim_{N\to\infty} &\sum_{\mathcal{V} \in \Xi_{R-,e_0}} \big|E_{\mathcal{V}} \cap \operatorname{supp} \gamma^0\big|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q^0)\big)\big) \\
&+ \sum_{\mathcal{V} \in \Xi_{R+,e_0}} \big|E_{\mathcal{V}} \cap \operatorname{supp} \gamma^R\big|^{-1} \Psi_\beta(\mathcal{V})\big(1 - \rho\big(\mathcal{V}(q^R)\big)\big).
\end{aligned}$$

Moreover, for any $n \geqslant 1$, we have

$$\begin{aligned}
\left| \frac{-\log\langle W_{\gamma_n}\rangle_\beta}{|\gamma_n|} - \hat{V}_\beta(R) \right| \leqslant &|\gamma_n|^{-1} \left(64 C_{\beta^*} \binom{m}{2} \sum_{j=2(m-1)}^{k-1} j \sum_{i=j}^{\infty} (i+1)^m e^{-4(\beta-\beta^*)i} \right. \\
&\left. + 16R\, C_{\beta^*} \binom{m}{2} \sum_{j=R}^{\infty} (j+1)^m e^{-4(\beta-\beta^*)j} \right).
\end{aligned} \tag{42}$$

Finally, we have

$$\left|\hat{V}_\beta(R) - \hat{V}_\beta\right| \leqslant 4\sum_{j=R}^{\infty}(3M)^{2j-1}(j+1)^m 2^j e^{-4\beta j} \tag{43}$$

and hence $\mathbf{lim}_{R\to\infty}\hat{V}_\beta(R) = \hat{V}_\beta$.

Proof. Note that for any $k \geqslant 1$ and N large enough to ensure that $\mathbf{dist}(e_0, \partial B_N) > k$, we have $\Xi_{1^+,k,e_0}(B_N) = \Xi_{1^+,k,e_0}(B_k)$. By Lemma 7.2, it follows that the sum

$$\hat{V}_{\beta,N}(R) := \sum_{\mathcal{V}\in\Xi_{R-,e_0}} \left|E_{\mathcal{V}} \cap \mathbf{supp}\,\gamma^0\right|^{-1}\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q^0)\big)\big) + \sum_{\mathcal{V}\in\Xi_{R+,e_0}} \left|E_{\mathcal{V}} \cap \mathbf{supp}\,\gamma^R\right|^{-1}\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q^R)\big)\big)$$

is well defined and absolute convergent, uniformly in N, and hence $\hat{V}_\beta(R) = \mathbf{lim}_{N\to\infty} V_{\beta,N}(R)$ is well defined.

Fix $n \geqslant 1$. By Proposition 3.5, we have

$$-\log\mathbb{E}_{\beta,N}[W_{\gamma_n}] = \sum_{\mathcal{V}\in\Xi}\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q_n)\big) = \sum_{j=1}^{\infty}\sum_{\mathcal{V}\in\Xi_j}\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q_n)\big)\big).$$

Consequently,

$$\begin{aligned}
&\left|-\log\mathbb{E}_{\beta,N}[W_{\gamma_n}] - |\gamma_n|\hat{V}_\beta(R)\right| \\
&\leqslant \sum_{\mathcal{V}\in\Xi_{T_n+}}\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q)\big)\big) + \left|\sum_{\mathcal{V}\in\Xi_{T_n-}}\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q)\big)\big) - |\gamma_n|\hat{V}_\beta(R)\right| \\
&\leqslant \sum_{\mathcal{V}\in\Xi_{T_n+}}\left|\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q)\big)\big)\right| \\
&\quad + \Bigg|\sum_{\mathcal{V}\in\Xi_{T_n-}}\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q)\big)\big) - |\gamma_n|\sum_{\mathcal{V}\in\Xi_{R-,e_0}}\left|E_{\mathcal{V}}\cap\mathbf{supp}\,\gamma^0\right|^{-1}\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q^0)\big)\big) \\
&\qquad - 2R\sum_{\mathcal{V}\in\Xi_{T_n-,e_0}\smallsetminus\Xi_{R-,e_0}}\left|E_{\mathcal{V}}\cap\mathbf{supp}\,\gamma^0\right|^{-1}\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q^0)\big)\big) \\
&\qquad - 2T_n\sum_{\mathcal{V}\in\Xi_{T_n-,e_0}\smallsetminus\Xi_{R-,e_0}}\left|E_{\mathcal{V}}\cap\mathbf{supp}\,\gamma^R\right|^{-1}\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q^R)\big)\big)\Bigg| \\
&\quad + |\gamma_n|\sum_{\mathcal{V}\in\Xi_{T_n+,e_0}\smallsetminus\Xi_{R-,e_0}}\left|E_{\mathcal{V}}\cap\mathbf{supp}\,\gamma^R\right|^{-1}\left|\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q^R)\big)\big)\right| \\
&\quad + 2R\sum_{\mathcal{V}\in\Xi_{T_n-,e_0}\smallsetminus\Xi_{R-,e_0}}\left|E_{\mathcal{V}}\cap\mathbf{supp}\,\gamma^R\right|^{-1}\left|\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q^R)\big)\big)\right| \\
&\quad + 2R\sum_{\mathcal{V}\in\Xi_{T_n-,e_0}\smallsetminus\Xi_{R-,e_0}}\left|E_{\mathcal{V}}\cap\mathbf{supp}\,\gamma^0\right|^{-1}\left|\Psi_\beta(\mathcal{V})\big(1-\rho\big(\mathcal{V}(q^0)\big)\big)\right|.
\end{aligned}$$

By applying Lemma 7.1 with $k = R$ and $k' = T_n$, we obtain

$$\begin{aligned}
&\left|-\log\mathbb{E}_{\beta,N}[W_{\gamma_n}] - |\gamma_n|\hat{V}_\beta(R)\right| \\
&\leqslant 4\sum_{j=0}^{\infty}|\gamma_{c,j}|\sum_{\mathcal{V}\in\Xi_{j,e_0}}\left|\Psi_\beta(\mathcal{V})\right| + 4|\gamma_n|\sum_{\mathcal{V}\in\Xi_{T_n+,e_0}}\left|\Psi_\beta(\mathcal{V})\right| + 8R\sum_{\mathcal{V}\in\Xi_{R+,e_0}}\left|\Psi_\beta(\mathcal{V})\right|.
\end{aligned}$$

Since γ_n is rectangular for each $n \geqslant 1$, we have $|\gamma_{c,j}| \leqslant 8j$ for each $j \geqslant 1$. Using Lemma 7.2, we thus obtain

$$\begin{aligned}
&\Big| -\log \mathbb{E}_{\beta,N}[W_{\gamma_n}] - |\gamma_n| \hat{V}_\beta(R) \Big| \\
&\leqslant 64 C_{\beta^*} \binom{m}{2} \sum_{j=2(m-1)}^{k-1} j \sum_{i=j}^{\infty} (i+1)^m e^{-4(\beta-\beta^*)i} + 8 C_{\beta^*} |\gamma_n| \binom{m}{2} \sum_{j=T_n}^{\infty} (j+1)^m e^{-4(\beta-\beta^*)j} \\
&\quad + 16 R\, C_{\beta^*} \binom{m}{2} \sum_{j=R}^{\infty} (j+1)^m e^{-4(\beta-\beta^*)j}.
\end{aligned}$$

Letting first N and then n tend to infinity, this completes the proof of (42).

To see that (43) holds, we first note that

$$\begin{aligned}
&\Big| \hat{V}_{\beta,N} - \hat{V}_{\beta,N}(R) \Big| \\
&\leqslant \Big| \sum_{\mathcal{V} \in \Xi_{R+,e_0}} \big| E_{\mathcal{V}} \cap \operatorname{supp} \gamma^0 \big|^{-1} \Psi_\beta(\mathcal{V}) \big(1 - \rho(\mathcal{V}(q^0))\big) - \sum_{\mathcal{V} \in \Xi_{R+,e_0}} \big| E_{\mathcal{V}} \cap \operatorname{supp} \gamma^R \big|^{-1} \Psi_\beta(\mathcal{V}) \big(1 - \rho(\mathcal{V}(q^R))\big) \Big| \\
&\leqslant 4 \sum_{\mathcal{V} \in \Xi_{R+,e_0}} \big| \Psi_\beta(\mathcal{V}) \big|.
\end{aligned}$$

Using Lemma 7.2, we thus obtain

$$\Big| \hat{V}_{\beta,N} - \hat{V}_{\beta,N}(R) \Big| \leqslant 8 C_{\beta^*} \binom{m}{2} \sum_{j=R}^{\infty} (j+1)^m e^{-4(\beta-\beta^*)j}.$$

Letting first N and then n tend to infinity, the desired conclusion immediately follows. ■

Proof of Theorem 1.4. By Proposition 7.4 applied with $T_n = n$, $T_n = R$, the limit

$$\lim_{T\to\infty} -\frac{1}{T} \log \langle W_{\gamma_{R,T}} \rangle_\beta = \lim_{T\to\infty} \lim_{N\to\infty} -\frac{1}{T} \log \mathbb{E}_{\beta,N}[W_{\gamma_n}] = \lim_{n\to\infty} \lim_{N\to\infty} -\frac{2}{|\gamma_n|} \log \mathbb{E}_{\beta,N}[W_{\gamma_n}]$$

exists and is equal to $2V_\beta(R)$. Using Theorem 1.2, the desired conclusion follows. ■

Proof of Theorem 1.7. By Proposition 7.3 applied with $R_n = Hn$ and $T_n = Ln$, the limit

$$\lim_{n\to\infty} -\frac{1}{R_n + T_n} \log \langle W_{\gamma_n} \rangle_\beta = \lim_{n\to\infty} \lim_{N\to\infty} -\frac{2}{|\gamma_n|} \log \mathbb{E}[W_{\gamma_n}]_{N,\beta}.$$

exists and is equal to $2\hat{V}_\beta$. We conclude using Theorem 1.2. ■

Funding

M.P.F. acknowledges support from the Ruth and Nils-Erik Stenbäck Foundation. F.V. acknowledges support from the Knut and Alice Wallenberg Foundation and the Swedish Research Council.

Acknowledgments

We thank the referee for their careful reading and for providing many very useful comments and suggestions. We also thank Juhan Aru and Christophe Garban for interesting discussions.

References

1. Adhikari, A. "Wilson loop expectations for non-abelian gauge fields coupled to a Higgs boson at low and high disorder." *Comm. Math. Phys.* **405** (2024): 117. https://doi.org/10.1007/s00220-024-04998-5.

2. Adhikari, A. and S. Cao. "Correlation decay for finite lattice gauge theories at weak coupling." *Ann. Probab.* **53**, no. 1 (2025): 140–74.
3. Cao, S. "Wilson loop expectations in lattice gauge theories with finite gauge groups." *Comm. Math. Phys.* **380** (2020): 1439–505.
4. Chatterjee, S. "Yang-Mills for Probabilists." *Probability and Analysis in Interacting Physical Systems, VAR75 2016. Springer Proc. Math. Stat., Vol. 283*, edited by Friz P., König W., Mukherjee C., and Olla S., 307–40. Cham: Springer, 2020.
5. Chatterjee, S. "Wilson loop expectations in Ising lattice gauge theory." *Comm. Math. Phys.* **377** (2020): 307–40.
6. Chatterjee, S. "A probabilistic mechanism for quark confinement." *Comm. Math. Phys.* **385** (2021): 1007–39.
7. Forsström, M. P. "Decay of correlations in finite abelian lattice gauge theories." *Comm. Math. Phys.* **393** (2022): 1311–46.
8. Forsström, M. P. "Wilson lines in the abelian lattice Higgs model." *Comm. Math. Phys.* **405** (2024): 275.
9. Forsström, M. P., J. Lenells, and F. Viklund. "Wilson loops in finite abelian lattice gauge theories." *Ann. Inst. H. Poincaré Probab. Statist.* **58**, no. 4 (2022): 2129–64.
10. Forsström, M. P., J. Lenells, and F. Viklund. "Wilson loops in the abelian lattice Higgs model." *Prob. Math. Phys* **4**, no. 2 (2023): 257–329.
11. Friedly, S. and Y. Velenik. *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction*. Cambridge: Cambridge University Press, 2017.
12. Fröhlich, J. and T. Spencer. "Massless phases and symmetry restoration in abelian gauge theories and spin systems." *Comm. Math. Phys.* **83** (1982): 411–54.
13. Garban, C. and A. Sepúlveda. "Improved spin-wave estimate for Wilson loops in $U(1)$ lattice gauge theory." *Int. Math. Res. Not. IMRN* **21** (2023): 18142–98.
14. Guth, A. "Existence proof of a nonconfining phase in four-dimensional $U(1)$ lattice gauge theory." *Phys. Rev. D* **21**, no. 8 (1980): 2291–307.
15. Guth, A., A. Ukawa, and P. Windey. "Dual variables for lattice gauge theories anti the phase structure of $Z(N)$ systems." *Phys. Rev. D* **21**, no. 4 (1980): 1013–36.
16. Göpfert, M. and G. Mack. "Proof of confinement of static quarks in 3-dimensional lattice gauge theory for all values of the coupling constant." *Comm. Math. Phys.* **82** (1982): 545–606.
17. Montway, I. and G. Munster. *Quantum Fields on a Lattice of Cambridge Monographs on Mathematical Physics*. Cambridge: Cambridge University Press, 1994.
18. Osterwalder, K. and E. Seiler. "Gauge field theories on a lattice." *Ann. Phys.* **110**, no. 2 (1978).
19. Seiler, E. "Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics." *Lecture Notes in Physics (LNP, volume 159)*. Springer, 1982.
20. Wegner, F. J. "Duality in generalized Ising models and phase transitions without local order parameters." *J. Math. Phys.* **12** (1971): 2259.
21. Wilson, K. G. "Confinement of quarks." *Phys. Rev. D* **10**, no. 8 (1974): 2445–59.

International Mathematics Research Notices, 2025, 1–27
https://doi.org/10.1093/imrn/rnaf163
Article