The Functor of Points Approach to Schemes in Cubical Agda

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Abstract

We present a formalization of quasi-compact and quasi-separated schemes (qcqs-schemes) in the `Cubical Agda` proof assistant. We follow Grothendieck's functor of points approach, which defines schemes, the quintessential notion of modern algebraic geometry, as certain well-behaved functors from commutative rings to sets. This approach is often regarded as conceptually simpler than the standard approach of defining schemes as locally ringed spaces, but to our knowledge it has not yet been adopted in formalizations of algebraic geometry. We build upon a previous formalization of the so-called Zariski lattice associated to a commutative ring in order to define the notion of compact open subfunctor. This allows for a concise definition of qcqs-schemes, streamlining the usual presentation as e.g. given in the standard textbook of Demazure and Gabriel. It also lets us obtain a fully constructive proof that compact open subfunctors of affine schemes are qcqs-schemes.

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1 Introduction

Algebraic geometry developed as the study of solutions to systems of polynomial equations. Objects of interest would e.g. be "affine complex varieties", subsets of $\mathbb{C}^n$ that can be described as the common roots of a finite system of polynomials $p_1, ..., p_m \in \mathbb{C}[x_1, ..., x_n]$. The discipline underwent a fundamental transformation during the latter half of the 20th century with the introduction of *schemes*. This development was spear-headed by Alexendre Grothendieck and led to many incredible achievements in geometry and number theory. Schemes can be seen as a generalization of varieties in several ways, but their standard presentation as "locally ringed spaces with an affine cover" somewhat blurs the connection to classical algebraic geometry, which can make it hard for students learning algebraic geometry to see in what sense schemes are "geometric" objects at all.

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There is, however, a different angle for generalization, where the original motivation of studying solutions to polynomials keeps a more prominent place. Take a polynomial with integer coefficients like $x^n + y^n - z^n \in \mathbb{Z}[x, y, z]$. Fermats last theorem tells us that this polynomial only has the trivial solution $x = y = z = 0$ for $n > 2$. This does of course only hold for solutions in the integers. We might interpret the same polynomial as living in a polynomial ring $A[x, y, z]$, where A is now any commutative ring (e.g. $\mathbb{C}$), and ask about solutions in A. The corresponding set of solutions is given by

$$V_{x^n+y^n-z^n}(A) = \{ (a_1, a_2, a_3) \in A^3 \mid a_1^n + a_2^n = a_3^n \}$$

Moreover, given a morphism of rings $\varphi \in \mathsf{Hom}(A, B)$, we can map a solution in A, $(a_1, a_2, a_3) \in V_{x^n+y^n-z^n}(A)$, to a solution $(\varphi(a_1), \varphi(a_2), \varphi(a_3)) \in V_{x^n+y^n-z^n}(B)$ in B. In categorical terms, our polynomial defines a *functor* from the category of commutative rings to the category of sets, mapping a ring A to the set $V_{x^n+y^n-z^n}(A)$ of solutions in A.

This functor $V_{x^n+y^n-z^n}(\_) : \mathsf{CommRing} \to \mathsf{Set}$ turns out to be a very familiar categorical object. For a ring A, homomorphisms $\mathsf{Hom}(\mathbb{Z}[x, y, z], A)$ are in bijection with A^3 (every morphism is determined by its values on x, y and z) and this induces a bijection of morphisms $\mathsf{Hom}(\mathbb{Z}[x,y,z]/\langle x^n+y^n-z^n \rangle, A)$ with $V_{x^n+y^n-z^n}(A)$. Now, a functor from $\mathsf{CommRing}$ to Set is nothing but a presheaf on the opposite category $\mathsf{CommRing}^{op}$. In this presheaf category we can look at the Yoneda embedding or representable of the quotient ring $R = \mathbb{Z}[x,y,z]/\langle x^n+y^n-z^n \rangle$, which we will denote by $\mathsf{Sp}(R)$. By the above argument, we get a natural isomorphism of presheaves $\mathsf{Sp}(R) = \mathsf{Hom}(R, \_) \cong V_{x^n+y^n-z^n}(\_)$.

In the category of functors from commutative rings to sets we can thus study solutions of systems of integer polynomials by looking at representable functors of quotients $\mathbb{Z}[x_1,\dots,x_n]/\langle p_1,\dots,p_m \rangle$. Algebraic geometers call these representables *absolute affine algebraic spaces* [16], which we can generalize to schemes (schemes over $\mathbb{Z}$ or absolute schemes to be more precise). Affine schemes are readily defined as representables of arbitrary commutative rings. From these we can build general schemes as presheaves on $\mathsf{CommRing}^{op}$ that are "local" and have an "open cover" by affine schemes in some appropriate sense.

Among the proponents of using the functor of points approach as the primary definition of schemes was Grothendieck himself [16], because, in the terms of Lawvere [22], it does not require "the baggage of prime ideals and the spectral space, sheaves of local rings, coverings and patchings, etc." Yet, most standard sources [13, 15, 17, 33] for students learning algebraic geometry start with precisely this "baggage". To our knowledge, the same can be said for existing formalizations of schemes [3, 4, 5, 7, 39]. We want to close this gap and present a first formalization of the functor of points approach.

Admittedly, part of the appeal of schemes as locally ringed spaces as a formalization target for proof assistants is that they are such a layered, involved notion, while at the same time being a point of departure for formalizing a plethora of interesting research level mathematics. The first full formalization of schemes in `Lean`'s `mathlib` by Buzzard et. al. [4] revealed certain bottlenecks that occur when defining schemes this way. As these bottlenecks might be addressed very differently in different proof assistants, schemes have become somewhat of a benchmark problem, inspiring a formalization in `Isabelle/HOL` [3], and partial formalizations in `Coq`'s `UniMath` library [5] and `Cubical Agda` [39].

It is worth noting that, except for the `Cubical Agda`-formalization [39], all of the above formalizations are non-constructive as they follow the presentation of Hartshorne's standard "Algebraic Geometry" [17]. In [39], the authors manage to stay constructive by using "ringed lattices" [8] instead of locally ringed spaces, but the formalization only includes affine schemes. The functor of points approach is often taken to be more amenable for constructive mathematics.[1] Indeed, to our knowledge we present the first fully constructive formalization of quasi-compact and quasi-separated schemes (qcqs-schemes), an important subclass of schemes that is sufficient for a large portion of modern algebraic geometry.[2]

Nowadays there exist extensive algebra and category theory libraries for many of the major proof assistants, providing a lot of the necessary tools to formally define schemes using the functor of points approach. The bottlenecks of defining schemes as locally ringed spaces, disappear when following the functor of points approach. One problem that occurs, however, is that the category of functors from rings to sets is not locally small, since arrows between two such functors are natural transformations, i.e. families of functions indexed by the "big" type of all rings in a given universe. As a result, one has to address size issues. Dealing with size issues in a predicative type theory like `Cubical Agda`'s, one is led to make certain finiteness assumptions, resulting in the aforementioned restriction to qcqs-schemes.

Our work is completely formalized in `Cubical Agda` and all results are integrated in the `agda/cubical` library.[3] We will comment on our usage of `Cubical Agda` in Section 2.1, but we want to stress that the formalization does not rely on cubical features. It should be possible to more or less directly translate the formalization into a system implementing Homotopy Type Theory and Univalent Foundations of the HoTT book [32] or into `UniMath` [35]. Our work can be understood as being in line with the goals of Voevodsky's Foundations library [38]: Developing a library of constructive set-level mathematics based on Univalent Foundations.

As a result of working fully constructive and predicative, our presentation deviates from the standard "Introduction to Algebraic Geometry and Algebraic Groups" by Demazure and Gabriel [12]. Our main contributions and design choices can be summarized as follows:

- In Section 3 we define the category of $\mathbb{Z}$-functors, differing slightly from Demazure and Gabriel. This is because `Agda`'s universes are not cumulative and we chose to work with a fully-faithful spectrum functor with the caveat that it only has a relative adjoint.
- In Section 4 we define the notion of coverage and sheaf wrt. a coverage. We define the Zariski coverage on $\mathsf{CommRing}^{op}$. Restricting from $\mathbb{Z}$-functors to Zariski sheaves can be seen as introducing a locality condition, akin to restricting from ringed to locally ringed spaces. We show that affine schemes are local, i.e. that representable presheaves are sheaves wrt. the Zariski coverage. For this one can reuse some key algebraic lemmas, first formalized in [39] to show the sheaf property of the structure sheaf of an affine scheme.
- In Section 5 we define the notions of compact open subfunctor, cover of compact opens and finally qcqs-scheme. It is in this section that we deviate substantially from the standard sources. We argue that the above notions are most conveniently defined by using an appropriate classifier in the topos theoretic sense. Since we have a small Zariski

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[1] See e.g. the discussion where the functor of points approach was first suggested as a formalization target for the `agda/cubical`-library: `https://github.com/agda/cubical/issues/657`

[2] In particular, every noetherian scheme is qcqs. When applying scheme theory to the classic motivating problems of algebraic geometry, Hartshorne notes that "practically all the schemes encountered in this way are noetherian" [17, p. 100]. Deligne's presentation of étale cohomology [10], a crucial tool for his proof of the Weil conjectures, assumes schemes to be qcqs throughout: "We consider only schemes that are quasi-compact (= finite union of open affines) and quasi-separated (= such that the intersection of two open affines is quasi-compact), and we simply call them schemes." [11, p. 1].

[3] The formalization is summarized in:
`https://github.com/agda/cubical/blob/60f18987bb1819a15fccc325343ef7b469bb2eeb/Cubical/Papers/FunctorialQcQsSchemes.agda`
This is a permalink to the library at the time of writing, which type-checks with `Agda` version 2.6.4.1.
A clickable rendered version that might be subject to change can be found here:
`https://agda.github.io/cubical/Cubical.Papers.FunctorialQcQsSchemes.html`

lattice but no small type of radical ideals in Cubical Agda, we can only classify *compact opens*. So far, these only appear in the literature on synthetic algebraic geometry ([2, Def. 19.15] and [6, Def. 4.2.1]), but they turn out to be very useful for our purposes as well.

- In Section 6 we prove that compact open subfunctors of affine schemes are qcqs-schemes. We give a point-free proof that the classifier for compact opens is separated, only using the universal property of the Zariski lattice. This gives us that compact opens of affine schemes are sheaves. The fact that compact opens of affines have an affine cover essentially follows from the Yoneda lemma.

2 Background

We begin by giving some helpful background. First, we discuss the Cubical Agda proof assistant and how it is used in the formalization. We then briefly present two algebraic constructions from the agda/cubical library, first formalized and described by Zeuner and Mörtberg in [39], that play a key role in this paper as well: localizations of commutative rings and the Zariski lattice.

2.1 Univalent type theory in Cubical Agda

For understanding the details of our formalization, it is worth knowing about certain particularities of the Cubical Agda proof assistant and its library. We will restrict ourselves to the features that are relevant for this paper. Readers familiar with Cubical Agda or Homotopy Type Theory and Univalent Foundations (HoTT/UF) can safely skim this section. Readers interested in more details are referred to [34].

Cubical Agda is a rather recent extension of the Agda proof assistant with fully constructive support of the univalence principle and higher inductive types (HITs). The notation used in this paper is inspired by Agda's syntax and the conventions of the agda/cubical library but we have taken the liberty to simplify the syntax and omit projections whenever possible in order to increase readability. For example we will write CommRing to denote both the type and the category of commutative rings and an element R : CommRing will denote both the ring with its structure and the carrier-type of R, i.e. we write $f : R$ for its elements. For the universe at level ℓ we write Type ℓ or Type_ℓ, and similarly $\mathsf{CommRing}_\ell$ for commutative rings whose carrier type lives in Type_ℓ. For a family $B : A \to$ Type ℓ, we denote the dependent pair type over this family as $\Sigma[\ x \in A\ ]\ B(x)$.

For definitional equalities we use $=$, while propositional equalities are written using $\equiv$. Note that Cubical Agda does not use Martin-Löf's inductive identity type [25] for expressing propositional equalities, but rather so-called *path* types. These path types are defined in terms of a primitive interval type I, which allows one to conveniently define *dependent* path types. In this formalization we will not make direct use of the interval or dependent path types. However, path types do entail function extensionality, the right behavior of equalities of dependent pairs and other useful principles, which we will use freely.[4]

Cubical Agda does not come with a designated universe of propositions and in fact we cannot generally expect propositional equality types, or rather path types, to be propositions in any sensible way. This is because Cubical Agda proves univalence and thus disproves *Uniqueness of Identity Proofs* (UIP), also known as Streicher's axiom K [31]. We can, however,

[4] These principles also follow from univalence albeit with a slightly different computational behavior.

internally define (proof-relevant) propositions as subsingleton types and sets as types whose equalities are propositions, i.e. as types satisfying UIP:

```
isProp : Type ℓ → Type ℓ                isSet : Type ℓ → Type ℓ
isProp A = (x y : A) → x ≡ y            isSet A = (x y : A) → isProp (x ≡ y)
```

The type (universe) of propositions at level ℓ is defined as hProp $\ell = \Sigma[\, A \in$ Type $\ell\,]$ (isProp A), A *subset* of A, where isSet A, is a function $S : A \to$ hProp ℓ. With some abuse of notation we will identify a subset S with the corresponding Σ-type $\Sigma[\, a \in A\,]$ $(a \in S)$, where $a \in S$ is the proposition (type of proofs) that a is actually in S. We thus write $a : S$ for elements of S when the proof of $a : A$ belonging to S can be ignored.

Univalence implies that there are types, which are neither propositions nor sets. These types are said to have a higher h-level (homotopy level [36]) than sets. One can use the so-called *structure identity principle* [32, Sec. 9.8] to prove that this holds true for types of algebraic or categorical structures like commutative rings or $\mathbb{Z}$-functors.[5] However, we want to stress that this does not affect the formalization presented in this paper.

We do make extensive use higher inductive types (HITs), the other main addition of HoTT/UF to dependent type theory alongside univalence. In particular, we require two HITs: set-quotients and propositional truncations. Set-quotients are needed to define localizations of rings and the Zariski lattice, which we will describe in Section 2.2. We will not go into details on how set-quotients are defined. It suffices to know that as long as we quotient sets by proposition-valued equivalence relations and only consider maps from those quotients into other sets, everything works as one would expect from quotients. The other HIT, propositional truncation, turns any type into a proposition:

```
data ‖_‖ (A : Type ℓ) : Type ℓ where
  |_| : A → ‖ A ‖
  squash : isProp ‖ A ‖
```

This is needed in HoTT/UF to express existential quantification, as using Σ-types is often too strong. We follow the convention and say “there *merely* exists $x : A$ such that $P(x)$”, if we have an inhabitant of

$$\exists[\, x \in A\,]\ P(x) = \|\, \Sigma[\, x \in A\,]\ P(x)\,\|$$

Note that in general this does not let us extract a witness $x : A$, satisfying $P(X)$. We will discuss an example showcasing the proper use of propositional truncation in Remark 18.

2.2 Localizations and the Zariski lattice

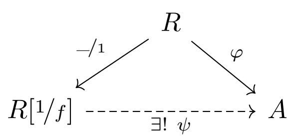
Our formalization builds on a lot of commutative algebra and category theory formalized in the `agda/cubical` library that we will presuppose in this paper. In particular, we will assume familiarity with presheaves, the Yoneda lemma and basic ideal theory of rings and we will not comment on their implementation in the `agda/cubical` library. There are two particular constructions, first described in [39], that are of special importance to this project and we will briefly describe them here.

[5] The implementation of the structure identity principle in the `agda/cubical`-library is described in [1].

The first, localizations of commutative rings, are a way of making elements invertible by adding fractions. In this paper we only need the special case of inverting a single element. For a ring R and $f : R$ the *localization of R away from f* is the ring $R[1/f]$ of fractions r/f^n where the denominator is a power of f. Equality of two fractions is slightly different than for fractions of integers and can be stated as:[6]

$$r/f^n \equiv r'/f^m \quad \text{iff} \quad \exists[\, k \in \mathbb{N} \,]\ (r f^{k+m} \equiv r' f^{k+n})$$

Localizations satisfy a universal property and in our special case it can be stated as: $R[1/f]$ is the initial R-algebra where f becomes invertible. This means that for any ring A with a homomorphism $\varphi : \mathsf{Hom}(R, A)$ such that $\varphi(f) \in A^\times$ (i.e. $\varphi(f)$ is a unit/invertible), there is a unique $\psi : \mathsf{Hom}(R[1/f], A)$ making the following diagram commute



where $\_/1 : \mathsf{Hom}(R, R[1/f])$ is the canonical morphism mapping $r : R$ to the fraction $r/1$. As shown in [39], formalizing localizations with the help of set-quotients is straightforward.

The second construction, the Zariski lattice associated to a ring is slightly more delicate. By a standard argument in classical algebraic geometry there is a one-to-one correspondence between Zariski open sets of $\mathsf{Spec}(R)$ and radical ideals of R. An ideal $I \subseteq R$ is radical if $I = \sqrt{I} = \{\, x \in R \mid \exists n > 0 : x^n \in I \,\}$. Furthermore, the *compact open* subsets of $\mathsf{Spec}(R)$ correspond radicals of *finitely generated* ideals. This correspondence is in fact an isomorphism of lattices. Set-theoretic union and intersection of compact opens correspond to addition and multiplication of finitely generated ideals.

This means that we can define this so-called *Zariski lattice* $\mathcal{L}_R$ without having to define $\mathsf{Spec}(R)$ and its topology first: Elements of $\mathcal{L}_R$ are generators $f_1, ..., f_n : R$ quotiented by the relation that relates another list of generators $g_1, ..., g_m : R$ if $\sqrt{\langle f_1, \ldots, f_n \rangle} \equiv \sqrt{\langle g_1, \ldots, g_m \rangle}$. The equivalence class of the generators $f_1, ..., f_n : R$ is denoted by $D(f_1, ..., f_n) : \mathcal{L}_R$ and the join on $\mathcal{L}_R$ is given by $D(f_1, ..., f_n) \vee D(g_1, ..., g_m) = D(f_1, ..., f_n, g_1, ..., g_m)$. The "basic open" $D(f)$ is the equivalence class corresponding to the radical of the principle ideal $\sqrt{\langle f \rangle}$, with $D(1)$ being the top element of $\mathcal{L}_R$ corresponding to the 1-ideal. The basic opens form a basis of $\mathcal{L}_R$, as $D(f_1, ..., f_n) = \bigvee_{i=1}^{n} D(f_i)$.

This definition is due to Español [14], but it has the disadvantage that it uses equality of ideals to define the quotienting relation. In the predicative type theory of `Cubical Agda` the type of ideals of R lives in the universe above R an so does the equality type between two ideals. This can be avoided by slightly rewriting the equivalence relation, as shown in [39], giving us $\mathcal{L}_R : \mathsf{DistLattice}_\ell$ for $R : \mathsf{CommRing}_\ell$.

Joyal [20] observed that the Zariski lattice has a certain universal property that can be stated in terms of *supports*. A map $d : R \to L$ from R into a (bounded) distributive lattice L is called a support if it satisfies:

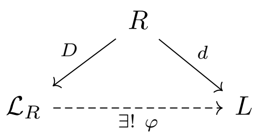
$$d(1) \equiv \top \ \text{ and } \ d(0) \equiv \bot \tag{1}$$

$$\forall (f\ g : R) \to d(fg) \equiv d(f) \wedge d(g) \tag{2}$$

$$\forall (f\ g : R) \to d(f + g) \leq d(f) \vee d(g) \tag{3}$$

[6] This is to account for zero-divisors and the case where f is nilpotent.

The map $D : R \to \mathcal{L}_R$ sending $f : R$ to the equivalence class $D(f)$ satisfies conditions (1)-(3) and it is a universal support in the sense for any other support $d : R \to L$ there is a unique lattice homomorphism $\varphi : \mathcal{L}_R \to L$ such that the following commutes



The partial order defined on the Zariski lattice is connected to localizations as for $f, g : R$

$$D(g) \leq D(f) \;\Leftrightarrow\; \sqrt{\langle g \rangle} \subseteq \sqrt{\langle f \rangle} \;\Leftrightarrow\; g \in \sqrt{\langle f \rangle} \;\Leftrightarrow\; {}^{f}/_{1} \in R[{}^{1}/_{g}]^{\times}$$

In the special case where $g = 1$, this gives us $D(1) \equiv D(f)$ iff $f \in R^{\times}$. We will utilize this fact in order to interpret the basic opens as affine subschemes in Definition 22. A slight generalization of this fact that we will use in Section 5 to informally justify that Definition 14 is sensible is that for $f_1, ..., f_n : R$

$$D(1) \equiv D(f_1, ..., f_n) \;\Leftrightarrow\; 1 \in \langle f_1, ..., f_n \rangle$$

This concludes our discussion of the preliminaries required to formalize qcqs-schemes following the functor of points approach.

3 $\mathbb{Z}$-Functors

Let us turn to our goal of defining qcqs-schemes as well-behaved functors from rings to sets. As size issues are unavoidable in the functor of points approach, we will be rather explicit about universe levels in this paper. For the remainder we will fix a universe level ℓ and work over commutative rings in the corresponding universe $\mathsf{CommRing}_\ell$.

▶ **Definition 1.** *The category of $\mathbb{Z}$-functors, denoted $\mathbb{Z}\mathsf{Functor}_\ell$, is the category of functors from $\mathsf{CommRing}_\ell$ to Set_ℓ. We write $\mathsf{Sp} : \mathsf{CommRing}_\ell^{op} \to \mathbb{Z}\mathsf{Functor}_\ell$ for the Yoneda embedding and $\mathbb{A}^1 : \mathbb{Z}\mathsf{Functor}_\ell$ for the forgetful functor from commutative rings to sets. We say that $X : \mathbb{Z}\mathsf{Functor}_\ell$ is an* affine scheme *if there merely exists $R : \mathsf{CommRing}_\ell$ such that $X \cong \mathsf{Sp}(R)$.*

▶ Remark 2. It is worth noticing that most modern algebraic geometry sources (see e.g. [13, 15, 26, 33]) usually omit any reference to universes when discussing the functor of points approach. The choice of taking functors from rings to sets in the same universe seems perhaps most natural, but actually differs from the standard reference on the functor of points approach by Demazure and Gabriel [12]. They essentially take $\mathbb{Z}$-functors to be functors from $\mathsf{CommRing}_\ell$ to $\mathsf{Set}_{\ell+1}$.[7] Their "big spectrum functor" $\mathsf{Sp} : \mathsf{CommRing}_{\ell+1}^{op} \to (\mathsf{CommRing}_\ell \to \mathsf{Set}_{\ell+1})$ is defined much like the Yoneda embedding as $\mathsf{Sp}(R) = \mathsf{Hom}(R, \_)$, but because of the universe level mismatch it is *not* fully faithful. However, this functor has a left adjoint, namely the functor that we will define in Definition 3. We decided to differ in our definition of $\mathbb{Z}$-functors since `Agda`'s non-cumulative universes would otherwise require explicit universe lifts in a lot of places, massively cluttering the code, and it seemed more convenient to use the fully-faithful Yoneda embedding as our Sp.

[7] They actually assume two Grothendieck universes $\mathcal{U} \subseteq \mathcal{V}$. As type theoretic universes are usually "lifted" from Grothendieck universes in presheaf models [18], our translation only seems natural.

▶ **Definition 3.** *Let X : $\mathbb{Z}$Functor$_\ell$, the* ring of functions *$\mathcal{O}(X)$ is the type of natural transformations $X \Rightarrow \mathbb{A}^1$ equipped with the canonical point-wise operations, i.e. for R : CommRing$_\ell$ and $x : X(R)$, addition and multiplication of $\alpha, \beta : X \Rightarrow \mathbb{A}^1$ are given by*

$$(\alpha + \beta)_R(x) \;=\; \alpha_R(x) + \beta_R(x) \qquad (\alpha \cdot \beta)_R(x) \;=\; \alpha_R(x) \cdot \beta_R(x)$$

This defines a functor $\mathcal{O}$: $\mathbb{Z}$Functor$_\ell \to$ CommRing$^{op}_{\ell+1}$, whose action on morphisms (natural transformations) is given by precomposition.

The universal property of schemes is often stated to be: The global sections functor Γ is left adjoint to Spec and the counit of this adjunction is an isomorphism. However, this is already true for locally ringed spaces. In a similar fashion we would like to have an adjunction $\mathcal{O} \dashv$ Sp, but unfortunately we run into a universe level mismatch. We still get something that looks a lot like an adjunction. The proof of the following proposition is straightforward.

▶ **Proposition 4.** *For R : CommRing$_\ell$ and X : $\mathbb{Z}$Functor$_\ell$ there is an isomorphism of types*

$$\mathsf{Hom}\big(R, \mathcal{O}(X)\big) \cong \big(X \Rightarrow \mathit{Sp}(R)\big)$$

which is natural in both R and X. Moreover, the induced "counit" ε_R : $\mathsf{Hom}\big(R, \mathcal{O}(\mathit{Sp}(R))\big)$, which is obtained by applying the inverse of above isomorphism to the identity transformation $\mathit{Sp}(R) \Rightarrow \mathit{Sp}(R)$, is an isomorphisms of rings for all R : CommRing$_\ell$.

▶ Remark 5. Proposition 4 type-checks because the type of ring homomorphisms is universe polymorphic, meaning it can take rings living in different universes as arguments. The same holds for the type of isomorphisms/equivalences between two types. From a categorical perspective, we get a so-called *relative coadjunction* [29], written $\mathcal{O} \dashv_i$ Sp, with respect to the inclusion, or lift functor i : CommRing$^{op}_\ell \to$ CommRing$^{op}_{\ell+1}$. This is why we only get a counit, but no unit.

4 Local $\mathbb{Z}$-functors

Functorial (qcqs-) schemes are sheaves with respect to the Zariski coverage. The notion of coverage (also called a Grothendieck pre-topology) generalizes point-set topologies to arbitrary categories. Roughly speaking, a coverage on a category $\mathcal{C}$ associates to each object $U : \mathcal{C}$ a family $\mathsf{Cov}(U)$ of covers. A cover $(U_i \to U)_{i:I} : \mathsf{Cov}(U)$ is a family of maps into U. These families $\mathsf{Cov}(U)$ should satisfy certain closure properties. If $\mathcal{C}$ has pullbacks then covers should be closed under pullbacks and a presheaf $\mathcal{F} : \mathcal{C}^{op} \to \mathsf{Set}$ can be defined to be a sheaf if for any $(U_i \to U)_{i:I} : \mathsf{Cov}(U)$ we get and equalizer diagram

$$\mathcal{F}(U) \to \prod_{i:I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j:I} \mathcal{F}(U_i \times_U U_j)$$

In the case where $\mathcal{C}$ is $\mathsf{Open}(X)$, the poset of open subsets of a topological space X, we get a canonical coverage: A family of opens $(U_i \subseteq U)_{i \in I}$ is in $\mathsf{Cov}(U)$ if and only if $\bigcup_{i \in I} U_i = U$. Pullbacks in $\mathsf{Open}(X)$ are given by set-theoretic intersection ∩ and we recover the usual definition of when a presheaf $\mathcal{F} : \mathsf{Open}(X)^{op} \to \mathsf{Set}$ is a sheaf.

The formalization of coverages and sheaves in the `agda/cubical` library follows the nLab [27] and Johnstone's classic "Sketches of an elephant" [19, C2]. The advantage of this approach is that it even works for categories without pullbacks. As it turns out, it also lets us conveniently define the Zariski coverage and prove that representables are Zariski sheaves. For now, let us fix an arbitrary category $\mathcal{C}$.

▶ **Definition 6.** *A* cover *on an object $c : \mathcal{C}$ consists of an index type I and for each $i : I$ an element in the slice category ${}^{\mathcal{C}}/_{c}$, i.e. an arrow $f_i : \mathcal{C}(c_i, c)$. A* coverage *on $\mathcal{C}$ consist of a family of covers for each $c : \mathcal{C}$ satisfying* pullback stability*: Given a cover $\{ f_i : \mathcal{C}(c_i, c) \}_{i:I}$ of c and arrow $f : \mathcal{C}(d, c)$, there merely exists a cover $\{ g_j : \mathcal{C}(d_j, d) \}_{j:J}$ of d such that for each index $j : J$ there merely exists an index $i : I$ and an arrow $h_{ij} : \mathcal{C}(d_j, c_i)$ with $f_i \circ h_{ij} \equiv f \circ g_j$.*

Pullback stability can also be stated as: Given an arrow $f : \mathcal{C}(d, c)$ and a cover on c, we can take the *sieve* generated by this cover and pull it back to a sieve on d. Then there exists a cover on d refining the pulled back sieve on d. Since sieves are not required for the remainder of the paper, we decided to unfold the definition of pullback stability and state it without recourse to sieves. We refer the interested reader to the formalization. We now define what it means to be sheaf with respect to a fixed coverage on $\mathcal{C}$. For a presheaf P on $\mathcal{C}$ and arrow $f : \mathcal{C}(c, d)$ we write $\_{\restriction_f} : P(d) \to P(c)$ for the restriction map, i.e. the action of P on f.

▶ **Definition 7.** *Let P be a presheaf on $\mathcal{C}$. Let $c : \mathcal{C}$ and $\{ f_i : \mathcal{C}(c_i, c) \}_{i:I}$ be a cover. A* compatible family *or* matching family *[28] is a dependent function $x : (i : I) \to P(c_i)$, i.e. a family of elements $x_i : P(c_i)$, such that for each pair of indices $i, j : I$ and arrows $g_i : \mathcal{C}(d, c_i)$ and $g_j : \mathcal{C}(d, c_j)$ with $f_j g_j \equiv f_i g_i$, we have $x_j{\restriction_{g_j}} \equiv x_i{\restriction_{g_i}}$ (in $P(d)$). We denote the type of compatible families over a cover $\{ f_i : \mathcal{C}(c_i, c) \}_{i:I}$ by $\mathsf{CompatibleFam}^P(\{ f_i : \mathcal{C}(c_i, c) \}_{i:I})$.*

For an element $x : P(c)$ we get an induced compatible family by taking the restrictions $x_i = x{\restriction_{f_i}}$ for $i : I$. The compatibility follows directly from the presheaf property of P. This construction gives us a map $\sigma_P : P(c) \to \mathsf{CompatibleFam}^P(\{ f_i : \mathcal{C}(c_i, c) \}_{i:I})$. We can now conveniently define sheaves in terms of the map σ.

▶ **Definition 8.** *A presheaf P is a* sheaf *if for all $c : \mathcal{C}$ and covers $\{ f_i : \mathcal{C}(c_i, c) \}_{i:I}$, the canonical map σ_P is an isomorphism.*

▶ **Definition 9.** *A coverage on $\mathcal{C}$ is called* subcanonical *if for all $c : \mathcal{C}$ the Yoneda embedding of c is a sheaf with respect to the coverage.*

In this paper we are interested in a particular example of a coverage on the opposite category of commutative rings. Covers of a ring R will come from finite lists of generators of the 1-ideal. Classically, this corresponds to the fact that any open cover of an affine scheme is of the form $\mathsf{Spec}(R) = \bigcup_{i=1}^n D(f_i)$ with $1 \in \langle f_1, \ldots, f_n \rangle$ (because $\mathsf{Spec}(R)$ is quasi-compact). We call a finite list of elements $f_1, \ldots, f_n : R$ such that $1 \in \langle f_1, \ldots, f_n \rangle$ a *unimodular vector*.

▶ **Definition 10.** *The* Zariski coverage *on $\mathcal{C} = \mathit{CommRing}_\ell^{op}$ is given by:*

- *For each $R : \mathit{CommRing}_\ell$, covers are indexed by the type of unimodular vectors over R.*
- *For each unimodular vector $f_1, \ldots, f_n : R$, the associated cover of R is given by the reversed canonical morphisms $\_/1 : R[1/f_i] \to R$, indexed by $i : \mathit{Fin}\ n$, the finite n-element type.*
- *For a unimodular vector $f_1, \ldots, f_n : R$ the pullback along a morphism $\varphi : \mathsf{Hom}(R, A)$ is the vector $\varphi(f_1), \ldots, \varphi(f_n) : A$, which is easily shown to be unimodular as well.*

A presheaf $X : \mathbb{Z}\mathit{Functor}_\ell$ is called local *if it is a sheaf wrt. the Zariski coverage.*

▶ **Lemma 11.** *The Zariski coverage is stable under pullbacks.*

Proof. Let $R, A : \mathsf{CommRing}_\ell$, $f_1, \ldots, f_n : R$ be a unimodular vector and $\varphi : \mathsf{Hom}(R, A)$. The universal property of localization induces ring morphisms $\psi_i : \mathsf{Hom}\big(R[1/f_i], A[1/\varphi(f_i)]\big)$ such that the following diagram commutes (in $\mathsf{CommRing}_\ell^{op}$)

$$\begin{array}{ccc} A[1/\varphi(f_i)] & \xrightarrow{\psi_i} & R[1/f_i] \\ \big\downarrow{\scriptstyle \_/1} & & \big\downarrow{\scriptstyle \_/1} \\ A & \xrightarrow{\varphi} & R \end{array}$$

◀

The key result of this section uses an algebraic fact that can be found in many textbooks, such as [23, p. 125], and was already formalized in `Cubical Agda` to prove [39, Lemma 15].

▶ **Theorem 12.** *The Zariski coverage is subcanonical, i.e.* $\mathsf{Sp}(A)$ *is local for* $A : \mathsf{CommRing}_\ell$.

Proof. Let $R : \mathsf{CommRing}_\ell$ and $f_1, ..., f_n : R$ a unimodular vector be given. For i, j in $1, ..., n$, we denote by $\chi^l_{ij} : R[1/f_i] \to R[1/f_if_j]$ and $\chi^r_{ij} : R[1/f_j] \to R[1/f_if_j]$ the canonical morphisms given by the universal property of localization. We use without proof that the map

$$R \;\to\; \Sigma[\; x \in (i : \mathsf{Fin}\; n) \to R[1/f_i] \;]\;\; \forall\; i\; j \to \chi^l_{ij}(x_i) \equiv \chi^r_{ij}(x_j)$$

sending $g : R$ to $g/1 : R[1/f_i]$ for $i = 1, ..., n$, is an isomorphism. Using this, one can construct a chain of isomorphisms

$$\begin{aligned} \mathsf{Hom}(A, R) &\cong \Sigma[\; \varphi \in (i : \mathsf{Fin}\; n) \to \mathsf{Hom}(A, R[1/f_i]) \;]\;\; \forall\; i\; j \to \chi^l_{ij} \circ \varphi_i \equiv \chi^r_{ij} \circ \varphi_j \\ &\cong \mathsf{CompatibleFam}^{\mathsf{Sp}(A)}\big(\{\, f_i \}_{i=1,...,n}\big) \end{aligned}$$

which factors through the canonical map $\sigma_{\mathsf{Sp}(A)}$. ◀

5 Compact opens and qcqs-schemes

The standard way to define open subfunctors follows a two step process. First, one defines them for representables using (radical) ideals. Then, one defines open subfunctors of general $\mathbb{Z}$-functors by pulling back to representables. Working predicatively in `Cubical Agda`, we need to restrict ourselves to *finitely generated* ideals, which gives us compact open subfunctors. Let us sketch the idea behind compact opens informally to see why this restriction is necessary: For a f.g. ideal $I \subseteq A$, we get the *affine compact open subfunctor* $\mathsf{Sp}(A)_I \hookrightarrow \mathsf{Sp}(A)$ given by

$$\mathsf{Sp}(A)_I(B) = \{\varphi \in \mathsf{Hom}(A, B) \mid \varphi^* I = B\} \subseteq \mathsf{Sp}(A)(B)$$

If $I = \langle f_1, ..., f_n \rangle$, then the "pullback" along $\varphi \in \mathsf{Hom}(A, B)$ is just $\varphi^* I = \langle \varphi(f_1), ..., \varphi(f_n) \rangle$. With this, we can define a subfunctor $U \hookrightarrow X$ to be *compact open* if pulling back along an A-valued point of X gives an affine compact open subfunctor of $\mathsf{Sp}(A)$, i.e. if for any ring A and $\phi : \mathsf{Sp}(A) \Rightarrow X$ there is a f.g. ideal $I \subseteq A$ such that the following is a pullback square

$$\begin{array}{ccc} \mathsf{Sp}(A)_I & \longrightarrow & U \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathsf{Sp}(A) & \xrightarrow{\;\phi\;} & X \end{array}$$

Note that the ideal I is not uniquely determined in this case. Indeed, if $I = \langle f_1, ..., f_n \rangle$ and $J = \langle g_1, ..., g_m \rangle$ are such that $\sqrt{I} = \sqrt{J}$, then for any $\varphi \in \mathsf{Hom}(A, B)$ we have

$$1 \in \langle \varphi(f_1), ..., \varphi(f_n) \rangle \quad \text{iff} \quad 1 \in \langle \varphi(g_1), ..., \varphi(g_m) \rangle$$

and thus $\mathsf{Sp}(A)_I \cong \mathsf{Sp}(A)_J$. In fact, one can prove that the converse also holds. This means that the compact open U and the A-valued point $\phi : \mathsf{Sp}(A) \Rightarrow X$ determine a finitely generated ideal $I = \langle f_1, ..., f_n \rangle$ up to equality of radical ideals, i.e. an element $D(f_1, ..., f_n)$ of the Zariski lattice $\mathcal{L}_A$. Note that we can describe $\mathsf{Sp}(A)_I$ purely in terms of $D(f_1, ..., f_n)$, as the B-valued points are given by

$$\begin{aligned} \mathsf{Sp}(A)_I(B) &= \{\varphi \in \mathsf{Hom}(A, B) \mid 1 \in \langle \varphi(f_1), ..., \varphi(f_n) \rangle\} \\ &= \{\varphi \in \mathsf{Hom}(A, B) \mid D\big(\varphi(f_1), ..., \varphi(f_n)\big) = D(1) \rangle\} \end{aligned}$$

The pullback condition ensures that this mapping is natural in A. In other words, the compact open subfunctors of X are in one-to-one correspondence with natural transformations from X to the $\mathbb{Z}$-functor $\mathcal{L}$ that sends a ring to its Zariski lattice. Note that we can define this $\mathbb{Z}$-functor $\mathcal{L}$ because of the "small" definition of Zariski lattice. If we drop the finiteness assumption on ideals to get open subfunctors we cannot hope to define the classifier in `Cubical Agda`.[8] We will discuss possibilities to do so in other systems in Section 7.1.

For a topos theorist it might not constitute a particularly deep insight that the compact open subfunctors (sub-objects) of a $\mathbb{Z}$-functor are *classified* by the "internal Zariski lattice" $\mathcal{L}$. This means that the compact opens are precisely given by pullbacks of the form

$$\begin{array}{ccc} U & \longrightarrow & \mathbf{1} \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle D(1)} \\ X & \longrightarrow & \mathcal{L} \end{array}$$

where $D(1) : \mathbf{1} \Rightarrow \mathcal{L}$ is the "constant" natural transformation, sending the point of the terminal $\mathbb{Z}$-functor $\mathbf{1}$ to the top element of the Zariski lattice. From a formal perspective however, we found it significantly more convenient to work with natural transformations into $\mathcal{L}$ and the induced subfunctors, as opposed to following the text-book strategy of defining compact-openness as a property of subfunctors through the two step process outlined above.[9] We will thus proceed to describe how compact opens can be formally defined as natural transformations and how this gives a concise definition of qcqs-schemes.

▶ **Definition 13.** *Let $\mathcal{L}$: ℤFunctor$_\ell$ be the $\mathbb{Z}$-functor mapping a ring R : CommRing$_\ell$ to the underlying set of the Zariski lattice $\mathcal{L}_R$. The action on morphisms is induced by the universal property of the Zariski lattice, i.e. for $\varphi : \mathsf{Hom}(A,B)$ we take the morphism $\varphi^{\mathcal{L}}$ induced by the support $D \circ \varphi$:*

$$\begin{array}{ccccc} & & A & & \\ & {\scriptstyle D}\swarrow & & \searrow{\scriptstyle D\circ\varphi} & \\ \mathcal{L}_A & & \underset{\exists!\ \varphi^{\mathcal{L}}}{\dashrightarrow} & & \mathcal{L}_B \end{array}$$

▶ **Definition 14.** *Let X : ℤFunctor$_\ell$, a* compact open *of X is a natural transformation $U : X \Rightarrow \mathcal{L}$. The* realization *$[\![U]\!]^{\mathsf{co}}$: ℤFunctor$_\ell$ of a compact open U of X, is given by*

$$[\![U]\!]^{\mathsf{co}}\,(R) \;=\; \Sigma[\; x \in X(R) \;]\; U(x) \equiv D(1)$$

A compact open U is called affine, *if its realization is affine, i.e. if there merely exists R : CommRing$_\ell$ such that $[\![U]\!]^{\mathsf{co}} \cong$ Sp(R).*

The reader may verify that for $U : X \Rightarrow \mathcal{L}$, R : CommRing$_\ell$ and $x : X(R)$ such that $U(x) = D(f_1, ..., f_n)$, we have

$$\begin{array}{ccccc} \mathsf{Sp}(R)_{\langle f_1,\ldots,f_n\rangle} & \longrightarrow & [\![U]\!]^{\mathsf{co}} & \longrightarrow & \mathbf{1} \\ \big\downarrow & \lrcorner & \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle D(1)} \\ \mathsf{Sp}(R) & \xrightarrow{\ \phi_x\ } & X & \xrightarrow{\ U\ } & \mathcal{L} \end{array}$$

[8] In fact, having such a classifier would imply propositional resizing. By a result of De Jong and Escardó [9, Cor. 28], the existence of a single ring R : CommRing$_\ell$, such that the frame of radical ideals of R (i.e. Zariski opens) is ℓ-small, would suffice to prove resizing for hProp$_\ell$.

[9] A rare exception to following the standard definition is a blog-post by Madore [24].

where ϕ_x corresponds to the R-valued point x by the Yoneda lemma.

Since $\mathcal{L}$ is a presheaf that takes values in distributive lattices and its restriction maps are lattice morphisms, it is an *internal* lattice in the presheaf topos of $\mathbb{Z}$-functors.[10] As such, it endows the compact opens with a distributive lattice structure.

▶ **Definition 15.** *Let* $X : \mathbb{Z}\mathsf{Functor}_\ell$*, the* lattice of compact opens *of* X*,* $\mathsf{CompOpen}(X)$*, is the type* $X \Rightarrow \mathcal{L}$ *equipped with the canonical point-wise operations, i.e. for* $R : \mathsf{CommRing}_\ell$ *and* $x : X(R)$*, top, bottom, join and meet are given by*

$$\top_R(x) = D(1), \quad \bot_R(x) = D(0)$$
$$(U \wedge V)_R(x) \;=\; U_R(x) \wedge V_R(x)$$
$$(U \vee V)_R(x) \;=\; U_R(x) \vee V_R(x)$$

This defines a functor $\mathsf{CompOpen} : \mathbb{Z}\mathsf{Functor}_\ell \to \mathsf{DistLattice}^{op}_{\ell+1}$*. With the action on morphisms given by pre-composition.*

▶ **Definition 16.** $X : \mathbb{Z}\mathsf{Functor}_\ell$ *is a* qcqs-scheme *if it is a local* $\mathbb{Z}$*-functor and has an affine cover by compact opens. That is, there merely exist compact opens* $U_1, ..., U_n : X \Rightarrow \mathcal{L}$ *such that each* U_i *is affine and* $\top \equiv \bigvee_{i=1}^{n} U_i$ *in the lattice* $\mathsf{CompOpen}(X)$*.*

As an immediate sanity check we get that affine schemes are qcqs-schemes:

▶ **Proposition 17.** $\mathsf{Sp}(R)$ *is a qcqs-scheme, for* $R : \mathsf{CommRing}_\ell$*.*

Proof. $\mathsf{Sp}(R)$ is local by Theorem 12. The top element $\top : \mathsf{CompOpen}(\mathsf{Sp}(R))$ is the "constant" natural transformation, sending everything to $D(1)$, which by the Yoneda lemma corresponds to $D(1) : \mathcal{L}_R$. It thus constitutes a trivial affine cover with $[\![\top]\!]^{\mathsf{co}} \cong \mathsf{Sp}(R)$. ◀

▶ Remark 18. Of course, a qcqs-scheme X can and will have multiple different covers. Definition 16 suggests that "having an affine cover" should be expressed using nested mere existential quantification. In practice, it is more convenient to define the record-type AffineCover, of all affine covers of X, consisting of a finite list or vector of compact opens and proofs that these compact opens are affine and cover X. The property of having an affine cover is then defined as the truncation of this record type:

```
record AffineCover (X : ℤFunctor ℓ)
                   : Type (ℓ-suc ℓ) where
  field
    n : ℕ
    U : FinVec (CompactOpen X) n
    covers : isCompactOpenCover X U
    isAffineU : ∀ i → isAffineCompactOpen (U i)
```

```
hasAffineCover : ℤFunctor ℓ → Type (ℓ-suc ℓ)
hasAffineCover X = ∥ AffineCover X ∥
```

If we want to prove a proposition about a qcqs-scheme X, we can assume we have a witness of type $\mathsf{AffineCover}(X)$. If we want to map from X into a set by using that X has an affine cover, we have to show that the mapping is independent of the choice of cover.[11] This is very much in line with informal mathematical practice.

[10] It is even an internal lattice the big Zariski topos, i.e. in local $\mathbb{Z}$-functors. However for our purposes, we do not need that $\mathcal{L}$ is a Zariski sheaf.

[11] This holds by the general *elimination* principle of the propositional truncation due to Kraus [21]. When mapping into types that are not sets, things get complicated very quickly.

▶ Remark 19. One big advantage of using the internal lattice $\mathcal{L}$ to classify compact opens is that we get the notion of cover for free from the induced lattice operations in Definition 15. In textbooks, a cover by open subfunctors is usually defined directly using addition of ideals [12, 26] or by taking the set-theoretic union at field-valued points [13]. The latter is not an option for our purposes, as the notion of field is not well-behaved constructively. In the `Cubical Agda` library, the join $\_ \vee \_ : \mathcal{L}_A \to \mathcal{L}_A \to \mathcal{L}_A$ is also defined in terms of ideal addition, but we can upstream the necessary constructions and do not have to concern ourselves with pullbacks of $\mathbb{Z}$-functors.

6 Open subschemes

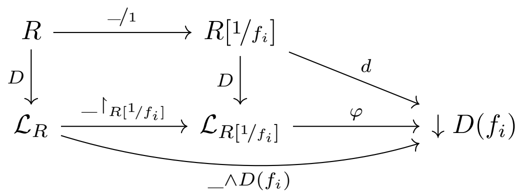
The benchmark for a workable formal definition of schemes as locally ringed spaces, as in [3, 4], usually consists of a proof of the "universal property", i.e. an adjunction $\Gamma \dashv \mathsf{Spec}$ where the counit is an isomorphism. Proposition 4, the functorial analogue is rather straightforward to prove. Instead, we give a proof that compact opens of affine schemes are qcqs-schemes. This can be seen as a constructive special case of the standard classical result that "open subfunctors of schemes are themselves schemes" [12, Ch. I, §1, 3.11]. We start by showing that compact opens of Zariski sheaves are Zariski sheaves. Essentially, this holds because compact opens are classified by $\mathcal{L}$, which is itself a Zariski sheaf. As it turns out, however, it is sufficient to prove something weaker.

For the remainder of the paper we adopt the following notation: For a ring R and elements $f : R$ and $u : \mathcal{L}_R$, we write $u{\restriction}_{R[1/f]} : \mathcal{L}_{R[1/f]}$ for the result of applying $\mathcal{L}_{(\_/1)}$, the $\mathcal{L}$-action on the canonical morphism. In particular we have $D(g_1, \dots, g_m){\restriction}_{R[1/f]} = D(g_1/1, \dots, g_m/1)$.

▶ **Lemma 20.** *$\mathcal{L}$ is Zariski-separated, i.e. for $R : \mathsf{CommRing}_\ell$ and $f_1, \dots, f_n : R$ unimodular the following holds: given $u, v : \mathcal{L}_R$, if $u{\restriction}_{R[1/f_i]} \equiv v{\restriction}_{R[1/f_i]}$ for all $i = 1, \dots, n$, then $u \equiv v$.*

Proof. Let $R : \mathsf{CommRing}_\ell$ and $f_1, \dots, f_n : R$ unimodular be given together with $u, v : \mathcal{L}_R$ satisfying $u{\restriction}_{R[1/f_i]} \equiv v{\restriction}_{R[1/f_i]}$ for all $i = 1, \dots, n$. Recall that for $i = 1, \dots, n$, the restriction $\_{\restriction}_{R[1/f_i]} : \mathcal{L}_R \to \mathcal{L}_{R[1/f_i]}$ is induced by the support $D(\_/1) : R \to \mathcal{L}_{R[1/f_i]}$. Now let us fix an $i = 1, \dots, n$. Much like in classical algebraic geometry, we can identify $\mathcal{L}_{R[1/f_i]}$ with $\downarrow D(f_i)$, the lattice of elements of $\mathcal{L}_R$ smaller than $D(f_i)$.[12] The map $d : R[1/f_i] \to \downarrow D(f_i)$ given by $d(r/f_i^n) = D(r) \wedge D(f_i)$ defines a support and thus induces a morphism $\varphi : \mathcal{L}_{R[1/f_i]} \to \downarrow D(f_i)$.

Now, consider the map $\_ \wedge D(f_i) : \mathcal{L}_R \to \downarrow D(f_i)$. We claim that $\_ \wedge D(f_i)$ factors through φ. By the universal property of $\mathcal{L}_R$, there is a unique $\psi : \mathcal{L}_R \to \downarrow D(f_i)$, such that $\psi \circ D \equiv d(\_/1)$. Both $\_ \wedge D(f_i)$ and $\varphi(\_{\restriction}_{R[1/f_i]})$ satisfy the same commutativity condition as ψ, which implies $\_ \wedge D(f_i) \equiv \psi \equiv \varphi(\_{\restriction}_{R[1/f_i]})$. Pictorially, this amounts to observing that the following diagram commutes



From our assumption it thus follows that

$$u \wedge D(f_i) \equiv \varphi(u{\restriction}_{R[1/f_i]}) \equiv \varphi(v{\restriction}_{R[1/f_i]}) \equiv v \wedge D(f_i)$$

[12] Showing that $\mathsf{Spec}(R[1/f])$ is homeomorphic to $D(f)$ is a standard exercise in algebraic geometry.

for all $i = 1, ..., n$. Since the f_i's are unimodular, we get $D(1) \equiv \bigvee_i D(f_i)$ and hence

$$u \equiv u \wedge D(1) \equiv \bigvee_{i=1}^{n} (u \wedge D(f_i)) \equiv \bigvee_{i=1}^{n} (v \wedge D(f_i)) \equiv v \wedge D(1) \equiv v$$ ◀

▶ **Lemma 21.** *If $X : \mathbb{Z}\mathsf{Functor}_\ell$ is local, then for any compact open $U : X \Rightarrow \mathcal{L}$ its realization $[\![U]\!]^{\mathsf{co}} : \mathbb{Z}\mathsf{Functor}_\ell$ is local.*

Proof. Let $R : \mathsf{CommRing}_\ell$ and $f_1, ..., f_n : R$ be unimodular. We need to construct an inverse to the map

$$\sigma_U : \Sigma[\ x \in X(R)\ ]\ U(x) \equiv D(1) \ \rightarrow\ \mathsf{CompatibleFam}^{[\![U]\!]^{\mathsf{co}}}\big(\{f_i\}_{i=1,...,n}\big)$$

For $x : X(R)$ with $U(x) \equiv D(1)$, $\sigma_U(x)$ is the family of elements $x{\restriction_{R[1/f_i]}}$. It is essentially the same map as the corresponding

$$\sigma_X : X(R) \ \rightarrow\ \mathsf{CompatibleFam}^{X}\big(\{f_i\}_{i=1,...,n}\big)$$

but it keeps track of the fact that for each $i = 1, ..., n$ one has $U(x{\restriction_{R[1/f_i]}}) \equiv D(1)$.

Now, any compatible family of elements $x_i : X(R[1/f_i])$ with $U(x_i) \equiv D(1)$ can be seen as a compatible family on X by forgetting that $U(x_i) \equiv D(1)$. To this family we apply the inverse map

$$\sigma_X^{-1} : \mathsf{CompatibleFam}^{X}\big(\{f_i\}_{i=1,...,n}\big) \ \rightarrow\ X(R)$$

that exists since X was assumed local. We claim that $U(\sigma_X^{-1}(\{x_i\}_{i=1,...,n})) \equiv D(1)$, thus allowing us to set $\sigma_U^{-1}(\{x_i\}_{i=1,...,n}) = \sigma_X^{-1}(\{x_i\}_{i=1,...,n})$. From this it also follows immediately that σ_U and σ_U^{-1} are mutually inverse. To prove the claim we use Lemma 20 and the fact that for each $i = 1, ..., n$:

$$\begin{aligned} U\big(\sigma_X^{-1}(\{x_i\}_{i=1,...,n})\big){\restriction_{R[1/f_i]}} &\equiv U\big(\sigma_X^{-1}(\{x_i\}_{i=1,...,n}){\restriction_{R[1/f_i]}}\big) \\ &\equiv U\big(\sigma_X\big(\sigma_X^{-1}(\{x_i\}_{i=1,...,n})\big)_i\big) \ \equiv U(x_i) \equiv D(1) \end{aligned}$$ ◀

It remains to prove that compact opens of affine schemes (merely) have an affine cover. Before treating arbitrary compact opens, we introduce the standard or basic opens of a representable $\mathbb{Z}$-functor with fair bit of abuse of notation.

▶ **Definition 22.** *Let $R : \mathsf{CommRing}_\ell$ and $f : R$, the standard open $D(f) : \mathsf{Sp}(R) \Rightarrow \mathcal{L}$ is given by applying the Yoneda lemma to the basic open $D(f) : \mathcal{L}_R$.*

▶ **Proposition 23.** *For $R : \mathsf{CommRing}_\ell$ and $f : R$, the standard open $D(f)$ is affine. In particular one has a natural isomorphism $[\![D(f)]\!]^{\mathsf{co}} \cong \mathsf{Sp}\big(R[1/f]\big)$.*

Proof. The universal properties of localization and Zariski lattice give us for A-valued points

$$\begin{aligned} \mathsf{Sp}\big(R[1/f]\big)(A) &= \mathsf{Hom}(R[1/f], A) \\ &\cong \Sigma[\ \varphi \in \mathsf{Hom}(R, A)\ ]\ \varphi(f) \in A^{\times} \\ &\cong \Sigma[\ \varphi \in \mathsf{Hom}(R, A)\ ]\ D(\varphi(f)) \equiv D(1) \\ &= [\![D(f)]\!]^{\mathsf{co}}(A) \end{aligned}$$

We omit the proof that this is natural in A. ◀

▶ **Theorem 24.** *The realization $[\![U]\!]^{\mathsf{co}}$ of a compact open $U : \mathsf{Sp}(R) \Rightarrow \mathcal{L}$ is a qcqs-scheme.*

Proof. We get that $[\![U]\!]^{co}$ is local from Lemma 21 and Theorem 12, the subcanonicity of the Zariski coverage. It remains to show that $[\![U]\!]^{co}$ (merely) has an affine cover. By the Yoneda lemma, the compact open U corresponds to an element $u : \mathcal{L}_R$. Every element of $\mathcal{L}_R$ can (merely) be expressed as a join of basic opens, i.e. we can assume $u \equiv \bigvee_i D(f_i)$ for some $f_1, ..., f_n : R$. Since the Yoneda lemma actually gives us an isomorphism of lattices between $\mathcal{L}_R$ and $\mathsf{CompOpen}(\mathsf{Sp}(R))$, we get a cover of compact opens $U \equiv \bigvee_i D(f_i)$ which is affine by Proposition 23. Note that this is an equality in the lattice $\mathsf{CompactOpen}(X)$. But since $D(f_i) \leq U$ in $\mathsf{CompactOpen}(X)$ for $i = 1, ..., n$, we may regard the $D(f_i)$ as affine compact opens of $[\![U]\!]^{co}$ covering of the top element of $\mathsf{CompactOpen}([\![U]\!]^{co})$. ◀

7 Conclusion

In this paper we presented a formalization of qcqs-schemes as a full subcategory of the category of $\mathbb{Z}$-functors. We defined the Zariski coverage on $\mathsf{CommRing}_\ell^{op}$ and proved it subcanonical. This let us define locality of $\mathbb{Z}$-functors and conclude that affine schemes, i.e. representable $\mathbb{Z}$-functors, are local. When formalizing the notion of an open covering, we introduced compact open subfunctors. We argued that compact opens can conveniently be classified by the $\mathbb{Z}$-functor that maps a ring to its Zariski lattice. We leveraged this fact to automatically obtain a notion of covering by compact opens and thus a formal definition of qcqs-schemes. Finally, we gave a fully constructive proof that compact opens of affine schemes are qcqs-schemes using only point-free methods.

As mentioned before, our formalization should be regarded as a univalent rather than a cubical formalization. We do not depend on cubical features of `Cubical Agda` such as the interval. However, we are adopting the univalent approach of distinguishing propositions, sets etc. internally and we do require the propositional truncation and the set-quotient HITs. Univalence is only used in the guise of its useful consequences like function extensionality.

7.1 Going classical

`Cubical Agda`'s type theory is fully constructive and predicative. Using set-quotients, we can define the Zariski lattice over a ring living in the same universe as the base ring, as shown in [39]. This predicative definition is essential for defining the classifier $\mathcal{L} : \mathbb{Z}\mathsf{Functor}_\ell$ of compact opens and thus plays a key role in our definition of functorial qcqs-schemes. This makes our approach easily extensible with the using additional logical assumptions. If one would want to formalize not only qcqs- but general schemes using the functor of points approach, this should be directly possible by using a classifier for opens, not only compact opens, instead.

Assuming impredicativity, e.g. in the form of Voevodsky's resizing axioms [37], one could define the classifier for open subfunctors as the $\mathbb{Z}$-functor sending a ring R to the *frame* of radical ideals of R.[13] Alternatively, assuming classical logic, one could use the frame of Zariski-open subsets of $\mathsf{Spec}(R)$ as the classifier. This also induces a notion of cover (not necessarily finite this time) and hence a notion of general functorial schemes. We expect that in this situation one can closely follow the approach of Section 6 to get a corresponding proof that open subfunctors of affine schemes are schemes. The only difference being perhaps the proof of Lemma 20, the fact that the classifier is separated wrt. the Zariski coverage.

[13] Impredicativity is needed to ensure that the type of ideals of a ring R lives in the same universe as R.

We decided to stick to qcqs-schemes not only because crucial tools like the Zariski lattice were already available in the `agda/cubical` library. We hope that the paper contains valuable insights for constructive mathematicians interested in the foundations of algebraic geometry, while still being usable as a blue-print for formalizing the functor of points approach in other (possibly classical) proof assistants.

7.2 Synthetic algebraic geometry

The functor of points approach allows one to develop algebraic geometry synthetically. Here, the word synthetic means "working in the internal language of a suitable topos". In our case this topos is the *big Zariski topos*, i.e. the sheaf topos of local $\mathbb{Z}$-functors. From the internal point of view, Zariski sheaves look like simple sets, which can make reasoning about them easier. The PhD thesis of Blechschmidt [2] contains an excellent introduction to synthetic algebraic geometry for interested readers familiar with classical algebraic geometry.

This approach can even be axiomatized. Recently, Cherubini, Coquand and Hutzler [6] have combined the axiomatic approach to synthetic algebraic geometry with HoTT/UF. By adding the axioms of synthetic algebraic geometry to a dependent type theory with univalence and HITs one can even study the cohomology of schemes synthetically. They give a model construction in a "higher" Zariski topos, where they restrict themselves to functors from *finitely presented algebras* to sets in order to avoid size issues. Finitely presented algebras over a ring R are of the form $R[x_1,\dots,x_n]/\langle p_1,\dots,p_m\rangle$. For a fixed R, the category of f.p. R-algebras is small and one can thus use it to develop functorial algebraic geometry without having to worry about universe levels. Repeating the steps outlined in this paper for f.p. algebras should give rise to a truly predicative formalization of schemes of finite presentation over R.

7.3 A constructive comparison theorem

For algebraic geometers, using the functor of points approach can sometimes be advantageous, but ultimately one wants to switch seamlessly between schemes as $\mathbb{Z}$-functors and schemes as locally ringed spaces. This is made possible by the so-called *comparison theorem* [12, p. 23], giving an adjunction between $\mathbb{Z}$-functors and locally ringed spaces, which becomes an equivalence of categories when restricted to the respective full subcategories of schemes.

Coquand, Lombardi and Schuster [8] give a point-free reconstruction of geometric qcqs-schemes that is suitable for constructive study. Instead of using locally ringed spaces, their "spectral schemes" are given as distributive lattices with a sheaf of rings. The affine scheme associated to a ring R is just the Zariski lattice $\mathcal{L}_R$ equipped with the usual structure sheaf. Classically, these spectral schemes are equivalent to conventional qcqs-schemes because the topology of a qcqs-scheme is *coherent* or *spectral*.[14]

Our definition of qcqs-scheme can be seen as the functorial counterpart to the lattice-based definition of spectral scheme due to Coquand, Lombardi and Schuster. One would hope that these turn out to be equivalent by a *constructive* comparison theorem à la Demazure and Gabriel. Proving such a theorem requires a decent amount of novel constructive mathematics to be developed first. One needs to introduce a point-free notion of *locally* ringed distributive lattices that contain spectral schemes as a full subcategory and construct a suitable adjunction with $\mathbb{Z}$-functors. In our setting, one would only get a relative coadjunction as in Remark 5. This could be a particularly interesting problem in a univalent setting.

[14] Stone's representation theorem for distributive lattices [30], tells us that all topological information of a coherent space is encoded in its lattice of compact open subsets.

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