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Reverse Hölder inequalities on the space of Kähler metrics of a Fano variety and effective openness

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Abstract

A reverse Hölder inequality is established on the space of Kähler metrics in the first Chern class of a Fano manifold X endowed with Darvas' L^p-Finsler metrics. The inequality holds under a uniform bound on a twisted Ricci potential and extends to Fano varieties with log terminal singularities. Its proof leverages a "hidden" log-concavity. An application to destabilizing geodesic rays is provided, which yields a reverse Hölder inequality for the speed of the geodesic. In the case of Aubin's continuity path on a K-unstable Fano variety, the constant in the corresponding Hölder bound is shown to only depend on p and the dimension of X. This leads to some intriguing relations to Harnack bounds and the partial C^0-estimate. In another direction, universal effective openness results are established for the complex singularity exponents (log canonical thresholds) of ω-plurisubharmonic functions on any Fano variety. Finally, another application to K-unstable Fano varieties is given, involving Archimedean Igusa zeta functions.

1 Introduction

1.1 Reverse Hölder inequalities on the space of Kähler metrics

Let X be an n-dimensional compact connected Kähler manifold. Consider the space $\mathcal{H}$ of all Kähler metrics on X, in a fixed cohomology class in $H^2(X,\mathbb{R})$. Assuming that $\mathcal{H}$ contains some Kähler metric ω, the space $\mathcal{H}$ may be identified with the quotient space $\mathcal{H}(X,\omega)/\mathbb{R}$, where $\mathcal{H}(X,\omega)$ denotes the space of all Kähler potentials (relative to ω) :

$$\mathcal{H}(X,\omega) := \left\{u \in C^\infty(X) : \omega_u := \omega + dd^c u > 0\right\} \quad \left(dd^c u := \frac{i}{2\pi}\partial\bar{\partial}u\right). \tag{1.1}$$

A canonical Riemannanian metric on $\mathcal{H}(X,\omega)$ was introduced in [48, 72, 83], turning $\mathcal{H}$ into an infinite dimensional Riemannian symmetric space of constant non-negative sectional curvature. More generally, a canonical L^p-Finsler metric on $\mathcal{H}(X,\omega)$ was put forth in [37], defined by

$$\left\|\frac{du_t}{dt}\right\|_p := \left(\int_X \left|\frac{du_t}{dt}\right|^p \frac{\omega_{u_t}^n}{V}\right)^{1/p}, \quad V := \int_X \omega^n$$

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This Finsler metric induces a bona fide metric d_p on $\mathcal{H}(X,\omega)$, as shown in [37] (generalizing the case $p=2$ established in [32]). From the point of view of quantization, the metric space $(\mathcal{H}(X,\omega),d_p)$ can be viewed as a limit of the finite dimensional spaces $GL(N,\mathbb{C})/U(N)$, endowed with an L^p-Finsler metric induced by the standard l^p-norm on $\mathbb{R}^N$, as $N\to\infty$ [42].

An important motivation for allowing $p\neq 2$ comes from the existence problem for canonical metrics on X, where $p=1$ plays a privileged role. For example, assuming for simplicity that X admits no non-trivial holomorphic vector fields, there exists, by [14, 33], a constant scalar curvature Kähler (CSCK) metric in $\mathcal{H}$ iff the Mabuchi functional $\mathcal{M}$ on $\mathcal{H}$, introduced in [72], admits a minimizer (namely the CSCK metric) iff $\mathcal{M}$ is coercive with respect to d_1, i.e. iff

$$\mathcal{M}(u)\geq Cd_1(u,0)-C$$

for some constant C (where $\mathcal{H}$ has been identified with the space $\mathcal{H}(X,\omega)_0$ of all Kähler potentials u satisfying $\sup_X u=0$). Moreover, in the *Fano case*—i.e when ω is in the first Chern class $c_1(X)$ of X—the coercivity in question is equivalent to (uniform) K-stability, by the solution of the Yau–Tian–Donaldson conjecture in this case [15, 34, 43, 71].

By Hölder's inequality, applied to $\left\|\frac{du_t}{dt}\right\|_p$,

$$d_1(u,0)\leq d_p(u,0)$$

However, in general, $d_p(u,0)$ can not be controlled by $d_1(u,0)$, since the metric completions $\overline{(\mathcal{H}(X,\omega),d_p)}$ are strictly decreasing with respect to p. This is illustrated by the toric case, where d_p may be identified with the standard L^p-norm for the space of convex functions on the moment polytope of X [56]. In contrast, a reverse Hölder inequality does hold in $\mathbb{R}^N$ (by the compactness of the unit-sphere),

$$\|\cdot\|_p\leq C_N\|\cdot\|_1\,, \tag{1.2}$$

but the constant C_N diverges as $N\to\infty$. Still, the following result shows that in the Fano case a reverse Hölder type inequality holds on $\mathcal{H}(X,\omega)$, under the assumption that the twisted Ricci potential $\rho_{u,\gamma}$ of u is uniformly bounded, for some $\gamma\in]0,1[$.

Theorem 1.1 *Let X be an n-dimensional Fano manifold. Given a Kähler form ω in $c_1(X)$, the following inequality holds on $\mathcal{H}(X,\omega)$, for any $p\in[1,\infty[$ and $\gamma\in]0,1[$*

$$d_p(u,0)\leq Ad_1(u,0)+B,$$

with

$$A=A_pe^{2\|\rho_{u,\gamma}\|_{L^\infty}},\quad B=B_p\left(\gamma^{-1}+(1-\gamma)^{-1}\right)e^{\|\rho_{u,\gamma}\|_{L^\infty}}$$

where A_p only depends on (p,n) and B_p also depends on (X,ω). Moreover, if $\sup_X u\leq 0$ then $A_p/(n+1)$ is independent of n and B_p only depends on (p,n).

We recall that the twisted Ricci potential $\rho_{u,\gamma}$, which depends on (ω_u,ω,γ), may be defined by the equations

$$dd^c\rho_{u,\gamma}=\mathrm{Ric}\omega_u-\gamma\omega_u-(1-\gamma)\omega,\quad \int_X e^{\rho_{u,\gamma}}\frac{\omega_u^n}{V}=1. \tag{1.3}$$

In particular, $\rho_{u,t}$ vanishes identically along *Aubin's continuity path* ω_t, defined by the following equations [1]:

$$\mathrm{Ric}\,\omega_t=t\omega_t+(1-t)\omega, \tag{1.4}$$

which for $t = 1$ is the Kähler Einstein equation. By [15, 31], the sup over all $t \in [0, 1[$ for which the equations are solvable coincides with $\min\{\delta(X), 1\}$, where $\delta(X)$ is the algebro-geometric invariant introduced in [52]. This invariant—known as the *delta-invariant* or the *stability threshold* of X - has the property that X is (uniformly) K-stable if and only if $\delta(X) > 1$ and K-semistable iff $\delta(X) \geq 1$ [22, 71].

1.2 Application to destabilizing geodesic rays

There is a range of deformation methods in Kähler geometry that—given an initial Kähler form ω_0 in $c_1(X)$—produce a path ω_t of Kähler metrics along which the Mabuchi functional $\mathcal{M}$ decreases. A notable example—apart from Aubin's continuity path ω_t—is the Kähler–Ricci flow. For these two examples it is well-known that, as t is increased, the d_1-distance $d_1(u_t, 0)$ at the level of Kähler potentials u_t stays bounded iff X admits a Kähler-Einstein metric, in which case u_t converges to a Kähler-Einstein potential. On the other hand, if $d_1(u_t, 0) \to \infty$ then u_t is weakly asymptotic to a geodesic ray in the metric completion $\overline{(\mathcal{H}(X, \omega), d_1)}$ along which $\mathcal{M}$ decreases. In fact, the existence of such a d_1-geodesic only uses that $\mathcal{M}(u_t)$ is decreasing (this result is implicit in [39]). However, for special deformations one should obtain geodesic rays v_t with advantageous properties. In the light of the Yau-Tian-Donaldson conjecture and its ramifications the best one can hope for is that the ray v_t be induced by a *test configuration* for X [34, 35, 43] (as we shall come back to below). In particular, such a ray is a d_p-geodesic ray in the metric completions $\overline{(\mathcal{H}(X, \omega), d_p)}$ for *any* $p \geq 1$. Here we show that the latter property holds under a uniform bound on the twisted Ricci potentials of u_t:

Corollary 1.2 *Let X be an n-dimensional Fano manifold and u_j a sequence in $\mathcal{H}(X, \omega)$ such that*

$$(i)\, d_1(u_j, 0) \to \infty, \quad (ii)\, \mathcal{M}(u_j) \leq C, \quad (iii)\, \left\| \rho_{u,\gamma_j} \right\|_{L^\infty(X)} \leq R$$

for some sequence γ_j contained in a compact subset of $]0, 1[$. *Then*

- *u_j is weakly asymptotic to a ray v_t which is a d_p-geodesic ray in $\overline{(\mathcal{H}(X, \omega), d_p)}$ for any $p \in [1, \infty[$ and $t \mapsto \mathcal{M}(v_t)$ is decreasing.*
- *the d_p- speed $\|\dot{v}\|_p$ of the geodesic v_t satisfies*

$$\|\dot{v}\|_p \leq A \|\dot{v}\|_1 , \tag{1.5}$$

for a constant A of the form $A_p e^{2R}$ where A_p only depends on (n, p).

When the bound on the Ricci potential is replaced by a uniform Harnack bound, the first item above was established in [39, Thm 3.2] and applied to the Kähler-Ricci flow. The proof in [39] uses the Harnack bound in [81], which also holds for Aubin's continuity path [2, 85, 87]. However, in general, Harnack bounds tend to require rather detailed control on ω_u, such as lower bounds on the Ricci curvature or uniform Sobelev constants (as discussed in Sect. 7). Accordingly, one advantage of Theorem 1.1 and its corollary is that they generalize to situations where such bounds are missing. In particular, as next discussed, the results apply to singular Fano varieties (see also Sect. 7.1 for an application to Aubin type equations on non-singular X in the presence of non-positive Ricci curvature).

By the solution of the Yau-Tian-Donaldson conjecture for singular Fano varieties [68, 70, 71], such a variety X admits a Kähler-Einstein metric if and only if it is K-polystable. For non-singular X this was originally shown in [34] using a singular version of Aubin's continuity path and then in [43] using Aubin's original continuity path ω_t. The proof is based on the partial C^0-estimate, which yields a detailed description of the blow-up behaviour of ω_t (as discussed below). However, for singular X the partial C^0-estimate is missing and the only proof of the Yau-Tian-Donaldson conjecture is variational [68, 70, 71], building on [12, 15]. In general, given a positive $(1, 1)$-current ω in $c_1(X)$, with locally bounded potentials the variational approach in [12] shows that there exists a solution ω_t to Aubin's continuity equation 1.3 for some $t > 0$ (in the weak sense of pluripotential theory) iff X has log terminal singularities. The following result describes the blow-up behaviour of ω_t for singular X in terms of the metric spaces $\overline{(\mathcal{H}(X, \omega), d_p)}$ (defined on singular varieties in [46]):

Corollary 1.3 *Assume that X is an n-dimensional K-unstable Fano variety with log terminal singularities, i.e.* $\delta(X) \in]0, 1[$. *Given a positive* $(1, 1)$*-current* ω *in* $c_1(X)$ *with locally bounded potentials, denote by* ω_t *the corresponding solutions to Aubin's continuity equation 1.4, defined for* $t \in [0, \delta(X)[$. *Then the curve* u_t *of the corresponding sup-normalized potentials* u_t *is weakly asymptotic—as t increases towards* $\delta(X)$*—to a non-trivial asymptotic ray* v_t*, which is a* d_p*-geodesic ray in* $\overline{(\mathcal{H}(X, \omega), d_p)}$ *for any* $p \in [1, \infty[$ *and* $t \mapsto \mathcal{M}(v_t)$ *is decreasing. Moreover, the* d_p*- speed* $\|\dot{v}\|_p$ *of the geodesic* v_t *satisfies*

$$\|\dot{v}\|_p \leq A_p(n+1)\,\|\dot{v}\|_1\,, \tag{1.6}$$

for a constant A_p *only depending on* p.

When X is non-singular we show that A_p can be taken as 1, using the Harnack type bound in [87] (see Sect. 7). It should be stressed that in the singular case there is an infinite number of deformation types of K-unstable Fano varieties with log terminal singularities in any given dimension n. Indeed, this is the case already for toric Fano varieties; see the examples 4.2 in [44, page 100], which are K-unstable by [9].

1.2.1 Relations to the partial C^0 -estimate

There are some intriguing relations between the inequality 1.6 and the partial C^0-estimate along Aubin's continuity path conjectured in [88] and established in [86], when X is non-singular. To explain this recall that the partial C^0-estimate says that the Kähler potential u_t of ω_t is of the form

$$u_t = \varphi_t + O(1), \tag{1.7}$$

where $O(1)$ is uniformly bounded in $L^\infty(X)$ and φ_t is a family of Bergman metrics associated to a some tensor power $K_X^{*\otimes k} \to X$. Embedding X in the projectivization of $H^0(X, K_X^{*\otimes k})$, identified with $\mathbb{P}^{N_k-1}$, this means that the corresponding Kähler forms ω_{φ_t} are the restriction to X of $G_t^*\omega_{\mathrm{FS}}$, for some curve $G_t \in \mathrm{GL}\,(N_k, \mathbb{C})$, where ω_{FS} denotes the Fubini-Study metric on $\mathbb{P}^{N_k-1}$. Since the space of Bergman metrics at level k—which is parameterized by $\mathrm{GL}\,(N_k, \mathbb{C})/U(N_k)$—is finite dimensional it seems thus natural to expect that the inequality 1.6 could be deduced from the standard reverse Hölder inequality in $\mathbb{R}^{N_k}$ (formula 1.2). This can be made more precise as follows. As shown in [34, III], a subsequence of a family G_t in $\mathrm{GL}\,(N_k, \mathbb{C})$—satisfying appropriate assumptions—is asymptotic to a one-parameter subgroup of $\mathrm{GL}\,(N_k, \mathbb{C})$, induced by a *special test configuration* $\mathcal{X}$ for X. If these assumption would apply, the previous corollary would follow from the fact that geodesic rays associated to a special test configurations also satisfy a universal reverse Hölder inequality of the form 1.6 (as observed in Sect. 8.1). However, as discussed in [43, Section 3.1], the assumptions in question have not yet been established for Aubin's original continuity method. But they do hold in the singular setup considered in [34], where the Kähler form ω is replaced by the positive current defined by an appropriate anti-canonical divisor Δ on X. Anyhow, the partial C^0-estimate appears to be wide open for singular Fano varieties X (since its proof requires, in particular, a uniform bound on the Sobolev constants; cf. [96]).

In the light of the previous discussion it seems natural to conjecture that the geodesic ray v_t appearing in the previous corollary is induced by a special test configuration $\mathcal{X}$ and that $\mathcal{X}$ computes the stability threshold $\delta(X)$ in the sense of [71, Thm 1.2] (a related conjecture is proposed in [94]).

1.3 Proof of Theorem 1.1 via log-concavity and moment bounds

In a nutshell, the idea of the proof of Theorem 1.1 is to relate $d_p(u, 0)$ to the moments of a somewhat hidden log-concave measure on $\mathbb{R}$ and use the reverse Hölder inequality for random variables with log-concave distribution [74, App.III] [66], known as the Kahane–Khinchin inequality. This leads to universal moment bounds (Theorem 4.1), from which Theorem 1.1 is deduced. In general, $d_p(u, 0)$ can also expressed directly as the p-th moment of a probability measure on $\mathbb{R}$, introduced in [21]. However, as pointed out in Sect. 8.2, this measure is not log-concave, nor is the analogous measure associated to a test configuration, studied in [24, 61, 93], in general.

1.4 Effective openness on Fano varieties

In another direction, the universal moment bounds alluded to above yield universal effective openness results for complex singularity exponents on Fano varieties. In order to state these results, we first recall some standard notation. Given a compact Kähler manifold X, denote by PSH (X, ω) the space of all ω-plurisubharmonic functions on X. This is the space of all $u \in L^1(X)$ such that $\omega_u \geq 0$, in the sense of currents on X (where ω_u is the current defined as in formula 1.1). The *complex singularity exponent* [45] of a function $u \in \mathrm{PSH}(X, \omega)$ also known as its *log canonical threshold*, is defined as the positive number

$$c_u := \sup \left\{ \gamma : Z_u(\gamma) := \int_X e^{-\gamma u} dV < \infty \right\}$$

for any fixed volume form dV on X. By the solution of Demailly–Kollar's openness conjecture [45] in [51], when $n \leq 2$ and [16, 19], in general,

$$Z_u(\gamma) < \infty \implies Z_u(\gamma + \epsilon) < \infty. \tag{1.8}$$

for some $\epsilon > 0$. More precisely, in [16, 19] a stronger local result on a ball in $\mathbb{C}^n$ is established which yields an effective bound on ϵ of the following form

$$\epsilon < \frac{\gamma}{C_{(X,\omega)} Z_u(\gamma)} \tag{1.9}$$

for a constant $C_{(X,\omega)}$ depending on (X, ω) (by covering X with a finite number of coordinate balls). See also [54, 56, 58, 76] for more general effective results on the strong form of Demailly–Kollar's openness conjecture.

Here we will concerned with the case when X is a Fano. More generally, we will allow X to be a Fano variety with (at worst) log terminal singularities. Fix a measure dV on X corresponding to a locally bounded metric on the anti-canonical line bundle K_X^* of X with positive curvature current, denoted by ω. The assumption that X has log terminal singularities ensures that dV has finite total mass. The following result involves the logarithmic derivative

$$\frac{d\log Z_u(\gamma)}{d\gamma} = \frac{\int_X (-u)e^{-\gamma u}dV}{\int_X e^{-\gamma u}dV}, \quad \left(Z_u(\gamma) := \int_X e^{-\gamma u}dV\right)$$

when $c_u < 1$.

Theorem 1.4 *Let X be a Fano variety, $\omega \in c_1(X)$ and assume that $u \in PSH(X,\omega)$ satisfies $u \leq 0$. If $Z_u(\gamma) < \infty$ and $Z_u(1-\delta) = \infty$ for some $\delta > 0$, then $Z_u(\gamma+\epsilon) < \infty$ for any ϵ satisfying*

$$\epsilon < \frac{1}{A\frac{d\log Z_u(\gamma)}{d\gamma} + C\left(\gamma^{-2} + \delta^{-2}\right)}$$

for universal constants A and C (i.e. independent of X and, in particular, on the dimension of X).

More precisely, it will be shown that A can be taken arbitrarily close to 16 (at the expense of increasing C), by using the effective Kahane–Khinchin inequality in [75]. Moreover, for a fixed value of A, the constant C could also readily be estimated.

The previous theorem may be reformulated as the following universal lower bound:

$$\frac{d\log Z_u(\gamma)}{d\gamma} \geq \frac{1}{A}\frac{1}{(c_u-\gamma)} - B\left(\gamma^{-2} + (1-c_u)^{-2}\right) \quad (C = AB). \tag{1.10}$$

Integrating this bound yields the following variant of the effective bound 1.9 in the present setup, which depends on $Z_u(\gamma)$ and γ in a universal manner.

Corollary 1.5 *Let X be a Fano variety and assume that $u \in PSH(X,\omega)$ satisfies $u \leq 0$ and that $\int_X dV = 1$. If $Z_u(\gamma) < \infty$ and $Z_u(1-\delta) = \infty$ for some $\delta > 0$, then $Z_u(\gamma+\epsilon) < \infty$ for any ϵ satisfying*

$$\epsilon < \frac{e^{-a\left(\gamma^{-1}+\gamma\delta^{-2}\right)}}{Z_u(\gamma)^A}$$

for universal constants A and a.

It should be stressed that ω-plurisubharmonic functions satisfying $c_u < 1$ exist on all Fano varieties X, that are not *weakly exceptional* (in the sense that lct $(X) \geq 1$, where lct (X)denotes the global log canonical threshold of $-K_X$, which coincides with the alpha invariant of $-K_X$) [30]. For example, among the non-singular Fano surfaces (i.e. del Pezzo surfaces) only those with degree one, and without anti-canonical cuspidal curves, are weakly exceptional [29]. Furthermore, according to a conjecture of Cheltsov [28], only two of the 105 deformation families of non-singular Fano threefolds contain weakly exceptional members.

To close the circle, an application of the lower bound 1.10 to K-unstable Fano varieties is given in the final section of the paper.

2 Preliminaries

Throughout the paper the precise values of the constants C etc appearing in the inequalities may change from line to line.

2.1 The space of Kähler potentials

Let (X, ω) be a compact Kähler manifold of dimension n and denote by $\mathcal{H}(X, \omega)$ the corresponding space of all Kähler potentials u on X:

$$\mathcal{H}(X, \omega) := \left\{u \in C^\infty(X) : \omega_u := \omega + dd^c u > 0\right\} \quad \left(dd^c u := \frac{i}{2\pi}\partial\bar{\partial}u\right).$$

Following [37] the Finsler L^p-metric on $\mathcal{H}(X, \omega)$ defined by

$$\left(\int_X \left|\frac{du_t}{dt}\right|^p \frac{\omega_{u_t}^n}{V}\right)^{1/p}, \quad V := \int_X \omega^n$$

induces a metric d_p on $\mathcal{H}(X, \omega)$ satisfying the following inequalities, for constants $c_{n,p}$ only depending on n and p (see [37, Thm 3]):

$$\begin{aligned} &c_{n,p}^{-1}\left(\int |u_0 - u_1|^p \left(\frac{\omega_{u_0}^n}{V} + \frac{\omega_{u_1}^n}{V}\right)\right)^{1/p} \\ &\leq d_p(u_0, u_1) \leq c_{n,p}\left(\int |u_0 - u_1|^p \left(\frac{\omega_{u_0}^n}{V} + \frac{\omega_{u_1}^n}{V}\right)\right)^{1/p} \end{aligned} \tag{2.1}$$

The first inequality, applied to $(u_0, u_1) = (0, u)$, implies that

$$\left(\int |u|^p \frac{\omega^n}{V}\right)^{1/p} \leq c_{n,p} d_p(u, 0) \tag{2.2}$$

and, as a consequence,

$$\left|\sup_X u\right| \leq c_{n,1} d_1(u, 0) + C_\omega \tag{2.3}$$

(see [37, Cor 4]). In the case when $(u_0, u_1) = (0, u)$ and $u \leq 0$ the following more precise estimates hold:

Lemma 2.1 *For $u \in \mathcal{H}(X, \omega)$ satisfying* $\sup u \leq 0$

$$d_p(u, 0) \leq \left(\int |u|^p \left(\frac{\omega_u^n}{V}\right)\right)^{1/p}, \quad \int |u| \frac{\omega_u^n}{V} \leq (n+1) d_1(u, 0). \tag{2.4}$$

Proof The first inequality is contained in [37, Lemma 4.1]. To prove the second one, note that since $-u \geq 0$ we have that $\int |u| \frac{\omega_u^n}{V} \leq -(n+1)\mathcal{E}(u)$, where

$$\mathcal{E}(u) := \frac{1}{V(n+1)} \int_X \sum_{j=0}^n (-u)\omega_u^j \wedge \omega^{n-j} \tag{2.5}$$

(which is the unique primitive of the one-form on $\mathcal{H}(X, \omega)$ defined by $u \mapsto \omega_u^n / V$ satisfying $\mathcal{E}(u) = 0$). Finally, by [37, Cor 4.14], $-\mathcal{E}(u) = d_1(u, 0)$, when $\sup_X u \leq 0$. □

2.1.1 Metric completions, finite energy spaces and geodesics

Denote by PSH (X, ω) the subspace of $L^1(X)$ consisting of all *ω-plurisubharmonic (ω-psh)* functions on X, i.e. all strongly upper-semicontinuous functions u such that $\omega_u \geq 0$ holds in the sense of currents [57]. We will denote by ω_u^n the *non-pluripolar Monge-Ampère measure* of u [58]. As shown in [37]—answering a conjecture of Guedjs when $p = 2$—the metric completion $\overline{(\mathcal{H}(X,\omega),d_p)}$ of the metric space $(\mathcal{H}(X,\omega),d_p)$ may be identified with the finite energy space

$$\mathcal{E}^p(X,\omega) := \left\{ u \in \text{PSH}\,(X,\omega) : \int_X \omega_u^n = \int_X \omega^n, \int_X |u|^p \omega_u^n < \infty \right\},$$

introduced in [58] and the inequalities 2.1 hold on all of $\mathcal{E}^p(X,\omega)$. Moreover, the spaces $\mathcal{E}^p(X,\omega)$ are strictly decreasing wrt p.

Any two elements u_0, u_1 in $\mathcal{E}^1(X,\omega)$ can be connected by a canonical path u_t called a *finite energy geodesic* in [37, 40] and a *psh geodesic* in [15], defined by the following envelope:

$$u_t(x) := \sup \left\{ v_t(x) : v_t \text{ is a subgeodesic } \limsup_{t\to 0} v_t \leq u_0, \limsup_{t\to 1} v_t \leq u_0 \right\}. \tag{2.6}$$

We recall that a *subgeodesic* v_t is defined as a curve in PSH (X,ω) with the following property: complexifying t the corresponding $i\mathbb{R}$-invariant function $V(x,t) := v_{\Re t}(x)$ on $X\times]0,1[\times i\mathbb{R}$ is in PSH $(X\times]0,1[\times i\mathbb{R},\omega)$, using the same notation ω for the pull-back of ω to the product $X \times (]0,1[\times i\mathbb{R})$.

Recall that, given a metric space (M,d), a curve u_t connecting two given points u_0 and u_1 in M is said to be a *d-geodesic* (also known as a *constant speed geodesic*) if

$$d(u_t,u_0) = td(u_1,u_0), \text{ when } t \in [0,1].$$

As shown in [37], building on [21], the psh geodesic u_t connecting any two given elements u_0, u_1 in $\mathcal{E}^p(X,\omega)$ is a d_p-geodesic. When $p > 1$ this is the unique d_p-geodesic connecting u_0, u_1 (by [40, Thm 3.3]). Given a d_p-geodesic v_t we will use the notation

$$\|\dot{v}\|_p := d_p(v_t,v_0)/t, \ t > 0$$

which is independent of t. When v_t is a geodesic ray emanating from 0 and v_0 is the constant geodesic 0, $\|\dot{v}\|_p$ is, in the terminology of [40], the distance between the d_p-geodesic v_t and v_0 wrt the cordal metric on the space of d_p-geodesic rays, introduced in [40].

2.2 The Fano setup

Henceforth, X will be assumed to be a compact *Fano manifold*. This means that the anti-canonical line bundle K_X^* is ample. Equivalently, X admits a volume form dV_X with positive Ricci curvature, i.e.

$$\omega := \text{Ric}\, dV_X \ \ (:= -dd^c \log dV_X)$$

defines a Kähler form on X in the first Chern class $c_1(X)$ of X. Conversely, any Kähler form ω in $c_1(X)$ may be expressed as in the previous equation and the corresponding volume form dV_X is uniquely determined by ω under the normalization condition $\int_X dV_X = 1$, which will henceforth be assumed. Denote by V the *volume* of X :

$$V := c_1(X)^n \left(= \int_X \omega^n \right)$$

To u in $\mathcal{H}(X,\omega)$ and $\gamma \in]0,\infty[$ we attach the following probability measure on X :

$$\mu_{\gamma u} := \frac{e^{-\gamma u} dV_X}{\int_X e^{-\gamma u} dV_X}. \tag{2.7}$$

(which only depends on the Kähler form $\omega + dd^c u$ when $\gamma = 1$). In terms of this measure, the *twisted Ricci potential* $\rho_{u,\gamma}$ of a Kähler potential u (definined in formula 1.3) may, alternatively, be defined by the relation

$$e^{\rho_{u,\gamma}} \frac{\omega_u^n}{V} = \mu_{\gamma u}. \tag{2.8}$$

$\rho_{u,1}$ is independent of ω and is usually simply called the *Ricci potential of* ω_u, denoted by ρ_{ω_u}. More generally, the twisted Ricci potential $\rho_{u,\gamma}$ is well-defined for any $u \in \mathrm{PSH}\,(X, \omega)$ satisfying

- the non-pluripolar Monge-Ampère measure ω_u^n is, locally, absolutely continious wrt Lesbegue measure
- $e^{-\gamma u} dV_X$ has finite total mass.

3 Log-concavity

Fix $u \in \mathcal{H}(X, \omega)$ such that $\sup_X u \leq 0$ and consider the following function on $\mathbb{R}$:

$$Z_u(\gamma) := \int_X e^{-\gamma u} dV_X.$$

The key analytic ingredient in the proofs in the coming sections is the following result, expressing $Z_u(\gamma)$ in terms of the Laplace transform of a *log concave measure* ν_0 on $\mathbb{R}$. That is to say that ν_0 is either a Dirac mass or ν_0 is absolutely continuous wrt Lesbegue measure on $\mathbb{R}$ and the logarithm of its density is concave.

Theorem 3.1 *For any $u \in \mathcal{H}(X, \omega)$ such that $\sup_X u \leq 0$ and any $\gamma \in]0, 1[$*

$$Z_u(\gamma) = G(\gamma) \int_{\mathbb{R}} e^{-t\gamma} \nu_0, \quad G(\gamma) := \frac{2\gamma}{\Gamma(1-\gamma)} \tag{3.1}$$

for a log-concave measure ν_0 on $\mathbb{R}$ (depending on u), where $\Gamma(s)$ denotes the Gamma-function. Moreover, the log-concave probability measures $\nu_\gamma := e^{-t\gamma}\nu_0 / \int_{\mathbb{R}} e^{-t\gamma}\nu_0$ satisfy the following inequality:

$$\int_{\mathbb{R}} |t| \nu_\gamma \leq \frac{d \log Z_u(\gamma)}{d\gamma} + C\left(\gamma^{-1} + (1-\gamma)^{-1}\right) \tag{3.2}$$

for a constant C independent of u and γ.

It should be stressed that the formula in the previous theorem does not hold with a *constant* function $G(\gamma)$, as can be checked in simple examples. This means that the measure on $\mathbb{R}$ defined as the push-forward of dV_X under u is not log-concave, in general. Still it would be interesting to know if $G(\gamma)$ can be taken to be bounded as $\gamma \to 1$? If so, this would eliminate the diverging factor $(1-\gamma)^{-1}$ appearing in Theorem 5.1.

3.1 Preparations for the proof of Thm 3.1

The proof is based on a lifting argument where X is replaced by the total space of $K_X \to X$. We start by setting up some notation. Given a Fano manfold X denote by Y the total space of the canonical line bundle $K_X \to X$ and by Y^* the space obtained by deleting the zero-section of $K_X \to X$. There is a canonical $\mathbb{C}^*$-action on Y, induced by the linear structure on the fibers of the line bundle $K_X \to X$. Moreover, the space Y comes with a canonical holomorphic top form Ω, which is one-homogeneous wrt the $\mathbb{C}^*$-action on Y. Indeed, Ω can be constructed as follows. Fix local holomorphic coordinates $z \in \mathbb{C}^n$ on X and trivialize, locally, $K_X \to X$ by the corresponding holomorphic section $dz(:= dz_1 \wedge \cdots \wedge dz_n)$. This induces local holomorphic coordinates $(z, w) \in \mathbb{C}^{n+1}$ on Y and the top form Ω defined by $dz \wedge dw$ glues to a globally well-defined holomorphic top form on Y.

Recall that we have fixed a volume form dV_X on X with positive Ricci curvature, denoted by ω (Sect. 2.2). By well-known general principles, a volume form dV on X induces a one-homogeneous function r on Y that is plurisubharmonic iff Ric $dV \geq 0$. Indeed, dV induces a metric on $K_X \to X$ and r may be defined as the corresponding "norm-function" on the total space Y of $K_X \to X$. But here it will be convenient to adopt the following alternative definition of the function r corresponding to dV_X : r is the function on Y determined by

$$i^{(n+1)}\Omega \wedge \bar{\Omega} = r dr \wedge dV_X \wedge d\theta, \text{ on } Y^* \tag{3.3}$$

where $dV_X \wedge d\theta$ denotes the fiber product of the measure dV_X on the base of the fibration $Y^* \to X$ with the family of standard S^1-invariant measures $d\theta$ on the fibers of the fibration. The function r is, indeed, one-homogeneous wrt the $\mathbb{R}_{>0}$-action on Y (since Ω is) and plurisubharmonic (since, by assumption, Ric $dV_X \geq 0$). Moreover, r extends to a psh function on Y (since it is bounded from above in a neighbourhood of the zero-section in K_X).

To any given $u \in \mathrm{PSH}(X, \omega)$ we associate the following psh function on Y :

$$\psi := u + \log r^2. \tag{3.4}$$

The psh functions ψ on Y that can be be expressed in this way are precisely the ones which are *log-homogeneous* under the $\mathbb{C}^*$-action on Y; $\psi(\lambda\cdot) = \log(|\lambda|^2) + \psi$ for any $\lambda \in \mathbb{C}^*$. In geometric terms, this follows form the fact that e^{ψ} is the one-homogenous function on Y corresponding to the volume form $dV := e^{-u} dV_X$ on X.

Proposition 3.2 *Assume that* $\psi_0 = f(\log r^2)$, *where* f *is an increasing convex function on* $]-\infty, \infty[$, *assumed bounded from below and that* ψ_1 *is psh and log-homogeneous on* Y. *Then the logarithm of the following function is concave on* $\mathbb{R}$:

$$V(t) := \int_{\{\psi_1 \leq t\}} e^{-\psi_0} i^{(n+1)} \Omega \wedge \bar{\Omega},$$

if it is finite for all t. *More generally, the result holds when* Y *is a (possibly singular) Fano variety and* r *is the one-homogeneous function corresponding to a (possibly singular) metric on* K_X^* *with positive curvature current, assuming that the corresponding measure* dV_X *on* X *gives finite total volume to* X.

Proof In the case when $Y = \mathbb{C}^m$ (which, after passing to a finite cover, corresponds to the case when $X = \mathbb{P}^n$) the result follows from [6, Prop 6.5], which is a slight generalization of [17, Thm 1.2] and [8, Thm 2.3]. The proof first uses the subharmonicity result for Bergman kernels established in [17, Thm 1.1]. Then the S^1-symmetry of ψ_0 and ψ_1 is used. In the case when X is smooth one could, presumably, proceed in a similar manner. Indeed, the proof of [17, Thm 1.1] is based on $\bar{\partial}$-estimates for $(n+1, 0)$-forms on the pseudoconvex domain $\{\{\psi_1 \leq t\}\}$ with weight $e^{-\psi_0}$, which still apply when Y is a complex manifold. But one technical difficulty in the proof in [17] stems from the fact that, in general, the domain $\{\psi_1 \leq t\}$ may be not have a smooth boundary, which is bypassed using an approximation procedure. Since this seems to lead to major technical difficulties when X (and thus also Y)

is singular we will instead give a more direct proof, which has the virtue that is also applies to singular X.

First assume that dV_X is a smooth volume form. Then r is a smooth coordinate on Y^*. Hence, using the factorization 3.3 and first performing the integration over r, we can express

$$V(t) = \int_X \left(\int_0^{e^{-u_t}} e^{-\psi_0(r)} r dr \right) dV_X, \quad u_t := u - t, \tag{3.5}$$

where u is the function on X corresponding to ψ_1 in formula 3.4. The change of variables $s = \log(r^2)$ yields

$$\frac{1}{2} \int_0^{e^{-u_t}} e^{-\psi_0(r)} r dr = \int_{\{s<-u_t\}} e^{-\phi(s)} ds, \quad \phi(s) := f(s) - s$$

Now, given a real variable u, define $\chi(u)$ by

$$\chi(u) = -\log \int_{\{s<-u\}} e^{-\phi(s)}.$$

Let us show that

$$(i)\ \chi''(u) \geq 0, \quad (ii)\ 0 \leq \chi'(u) \leq 1. \tag{3.6}$$

Since $\phi(s)$ is convex the first item follows directly from Prekopa's theorem [79] (or the Brunn–Minkoski inequality). Next, by definition,

$$\chi'(u) := \frac{e^{-\phi(-u)}}{\int_{\{s<-u\}} e^{-\phi(s)}},$$

which is manifestly non-negative. In order to show that $\chi'(u) \leq 1$ it is—since $\chi(u)$ is convex, by (i)—enough to consider the limit when $u \to \infty$. By assumption, we have that $\phi(s) = f(-\infty) - s + o(1)$, uniformly as $s \to -\infty$. Hence, in the limit where $u \to \infty$ the quotient above coincides with

$$\lim_{u\to\infty} \frac{e^{-u}}{\int_{\{s<-u\}} e^s} = 1,$$

using that the denominater equals e^{-u}. This proves formula 3.6.

Next note that, if χ satisfies the conditions in formula 3.6, then

$$u \in \text{PSH}\,(X, \omega), \implies \chi(u) \in \text{PSH}\,(X, \omega). \tag{3.7}$$

Indeed,

$$dd^c\chi(u) = \chi'' du \wedge d^c u + \chi' dd^c u \geq \chi'\omega_u - \chi'\omega \geq -\omega.$$

Now complexify t and consider the function $U := u - \Re t$ on $X \times (\mathbb{R} \times i\mathbb{R})$. Then $dd^c U \geq -\omega$, where we have identified ω with its pull-back to $X \times (\mathbb{R} \times i\mathbb{R})$. Applying the implication 3.7 with u replaced by U thus reveals that $\chi(U) \in \text{PSH}\,(X \times (\mathbb{R} \times i\mathbb{R}), \omega)$. This means that $v_t := \chi(u_t)$ is a subgeodesic in PSH(X, ω) (as defined in Sect. 2.1.1). Hence, the proposition follows from the following complex generalization of the Prekopa theorem for convex functions established in [20]: the function

$$t \mapsto -\log \int_X e^{-v_t} dV_X \tag{3.8}$$

is convex for any subgeodesic v_t.

Next, consider the case when X is non-singular, but dV_X corresponds to a (possibly singular) metric on K_X^*. This means that the local density of dV_X is of the form $e^{-\phi}$ where ϕ locally represents the metric on K_X^* in question, in additive notation. In the case that ϕ is locally bounded the same proof as in the case when dV_X is a volume form (i.e. ϕ is smooth) still applies. In the general case we can express ϕ as a decreasing limit of locally bounded metrics ϕ_j on K_X^* with positive curvature current and conclude using the monotone convergence theorem. Finally, when X is non-singular we proceed in essentially the same way, using that the convexity of the function 3.8 still holds, as observed in [12] (and the decomposition 3.5 still holds, since it can be applied on the regular locus of X). $\square$

3.2 Conclusion of the proof of Thm 3.1

We will apply Prop 3.2 with $\psi_0 = r^2$. Expressing u in terms of ψ on Y we can write

$$\int_X e^{-\gamma u} dV_X = \frac{1}{\int_0^\infty r^{-2\gamma} e^{-r^2} r dr} \int_Y e^{-\gamma\psi} dV_Y, \quad dV_Y := e^{-\psi_0} i^{(n+1)} \Omega \wedge \bar{\Omega} \tag{3.9}$$

Hence, pushing forward the integration over Y to $\mathbb{R}$ and using the factorization 3.3 gives

$$Z_u(\gamma) = \frac{1}{\int_0^\infty r^{-2\gamma} e^{-r^2} r dr} \int_{\mathbb{R}} e^{-\gamma t} \psi_* dV_Y, \tag{3.10}$$

for any γ. Next, note that for $\gamma > 0$

$$\int_{\mathbb{R}} e^{-\gamma t} \psi_* dV_Y = \gamma \int_{\mathbb{R}} e^{-t\gamma} \nu_0, \quad \nu_0 = V(t) dt, \quad V(t) := dV_Y(\{\psi < t\}). \tag{3.11}$$

Indeed, since $\psi_* dV_Y = V'(t) dt$ (in the sense of distributions) integrating by parts reveals that the previous formula holds if

$$\lim_{t \to \pm\infty} e^{-t\gamma} V(t) = 0, \quad (\gamma > 0)$$

But, by Chebyshev's inequality, $V(t) \leq C_\epsilon e^{\epsilon t}$ for any $\epsilon \in]0, 1[$ (with $C_\epsilon = \int_Y e^{-\epsilon\psi} dV_Y$). The vanishing in the previous equation thus follows when $t \to \infty$ and $t \to -\infty$ by taking ϵ sufficiently close to 0 and 1, respectively. This concludes the proof of formula 3.11 and shows that formula in the Theorem holds with

$$G(\gamma) := \frac{\gamma}{\int_0^\infty r^{-2\gamma} e^{-r^2} r dr} \left(= \frac{\gamma}{\Gamma(1-\gamma)/2}\right),$$

where $\Gamma(s)$ is the classical Gamma-function. Finally, the log-concavity of ν_0 follows from Prop 2.7, applied to the functions $(\psi_0, \psi_1) = (r^2, \psi)$ on Y.

3.2.1 Proof of the inequality 3.2

Let us first show that

$$\int_{\mathbb{R}} |t| d\nu_\gamma \leq \int_Y |\psi| \frac{e^{-\gamma\psi} dV_Y}{\int_Y e^{-\gamma\psi} dV_Y} + \gamma^{-1}. \tag{3.12}$$

First, integrating by parts (just as in the proof of formula 3.11) yields, with $\sigma(t)$ denoting the L^∞-function defined by the sign of t, which is the distributional derivative of $|t|$:

$$\gamma \int_{\mathbb{R}} |t| e^{-\gamma t} V(t) dt = \int_{\mathbb{R}} e^{-\gamma t} (|t| V(t))' dt = \int_{\mathbb{R}} e^{-\gamma t} \big(|t| V'(t) + \sigma(t) V(t)\big) dt$$

Hence,

$$\int_{\mathbb{R}} |t| e^{-\gamma t} V(t) dt \le \gamma^{-1} \int_{\mathbb{R}} e^{-\gamma t} |t| V'(t) dt + \gamma^{-1} \int_{\mathbb{R}} e^{-\gamma t} V(t) dt.$$

Dividing both sides with $\int e^{-\gamma t} V(t) dt$ and invoking formulas 3.1, 3.11, thus proves formula 3.12. Next, using that that $|\psi| \le |u| + |\log r^2|$ the integral over Y in formula 3.12 may be estimated by

$$\int_Y |u| \frac{e^{-\gamma\psi} dV_Y}{\int_Y e^{-\gamma\psi} dV_Y} + \int_Y |\log r^2| \frac{e^{-\gamma\psi} dV_Y}{\int_Y e^{-\gamma\psi} dV_Y},$$

where dV_Y was defined in formula 3.9. Expressing dV_Y in terms of dV_X, the first integral equals $\int_Y |u| \mu_\gamma$ and the second one is equal to the following constant, only depending on γ,

$$C_\gamma := \frac{\int_0^\infty e^{-r^2} |\log r^2| r^{-2\gamma} r dr}{\int_0^\infty e^{-r^2} r^{-2\gamma} r dr},$$

which satisfies $C_\gamma \le C(1-\gamma)^{-1}$ (using that C_γ is comparable to the first derivative of $\log \Gamma(1-\gamma)$ at $\gamma = 1$).

4 Moment bounds on Fano manifolds

Let X be a Fano manifold. In this section, we prove the following general dimension-free moment bounds, expressed in terms of the probability measure $\mu_{\gamma u}$, defined by formula 2.7. They will be used in the proof of Theorem 1.1, given in the next section.

Theorem 4.1 *Let X be an n-dimensional Fano manifold and ω a Kähler form in $c_1(X)$. Given $p \in [1, \infty[$ and $\gamma \in]0, 1[$ the following inequality holds for any u in $\mathcal{H}(X, \omega)$ such that* $\sup_X u \le 0$:

$$\left(\int_X (-u)^p \mu_{\gamma u} \right)^{1/p} \le A_p \int_X (-u) \mu_{\gamma u} + B_p \left(\gamma^{-1} + (1-\gamma)^{-1} \right)$$

where the constants A_p and B_p only depend on p. More generally, given $\gamma \in]0, 1[$ the inequality holds for any $u \in PSH(X, \omega)$ such that $\sup_X u \le 0$, *if* $\int e^{-\gamma u} dV_X < \infty$.

Remark 4.2 The inequality above does not hold with A_p and B_p independent of p. Indeed, otherwise one could, by letting $p \to \infty$, replace the lhs in the inequality with $\|u\|_{L^\infty(X)}$. But this contradicts the fact that there exist $u \in \mathrm{PSH}(X, \omega)$ which are unbounded, while $\int_X (-u) \mu_{\gamma u}$ is finite for any γ (for example, any unbounded $u \in \mathrm{PSH}(X, \omega)$ with vanishing Lelong numbers).

The idea of the proof is to combine Thm 3.1 with the following well-known Kahane–Khinchin inequality for log-concave probability measure ν on $\mathbb{R}$ and $p \ge 1$:

$$\left(\int |t|^p d\nu \right)^{1/p} \le C_p \left(\int |t| d\nu \right) \tag{4.1}$$

with a universal constant C_p (only depending on p) [74, App.III] [66]. In order to use this inequality we will apply the following elementary lemma that relates moments to higher order derivatives of the cumulant-generating function:

Lemma 4.3 *Let σ be a measure on a topological space S and f a measurable function on (S,σ). Given $\gamma\in\mathbb{R}$ denote by $\langle\cdot\rangle_\gamma$ integration wrt the probability measure $e^{\gamma f}\sigma/\int_S e^{\gamma f}\sigma$, assuming that $\int_S e^{(\gamma+\epsilon)f}\sigma<\infty$ for all sufficiently small ϵ. Then there exist universal coefficents $a_{j_1,\ldots,j_{p-1}}\in\mathbb{R}$ such that*

$$\left\langle f^p\right\rangle_\gamma=\frac{d^p}{d\gamma^p}\log\left\langle e^{\gamma f}\right\rangle_\gamma+\sum_{j_1,\ldots,j_{p-1}}a_{j_1,\ldots,j_{p-1}}\left\langle f^{j_1}\right\rangle_\gamma\left\langle f^{j_2}\right\rangle_\gamma\cdots\left\langle f^{j_{p-1}}\right\rangle_\gamma,$$

where the sum ranges over all non-negative integer indices $(j_1,\ldots,j_{p-1})$ such that $j_i\leq p-1$, $j_1+\cdots+j_{p-1}=p$.

Proof This is well-known and can be shown using direct differentiation, by rewriting $e^{\gamma f}\sigma/\int_S e^{\gamma f}\sigma=e^{\gamma f-\log\int_S e^{\gamma f}\sigma}\sigma$. Alternatively, the formula follows from general result expressing $a_{j_1,\ldots,j_{p-1}}$ in terms of partial Bell polynomials [92]. For future reference, let us record the following explicit expressions when $p\leq 2$:

$$\frac{d}{d\gamma}\log\left\langle e^{\gamma f}\right\rangle_\gamma=\langle f\rangle_\gamma,\quad\frac{d^2}{d\gamma^2}\log\left\langle e^{\gamma f}\right\rangle_\gamma=\left\langle f^2\right\rangle_\gamma-\langle f\rangle_\gamma^2. \tag{4.2}$$

□

We will prove Theorem 4.1 by induction over p. We thus assume that $p\geq 2$ and that Theorem 4.1 holds for $p-1$. Applying the previous lemma to (S,σ,f) given by $(X,dV_X,-u)$ and using the assumption that the theorem holds for $p-1$ gives

$$\begin{aligned}\left\langle(-u)^p\right\rangle_\gamma&\leq\frac{d^p}{d\gamma^p}\log\left\langle e^{-u\gamma}\right\rangle\\&\quad+\sum_{j_1,\ldots,j_{p-1}}\left|a_{j_1,\ldots,j_{p-1}}\right|\left(A_{p-1}\langle(-u)\rangle_\gamma+B_{p-1}\left(\gamma^{-1}+(1-\gamma)^{-1}\right)\right)^p\\&\leq\frac{d^p}{d\gamma^p}\log\left\langle e^{-u\gamma}\right\rangle+A'_p\langle(-u)\rangle_\gamma^p+B'_p\left(\gamma^{-1}+(1-\gamma)^{-1}\right)^p,\end{aligned}$$

for some constants A'_p and B'_p only depending on p. Next, by the formula in Theorem 3.1, we can express

$$\log\left\langle e^{-u\gamma}\right\rangle=\log\int_{\mathbb{R}}e^{-t\gamma}\nu_0+\log G(\gamma),$$

for a log-concave measure ν_0. Hence,

$$\begin{aligned}\left\langle(-u)^p\right\rangle_\gamma\leq\frac{d^p}{d\gamma^p}\log\int_{\mathbb{R}}e^{-t\gamma}\nu_0+\left|\frac{d^p}{d\gamma^p}\log G(\gamma)\right|+A'_p\langle(-u)\rangle_\gamma^p\\+B'_p\left(\gamma^{-1}+(1-\gamma)^{-1}\right)^p.\end{aligned}$$

Since $\Gamma(s)$ has a simple pole at $s=0$ we have $\left|\frac{d^p}{d\gamma^p}\log G(\gamma)\right|\leq c_p\left(\gamma^{-1}+(1-\gamma)^{-1}\right)^p$, giving

$$\left\langle(-u)^p\right\rangle_\gamma\leq\frac{d^p}{d\gamma^p}\log\int_{\mathbb{R}}e^{-t\gamma}\nu_0+A'_p\langle(-u)\rangle_\gamma^p+B''_p\left(\gamma^{-1}+(1-\gamma)^{-1}\right)^p,$$

for $B''_p=B'_p+c_p$. Next, applying Lemma 4.3 to $(\mathbb{R},\nu_0,t)$, combined with the Kahane–Khinchin inequality 4.1 for the log-concave probability measures ν_γ on $\mathbb{R}$ defined by

$$\nu_\gamma:=\frac{e^{-t\gamma}\nu_0}{\int_{\mathbb{R}}e^{-t\gamma}\nu_0},$$

gives

$$\frac{d^p\gamma}{d\gamma^p}\log\int_{\mathbb{R}} e^{-t\gamma}\nu_0 \leq C_p\left(\int |t|\nu_\gamma\right)^p.$$

By the inequality in Theorem 3.1 (combined with formula 4.2 for $p = 1$),

$$\int_{\mathbb{R}} |t|\nu_\gamma \leq \langle(-u)\rangle_\gamma + C\left(\gamma^{-1} + (1-\gamma)^{-1}\right).$$

All in all, this means that

$$\left\langle(-u)^p\right\rangle_\gamma \leq C_p\left(\langle(-u)\rangle_\gamma + C\left(\gamma^{-1} + (1-\gamma)^{-1}\right)\right)^p + A'_p\langle(-u)\rangle_\gamma^p$$
$$+B''_p\left(\gamma^{-1} + (1-\gamma)^{-1}\right)^p.$$

The induction step is thus concluded by combining the inequality in Theorem 3.1 with the standard reverse Hölder inequality in $\mathbb{R}^3$ (appearing in formula 1.2).

4.1 The proof for general *u*

Finally, given $\gamma \in]0, 1[$ we will show that the inequality in the theorem holds in the general case where $u \in \mathrm{PSH}(X, \omega)$ and $\int e^{-\gamma u} dV_X < \infty$ (assuming that $\sup_X u \leq 0$, as before). First observe that both sides in the inequality to be shown are still finite in the general case. Indeed, by the resolution of Demailly–Kollar's openness conjecture (that we shall come back to in Sect. 9) there exists $\epsilon > 0$ such that

$$\int e^{-(\gamma+\epsilon)u} dV_X < \infty. \tag{4.3}$$

Given this integrability 4.3, the proof of the theorem proceeds exactly as in the previous case.

5 Reverse Hölder inequalities on the space of Kähler metrics

We next turn to the proof of the following slightly more general formulation of Theorem 1.1, stated in the introduction, where $\rho_{u,\gamma}$ denotes the twisted Ricci potential, defined by formula 2.8.

Theorem 5.1 *Given a Kähler form ω in $c_1(X)$ and $p \in [1, \infty[$ the following inequality holds for any $u \in PSH(X, \omega)$ such that ω_u^n is absolutely continuous wrt Lesbesgue measure and any $\gamma \in]0, 1[$ such that $\int e^{-\gamma u} dV_X < \infty$:*

$$d_p(u, 0) \leq Ad_1(u, 0) + B,$$

with

$$A = A_p e^{2\|\rho_{u,\gamma}\|_{L^\infty}}, \quad B = B_p\left(\gamma^{-1} + (1-\gamma)^{-1}\right)e^{\|\rho_{u,\gamma}\|_{L^\infty}},$$

where A_p only depends on (p, n) and B_p also depends on (X,ω). Moreover, if $\sup_X u \leq 0$ then $A_p/(n+1)$ is independent of n and B_p only depends on (p, n).

First assume that $\sup_X u \leq 0$. Then Theorem 4.1 implies, by the very definition of the twisted Ricci potential (Definition 2.8), that

$$\left(\int_X (-u)^p \frac{\omega_u^n}{V}\right)^{1/p} \leq A_p e^{\sup_X \rho_{u,\gamma} - p^{-1}\inf_X \rho_{u,\gamma}} \int_X (-u)\frac{\omega_u^n}{V} + Be^{-p^{-1}\inf_X \rho_{u,\gamma}}, \tag{5.1}$$

where $B = B_p\left(\gamma^{-1} + (1-\gamma)^{-1}\right)$. Invoking the inequalities in Lemma 2.1 thus concludes the proof when $\sup_X u \leq 0$. Finally, in the general case we may decompose $u = \tilde{u} + \sup_X u$ where $\sup_X \tilde{u} = 0$. Combining the previous case with the triangle inequality for d_p yields

$$d_p(u,0) \leq d_p(\tilde{u},0) + \left|\sup_X u\right| \leq Ad_1(\tilde{u},0) + B + \left|\sup_X u\right| \leq Ad_1(u,0) + B + 2\left|\sup_X u\right|.$$

The proof is thus concluded by invoking the inequality 2.3 for $p = 1$.

5.1 Application to destablizing geodesic rays

We next deduce the following slighltly more general formulation of Cor 1.2.

Corollary 5.2 *Let u_j be a sequence in $\mathcal{E}^1(X,\omega)$ such that $\omega_{u_j}^n$ is absolutely continuous wrt Lesbesgue measure and $\int e^{-\gamma_j u} dV_X < \infty$ for some sequence γ_j contained in a compact subset of* $]0,1[$. *Assume that*

$$(i)\, d_1(u_j,0) \to \infty, \quad (ii)\, \mathcal{M}(u_j) \leq C, \quad (iii)\, \left|\rho_{u_j,\gamma_j}\right| \leq R$$

Then

- *u_j is weakly asymptotic to a ray v_t which is a d_p-geodesic ray in $\overline{(\mathcal{H}(X,\omega), d_p)}$ for any $p \in [1,\infty[$ and $t \mapsto \mathcal{M}(v_t)$ is decreasing.*
- *the d_p- speed $\|\dot{v}\|_p$ of the geodesic v_t satisfies*

$$\|\dot{v}\|_p \leq A\,\|\dot{v}\|_1\,, \tag{5.2}$$

for a constant A of the form $A_{p,n}e^{2R}$ where $A_{p,n}$ only depends on (n,p). Moreover, if $\sup_X u_j \leq 0$, *then $A_{p,n} = A_p(n+1)$, where A_p only depends on p.*

Given Theorem 1.1 the proof is similar to the proof of [39, Thm 3.2] (see also [94] for general results in Hadamard spaces covering the case $p = 2$). Let $v_j(t)$ be the psh geodesic

$$[0, d_1(u_j,0)] \to \mathcal{E}^1(X,\omega), \quad t \mapsto v_j(t) \tag{5.3}$$

coinciding with 0 and u_j at $t = 0$ and at $t = d_1(u_j,0)$, respectively (the parametrization has been made so that $v_j(t)$ has unit d_1-speed). We will first show (assuming (i) and (ii)) that there exists a geodesic ray $v(t)$ in $\mathcal{E}^1(X,\omega)$ such that $v_j(t) \to v(t)$ in $\mathcal{E}^1(X,\omega)$ uniformly on $[0,T]$, for any given finite T—after perhaps passing to a subsequence. Moreover, if the Ricci potential of u_j is uniformly bounded, then the construction will show that v_t is a d_p-geodesic ray for any $p \in [1,\infty[$. To this end fix $T > 0$ and consider $v_j(t)$ on $[0,T]$. By construction

$$t \in [0,T] \implies d_1(v_j(t),0) = t \leq T \quad (\implies \left|\sup v_j(t)\right| \leq C_T) \tag{5.4}$$

Moreover, by assumption (ii)

$$\mathcal{M}(v_j(t)) \leq C_0, \quad C_0 := \max\{C, \mathcal{M}(0)\} \tag{5.5}$$

Indeed, by assumption the estimate holds at $t = T_j$ and hence it holds for $t \leq T_j$ by the convexity of $\mathcal{M}$ on $\mathcal{E}^1(X,\omega)$ [13]. The previous two estimates imply, by the compactness theorem in [12], that the maps 5.3, defined by the geodesic $v_j(t)$, when restricted to $[0,T]$, take values in a fixed compact subset K_T of $\mathcal{E}^1(X,\omega)$. Since the maps are 1-Lipschitz (by the very definition of d_1-geodesics) it thus follows from the Arzela-Ascoli theorem in metric spaces that $v_j(t)$ converges uniformly on $[0,T]$ to a curve $v(t)$ in $\mathcal{E}^1(X,\omega)$, after perhaps passing to a subsequence. As a consequence, by [15, Prop 1.11], $v(t)$ is a *psh geodesic* and, in particular, a d_1-geodesic in $\mathcal{E}^1(X, \omega)$. Using a diagonal argument this yields a d_1-geodesic ray $v(t)$ with the required properties.

Next, assume a uniform bound R on the twisted Ricci ρ_{u_j,γ_j} potentials of u_j and that γ_j is contained in a compact subset K of $]0, 1[$. Then, by Theorem 5.1, the bound on $d_1(v_j(t), 0)$ in formula 5.4 yields, since any psh geodesic is a d_p-geodesic,

$$t \leq T \implies d_p(v_j(t), 0) = t\frac{d_p(u_j, 0)}{d_1(u_j, 0)} \leq TA + \frac{B}{d_1(u_j, 0)}) \tag{5.6}$$

(where A and B depend on R and p and B also depends on the compact subset K). Hence, by [13, Prop 2.7], the d_1-limit point $v(t)$ is in $\mathcal{E}^p(X, \omega)$ for any $p \geq 1$. Since the previous bound holds for any $p \geq 1$ it follows from Lemma 5.3 below that, for t fixed, $v_j(t)$ d_p-converges towards $v(t)$ for any $p \geq 1$. Next, recall that $v_j(t)$ has unit d_1-speed, i.e. $T = d_1(v_j(T), 0) = T$. Hence, letting $j \to \infty$ in the bound 5.6 at $t = T$ gives

$$d_p(v(T), 0) \leq d_1(v(T), 0)A,$$

which proves the inequality 5.2, by taking $T = 1$. Finally, the bound 5.5 on $\mathcal{M}(v_j(t))$ implies, since $\mathcal{M}$ is d_1-lower semi continuous [13], that $\mathcal{M}(v(t)) \leq C_0$. It thus follows from the convexity of $t \mapsto \mathcal{M}(v(t))$ that $\mathcal{M}(v(t))$ is decreasing in t.

Lemma 5.3 *Assume that v_j d_1-converges towards v and that there exists $\epsilon > 0$ such that $d_{p+\epsilon}(v_j, 0)$ is uniformly bounded. Then v_j d_p-converges towards v.*

Proof We will use the inequalities 2.1. Given $R > 0$ decompose

$$\int_X |v_j - v|^p \left(\frac{\omega_{v_j}^n}{V} + \frac{\omega_v^n}{V}\right) = \int_{|v_j - v| \leq R} |v_j - v|^p \left(\frac{\omega_{v_j}^n}{V} + \frac{\omega_v^n}{V}\right) + \int_{|v_j - v| > R} |v_j - v|^p \left(\frac{\omega_{v_j}^n}{V} + \frac{\omega_v^n}{V}\right).$$

Rewriting $|v_j - v|^p = R^P(|v_j - v|/R)^p$, the first term may, since $(|v_j - v|/R)^p \leq (|v_j - v|/R)^1$, be estimated as

$$\int_{|v_j - v| \leq R} |v_j - v|^p \left(\frac{\omega_{v_j}^n}{V} + \frac{\omega_v^n}{V}\right) \leq R^{p-1} \int_X |v_j - v|^1 \left(\frac{\omega_{v_j}^n}{V} + \frac{\omega_v^n}{V}\right),$$

which converges to zero when $j \to \infty$, since v_j d^1-converges towards v. Next, using that $1 \leq |v_j - v|^\epsilon R^{-\epsilon}$, when $|v_j - v| > R$ the second term above may be estimated by

$$R^{-\epsilon} \int_X |v_j - v|^{p+\epsilon} \left(\frac{\omega_{v_j}^n}{V} + \frac{\omega_v^n}{V}\right) \leq R^{-\epsilon} C_\epsilon.$$

Hence, letting first j and then R tend to infinity concludes the proof. □

6 Generalization to singular Fano varieties

In this section we explain how to extend the previous results to singular Fano varieties and prove Corollary 1.3. The results also extend—with essentially the same proofs—to the more general twisted setting (considered in [15] when X is smooth).

We thus let X be a singular Fano variety. This means that X is a normal complex variety such that K_X^* is defined as an ample $\mathbb{Q}$-line bundle over X (see [12, 46, 68, 70]). Fix a measure dV_X on X corresponding to a locally bounded metric on K_X^* with positive curvature current, denoted by ω. We will assume that dV_X has finite total mass. As is well-known [12] this is equivalent to X having *log terminal singularities* which will henceforth be assumed.

We recall that a Kähler form on a normal complex variety X is, by definition, locally the restriction to X of a Kähler form on $\mathbb{C}^M$ under a local embedding of X as a variety in $\mathbb{C}^M$. The space $\mathcal{H}(X, \omega)$ of Kähler potentials (relative to ω) is defined exactly as in the case when X is singular. Theorem 4.1 now extends directly to singular Fano varieties, using that Prop 3.2 applies to singular Fano varieties. By [46], all the results in Sect. 2.1 on Darvas' L^p-distances extend to singular normal varieties. Hence, Theorem 4.1 implies, precisely as before, that Theorem 5.1 holds for any singular Fano variety with log terminal singularities. In turn, just as before, the latter theorem implies that Corollary 5.2 applies to singular Fano varieties (using the results on singular Fano varieties in [12, 46]).

6.0.1. Proof of Corollary 1.3

Consider now Aubin's equations 1.4 on X, defined in the weak sense of pluripotential theory [12], i.e. the sup-normalized potential u_t of ω_t, which is in PSH $(X, \omega) \cap L^\infty(X)$, satisfies

$$\omega_{u_t}^n / V = \mu_{u_t,t} \tag{6.1}$$

By results in [12] this means, equivalently, that u_t minimizes a twisted Mabuchi functional that we shall denote by $\mathcal{M}_t$. Moreover, it follows from results in [68, 70] (or more precisely the proof of the main results) that for any $\{1, \delta(X)\}$ is the sup over all $t \in [0, 1[$ for which the Eq. 6.1 admits a solution u_t in $\mathcal{E}^1(X, \omega)$ (the assumption that X has log terminal singularities ensures that $\delta(X) > 0$). The Eq. 6.1 says, in particular, that $\omega_{u_t}^n$ is, locally on the regular locus of X, absolutely continuous wrt Lebesgue measure, $\int_X e^{-tu_t} dV_X < \infty$ and $\rho_{u,t} \equiv 0$. Hence, Cor 1.3 follows from Cor 5.2 on Fano varieties, once we have verified that

$$d_1(u_t, 0) \to \infty \tag{6.2}$$

as $t \to \delta(X)$ (since $\sup_X u_t = 0$ the corresponding geodesic ray v_t is then non-trivial in the sense that v_t is not of the form ct for any constant c). Assume, in order to get a contractiction, that $d_1(u_t, 0) \leq C$, , after passing to a subsequence. Then it follows from results in [12, 46] that, after passing to a subsequence, u_t d_1-converges to u_δ in $\mathcal{E}^1(X, \omega)$, minimizing $\mathcal{M}_{\delta(X)}$ and thus u_δ satisfies the Eq. 6.1 for $t = \delta(X)$. By, assumption, $\delta(X) < 1$ and, as a consequence, any solution of the equation 6.1 for $t = \delta(X)$ is uniquely determined. Indeed, since ω is a positive current with locally bounded potentials [12, Thm 11.1] implies, just as in the case of $t = 1$ considered in [12, Thm 5.1], that u_δ is uniquely determined modulo the flow of a holomorphic vector field W on X, preserving ω. But this can only happen if W vanishes identically, as follows from [20, Prop 8.2] applied to any non-singular resolution of X. Thus u_δ is uniquely determined. As a consequence—just as in the case that $t = 1$ and there are no holomorphic vector fields, considered in [46]—$\mathcal{M}_t$ is coercive on $\mathcal{E}^1(X, \omega)$ when $t = \delta(X)$. Since coercivity is preserved when t is replaced by $t + \epsilon$ for any sufficently small number ϵ this means that $\mathcal{M}_{\delta(X)+\epsilon}$ is coercive for any sufficently small positive number ϵ. It thus follows from [12] that $\mathcal{M}_{\delta(X)+\epsilon}$ has a minimizer, which satisfies the Eq. 6.1 for $t = \delta(X) + \epsilon$. But this contradicts the fact that $\min\{1, \delta(X)\}$ is the sup over all $t \in [0, 1[$ for which the Eq. 6.1 is solvable. This proves the divergence 6.2.

7 Comparison with Harnack type bounds

In this section we compare Theorems 1.1 and Corollary 1.2 with the Harnack type bounds in [2, 85, 87] for non-singular X. We start with the following analogs of the reversed Hölder inequalities in Theorems 4.1, 1.1.

Proposition 7.1 *Let X be a Fano manifold of dimension and $\omega \in c^1(X)$. There exists a constant B_n, depending on n, such that*

$$\|u\|_{L^\infty(X)} \leq \int_X (-u)\frac{\omega_u^n}{V} + \delta^{-1}B_n \tag{7.1}$$

for any $u \in \mathcal{H}(X,\omega)$ satisfying $\sup_X u \leq 0$ *and*

$$Ric\ \omega_u \geq \delta\omega_u. \tag{7.2}$$

As a consequence,

$$d_p(u,0) \leq (n+1)d_1(u,0) + \delta^{-1}B_n. \tag{7.3}$$

for any $p \in [1,\infty[$.

Proof The first inequality is shown in [2, Prop 3.6] (using a uniform lower bound on the Green function for the Laplacian of (X,ω_u), in terms of the diameter, deduced from the uniform bound on the L^2-Sobolev constant of ω_u [63]). The second inequality then follows from Lemma 2.1. □

Remark 7.2 The inequality 7.1 is also shown in [87] (in the course of the proof of [87, Thm 2.1]), but with a constant A_n in front of the integral over X. The proof uses that a strict lower bound on the Ricci curvature of ω_u implies a uniform upper bound on the Sobolev constant and the Poincaré constant of (X,ω_u) (so that Moser iteration can be applied). As shown in [81] the latter upper bounds hold along the Kähler-Ricci flow. As a consequence, so does the inequality 7.1 (see Step 3 in the proof of the main result in [81]). This means, by the proof of the previous proposition, that the reversed Hölder inequality 7.3 holds along the Kähler-Ricci flow (with A and B depending on X). This was first shown in [39, Thm 1] for a particular normalization of the potentials, using the uniform Harnack bound discussed below.

Now consider ω_{u_t} satisfying Aubin's continuity equation. Then the Ricci curvature bound in the previous proposition automatically holds when $t \geq \delta > 0$. The reversed Hölder bound 7.3 thus follows for u_t when $t \geq \delta > 0$. As a consequence, the proof of 5.2 yields a strong reverse Hölder inequality for the corresponding destabilizing geodesic ray appearing in Cor 1.2, for non-singular X :

$$\|\dot{v}\|_p \leq (n+1)\,\|\dot{v}\|_1\,. \tag{7.4}$$

While the potential u_t is only determined up to an additive constant (depending on t), the constant is often fixed by demanding that

$$\int_X e^{-tu_t}dV_X = 1, \tag{7.5}$$

(where, as before, Ric $dV_X := \omega$). Equivalently, this means that u_t is the unique solution to Aubin's Monge-Ampère equation [1]:

$$\frac{\omega_{u_t}^n}{V} = e^{-tu_t}dV_X. \tag{7.6}$$

Note that, by Jensen's inequality, $\sup u_t \geq 0$.

Proposition 7.3 *Let X be a Fano manifold of dimension. The following* Harnack bound

$$-\inf_X u \leq A\sup_X u + \delta^{-1}B_n \tag{7.7}$$

holds for u_t satisfying Eq. 7.6 for $t \geq \delta > 0$, where $A = n$ and B depends on (X, ω, δ).

Proof A slightly weaker inequality appears in [85, Prop 2.1] and closely related inequalities also appear in [87]. Here we note that the proposition also follows from the inequality 7.1, applied to the function $\sup_X u_t - u_t$, which gives

$$\sup_X u_t - \inf_X u_t \leq (n+1)(\sup_X u_t - \mathcal{E}(u_t)) + B$$

where $\mathcal{E}(u)$ is the functional appearing in formula 2.5. The proof is thus concluded by noting that $-\mathcal{E}(u_t)$ is uniformly bounded from above, by the bound 7.10 below. □

In general, as shown [39] (in the context of the Kähler-Ricci flow), a uniform Harnack bound implies that

$$d_p(u, 0) \leq A_{p,\delta}\sup_X u + B_{p,\delta}, \tag{7.8}$$

using the inequalities 2.1; see [39, Thm 3.1] and its proof. However, for singular X the problem of establishing a Harnack bound along Aubin's complex Monge-Ampère equation appears to be wide open. Still, Theorem 4.1 implies that a weak Harnack bound holds, which implies the inequality 7.8, assuming that $\delta(X) < 1$:

Proposition 7.4 *Let X be Fano variety with log terminal singularities and let $u_t \in L^\infty(X) \cap PSH(X, \omega)$ be the solution to Aubin's Monge–Ampère equation 7.6, for $t \in]0, \delta(X)[$. Then, for any given $p \in [1, \infty[$, there exists constants a_p and b_p such that the following* weak Harnack bound *holds:*

$$\left(\int_X |u_t|^p \frac{\omega_{u_t}^n}{V}\right)^{1/p} \leq a_p(n+1)\sup u_t + b_p\left(t^{-1} + (1-t)^{-1}\right), \tag{7.9}$$

where a_p depends only on p and b_p on (p, X, ω). As a consequence

$$B^{-1}\sup_X u_t - B \leq d_p(u_t, 0) \leq A\sup_X u_t + B\left(t^{-1} + (1-t)^{-1}\right)$$

for a constant A only depending on (p, n) and a constant B also depending on (X, ω).

Proof According to the inequalities 2.1 it will be enough to establish the first inequality. Applying Theorem 5.1 to $u_t - \sup_X u_t$ and using the triangle inequality yields

$$\left(\int_X |u_t|^p \frac{\omega_{u_t}^n}{V}\right)^{1/p} \leq \left|\sup_X u_t\right| + A_p(n+1)\left(\sup_X u - \mathcal{E}(u)\right) + B_p\left(t^{-1} + (1-t)^{-1}\right),$$

where $\mathcal{E}(u)$ is the functional appearing in formula 2.5. All that remains is thus to show that

$$-\mathcal{E}(u_t) \leq C. \tag{7.10}$$

To this end consider the *twisted Ding functional* defined by

$$\mathcal{D}_t(u) := -\mathcal{E}(u) - t^{-1}\log\int_X e^{-tu}dV_X \tag{7.11}$$

for a given $t \in [0, 1[$. We define $\mathcal{D}_0(u)$ as the limit of $\mathcal{D}_t(u)$ as t decreases to 0, which amounts to replacing the second term in the definition of $\mathcal{D}_t(u)$ by $\int_X u dV_X$. Note that $\mathcal{D}_t(u)$ is decreasing in t, as follows directly from Hölder's inequality. Moreover, u_t minimizes $\mathcal{D}_t(u)$ (see [12]). Hence,

$$\mathcal{D}_1(u_t) \leq \mathcal{D}_t(u_t) \leq \mathcal{D}_t(u_0) \leq C := \mathcal{D}_0(u_0)$$

By formula 7.5, this proves the bound 7.10. □

Remark 7.5 The terminilogy of Harnack bounds and weak Harnack bounds adopted here mimics the corresponding terminilogy for positive solutions and supersolutions to linear elliptic equations, where the role of u is played by $-u$ (cf. [90, Cor 10] and [90, Thm 9], respectively). However, in the present setup u does not have a fixed sign, which effects the formulation of the bounds.

There are some intruiging connections between the proof of the Harnack bound 7.7 and the weak Harnack bound in Prop 7.4. As discussed in Remark 7.2 the Harnack bound follows from bounds on the Sobolev and Poincaré constants, which in turn follow from a strictly positive uniform lower bound on the Ricci curvature of (X, ω_u). The latter bounds have been extended to complete metric spaces (X, d) using various generalized notions of Ricci curvature, defined in terms of the convexity of the entropy functional on the space of all probability measures on X, endowed with the L^2-Wasserstein metric [91, Section 30] [80, Prop 3.3]. When X is a singular Fano variety and ω is taken to be a Kähler form the positive current ω_{u_t} defines a Kähler metric on the regular locus X_{reg} of X. However, $(X_{\mathrm{reg}}, \omega_t)$ is not complete and its seems to be unknown whether the metric completion of $(X_{\mathrm{reg}}, \omega_t)$ satisfies any kind of Ricci curvature bound in the sense of metric spaces. From this point of view the key advantage of the method of proof of Prop 7.4 is that it is based on the convexity of the functional appearing in formula 3.8, which holds when $\mathrm{Ric}\, dV \geq 0$ in the general setup of singular Fano varieties. Incidently, this functional is precisely the Legendre-Fenchel tranform of the entropy functional appearing in the definition of Ricci curvature of metric spaces.

7.1 Aubin type equations in the absence of positive Ricci curvature

Given a Kähler form ω in $c_1(X)$, $F \in C^\infty(X)$ and $t > 0$ consider the following complex Monge-Ampère equations for $u_t \in \mathcal{H}(X, \omega)$:

$$\omega_{u_t}^n = e^{-tu_t} e^F \omega^n. \tag{7.12}$$

This equation has been studied extensively when $n = 1$ motivated, in particular, by Nirenberg's problem of prescribing the scalar curvature of conformal metrics on the two-sphere and the Chern-Simons Higgs model (see, for example, the blow-up analysis in [47, 67]).

When F is the Ricci potential of ω, $F = \rho_\omega$, the Eq. 7.12 is precisely Aubin's Monge–Ampère equation 7.6. However, in general, unless F is constant, ω_{u_t} does not have positive Ricci curvature. As a consequence, the inequalities in Proposition 7.1, do not apply. But Theorem 4.1 directly yields

$$\left(\int_X (\sup u_t - u_t)^p \frac{\omega_{u_t}^n}{V} \right)^{1/p} \leq e^{2\|F - \rho_\omega\|_{L^\infty}} \left(A \int_X (\sup u_t - u_t) \frac{\omega_{u_t}^n}{V} + B \right),$$

where $A = A_p$ and $B = B_p \left(\gamma^{-1} + (1-\gamma)^{-1}\right)$ for the constants A_p and B_p appearing in Theorem 4.1. As a consequence, if u_t minimizes the corresponding twisted Ding functional

$\mathcal{D}_t$ (defined by replacing dV_X in formula 7.11 with $e^F\omega^n$) then the proof of Proposition 7.4 reveals that u_t satisfies a weak Harnack bound of the form 7.9, obtained by multiplying a_p and b_p with $e^{2\|F-\rho_\omega\|_{L^\infty}}$. It should be stressed that the minimizing property in question is not automatic, unless F is constant (but, by [5], any minimizer u_t satisfies the Eq. 7.12, under the normalization condition $\int e^{-tu_t}e^F\omega^n = 1$). When $n = 1$ the stronger Harnack bound 7.7 holds, with constants A and B depending on $\|F\|_{L^\infty}$. This follows from the local results in [84] , confirming a conjecture in [25]. The proof in [84] uses the Alexandrov–Bol isoperimetric inequality for surfaces.

8 Comparison with Duistermaat–Heckman type measures and log-concavity

8.1 Test configurations and K-stability

Let L be an ample line over a compact complex manifold X. Recall that a *test configuration* $(\mathcal{X}, \mathcal{L})$ for (X, L) (as appearing the definition of K-stability, discussed below) may be defined as a $\mathbb{C}^*$-equivariant embedding

$$(X \times \mathbb{C}^*, L) \hookrightarrow (\mathcal{X}, \mathcal{L})$$

of the polarized trivial fibration $(X \times \mathbb{C}^*, L)$ over $\mathbb{C}^*$ into a normal variety $\mathcal{X}$, fibered over $\mathbb{C}$, endowed with a relatively ample $\mathbb{Q}$-line bundle $\mathcal{L}$ and a $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ covering the standard $\mathbb{C}^*$-action on $\mathbb{C}$. In particular, there is a $\mathbb{C}^*$-action on the scheme defined by the central fiber $(\mathcal{X}_0, \mathcal{L}_0)$. Given $\omega \in c_1(L)$ a test configuration induces, by [78], a psh geodesic ray u_t (emanating from 0) which is in $\mathcal{E}^p(X, \omega)$ and thus defines a d_p-geodesic, for any p. Indeed, u_t may be defined as an envelope (as in formula 2.6) over all v_t such that the corresponding ω-psh function $V(x, \tau)$ on $X \times \mathbb{C}^*_\tau$, where $\tau := e^{-t}$, has the property that $dd^cV + \omega$ is the restriction to $X \times \mathbb{C}^*$ of the curvature form of a locally bounded and positively curved metric on $\mathcal{L} \to \mathcal{X}$. Set

$$\dot{u} := \frac{du_t}{dt}|_{t=0}, \quad \mu := \dot{u}_*(\frac{\omega^n}{V}),$$

By [36], $\dot{u} \in L^\infty(X)$, which ensures that the push-forward $\dot{u}_*(\frac{\omega^n}{V})$ is well-defined. As in [21], the d_p-speed of the geodesic u_t may be expressed as

$$\|\dot{u}\|_p := d_p(u_1, 0) = \left(\int_X |\dot{u}|^p \frac{\omega^n}{V}\right)^{1/p} = \left(\int_{\mathbb{R}} |t|^p d\mu\right)^{1/p}. \tag{8.1}$$

By the main result of [61]—proving a conjecture in [93] - the probability measure μ on $\mathbb{R}$ can be expressed as the following weak limit of weight measures:

$$\mu = \lim_{k\to\infty} \frac{1}{N_k}\sum_{i=1}^{N_k} \delta_{\lambda_i^{(k)}/k}$$

where the real numbers $\lambda_1^{(k)}, \dots, \lambda_{N_k}^{(k)}$ are the weights of the $\mathbb{C}^*$-action on the complex vector space $H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k})$. This limit is called the *Duistermaat–Heckman measure* of $(\mathcal{X}, \mathcal{L})$ in [24]. In the terminology of [24, 49] $\|\dot{u}\|_p$ thus coincides with the L^p-norm $\|(\mathcal{X}, \mathcal{L})\|_{L^p}$ of the test configuration $(\mathcal{X}, \mathcal{L})$.

When the central fiber of $\mathcal{X}_0$ is reduced and irreducible it follows from the main result of [77] that the probability measure μ on $\mathbb{R}$ is log-concave. More precisely, it is shown in [77] that μ is the push-forward to $\mathbb{R}$ of the uniform measure on a convex body in $\mathbb{R}^n$ under a linear map. Log-concavity then follows directly from the classical Brunn–Minkowski theorem. In particular, by the reverse Hölder inequality 4.1 for log-concave measures, there exists, for any given $p \in [1, \infty[$, a constant C_p (only depending on p) such that

$$\|(\mathcal{X}, \mathcal{L})\|_{L^p} \leq C_p \|(\mathcal{X}, \mathcal{L})\|_{L^1} \tag{8.2}$$

This inequality does not seem to have been observed before in this context. But as pointed out the author by Sébastien Boucksom and Mattias Jonsson, it is closely related to the inequality in [23, Lemma 3.14]. Indeed, the latter inequality yields the strong reverse Hölder bound,

$$\|(\mathcal{X}, \mathcal{L})\|_{L^p} \leq (n+1) \|(\mathcal{X}, \mathcal{L})\|_{L^1} \tag{8.3}$$

for all $p \geq 1$ if μ is supported in $[0, \infty[$. In particular, the corresponding constant is uniformly bounded with respect to p. But in contrast to the inequality 8.2 the bound depends on n. Moreover, similar inequalities also appear in [97], as pointed out the the author by Tamas Darvas. The proof of the bound 8.3 exploits that the nth root of $\mu(]t, \infty[)$ is concave, as a consequence of the Brunn–Minkowski theorem.

Remark 8.1 Note that the constant $(n+1)$ in the bound 8.3 for $(\mathcal{X}, \mathcal{L})$ also appears in the reverse Hölder bound 7.4 for the destabilizing geodesic ray v_t that is weakly asymptotic to Aubin's continuity path. This is in line with the discussion on the partial C^0-estimate, following Corollary 1.3. Indeed, if the assumptions on the corresponding curve G_t in $\mathrm{GL}(N_k, \mathbb{C})$ would apply, then the bound 7.4 for v_t would follow from the bound 8.3 applied to the special test-configuration associated to G_t.

It should be stressed that the bound 8.2 does not hold for *all* test configurations. Indeed, by the example in [24, Prop 8.5], the inequality fails for $p > n/(n-1)$ when $\mathcal{X}$ is taken to be the deformation to the normal cone of a given point x on X, i.e. $p : \mathcal{X} \to X \times \mathbb{C}$ is the blow-up of $\{x\} \times \{0\}$ in $X \times \mathbb{C}$ and $\mathcal{L}_\epsilon := p^*L - \epsilon E$, where E denotes the exceptional divisor over p and ϵ is a given sufficiently small positive rational number. In particular, in this example (where$\mathcal{X}_0$ is reduced, but has two components) μ can not be log-concave.

8.1.1 Relations to K-stability

Recall that, in the context of the Yau-Tian-Donaldson conjecture [15, 34, 43], a Fano manifold (X, K_X^*) is said to be *K-stable* if the Donaldson-Futaki invariant $\mathrm{DF}(\mathcal{X}, \mathcal{L})$ is strictly positive for all non-trivial test configurations and *uniformly K-stable* if there exists a constant $\epsilon > 0$ such thatl

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \epsilon \|(\mathcal{X}, \mathcal{L})\|_1 ,$$

where $\|(\mathcal{X}, \mathcal{L})\|_p$ is defined, in terms of the Duistermaat–Heckman μ, discussed above, as

$$\|(\mathcal{X}, \mathcal{L})\|_p := \left(\int_{\mathbb{R}} |t-c|^p \mu \right)^{1/p}, \quad c := \int_{\mathbb{R}} \mu.$$

By [71], (X, K_X^*) is, in fact, K-stable iff it is uniformly K-stable. Moreover, by [69], one may when testing (uniform) K-stability restrict to *special* test konfigurations $(\mathcal{X}, \mathcal{L})$, i.e. such that $\mathcal{X}_0$ has log terminal singularities (and, in particular, is reduced and irreducible). But in this case μ is, as pointed out above, log-concave and, as a consequence, so is the translation of μ by c. Hence, as explained above, for any $p \in [1, \infty[$ there exists a constant C_p such that

$$\|(\mathcal{X}, \mathcal{L})\|_p \leq C_p \|(\mathcal{X}, \mathcal{L})\|_1$$

for all special test configurations. All in all, this means that

$$(X, K_X^*) \text{ is K-stable } \iff \forall p \in [1, \infty[\, \exists \epsilon_p \in]0, \infty[: \text{ DF } (\mathcal{X}, \mathcal{L}) \geq \epsilon_p \|(\mathcal{X}, \mathcal{L})\|_p$$

for all non-trivial special test configurations

8.2 Failure of log-concavity for general geodesic segments

Given $u \in \mathcal{H}(X, \omega)$, let u_t be the psh geodesic coinciding with 0 and u at $t = 0$ and $t = 1$, respectively. Then

$$d_p(u, 0) = \left(\int_X |\dot{u}|^p \frac{\omega^n}{V} \right)^{1/p} = \left(\int_{\mathbb{R}} |t|^p d\mu \right)^{1/p}$$

using that $\dot{u} \in L^\infty(X)$ [21]. However, in general, μ is *not* log-concave. Indeed, assume in order to get a contraction that μ is log-concave. Then the reverse Hölder inequality 4.1 for log-concave measures implies that

$$d_p(u, 0) \leq C_p d_1(u, 0)$$

for a C_p only depending on p. But such a reverse Hölder inequality does not hold, in general, as stressed in the introduction of the paper.

9 Effective openness

The non-effective openness result 1.8 may be reformulated as

$$\lim_{\gamma \to c_u} Z_u(\gamma) = \infty, \quad Z_u(\gamma) := \int_X e^{-\gamma u} dV \tag{9.1}$$

We will prove the following reformulation of the effective openness in Theorem 1.4:

Theorem 9.1 *Let X be a Fano variety. Assume that $u \in PSH(X, \omega)$ satisfies $c_u < 1$ and* $\sup u \leq 0$. *Then there exist universal constants A and B such that*

$$\frac{d \log Z_u(\gamma)}{d\gamma} \geq \frac{1}{A} \frac{1}{(c_u - \gamma)} - B \left(\gamma^{-2} + (1 - c_u)^{-2} \right),$$

for any positive γ such that $Z_u(\gamma) < \infty$. In other words, setting $C := AB$,

$$(c_u - \gamma) \geq \frac{1}{A \frac{d \log Z_u(\gamma)}{d\gamma} + C \left(\gamma^{-2} + (1 - c_u)^{-2} \right)}.$$

The constant A can be taken arbitrarily close to 16.

The key ingredient in the proof is the following effective refinement of the moment inequalities in Theorem 4.1 (in the case $p = 2$):

Lemma 9.2 *There exists a universal constant B such that*

$$\left(\frac{d^2 \log Z_u(\gamma)}{d^2\gamma}\right)^{1/2} \leq 2C_2 \frac{d \log Z_u(\gamma)}{d\gamma} + B\left(\gamma^{-1} + (1-\gamma)^{-1}\right)$$

where $C_2 \leq 2$.

Proof By Thm 3.1 (and its proof) the function $Z_u(\gamma)$ on $]0, c_u[$ admits the following representation, where $\Gamma(s)$ denotes the classical Gamma-function:

$$Z_u(\gamma) = G_0(\gamma) \int_{\mathbb{R}} e^{-t\gamma} \nu_0, \quad G_0(\gamma) := \frac{\gamma}{\Gamma(1-\gamma)/2}, \tag{9.2}$$

for a log-concave measure ν_0 on $\mathbb{R}$ (depending on u). Moreover,

$$\int_{\mathbb{R}} |t| d\nu_\gamma \leq \frac{d \log Z_u(\gamma)}{d\gamma} + \gamma^{-1} + C_\gamma, \quad C_\gamma := \frac{\int_0^\infty e^{-r^2} |\log r^2| r^{-2\gamma} r dr}{\int_0^\infty e^{-r^2} r^{-2\gamma} r dr} \leq C(1-\gamma)^{-1} \tag{9.3}$$

where ν_γ denotes the log-concave probability measure $e^{-t\gamma}\nu_0 / \int_{\mathbb{R}} e^{-t\gamma}\nu_0$ on $\mathbb{R}$ and C is a universal constant (that can be estimated using [60, Thm 1]). Now, applying formula 9.2 gives (using formula 4.2):

$$\frac{d^2 \log Z_u}{d^2\gamma} = \frac{d^2 \log G_0}{d^2\gamma} + \left\langle \left(t - \langle t \rangle_\gamma\right)^2 \right\rangle_\gamma$$

where $\langle \cdot \rangle_\gamma$ denotes integration wrt the probability measure ν_γ on $\mathbb{R}$. Denote by C_2 the best constant in the following inequality

$$\left(\int_{\mathbb{R}} |t|^2 \, \nu\right)^{1/2} \leq C_2 \left(\int_{\mathbb{R}} |t| \, \nu\right)$$

for centered log-concave probability measures ν. By the recent result [75, Thm 1.2] on effective Kahane–Khinchin inequalities the following explicit bound holds:

$$C_2 \leq 2$$

(see also [50] for related results). Since ν_γ is log-concave it thus follows that

$$\left\langle \left(t - \langle t \rangle_\gamma\right)^2 \right\rangle_\gamma \leq C_2 \left\langle \left(t - \langle t \rangle_\gamma\right) \right\rangle_\gamma \leq 2C_2 \left\langle |t| \right\rangle_\gamma .$$

The proof is thus concluded by applying the inequality 9.3 and noting that $\frac{d^2 \log G_0}{d^2\gamma} \leq 0$ (since both $\log \gamma$ and $-\log \Gamma(1-\gamma)$ are convex). Alternatively, by including $\frac{d^2 \log G_0}{d^2\gamma}$ the value of the constant B could be decreased slightly. □

9.0.1. Proof of Theorem 9.1

Set $g(\gamma) := d \log Z_u(\gamma)/d\gamma$. First observe that, regardless of the assumption that $u \leq 0$,

$$g(\gamma) \to \infty, \quad \gamma \to c_u. \tag{9.4}$$

Indeed, $g(\gamma)$ is increasing, since $\log Z(\gamma)$ is convex. Hence, if the divergence in question does not hold, then $Z(\gamma)$ is bounded as $\gamma \to c_u$, which contradicts that $Z(\gamma) \to \infty$.

By the previous lemma

$$\left(\frac{dg(\gamma)}{d\gamma}\right)^{1/2} \leq 2C_2 g(\gamma) + B_\gamma, \;\; B_\gamma := B\left(\gamma^{-1} + (1-\gamma)^{-1}\right)$$

Now assume that $\gamma \in [\gamma_0, c_u[$. Then

$$B_\gamma \leq b_0 := B\left(\gamma_0^{-1} + (1-c_u)^{-1}\right).$$

Thus, if we assume that $b_0 \leq g$, then

$$\left(\frac{dg(\gamma)}{d\gamma}\right)^{1/2} \leq (2C_2+1)g(\gamma),$$

i.e.

$$\frac{dg(\gamma)}{d\gamma} \leq Ag(\gamma)^2, \;\; A := (2C_2+1)^2$$

Setting $t := c_u - \gamma$ this means that

$$\frac{d(g^{-1})}{dt} \leq A$$

when $t \in]0, c_u - \gamma_0[$. Since $g^{-1} \to 0$ as $t \to 0$ (by 9.4) it thus follows that $g \geq (At)^{-1}$ when $t \in]0, c_u - \gamma_0[$, under the assumption that $g \geq b_0$. Finally, replacing a general given u, satisfying $u \leq 0$, with $u - b_0$ gives $g_{u-b_0} = g_u + b_0 \geq b_0$. Hence, applying the previous bound to $u - b_0$ gives

$$g_u + b_0 \geq (At)^{-1}, \quad t \in]0, c_u - \gamma_0[$$

In other words, for $\gamma \in [\gamma_0, c_u[$

$$g_u(\gamma) \geq \frac{1}{A}\frac{1}{(c_u-\gamma)} - b_0, \;\; \gamma \in [\gamma_0, c_u[$$

In particular, taking $\gamma = \gamma_0$ concludes the proof. In fact, A can be taken arbitrarily close to $(2C_2)^2$ (at the expense of increasing B) if one applies the previous argument to $u - \epsilon^{-1}$ for ϵ sufficiently small. In particular, since $C_2 \leq 2$ this shows that A can be taken arbitrarily close to 16.

9.1 Proof of Cor 1.2

We next deduce the following reformulation of Cor 1.2:

Corollary 9.3 *Let X be a Fano variety assume that $u \in PSH(X,\omega)$ satisfies $u \leq 0$ and that $\int_X dV = 1$. If $Z_u(\gamma) < \infty$ and $Z_u(1-\delta) = \infty$ for some $\delta > 0$. Then there exist universal constants A and b such that*

$$Z_u(\gamma) \geq \frac{e^{-b(\gamma^{-1}+\gamma\delta^{-2})}}{(c_u-\gamma)^{1/A}},$$

when $\gamma \in]0, c_u[$.

Proof Integrating $d\log Z_u(\gamma)/d\gamma$ over $[\gamma/2,\gamma]$ and using the bound in the previous theorem gives

$$\log Z_u(\gamma)-\log Z_u(\frac{\gamma}{2})\geq\frac{1}{A}\log\frac{1}{(c_u-\gamma)}+\frac{1}{A}\log(c_u-\gamma/2)-B\left(\gamma^{-1}+\frac{1}{2}\delta^{-2}\gamma\right).$$

Since $u\leq 0$ we have $Z_u(\frac{\gamma}{2})\geq\int_X dV=1$. Moreover, $c_u-\gamma/2\geq\gamma/2$, using that $\gamma<c_u$. Hence,

$$\log Z_u(\gamma)\geq\frac{1}{A}\log\frac{1}{(c_u-\gamma)}+\frac{\log(\gamma/2)}{A}-B\left(\gamma^{-1}+\frac{1}{2}\delta^{-2}\gamma\right).$$

Finally, since $\gamma\leq 1$ we have $\frac{\log(\gamma/2)}{A}-B\gamma^{-1}\geq -b\gamma^{-1}$ for b sufficiently large (depending on A and B). □

10 Universal bounds on Archimedean zeta functions and K-unstable Fanos

The logarithmic derivative of $Z_u(\gamma)$, appearing in Theorem 9.1, does not seem to have appeared previously in the context of Demailly–Kollar's openness conjecture. But its lower bound can be motivated as follows. Consider the case when u corresponds to an anti-canonical effective $\mathbb{Q}$-divisor D on X. This means that kD is cut out by some section $f_k\in H^0(X,K_X^{*\otimes k})$, for a positive integer k and

$$u=k^{-1}\log\|f_k\|^2,\quad f_k\in H^0(X,K_X^{*\otimes k})\tag{10.1}$$

where $\|\cdot\|$ denotes the metric on $K_X^{*\otimes k}$, corresponding to dV. Hence,

$$Z_u(\gamma)=\int_X\|f_k\|^{-2\gamma/k}\,dV.\tag{10.2}$$

By scaling f_k we may assume that $u\leq 0$, i.e. that

$$\sup_X\|f_k\|\leq 1.\tag{10.3}$$

The complex singularity c of u coincides with the log-canonical threshold of the divisor D:

$$c_u=\text{lct (D)}$$

(the assumption that $c_u<1$ equivalently means that the divisor D is *non-log canonical* in the standard sense of birational algebraic geometry [65]). It is well-known that $Z_u(\gamma)$ extends to a meromorphic function on $\mathbb{C}$. In fact, $Z_u(\gamma)$ is an instance of the *geometric Igusa zeta functions* for local fields studied in [26, 27] (after making the change of variables $s=-\gamma/k$. More precisely, the present setup concerns the field $\mathbb{C}$ endowed with its standard Archimedean absolute value. All the poles of $Z_u(\gamma)$ are located at negative rational numbers and $-c_u$ is the largest pole of $Z_u(\gamma)$ (as follows from resolution of singularities; see [62, Thm 5.4.1] and [27, Prop 4.2.4]). In particular, there exists a positive integer m (the order of the largest pole) such that, for γ close to c_u:

$$Z_u(\gamma)=\frac{F_u(\gamma)}{(c_u-\gamma)^m},$$

where $F_u(\gamma)$ is holomorphic function which is non-vanishing for $\gamma = c_u$. Differentiating thus gives

$$\frac{d\log Z_u(\gamma)}{d\gamma} = m\frac{1}{(c_u-\gamma)} - G_u(\gamma)$$

for a local holomorphic function $G_u(\gamma)$. Since $m \geq 1$ this means that for any fixed divisor the lower bound 1.10 holds with $A = 1$, but with B depending on $G_u(\gamma)$ and thus on u (i.e. on the divisor D). From this point of view, the main point of the lower bound 1.10 is thus the universality.

10.1 Application to K-unstable Fano varieties

Given a Fano variety X, set

$$N_k := \dim H^0(X, K_X^{*\otimes k}),$$

which tends to infinity as k is increased. Let $\mathcal{D}_k$ be the anti-canonical effective $\mathbb{Q}$-divisor on X^{N_k} whose support consists of all configurations $(x_1, \ldots, x_{N_k})$ of N_k points on X,which are in "bad position" with respect to $H^0(X, K_X^{*\otimes k})$:

$$\text{Supp}\mathcal{D}_k := \left\{(x_1, \ldots, x_{N_k}) : \exists s_k \in H(X, K_X^{*\otimes k}) : \ s_k(x_i) = 0, \ \ \forall i, \ \ s_k \not\equiv 0\right\}.$$

Denote by $Z_k(\gamma)$ the corresponding Archimedean zeta function 10.2. In the context of the probabilistic approach to the construction of Kähler-Einstein metrics, it is conjectured in [4] that X admits a unique Kähler-Einstein metric iff $\text{lct}(\mathcal{D}_k) \geq 1+\epsilon$ for k sufficiently large, for some $\epsilon > 0$. The "if direction" was established in [52]. More precisely, combining results in [52, 82] yields

$$\limsup_{k\to\infty} \text{lct}(\mathcal{D}_k) \leq \delta(X)$$

(see [7] for a direct analytic proof). In particular, if X is *K-unstable* (i.e. not K-semistable)—which by [22, Thm B] is equivalent to $\delta(X) < 1$—then

$$\limsup_{k\to\infty} \text{lct}(\mathcal{D}_k) < 1.$$

As a consequence, if X is K-unstable, the lower bound 1.10 applies to the Archimedean zeta function $Z_k(\gamma)$ corresponding to the divisor $\mathcal{D}_k$ on X^{N_k} :

$$\frac{d\log Z_k(\gamma)}{d\gamma} \geq \frac{1}{A}\frac{1}{(\text{lct}\,(\mathcal{D}_k)-\gamma)} - B\left(\gamma^{-2} + (1-\text{lct}\,(\mathcal{D}_k))^{-2}\right), \quad \gamma \in]0, \text{lct}\,(\mathcal{D}_k)[\quad (10.4)$$

for k sufficiently large. This yields a quantitative universal lower bound on the rate of the blow-up of the logarithmic derivative of $Z_k(\gamma)$ as γ increases towards lct $(\mathcal{D}_k)$.

Concluding speculations

Under the hypothesis that the meromorphic function $Z_k(\gamma)$ on $\mathbb{C}$ is zero-free and holomorphic in a k-independent neighborhood of $[0, \text{lct}\,(\mathcal{D}_k)[$ in $\mathbb{C}$, it follows from results in [3] that $\text{lct}(\mathcal{D}_k)$ converges towards $\delta(X)$ and

$$\lim_{k\to\infty} N_k^{-1}\frac{d\log Z_k(\gamma)}{d\gamma} = E\left(\frac{\omega_{u_\gamma}^n}{V}\right), \ \ \gamma < \delta(X),$$

where $E(\mu)$ denotes the *pluricomplex energy* of a probability measure μ on X, relative to ω (using the notation in [3]) and u_γ denotes the unique potential of ω_t solving Aubin's continuity equation 1.4 for $t = \gamma$, normalized so that $\sup_X u_\gamma = 0$.[1] Since $E(\omega^n_{u_\gamma}/V)$ is comparable to $d_1(u_\gamma, 0)$ it follows from formula 6.2 that

$$\lim_{\gamma\to\delta(X)} E(\omega^n_{u_\gamma}/V) = \infty.$$

Comparing with the lower bound 10.4 leads one to wonder if the blow-up rate of $E(\omega^n_{u_\gamma}/V)$ can also be quantified? However, no such information can be deduced from the lower bound 10.4, since the bound is suppressed when it is divided by N_k and $k \to \infty$. But one can get an idea of what to expect by looking at the "local" setup where X is replaced with the unit-ball B_1 in $\mathbb{C}^n$ and the reference form ω is assumed to vanish identically on B_1, considered in [10, 59]. Then u_γ is taken to be a continuous plurisubharmonic function solving

$$(dd^c u_\gamma)^n = \frac{e^{-\gamma u} dz \wedge d\bar{z}}{\int_B e^{-\gamma u} dz \wedge d\bar{z}} \text{ on } B_1, \quad u = 0 \text{ on } \partial B_1 = 0. \tag{10.5}$$

Compared to Eq. 6.1 we have set $V = 1$, which in this local setup, can always be arranged by rescaling γ. If we impose the condition that u be rotationally invariant the boundary condition on u is equivalent to the condition that $\sup_{B_1} u = 0$, which is inline with the condition on u_γ employed in the global setup of Fano manifolds. Under this symmetry condition there is a unique solution u_γ to Eq. 10.5 when $\gamma < n+1$ and the explicit computations in [10, Section 3.3] reveal that there exists a constant b_n such that

$$E(\omega^n_{u_\gamma}/V) = \frac{n}{n+1} \log\left(\frac{1}{\delta(B_1) - \gamma}\right) + b_n, \quad \gamma \to \delta(B_1)(:= n+1).$$

Accordingly, it seems natural to ask if for any K-unstable Fano manifold X there exists a positive constant a such that, as γ is increased towards $\delta(X)$,

$$E(\omega^n_{u_\gamma}/V) = a \log\left(\frac{1}{\delta(X) - \gamma}\right) + O(1)?$$

However, it may be that a not only depends on X but also on the fixed Kähler form ω in $c_1(X)$.

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[1] In the case $n = 1$ the previous asymptotics were established unconditionally in [3].

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